Hop Constrained Connected Facility Location

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December 23, 2009

Abstract

The Connected Facility Location (ConFL) problem combines facility location and Steiner trees: given a set of customers, a set of potential facility locations and some inter-connection nodes, ConFL searches for the minimum-cost way of assigning each customer to exactly one open facility, and connecting the open facilities via a Steiner tree. The costs needed for building the Steiner tree, facility opening costs and the assignment costs need to be minimized. In some telecommunications applications, the number of nodes between the root and open facilities is limited. This leads to the Hop Constrained Facility Location Problem. This is a new problem not studied in the literature yet. We develop 17 mixed integer programming models for this problem. Some of these models are extensions from corresponding models for the ConFL, some extend ideas for related problems like the Minimum Spanning or Steiner Tree problem with hop constraints. In branch-and-bound frameworks the quality of linear programming lower bounds of these formulations is of particular interest. We compare the relative quality of these relaxations and provide a hierarchy of the corresponding models. We also show how the Hop Constrained ConFL can be modelled as ConFL or the Steiner Tree problem on three different layered graphs. Finally we prove, that a disaggregation of variables indicating facilities or customers in the layered graphs does not improve the quality of the lower bounds.

Keywords: Hop constrained Steiner trees, Connected Facility Location, Mixed Integer Programming Models, LP-relaxations.

1 Introduction

In the field of designing the last mile of telecommunication networks a wide range of combinatorial optimization problems occur. In some applications, Fiber-to-the-Curb networks are built such that the central office lies within a 2-connected backbone network and the number of nodes between the central office and the cabinets is limited. In Gollowitzer and Ljubić [2009] we have shown that the Fiber-to-the-Curb strategy is modelled by the Connected Facility Location problem (ConFL): Fiber optic cables run to a cabinet serving a neighborhood. End users connect to this cabinet using the existing copper connections. Expensive switching devices are installed in these cabinets. The problem is to minimize the costs by determining positions of cabinets, deciding which customers to connect to them, and how to reconnect cabinets among each other and to the backbone.

In such simply connected graphs reliability against single arc failures is not provided. Economic arguments do not allow the installation of 2-connected last mile networks. Therefore, the reliability of end-user connections is maintained by limiting the number of nodes between them and the 2-connected backbone network. We model these reliability constraints within the Fiber-to-the-Curb strategy by generalizing the ConFL to the Hop Constrained ConFL.

1.1 Problem Definition

In Gollowitzer and Ljubić [2009] we have discussed a number of issues regarding the exact definition of the Connected Facility Location problem. We assume that a root node (i.e., a central office) is given in advance and needs to be included in any feasible solution.

*Supported by the Hertha-Firnberg Fellowship of the Austrian Science Foundation (FWF)

Definition 1 (rooted ConFL)**.** We are given an undirected graph (V, E) with edge costs $c_e \geq 0, e \in E$, facility opening costs $f_i \geq 0, i \in F$, a disjoint partition $\{S, R\}$ of V with $R \subset V$ being the set of *customers*, $S \subset V$ the set of possible *Steiner nodes*, $F \subseteq S$ the set of *facilities*, and the *root* node $r \in F$. Find a subset of open facilities such that:

- each customer is assigned to the closest open facility,
- a Steiner tree connects all open facilities, and
- the sum of assignment, facility opening and Steiner tree costs is minimized.

Any optimal solution for the ConFL, as has just been described, is a tree. In this tree, the number of edges on the path between the root node and an open facility is termed *Hops*. Based on this definition the *Hop Constrained Connected Facility Location Problem* is:

Definition 2 (HC ConFL)**.** Given an instance of the rooted ConFL, find a solution that is valid for ConFL and in which there are at most H hops between the root and any open facility.

Observation 1. *Using the transformation given in Gollowitzer and Ljubić [2009], any HC ConFL instance, in which $S \cap R \neq \emptyset$, can be transformed into an equivalent one such that $\{S, R\}$ is a proper partition of V.*

HC ConFL is not in APX, i.e. it not possible to have polynomial time heuristics that guarantee a constant approximation ratio. This result can be obtained by applying an error-preserving reduction from SET COVER. (see, e.g. Manyem and Stallmann [1996]). Manyem [2009] shows that the related hop-constrained Steiner tree problem (HCSTP) is not in APX, even if the edge weights satisfy the triangle inequality. Obviously, HCSTP is a special case of HC ConFL, in which every facility supplies exactly one customer. Therefore, this non-approximability results apply to HC ConFL as well.
The remainder of this paper is organized as follows: The following section will provide a literature review on some problems related to HC ConFL. In Section 3 we propose 17 mixed integer programming models for HC ConFL. In Section 4 we provide a full hierarchy of the models based on the theoretical comparison of the quality of their lower bounds. Finally, we give some concluding remarks.

2 Literature Review

The Hop Constrained Connected Facility Location Problem has not yet been studied in the literature. However, it is closely related to two other problems. First, the Connected Facility Location problem. Second, the Steiner tree problem with hop constraints.

Connected Facility Location The Connected Facility Location problem has attracted stronger interest lately. Both, the computer science and operations research community work on this topic. A number of authors have developed approximation algorithms of varying quality (see, e.g. Eisenbrand et al. [2008], Karger and Minkoff [2000]). Heuristics and exact methods are described in Gollowitzer and Ljubić [2009], Ljubić [2007], Tomazic and Ljubić [2008]. A more general version of ConFL can be found in Bardossy and Raghavan [2009]. For a more detailed literature review for ConFL we refer the reader to Gollowitzer and Ljubić [2009].

The Steiner tree problem with hop constraints (HCSTP) There has been intensive research on the Minimum Spanning Tree problem with hop constraints (HCMST), a special case of the HCSTP where each node in the graph is a terminal. A recent survey for the HCMST can be found in Dahl et al. [2006].
Much less has been said about the Steiner tree problem with hop constraints: The problem was first mentioned by Gouveia [1998], who develops a strengthened version of a multi-commodity flow model for the Minimum Spanning and Steiner tree problem. The LP lower bounds of this model are equal to the ones from a Lagrangean relaxation approach of a weaker MIP model introduced in Gouveia [1996]. Results for instances with up to 100 nodes and 350 edges are presented.
Voß [1999] presents MIP formulations based on Miller-Tucker-Zemlin subtour elimination constraints.

The formulation is then strengthened by disaggregation of the variables indicating used arcs. The author develops a simple heuristic to find starting solutions and improves these with an exchange procedure based on tabu search. Numerical results are given for instances with up to 2500 nodes and 65000 edges.
Gouveia [1999] gives a survey of hop-indexed tree and flow formulations for the hop constrained spanning and Steiner tree problem.
Costa, Cordeau, and Laporte [2008] give a comparison of three heuristic methods for a generalization of the HCSTP, namely the Steiner tree problems with revenues, budget and hop constraints (STPRBH). The considered methods comprise a greedy algorithm, a destroy-and-repair method and a tabu search approach. Computational results are reported for instances with up to 500 nodes and 12500 edges. In Costa et al. [2009] the authors introduce two new models for the STPRBH. They are both based on the generalized sub-tour elimination constraints and a set of hop constraints of exponential size. The authors provide a theoretical and computational comparison with the two models based on Miller-Tucker-Zemlin constraints presented in Gouveia [1999], Voß [1999].

3 (M)ILP Formulations for HC ConFL

Problem formulations on directed graphs often give better lower bounds than their undirected equivalents (see, e.g., Magnanti and Wolsey [1995]). By replacing edges between nodes in S by two directed arcs of the same cost and each edge between a facility and a customer by an arc directed from the facility towards the customer, undirected instances can be transformed into directed ones [Gollowitzer and Ljubić, 2009]. In the remainder of this paper we will focus on the Hop Constrained Connected Facility Location problem on directed graphs defined as follows:

Definition 3 (HC ConFL on directed graphs)**.** We are given a directed graph (V, A) with edge costs $c_{ij} \geq 0, ij \in A$, facility opening costs $f_i \geq 0, i \in F$ and a disjoint partition $\{S, R\}$ of V with $R \subset V$ being the set of customers, $S \subset V$ the set of possible Steiner tree nodes, $F \subset S$ the set of facilities, and the root node $r \in F$. Find a subset of open facilities such that

- each customer is assigned to exactly one open facility,
- a *Steiner arborescence* rooted in r connects all open facilities,
- the cost defined as the sum of assignment, facility opening and Steiner arborescence cost, is minimized and
- there are *at most H hops* between the root and any open facility.

To model the problem, we will use the following binary variables:

$$x_{ij} = \begin{cases} 1, & \text{if } ij \text{ belongs to the solution} \\ 0, & \text{otherwise} \end{cases} \quad \forall ij \in A \qquad z_i = \begin{cases} 1, & \text{if } i \text{ is open} \\ 0, & \text{otherwise} \end{cases} \quad \forall i \in F$$

We will use the following notation: $A_R = \{ij \in A \mid i \in F, j \in R\}$, $A_S = \{ij \in A \mid i, j \in S\}$. We will refer to A_R as *assignment arcs* and to A_S as *core arcs*. Furthermore, for any $W \subset V$ we denote by $\delta^-(W) = \{ij \in A \mid i \notin W, j \in W\}$, $\delta^+(W) = \{ij \in A \mid i \in W, j \notin W\}$ and $x(D) = \sum_{ij \in D} x_{ij}$, for every $D \subseteq A$. The examples described in the following sections use the following symbols: $\blacksquare$ represents the root node, $\circ$ represents a Steiner node. $\square^l$ represents a facility with label l. $\star$ represents a customer. In these examples the default arc values, facility opening and assignment costs are all set to one. Costs different from one are displayed next to the respective arc / node. The core network is presented as undirected graph.

3.1 Cut-Based Formulations

Two types of cut set formulations for the HCSTP and HCMST can be found in the literature: path-based and jump-based, respectively. The earlier have been mentioned by Costa et al. [2009], the latter are a development of Dahl et al. [2006].

3.1.1 Cut Set Formulations with Path Constraints

Cut Set Formulation based on the one in Gupta et al. [2001] An undirected cut set ILP formulation for ConFL was introduced in Gupta et al. [2001]. By adding an exponential number of constraints that limit the number of hops we can generalize it to model HC ConFL:
Let $P = \{(i_1, j_1), \ldots, (i_l, j_l)\}$ with $i_1 = r$ and $j_{k-1} = i_k,\ k = 2 \ldots l$ denote a path originating at the root node with l arcs. For a given number l, let $\mathcal{P}_l$ be the set of all such paths P consisting of l arcs.
Then we can formulate HC ConFL as follows:

$$(CUT_R^P) \quad \min \sum_{ij \in A} c_{ij} x_{ij} + \sum_{i \in F} f_i z_i$$

$$\text{s.t.} \quad \sum_{uv \in \delta^-(U)} x_{uv} + \sum_{jk \in A_R : j \notin U} x_{jk} \geq 1 \qquad \forall U \subseteq S \setminus \{r\}, U \cap F \neq \emptyset,\ \forall k \in R \tag{1}$$

$$\sum_{uv \in P} (x_{uv} + x_{vu}) \leq H \qquad \forall P \in \mathcal{P}_{H+1}, P \subseteq A_S \tag{2}$$

$$\sum_{jk \in A_R} x_{jk} = 1 \qquad \forall k \in R \tag{3}$$

$$x_{jk} \leq z_j \qquad \forall jk \in A_R \tag{4}$$

$$z_r = 1 \tag{5}$$

$$x_{ij} \in \{0,1\} \quad \forall ij \in A_S \tag{6}$$

$$x_{jk} \in [0,1] \quad \forall jk \in A_R \tag{7}$$

$$z_i \in \{0,1\} \quad \forall i \in F \tag{8}$$

The objective comprises the cost for the Steiner arborescence ($\sum_{ij \in A_S} c_{ij} x_{ij}$), the cost to connect customers to facilities (that we also refer to as *assignment cost*, i.e. $\sum_{ij \in A_R} c_{ij} x_{ij}$) and the facility opening cost ($\sum_{i \in F} f_i z_i$). Inequalities (1) represent the set of connectivity cuts. For every subset $U \subseteq S \setminus \{r\}$ and for each customer $k \in R$, an open arc from a facility in U toward j, necessitates a directed path from r towards U. Constraints (2) are path-based hop constraints. For any path consisting of $H + 1$ arcs at most H arcs of the core graph are allowed to be open in a valid solution. Constraints (3) ensure that every customer is connected to one facility, constraints (4) ensure that each facility is opened if customers are assigned to it, equation (5) defines the root node. Constraints (3) can be replaced by inequality in case that $c_{ij} > 0$, for all $ij \in A_R$.

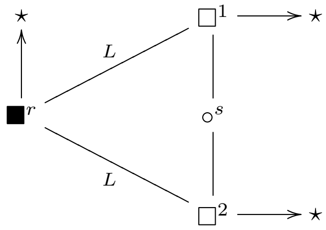
Lemma 1. *The path constraints* (2) *dominate the following classes of inequalities:*

$$a) \qquad \sum_{uv \in P} x_{uv} \leq H \qquad \forall P \in \mathcal{P}_{H+1},\ P \subseteq A_S \tag{9}$$

$$b) \qquad \sum_{uv \in P} (x_{uv} + x_{vu}) \leq H + 1 \qquad \forall P \in \mathcal{P}_{H+2} \tag{10}$$

$$c) \qquad \sum_{uv \in P} x_{uv} \leq H + 1 \qquad \forall P \in \mathcal{P}_{H+2} \tag{11}$$

Proof. a) Obviously every solution that satisfies constraints (2) also satisfies (9). To show that the latter are strictly dominated, consider the following example: The hop limit H is 2. The solution in which $x_{r1} = x_{r2} = x_{1s} = x_{s2} = x_{2s} = x_{s1} = 0.5$ is only valid for model CUT_R^P where constraints (2) are replaced by the weaker constraints (9).



b) The relation follows from the fact that any path in $\mathcal{P}_{H+2}$ can be derived by adding a single assignment arc to a path in $\mathcal{P}_{H+1}$.

c) Follows from a) and b).

$\square$

An Adaption of Ljubić' Cut Set Formulation If we replace (1) by the following constraints,

$$\sum_{uv \in \delta^-(W)} x_{uv} \geq z_i \qquad \forall W \subseteq S \setminus \{r\},\ \forall i \in W \cap F \neq \emptyset \tag{12}$$

we obtain a hop constrained extension of Ljubić' cut set model. We refer to it as CUT_F^P. We have shown in Gollowitzer and Ljubić [2009] that the lower bounds of this formulation are up to $|F|-1$ times worse than the bounds of CUT_R^P in the absence of hop constraints. HC ConFL contains ConFL as a special case. Thus, this results still holds for HC ConFL.
Connectivity constraints are separated using the maximum flow algorithm (see, e.g., Gollowitzer and Ljubić [2009]). A separation algorithm for path constraints is given in Costa et al. [2009].

3.1.2 Cut Set Formulations with Jump Constraints

To formulate cut set based models for HC ConFL with jump constraints we borrow the notation proposed in Dahl et al. [2006]:

Facility-Based Jump Formulation Let $S_0, S_1, \ldots, S_{H+1}$ be a partition of S, such that none of the subsets is empty and that the root node $r \in S_0$ and $S_{H+1} \cap F \neq \emptyset$. We call $J = J(S_0, S_1, \ldots, S_{H+1}) = \bigcup_{(i,j):i<j-1} [S_i, S_j]$ where $[S_i, S_j] = \{uv \in A_S : u \in S_i, v \in S_j\}$ a H-jump. Using J_H, the set of all possible H-jumps, we can formulate hop constraints on the core graph by using the following jump inequalities:

$$\sum_{ij \in J} x_{ij} \geq z_l \quad \forall J \in J_H,\ l \in F \cap S_{H+1}. \tag{13}$$

In the following, let CUT_F^J denote the formulation given by replacing constraints (2) by (13) in formulation CUT_F^P.
These particular jump constraints represent *a new way to model hop constraints*. These inequalities can be applied to all hop constrained network design problems with node variables, like the hop constrained prize-collecting STP or STPRBH.

Customer-Based Jump Formulation Let $S_0, S_1, \ldots, S_{H+2}$ be a partition of S, such that none of the subsets is empty and that the root node $r \in S_0$ and $S_{H+2} \cap R \neq \emptyset$. W.l.o.g. we also assume, that $S_{H+2} = R$. We call $J = J(S_0, S_1, \ldots, S_{H+2}) = \bigcup_{(i,j):i<j-1} [S_i, S_j]$ a $(H+1)$-jump. Using J_{H+1}, the set of all possible $(H+1)$-jumps, we can formulate hop constraints on the core and assignment graph by using the following jump inequalities

$$\sum_{ij \in J} x_{ij} \geq 1 \quad \forall J \in J_{H+1}. \tag{14}$$

The formulation derived by replacing inequalities (2) by (14) in CUT_R^P we denote by CUT_R^J.
An illustration of these jump sets is given in Figure 1.
It is an open question whether the LP relaxation of the models with jump constraints is polynomially solvable. There is a conjecture, that the separation of jump constraints is an NP hard problem.

3.2 Flow-based Formulations

3.2.1 Multi-Commodity Flow Formulations

Balakrishnan and Altinkemer [1992] and Gouveia [1996] have used multi-commodity flow formulations for network design problems with hop constraints. In both papers the authors limit the amount of flow for each commodity by the hop limit. Together with flow preservation constraints, this idea can be used to derive two valid MIP models for HC ConFL.

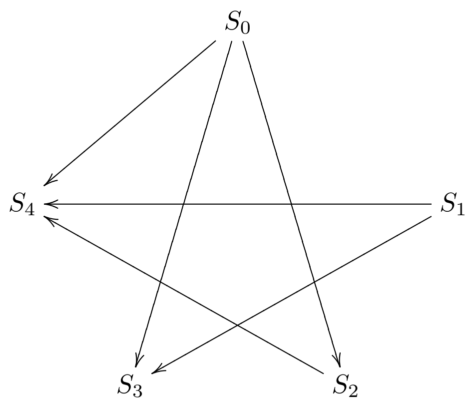


Figure 1: Illustration of the arcs contained in a jump for $H = 3$.

Multi-Commodity Flow with One Commodity per Facility Choosing one commodity per facility, each variable indicating an open facility is linked to a distinct commodity. A multi-commodity flow formulation with one commodity per facility is given by:

$$(MCF_F) \quad \min \sum_{ij \in A} c_{ij} x_{ij} + \sum_{i \in F} f_i z_i$$

$$\text{s.t.} \quad \sum_{ji \in A_S} g_{ji}^k - \sum_{ij \in A_S} g_{ij}^k = \begin{cases} z_k & i = k \\ -z_k & i = r \\ 0 & i \neq k, r \end{cases} \qquad \forall i \in S \quad \forall k \in F \setminus \{r\} \tag{15}$$

$$0 \leq g_{ij}^k \leq x_{ij} \qquad \forall ij \in A_S,\ \forall k \in F \setminus \{r\} \tag{16}$$

$$\sum_{ij \in A_S} g_{ij}^k \leq H \qquad \forall k \in F \setminus \{r\} \tag{17}$$

$$(3)\text{ - }(8)$$

Equations (15) are the flow preservation constraints defining the flow from the root node to each facility. These constraints ensure the existence of a connected path from r to every open facility. The coupling constraints (16) ensure that the arc is open if a flow is sent through it. The maximum number of hops on the path from r to k is modelled by inequalities (17)
Formulation MCF_F comprises $O(|A_S||F| + |A_R|)$ constraints, $O(|A_S||F|)$ continuous and $O(|A|)$ binary variables.
Constraints (17) can be replaced by stronger ones,

$$\sum_{ij \in A_S} g_{ij}^k \leq H \cdot z_j \qquad \forall k \in F \setminus \{r\}. \tag{18}$$

Thereby we obtain a formulation that we denote by MCF_F^+.

Multi-Commodity Flow with One Commodity per Customer Another choice for the commodities we use, is the set of customers. Assigning a commodity of demand 1 to each customer allows to remove the **z** variables from the flow preservation constraints. Using one commodity per customer, HC

ConFL can be stated as:

$$
\begin{aligned}
(MCF_R) \quad & \min \sum_{ij \in A} c_{ij} x_{ij} + \sum_{i \in F} f_i z_i \\
\text{s.t.} \quad & \sum_{ji \in A} f_{ji}^k - \sum_{ij \in A} f_{ij}^k = \begin{cases} 1 & i = k \\ -1 & i = r \\ 0 & i \neq k, r \end{cases} \qquad \forall i \in V \quad \forall k \in R && (19) \\
& 0 \leq f_{ij}^k \leq x_{ij} \qquad \forall ij \in A,\ \forall k \in R && (20) \\
& \sum_{ij \in A} f_{ij}^k \leq H + 1 \quad \forall k \in R && (21) \\
& (4)\ \text{-}\ (8)
\end{aligned}
$$

Constraints (19) and (20) guarantee the existence of a directed path from the root r to customer k. Together with constraints (21) this path contains at most $H+1$ arcs. Formulation MCF_R comprises $O(|A||R|)$ constraints, $O(|A||R|)$ continuous and $O(|A|)$ binary variables.
Note that in this formulation variables x_{jk} can be replaced by flows f_{jk}^k for all jk in A_R, as we have already shown in Gollowitzer and Ljubić [2009]. Also note that constraints (21) are equivalent to the following:

$$
\sum_{ij \in A_S} f_{ij}^k \leq H \qquad \forall k \in R \tag{22}
$$

3.2.2 Hop Indexed Multi-Commodity Flow Formulations

Gouveia [1998] develops a hop indexed formulation for the HCMST and HCSTP. It uses the usual multi-commodity flow variables with an additional hop-index. We use a similar formulation in which we reduce the number of backbone variables to handle HC ConFL.
As for the MCF models, there are two choices on the commodities considered, facilities or customers. The variant, where facilities resemble commodities, is an extension of MCF_F, the other one is based on MCF_R.

Hop Indexed Multi-Commodity Flow Between Root and Facilities Let g_{ij}^{kp} denote the flow towards facility $k \in F$, over arc ij, at position p of the path from r to k. Then a MIP formulation of HC ConFL using hop-indexed multi-commodity flows from the root to facilities is given by:

$$
\begin{aligned}
(HD_F) \quad & \min \sum_{ij \in A} c_{ij} x_{ij} + \sum_{i \in F} f_i z_i \\
\text{s.t.} \quad & \sum_{ji \in A_S} g_{ji}^{k,p-1} - \sum_{ij \in A} g_{ij}^{kp} = 0 \qquad \forall k \in F \setminus \{r\},\ i \in S \setminus \{r, k\},\ p = 2, \ldots, H && (23) \\
& \sum_{rj \in A_S} g_{rj}^{k1} = z_k \qquad \forall k \in F \setminus \{r\} && (24) \\
& \sum_{p=1}^{H} \sum_{jk \in A_S} g_{jk}^{kp} = z_k \qquad \forall k \in F \setminus \{r\} && (25) \\
& g_{ij}^{kp} = 0 \qquad \forall ij \in A_S,\ k \in F \setminus \{r\}, \begin{cases} i \notin \{r, k\},\ p = 1 \\ i = r,\ p = 2, \ldots, H \end{cases} && (26) \\
& \sum_{p=1}^{H} g_{ij}^{kp} \leq x_{ij} \qquad \forall ij \in A_S,\ k \in F \setminus \{r\} && (27) \\
& g_{ij}^{kp} \geq 0 \qquad \forall ij \in A_S,\ k \in F \setminus \{r\},\ p = 1, \ldots, H && (28) \\
& (3) - (8)
\end{aligned}
$$

Equations (23) - (25) are flow conservation constraints. Equalities (23) set the outflows of a commodity equal to the inflows of the same commodity one position earlier. Constraints (24) ensure that z_k units of

commodity k leave the root, constraints (25) ensure they terminate in the respective facility. Constraints (26) fix some flows to zero: Flows at position one are limited to arcs emanating from the root, flows at a higher position than one don't emanate from the root. Inequalities (27) ensure an arc is in the solution if flow is sent through it.
In contrast to the model in Gouveia [1998] we do not consider variables g^{kp}_{kk} in our model. Thus, commodity flows can end in the respective facility at any position. All flows fixed to zero in (26) could be removed from the model but are kept to simplify the notation of constraints (23) - (25).
Formulation HD_F comprises $O(H|S||F| + |A_S||F|)$ constraints, $O(H|A_S||F|)$ continuous and $O(|A|)$ binary variables.

Hop Indexed Multi-Commodity Flow Between Root and Customers Based on the MCF_R model, we can now derive a different hop-indexed formulation. Let f^{kp}_{ij} denote the flow towards customer $k \in R$, over arc ij, at position p of the path from r to k. The formulation using hop-indexed multi-commodity flows from the root to customers is then given by:

$$(HD_R) \quad \min \sum_{ij \in A} c_{ij} x_{ij} + \sum_{i \in F} f_i z_i$$

$$\text{s.t.} \quad \sum_{ji \in A_S} f^{k,p-1}_{ji} - \sum_{ij \in A} f^{kp}_{ij} = 0 \qquad \forall i \in S \setminus \{r\},\ k \in R,\ p = 2, \dots, H+1 \tag{29}$$

$$\sum_{rj \in A} f^{k1}_{rj} = 1 \qquad \forall k \in R \tag{30}$$

$$\sum_{p=1}^{H+1} \sum_{jk \in A_R} f^{kp}_{jk} = 1 \qquad \forall k \in R \tag{31}$$

$$f^{kp}_{ij} = 0 \qquad \forall ij \in A,\ k \in R, \begin{cases} i \neq r,\ p = 1 \\ i = r,\ p = 2, \dots, H+1 \end{cases} \tag{32}$$

$$\sum_{p=1}^{H+1} f^{kp}_{ij} \leq x_{ij} \qquad \forall ij \in A,\ k \in R \tag{33}$$

$$f^{kp}_{ij} \geq 0 \qquad \forall ij \in A,\ k \in R,\ p = 1, \dots, H+1 \tag{34}$$

$$(4) - (8)$$

Constraints (29), (30) and (31) are flow preservation constraints similar to the ones in HD_F. Constraints (32) fix some flows to zero as in HD_F: Flows at position one are only allowed to emanate from the root node. No flows in a later position can occur on arcs leaving the root. Inequalities (33) ensure an arc is in the solution if there is flow on it.
Formulation HD_R comprises $O(|S||R|H + |A||R|)$ constraints, $O(|A||R|H)$ continuous and $O(|A|)$ binary variables.

3.3 A Formulation Based on Sub-tour Elimination Constraints

Miller-Tucker-Zemlin Formulation Miller-Tucker-Zemlin constraints Miller et al. [1960] have been applied to a number of problems. Besides Connected Facility Location Gollowitzer and Ljubić [2009] we shall mention the models for the Hop Constrained Minimum Spanning and Steiner Tree Problem Costa et al. [2009], Gouveia [1995]. In addition to x and z variables, we now introduce *hop variables* $u_i \geq 0$, for all $i \in S$. These indicate the distance in hops of each node i from the root. The root node has a distance of zero.
Using the Miller-Tucker-Zemlin (MTZ) constraints (see, e.g., Gouveia [1999]), HC ConFL can be stated

as:

$$
\begin{aligned}
(MTZ) \quad & \min \sum_{ij\in A} c_{ij}x_{ij} + \sum_{i\in F} f_i z_i \\
& \sum_{ij\in A_S} x_{ij} \geq x_{jk} && \forall j \in S\setminus\{r\},\ k\in V && (35)\\
& (H+1)x_{ij} + u_i \leq u_j + H && \forall ij \in A_S && (36)\\
& u_r = 0 && && (37)\\
& u_i \geq 0 && \forall i \in S\setminus\{r\} && (38)\\
& (3)-(8)
\end{aligned}
$$

Constraints (35) limit the out-degree of a node by its in-degree. Constraints (36) are Miller-Tucker-Zemlin sub-tour elimination constraints, setting the difference $u_j - u_i$ for an open arc ij to at least 1. They thereby eliminate cycles in the Steiner tree connecting the facilities and paths on the core graph with more than H arcs. Constraint (37) sets the hop variable to zero for the root node. Formulation MTZ comprises $O(|A|)$ constraints, $O(|S|)$ continuous and $O(|A|)$ binary variables.
In Gollowitzer and Ljubić [2009] we show that for ConFL the formulation based on generalized sub-tour elimination constraints is equivalent to the cut set model with facility based cuts. Therefore, we do not consider it separately in this paper.

3.4 Hop-indexed Tree Formulations

Gouveia [1999] proposes a hop-indexed tree model for the Hop Constrained STP. Voß [1999] states that this is a disaggregation of the formulation MTZ (see Section 4).
To model HC ConFL, there are two options for the hop-indexed variables. We can consider them on the whole graph or alternatively we can separate core and assignment graph and link them by the z-variables indicating the use of facilities.

Hop Constraints on the Entire Graph Let X_{ij}^p indicate whether arc $ij \in A$ is used at the p-th position from the root node. Then we can model HC ConFL as follows:

$$
\begin{aligned}
(HOP_R) \quad & \min \sum_{p=1}^{H+1}\sum_{ij\in A} c_{ij}X_{ij}^p + \sum_{i\in F} f_i z_i \\
& \sum_{\substack{i\in S\setminus\{k\}:\\ ij\in A_S}} X_{ij}^{p-1} \geq X_{jk}^p && \forall jk \in A,\ j\neq r,\ p=2,\dots,H+1 && (39)\\
& \sum_{p=1}^{H+1}\sum_{jk\in A_R} X_{jk}^p = 1 && \forall k \in R && (40)\\
& \sum_{p=1}^{H+1} X_{jk}^p \leq z_j && \forall jk \in A_R,\ j\neq r && (41)\\
& X_{ij}^p = 0 && \forall ij \in A, \begin{cases} i=r,\ p=2,\dots,H+1\\ i\neq r,\ p=1\end{cases} && (42)\\
& X_{ij}^p \in \{0,1\} && \forall ij\in A,\ p=1,\dots,H+1 && (43)\\
& (8)
\end{aligned}
$$

Constraints (39) are connectivity constraints. As X_{ij}^p are binary, they eliminate cycles as well. Constraints (41) ensure a facility is open if it serves a customer. Constraints (40) ensure that each customer is served. Equations (42) fix some of the X_{ij}^p to zero: Arcs emanating from the root can only be 1 hop away from it. Conversely, all other arcs are at least two hops away from the root.
For the polyhedral comparison in Section 4 we define the projection of $(\mathbf{X}', \mathbf{z}') \in \mathcal{P}_{HOP_R}$ onto the space of $(\mathbf{x}, \mathbf{z})$ as follows: $x_{ij} := \sum_{p=1}^{H+1} X'^p_{ij}$ for all ij in A and $z_i := z'_i$ for all i in F.

Hop Constraints on the Core Graph We separate core and assignment graph and link them by variables $z_j, j \in F$. After replacing variables $X^p_{ij}, ij \in A_R$ from formulation HOP_R by assignment variables $x_{ij}, ij \in A_R$, we can formulate HC ConFL using hop constraints only on the core graph:

$$(HOP_F) \quad \min \sum_{p=1}^{H} \sum_{ij \in A_S} c_{ij} X^p_{ij} + \sum_{jk \in A_R} c_{jk} x_{jk} + \sum_{i \in F} f_i z_i$$

$$\sum_{\substack{i \in S \setminus \{k\}: \\ ij \in A_S}} X^{p-1}_{ij} \geq X^p_{jk} \qquad \forall jk \in A_S,\ j \neq r,\ p = 2, \ldots, H \tag{44}$$

$$\sum_{ij \in A_S} \sum_{p=1}^{H} X^p_{ij} \geq z_j \qquad \forall j \in F \setminus \{r\} \tag{45}$$

$$X^p_{ij} = 0 \qquad ij \in A_S, \begin{cases} i = r,\ p = 2, \ldots, H \\ i \neq r,\ p = 1 \end{cases} \tag{46}$$

$$X^p_{ij} \in \{0, 1\} \qquad \forall ij \in A_S,\ p = 1, \ldots, H \tag{47}$$

$$(3) - (5), (7), (8)$$

Constraints (44) are connectivity constraints like (39). Similarly, inequalities (45) link opening facilities to their in-degree. Constraints (46) are similar to (42).
We define the projection of $(\mathbf{X}', \mathbf{z}', \mathbf{x}') \in \mathcal{P}_{HOP_F}$ onto the space of $(\mathbf{x}, \mathbf{z})$ as follows: $x_{ij} := \sum_{p=1}^{H} X'^p_{ij}$ for all ij in A_S; $x_{jk} := x'_{jk}$ for all jk in A_R; $z_i := z'_i$ for all i in F.

3.5 Modelling Hop Constraints on Layered Graphs

Gouveia et al. [2010] model the Minimum Spanning Tree problem with hop constraints as Steiner tree problem (STP) on a so-called layered graph. This allows to apply all algorithms developed for the STP to the HCMST. Additionally, the directed cut model on this layered graph turns out to be stronger than the models considered before.
We extend this idea and develop three variants of a layered graph to model the HC ConFL as ConFL on a directed graph. In the first one we transform only the core graph into the layered graph, define nodes at the level H as potential facilities and leave the assignment graph unchanged. We denote the models on this graph by LG_x. For the second variant we build a layered graph in a similar fashion, but now we disaggregate the assignment graph by allowing assignments between a customer and each potential facility at level h, $1 \leq h \leq H$. The models on this graph are denoted by $LG_{x,z}$. Finally, we build a layered graph by introducing facilities and customer nodes at each level $1 \leq h \leq H$ and $1 \leq h \leq H + 1$, respectively. The latter models we denote by $LG_{x,z,x}$.

3.5.1 Layered Core Graph LG_x

Consider a graph $LG_x = (V_x, A_x)$ representing an instance of directed ConFL with the set of customers R defined a above and the set of potential facilities F_x and the set of core nodes S_x as follows:

$$
\begin{aligned}
V_x &:= \{r\} \cup S_x \cup R \text{ where} \\
&\quad F_x = \{(i,H) : i \in F \setminus \{r\}\}\,, \\
&\quad S_x = F_x \cup \{(i,p) : 1 \le p \le H-1, i \in S\} \text{ and} \\
A_x &:= \bigcup_{i=1}^{6} A_i \text{ where} \\
&\quad A_1 = \{(r,(j,1)) : rj \in A_S\}, \\
&\quad A_2 = \{((i,p),(j,p+1)) : 1 \le p \le H-2, (i,j) \in A_S\}, \\
&\quad A_3 = \{((i,H-1),(j,H)) : ij \in A_S, i \in S \setminus \{r\}, j \in F \setminus \{r\}\}, \\
&\quad A_4 = \{((i,p),(i,H)) : 1 \le p \le H-1, i \in F \setminus \{r\}\}, \\
&\quad A_5 = \{((j,H),k) : jk \in A_R, j \ne r\} \text{ and} \\
&\quad A_6 = \{rk : rk \in A_R\}
\end{aligned}
$$

The facility opening and assignment costs are left unchanged. The arc costs between (i,p) and $(j,p+1)$ are given as c_{ij}. Finally, arcs between (i,p) and (i,H) are assigned costs of 0 for all $p = 1,\dots,H-1$ and i in F.

Lemma 2. *Any HC ConFL instance can be transformed into an equivalent directed ConFL instance on the layered graph LG_x as described above.*

Figure 2 illustrates the transformation of an instance for HC ConFL to an instance for ConFL on LG_x.

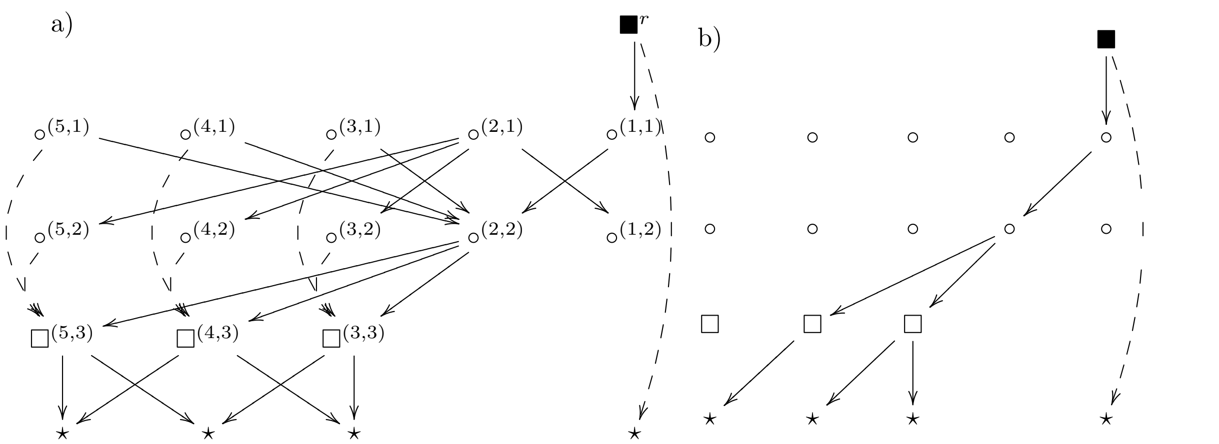


Figure 2: a) Layered graph (V_x, A_x) for Example 1 on page 15; b) An optimal integer solution.

We link binary variables to the arcs in A_x as follows: X^1_{rj} corresponds to $(r,(j,1)) \in A_1$, X^p_{ij} to $((i,p-1),(j,p)) \in A_2$, X^H_{ij} to $((i,H-1),(j,H)) \in A_3$, X^p_{ii} to $((i,p-1),(i,H)) \in A_4$, X_{jk} to $((j,H),k) \in A_5$ and X^1_{rk} to $rk \in A_6$.
Let $X[V_x \setminus W, W]$ denote the sum of all variables X^p_{ij} and X_{jk} in the cut $\delta^-(W)$ in LG_x defined by $W \subset V_x$ and $r \notin W$. Based on the two cut set based models for ConFL, CUT_F and CUT_R (see Section

3.1.1) we derive two new formulations, LG_xCUT_F and LG_xCUT_R, as follows:

$$(LG_xCUT_F)\ \min \sum_{rj\in A} c_{rj}X^1_{rj} + \sum_{ij\in A, i\neq r} c_{ij}\sum_{p=2}^{H} X^p_{ij} \quad + \sum_{jk\in A_R, j\neq r} c_{jk}X_{jk} + \sum_{i\in F} f_i z_i$$

$$X[V_x \setminus W, W] \geq z_i \qquad \forall W \subset S_x,\ r\notin W,\ W\cap F_x \neq \emptyset \tag{48}$$

$$X^1_{rk} + \sum_{jk:((j,H),k)\in A_5} X_{jk} = 1 \qquad \forall k \in R \tag{49}$$

$$X_{jk} \leq z_j \qquad \forall ((j,H),k)\in A_5 \tag{50}$$

$$\mathbf{X} \in \{0,1\}^{|A_x|} \tag{51}$$

$$(5), (8)$$

Constraints (48) are cuts on LG_x between sets containing the root and a facility i respectively. Equalities (49) ensure that each customer is assigned to a facility. Inequalities (50) necessitate a facility to be open if customers are assigned to it.
If we replace constraints (48) and (49) by the following ones, we obtain a stronger formulation that we denote by LG_xCUT_R:

$$X[V_x\setminus W, W] \geq 1 \quad \forall W \subset V_x \setminus \{r\}, W\cap R \neq \emptyset \tag{52}$$

Inequalities (52) are cuts on LG_x between sets that contain the root and at least one customer respectively.
We define the projection of $(\mathbf{X}', \mathbf{z}') \in \mathcal{P}_{LG_xCUT_F} \cup \mathcal{P}_{LG_xCUT_R}$ onto the space of $(\mathbf{x},\mathbf{z})$ variables as follows: $x_{rj} := {X'}^1_{rj}$ for all rj in A_S; $x_{ij} := \sum_{p=1}^{H} {X'}^p_{ij}$ for all ij in A_S with j in $F\setminus\{r\}$; $x_{ij} := \sum_{p=1}^{H-1} {X'}^p_{ij}$ for all ij in A_S with j in $S\setminus F$; $x_{jk} := X'_{jk}$ for all jk in A_R, $j\neq r$; $x_{rj} := {X'}^1_{rj}$ for all rj in A_R and $z_i := z'_i$ for all i in $F\setminus\{r\}$.

3.5.2 Layered Core and Assignment Graph $LG_{x,z}$

Consider a graph $LG_{x,z} = (V_{x,z}, A_{x,z})$ defined as an instance of directed ConFL with the set of potential facilities $F_{x,z}$ and the set of core nodes $S_{x,z}$ as follows:

$$\begin{aligned}
V_{x,z} &:= \{r\} \cup S_{x,z} \cup R \text{ where}\\
&\quad F_{x,z} = \{(i,p) : i \in F\setminus\{r\}, 1\leq p\leq H\},\\
&\quad S_{x,z} = F_{x,z} \cup \{(i,p) : 1 \leq p \leq H-1, i \in S\} \text{ and}\\
A_{x,z} &:= \bigcup_{i=1}^{3} A_i \cup A_6 \cup A_7 \text{ where}\\
&\quad A_1, A_2, A_3 \text{ and } A_6 \text{ are defined as for } A_x \text{ and}\\
&\quad A_7 = \{((i,p),k) \mid (i,p)\in F_{x,z}, k \in R\}.
\end{aligned}$$

The arc costs for the latter set are defined as c_{ik} for all $i \in F\setminus\{r\}$ and $k\in R$. The facility opening costs are f_i for all (i,p) with $p = 1,\ldots,H$, $i\in F\setminus\{r\}$.

Lemma 3. *Any HC ConFL instance can be transformed into an equivalent directed ConFL instance on the layered graph $LG_{x,z}$ as described above.*

Figure 3 illustrates the transformation of an instance for HC ConFL to an instance for ConFL on $LG_{x,z}$.

We link binary variables X to the arcs in A_1 to A_3 and A_6 as above and to arcs in A_7 as follows: X^p_{jk} corresponds to $((j,p),k)\in A_7$. Additionally, we link variables Z^p_i to each (i,p) in F_x.
Let $X[V_{x,z}\setminus W, W]$ denote the sum of all variables X^p_{ij} in the cut $\delta^-(W)$ in $LG_{x,z}$ defined by $W\subset V_{x,z}$ and $r\notin W$. Again, we derive two formulations for HC ConFL on $LG_{x,z}$:

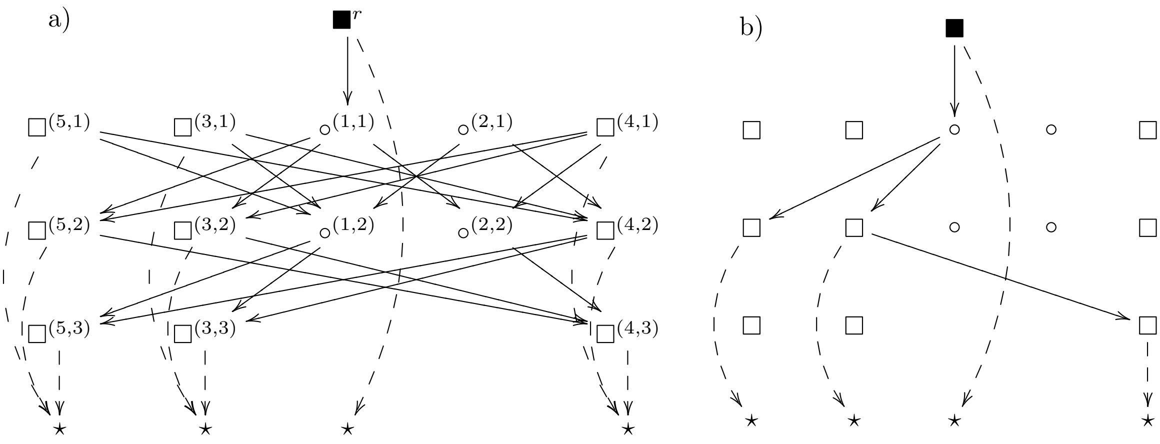


Figure 3: a) Layered graph $(V_{x,z}, A_{x,z})$ for Example 2 on page 15; b) An optimal solution.

$$(LG_{x,z}CUT_F) \quad \min \sum_{rj \in A} c_{rj} X_{rj}^1 + \sum_{ij \in A, i \neq r} c_{ij} \sum_{p=1}^{H} X_{ij}^p + \sum_{i \in F \setminus \{r\}} f_i \sum_{p=1}^{H} Z_i^p + f_r z_r$$

$$X[V_{x,z} \setminus W, W] \geq Z_i^p \qquad \forall (i,p) \in F_x \cap W \neq \emptyset, r \notin W \tag{53}$$

$$X_{rk}^1 + \sum_{jk \in A_R, j \neq r} \sum_{p=1}^{H} X_{jk}^p = 1 \qquad \forall k \in R \tag{54}$$

$$X_{jk}^p \leq Z_j^p \qquad \forall jk \in A_R,\ p = 1, \ldots, H,\ j \neq r \tag{55}$$

$$\mathbf{X} \in \{0,1\}^{|A_{x,z}|} \tag{56}$$

$$(5)$$

Constraints (53) are cuts on $LG_{x,z}$ between the root r and each facility at a level p, (i,p). Equalities (54) are assignment constraints. Inequalities (55) necessitate a facility at a level p to be open if customers are assigned to it.

Observation 2. *Constraints*

$$\sum_{p=1}^{H} Z_j^p \leq 1 \qquad \forall i \in F \setminus \{r\}$$

are implied by the previous model if $f_i \geq 0$ for all $i \in F \setminus \{r\}$ and $c_{ij} \geq 0$ for all $ij \in A_R$.

Lemma 4. *We can replace connectivity cuts* (53) *by the following stronger ones:*

$$X[V_{x,z} \setminus W, W] \geq \sum_{p=1}^{H} Z_i^p \qquad \forall i \in F \setminus \{r\} : (i,p) \in F_x \cap W, r \notin W \tag{57}$$

Proof. The validity of these constraints follows from the fact that for each $i \in F$, the facilities (i,p) with $p = 1, \ldots, H$ serve the same subset of customers with the same assignment costs. Therefore, any optimal solution will open at most one among those facilities, i.e. $\sum_{p=1}^{H} Z_i^p \leq 1$ for all i in $F \setminus \{r\}$. □

$$(LG_{x,z}CUT_R) \quad \min \sum_{rj\in A} c_{rj}X^1_{rj} + \sum_{ij\in A, i\neq r} c_{ij}\sum_{p=1}^{H} X^p_{ij} + \sum_{i\in F\setminus\{r\}} f_i \sum_{p=1}^{H} Z^p_i + f_r z_r$$

$$X[V_{x,z}\setminus W, W] \geq 1 \qquad \forall W \subset V_{x,z}\setminus\{r\}, W\cap R \neq \emptyset \tag{58}$$

$$(5), (55), (56)$$

Inequalities (58) are cuts on $LG_{x,z}$ between sets containing the root and a customer respectively. We define the projection of $(\mathbf{X}', \mathbf{Z}') \in \mathcal{P}_{LG_{x,z}CUT_F} \cup \mathcal{P}_{LG_{x,z}CUT_R}$ onto the space of $(\mathbf{x}, \mathbf{z})$ variables as follows: $x_{rj} := {X'}^1_{rj}$ for all rj in A_S; $x_{ij} := \sum_{p=1}^{H} {X'}^p_{ij}$ for all ij in A_S with j in $F\setminus\{r\}$; $x_{ij} := \sum_{p=1}^{H-1} {X'}^p_{ij}$ for all ij in A_S with j in $S\setminus F$; $x_{jk} := \sum_{p=1}^{H} {X'}^p_{jk}$ for all jk in A_R, $j\neq r$; $x_{rk} := {X'}^1_{rk}$ for all rk in A_R and $z_i := \sum_{p=1}^{H} {Z'}^p_i$ for all i in $F\setminus\{r\}$.

3.5.3 Layered Core and Assignment Graph with Split Customers $LG_{x,z,x}$

Consider a graph $LG_{x,z,x} = (V_{x,z,x}, A_{x,z,x})$ with the following definitions:

$V_{x,z,x} := \{r\} \cup S_{x,z} \cup R_x$ where
$F_{x,z}$ and $S_{x,z}$ are defined as above and
$R_x = \{(k,p) : k \in R, 1 \leq p \leq H+1\}$.
$A_{x,z,x} := \bigcup_{i=1}^{3} A_i \cup \bigcup_{i=8}^{10} A_i$ where
A_1, A_2 and A_3 are defined as for A_x,
$A_8 = \{(r,(k,1)) : rk \in A_R\}$,
$A_9 = \{((j,p),(k,p+1)) : (j,p) \in F_x, jk \in A_R\}$ and
$A_{10} = \{((k,p),(k,H+1)) : k \in R, 1 \leq p \leq H\}$

In this directed graph the set of customers is given by $\{(k,H+1) : k \in R\}$. The set of facilities $F_{x,z}$ as well as arc and facility opening costs are the same as in $LG_{x,z}$. Assignment costs are c_{rk} for all $(r,(k,1))$ in A_8 and c_{jk} for all $((j,p),(k,p+1))$ in A_9. Arcs in A_{10} are assigned costs of zero.

Lemma 5. *Any HC ConFL instance can be transformed into an equivalent directed ConFL instance on the layered graph $LG_{x,z,x}$ as described above.*

Figure 4 illustrates the difference between $LG_{x,z}$ and $LG_{x,z,x}$ for a detail of an instance.

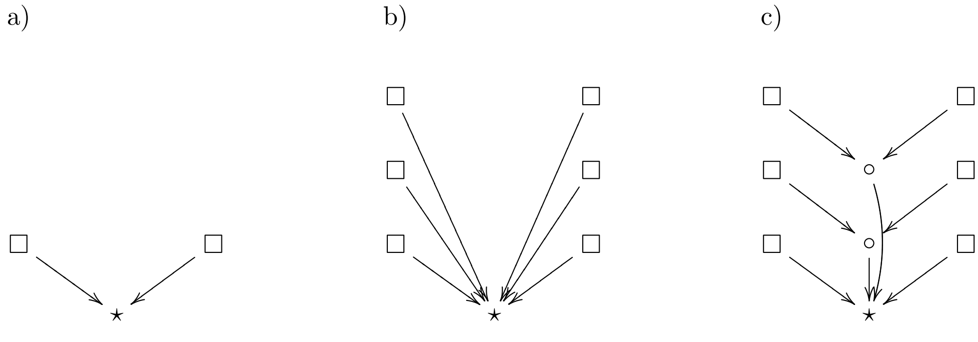


Figure 4: a) Detail of an HC ConFL instance; b) Detail on the Layered graph $LG_{x,z}$; c) Detail on the Layered graph $LG_{x,z,x}$.

We link binary variables X to the arcs in A_1 to A_3 as above and to arcs in A_8, A_9 and A_{10} as follows: X^1_{rk} corresponds to $(r,(k,1)) \in A_8$. X^p_{jk} corresponds to $((j,p),(k,p+1)) \in A_9$ and X^p_{kk} corresponds to $((k,p),(k,H+1)) \in A_{10}$. Analogously to $LG_{x.z}$ we link variables Z^p_i to each (i,p) in F_x.

Let $X[V_{x,z,x} \backslash W, W]$ denote the sum of all variables X_{ij}^p in the cut $\delta^-(W)$ in $LG_{x,z,x}$ defined by $W \subset V_{x,z,x}$ and $r \notin W$. We once again derive two formulations for HC ConFL:

$$
\begin{aligned}
(LG_{x,z,x}CUT_F)\ \min \sum_{rj \in A} c_{rj} X_{rj}^1 + \sum_{ij \in A, i \neq r} c_{ij} \sum_{p=1}^{H} X_{ij}^p + \sum_{i \in F \setminus \{r\}} f_i \sum_{p=1}^{H} Z_i^p + f_r z_r \\
X[V_{x,z,x} \setminus W, W] \geq Z_i^p \qquad & \forall (i,p) \in F_{x,z} \cap W \neq \emptyset, r \notin W && (59)\\
X_{rk}^1 + \sum_{jk \in A_R, j \neq r} \sum_{p=2}^{H+1} X_{jk}^p = 1 \qquad & \forall k \in R && (60)\\
X_{jk}^{p+1} \leq Z_j^p \qquad & \forall jk \in A_R,\ p = 1, \dots, H,\ j \neq r && (61)\\
\mathbf{X} \in \{0,1\}^{|A_{x,z,x}|} \qquad & && (62)\\
(5) \qquad & &&
\end{aligned}
$$

Constraints (59) are cuts on $LG_{x,z,x}$ between the root r and each facility at a level p, (i,p). Equalities (60) are assignment constraints. Inequalities (61) necessitate a facility at a level p to be open if customers are assigned to it.

$$
\begin{aligned}
(LG_{x,z,x}CUT_R)\ \min \sum_{rj \in A} c_{rj} X_{rj}^1 + \sum_{ij \in A, i \neq r} c_{ij} \sum_{p=1}^{H} X_{ij}^p + \sum_{i \in F \setminus \{r\}} f_i \sum_{p=1}^{H} Z_i^p + f_r z_r \\
X[V_x \setminus W, W] \geq 1 \qquad & \forall W \subset V_x \setminus \{r\}, W \cap \{(k, H+1) : k \in R\} \neq \emptyset && (63)\\
(5), (61), (62) \qquad & &&
\end{aligned}
$$

Inequalities (63) are cuts on $LG_{x,z,x}$ between sets containing the root and a customer respectively. The projection of $(\mathbf{X}', \mathbf{Z}') \in \mathcal{P}_{LG_{x,z,x}CUT_F} \cup \mathcal{P}_{LG_{x,z,x}CUT_R}$ onto the space of $(\mathbf{x}, \mathbf{z})$ variables is defined as follows: $x_{rj} := {X'}_{rj}^1$ for all rj in A_S; $x_{ij} := \sum_{p=1}^{H} {X'}_{ij}^p$ for all ij in A_S with j in $F \setminus \{r\}$; $x_{ij} := \sum_{p=1}^{H-1} {X'}_{ij}^p$ for all ij in A_S with j in $S \setminus F$; $x_{jk} := \sum_{p=1}^{H} {X'}_{jk}^p$ for all jk in A_R, $j \neq r$; $x_{rk} := {X'}_{rk}^1$ for all rk in A_R and $z_i := \sum_{p=1}^{H} {Z'}_i^p$ for all i in $F \setminus \{r\}$.

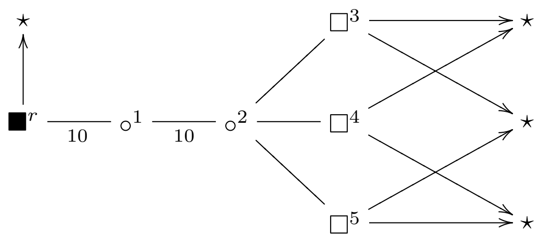
3.5.4 Modelling HC ConFL as STP on a Layered Graph

In Gollowitzer and Ljubić [2009] we have shown that by splitting facility nodes one can model ConFL as the Steiner arborescence problem on the transformed graph. If we apply this transformation to the corresponding instances on the layered graphs LG_x, $LG_{x,z}$ and $LG_{x,z,x}$, we end up with three ways of formulating HC ConFL as the Steiner arborescence problem.
However, we have shown that for ConFL this transformation does not lead to improved LP lower bounds. Thus, we do not consider the Steiner arborescence models in the theoretical discussion provided below.

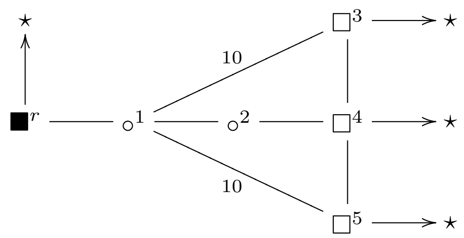
4 Polyhedral Comparison

In this section we provide a theoretical comparison of the MIP models described above with respect to optimal values of their LP-relaxations. The examples given below are used in the proofs of this section. In Example 3 $H = 5$, in all other examples $H = 3$. Recall, that the default arc, facility opening and assignment costs are set to 1.

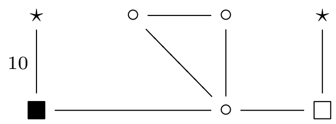
Example 1.



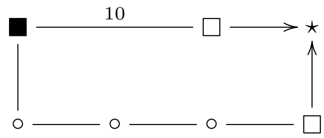
Example 2.



Example 3.



Example 4.



	Ex. 1	Ex. 2	Ex. 3	Ex. 4
OPT	29.00	30.00	15.00	13.00
MTZ	18.00	13.54	8.00	7.38
$HOP_{\{F,R\}}$	18.00	30.00	10.33	13.00
CUT_F^P	18.00	19.80	15.00	8.50
CUT_R^P	28.00	19.80	15.00	8.50
MCF_F	18.00	21.00	15.00	8.50
MCF_F^+	18.00	21.00	15.00	13.00
MCF_R	28.00	21.00	15.00	9.00
HD_F	18.00	30.00	15.00	13.00
HD_R	28.00	30.00	15.00	13.00

Table 1: Optimal LP solutions for Examples 1 - 4.

Let $v_{LP}(.)$ denote the optimal solution value of the LP relaxation of a given model. By comparing the optimal LP solution values for the aforementioned examples, provided by the models in Section 3, we can state the following

Lemma 6. *The following pairs of formulations are incomparable with respect to the quality of lower bounds:*

a) $HOP_\cdot$ *and* $CUT_\cdot^P$*,*

b) $HOP_\cdot$ *and* $MCF_\cdot^{(+)}$*, where* $\cdot$ *is to be replaced by* F *or* R *and*

c) MCF_F^+ *and* MCF_R*.*

Proof. Consider the optimal LP solution values for the Examples in Table 1. □

Denote by $\mathcal{P}_\cdot$ the polytope of the LP-relaxation of any of the MIP models described above, and by $Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_\cdot)$ the natural projection of that polytope onto the space of variables $\mathbf{x}$ and $\mathbf{z}$.

Lemma 7. *The projections* $Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_\cdot)$ *of sets of feasible LP solutions of the following formulations are identical:*

a) $LG_{x,z,x}CUT_R$*,* $LG_{x,z}CUT_R$ *and* LG_xCUT_R

b) $LG_{x,z,x}CUT_F$*,* $LG_{x,z}CUT_F$ *and* LG_xCUT_F

Proof. To prove the relation in a), we describe bidirectional mappings between arbitrary vectors in $\mathcal{P}_{LG_xCUT_R}$ and $\mathcal{P}_{LG_{x,z}CUT_R}$ as well as in $\mathcal{P}_{LG_{x,z}CUT_R}$ and $\mathcal{P}_{LG_{x,z,x}CUT_R}$. These mappings will be in a way, such that both vectors have the same objective function value and that their projections onto the space of $(\mathbf{x},\mathbf{z})$ are the same. The proof for b) uses the same arguments.

$Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{LG_{x,z,x}CUT_R}) \subseteq Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{LG_{x,z}CUT_R})$**:** Let $(\mathbf{X}',\mathbf{Z}') \in \mathcal{P}_{LG_{x,z,x}CUT_R}$ and $(\mathbf{X},\mathbf{Z})$ be defined as follows: $X_{ij}^p := X'^p_{ij}$ for all arcs in A_1, A_2, A_3 and A_6; $X_{jk}^p := X'^p_{jk}$ for all arcs in A_7. In addition $Z_i^p := Z'^p_i$ for all (i,p) in F_x. Then $(\mathbf{X},\mathbf{Z}) \in \mathcal{P}_{LG_{x,z}CUT_R}$, and the projections of $(\mathbf{X}',\mathbf{Z}')$ and $(\mathbf{X},\mathbf{Z})$ coincide.

$Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{LG_{x,z}CUT_R}) \subseteq Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{LG_{x,z,x}CUT_R})$**:** Let $(\mathbf{X},\mathbf{Z}) \in \mathcal{P}_{LG_{x,z}CUT_R}$ and $(\mathbf{X}',\mathbf{Z}')$ be defined as follows: $X'^p_{ij} := X_{ij}^p$ for all arcs in A_1, A_2, A_3, A_8 and A_9; $X'^p_{kk} := \sum_{j \in F} X_{jk}^p$ for all arcs in A_{10}. $Z_i^p := Z'^p_i$ for all (i,p) in F_x. Then $(\mathbf{X}',\mathbf{Z}') \in \mathcal{P}_{LG_{x,z,x}CUT_R}$, and the projections of $(\mathbf{X},\mathbf{Z})$ and $(\mathbf{X}',\mathbf{Z}')$ coincide.

$Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{LG_{x,z}CUT_R}) \subseteq Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{LG_x CUT_R})$**:** Let $(\mathbf{X},\mathbf{Z}) \in \mathcal{P}_{LG_{x,z}CUT_R}$ and $(\mathbf{X}',\mathbf{z}')$ be defined as follows: $X'^p_{ij} := X^p_{ij}$ for all arcs in A_1, A_2 and A_3; $X'^p_{jj} := Z^p_j (= \max_{k\in R} X^p_{jk})$ for all arcs in A_4; ${X'}_{jk} := \sum_{p=1}^H X^p_{jk}$ for all edges in A_5; ${X'}_{rk} := X^1_{rk}$ for all edges in A_6; $z_i := \sum_{p=1}^H Z^p_i$. Then $(\mathbf{X}',\mathbf{z}') \in \mathcal{P}_{LG_x CUT_R}$, and the projections of $(\mathbf{X}',\mathbf{z}')$ and $(\mathbf{X},\mathbf{Z})$ coincide.

$Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{LG_x CUT_R}) \subseteq Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{LG_{x,z}CUT_R})$**:** Let $(\mathbf{X}',\mathbf{z}') \in \mathcal{P}_{LG_x CUT_R}$, we define $(\mathbf{X},\mathbf{Z})$ as follows: $X^p_{ij} := X'^p_{ij}$ for all edges in A_1, A_2 and A_3; $X^1_{rk} := {X'}_{rk}$ for all edges in A_6. Furthermore, we set $Z^p_j := X^p_{jj}$, for all arcs from A_4, for $p = 1,\dots,H-1$, and $Z^H_j := z'_j - \sum_{p=1}^{H-1} Z^p_j$, for all $j \in F\setminus\{r\}$. We then recursively define $X^p_{jk} := \min(Z^p_j, {X'}_{jk} - \sum_{q=p+1}^H {X'}^q_{jk})$ starting from $p = H,\dots,1$. Then $(\mathbf{X},\mathbf{Z}) \in \mathcal{P}_{LG_{x,z}CUT_R}$, and the corresponding projections onto the space of $(\mathbf{x},\mathbf{z})$ variables coincide.

□

Lemma 8. *The following results hold:*

a)	$Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{LG_x CUT_R}) \subset Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{LG_x CUT_F})$,	*d)*	$Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{MCF_R}) \subset Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{MCF_F})$,
b)	$Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{HD_R}) \subset Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{HD_F})$,	*e)*	$\mathcal{P}_{CUT^P_R} \subset \mathcal{P}_{CUT^P_F}$ *and*
c)	$Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{HOP_R}) = Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{HOP_F})$,	*f)*	$\mathcal{P}_{CUT^J_R} \subset \mathcal{P}_{CUT^J_F}$.

Proof. **a)** The inclusions follow directly from the results for ConFL in Gollowitzer and Ljubić [2009].

b) Every $(\mathbf{x},\mathbf{z},\mathbf{f})$ in $\mathcal{P}_{HD_R}$ can be projected into $\mathcal{P}_{HD_F}$ by decomposing the flows to the customers into facility flows. A similar proof is used for the common flow models for ConFL in Gollowitzer and Ljubić [2009].

c) To prove the relation we describe a mapping from any solution of HOP_F to a solution of HOP_R of the same objective value and vice versa.

$Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{HOP_R}) \subseteq Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{HOP_F})$**:** Let $(\mathbf{X},\mathbf{z}) \in \mathcal{P}_{HOP_R}$. Then, w.l.og., for any facility j equation $z_j = \max_{k\in R} \sum_{p=1}^{H+1} X^p_{jk}$ holds. Let $(\mathbf{X}',\mathbf{x}',\mathbf{z}')$ be defined as: $z'_j := z_j$ for all j in F; $X'^p_{ij} := X^p_{ij}$ for all ij in A_S, $p = 1,\dots,H$ and ${x'}_{jk} := \sum_{p=1}^{H+1} X^p_{jk}$ for all jk in A_R. Then $(\mathbf{X}',\mathbf{x}',\mathbf{z}') \in \mathcal{P}_{HOP_F}$: Inequalities (44) and (46) follow from (39) and (42). Constraints (3) and (4) are implied by (41) and (40) respectively. For all $j \in F \setminus \{r\}$ let $k^j := \arg\max_{k\in R} \sum_{p=2}^{H+1} X^p_{jk}$. Then, w.l.o.g., we have $z_j = \sum_{p=2}^{H+1} X^p_{jk^j}$ and further we have

$$z'_j = z_j = \sum_{p=2}^{H+1} X^p_{jk^j} \overset{(39)}{\leq} \sum_{p=1}^{H} \sum_{ij\in A_S} X^p_{ij} = \sum_{p=1}^{H} \sum_{ij\in A_S} X'^p_{ij},$$

hence equations (45) also hold for $(\mathbf{X}',\mathbf{x}',\mathbf{z}')$.

$Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{HOP_F}) \subseteq Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{HOP_R})$**:** Let $(\mathbf{X}',\mathbf{x}',\mathbf{z}') \in \mathcal{P}_{HOP_F}$ and $(\mathbf{X},\mathbf{z})$ defined as $z_j := z'_j$ for all j in F and $X^p_{ij} := X'^p_{ij}$ for all ij in A_S, $p = 1,\dots,H$. Let $x'_{jk} > 0$ with $j \in F\setminus\{r\}, k \in R$. From equations (45) and (4) we have $x'_{jk} \leq z'_j \leq \sum_{ij\in A_S} \sum_{p=1}^H X'^p_{ij}$. From the right hand side we can select X^{*p}_{ij} with $X^{*p}_{ij} \leq X'^p_{ij}$ such that $\sum_{ij\in A_S} \sum_{p=1}^H X^{*p}_{ij} = x'_{jk}$. We can do so for all $jk \in A_R$. Let

$$X^{p+1}_{jk} := \sum_{ij\in A_S} X^{*p}_{ij} \qquad \forall j \in F\setminus\{r\}, k \in R,\ p = 1,\dots,H \tag{64}$$

and

$$X^1_{rk} := x'_{rk} \qquad \forall k \in R.$$

Then we can show that $(\mathbf{X},\mathbf{z}) \in \mathcal{P}_{HOP_R}$:
Equations (39) follow from the definitions. Constraints (41) follow from

$$\sum_{p=1}^{H+1} X^p_{jk} \overset{(46)}{=} 0 + \sum_{p=2}^{H+1} X^p_{jk} \overset{(64)}{=} \sum_{ij\in A_S} \sum_{p=1}^{H} X^{*p}_{ij} = x'_{jk} \leq z'_j = z_j \qquad \forall jk \in A_R.$$

Constraints (40) follow from

$$\sum_{p=1}^{H+1}\sum_{jk\in A_R} X^p_{jk} = X^1_{rk} + \sum_{p=1}^{H}\sum_{ij\in A_S} X^{*p}_{ij} = \tag{65}$$

$$= X^1_{rk} + \sum_{ij\in A_S}\sum_{p=1}^{H+1} X^{*p}_{ij} = \tag{66}$$

$$= \sum_{jk\in A_R} x'_{jk} \overset{(3)}{=} 1. \tag{67}$$

d) The relation holds for the case without hop constraints (cf. Gollowitzer and Ljubić [2009]). From equations (19) we have $\sum_{ik\in A_R} f^k_{ik} = 1$ for all $k \in R$. Thus, we have $\sum_{ij\in A_S} f^k_{ij} \leq H$ for all $k \in R$. These constraints are at least as strong as (17). Therefore, $Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{MCF_R}) \subset Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{MCF_F})$.

e) We have shown the relation for the case without hop constraints (i.e. $H = |S| - 1$) in Gollowitzer and Ljubić [2009]. Constraints (2) are common in both models. Thus $\mathcal{P}_{CUT^P_R} \subset \mathcal{P}_{CUT^P_F}$ holds for the hop constrained case as well.

f) Connectivity and jump inequalities of CUT^J_R imply the respective inequalities of CUT^J_F.

$\square$

Lemma 9. *The following results hold:*

a) $Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{LG_xCUT_R}) \subset Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{HD_R}) \subset Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{MCF_R}) \subset \mathcal{P}_{CUT^P_R}$,

b) $Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{LG_xCUT_F}) \subset Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{HD_F}) \subset Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{MCF^+_F}) \subset Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{MCF_F}) \subset \mathcal{P}_{CUT^P_F}$,

c) $Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{HD_R}) \subset \mathcal{P}_{CUT^J_R} \subset \mathcal{P}_{CUT^P_R}$,

d) $Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{HD_F}) \subset \mathcal{P}_{CUT^J_F} \subset \mathcal{P}_{CUT^P_F}$ *and*

e) $Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{HD_F}) \subset Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{HOP_F}) \subset Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{MTZ})$.

Proof. Strict inclusions in a) to e) follow from the optimal LP solution values in Table 1.

a) $Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{LG_xCUT_R}) \subset Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{HD_R})$**:** The inclusion can be shown by adapting the proof in Gouveia et al. [2010]. It is strict because of the example in the same paper. Note that by disaggregating constraints (33) and introducing variables X^p_{ij} formulation HD_R becomes an equivalent of formulation LG_R.

$Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{HD_R}) \subset Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{MCF_R})$**:** MCF_R is an aggregation of HD_R ($f^k_{ij} := \sum_{p=1}^{H+1} f^{kp}_{ij}$).

$Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{MCF_R}) \subset \mathcal{P}_{CUT^P_R}$**:** Let $(\mathbf{f},\mathbf{x},\mathbf{z}) \in \mathcal{P}_{MCF_R}$ and assume that there exists a path P of length $H+1$ such that $\sum_{ij\in P} x_{ij} > H$, i.e. $(\mathbf{x},\mathbf{z}) \notin \mathcal{P}_{CUT^P_R}$. Let further $i'j' := \arg\max_{ij\in P} x_{ij}$. Then there exists $k \in R$ such that $f^k_{i'j'} = x_{i'j'}$. We denote this amount of flow by $a > 0$. In the worst case, a units of flow are sent through the whole path P. The complementary flow of commodity k is sent outside of P. Even if only a single edge is used on this complementary route, constraints (22) imply $(H+1)a + 1 - a = Ha + 1 \leq H$. Therefore, $a \leq \frac{H-1}{H}$. But then $\sum_{ij\in P} x_{ij} \leq (H+1)a < H$, which is a contradiction.

b) The inclusions follow from similar arguments as used in a), except for

$Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{MCF^+_F}) \subset Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{MCF_F})$**:** MCF^+_F is an aggregation of HD_F, except for constraints (18). These are implied by the flow conservation constraints (15) together with the fact that variables g^{kp}_{ij} are only defined for $p = 1,\ldots,H$.

c) $Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{HD_R}) \subset \mathcal{P}_{CUT^J_R}$**:** Assume that $(\mathbf{x},\mathbf{z}) \in Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{HD_R})$ and $(\mathbf{x},\mathbf{z}) \notin Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{CUT^J_R})$. Then there exists a $(H+2)$-jump J (where $S_{H+1} = R$) and such that $\sum_{ij\in J} x_{ij} = 1 - \epsilon$, and $\epsilon > 0$. Because of the flow preservation constraints (29) - (31) there needs to be a flow of ϵ on the path $P = \{ij : i \in S_i, j \in S_{i+1}, i = 0,\ldots,H+1\}$. This flow uses $H+2$ hops and cannot be composed of flow variables f^{kp}_{ij}, $p = 1,\ldots,H+1$, which is a contradiction.

$\mathcal{P}_{CUT^J_R} \subset \mathcal{P}_{CUT^P_R}$**:** Assume that $(\mathbf{x},\mathbf{z}) \in Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{CUT^J_R})$ and optimal and $(\mathbf{x},\mathbf{z}) \notin Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{CUT^P_R})$. Let P' be the path for which constraint (2) is violated, i.e. inequality $\sum_{ij\in P'}(x_{ij}+x_{ji}) > H$ holds. For the jump J with $S_0 = \{r\}$, $S_1 = \{i_1\}$, $S_2 = \{i_2\}$, $\ldots$, $S_H = \{i_H\}$, $S_{H+1} = V \setminus \{r, i_1, \ldots, i_H\}$ we have $\sum_{ij\in J} x_{ij} \geq 1$. By adding these two inequalities we get $\sum_{j:ij\in P'} x(\delta^-(j)) \geq \sum_{ij\in J\cup P'} x_{ij} > H+1$, thus $x(\delta^-(j)) > 1$ for some $j \in \{i_1, \ldots, i_{H+1}\}$, which is a contradiction to $(\mathbf{x},\mathbf{z})$ being an optimal LP solution.
In a similar way one can show that the jump formulation for HCSTP is stronger than the path formulation [Gouveia, 2009].

d) See the arguments in c).

e) $Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{HD_F}) \subset Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{HOP_F})$ Let $(\mathbf{x},\mathbf{z},\mathbf{f})$ be an arbitrary solution for the LP-relaxation of HD_F and let $(\mathbf{X}',\mathbf{x}',\mathbf{z}')$ be defined as follows: $x'_{jk} := x_{jk}$ for all jk in A_R; $z'_j := z_j$ for all j in F and $X'^p_{ij} := \max_{k\in F} f^{kp}_{ij}$ for all ij in A_S, $p = 1, \ldots, H$.
Then $(\mathbf{X}',\mathbf{x}',\mathbf{z}') \in \mathcal{P}_{HOP_F}$: From equations (25) and the definition of X' we have

$$z_j = \sum_{p=1}^{H} \sum_{ij\in A_S} f^{jp}_{ij} \leq \sum_{p=1}^{H} \sum_{ij\in A_S} \max_{k\in F} f^{kp}_{ij} = \sum_{p=1}^{H} \sum_{ij\in A_S} X'^p_{ij}$$

for compliance with equations (45). With $k^* := \arg\max_{k\in F} f^{kp}_{ij}$ estimations

$$X'^p_{ij} = \max_{k\in F} f^{kp}_{ij} \overset{(23)}{\leq} \sum_{\substack{l\in S\setminus\{j\}:\\ li\in A_S}} f^{k^*,p-1}_{li} \leq \sum_{\substack{l\in S\setminus\{j\}:\\ li\in A_S}} \max_{k\in F} f^{k,p-1}_{li} = \sum_{\substack{l\in S\setminus\{j\}:\\ li\in A_S}} X'^{p-1}_{li}$$

give equations (44). Constraints (3) - (5), (7), (8) are common in both models and thus met trivially.

$Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{HOP_F}) \subset Proj_{\mathbf{x},\mathbf{z}}(\mathcal{P}_{MTZ})$ MTZ is an aggregation of HOP_F (cf. Voß [1999]; $u_j := \sum_{p=1}^{H} pX^p_{ij}$).

□

4.1 Full Hierarchy of Formulations

The hierarchical scheme given in Figure 4.1 summarizes the relationships between the LP relaxations of the MIP models considered throughout this paper. A filled arrow specifies that the target formulation is strictly stronger than the source formulation. A double-headed arrow denotes formulations of equal strength. Whenever formulations are not comparable or we do not know their relation, this is not indicated in the figure for the sake of simplicity.

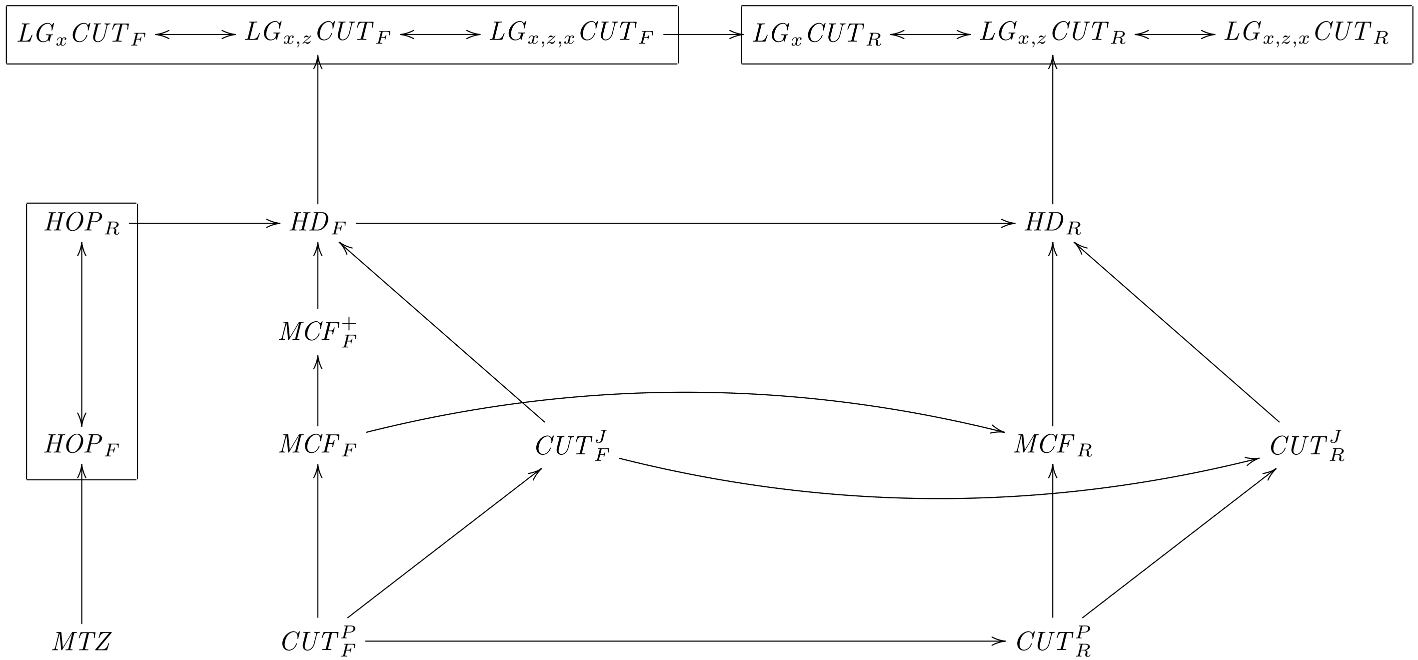


Figure 5: Relations between LP-relaxations of MIP models for ConFL.

5 Conclusions and Future Work

In this paper we provide an extensive theoretical comparison of LP relaxations of 17 MIP models for HC ConFL. We also introduce new sets of inequalities to model the corresponding (prize-collecting) HCSTP and HCMST. In particular, directed path constraints are shown to be strictly stronger than those originally proposed by Costa et al. [2009]. To model the prize-collecting HCSTP we introduce a new set of jump inequalities.

For HC ConFL we follow the concept presented in Gollowitzer and Ljubić [2009] to basically derive two groups of models. Our "F" models require connectivity among open facilities and the root node, and in addition a proper assignment of customers. We derive the stronger "R" models by requiring connectivity between customers and the root node. We describe a transformation of the HC ConFL into the ConFL on three variants of a layered graph. This leads to the strongest models in the presented hierarchy. A disaggregation of variables indicating facilities or customers in the layered graphs is found not to improve the quality of the LP lower bounds.

Our result regarding the relation between jump formulation and the two models based on path constraints and hop indexed multi-commodity flows extends to HCSTP and HCMST as well. The relation between jump constraints and the ones derived for the multi-commodity flow formulation remains an open question for all three problems, HC ConFL, HCSTP and HCMST. We believe that formulation MTZ is weaker than CUT_F^P. This relation is not known for HCSTP and HCMST either.

Future work on HC ConFL comprises questions of practical applicability of the studied MIP models. Our preliminary computational study (see Ljubić and Gollowitzer [2009]) shows that the facility based cut model on the layered graph outperforms its customer based counterpart. A further study of valid and facet defining inequalities that can improve the computational efficacy is of particular interest as well.

Acknowledgements

The authors thank to Luis Gouveia for very useful discussions on topics related to HCMST and HCSTP.

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