Problem 1

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1 Part a, Problem Statement

A graph (or a network) is a structure consisting of a set of objects (also known as vertices or nodes), some pairs of which are connected via edges. Assume that there are no self-edges (or loops). The degree of a vertex is the number of other vertices that share an edge with it. Say we are given a set of k colors. A graph is said to be k-colorable if each vertex can be assigned one of the k colors such that all neighboring vertices (i.e., any pair of vertices that share an edge) have different colors. (Some of the k colors may be left unused.) Prove that any graph with maximum degree d is (d + 1)-colorable.

1.1 Some Important Definitions

Degree:= Defined for a vertex, is the number of other vertexes connected to the vertex.
k-colorable:= If each vertex can be assigned one of the k colors in a way that no adjacent vertex has the same color, we say the graph is k-colorable.

1.2 Main Proof

To Prove that any graph of max degree d is (d+1)-colorable, we use proof by mathematical induction

we say, $p(x)$ = graph of max x degree is (x+1)-colorable

$$(p(1) \wedge (\forall x \in \mathbb{N}, p(x) \longrightarrow p(x+1))) \longrightarrow (x \in \mathbb{N}, p(x))$$

1.2.1 For p(1)

graph must be 2-colorable
Let the colors be in the set

$$S = \{c_1, c_2\}$$

And the nodes be;

$$a_1, a_2, ..., a_n \text{ for } n \in \mathbb{N}$$

Without loss of generality, we may choose, a_i for some $i \in \mathbb{N}$

now, there are two cases, either a_i has the color c_1 or c_2;
Again without loss of generality, let's choose the color to be c_1
We, may choose the colours of a_{i-1} and a_{i+1} to be c_2

And thus the predicate $p(1)$ is true

1.2.2 Assuming p(n-1), for p(n)

For $p(n)$, let's assume that $p(n-1)$ is true
That is for some set of colors

$$S = \{c_1, c_2, ..., c_n\}$$

we have nodes

$$a_1, a_2, ..., a_{n-1} \text{ for } n \in \mathbb{N}$$

such that no adjacent edges have the same color.

Now, we add one edge to the graph, and to make the graph have n degrees, we connect the node to every vertex of the rest of the graph,

we have the set of colors

$$S_1 = \{c_1, c_2, ..., c_{n+1}\}$$

the new vertex will have adjacent vertexes with colors from some set S_2 and clearly, $S_2 \subset S$
clearly, we can give the new edge the color from the set S_1 S_2, and we know that there exists at least one element in the set $S_1 \backslash S_2$ as $\{c_{n+1}\} = S_1 \backslash S$, hence $p(n)$ is true;

thus proved through **PMI** that $p(n)$ is true

2 Part b, Problem Statement

The number of subsets of an n-element set is 2^n

2.1 Main Proof

To Prove that the number of subsets of an n-element set is 2^n, we use proof by mathematical induction we say, $p(x) =$ set of x elements as 2^x elements

$$(p(1) \wedge (\forall x \in \mathbb{N}, p(x) \longrightarrow p(x+1))) \longrightarrow (x \in \mathbb{N}, p(x))$$

2.1.1 For p(1)

We can easily observe for a set with one element, let's say

$$S = \{a_1\}$$

clearly, the possible subsets are formed by taking or not taking one element. Thus the subsets formed are

$$\{\} \text{ and } \{a_1\}$$

Thus p(1) is true

2.1.2 Assuming p(n-1), for p(n)

Now, we assume p(n-1) to be true, that means

$$S = \{a_1, a_2, ..., a_{n-1}\} \text{ has } 2^{n-1} \text{ subsets } \forall n \in \mathbb{N}$$

now, we add a new element a_n to the set S, now for each subset, let's name S_i as the i^{th} subset; we have two subsets $S_i \cup \{a_n\}$ and S_i $\forall i \ \in \mathbb{N} \ \wedge i \leq n$

clearly, for every 2^{n-1} subset of the previous set, we have 2 new subsets; after adding a_n, thus we have $2 \times 2^{n-1}$ subsets, that is 2^n subsets

Hence Proven

3 Part c, Problem Statement

The number of ways of ranking n different objects is n!

3.1 Some Important definitions

$$\text{for any } n \in \mathbb{N}$$

$$n! = 1 \cdot 2 \cdot ... \cdot n$$

3.2 Main Proof

To Prove that the number of ways of ranking n different objects is n!
we say, $p(x) =$ number of ways of ranking x different objects is x!

$$(p(1) \wedge (\forall x \in \mathbb{N}, p(x) \longrightarrow p(x+1))) \longrightarrow (x \in \mathbb{N}, p(x))$$

3.2.1 For p(1)

clearly, the number of ways of ranking 1 object is 1, which is equal to 1!
thus we can say p(1) is true

3.2.2 Assuming p(n-1), for p(n)

now assuming, p(n-1) is true, that implies
there are $(n-1)!$ ways of ranking $n-1$ different objects

we add that n^{th} object in the order, we can clearly observe there are n spaces where the object could be placed, thus the number of ways become

$$\text{number of ways} = n \cdot (n-1)!$$

indeed this is the definition of n! thus we may write

$$\text{number of ways} = n!$$

Hence proved