Peridynamics

Research Paper
Notations

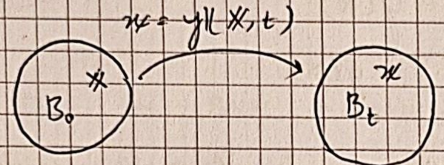
PD - Peridynamics
CCM - Classical Continuum Mechanics
CPD - Continuum - Kinematics Inspired Peridynamics.

↓ 1, 2, 3 neighbour interactions.

↓ Geometrically exact framework + captures Poisson effect.

no concept of stress & strains

non-locality assumption
⇒ any point $X \in B_0$ can interact with points within its finite neighbourhood $H_0(X)$ which is the <u>horizon</u>



$H_0(X)$

$H_t = y(H_0(X), t) \Rightarrow$ mapping of H_0

$\delta_0 =$ measure (H_0)
$\Rightarrow$ Radius, centred at X.

for $\delta_0 \to 0$ (infinitesimal neighbourhood) $\Rightarrow H_0 \to X,\ H_t \to x$
↳ local Continuum Mechanics.

<u>Notations</u>

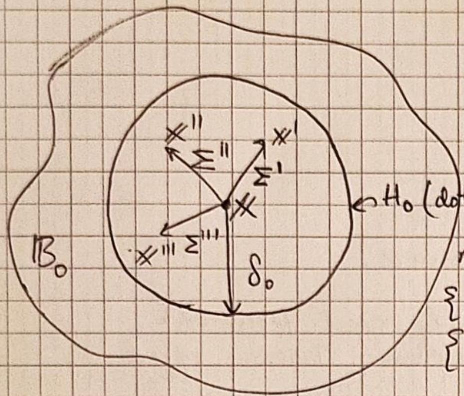
- $X' \in H_0(X) \longrightarrow$ neighbour of X.
- $x' \in H_t(x) \longrightarrow$ point within the horizon x mapped to X'

$H_t = y(H_0(X), t)$ can be rewritten as

$x' = y(X', t)$

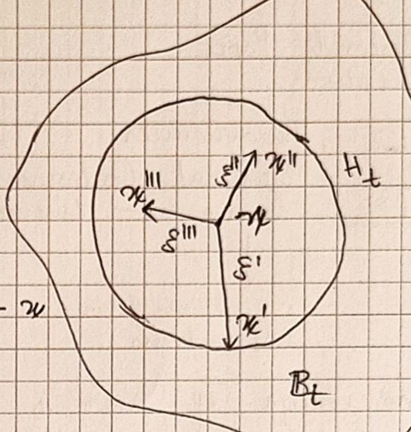
⇒ Similarly, neighbour sets for 1, 2, 3 neighbours
→ X', X'', X'''
mapped to x', x'', x''' respectively

- $\Xi^{\{\cdot\}}$ not the summation
  $\xi^{\{\cdot\}}$ ⇒ finite elements (relative position) in material & spatial configuration
  → $\{\cdot\}$ ~~denotes~~ identifies the neighbour



$x = y(X,t)$

neighbour sets
$\{X', X'', X'''\}$
$\{x', x'', x'''\}$ $\quad x^{',\,'',\,'''} - x$



$\Sigma' = X' - X$ $\qquad \xi' = y(X') - y(X)$

$\Sigma'' = X'' - X$ $\qquad \xi'' = y(X'') - y(X)$

$\Sigma''' = X''' - X$ $\qquad \xi''' = y'(X''') - y(X)$

Local Kinematic Measures

i) Deformation Gradient, $\mathbb{F} = \nabla_X y$

ii) Cofactor, $\mathbb{K} = \mathrm{cof}(\mathbb{F})$

iii) Determinant / Jacobian, $J = \det(\mathbb{F})$

They both mimic each other but with additional information. A different notation.

Non-Local Kinematic Measures

Relative Deformation measures.

(i) $|\xi'| \Rightarrow x' - x$ mimics $\mathbb{F}$ (Line element) $dX \longrightarrow dx$

→ Prescribes 1 neighbour interaction

ii) $a^{'/''}$ → area element, Nanson's formula
→ Prescribes 2 neighbour interaction

$\Rightarrow [x' - x] \times [x'' - x]$

$dA \longrightarrow da$

iii) $v^{'/''/'''}$
→ Volume element
→ Similar to J
→ 3-neighbour interaction

$\Rightarrow \left[[x' - x] \times [x'' - x]\right] \cdot [x''' - x]$

Balance Equations

Imp note: ★ The localisation procedure on global forms renders a point-wise formulation containing integrals over the horizon, thus they are non-local.

Balance of linear Momentum.

t_0^{ext} = External traction on the boundary N/m^2

Global ⇒ $$\int_{\partial B_0} t_0^{ext}\, dA + \int_{B_0} b_0^{ext}\, dV = 0$$

b_0^{ext} ↳ External force density per volume in material configuration ⇒ N/m^3

Comparison of CCM with CPD.

KM (above CCM), PD (above CPD)

	BOLM	BOAM
CCM	$\mathrm{Div}\,\mathbb{P} + \mathbb{b}_0^{ext} = \mathbb{0}$	$\mathbb{E} : [\mathbf{F} \cdot \mathbb{P}^T] = \mathbb{0}$
CPD	$\int_{H_0} \mathbb{P}'\, dV' + \mathbb{b}_0^{ext} = \mathbb{0}$	$\int_{H_0} \mathbb{E}' \times \mathbb{P}'\, dV' = \mathbb{0}$

$\mathbb{E} \rightarrow$ Third order Permutation Tensor

$\mathbb{P} \rightarrow$ Piola stress.

$\mathbb{P}' \rightarrow$ Force Density per volume squared $= N/(m^3)^2 = N/m^6$

BOLM in volume integral $\Rightarrow \int_{B_0} (\mathbb{b}_0^{int} + \mathbb{b}_0^{ext})\, dV = \mathbb{0}$.

$$\mathbb{b}_0^{int} = \int_{H_0} \mathbb{P}'\, dV'$$

Using the the BOLM is derived for CPD.

Similarly for BOAM:

Global form of quasi-static balance of momentum $\Rightarrow$

$$\int_{\partial B_0} y \times t_0^{ext}\, dA + \int_{B_0} y \times \mathbb{b}_0^{ext}\, dV = \mathbb{0}$$

$$\xrightarrow[\text{, localisation}]{\text{using BOLM}} \int_{H_0} \mathbb{E}' \times \mathbb{P}'\, dV' = \mathbb{0}$$

Constitutive Laws:

ψ_1', $\psi_2^{'/''}$, $\psi_3^{'/''/'''}$ $\Rightarrow$ Stored energy densities corresponding to 1, 2, 3 neighbours.

Point wise stored energy

$$\psi \Rightarrow \int_{H_0} \tfrac{1}{2}\, \psi_1'\, dV' + \int_{H_0}\int_{H_0} \tfrac{1}{3}\, \psi_2^{'/''}\, dV''\, dV' + \iiint_{H_0 H_0 H_0} \frac{1}{4}\, \psi_3^{'/''/'''}\, dV'''\, dV''\, dV'$$

$$\dot{\psi} = \int_{H_0} [\mathbb{P}_1' + \mathbb{P}_2' + \mathbb{P}_3'] \cdot \dot{\mathbb{E}}'\, dV'.$$

$$\mathbb{P}_1' = \frac{\partial \psi_1'}{\partial \mathbb{E}'} \; ; \; \mathbb{P}_2' = \int_{H_0} 2\, \mathbb{E}'' \times \frac{\partial \psi_2^{'/''}}{\partial a^{'/''}}\, dV''$$

$$\mathbb{P}_3' = \iint_{H_0 H_0} 3\, \mathbb{E}'' \times \mathbb{E}''' \frac{\partial \psi_3^{'/''/'''}}{\partial v^{'/''/'''}}\, dV'''\, dV''$$

Stuff to be calculated (1D, 2D, 3D) $\Rightarrow$ From PDF $\rightarrow$ Like a Summary.

[all integrals are over $\mathcal{H}_0 \rightarrow \int_{\mathcal{H}_0}$]

$\psi \rightarrow$ Point-wise stored energy density.

$$\psi = \int \frac{1}{2} \underbrace{\psi_1'}\, dv' + \iint \frac{1}{3} \underbrace{\psi_2^{1/11}}\, dv'' dv' + \iiint \frac{1}{4} \underbrace{\psi_3^{1/11/111}}\, dv''' dv'' dv'$$

1-neighbour interaction; 2-neighbour (1 & 2); 3-neighbour [1, 2 & 3] interaction.

Rate

$$\dot{\psi} = \int [P_1' + P_2' + P_3'] \cdot \dot{\xi}' \, dV \quad ; \quad P \rightarrow \frac{\text{Force density}}{(\text{volume})^2}$$

where, P_1', P_2' & P_3' [vectors] are defined as:

same as dx in NLKM. $\Leftarrow \xi' \Rightarrow$ 1st relative deformation measure $\rightarrow$ line measure.

$$P_1' = \frac{\partial \psi_1'}{\partial \xi'} \Rightarrow \text{corresponds to 1-neighbour interaction.}$$

$$P_2' = \int 2\xi'' \times \frac{\partial \psi_2^{1/11}}{\partial a^{1/11}} \, dV'' \Rightarrow \text{2-neighbour interaction}$$

Cross-product

$$P_3' = \iint 3\xi'' \times \xi''' \frac{\partial \psi_3^{1/11/111}}{\partial v^{1/11/111}} \, dV''' \, dV'' \Rightarrow \text{3-neighbour interaction.}$$

$$\dot{\psi} = \int P' \cdot \dot{\xi}' \, dV' \quad ; \quad P' = P_1' + P_2' + P_3'$$

Scalar-valued line, area & volume elements.

Both material & spatial Configurations

$l = |\xi'|$ $\qquad L = |\Sigma'|$

$a = |a^{1/11}| = |\xi' \times \xi''|$ $\qquad A = |A^{1/11}| = |\Sigma' \times \Sigma''|$

$v = |v^{1/11/111}| = |\xi' \cdot [\xi'' \times \xi''']|$ $\qquad V = |V^{1/11/111}| = |\Sigma' \cdot [\Sigma'' \times \Sigma''']|$

$\psi_1' = \psi_1(\xi') = \psi_1(l)$

$\psi_2' = \psi_2(a^{1/11}) = \psi_2(a)$

$\psi_3' = \psi_3(v^{1/11/111}) = \psi_3(v)$

Stored Energy Density per volume squared in material Configurations:

$$\Rightarrow \psi_1^{I} = \psi_1(l; L) = \frac{1}{2} C_1 L (s_1 - 1)^2 \Rightarrow$$ 1-neighbour interaction.

$$[C_1] = \frac{N.m}{m^7} \wedge s_1 = \frac{l}{L}.$$

$$\Rightarrow \psi_2^{I/II} = \psi_2(a; A) = \frac{1}{2} C_2 A (s_2 - 1)^2 \Rightarrow$$ 2-neighbour interaction.

$$[C_2] = \frac{Nm}{m^{11}} \wedge s_2 = \frac{a}{A}$$

$$\Rightarrow \psi_3^{I/II/III} = \psi_3(v; V) = \frac{1}{2} C_3 V (s_3 - 1)^2 \Rightarrow$$ 3-neighbour interaction

$$[C_3] = \frac{Nm}{m^{15}} \wedge s_3 = \frac{v}{V}.$$

Summary pasted (SS)

Table 2 Unification of concepts and fundamental relations of CPD

One-neighbour	Two-neighbour	Three-neighbour
Force density per volume squared in the material configuration with dimension N/m^6		
$p_1^{I} := \frac{\partial \psi_1^{I}}{\partial \xi^{I}}$	$p_2^{I} := \int_{\mathcal{H}_0} 2\, \xi^{II} \times \frac{\partial \psi_2^{I/II}}{\partial a^{I/II}} \, dV^{II}$	$p_3^{I} := \int_{\mathcal{H}_0} \int_{\mathcal{H}_0} 3\, \xi^{II} \times \xi^{III} \frac{\partial \psi_3^{I/II/III}}{\partial v^{I/II/III}} \, dV^{III}\, dV^{II}$
Angular momentum balance		
$\int_{\mathcal{H}_0} \xi^{I} \times p_1^{I} \, dV^{I} \overset{!}{=} \mathbf{0}$	$\int_{\mathcal{H}_0} \xi^{I} \times p_2^{I} \, dV^{I} \overset{!}{=} \mathbf{0}$	$\int_{\mathcal{H}_0} \xi^{I} \times p_3^{I} \, dV^{I} \overset{!}{=} \mathbf{0}$
Suitable deformation measures		
$l := \lvert\xi^{I}\rvert \quad , \quad L := \lvert\Xi^{I}\rvert$	$a := \lvert a^{I/II}\rvert \quad , \quad A := \lvert A^{I/II}\rvert$	$v := \lvert v^{I/II/III}\rvert \quad , \quad V := \lvert V^{I/II/III}\rvert$
Generic examples of interaction energy densities		
$\psi_1^{I} = \psi_1(l; L)$	$\psi_2^{I/II} = \psi_2(a; A)$	$\psi_3^{I/II/III} = \psi_3(v; V)$
Specific examples of interaction energy densities		
$\psi_1^{I} = \frac{1}{2} C_1 L \left[\frac{l}{L} - 1\right]^2$	$\psi_2^{I/II} = \frac{1}{2} C_2 A \left[\frac{a}{A} - 1\right]^2$	$\psi_3^{I/II/III} = \frac{1}{2} C_3 V \left[\frac{v}{V} - 1\right]^2$
$\frac{\partial \psi_1^{I}}{\partial \xi^{I}} = C_1 \left[\frac{l}{L} - 1\right] \frac{\xi^{I}}{\lvert\xi^{I}\rvert}$	$\frac{\partial \psi_2^{I/II}}{\partial a^{I/II}} = C_2 \left[\frac{a}{A} - 1\right] \frac{a^{I/II}}{\lvert a^{I/II}\rvert}$	$\frac{\partial \psi_3^{I/II/III}}{\partial v^{I/II/III}} = C_3 \left[\frac{V}{V} - 1\right] \frac{v^{I/II/III}}{\lvert v^{I/II/III}\rvert}$

Computational Implementation:

→ Replace integral equations with Quadrature

→ Discretized form of linear momentum balance

→ Solve, non-linear equation of the form $\mathbb{R} = 0$ using Newton - Raphson scheme.

→ Compute Tangent matrix $\mathbb{K}$ as linearised $\mathbb{R}$

→ Get $\mathbb{R}$ & $\mathbb{K}$ from force densities from constitutive laws. associated with stored energy density

Assumption:

⇒ The quadrature points & the points of evaluation coincides precisely ⇒ Collectively referred as "grid points" or just "points".

Process:

(i) → Discretise domain by a set of "points" [might refer as "grid points" also later]

→ It represents a continuum point as opposed to physical property.

→ Defines a neighbourhood $\mathcal{H}_0 \subset \mathcal{B}_0$

→ Grid point (P^a) is identified by its coordinates $\mathcal{X}^a$ & $\mathcal{x}^a$ for material and spatial configurations respectively.

→ The integral over the domain of an arbitrary quantity $\{\cdot\}$ is approximated by:

$$\int_{\mathcal{B}_0} \{\cdot\} \, dV = \sum_{a=1}^{\text{no. of pts } (\#P)} \{\cdot\}^a \, V^a \; ;$$

V^a ⇒ Volume of P^a

$\#P$ ⇒ total no. of grid Points P^a

[**Note:** V^a is usually calculated by the discretization strategy used (in our case its fixed 1, so no involvement).]

We also use uniform grid spacing so, l & L and other parameters can be directly used. ⇒ missed assumption.

Discretisation of the integrals involved.

→ $\# N$ ⇒ no. of points within the horizon of P^a

→ V_1 ⇒ The effective volume of neighbouring part i to one-neighbour interaction with P^a

Assuming all neighbours equally contribute

$$V_1 = \frac{\mathcal{V}_H}{\# N_1} \; ; \; \mathcal{V}_H = \int_{H_0} dV'$$

$\mathcal{V}_H$ → volume of neighbourhood of P^a

(eg) square with δ_0 radius

→ V_2 ⇒ The effective volume squared contributing to 2-neighbour interaction.

$$V_2 = \frac{[V_H]^2}{\# N_2} \; ; \; [V_H]^2 = \int_{H_0}\int_{H_0} dV'' dV'$$

→ V_3 ⇒ The effective volume cubed contributing to three neighbour interactions.

$$V_3 = \frac{[V_H]^3}{\# N_3} \; ; \; [V_H]^3 = \int_{H_0}\int_{H_0}\int_{H_0} dV''' dV'' dV'$$

Note: We assume the point P^a is completely inside the horizon irrespective of its size. [i.e if its a circle, the centre is the point P^a if only till the $3/4$ part is there but the centre pt is inside ⇒ We consider the whole body is inside; i.e. It is either inside (or) not]

The contributing Neighbours:

In 1-neighbour interaction by default all neighbours are contributing neighbours. But for 3- & 2- neighbour interactions the points $\underbrace{\{a, i, j, k\}}_{\text{3-neighbour}}$ & $\underbrace{\{a, i, j\}}_{\text{2-neighbour}}$ respectively the points and the neighbours should be non-collinear.

Why? ⇒ If they are collinear, it leads to a singular stiffness matrix.

The Integrals:

1- neighbour $\Rightarrow \int_{B_0}\int_{H_0} \{\cdot\}\, dV' dV = \sum_{a=1}^{\#P} \sum_{\substack{i=1 \\ i\neq a}}^{\#N} \{\cdot\}_i^a\, V_1 V^a$

Over the Body ↲ ↳ Neighbour

2- neighbour $\Rightarrow \int_{B_0}\int_{H_0}\int_{H_0} \{\cdot\}\, dV'' dV' dV = \sum_{a=1}^{\#P} \sum_{\substack{i=1 \\ i\neq a}}^{\#N} \sum_{\substack{j=1 \\ j\neq a \\ j\neq i}}^{\#N} \{\cdot\}_{ij}^a\, V_2 V^a$

3- neighbour $\Rightarrow \int_{B_0}\int_{H_0}\int_{H_0} \{\cdot\}\, dV''' dV'' dV' dV = \sum_{a=1}^{\#P} \sum_{\substack{i=1 \\ i\neq a}}^{\#N} \sum_{\substack{j=1 \\ j\neq a \\ j\neq i}}^{\#N} \sum_{\substack{k=1 \\ k\neq a \\ k\neq i \\ k\neq j}}^{\#N} \{\cdot\}_{ijk}^a\, V_3 V^a$

Discretised Balance of Linear Momentum:

Global form: $\int_{H_0} \mathbb{P}' dV' = \mathbb{0}$ $\Rightarrow$ Same P ; just to give a tensor 0, tensor P is used.

(Non-local) not exactly Global

Local form: Reduced BOLM

$$\int_{H_0} \mathbb{P}_1' dV' + \int_{H_0} \mathbb{P}_2' dV' + \int_{H_0} \mathbb{P}_3' dV' = \mathbb{0}$$

Point-Wise Balance of Linear Momentum expressed as (In Residual form).

$\mathbb{R} = \mathbb{0}$; $\mathbb{R} = \mathbb{R}_1 + \mathbb{R}_2 + \mathbb{R}_3 = \mathbb{0}$; $\mathbb{R}, \mathbb{R}_1, \mathbb{R}_2, \mathbb{R}_3$ are Residual vectors. (tensors) and represent 1, 2, 3- neighbour interaction respectively.

mostly it'll be referred like this

R_1 $\quad \mathbb{R}_1 = \int_{H_0} \mathbb{P}_1' dV' = \int_{H_0} \frac{\partial \psi_1'}{\partial \xi'} dV'$

$$\mathbb{R}_2 = \int_{H_0} \mathbb{P}_2' dV' = \int_{H_0}\int_{H_0} 2\, \xi'' \times \frac{\partial \psi_2^{\,1/11}}{\partial a^{1/11}} dV'' dV'$$

$$\mathbb{R}_3 = \int_{H_0} \mathbb{P}_3' dV' = \int_{H_0}\int_{H_0}\int_{H_0} 3\, \xi'' \times \xi''' \frac{\partial \psi_3^{\,1/11/111}}{\partial a^{1/11/111}} dV''' dV'' dV'$$

Residual Vector Discretization:

$$\mathbb{R} = \begin{bmatrix} \mathbb{R}^1 \\ \mathbb{R}^2 \\ \vdots \\ \mathbb{R}^a \\ \vdots \\ \mathbb{R}^{\#P} \end{bmatrix} = \begin{bmatrix} \mathbb{R}^1_1 + \mathbb{R}^1_2 + \mathbb{R}^1_3 \\ \mathbb{R}^2_1 + \mathbb{R}^2_2 + \mathbb{R}^2_3 \\ \vdots \\ \mathbb{R}^a_1 + \mathbb{R}^a_2 + \mathbb{R}^a_3 \\ \vdots \\ \mathbb{R}^{\#P}_1 + \mathbb{R}^{\#P}_2 + \mathbb{R}^{\#P}_3 \end{bmatrix}$$

The point-wise discretised residual vector $\mathbb{R}^a$ of the point p^a is comprised of point-wise discretized residual vectors $\mathbb{R}^a_1$, $\mathbb{R}^a_2$, $\mathbb{R}^a_3$ corresponding to 1, 2, 3- neighbour interaction respectively.

$$\mathbb{R}^a_1 = \sum_{\substack{i=1 \\ i=a}}^{\#N} \frac{\partial \psi_1'}{\partial \xi'} V_1$$

$$\mathbb{R}^a_2 = \sum_{\substack{i=1 \\ i\neq a}}^{\#N} \sum_{\substack{j=1 \\ j=a \\ j=i}}^{\#N} 2\xi'' \times \frac{\partial \psi_2^{1/11}}{\partial a^{1/11}} V_2$$

$$\mathbb{R}^a_3 = \sum_{\substack{i=1 \\ i\neq a}}^{\#N} \sum_{\substack{j=1 \\ j\neq a \\ j\neq i}}^{\#N} \sum_{\substack{k=1 \\ k\neq a \\ k\neq i \\ k\neq j}}^{\#N} 3\xi'' \times \xi''' \frac{\partial \psi_3^{1/11/111}}{\partial v^{1/11/111}} V_3 .$$

$$\xi' = x^i - x^a$$
$$\xi'' = x^j - x^a$$
$$\xi''' = x^k - x^a$$

Global deformation vector "x"

$$[x] = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^a \\ \vdots \\ x^{\#P} \end{bmatrix}$$

The approximate solution of $\mathbb{R} = 0$ is obtained by an iterative Newton-Raphson scheme. The consistent linearisation of the resulting system at iteration k reads

$$\Rightarrow \mathbb{R}_{k+1} = 0 \quad \text{with} \quad \mathbb{R}_{k+1} = \mathbb{R}_k + \left.\frac{\partial \mathbb{R}}{\partial x}\right|_k \cdot \Delta x_k$$

$$\Rightarrow \mathbb{R}_k + \left.\frac{\partial \mathbb{R}}{\partial x}\right|_k \Delta x_k = 0$$

(A)

$$\Delta x_k = -\mathbb{K}_k^{-1} \mathbb{R}_k \quad \text{with} \quad \mathbb{K}_k = \left.\frac{\partial \mathbb{R}}{\partial x}\right|_k$$

where $\mathbb{K}_k$ is algorithmic tangent stiffness at iteration k.

A Δx is updated everytime

$$x_{k+1} = x_k + \Delta x_k$$

Tangent Stiffness Matrix.

$\mathbb{K}^{ab}$ = Point-wise contributions.

$$[\mathbb{K}] = \begin{bmatrix} \mathbb{K}^{11} & \mathbb{K}^{12} & \cdots & \mathbb{K}^{1b} & \cdots & \mathbb{K}^{1\#P} \\ \mathbb{K}^{21} & \mathbb{K}^{22} & \cdots & \mathbb{K}^{2b} & \cdots & \mathbb{K}^{2\#P} \\ \vdots & \vdots & & \vdots & & \vdots \\ \mathbb{K}^{a1} & \mathbb{K}^{a2} & \cdots & \mathbb{K}^{ab} & \cdots & \mathbb{K}^{a\#P} \\ \vdots & \vdots & & \vdots & & \vdots \\ \mathbb{K}^{\#P1} & \mathbb{K}^{\#P2} & \cdots & \mathbb{K}^{\#Pb} & \cdots & \mathbb{K}^{\#P\#P} \end{bmatrix} \quad \text{with} \quad \boxed{\mathbb{K}^{ab} = \frac{\partial \mathbb{R}^a}{\partial x^b}}$$

It is further decomposed as

$$\mathbb{K}^{ab} = \mathbb{K}_1^{ab} + \mathbb{K}_2^{ab} + \mathbb{K}_3^{ab}$$

with, $\mathbb{K}_1^{ab} = \dfrac{\partial \mathbb{R}_1^a}{\partial x^b}$; $\mathbb{K}_2^{ab} = \dfrac{\partial \mathbb{R}_2^a}{\partial x^b}$; $\mathbb{K}_3^{ab} = \dfrac{\partial \mathbb{R}_3^a}{\partial x^b}$

Final Steps of Computation:

(i) Express discretised Residual and Tangent interms of P^a & its neighbours

(ii) Compute their associated tangents.

Recall: $\Sigma^{I/II/III} = X^{i,j,k} - X^a$; $\xi^{I/II/III} = x^{i,j,k} - x^a$

$$\mathbb{R}_1^a = \sum_{\substack{i=1 \\ i\neq a}}^{\#N} C_1 \left[\frac{1}{|\Sigma'|} - \frac{1}{|\xi'|}\right] \xi' \, V_1$$

$$\mathbb{R}_2^a = \sum_{\substack{i=1 \\ i\neq a}}^{\#N} \sum_{\substack{j=1 \\ j\neq a \\ j\neq i}}^{\#N} 2C_2 \left[\frac{1}{|\Sigma'\times\Sigma''|} - \frac{1}{|\xi'\times\xi''|}\right] \left[\xi'[\xi''\cdot\xi''] - \xi''[\xi''\cdot\xi']\right] \cdot V_2$$

$$\mathbb{R}_3^a = \sum_{\substack{i=1 \\ i\neq a}}^{\#N} \sum_{\substack{j=1 \\ j\neq a \\ j\neq i}}^{\#N} \sum_{\substack{k=1 \\ k\neq a \\ k\neq i \\ k\neq j}}^{\#N} 3C_3 \, [\xi'\times\xi''] \left[\frac{1}{[[\Sigma'\times\Sigma'']\cdot\Sigma''']} - \frac{1}{[[\xi'\times\xi'']\cdot\xi''']}\right] \left[\xi'''\cdot[\xi'\times\xi'']\right] \cdot V_3$$

$$\mathbb{K}_1^{ab} = \frac{\partial R^a_1}{\partial x^b} = \sum_{\substack{i=1\\ i\neq a}}^{\#N} C_1 \left[\delta^{ib} - \delta^{ab}\right] \left[\frac{1}{|\xi'|^3}\, \xi' \otimes \xi' + \left[\frac{1}{|\Sigma'|} - \frac{1}{|\xi'|}\right] \mathbf{i} \right] \cdot V_1$$

$$\mathbb{K}_2^{ab} = \sum_{\substack{i=1\\ i\neq a}}^{\#N} \sum_{\substack{j=1\\ j\neq a\\ j\neq i}}^{\#N} 2C_2 \left[\delta^{ib} - \delta^{ab}\right] \frac{1}{|\xi' \times \xi''|^3} \left[\left[\xi''\cdot\xi'\right]\xi'' - \left[\xi''\cdot\xi''\right]\xi'\right] \otimes \left[\left[\xi''\cdot\xi'\right]\xi'' - \left[\xi''\cdot\xi''\right]\xi'\right] V_2$$

$$+ \sum_{\substack{i=1\\ i\neq a}}^{\#N} \sum_{\substack{j=1\\ j\neq a\\ j\neq i}}^{\#N} 2C_2 \left[\delta^{ib} - \delta^{ab}\right] \left[\frac{1}{|\xi' \times \xi''|} - \frac{1}{|\Sigma' \times \Sigma''|}\right] \left[\xi'' \otimes \xi'' - \left[\xi''\cdot\xi''\right]\mathbf{i}\right] V_2$$

$$+ \sum_{\substack{i=1\\ i\neq a}}^{\#N} \sum_{\substack{j=1\\ j\neq a\\ j\neq i}}^{\#N} 2C_2 \left[\delta^{jb} - \delta^{ab}\right] \frac{1}{|\xi' \times \xi''|^3} \left[\left[\xi''\cdot\xi'\right]\xi'' - \left[\xi''\cdot\xi''\right]\xi'\right] \otimes \left[\left[\xi'\cdot\xi''\right]\xi' - \left[\xi'\cdot\xi'\right]\xi''\right] V_2$$

$$+ \sum_{\substack{i=1\\ i\neq a}}^{\#N} \sum_{\substack{j=1\\ j\neq a\\ j\neq i}}^{\#N} 2C_2 \left[\delta^{jb} - \delta^{ab}\right] \left[\frac{1}{|\xi' \times \xi''|} - \frac{1}{|\Sigma' \times \Sigma''|}\right] \left[\xi'' \otimes \xi' + \left[\xi''\cdot\xi'\right]\mathbf{i} - 2\xi' \otimes \xi''\right] V_2$$

$$\mathbb{K}_3^{ab} = \sum_{\substack{i=1\\ i\neq a}}^{\#N} \sum_{\substack{j=1\\ j\neq a\\ j\neq i}}^{\#N} \sum_{\substack{k=1\\ k\neq a\\ k\neq i\\ k\neq j}}^{\#N} 3C_3 \left[\delta^{ib} - \delta^{ab}\right] \frac{1}{|[\Sigma' \times \Sigma'']\Sigma'''|} \left[\left[\xi'' \times \xi'''\right] \otimes \left[\xi'' \times \xi'''\right]\right] V_3$$

$$+ \sum_{\substack{i=1\\ i\neq a}}^{\#N} \sum_{\substack{j=1\\ j\neq a\\ j\neq i}}^{\#N} \sum_{\substack{k=1\\ k\neq a\\ k\neq i\\ k\neq j}}^{\#N} 3C_3 \left[\delta^{jb} - \delta^{ab}\right] \left[\frac{1}{|[\Sigma' \times \Sigma'']\cdot\Sigma'''|} - \frac{1}{|[\xi' \times \xi'']\cdot\xi'''|}\right] \left[\left[\xi' \times \xi''\right]\cdot\xi'''\right]\left[\varepsilon\cdot\xi'''\right] V_3$$

3rd order Permutation Tensor.

$$+ \cdots$$

$$\ldots + \sum_{\substack{i=1\\ i\neq a}}^{\#N} \sum_{\substack{j=1\\ j\neq a\\ j\neq i}}^{\#N} \sum_{\substack{k=1\\ k\neq a\\ k\neq i\\ k\neq j}}^{\#N} 3C_3 \left[\delta^{jb} - \delta^{ab}\right] \left[\frac{1}{\left|\left[\Sigma' \times \Sigma''\right]\cdot \Sigma'''\right|}\right] \left[\left[\xi'' \times \xi'''\right] \otimes \left[\xi'' \times \xi'\right]\right] \cdot V_3$$

$$- \sum_{\substack{i=1\\ i\neq a}}^{\#N} \sum_{\substack{j=1\\ j\neq a\\ j\neq i}}^{\#N} \sum_{\substack{k=1\\ k\neq a\\ k\neq i\\ k\neq j}}^{\#N} 3C_3 \left[\delta^{kb} - \delta^{ab}\right] \left[\frac{1}{\left|\left[\Sigma' \times \Sigma''\right]\cdot \Sigma'''\right|} - \frac{1}{\left|\left[\xi' \times \xi''\right]\cdot \xi'''\right|}\right] \left[\frac{\left[\xi' \times \xi''\right]\cdot \xi'''}{\left[\xi \cdot \xi''\right]}\right] . V_3$$

$$+ \sum_{\substack{i=1\\ i\neq a}}^{\#N} \sum_{\substack{j=1\\ j\neq a\\ j\neq i}}^{\#N} \sum_{\substack{k=1\\ k\neq a\\ k\neq i\\ k\neq j}}^{\#N} 3C_3 \left[\delta^{kb} - \delta^{ab}\right] \frac{1}{\left|\left[\Sigma' \times \Sigma''\right]\cdot \Sigma'''\right|} \left[\left[\xi'' \times \xi'''\right] \otimes \left[\xi' \times \xi''\right]\right] . V_3$$

all these having repeated terms ⇒ can be made compact, but for simplicity, the expanded form.

The stiffness components of 1-, 2-, 3- neighbour interactions are all symmetric.