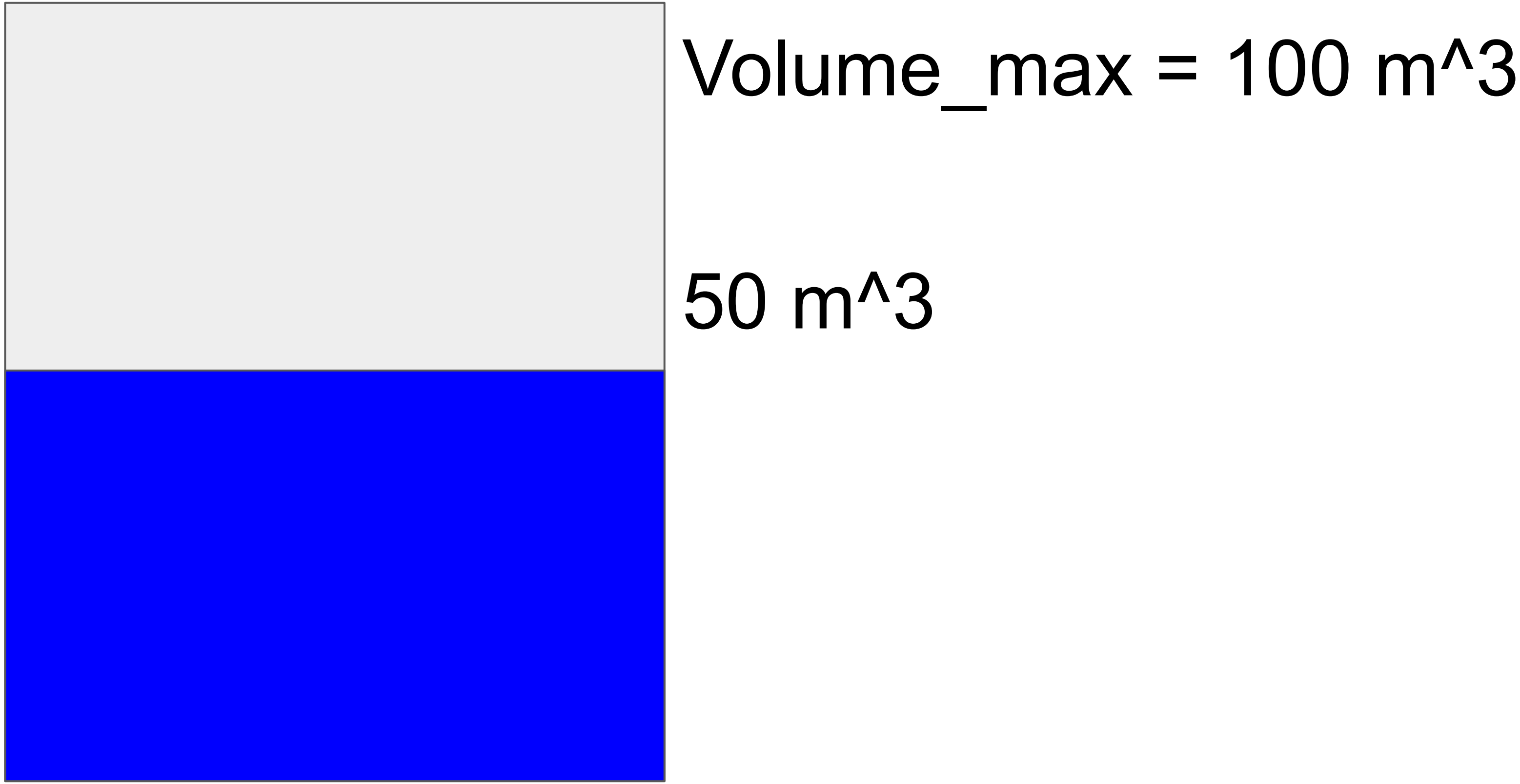
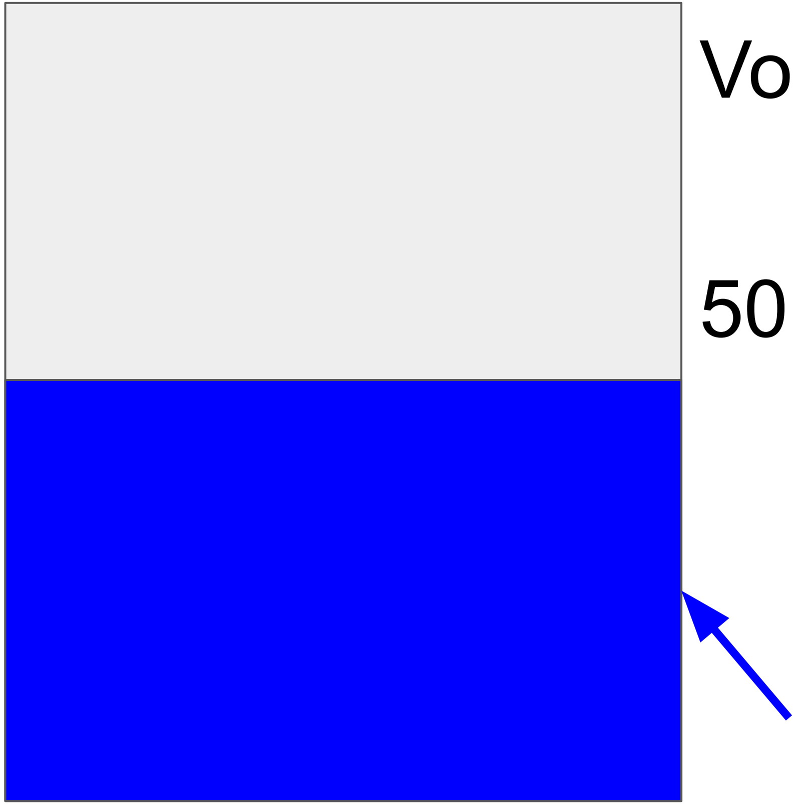
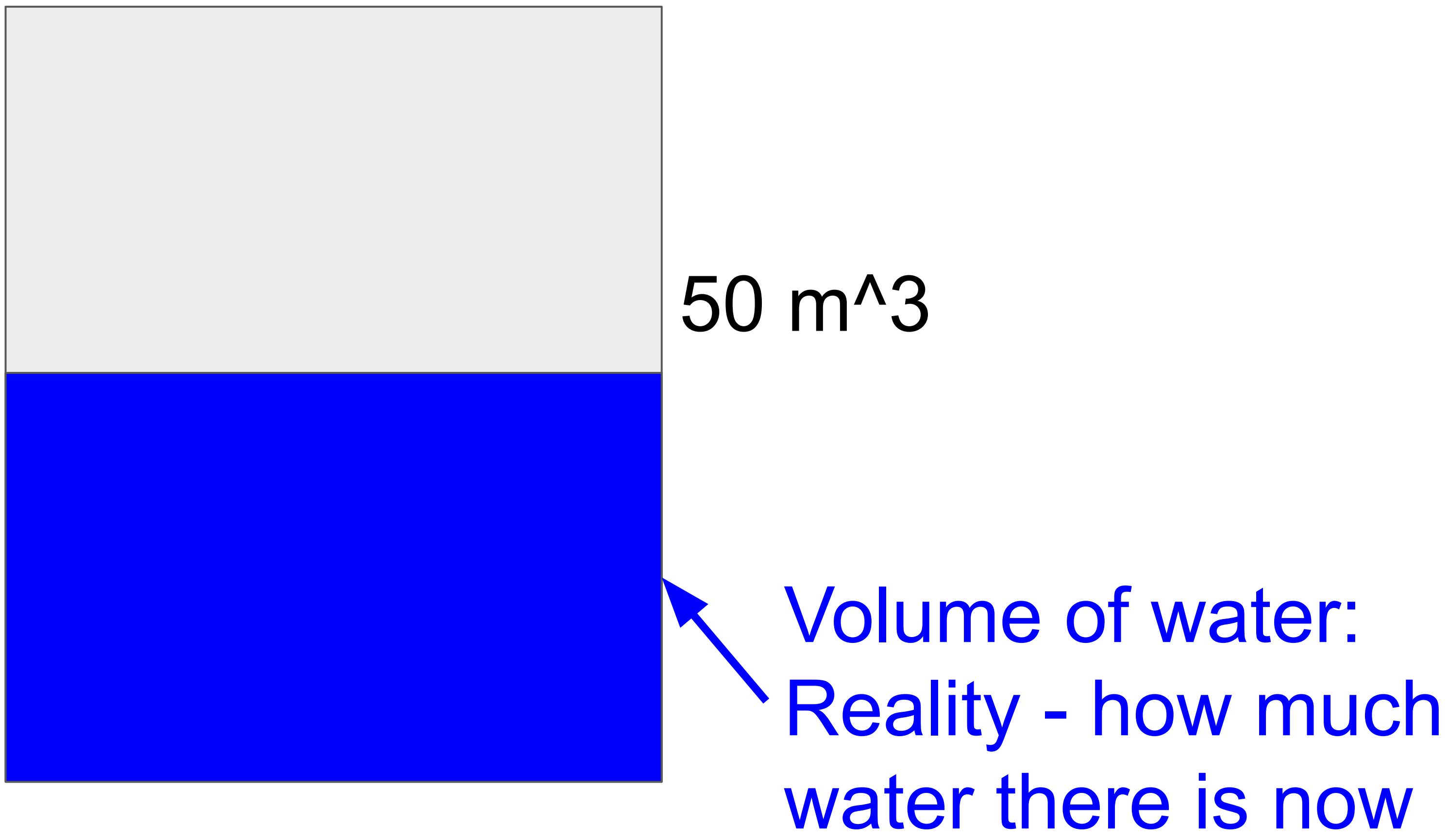
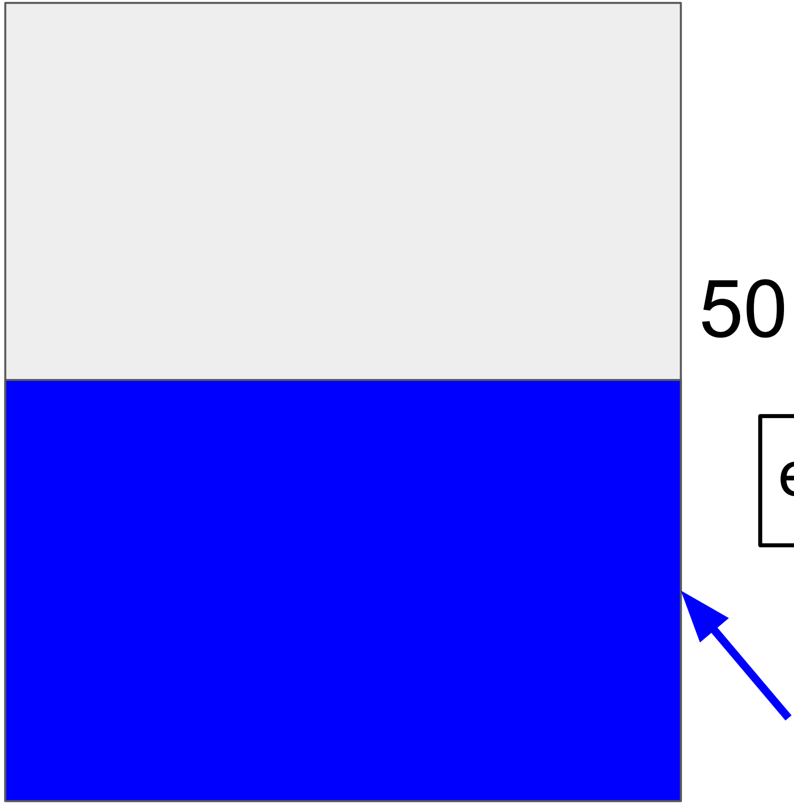
Proportional Controller

Mark Misin


Volume_max = 100 m^3
50 m^3


Volume_max = 100 m^3
50 m^3
Volume of water:
Reality - how much
water there is now

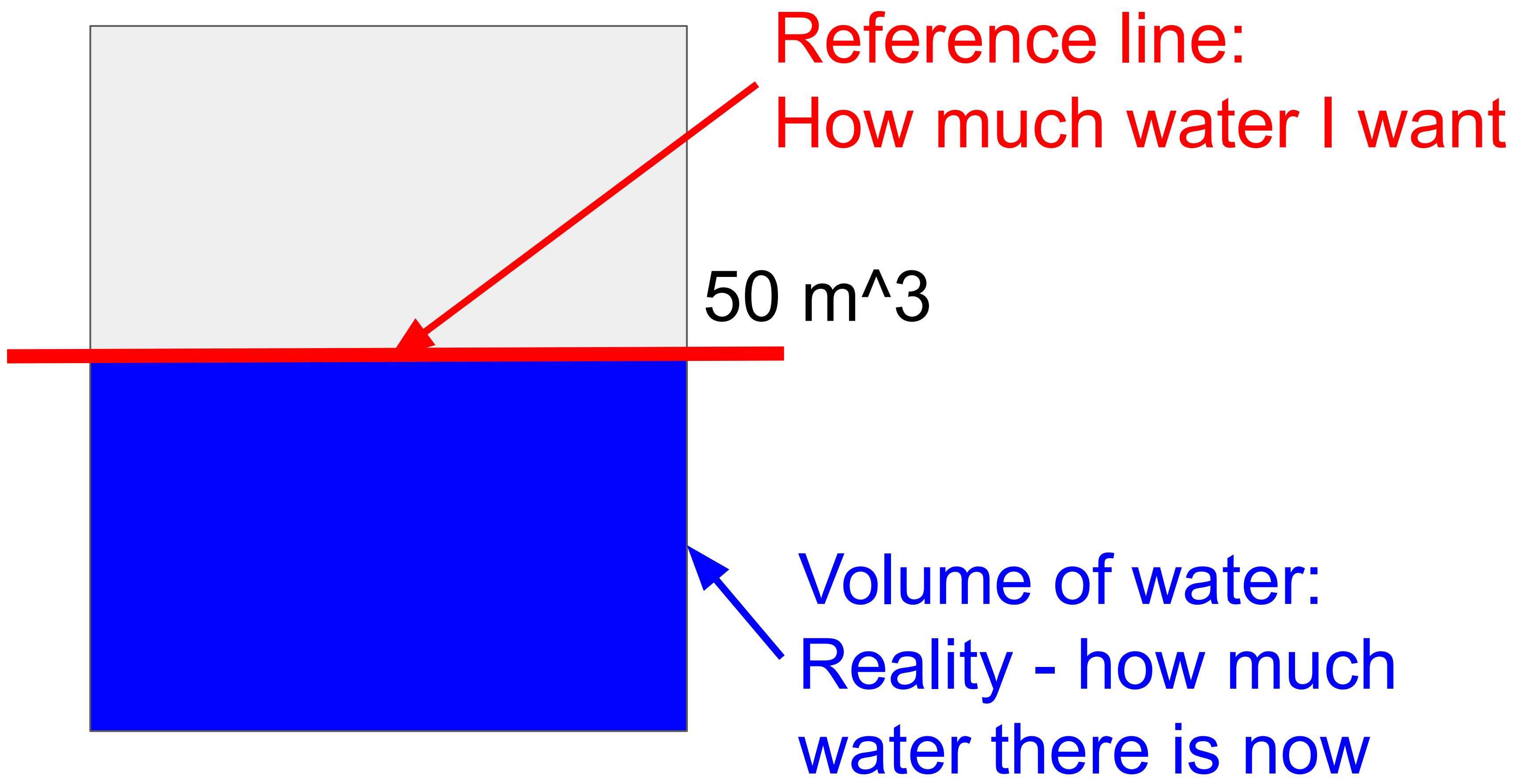

50 m^3
Volume of water:
Reality - how much
water there is now



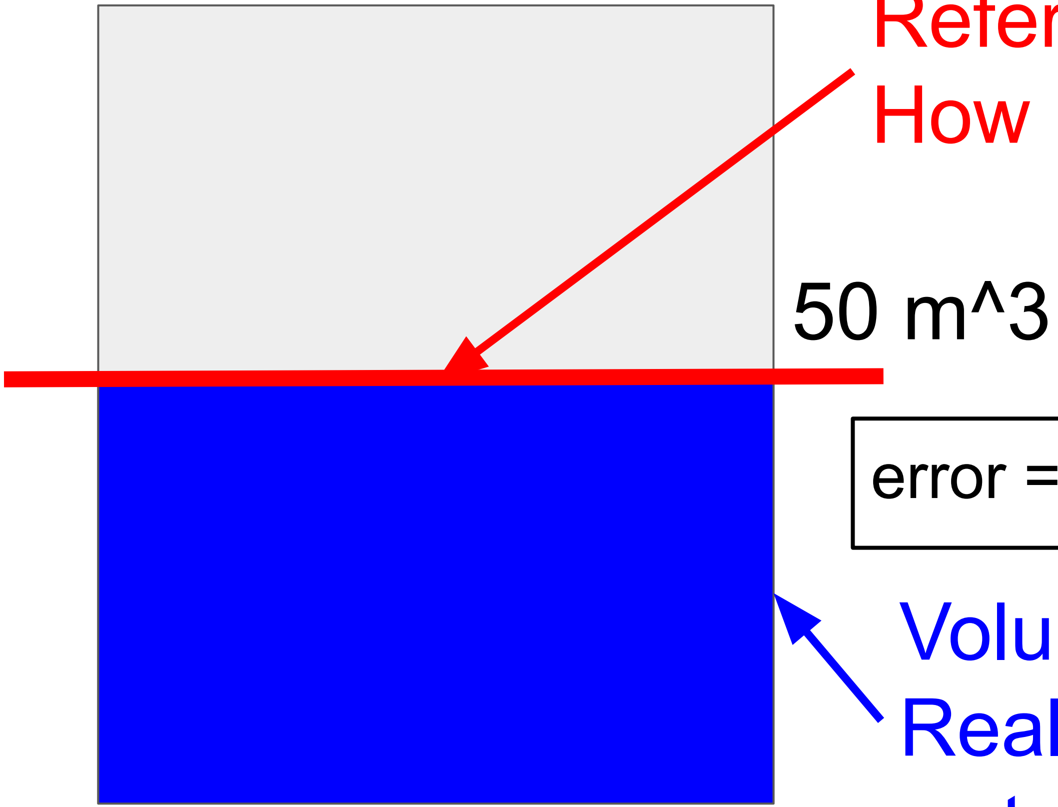
50 m^3

error = reference - volume

Volume of water:
Reality - how much water there is now

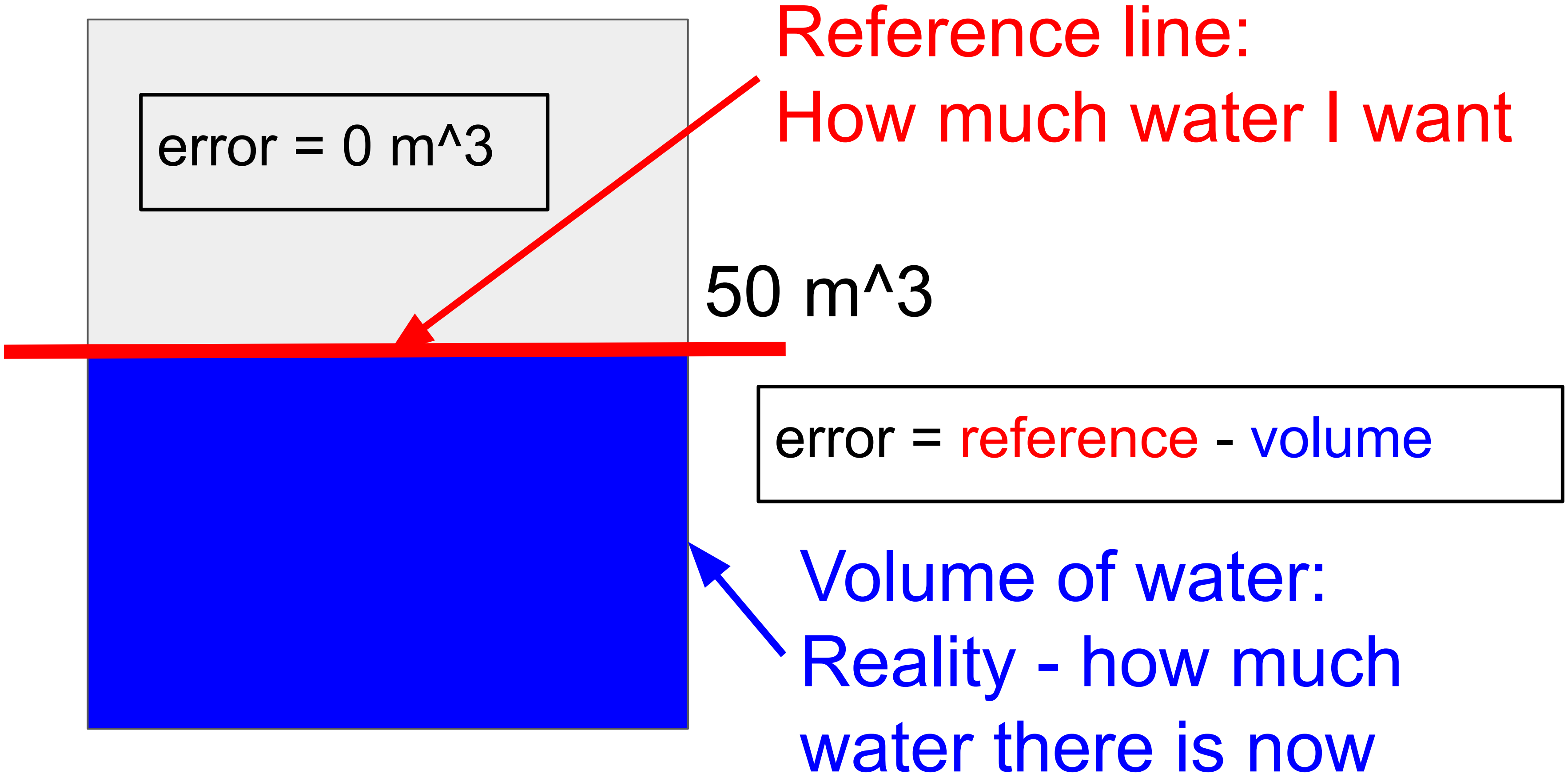

Reference line:
How much water I want
50 m^3
Volume of water:
Reality - how much
water there is now

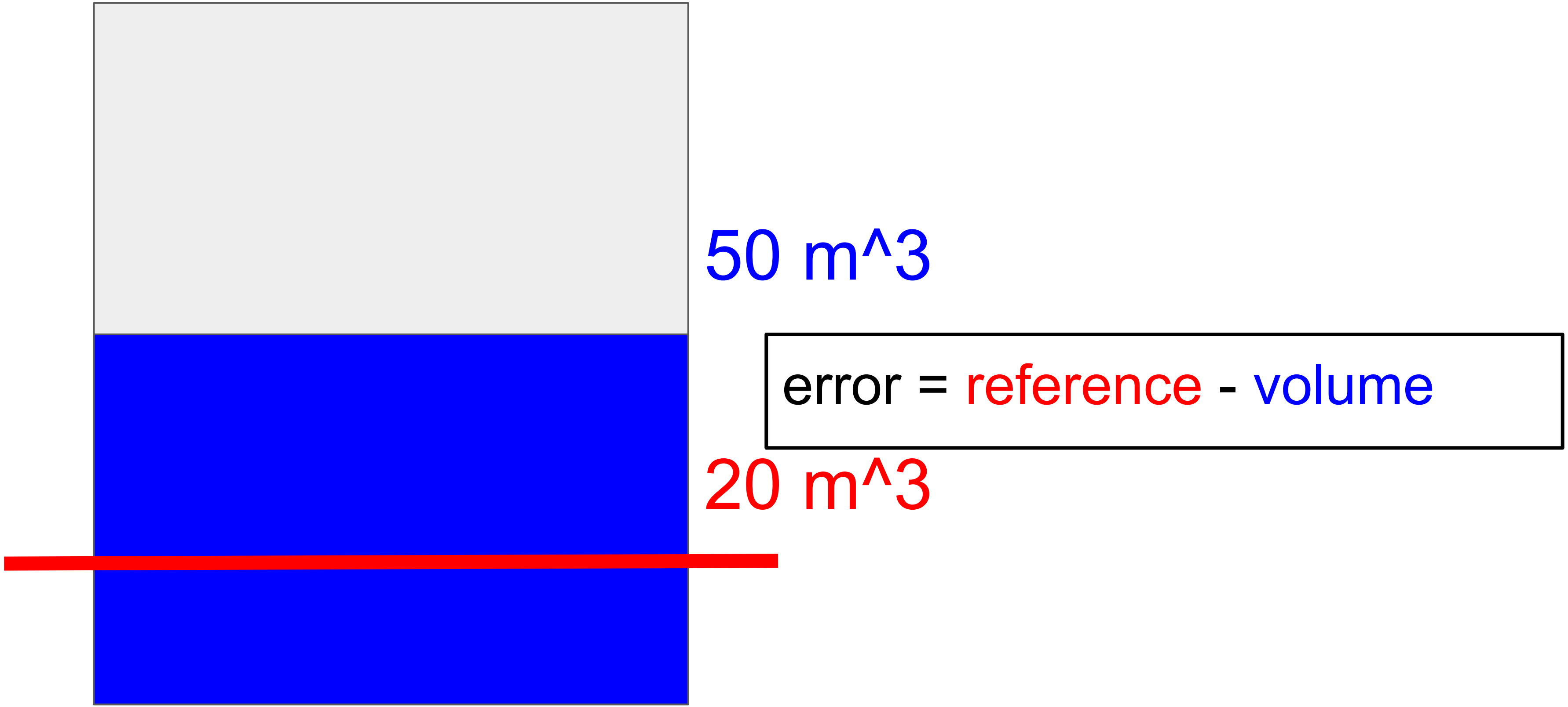
Reference line:
How much water I want

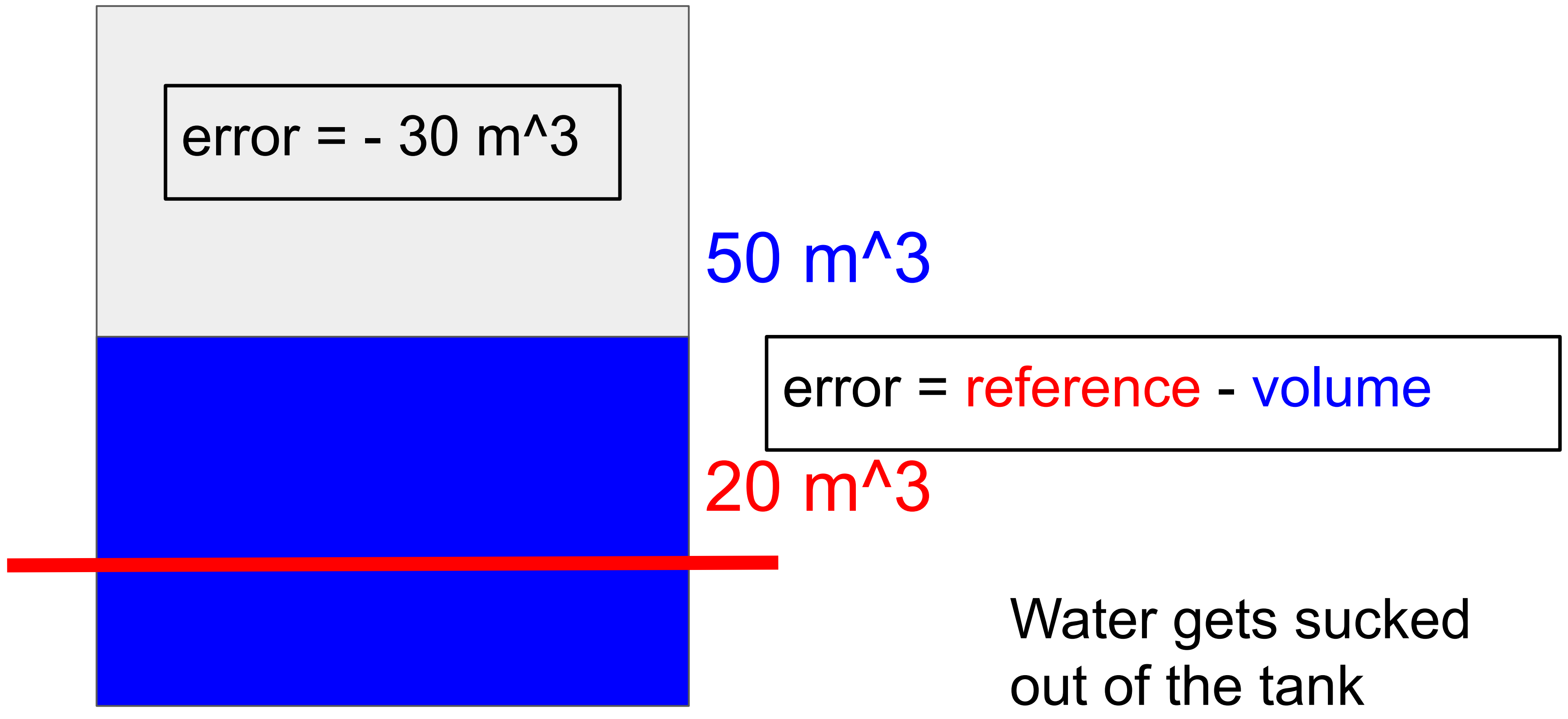


error = reference - volume

Volume of water:
Reality - how much
water there is now

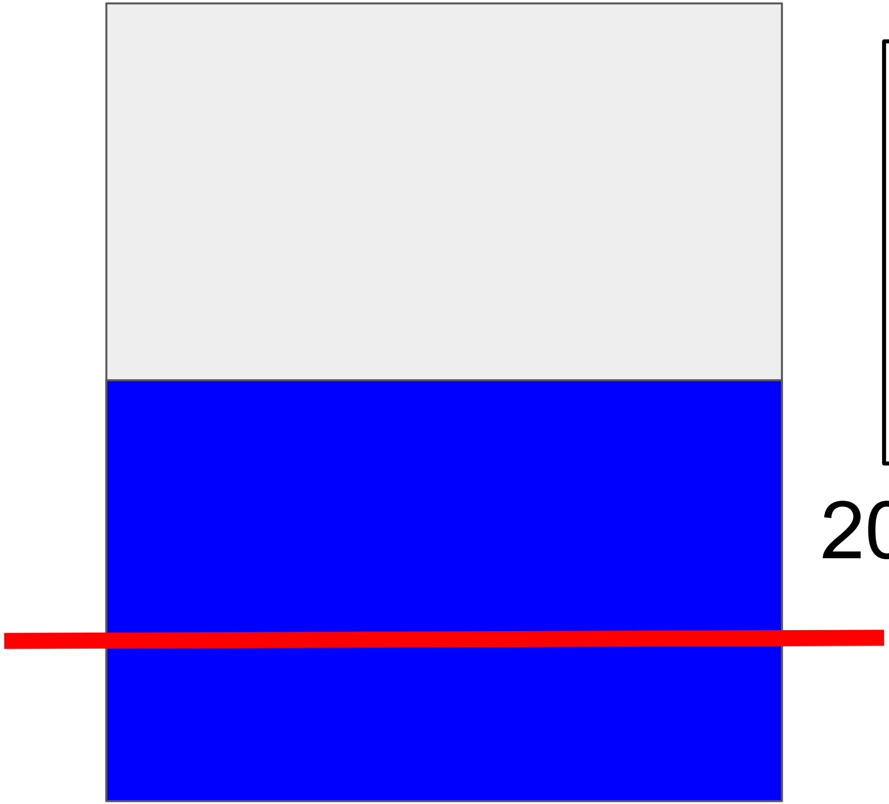

error = 0 m^3
Reference line:
How much water I want
50 m^3
error = reference - volume
Volume of water:
Reality - how much
water there is now


50 m^3
error = reference - volume
20 m^3

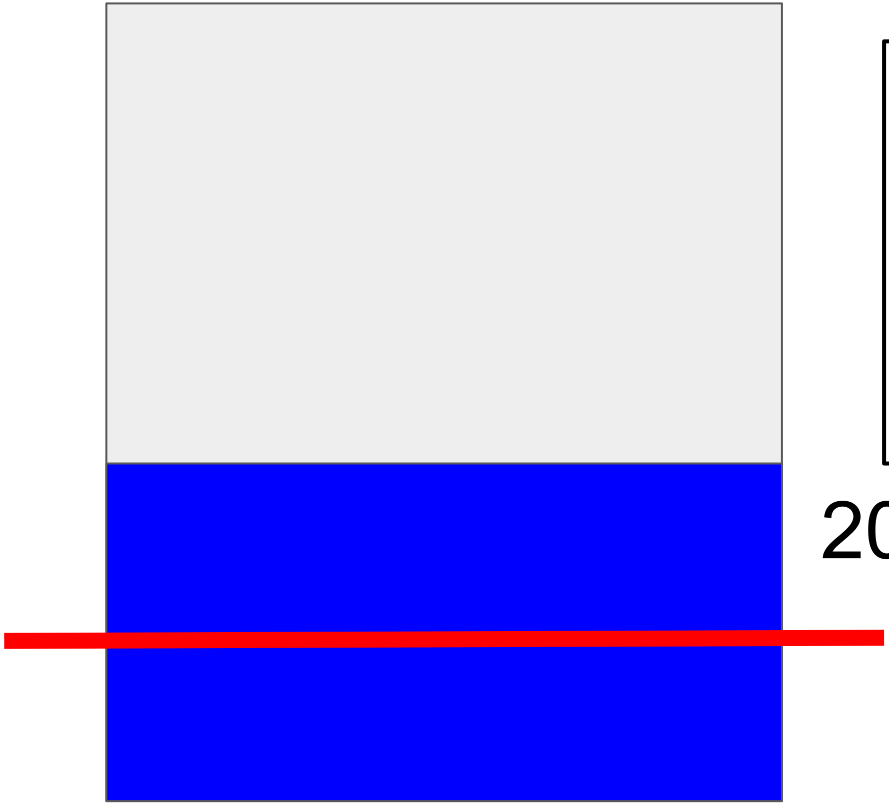

error = - 30 m^3
50 m^3
error = reference - volume
20 m^3
Water gets sucked
out of the tank

A sensor measures the volume of water every 0.02 seconds OR @ 50 Hz

$f = 1/0.02$ [1/s = Hz]



20 m^3



20 m^3

A sensor measures the volume of water every 0.02 seconds OR @ 50 Hz

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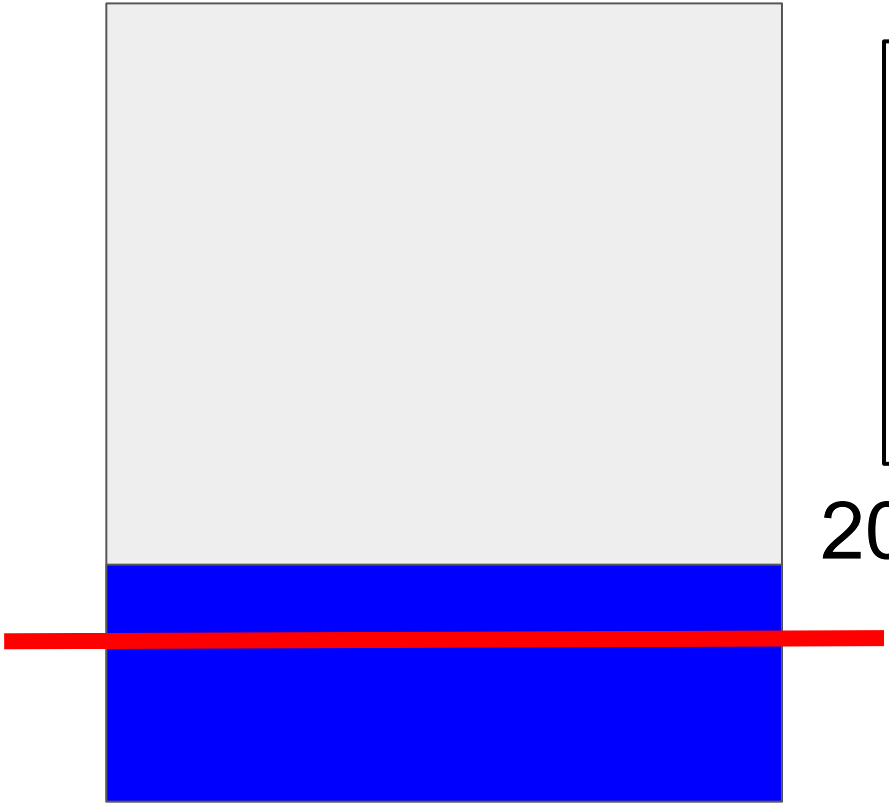
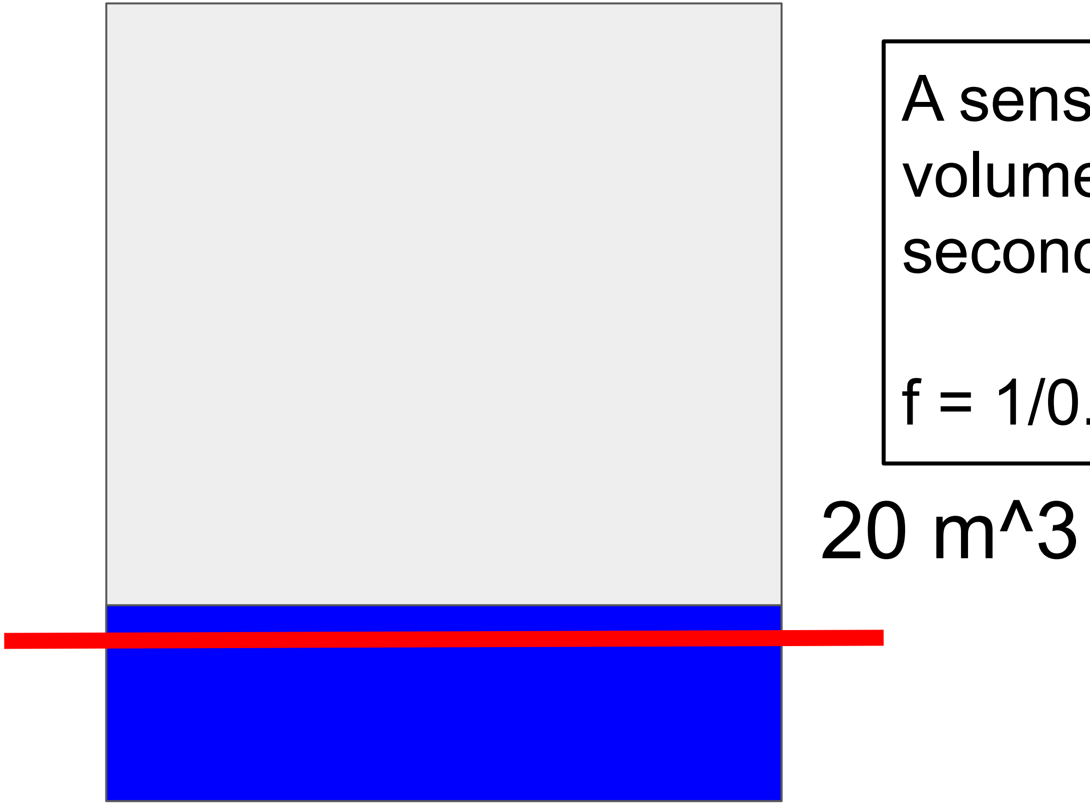


A sensor measures the volume of water every 0.02 seconds OR @ 50 Hz

f = 1/0.02 [1/s = Hz]

20 m^3



A sensor measures the volume of water every 0.02 seconds OR @ 50 Hz

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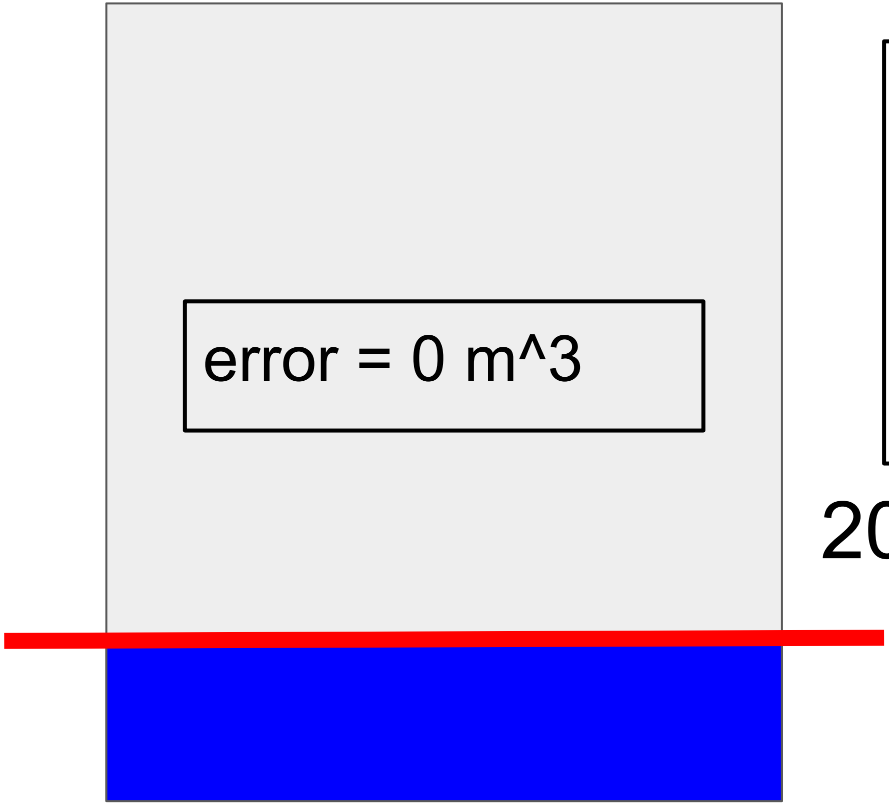
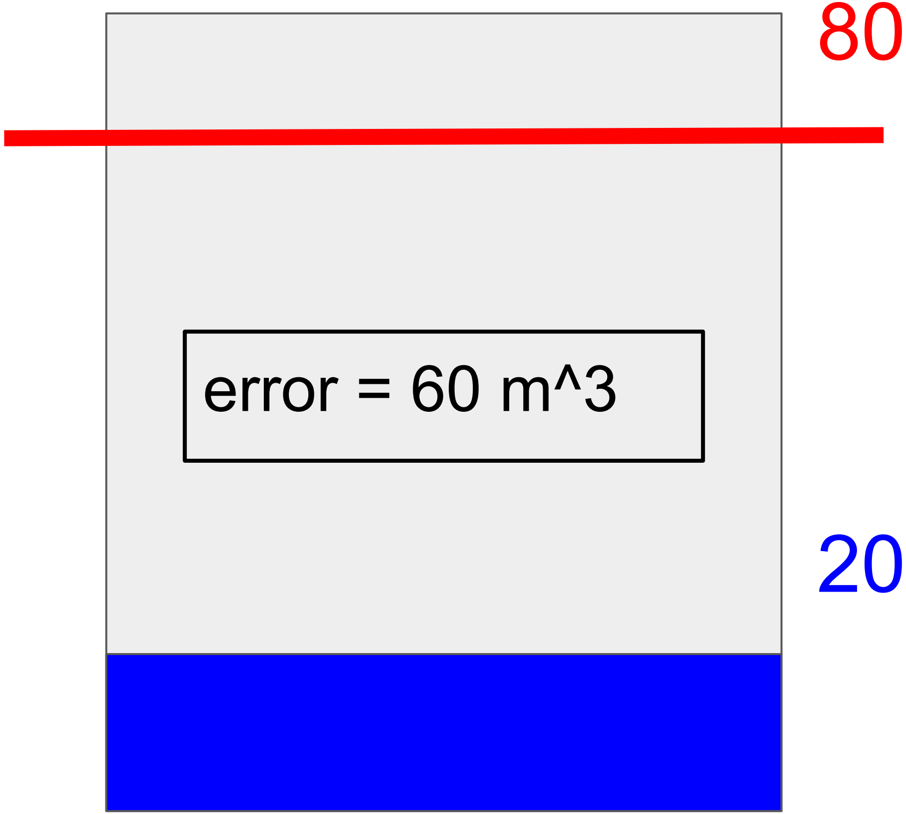


A sensor measures the volume of water every 0.02 seconds OR @ 50 Hz

$f = 1/0.02$ [1/s = Hz]

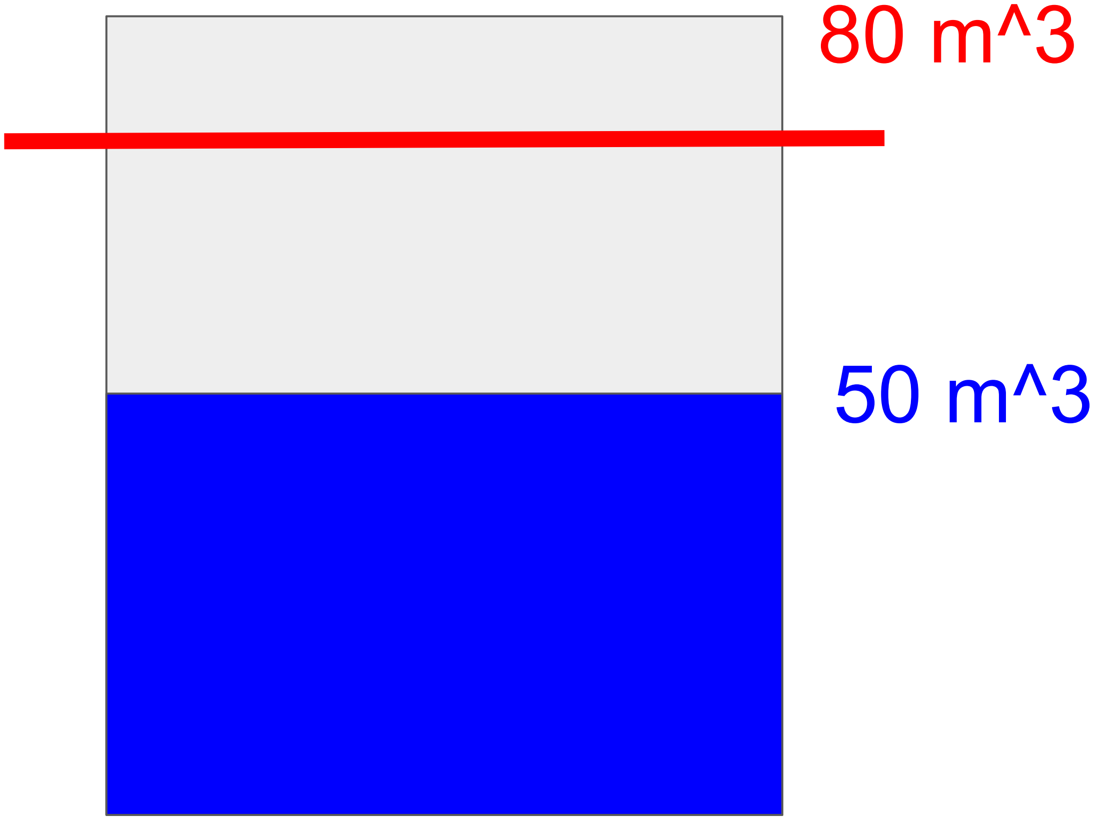
20 m^3

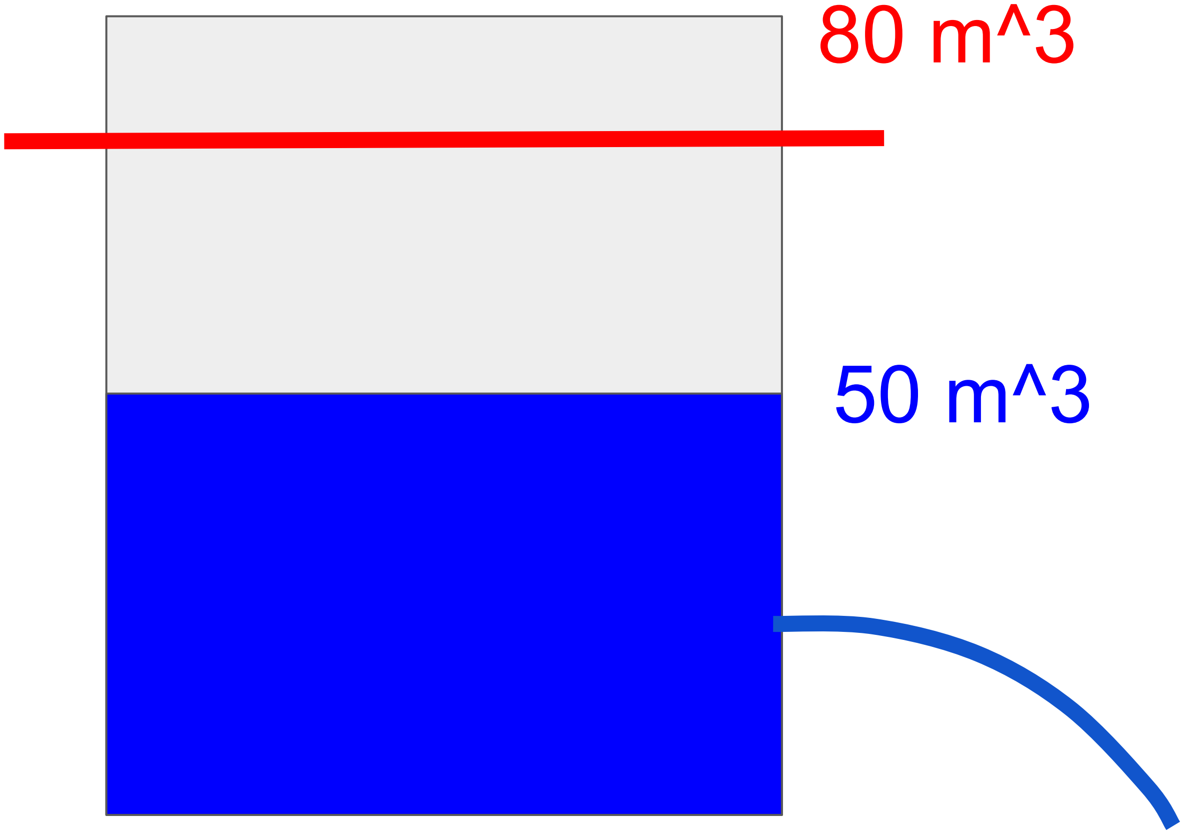


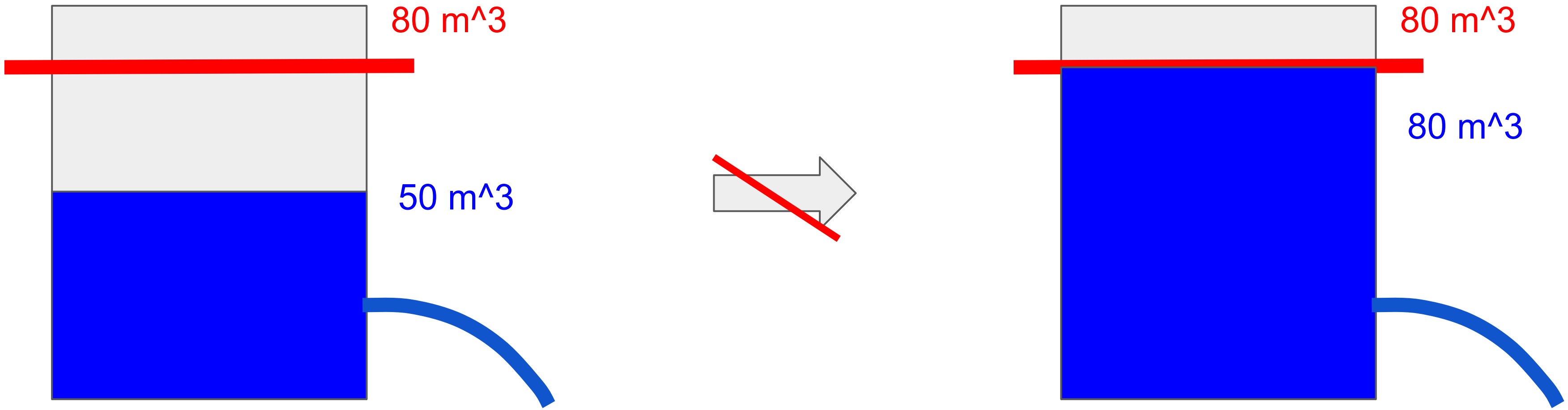
A sensor measures the volume of water every 0.02 seconds OR @ 50 Hz

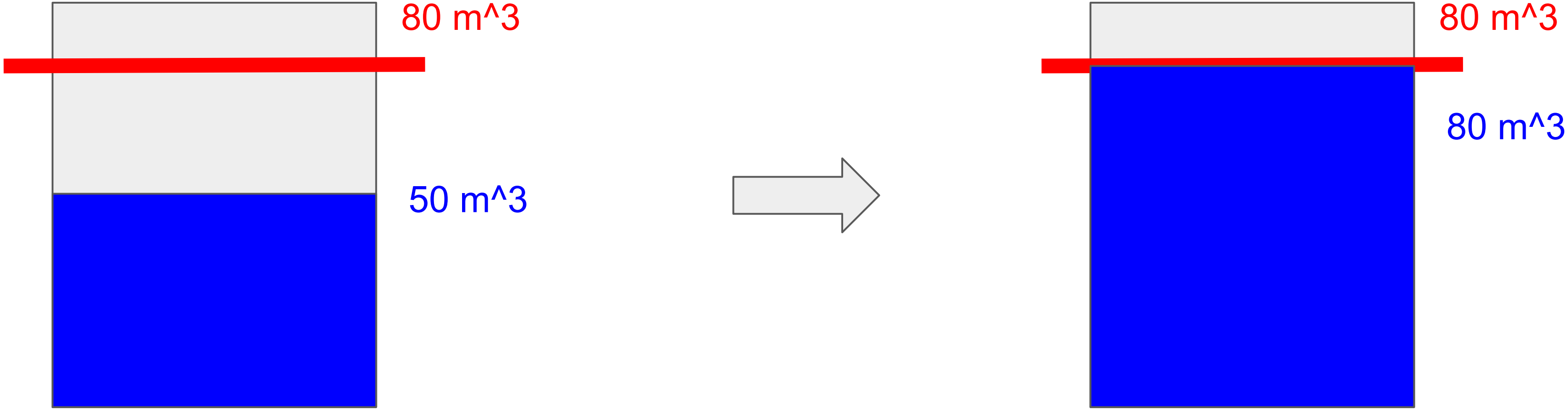
f = 1/0.02 [1/s = Hz]

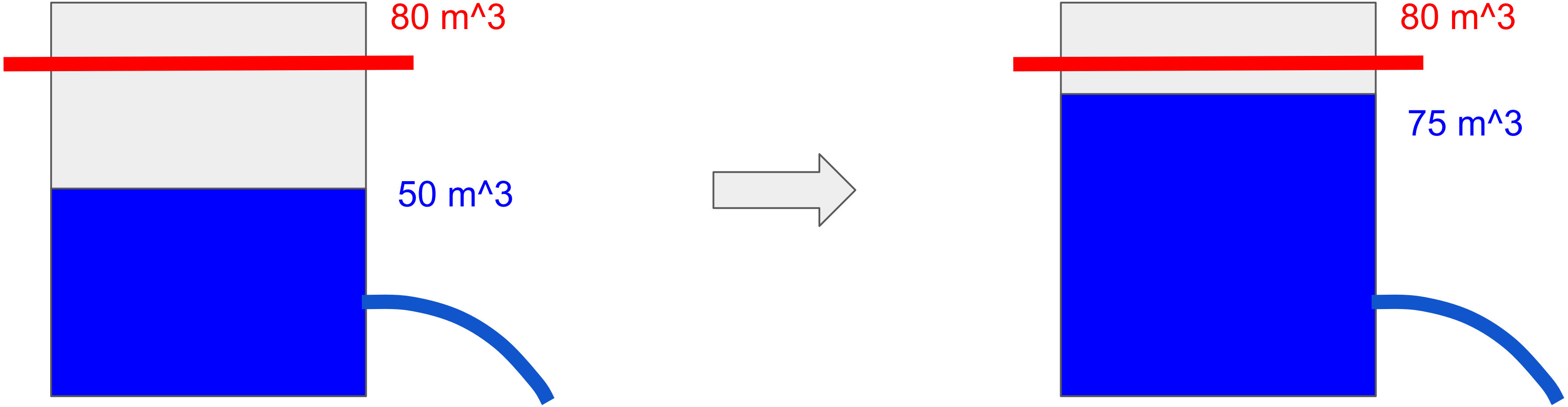
Water gets pumped into the tank

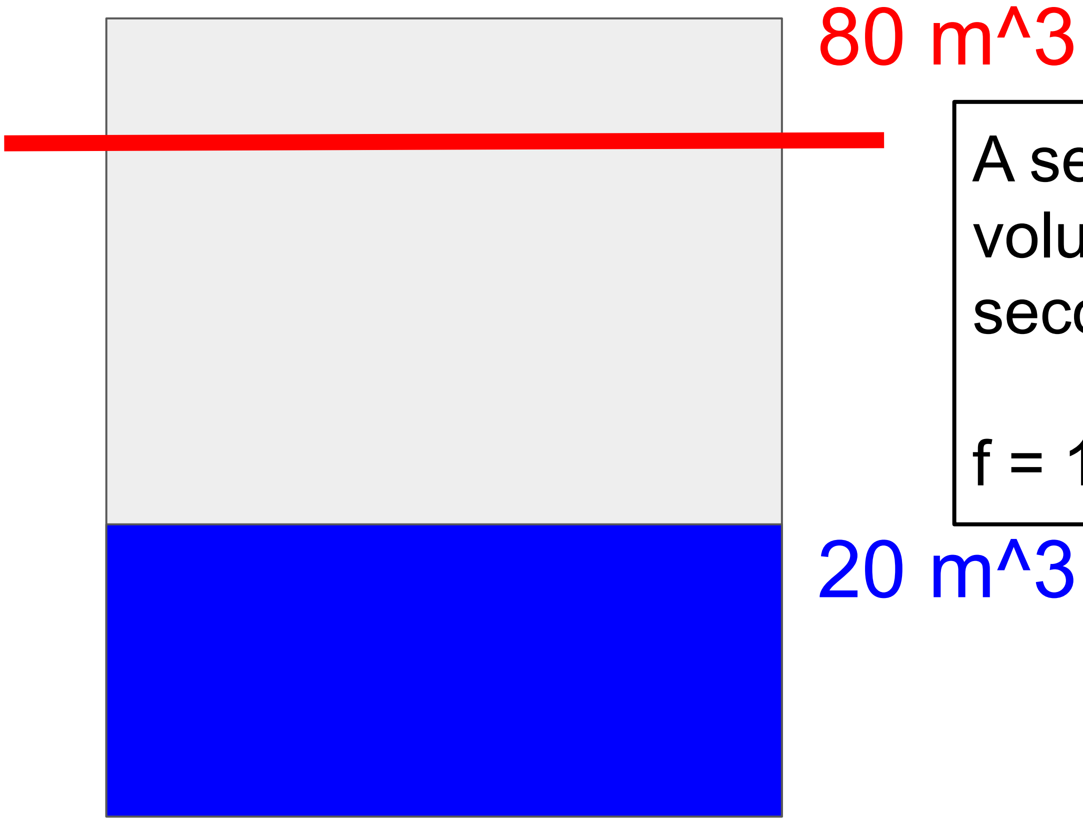

80 m^3
50 m^3


80 m^3
50 m^3


80 m^3
50 m^3
80 m^3
80 m^3


80 m^3
50 m^3
80 m^3
80 m^3

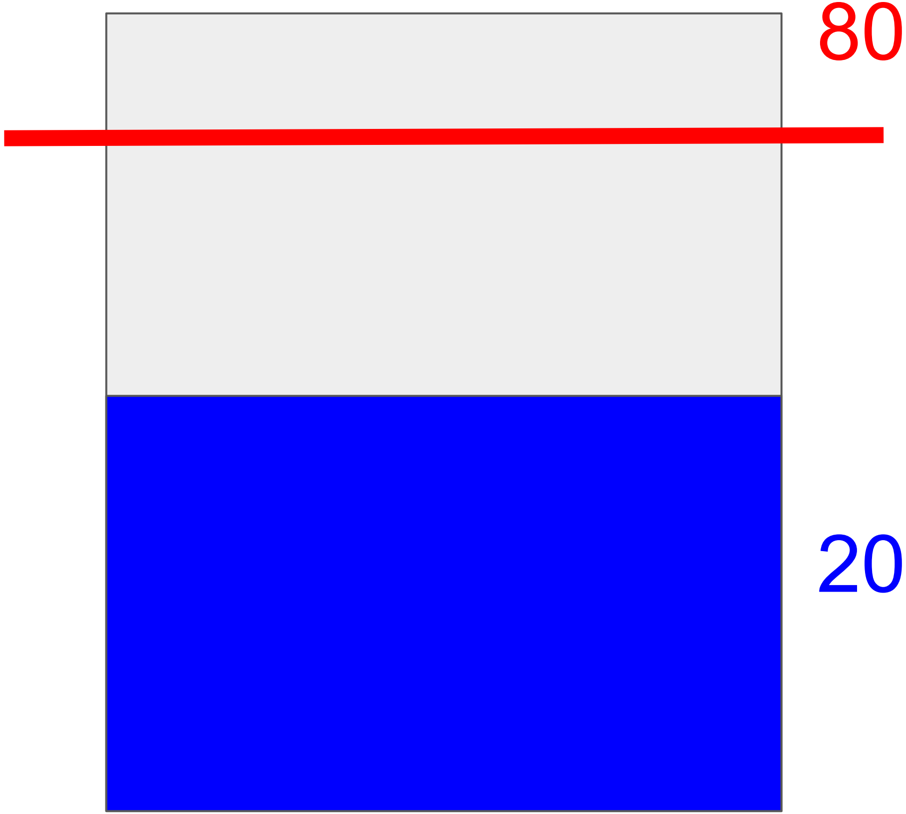

80 m^3
50 m^3
80 m^3
75 m^3



A sensor measures the volume of water every 0.02 seconds OR @ 50 Hz

f = 1/0.02 [1/s = Hz]

Water gets pumped into the tank



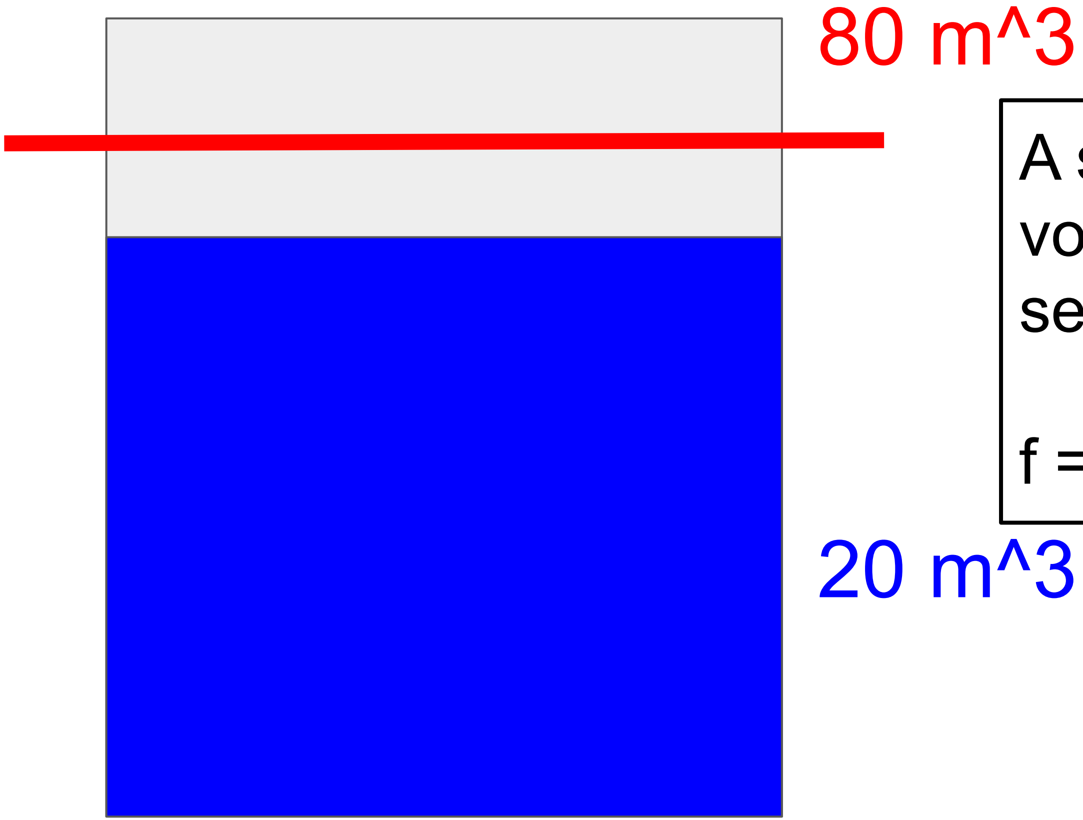
80 m^3

20 m^3

A sensor measures the volume of water every 0.02 seconds OR @ 50 Hz

f = 1/0.02 [1/s = Hz]

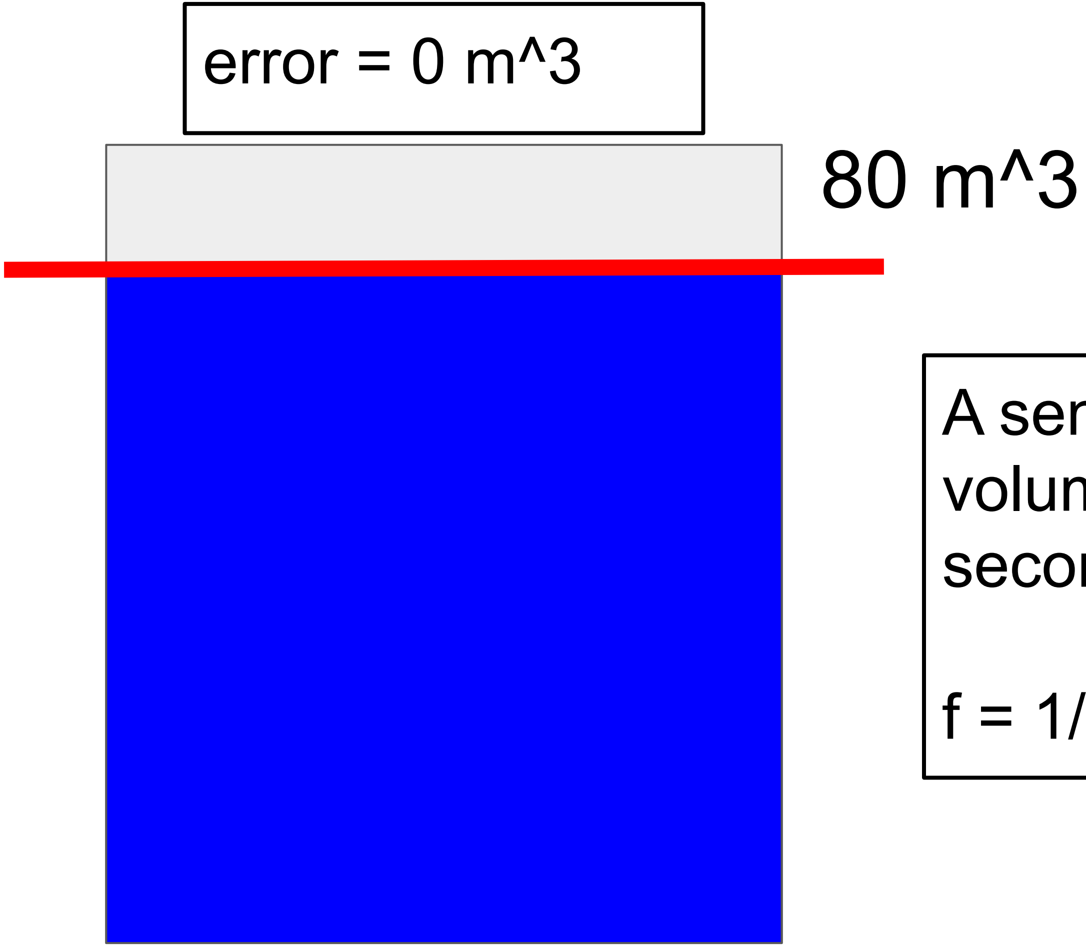
Water gets pumped into the tank



A sensor measures the volume of water every 0.02 seconds OR @ 50 Hz

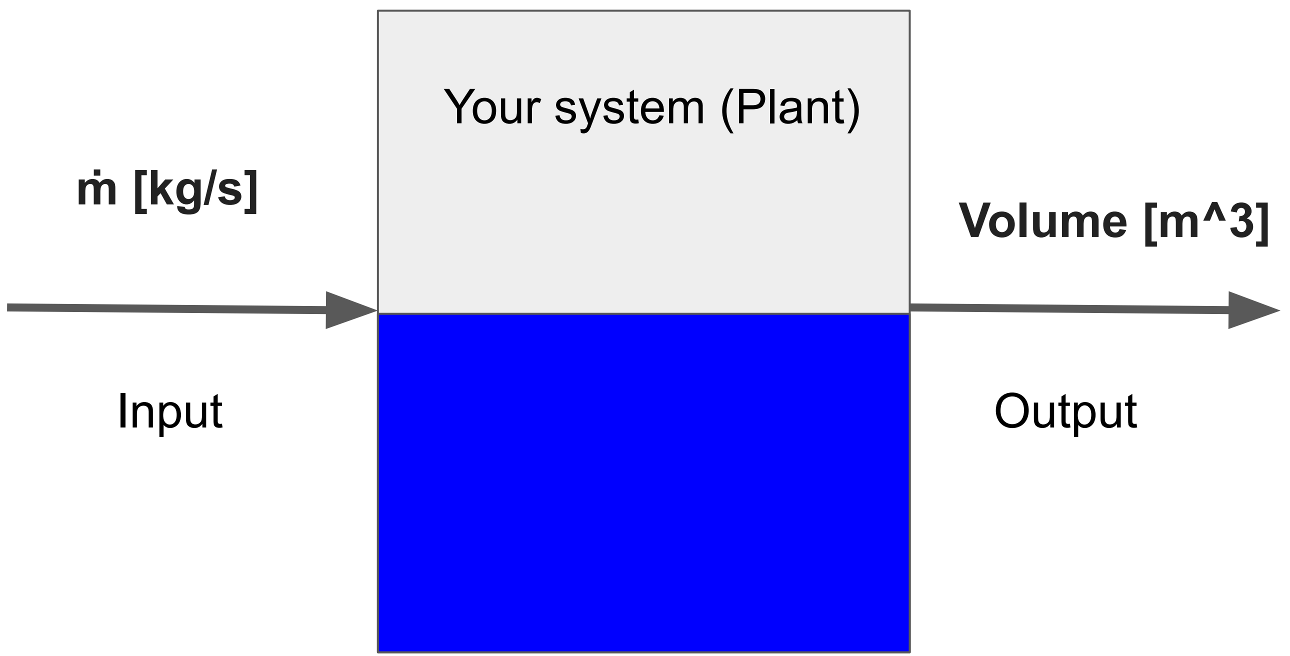
f = 1/0.02 [1/s = Hz]

Water gets pumped into the tank

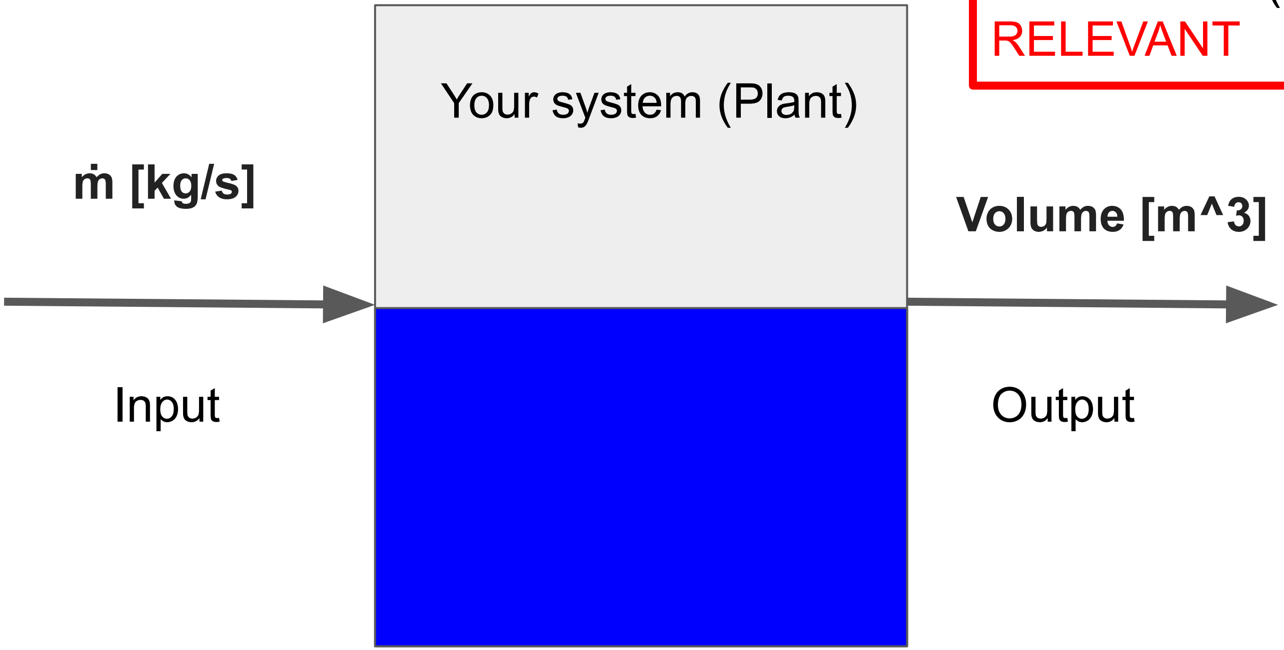


A sensor measures the volume of water every 0.02 seconds OR @ 50 Hz

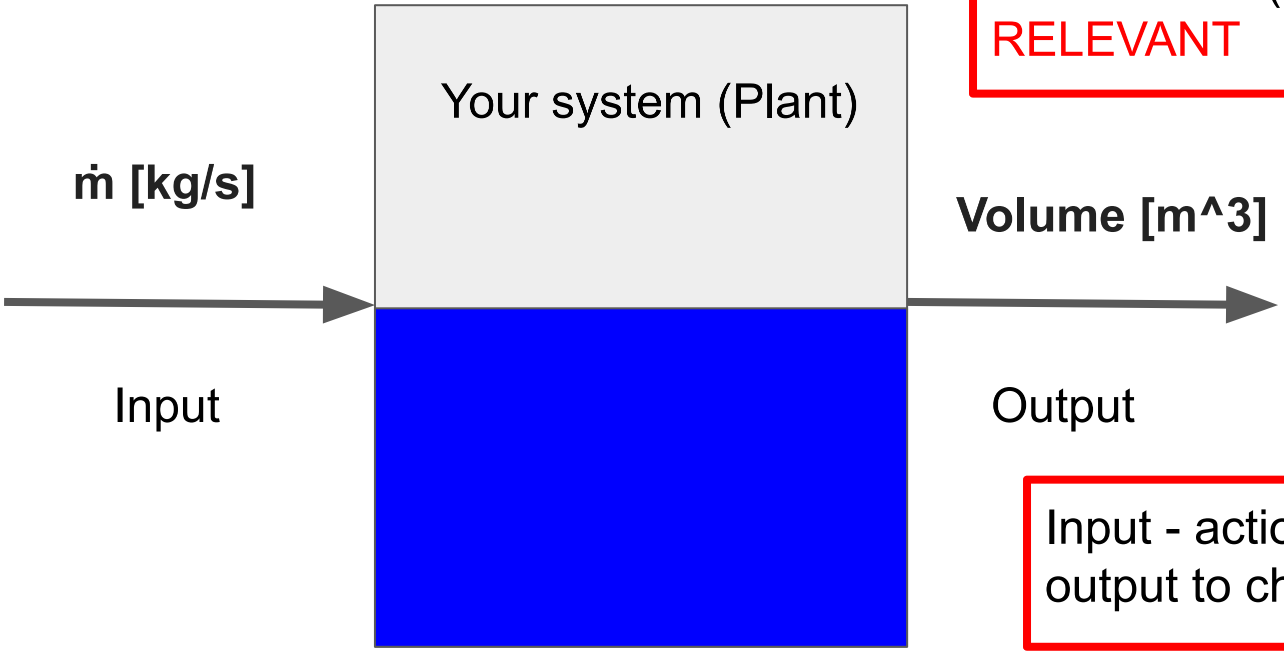
f = 1/0.02 [1/s = Hz]


ṁ [kg/s]
Input
Your system (Plant)
Volume [m^3]
Output

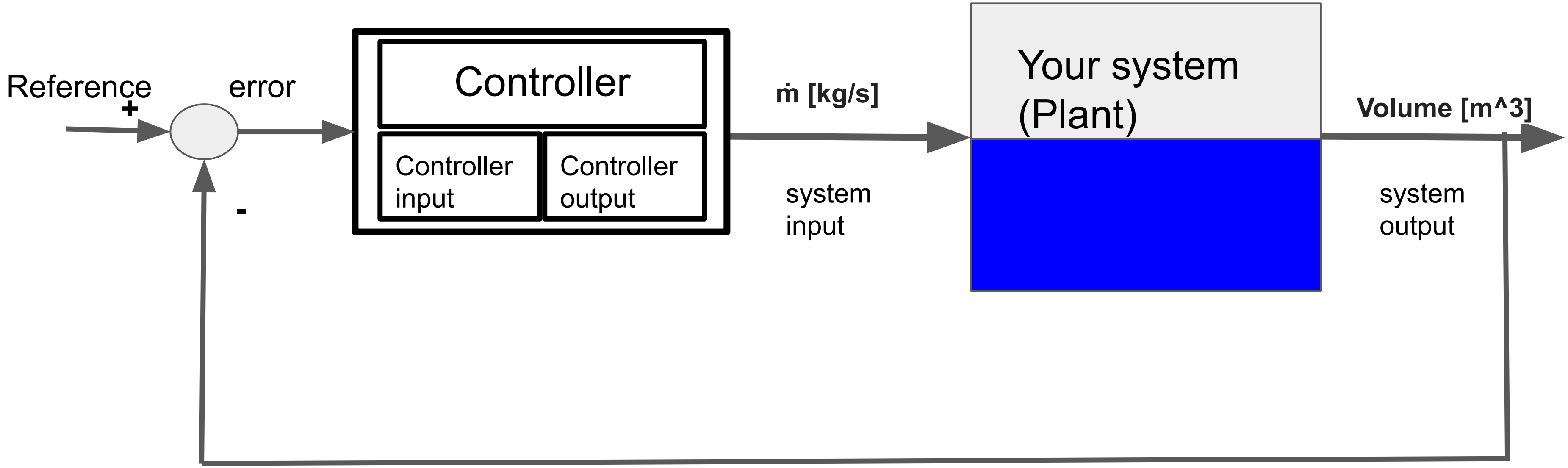
Output - something that we can observe (measure).
RELEVANT

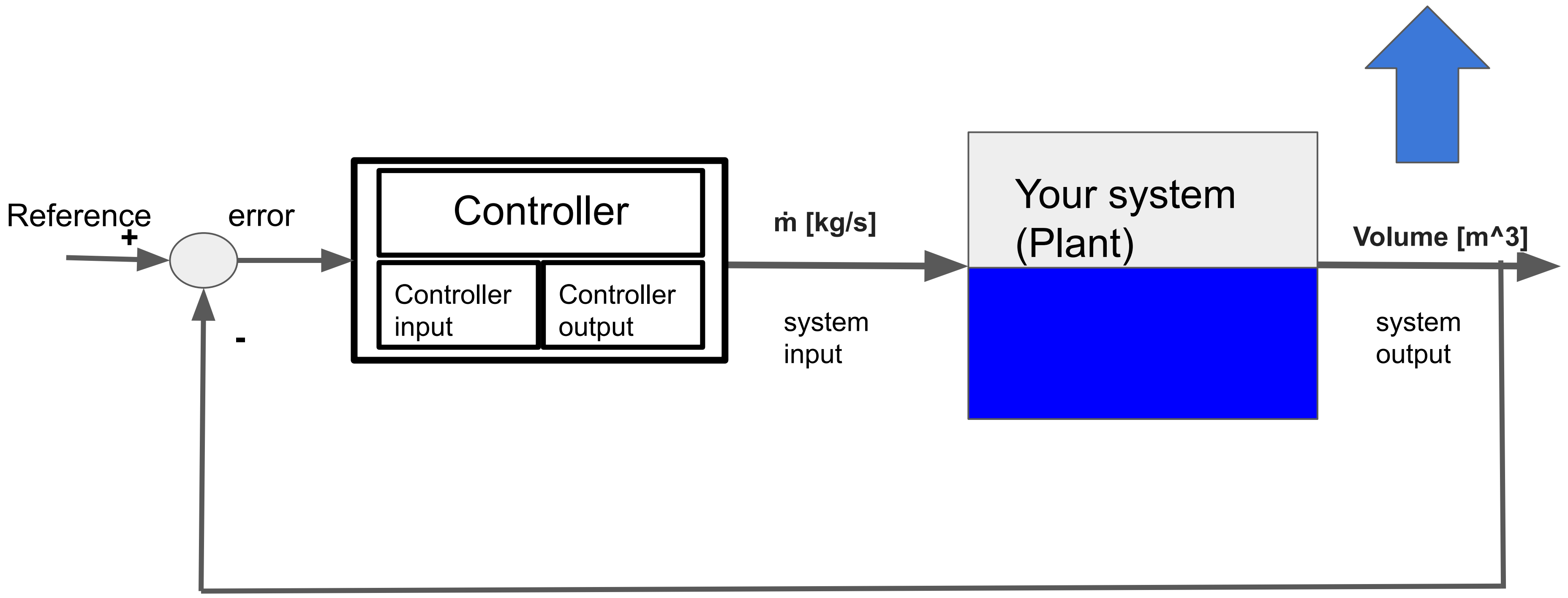


Output - something that we can observe (measure).
RELEVANT

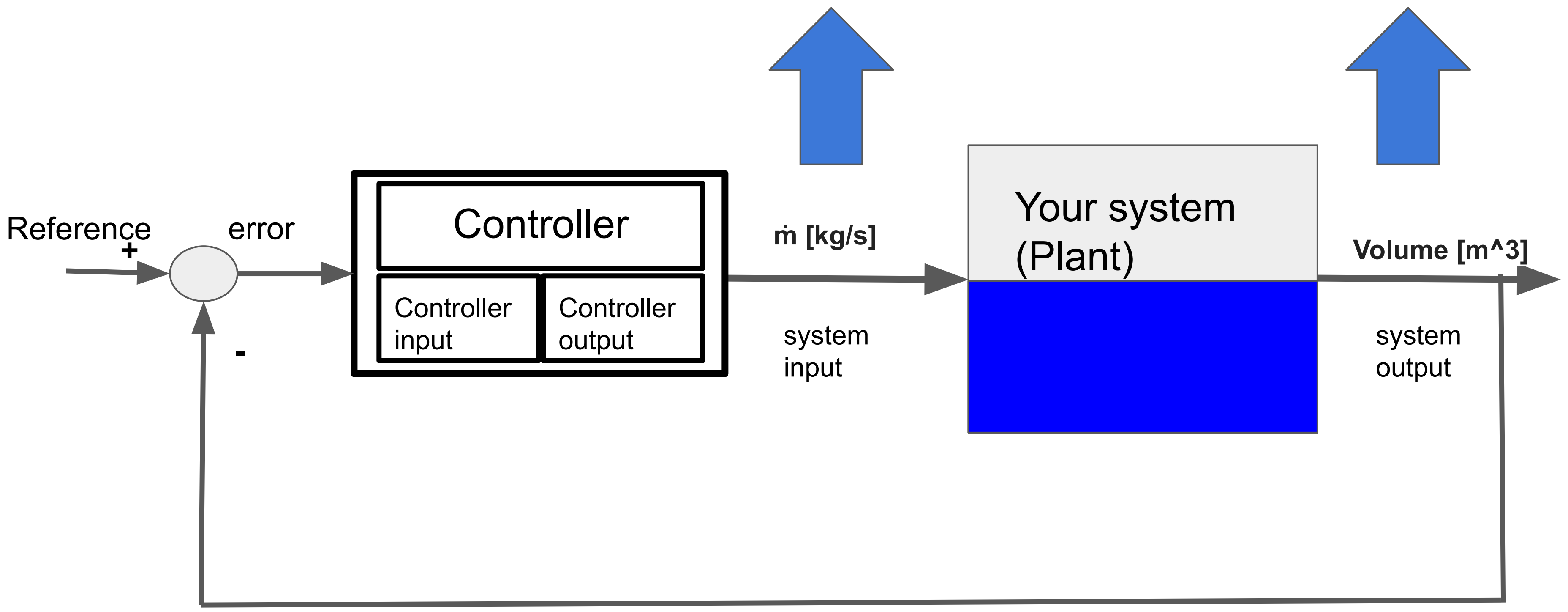


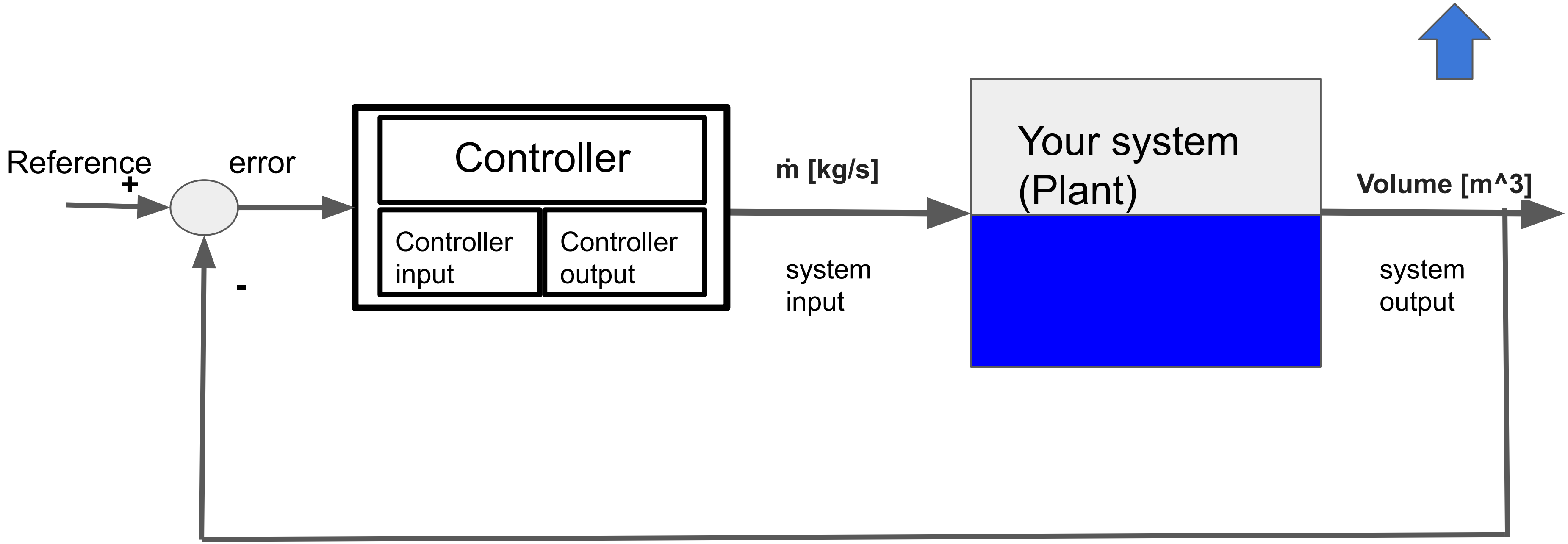
Input - action that causes output to change

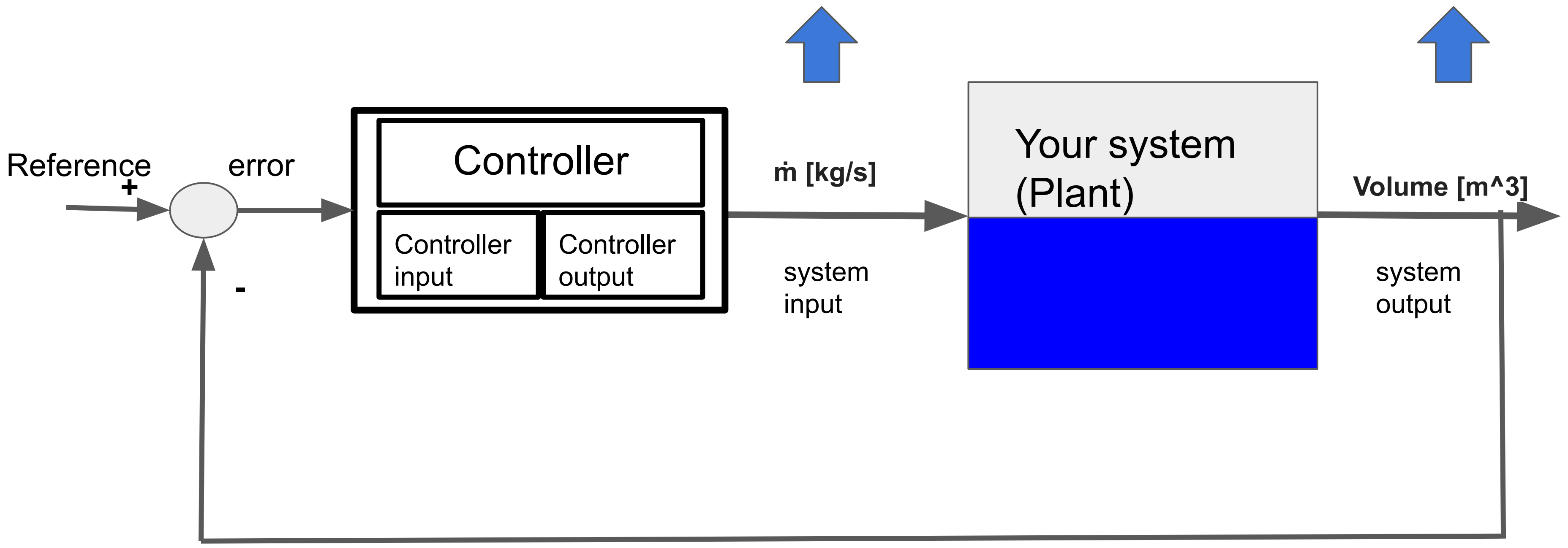

Reference
+
error
-
Controller
Controller input
Controller output
ṁ [kg/s]
system input
Your system (Plant)
Volume [m^3]
system output



Reference
+
error
-
Controller
Controller input
Controller output
ṁ [kg/s]
system input
Your system (Plant)
Volume [m^3]
system output


Reference
+
error
-
Controller
Controller input
Controller output
ṁ [kg/s]
system input
Your system (Plant)
Volume [m^3]
system output


Reference
+
error
-
Controller
Controller input
Controller output
ṁ [kg/s]
system input
Your system (Plant)
Volume [m^3]
system output



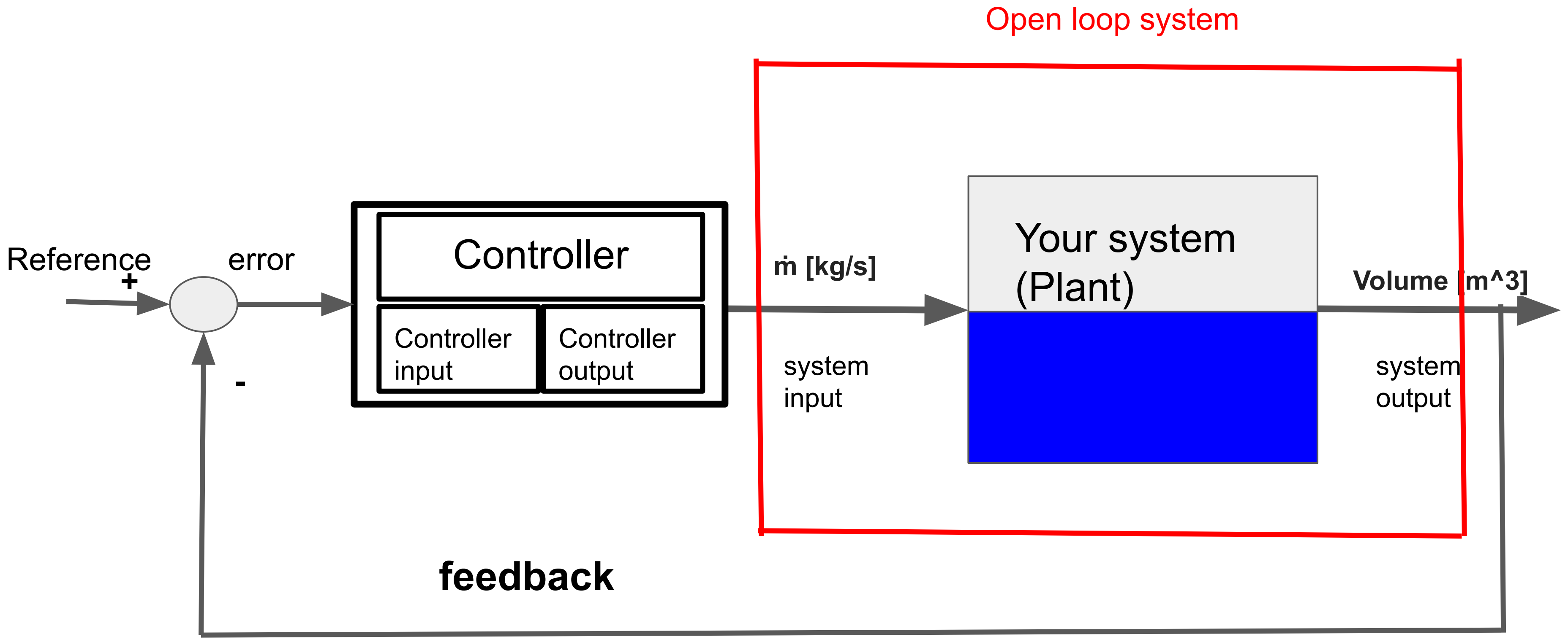
Reference
+
error
-
Controller
Controller input
Controller output
ṁ [kg/s]
system input
Your system (Plant)
Volume [m^3]
system output

If e<0, $\dot{m}$=-10 kg/s, elseif e>0, $\dot{m}$=10 kg/s, else $\dot{m}$=0 kg/s

$$\dot{m} = \begin{cases} -10 \quad [kg/s], & e < 0 \\ 0 \quad [kg/s], & e = 0 \\ 10 \quad [kg/s], & e > 0 \end{cases}$$

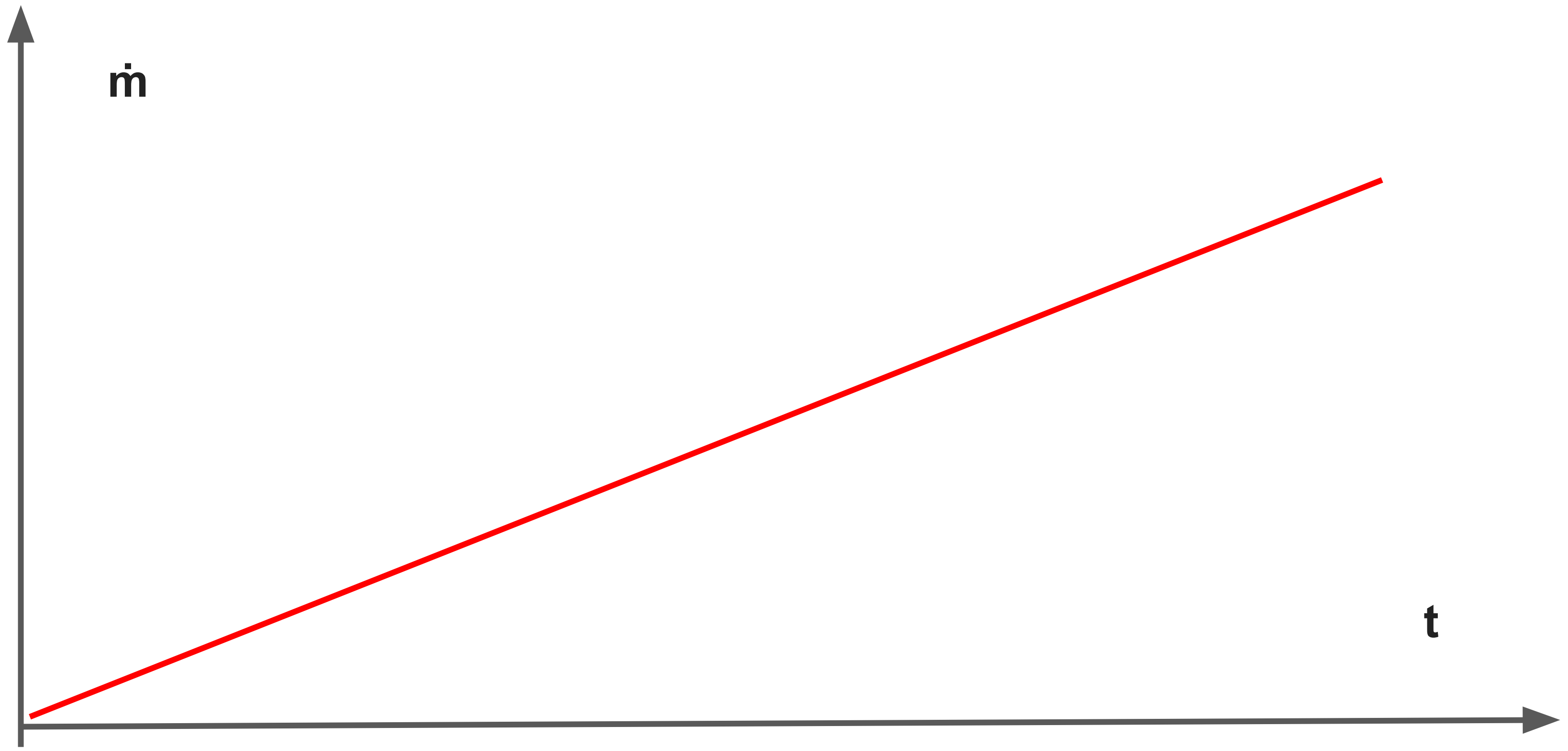
If e<0, ṁ=-100 kg/s, elseif e>0, ṁ=100 kg/s, else ṁ=0 kg/s

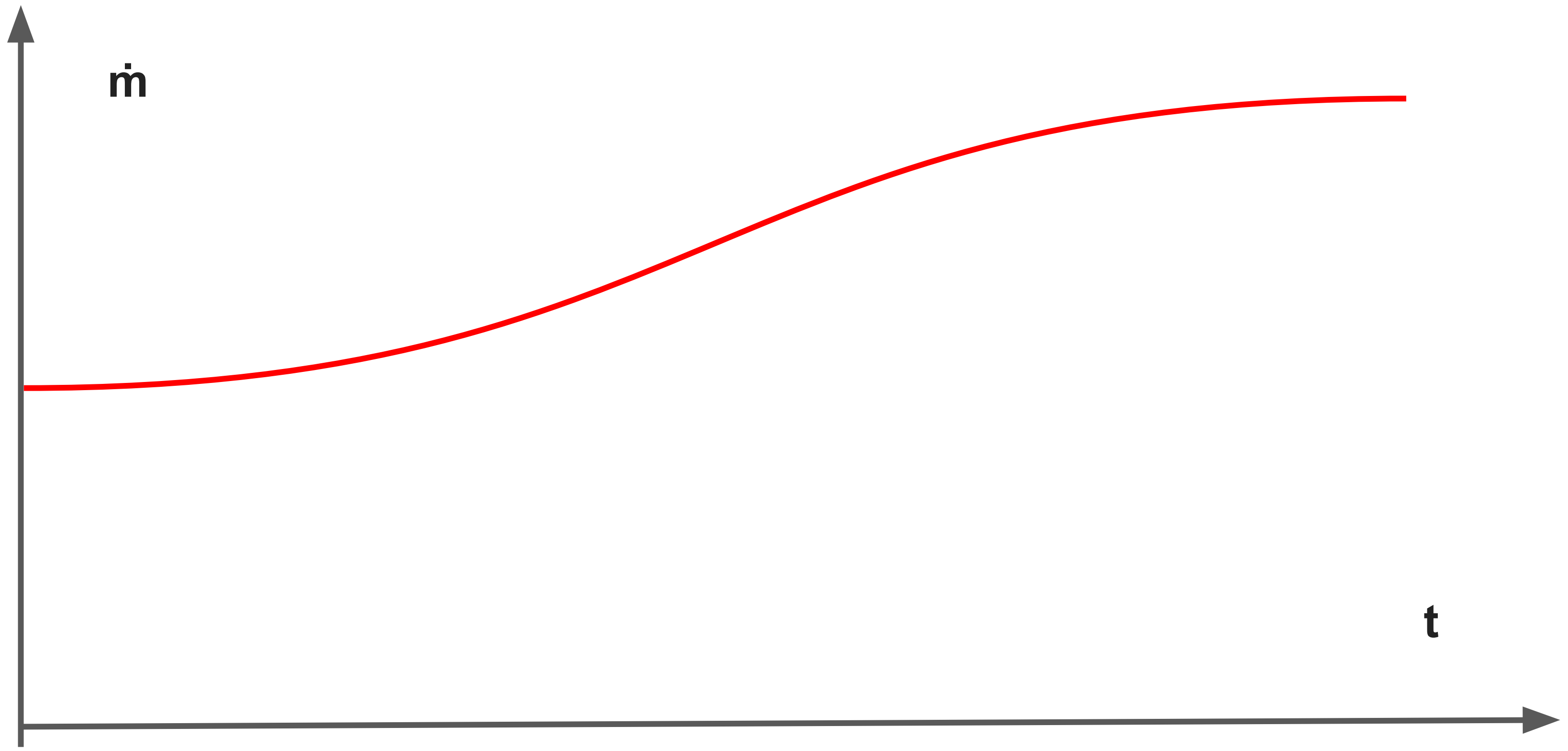
$$\dot{m} = \begin{cases} -100 \quad [kg/s], & e < 0 \\ 0 \quad [kg/s], & e = 0 \\ 100 \quad [kg/s], & e > 0 \end{cases}$$

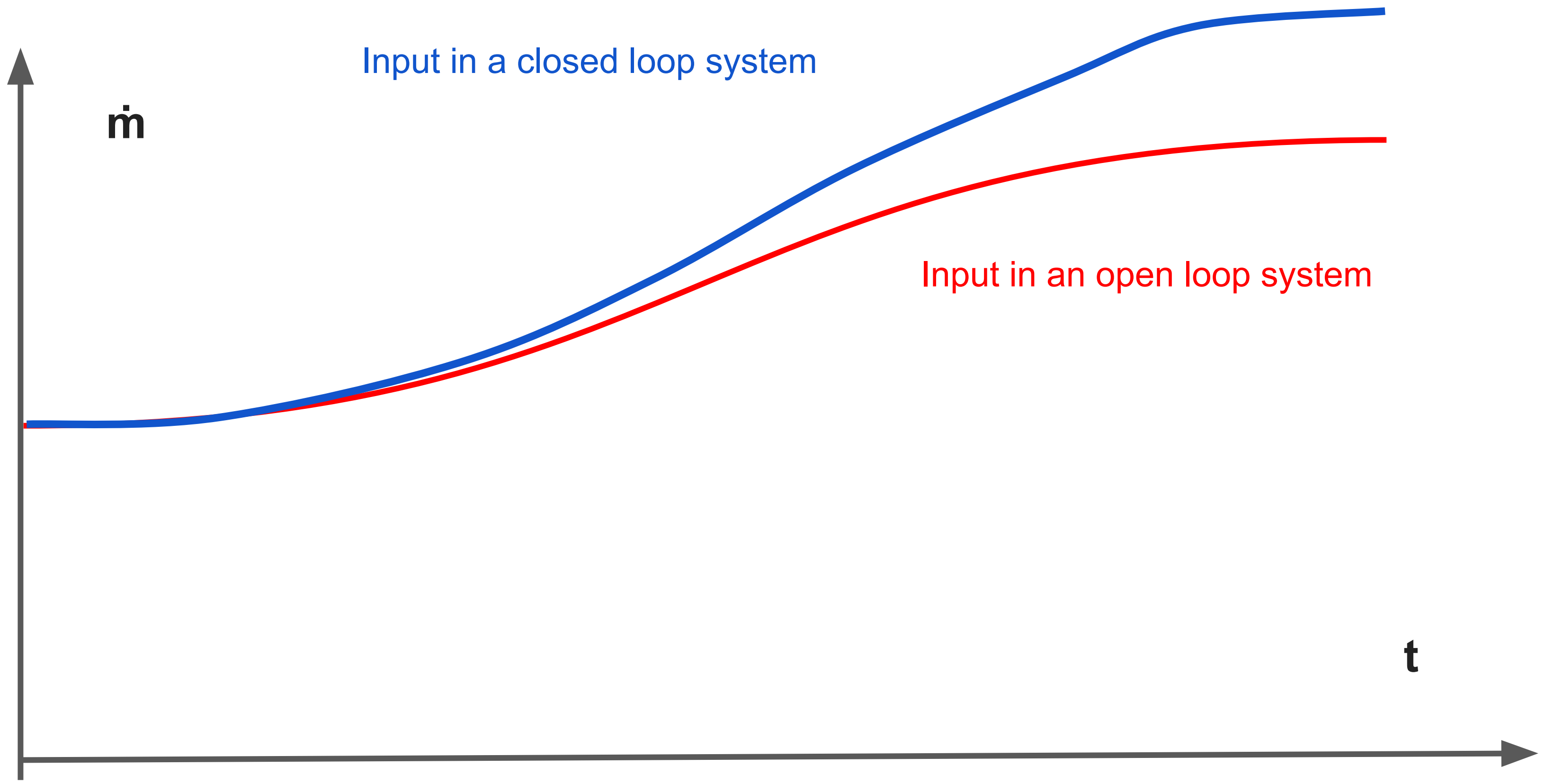


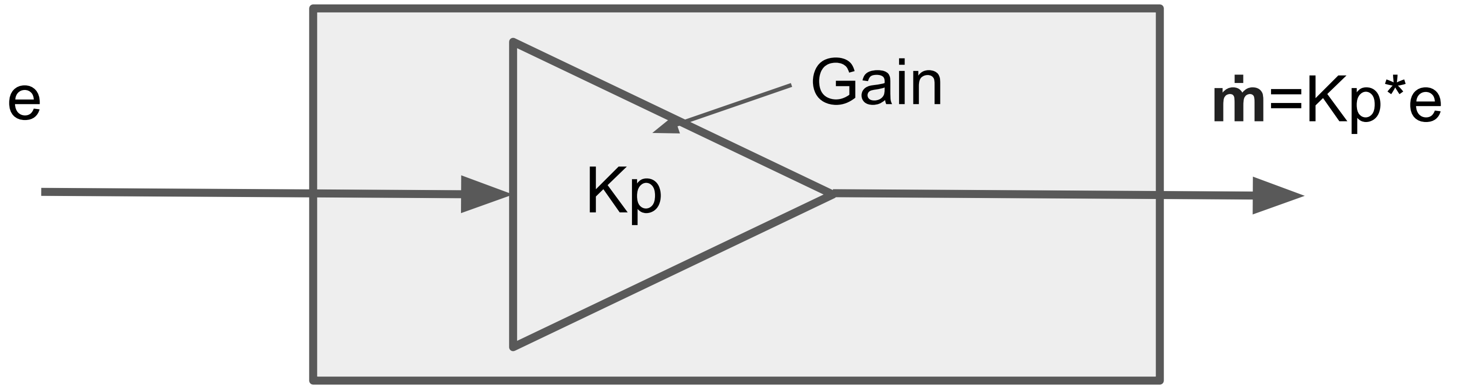
CLOSED LOOP SYSTEM


$\dot{m}$
t


ṁ
t


ṁ
t

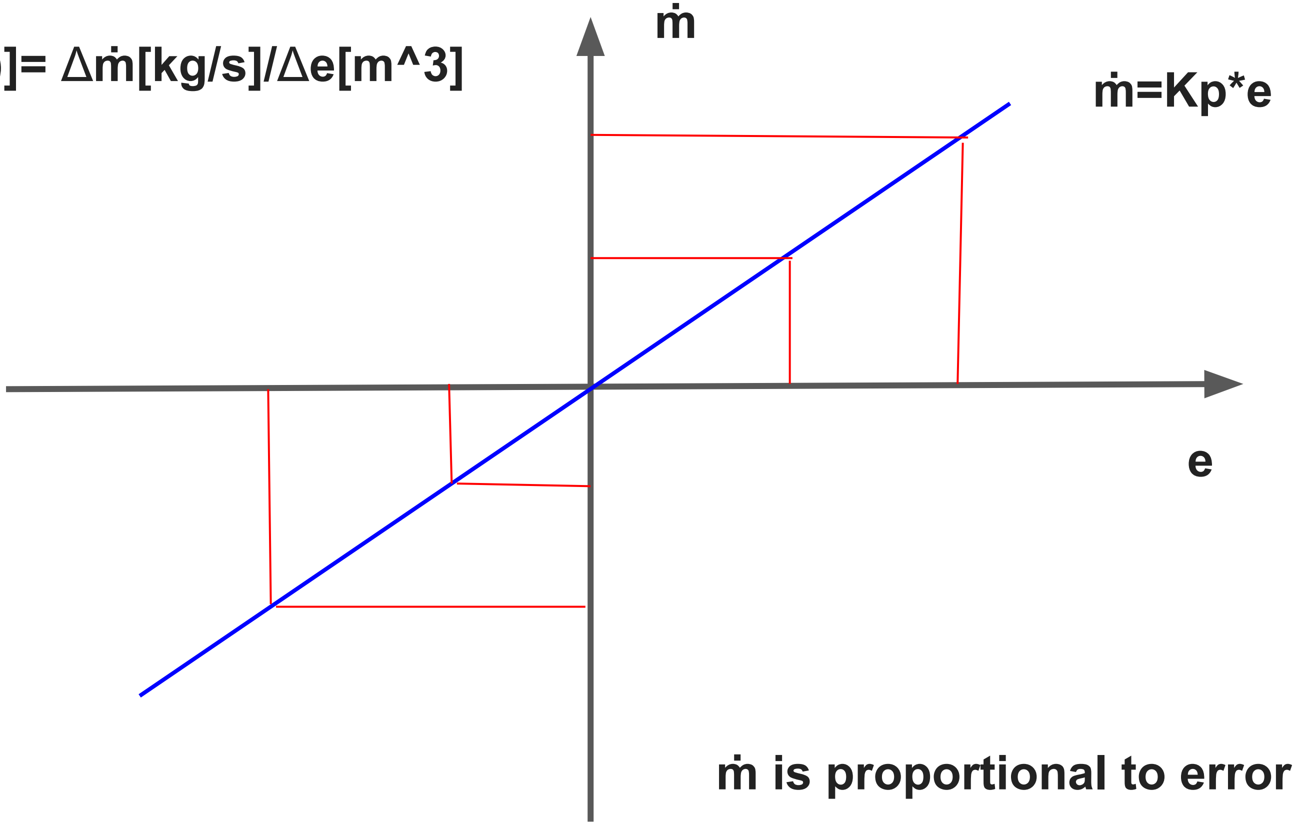

Input in a closed loop system
ṁ
Input in an open loop system
t



e
Gain
Kp
ṁ=Kp*e

Kp [kg/(s*m^3)]= $\Delta\dot{m}$[kg/s]/Δe[m^3]

Kp→ slope

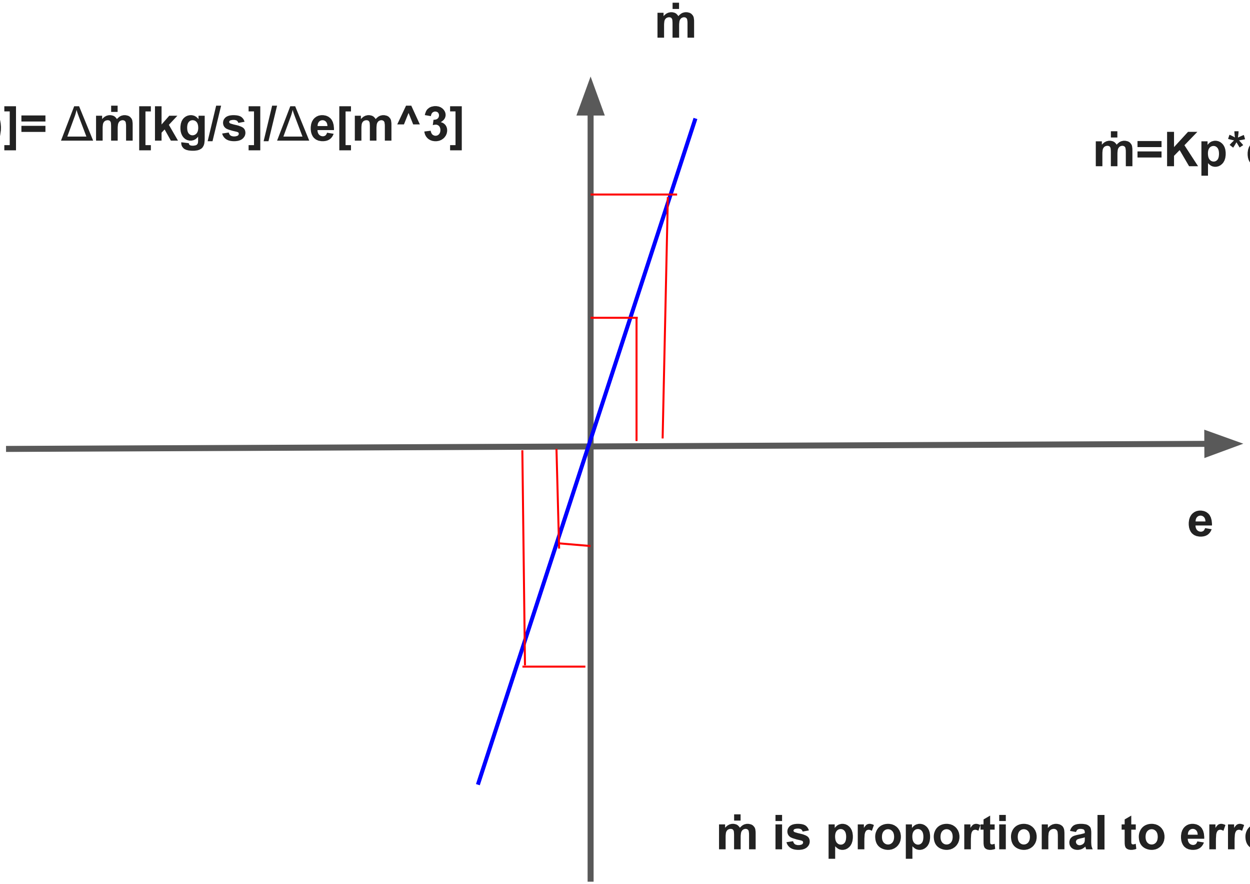


$\dot{m}$ is proportional to error

Kp [kg/(s*m^3)]= $\Delta\dot{m}$[kg/s]/Δe[m^3]

Kp→ slope

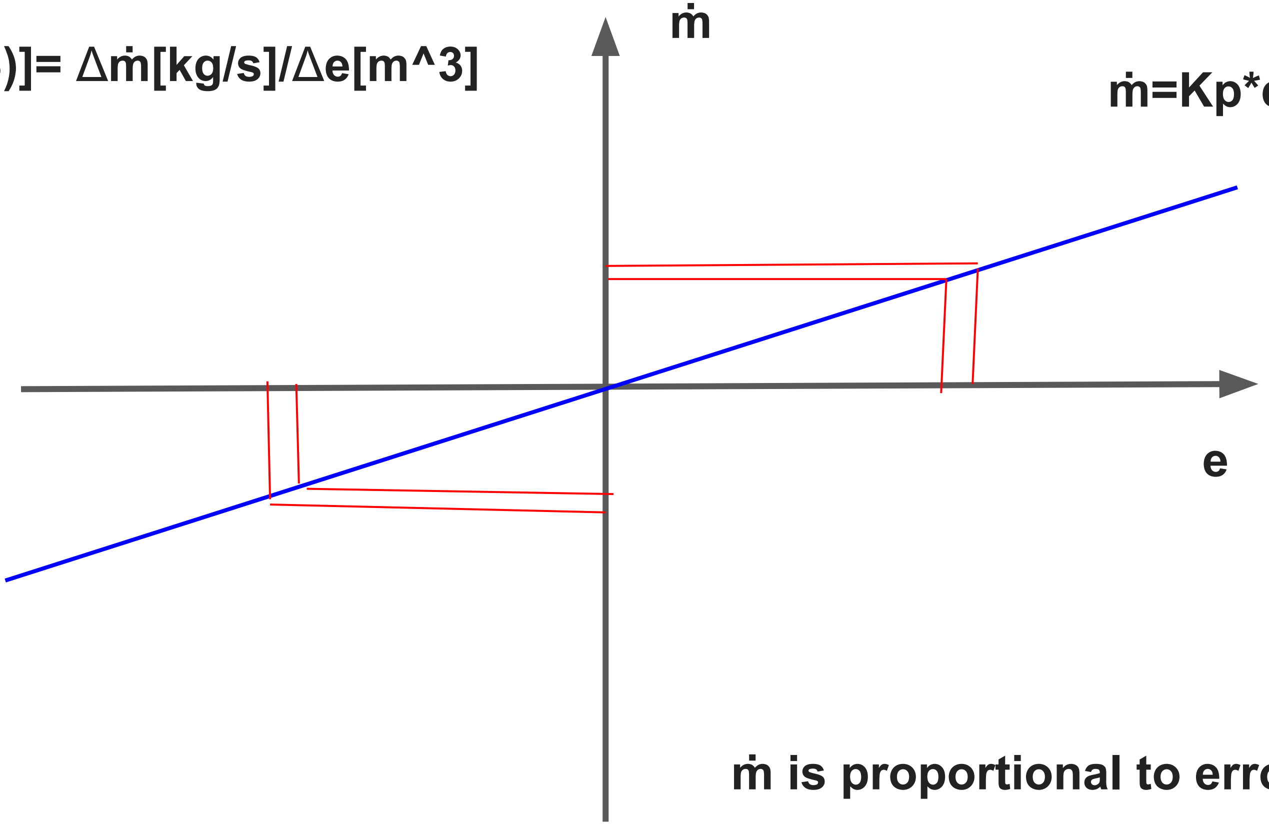
$\dot{m}$=Kp*e



$\dot{m}$ is proportional to error

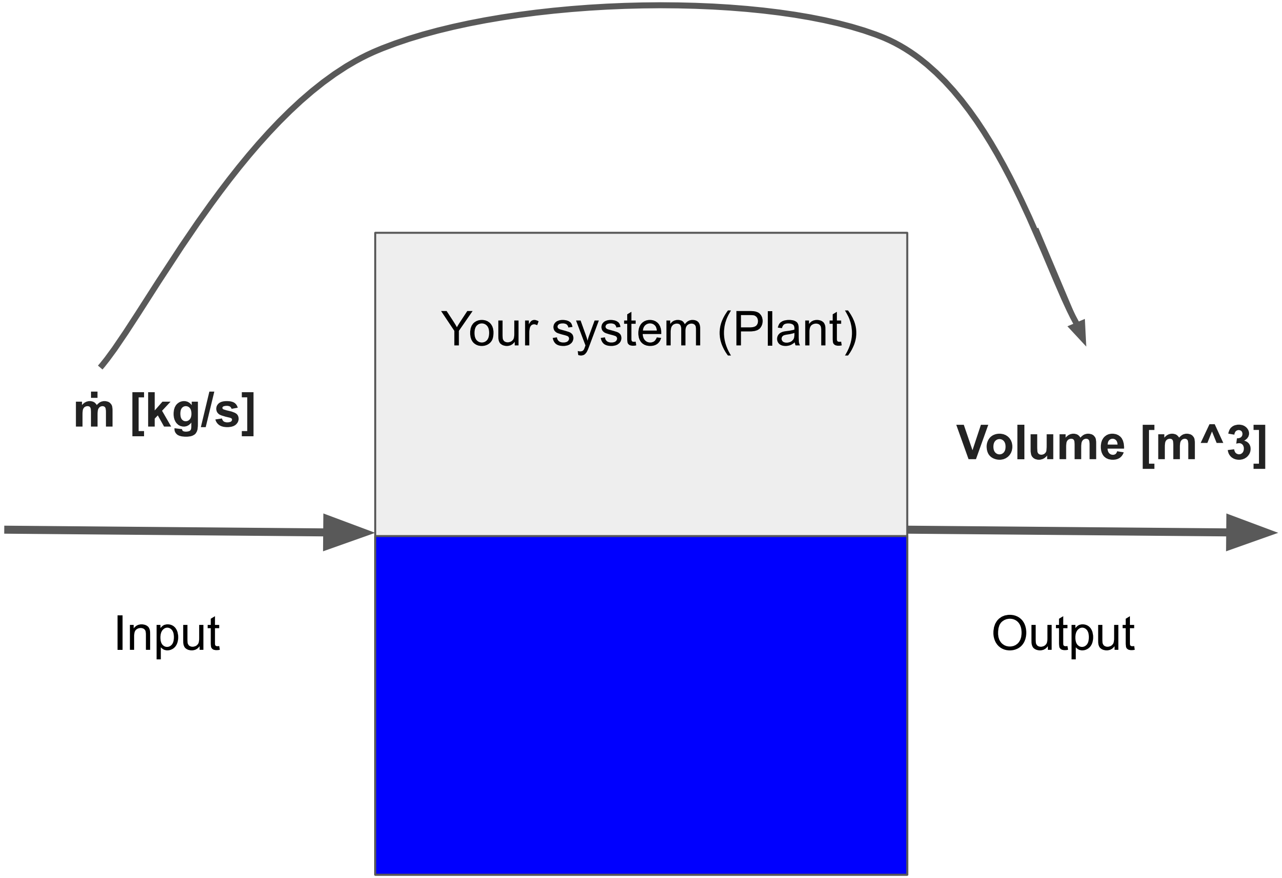
K_p [kg/(s*m^3)]= $\Delta\dot{m}$[kg/s]/Δe[m^3]

$K_p \rightarrow$ slope



$\dot{m}$ is proportional to error

Simulating a water tank and its proportional controller



Let’s create a mathematical model!

$$\dot{m} = \frac{\mathrm{d}(mass_w)}{\mathrm{d}t}$$

$$mass_w = vol_w * density_w = vol_w * \rho_w$$

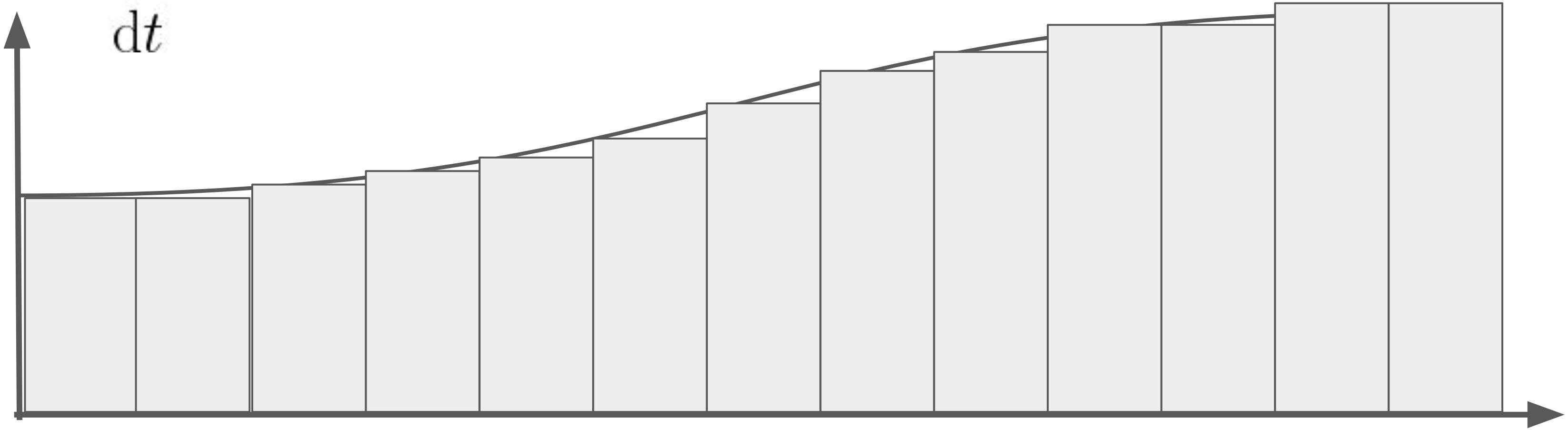
$$\dot{m} = \frac{\mathrm{d}(mass_w)}{\mathrm{d}t} = \frac{\mathrm{d}(vol_w * \rho_w)}{\mathrm{d}t} =$$

$$= \frac{\mathrm{d}(vol_w)}{\mathrm{d}t} * \rho_w + \frac{\mathrm{d}(\rho_w)}{\mathrm{d}t} * vol_w = \frac{\mathrm{d}(vol_w)}{\mathrm{d}t} * \rho_w$$

$$\frac{\mathrm{d}(vol_w)}{\mathrm{d}t} = \frac{1}{\rho} * \dot{m}$$

$$\frac{\mathrm{d}(vol_w)}{\mathrm{d}t} = \frac{1}{\rho} * \dot{m}(t)$$

$$\frac{\mathrm{d}(vol_w)}{\mathrm{d}t}$$ [m^3/s]



t

$$\rho_w = constant, \quad \frac{\mathrm{d}(\rho_w)}{\mathrm{d}t} = 0$$

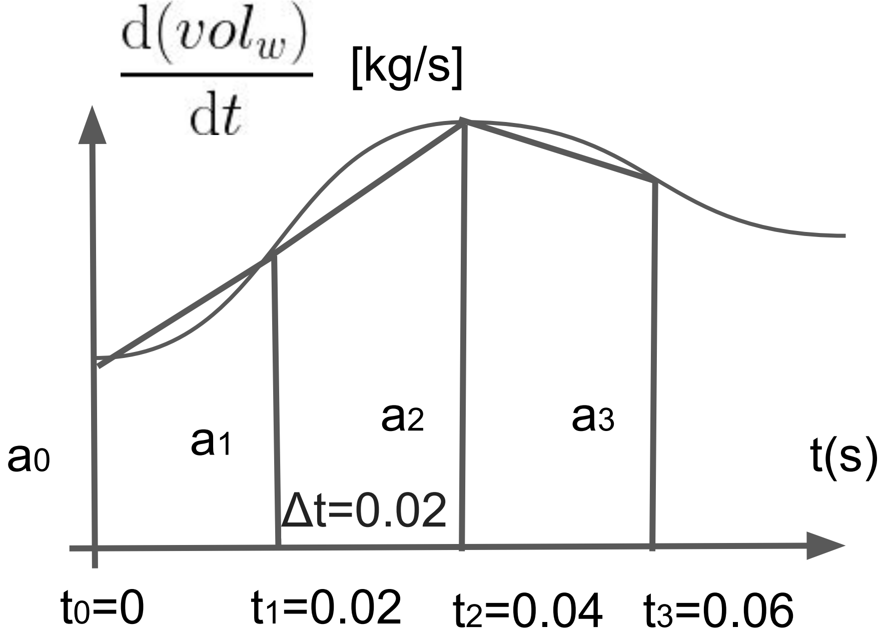
$$\dot{m}(t) = not \quad constant, \quad \frac{\mathrm{d}(\dot{m}(t))}{\mathrm{d}t} \neq 0$$

$$\int_{vol_i}^{vol(t)} d(vol) = \int_0^t \frac{1}{\rho}\dot{m} * dt = \frac{1}{\rho}\int_0^t \dot{m} * dt$$

$$Vol(t) - Vol_i = \frac{1}{\rho}\int_0^t \dot{m} * dt$$

$$Vol(t) = Vol_i + \boxed{\frac{1}{\rho}\int_0^t \dot{m} * dt}$$

More or less: Numerical integration is an approximation



$$\frac{d(vol_w)}{dt} = \frac{1}{\rho} * \dot{m}(t)$$

Trapezoidal rule

$$vol = vol_i + \left(\frac{a_0 + a_1}{2}\right)\Delta t + \left(\frac{a_1 + a_2}{2}\right)\Delta t + \left(\frac{a_2 + a_3}{2}\right)\Delta t$$

vol[t1]

vol[t2]

vol[t3]

$$vol = vol_i + \left(\frac{a_0 + a_1}{2}\right)\Delta t + \left(\frac{a_1 + a_2}{2}\right)\Delta t + \left(\frac{a_2 + a_3}{2}\right)\Delta t$$

vol[t1]

vol[t2]

vol[t3]

$$vol[t_3] = vol[t_2] + \left(\frac{a_2 + a_3}{2}\right)\Delta t$$

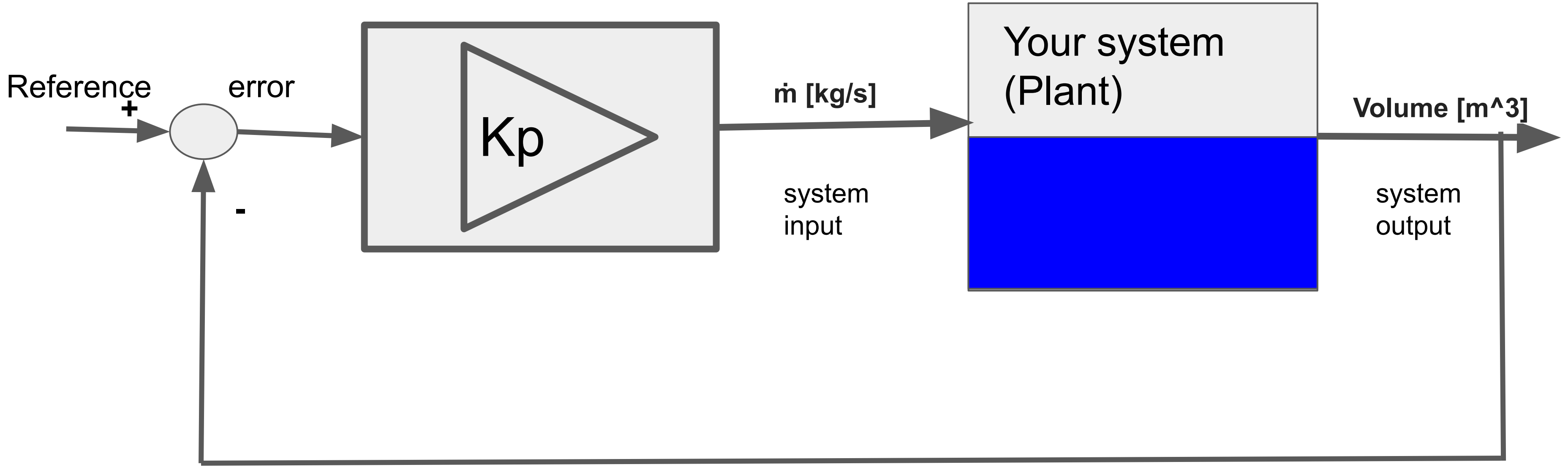
Generalized algorithm:

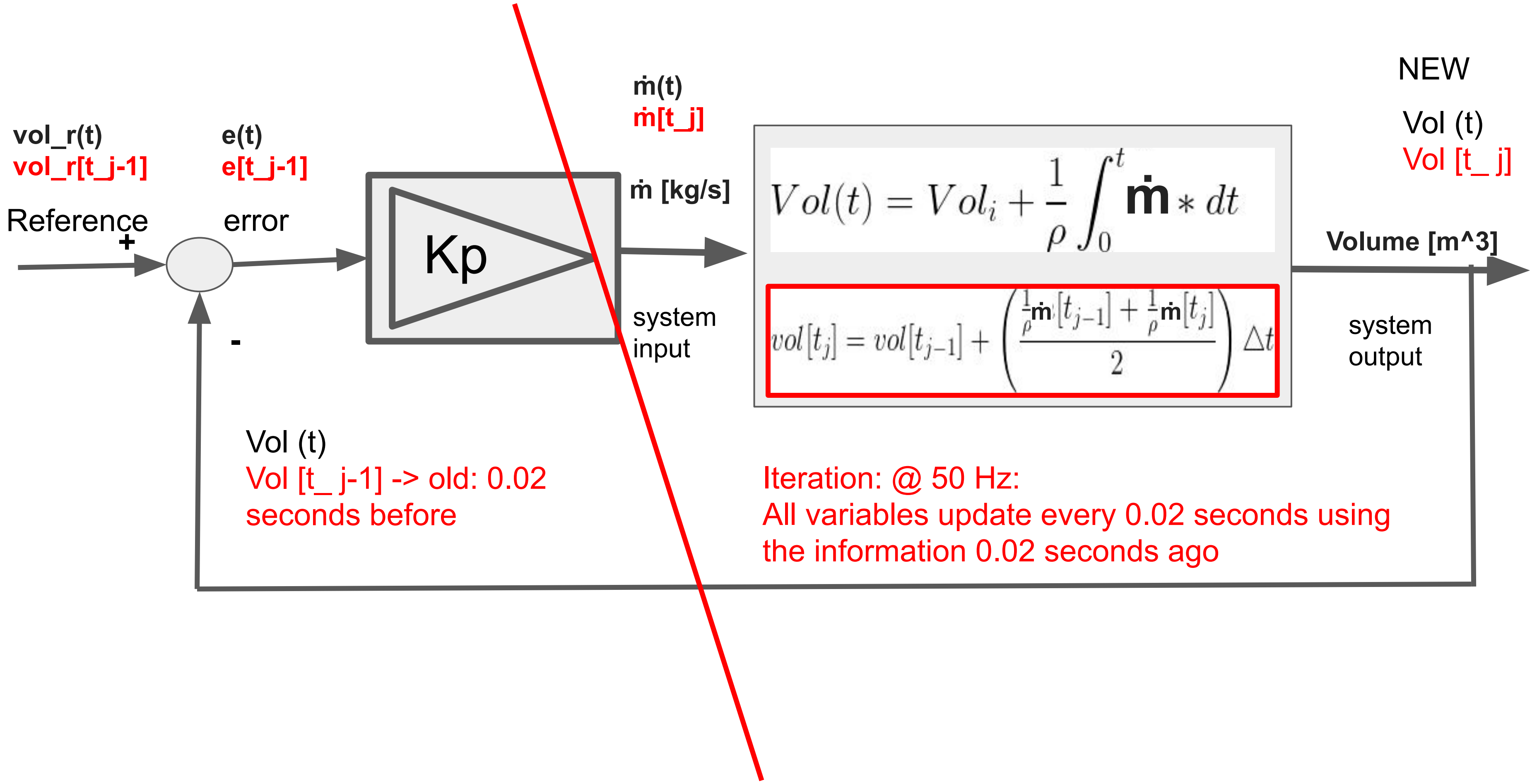
$$vol[t_j] = vol[t_{j-1}] + \left(\frac{\frac{1}{\rho}m[t_{j-1}] + \frac{1}{\rho}m[t_j]}{2}\right)\Delta t$$

$$vol[t_1] = vol[t_0] + \left(\frac{\frac{1}{\rho}m[t_0] + \frac{1}{\rho}m[t_1]}{2} \right) \triangle t$$

$$vol[t_2] = vol[t_1] + \left(\frac{\frac{1}{\rho}m[t_1] + \frac{1}{\rho}m[t_2]}{2} \right) \triangle t$$

$$vol[t_3] = vol[t_2] + \left(\frac{\frac{1}{\rho}m[t_2] + \frac{1}{\rho}m[t_3]}{2} \right) \triangle t$$


Reference
+
error
-
Kp
ṁ [kg/s]
system
input
Your system
(Plant)
Volume [m^3]
system
output



NEW
Vol (t)
Vol [t_ j]
vol_r(t)
vol_r[t_j-1]
e(t)
e[t_j-1]
ṁ(t)
ṁ[t_j]
Reference
+
error
Kp
ṁ [kg/s]
system
input
$Vol(t) = Vol_i + \frac{1}{\rho}\int_0^t \dot{m} * dt$
$vol[t_j] = vol[t_{j-1}] + \left(\frac{\frac{1}{\rho}\dot{m}[t_{j-1}] + \frac{1}{\rho}\dot{m}[t_j]}{2}\right)\Delta t$
Volume [m^3]
system
output
-
Vol (t)
Vol [t_ j-1] -> old: 0.02
seconds before
Iteration: @ 50 Hz:
All variables update every 0.02 seconds using
the information 0.02 seconds ago

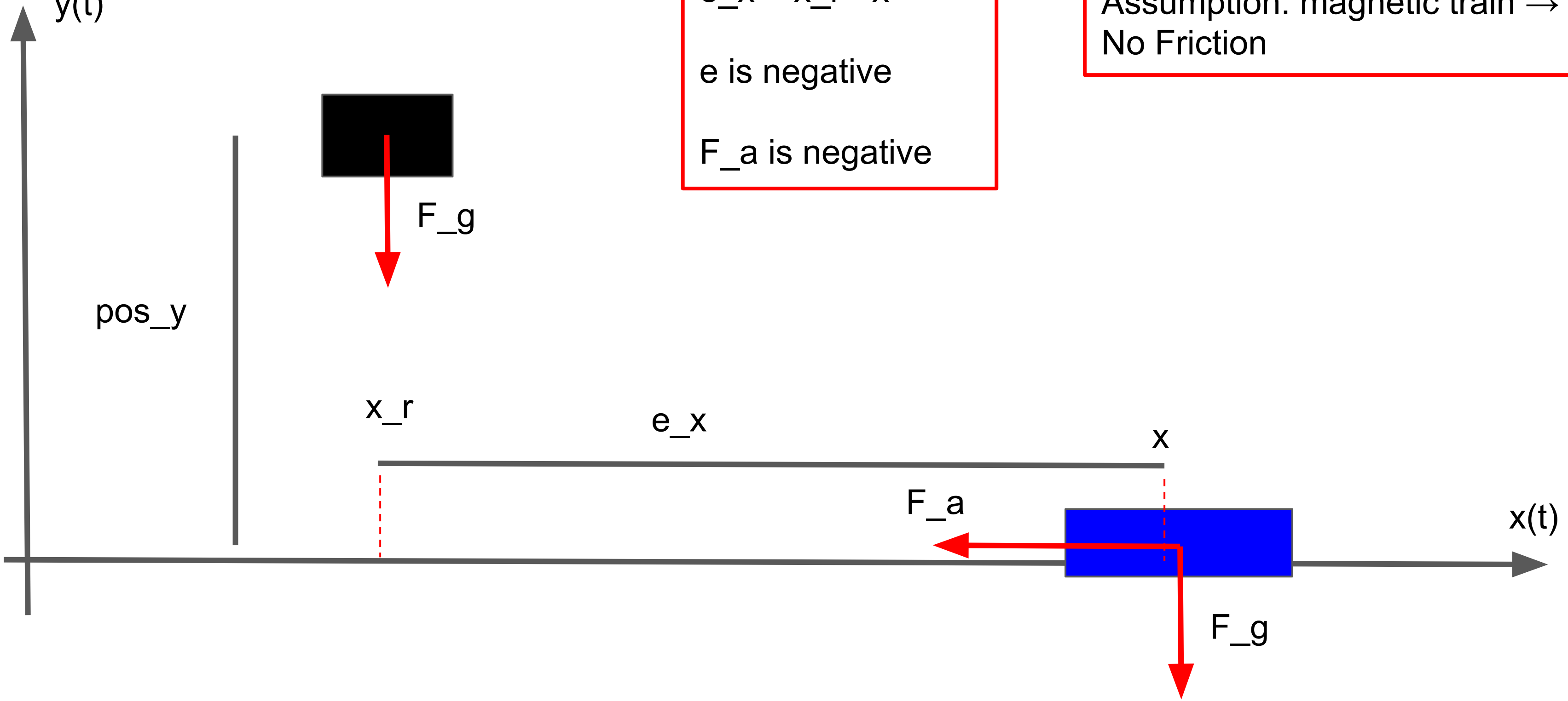
PID - controller

e_x = x_r - x

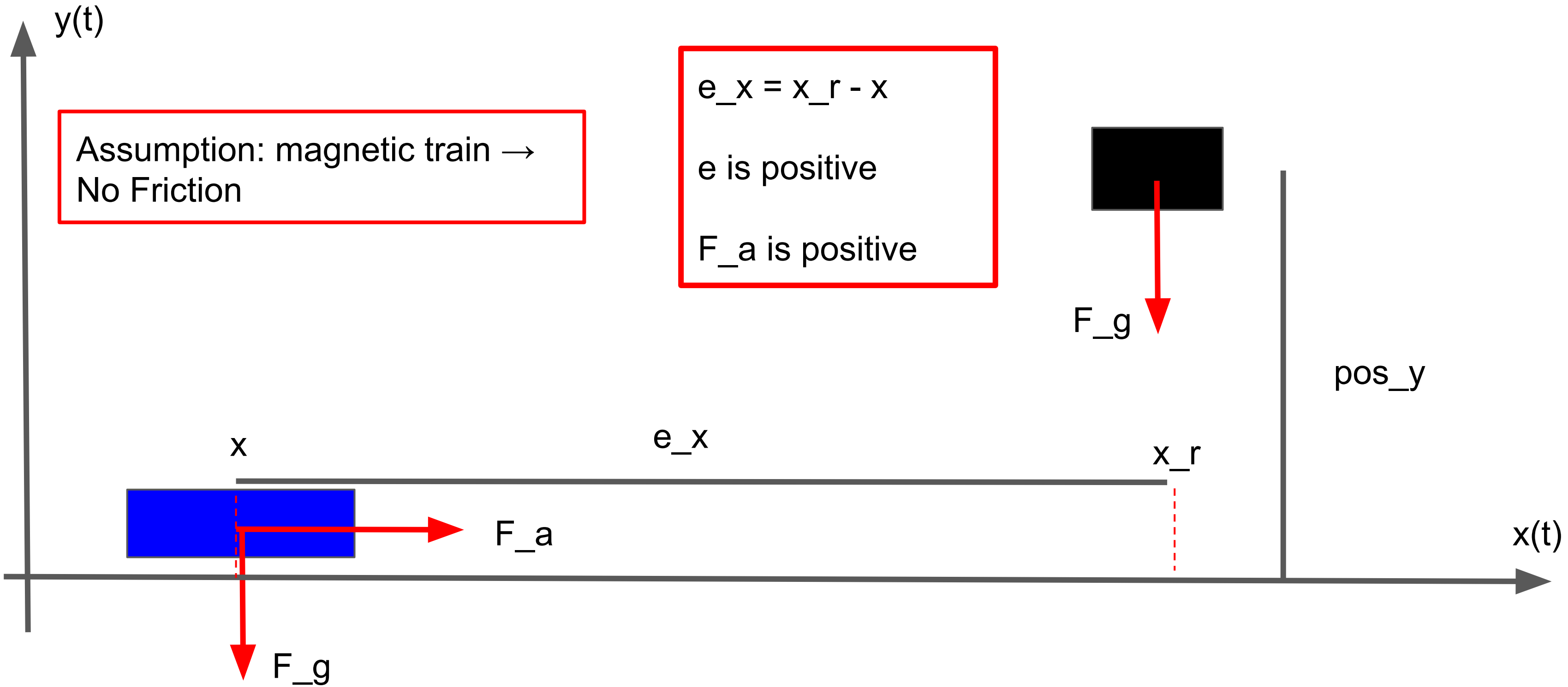
e is negative

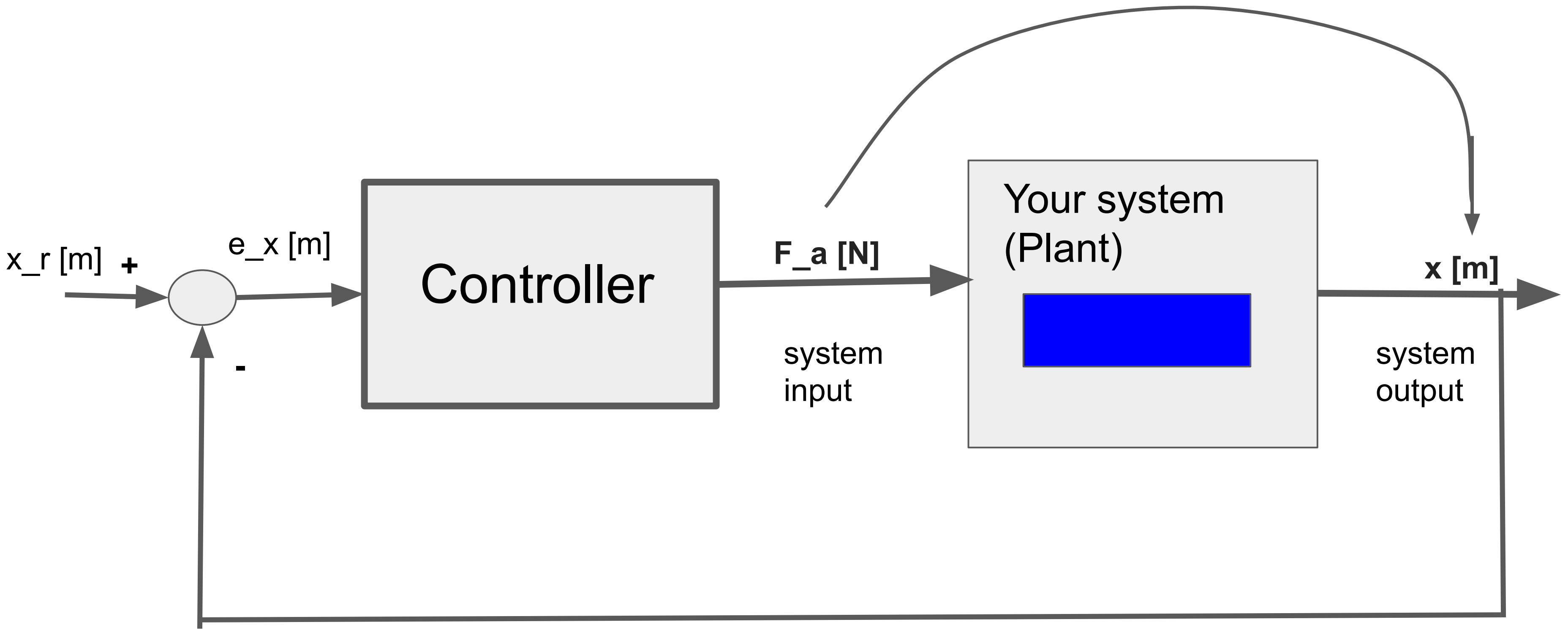
F_a is negative

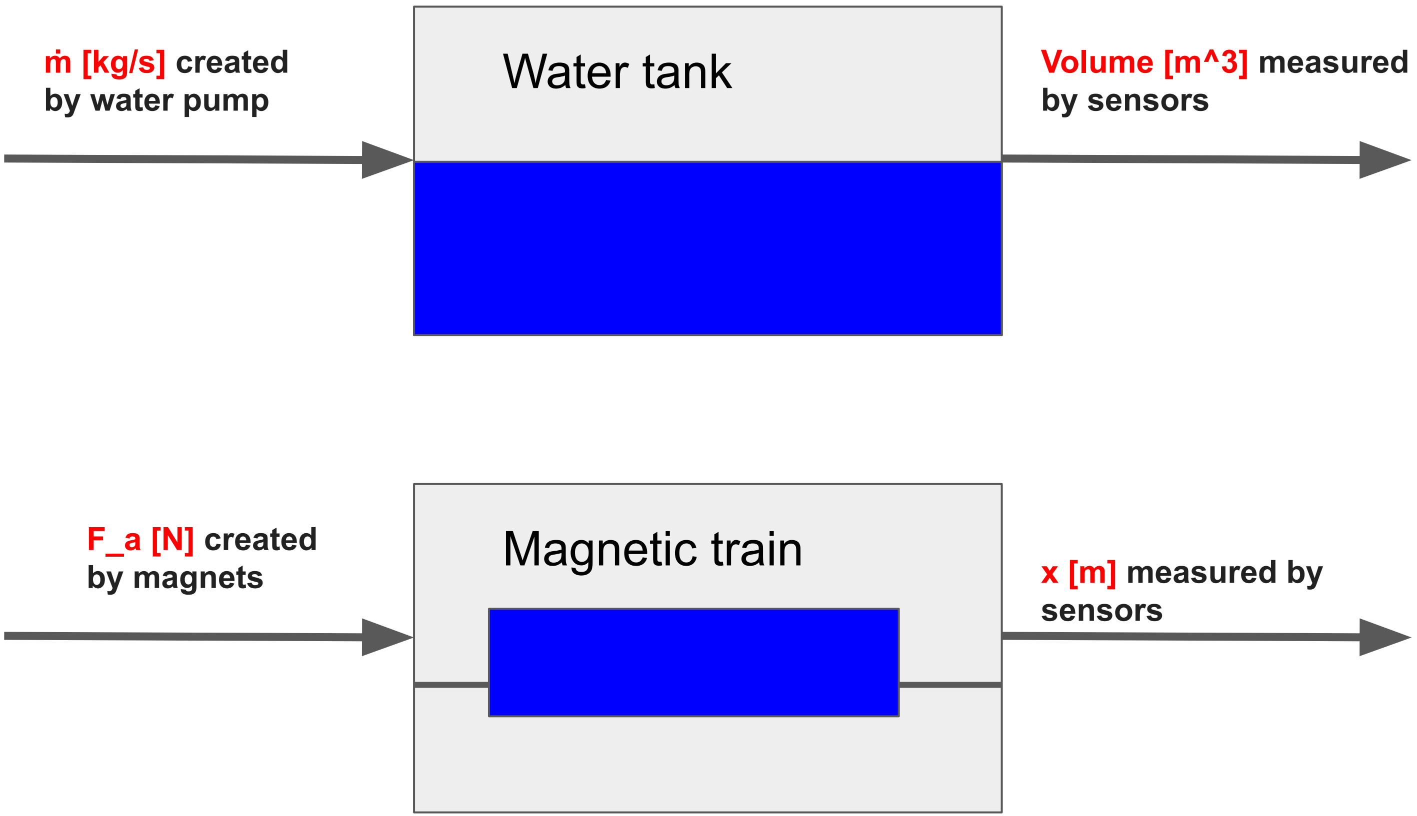
Assumption: magnetic train → No Friction



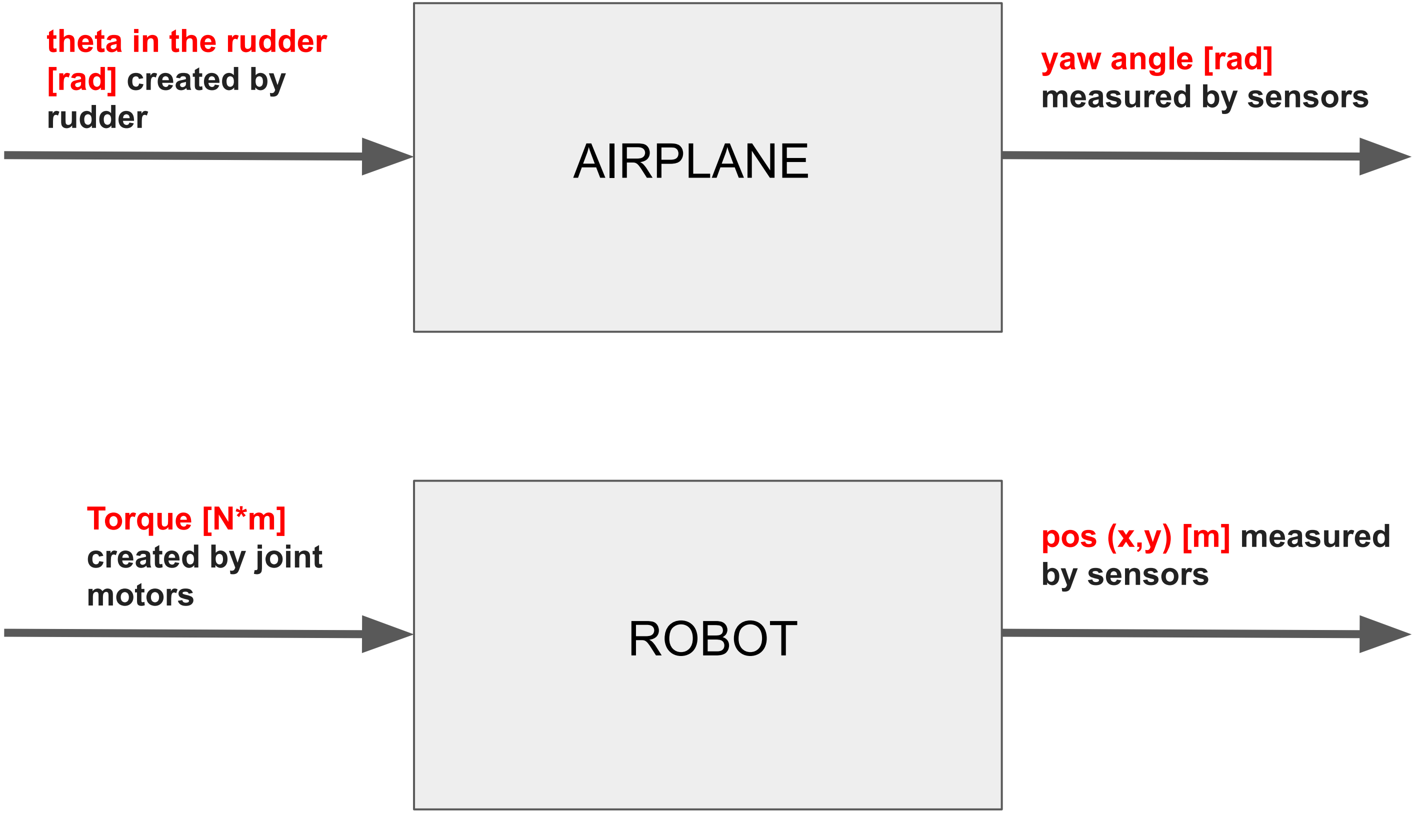
PID - controller




x_r [m]
+
e_x [m]
-
Controller
F_a [N]
system
input
Your system
(Plant)
x [m]
system
output



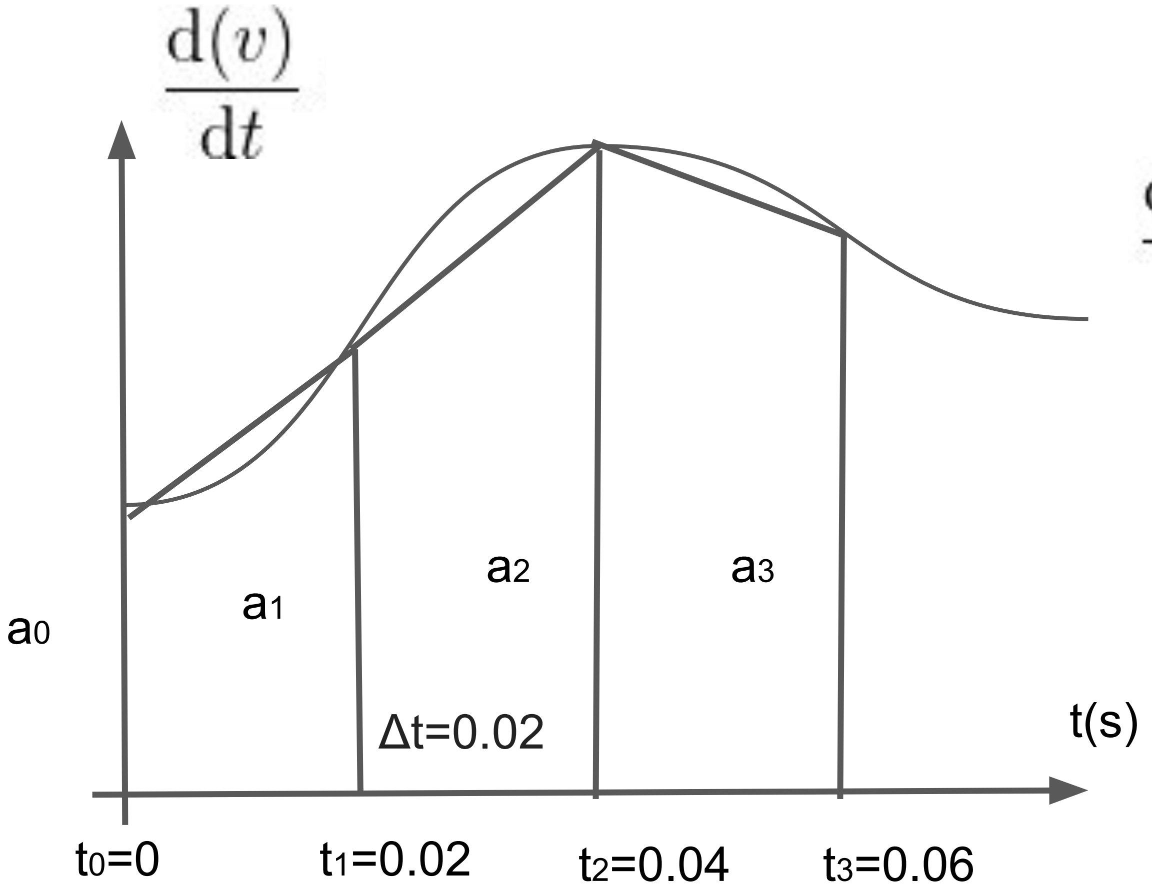
ṁ [kg/s] created by water pump
Water tank
Volume [m^3] measured by sensors
F_a [N] created by magnets
Magnetic train
x [m] measured by sensors


theta in the rudder [rad] created by rudder
AIRPLANE
yaw angle [rad] measured by sensors
Torque [N*m] created by joint motors
ROBOT
pos (x,y) [m] measured by sensors

$$F_a(t) = \frac{\mathrm{d}(m * v)}{\mathrm{d}t} = \frac{\mathrm{d}(m)}{\mathrm{d}t} * v + \frac{\mathrm{d}(v)}{\mathrm{d}t} * m$$

0

$$F_a(t) = m * \frac{\mathrm{d}(v)}{\mathrm{d}t} \rightarrow \frac{\mathrm{d}(v)}{\mathrm{d}t} = \frac{1}{m} * F_a(t)$$



$$\frac{d(v)}{dt} = \frac{1}{m} * F_a(t)$$

$$\int_{v_i}^{v(t)} dv = \frac{1}{m} * \int_0^t F_a(t)dt$$

$$\int_{v_i}^{v(t)} dv = \frac{1}{m} * \int_0^t F_a(t)dt$$

$$v(t) - v_i = \frac{1}{m} * \int_0^t F_a(t)dt$$

$$\int_{v_i}^{v(t)} dv = \frac{1}{m} * \int_0^t F_a(t)dt$$

$$v(t) - v_i = \frac{1}{m} * \int_0^t F_a(t)dt$$

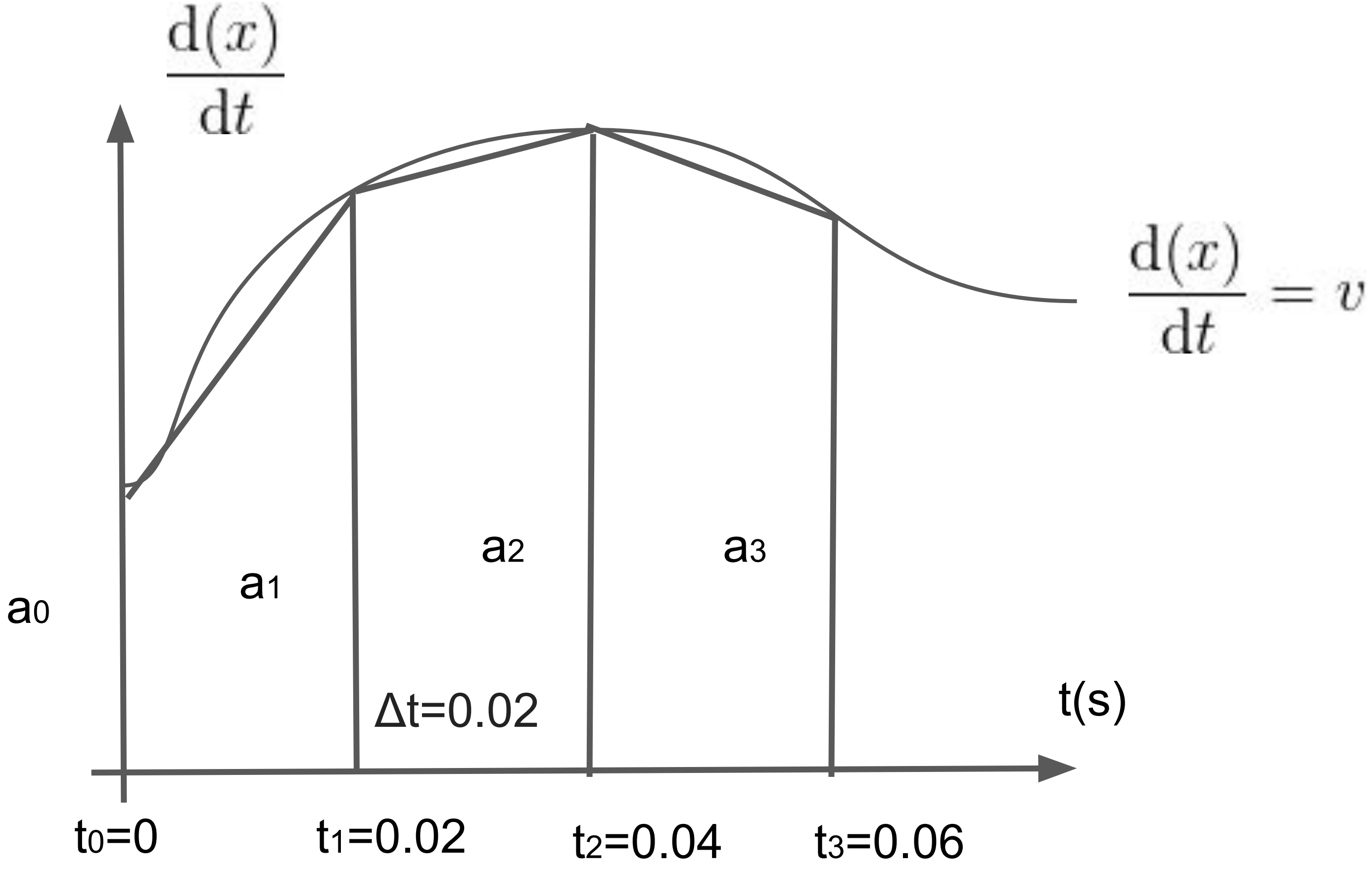
$$v(t) = v_i + \frac{1}{m} * \int_0^t F_a(t)dt$$

$$\int_{v_i}^{v(t)} dv = \frac{1}{m} * \int_0^t F_a(t)dt$$

$$v(t) - v_i = \frac{1}{m} * \int_0^t F_a(t)dt$$

$$v(t) = v_i + \frac{1}{m} * \int_0^t F_a(t)dt$$

$$v(t_j) = v(t_{j-1}) + \frac{1}{m} * (\frac{F_a(t_{j-1}) + F_a(t_j)}{2})\Delta t$$


$\frac{d(x)}{dt}$
$\frac{d(x)}{dt} = v$
a_0
a_1
a_2
a_3
$\Delta t = 0.02$
t(s)
$t_0=0$
$t_1=0.02$
$t_2=0.04$
$t_3=0.06$

$$\int_{x_i}^{x(t)} dx = \int_0^t v(t)dt$$

$$\int_{x_i}^{x(t)} dx = \int_0^t v(t)dt$$

$$x(t) - x_i = \int_0^t v(t)dt$$

$$\int_{x_i}^{x(t)} dx = \int_0^t v(t)dt$$

$$x(t) - x_i = \int_0^t v(t)dt$$

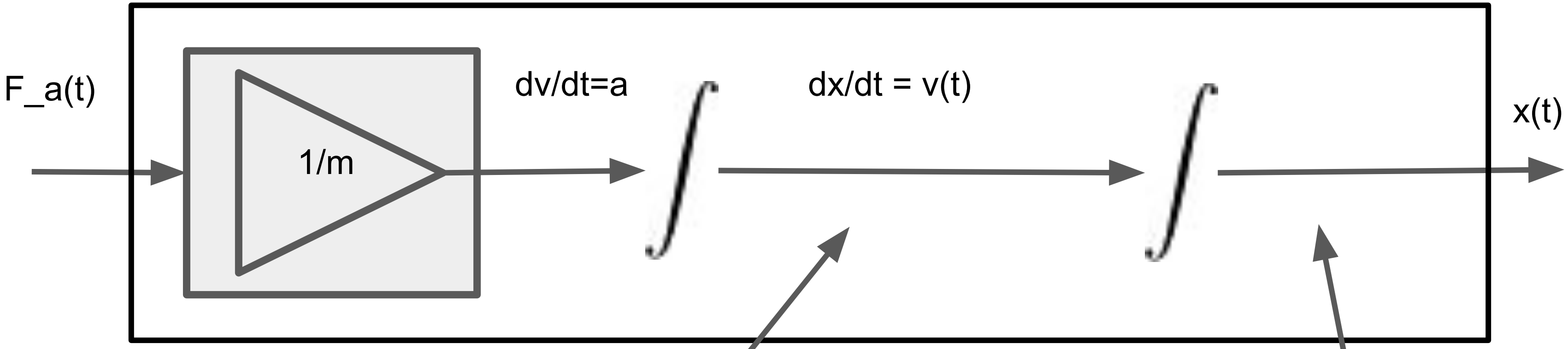
$$x(t) = x_i + \int_0^t v(t)dt$$

$$\int_{x_i}^{x(t)} dx = \int_0^t v(t)dt$$

$$x(t) - x_i = \int_0^t v(t)dt$$

$$x(t) = x_i + \int_0^t v(t)dt$$

$$x(t_j) = x(t_{j-1}) + (\frac{v(t_{j-1}) + v(t_j)}{2})\Delta t$$



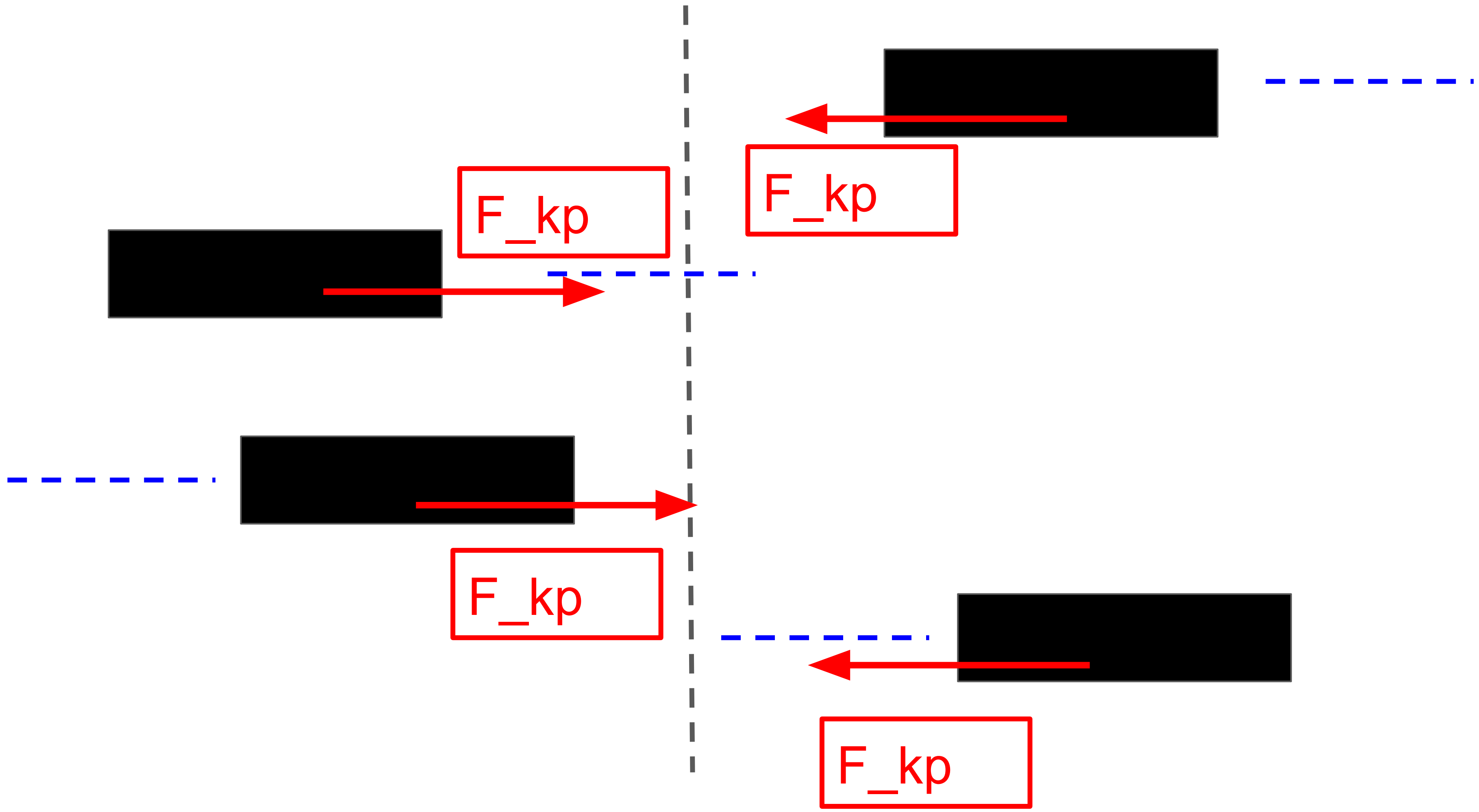
Mathematical model of our train.
Let's try K_p now

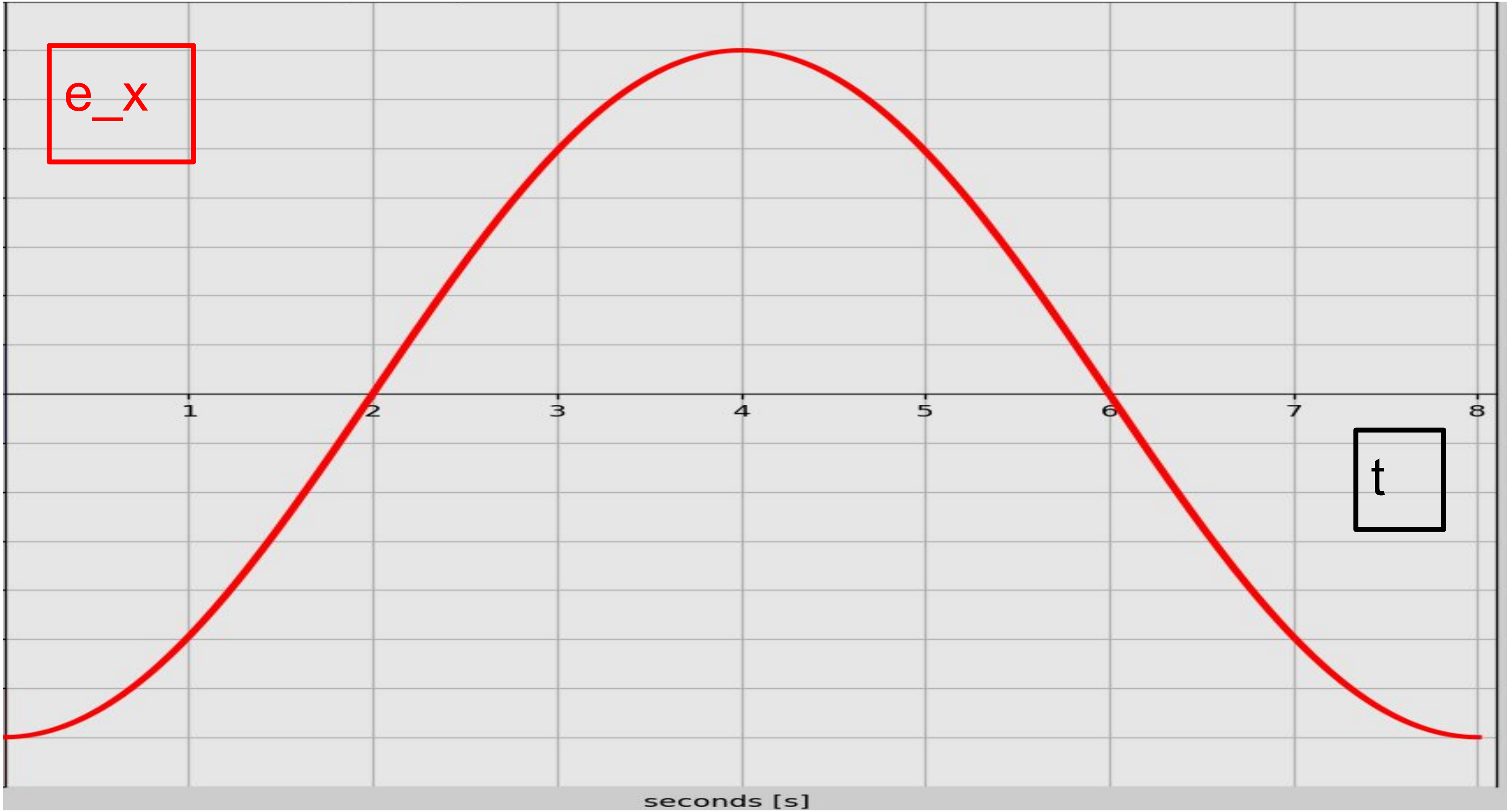
$$v(t) = v_i + \frac{1}{m} * \int_0^t F_a(t)dt$$

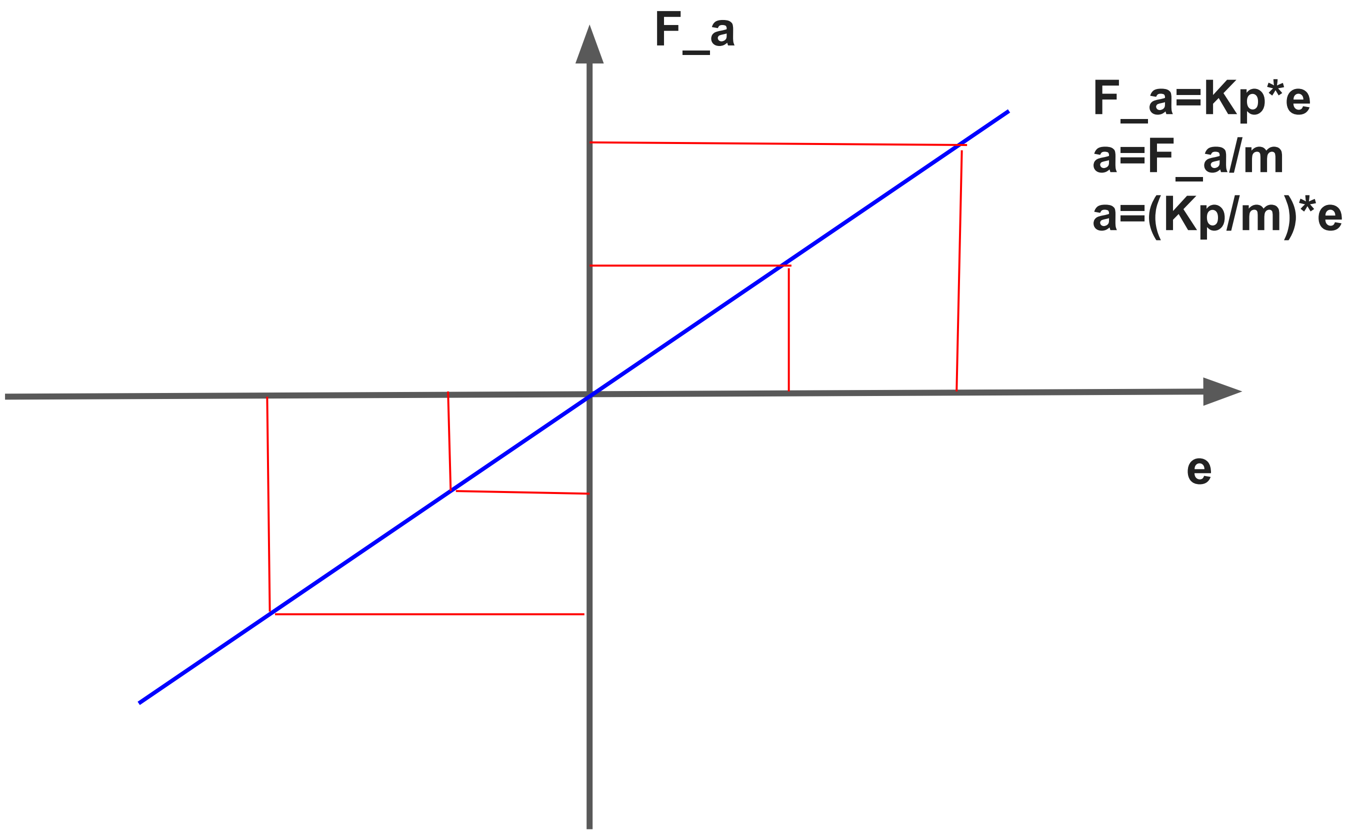
$$v(t_j) = v(t_{j-1}) + \frac{1}{m} * (\frac{F_a(t_{j-1}) + F_a(t_j)}{2})\Delta t$$

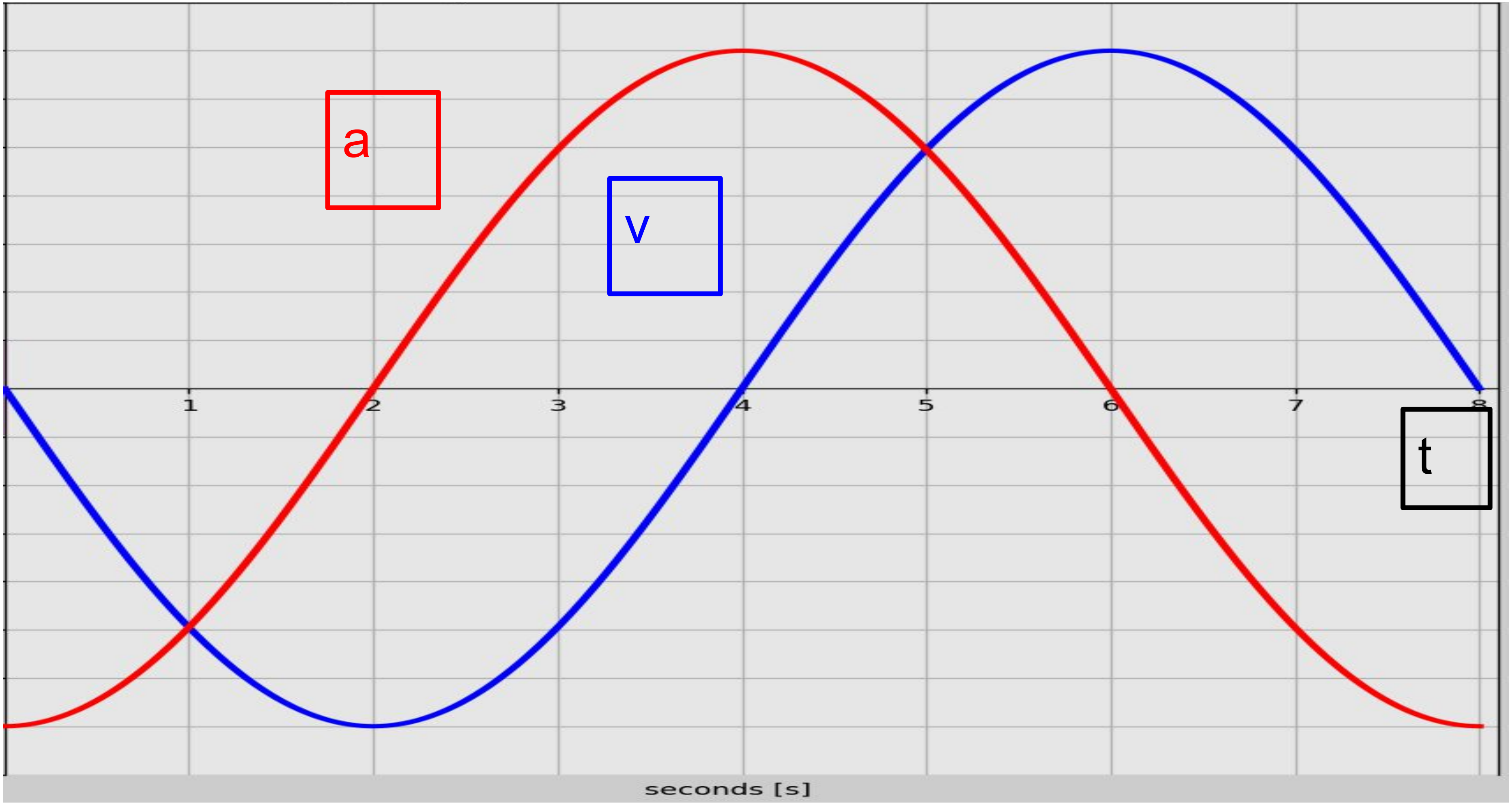
$$x(t) = x_i + \int_0^t v(t)dt$$

$$x(t_j) = x(t_{j-1}) + (\frac{v(t_{j-1}) + v(t_j)}{2})\Delta t$$

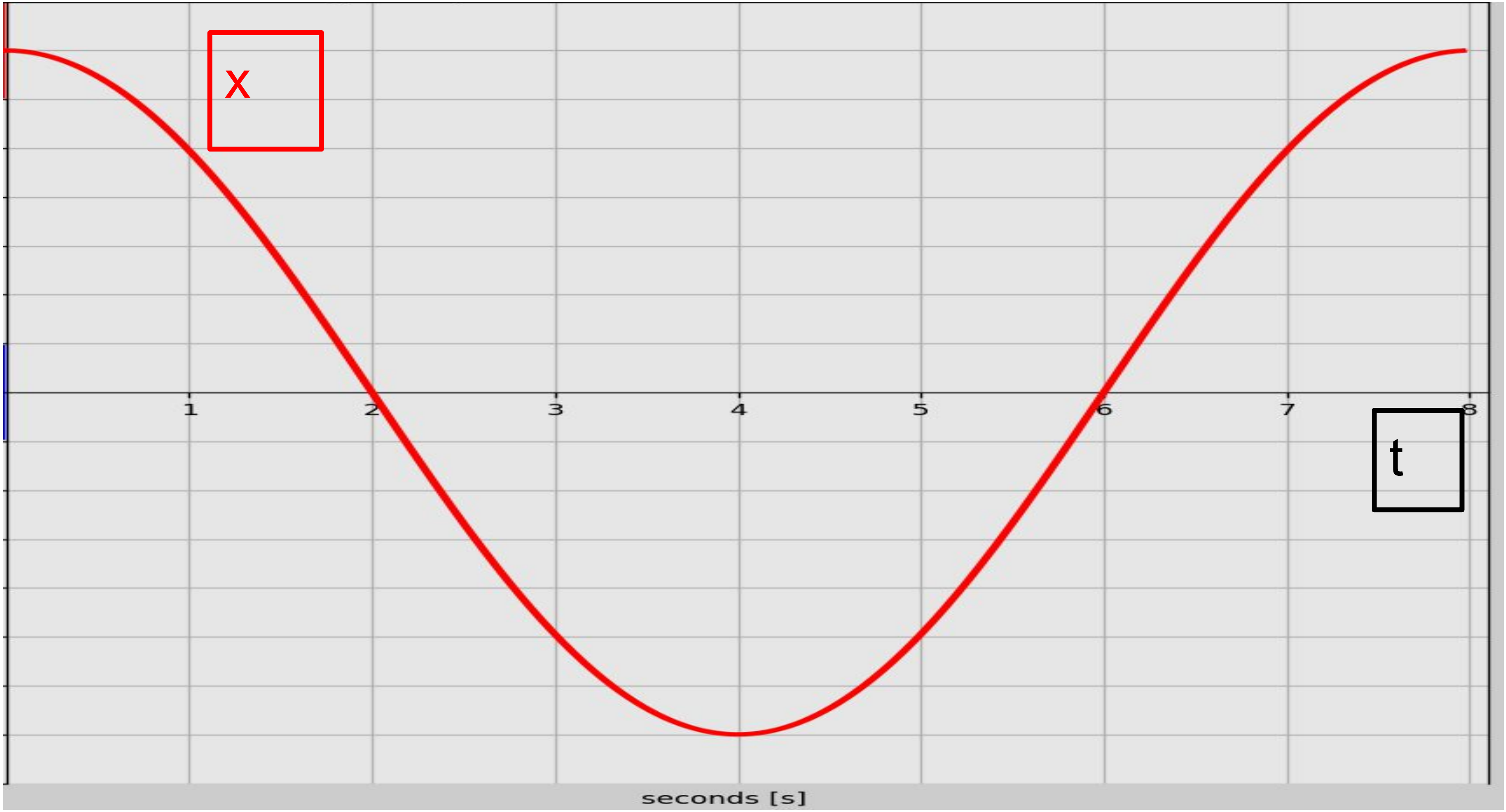

F_kp
F_kp
F_kp
F_kp

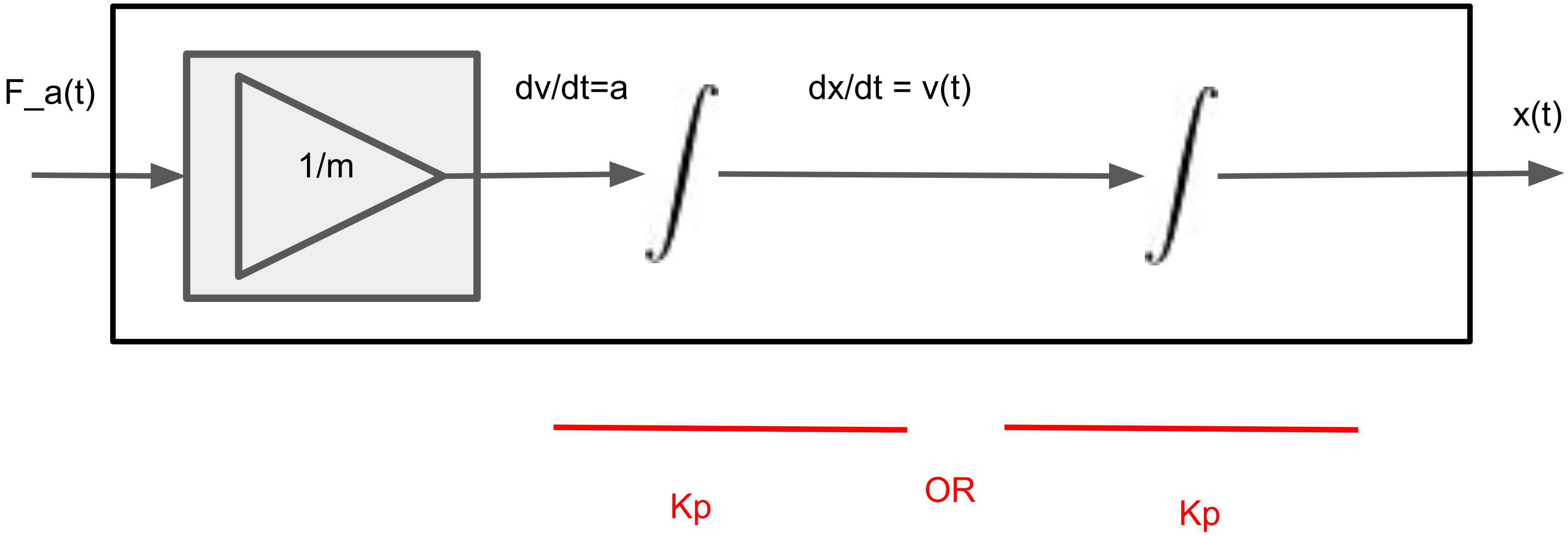

e_x
t
1
2
3
4
5
6
7
8
seconds [s]


F_a
F_a=Kp*e
a=F_a/m
a=(Kp/m)*e
e

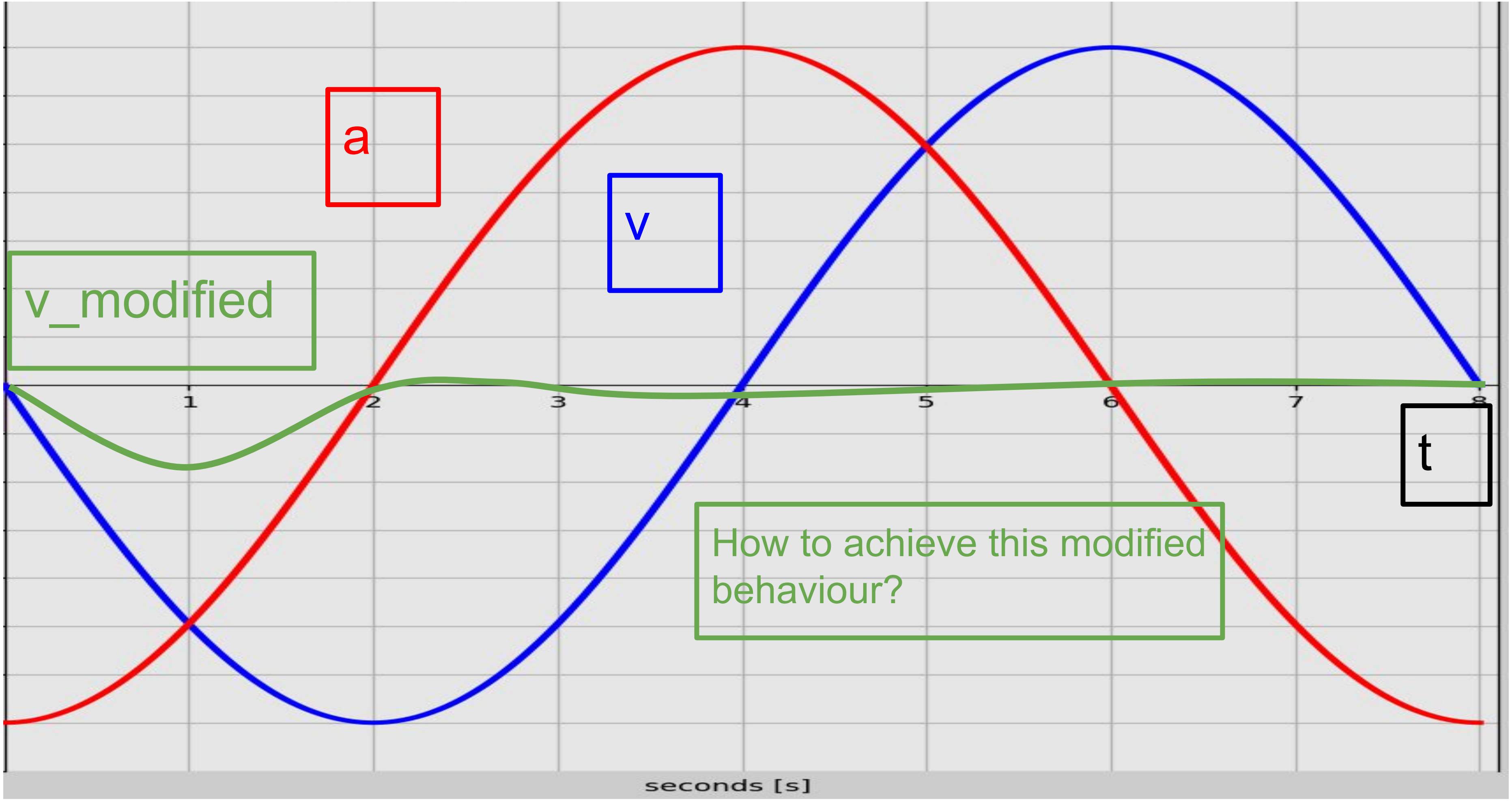


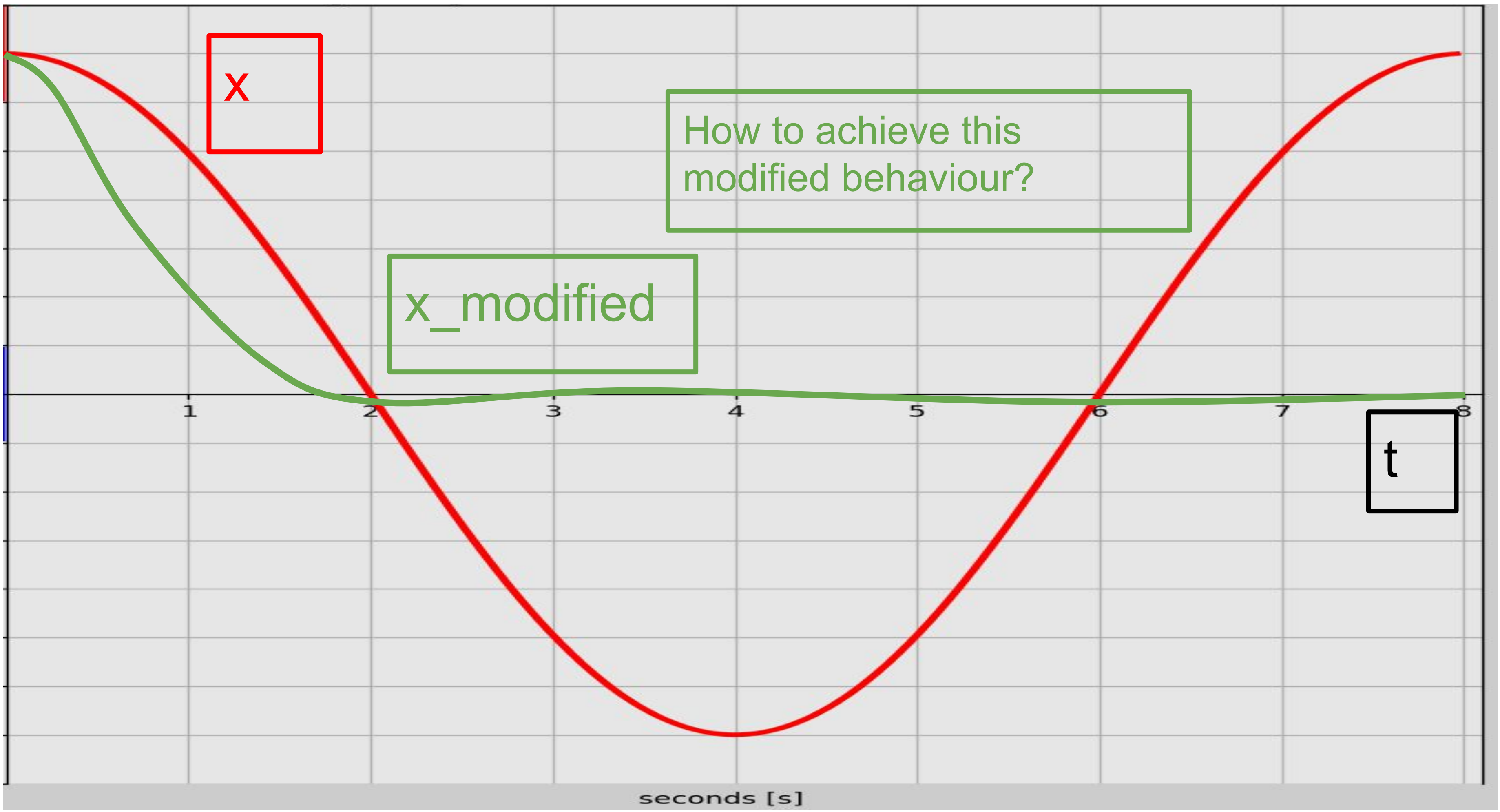
a
v
t
1
2
3
4
5
6
7
8
seconds [s]

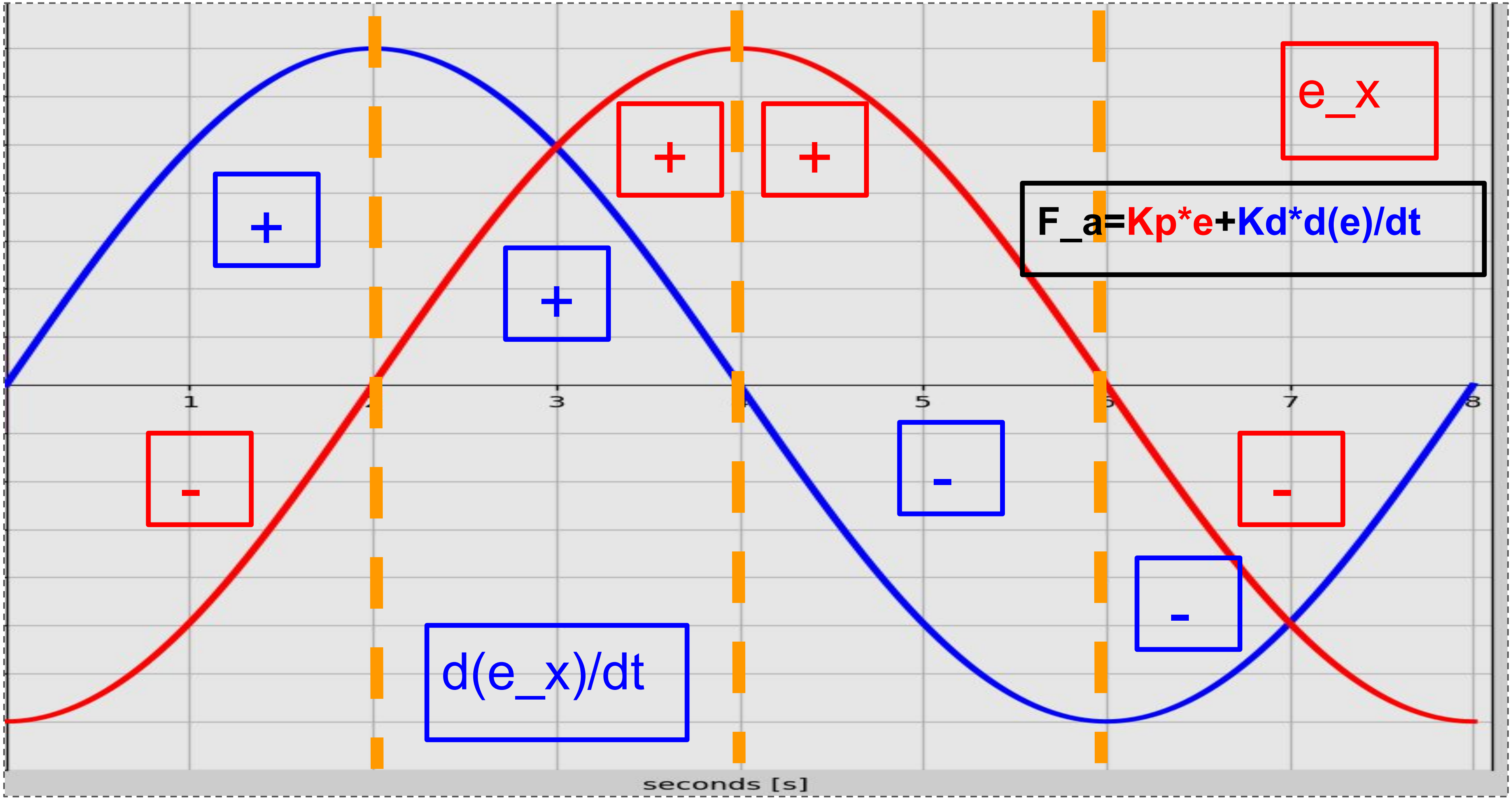

x
1
2
3
4
5
6
7
8
t
seconds [s]

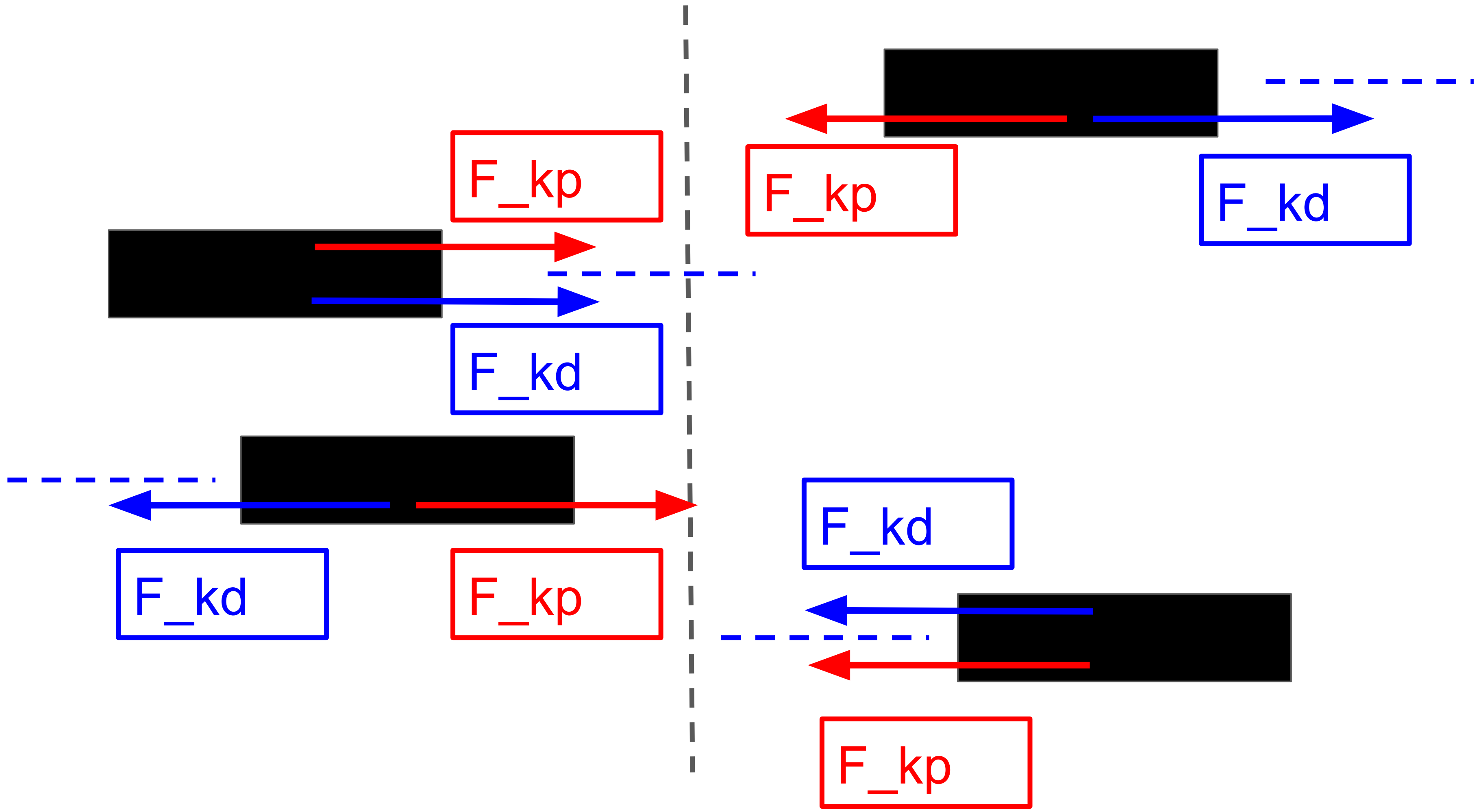


A proportional controller can only manage one integral operator, not more


a
v
v_modified
1
2
3
4
5
6
7
8
t
How to achieve this modified behaviour?
seconds [s]

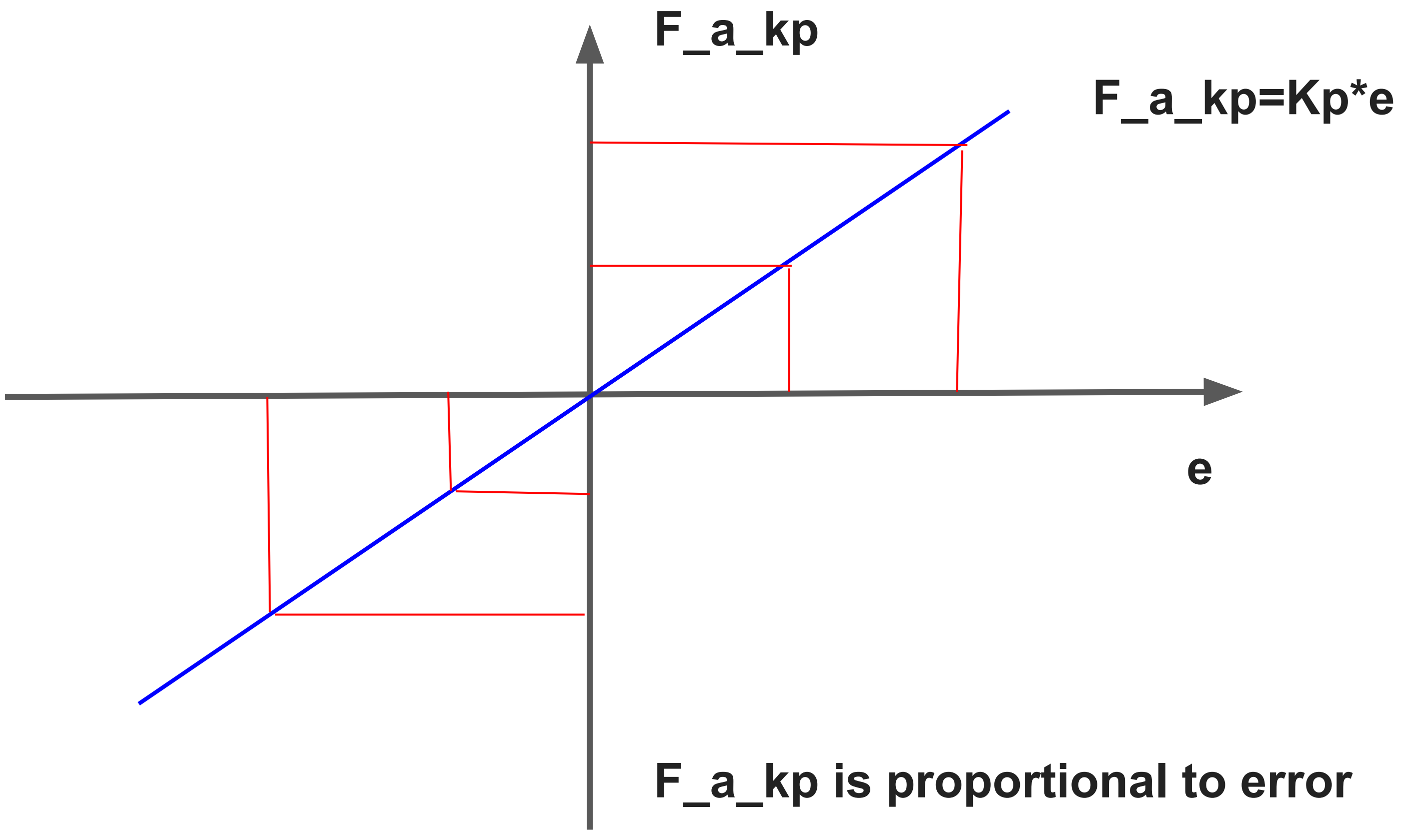

x
How to achieve this
modified behaviour?
x_modified
1
2
3
4
5
6
7
8
t
seconds [s]


e_x
F_a=Kp*e+Kd*d(e)/dt
+
+
+
+
-
-
-
-
d(e_x)/dt
1
3
5
7
8
seconds [s]


F_kp
F_kp
F_kd
F_kd
F_kd
F_kd
F_kp
F_kp

Kp [N/m]

Kp→ slope

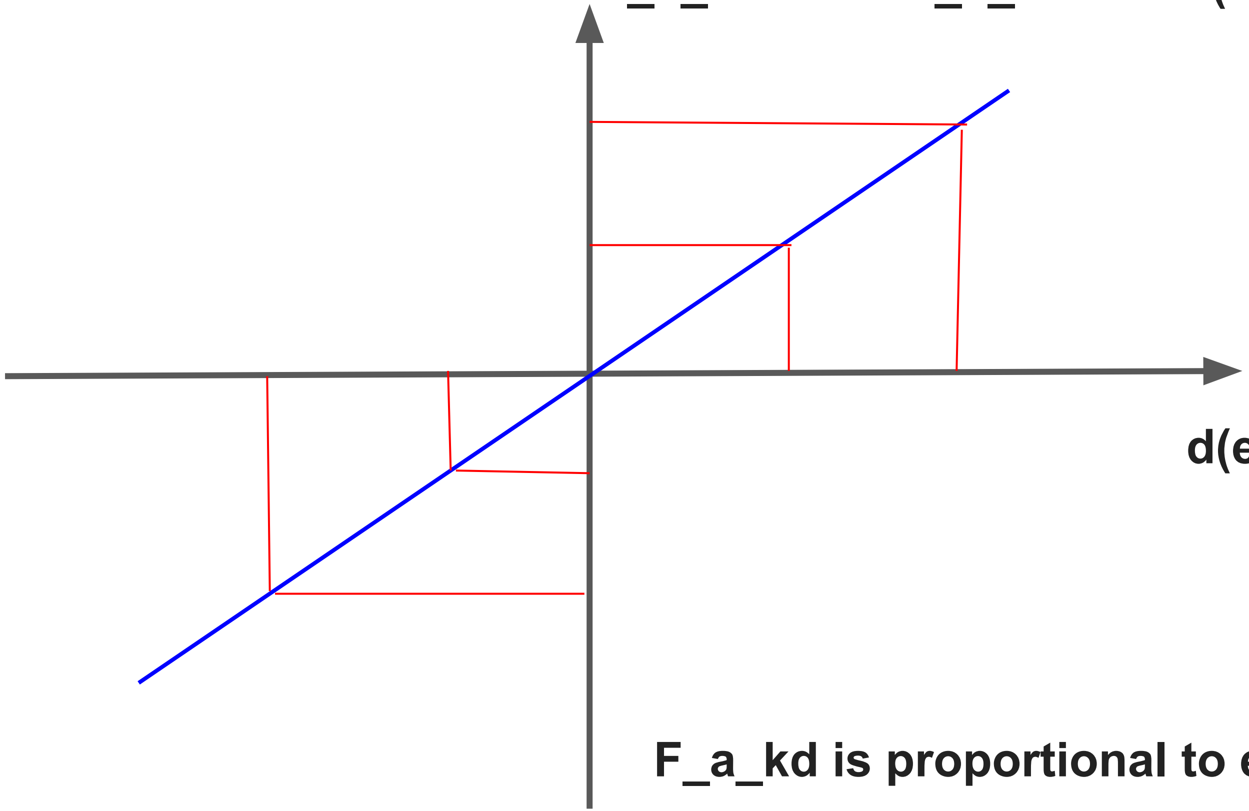


Kd [N/(m/s)]

Kd→ slope

F_a_kd

F_a_kd=Kd*d(e)/dt

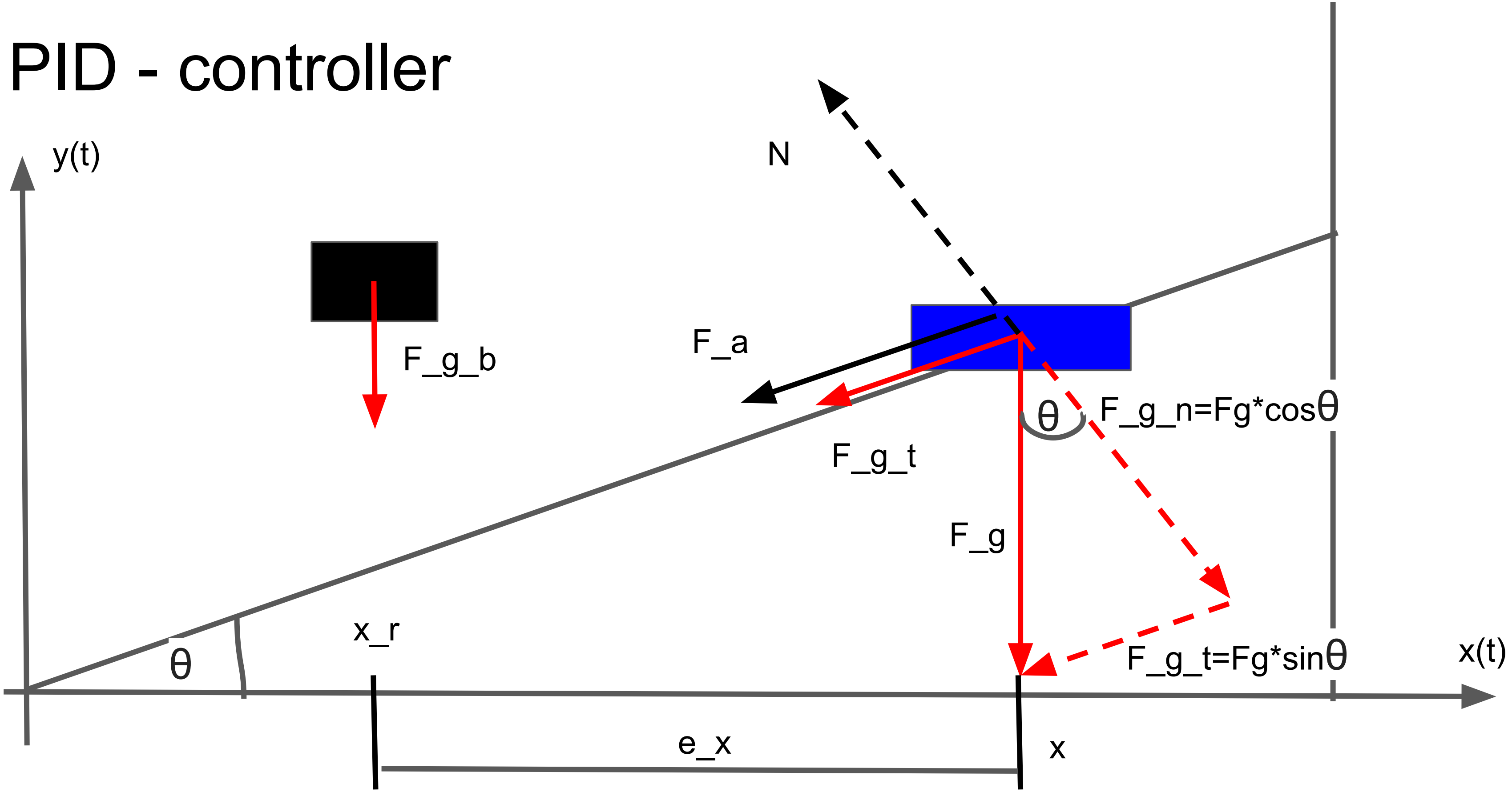


d(e)/dt

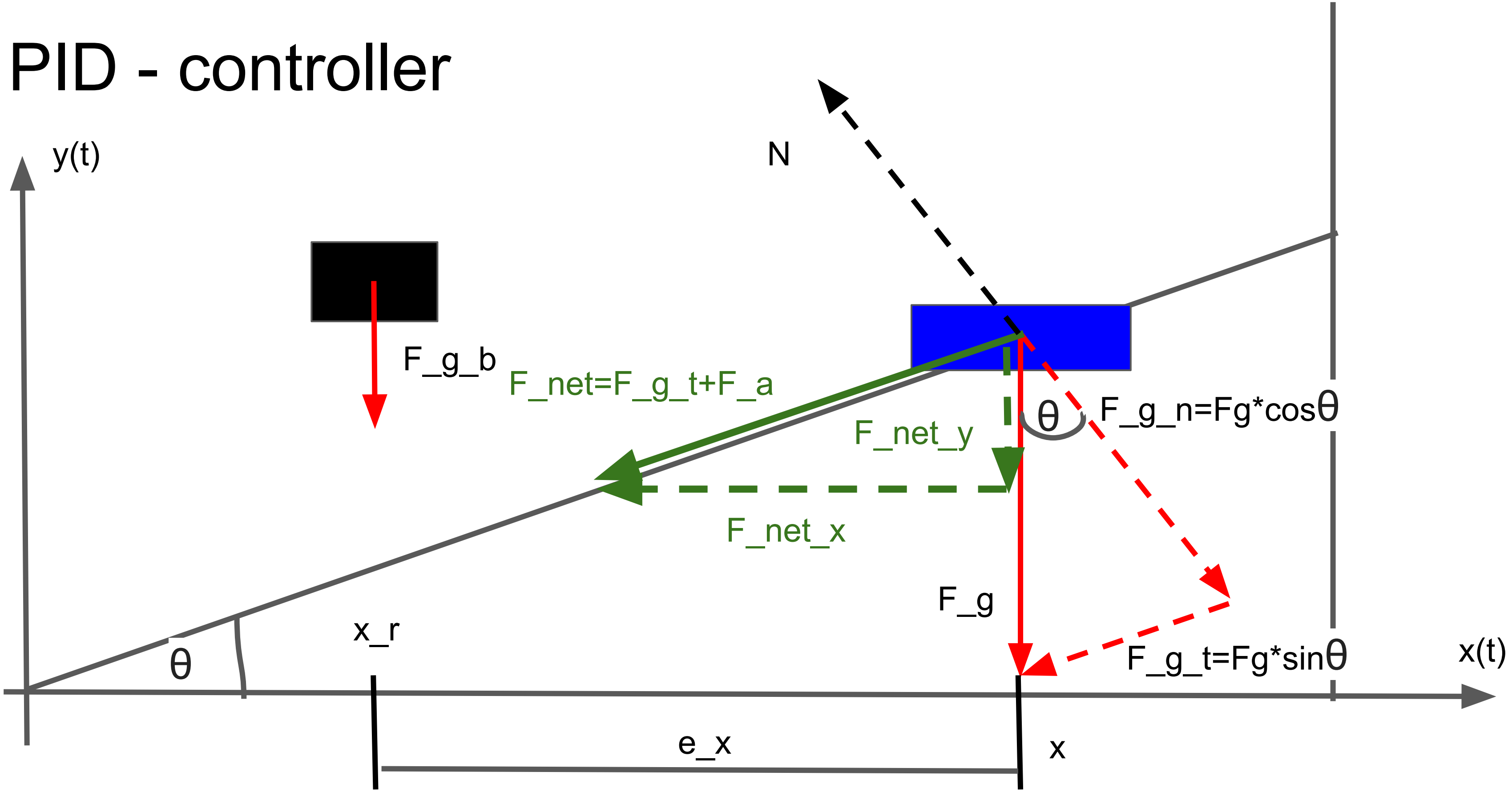
F_a_kd is proportional to error

Depending on the system, you tune Kp and Kd. You can make the proportional part stronger by making Kp bigger. Or, you can make the derivative part stronger by making Kd stronger.

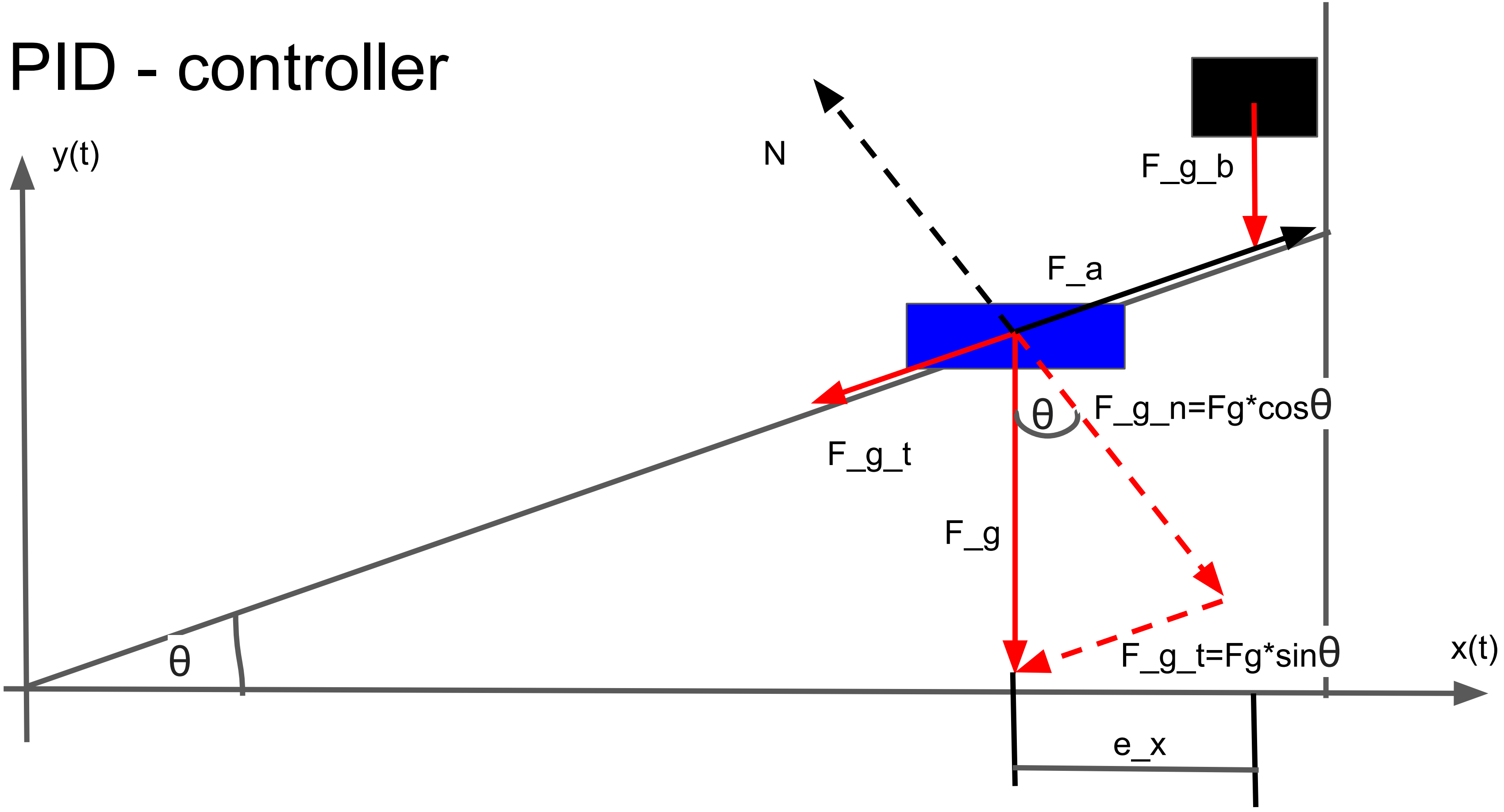
PID - controller



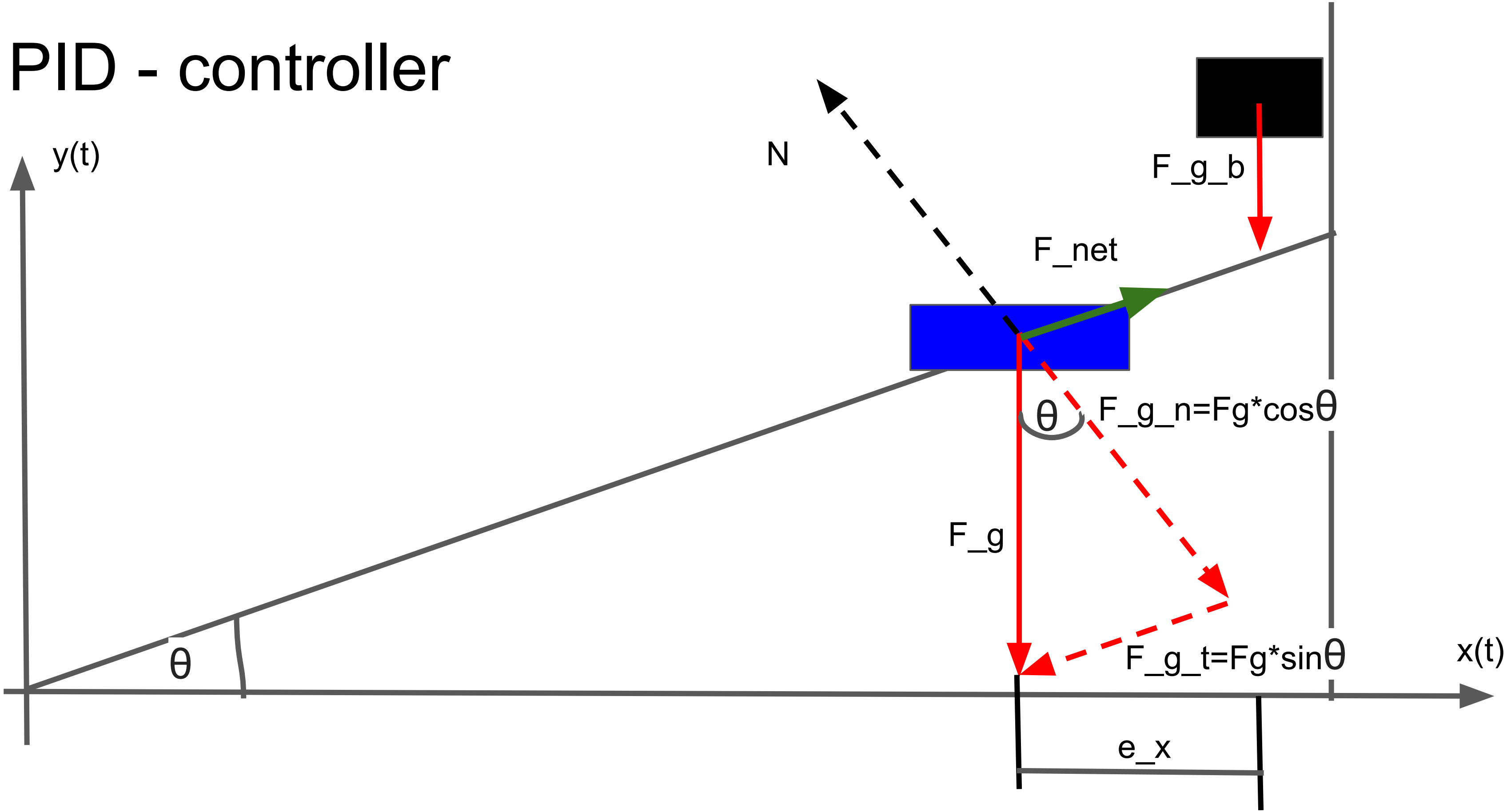
PID - controller



PID - controller



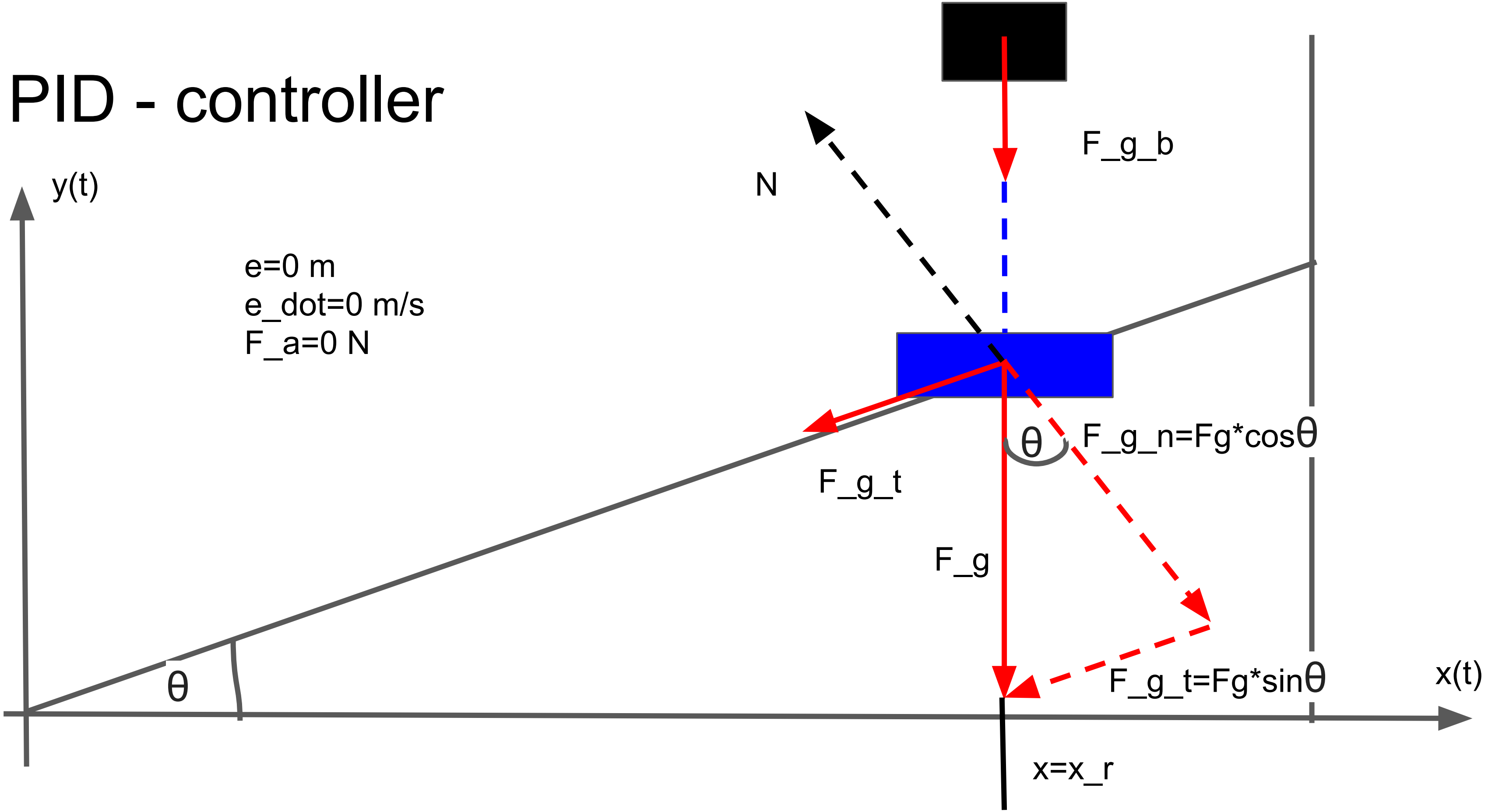
PID - controller



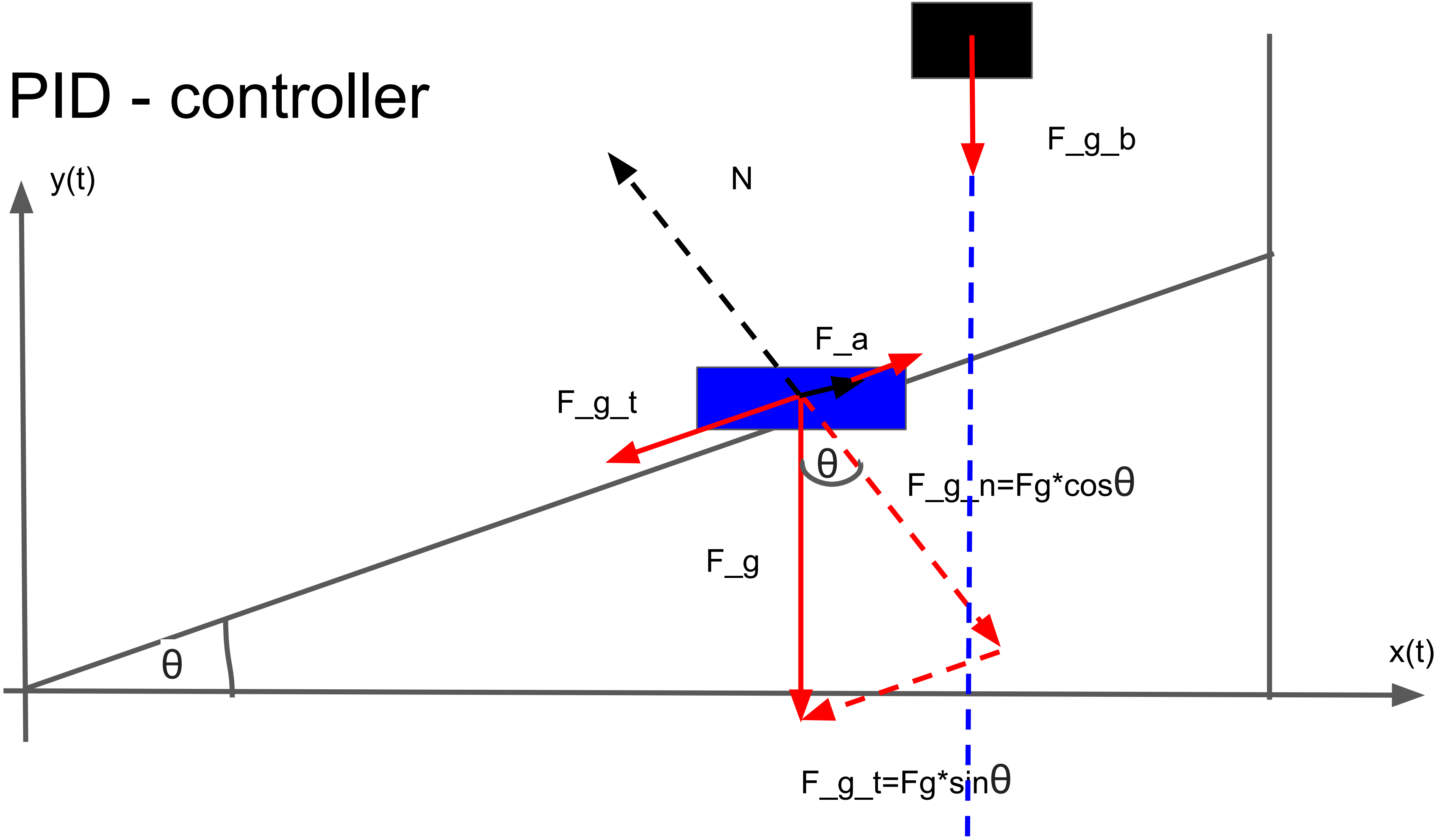
$$\int_{v_{ix}}^{v_x(t)} dv_x = \frac{1}{m} * \int_0^t F_{net_x}(t)dt$$

$$\int_{v_{iy}}^{v_y(t)} dv_y = \frac{1}{m} * \int_0^t F_{net_y}(t)dt$$

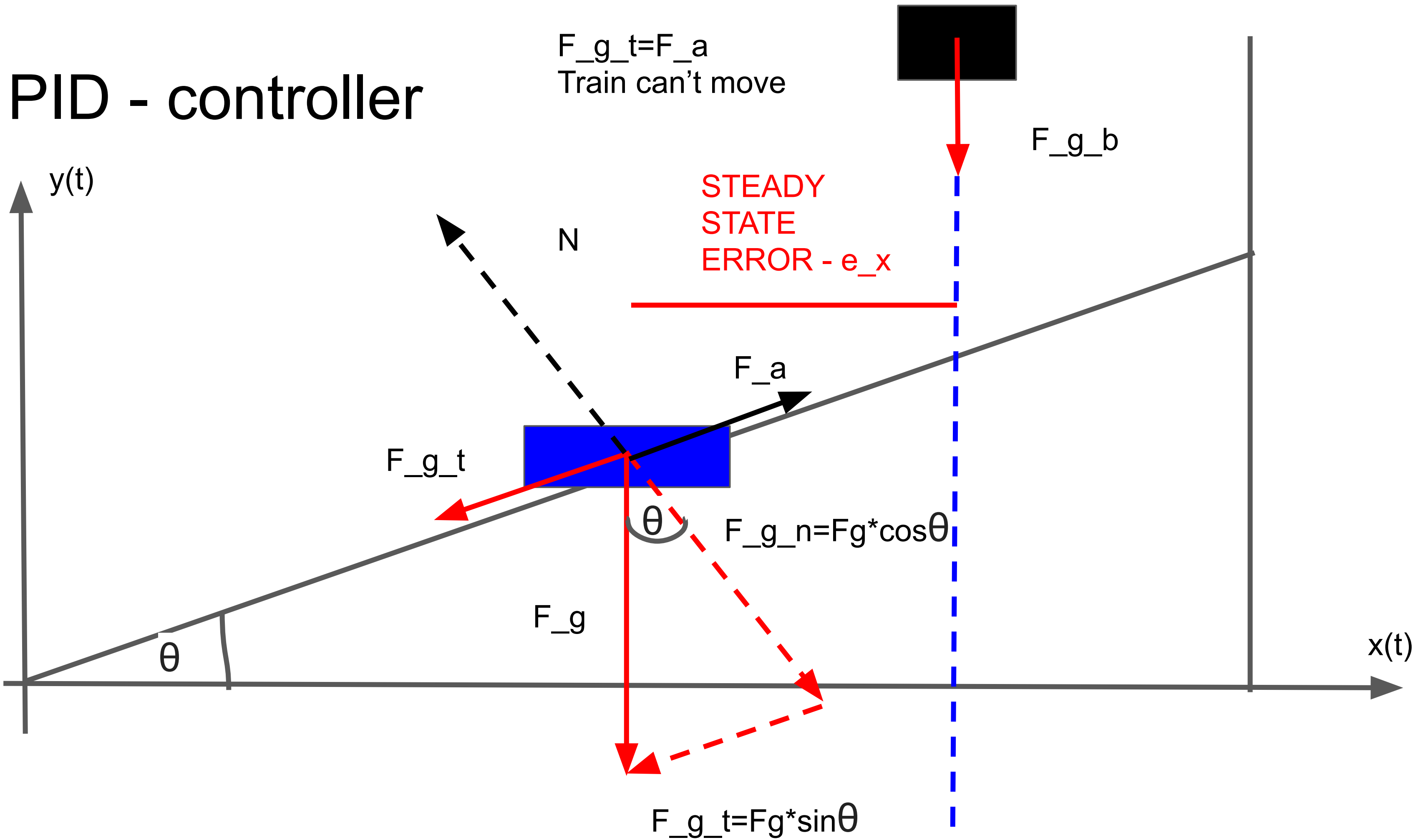
PID - controller

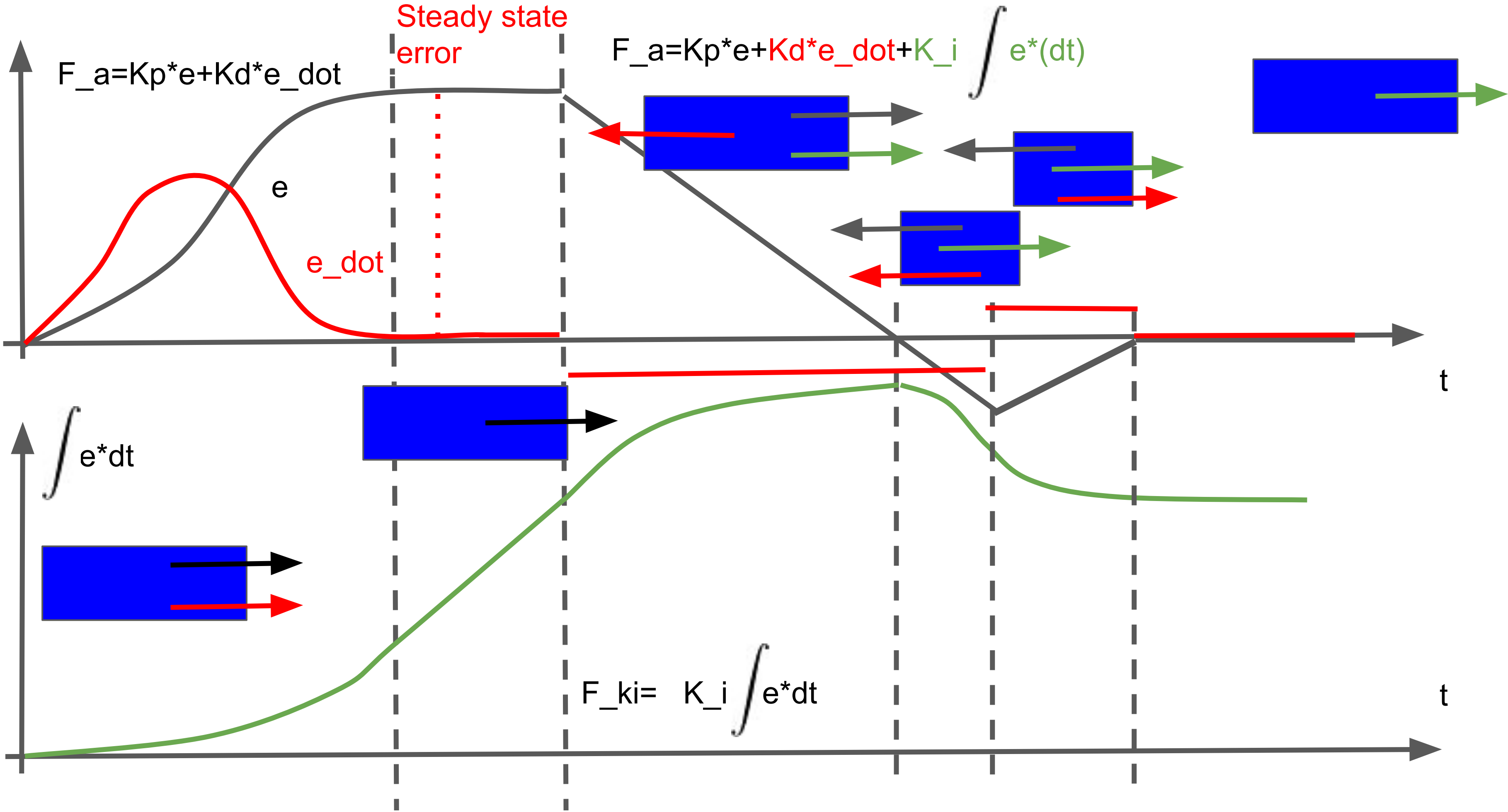


PID - controller



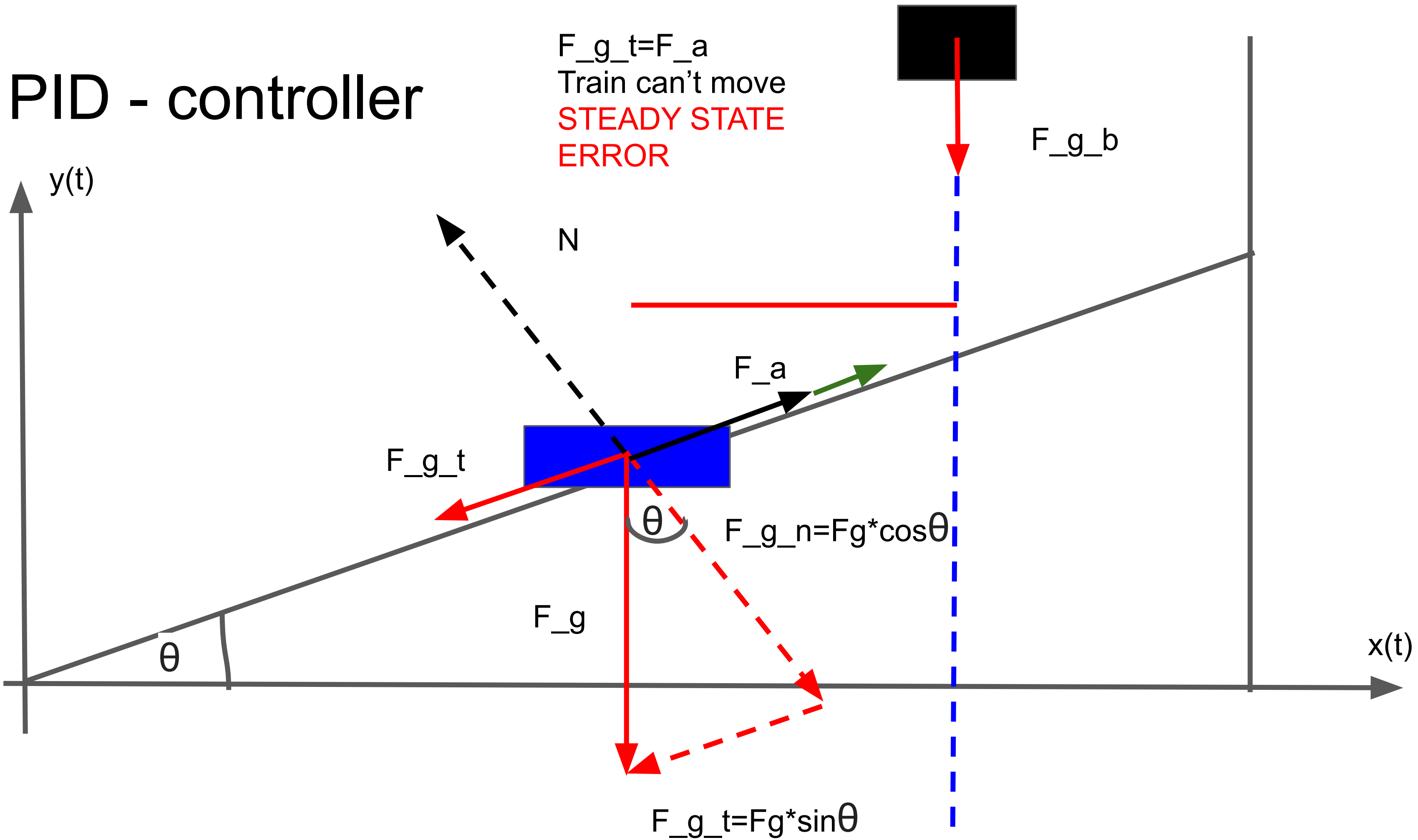
PID - controller





Steady state error
F_a=Kp*e+Kd*e_dot
F_a=Kp*e+Kd*e_dot+K_i ∫ e*(dt)
e
e_dot
t
∫ e*dt
F_ki= K_i ∫ e*dt
t

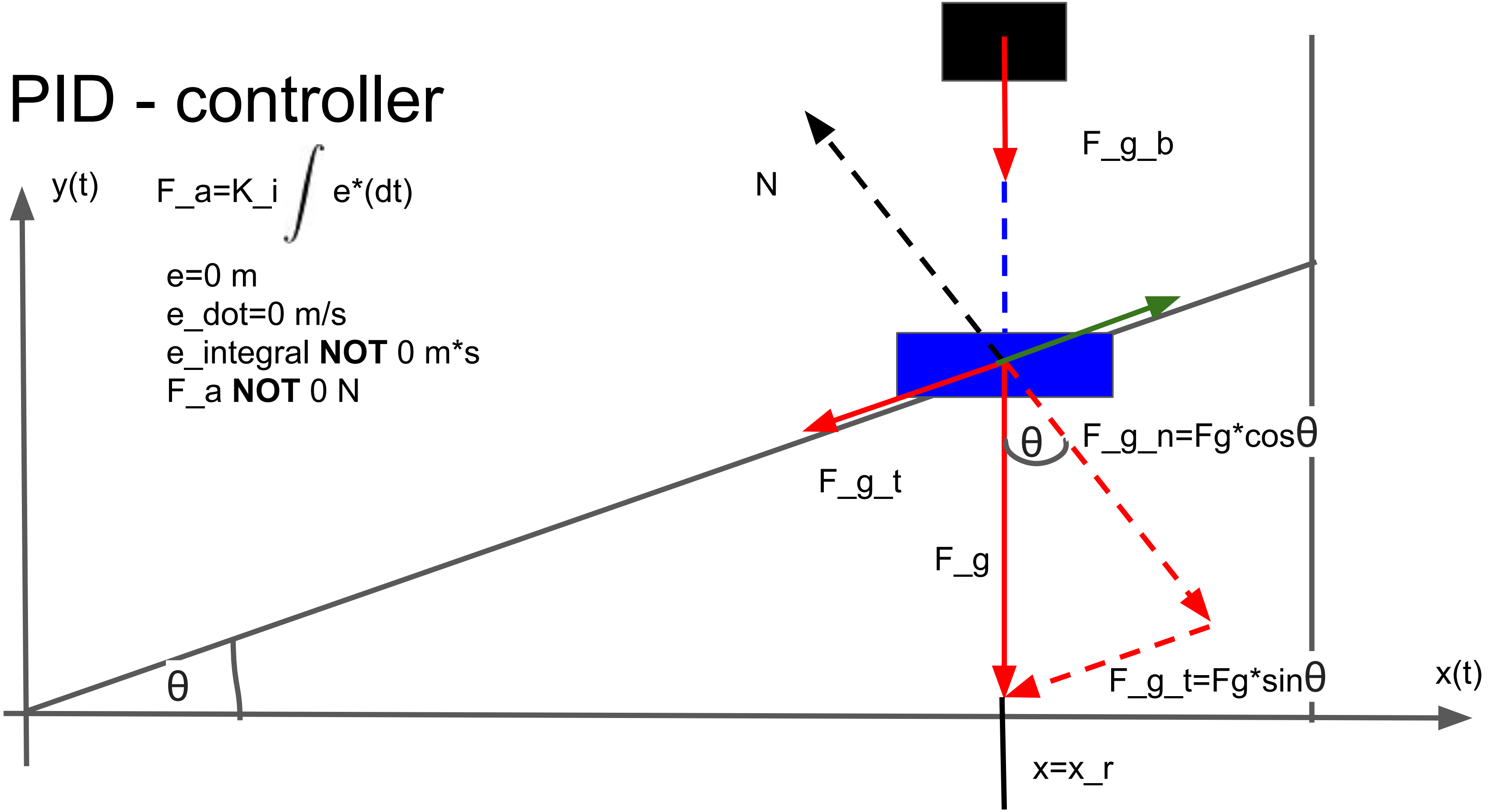
PID - controller

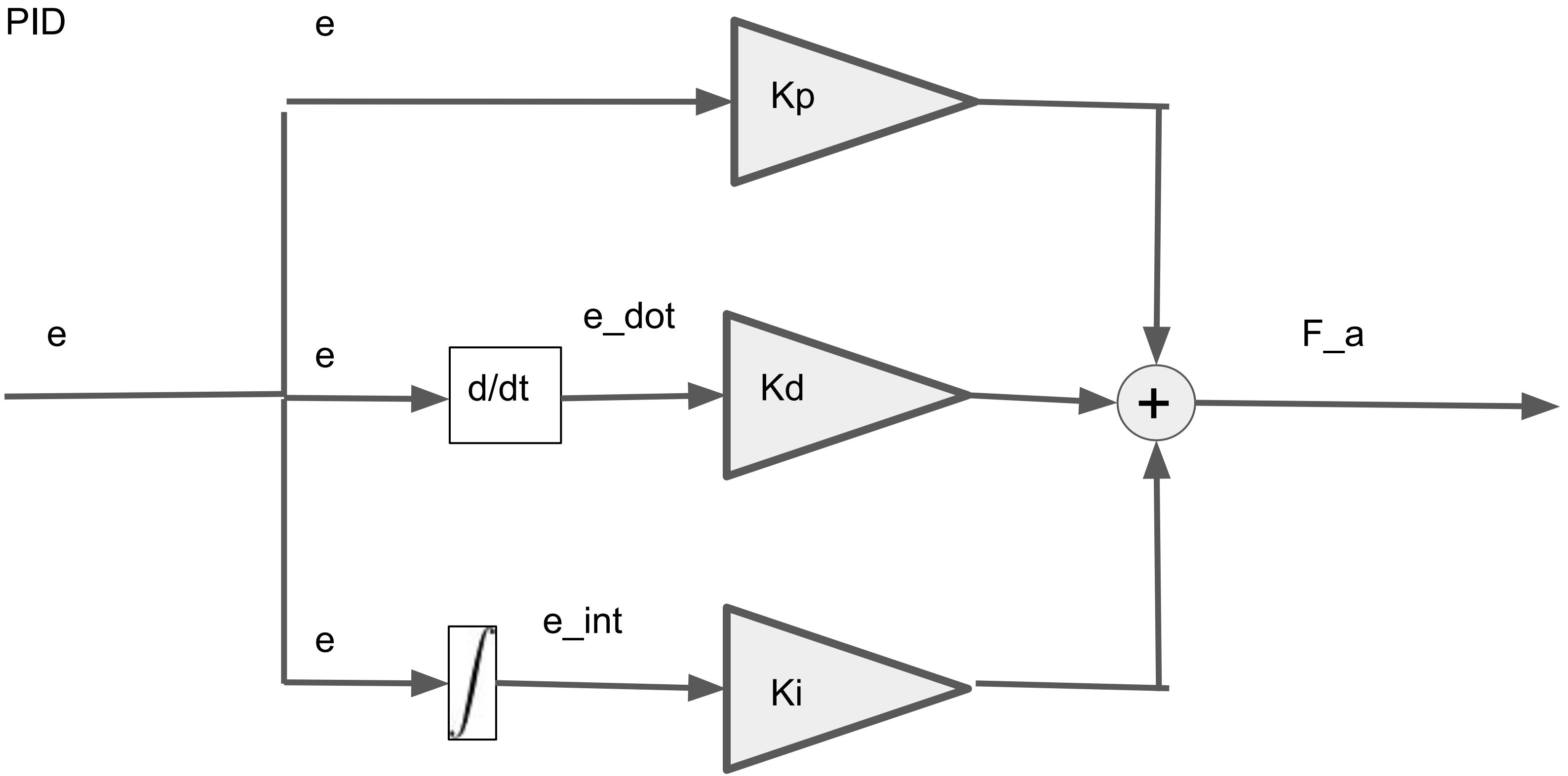


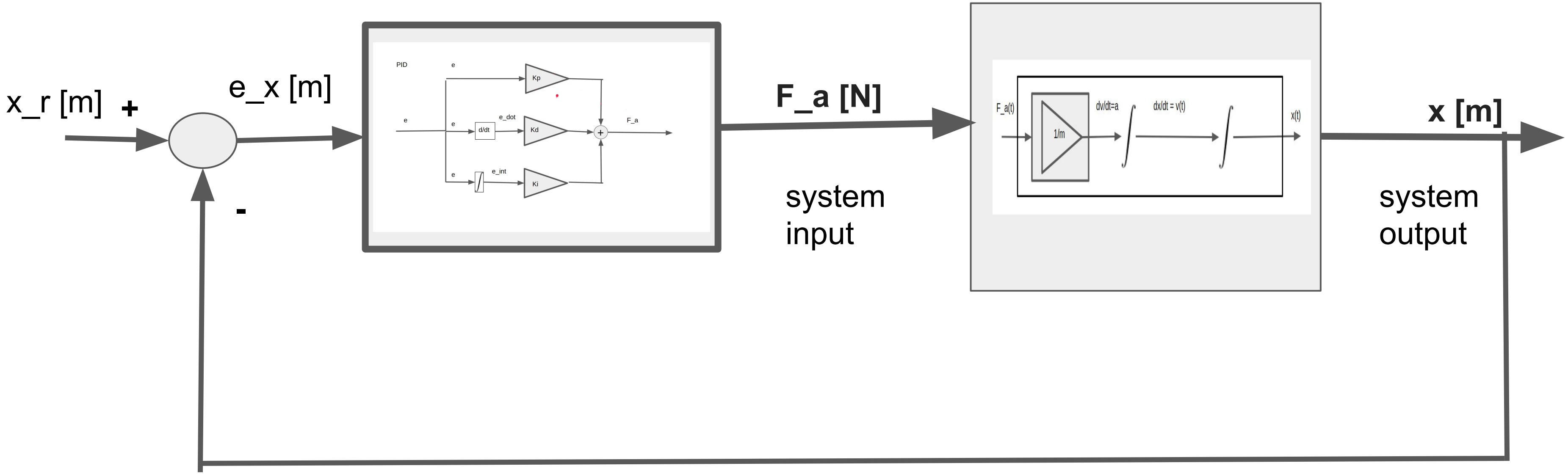
PID - controller

$F\_a = K\_i \int e^*(dt)$

e=0 m
e_dot=0 m/s
e_integral **NOT** 0 m*s
F_a **NOT** 0 N




PID
e
Kp
e_dot
F_a
e
e
d/dt
Kd
+
e_int
e
Ki


x_r [m]
+
-
e_x [m]
PID
e
e
e
e
d/dt
e_dot
e_int
Kp
Kd
Ki
+
F_a
F_a [N]
system
input
F_a(t)
1/m
dv/dt=a
dx/dt = v(t)
x(t)
x [m]
system
output