高级算法 (Fall 2019)/Problem Set 3

- 作业电子版于2019/11/26 23:59 之前提交到邮箱 njuadvalg@163.com
- 每道题目的解答都要有完整的解题过程。中英文不限。

Contents

Problem 1

Let $G(V,E)$ be an undirected graph with positive edge weights $w:E\to\mathbb{Z}^+$. Given a partition of V into k disjoint subsets $S_1,S_2,\ldots,S_k$, we define

$$w(S_1,S_2,\ldots,S_k)=\sum_{\substack{uv\in E\\ \exists i\neq j: u\in S_i, v\in S_j}} w(uv)$$

as the cost of the **k-cut** $\{S_1,S_2,\ldots,S_k\}$. Our goal is to find a k-cut with maximum cost.

1. Give a poly-time greedy algorithm for finding the weighted max k-cut. Prove that the approximation ratio is $(1-1/k)$.
2. Consider the following local search algorithm for the weighted max cut (max 2-cut).

   Fill in the blank parenthesis. Give an analysis of the running time of the algorithm. And prove that the approximation ratio is 0.5.

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start with an arbitrary bipartition of V into disjoint S_0, S_1;
while (true) do
   if ∃i ∈ {0,1} and v ∈ S_i such that (______________)
      then v leaves S_i and joins S_{1-i};
      continue;
   end if
   break;
end
```

Problem 2

Given m subsets $S_1,S_2,\ldots,S_m\subseteq U$ of a universe U of size n, we want to find a $C\subseteq\{1,2,\ldots,m\}$ of fixed size $k=|C|$ with the maximum **coverage** $\left|\bigcup_{i\in C}S_i\right|$.

- Give a poly-time greedy algorithm for the problem. Prove that the approximation ratio is $1-(1-1/k)^k>1-1/e$.

Problem 3

In the *maximum directed cut* (MAX-DICUT) problem, we are given as input a directed graph $G(V,E)$. The goal is to partition V into disjoint S and T so that the number of edges in $E(S,T)=\{(u,v)\in E\mid u\in S, v\in T\}$ is maximized. The following is the integer program for MAX-DICUT:

$$\begin{aligned}\text{maximize} \quad & \sum_{(u,v)\in E} y_{u,v} \\ \text{subject to} \quad & y_{u,v}\le x_u, & \forall (u,v)\in E, \\ & y_{u,v}\le 1-x_v, & \forall (u,v)\in E, \\ & x_v\in\{0,1\}, & \forall v\in V, \\ & y_{u,v}\in\{0,1\}, & \forall (u,v)\in E.\end{aligned}$$

Let $x_v^*, y_{u,v}^*$ denote the optimal solution to the **LP-relaxation** of the above integer program.

- Apply the randomized rounding such that for every $v\in V$, $\hat{x}_v=1$ independently with probability x_v^*. Analyze the approximation ratio (between the expected size of the random cut and OPT).
- Apply another randomized rounding such that for every $v\in V$, $\hat{x}_v=1$ independently with probability $1/4+x_v^*/2$. Analyze the approximation ratio for this algorithm.

Problem 4

Recall the MAX-SAT problem and its integer program:

$$\begin{aligned}\text{maximize} \quad & \sum_{j=1}^{m} y_j \\ \text{subject to} \quad & \sum_{i\in S_j^+} x_i + \sum_{i\in S_j^-}(1-x_i)\ge y_j, & 1\le j\le m, \\ & x_i\in\{0,1\}, & 1\le i\le n, \\ & y_j\in\{0,1\}, & 1\le j\le m.\end{aligned}$$

Recall that $S_j^+, S_j^-\subseteq\{1,2,\ldots,n\}$ are the respective sets of variables appearing positively and negatively in clause j.

Let x_i^*, y_j^* denote the optimal solution to the **LP-relaxation** of the above integer program. In our class we learnt that if $\hat{x}_i$ is round to 1 independently with probability x_i^*, we have approximation ratio $1-1/\mathrm{e}$.

We consider a generalized rounding scheme such that every $\hat{x}_i$ is round to 1 independently with probability $f(x_i^*)$ for some function $f:[0,1]\to[0,1]$ to be specified.

- Suppose $f(x)$ is an arbitrary function satisfying that $1-4^{-x}\le f(x)\le 4^{x-1}$ for any $x\in[0,1]$. Show that with this rounding scheme, the approximation ratio (between the expected number of satisfied clauses and OPT) is at least $3/4$.
- Derandomize this algorithm through conditional expectation and give a deterministic polynomial time algorithm with approximation ratio $3/4$.
- Is it possible that for some more clever f we can do better than this? Try to justify your argument.

Problem 5

The following is the weighted version of set cover problem:

Given m subsets $S_1, S_2, \ldots, S_m \subseteq U$, where U is a universe of size $n = |U|$, and each subset S_i is assigned a positive weight $w_i > 0$, the goal is to find a $C \subseteq \{1, 2, \ldots, m\}$ such that $U = \bigcup_{i \in C} S_i$ and the total weight $\sum_{I \in C} w_i$ is minimized.

- Give an integer program for the problem and its LP relaxation.
- Consider the following idea of randomized rounding: independently round each fractional value to $\{0, 1\}$ with the probability of the fractional value itself; and repeatedly apply this process to the variables rounded to 0 in previous iterations until U is fully covered. Show that this can return a set cover with $O(\log n)$ approximation ratio with probability at least 0.99.