

1 Problem 1

1.1 Modified Algorithm

Given weighted graph $G(V, E)$, $|V| = n$. Assuming the min-cut is C , and the weights of all edges in C is W . We define the weight of the edge e_k to $w_k, k = 1, 2, \dots, |E|$. The original contraction algorithm can be modified as below:

Algorithm 1 Modified Karger's Contraction Algorithm

Input: $G(V, E)$

Output: Min-cut C

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1: function Modified-Contraction( $G$ )
2:   while  $|V| > 2$  do
3:     Select edge  $e_k \in E$  with probability  $\frac{w_k}{\sum_{j=1}^{|E|} w_j}, k = 1, 2, \dots, |E|$ ;
4:     Merge the endpoints of  $e_k$ , producing graph  $G'$ ;
5:     Modified-Contraction( $G'$ );
6:   return sets of edges remaining in the graph as  $C$ 
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1.2 Proof for the success probability of algorithm

In the i -th ground contraction, the probability to choose an edge which is in the weighted min-cut C is **at most**:

$$\frac{W}{\frac{W(n-i)}{2}} = \frac{2}{n-i}$$

The algorithm perform $n-2$ contractions on the graph, we get this lower bound:

$$\begin{aligned} Pr[\text{return weighted min-cut } C] &\geq \prod_{i=0}^{n-3} \left(1 - \frac{2}{n-i}\right) \\ &= \prod_{i=0}^{n-3} \frac{n-i-2}{n-i} \\ &= \frac{2}{n(n-1)} \end{aligned} \tag{1}$$

2 Problem 2

2.1 Calculate the expected value

For any $k \in \{0, 1, 2, \dots, m\}$, the probability of $|E(S, T)| = k$ should be:

$$Pr[|E(S, T)| = k] = \frac{\binom{m}{k}}{2^m} \quad (2)$$

The expected value can be calculated as below:

$$\begin{aligned} \mathbb{E}[|E(S, T)|] &= \sum_{k=0}^m Pr[|E(S, T)| = k] \cdot k \\ &= \sum_{k=0}^m \frac{\binom{m}{k}}{2^m} \cdot k \\ &= \frac{1}{2^m} \sum_{k=0}^m \binom{m}{k} \cdot k \\ &= \frac{1}{2^m} 2^{m-1} m \\ &= \frac{m}{2} \end{aligned} \quad (3)$$

2.2 Algorithm to generate subset

Algorithm 2 Subroutine to generate Subset

Input: $\mathcal{R}(\cdot)$, \mathbf{V}

Output: \mathbf{S}

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1: function GENERATE( $\mathcal{R}(\cdot)$ ,  $\mathbf{V}$ )
2:   Initialize  $\mathbf{S} = \emptyset$ 
3:   while  $|\mathbf{S}| < \frac{n}{2}$  do
4:     for  $v \in \mathbf{V}$  do
5:        $p = \mathcal{R}(v)$ 
6:       if  $p \geq \frac{1}{2}$  then
7:          $\mathbf{S} = \mathbf{S} \cup \{v\};$ 
8:   return  $\mathbf{S}$ 

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To analysis the correctness of the algorithm, we need to prove that:
 $\forall S \in \mathcal{F}$, the uniformal probability is given below:

$$Pr[\text{Choose } \mathbf{S}] = \frac{1}{\binom{n}{\frac{n}{2}}}, \quad \forall S \in \mathcal{F}.$$

Given random variables generated by the algorithm, $\Pr[X_i = 1] = \frac{1}{2}, i = 1, 2, \dots, n$, For any \mathbf{S} is chosen only and if only when there exists $\frac{n}{2}$ random variables is equal to 1. Define $Y_i = \sum_{k=0}^n X_k, i = 1, 2, \dots, \left(\frac{n}{2}\right)$.

$$\begin{aligned}\mathbb{E}[Y_i = \frac{n}{2}] &= \mathbb{E}[\sum_{k=0}^n X_k = \frac{n}{2}] \\ &= \binom{n}{\frac{n}{2}} \mathbb{E}[X_i] \\ &= \frac{1}{2} \binom{n}{\frac{n}{2}}\end{aligned}\tag{4}$$

This is exactly the same with uniform distribution.

In the procedure of the algorithm, if we have generate a qualified subset S , the algorithm finish immediately, expected number of times to call the $\mathcal{R}(\cdot)$ subroutine can be derived as below:

$$\begin{aligned}\mathbb{E}[N] &= \sum_{k=\frac{n}{2}}^n \Pr[N_k = k] \cdot k \\ &= \sum_{k=\frac{n}{2}}^n \binom{k-1}{\frac{n}{2}-1} \left(\frac{1}{2}\right)^{k-1} \cdot \frac{1}{2} \cdot k \\ &= \sum_{k=\frac{n}{2}}^n \binom{k-1}{\frac{n}{2}-1} \left(\frac{1}{2}\right)^k \cdot k\end{aligned}\tag{5}$$

3 Problem 3

As vertices in the rooted tree has recursive structure. We consider to assign a polynomial to each vertex in the tree. For each vertex $v \in T$ with height h , the polynomial is defined as :

$$P_v = \begin{cases} x_0, & h = 0, \\ \prod_{i=1}^k (x_h - P_{u_i}), & h \geq 1 \end{cases}$$

Given rooted tree T with height h and n vertices, according to the polynomial defined above, there are $h+1$ variables in $P_{root(T)} \rightarrow \{x_0, x_1, \dots, x_h\}$, it's necessary for us to figure out the degree of $P_{root(T)}$. Expand $P_{root(T)}$, we get:

$$\begin{aligned}P_{root(T)} &= (x_h - P_{v_1})(x_h - P_{v_2}) \cdots (x_h - P_{v_k}) \\ &= x_h^k + \dots + \prod_{i=1}^k P_{v_i}\end{aligned}\tag{6}$$

$P_{root(T)}$ is then divided into 3 parts. We analysis the degree one by one.

$$\mathbf{Deg}(x_h^k) = k$$

$$\mathbf{Deg}(\dots) = \max_{1 \leq i \leq k-1} \sum_{k=1}^i \mathbf{Deg}(P_{v_k}) + k - i$$

$$\mathbf{Deg}\left(\prod_{i=1}^k P_{v_i}\right) = \sum_{i=1}^k \mathbf{Deg}(P_{v_i}) = n$$

As $\forall i \in \{1, 2, \dots, k\}$, $\mathbf{Deg}(P_{v_i}) \geq 1$. It's easy to prove that:

$$n = \mathbf{Deg}\left(\prod_{i=1}^k P_{v_i}\right) \geq \mathbf{Deg}(\dots) \geq \mathbf{Deg}(x_h^k) = k$$

where n is the number of vertices in rooted tree T .

Now it's clear that $P_{root(T)}$ is a $h+1$ multivariate polynomial with degree n . We are going to construct polynomial for rooted trees T_1 and T_2 , and prove that T_1 and T_2 are isomorphic is equivalent to $P_{root(T_1)} \equiv P_{root(T_2)}$.

Applying induction on the height of rooted trees.

- 1 If $h = 0$, it's trivial that $P_{root(T_1)} = x_0$ and $P_{root(T_2)} = x_0$, it's obvious T_1 and T_2 are isomorphic.
- 2 Assume that $\forall h \leq k$, we have $P_{root(T_1)} \equiv P_{root(T_2)}$ equivalent to the isomorphism between T_1 and T_2 .
- 3 For $h = k + 1$, expand two polynomials, we have all terms are children vertices of $root(T_1)$ and $root(T_2)$, they are with heights $h = k$, can be verified by our hypothesis. Thus, it still works.

Combining all of above, **we can test the isomorphism of rooted trees by test their polynomial identity.**

We construct a polynomial as the subtraction of two polynomials:

$$Q(x_0, x_1, \dots, x_h) = P_{root(T_1)} - P_{root(T_2)} \quad (7)$$

We can test its identity on field \mathbb{Z}_p , where prime $p \geq 2n$.

According to **Schwartz-Zippel Theorem**, the error probability:

$$\Pr[Q(r_0, r_1, \dots, r_h) = 0] \leq \frac{n}{p} \leq \frac{n}{2n} = \frac{1}{2}$$

We can independently perform such identity test for $\log n$ times, thus we can reduce the error probability to:

$$\left(\frac{1}{2}\right)^{\log n} = O\left(\frac{1}{n}\right)$$

4 Problem 4

Consider we have a bijection $\sigma \in \mathbf{S}_n$, where \mathbf{S}_n is the universe of permutations of set $\{1, 2, \dots, n\}$, σ maps a_i to b_{σ_i} , $1 \leq i \leq n$. That's to say, if a_1, \dots, a_n is a permutation of b_1, \dots, b_n , a σ should exist. Therefore, we can define **Edmonds Matrix A**:

$$\mathbf{A}_{ij} = \begin{cases} x_{ij}, & \text{if } a_i = b_{\sigma_i} (\sigma_i = j), \\ 0, & \text{otherwise.} \end{cases}$$

Matrix **A** has determinant:

$$\det(A) = \sum_{\sigma \in \mathbf{S}_n} \text{sgn}(\sigma) \prod_{i=1}^n A_{i, \sigma_i}$$

The determinant is a summation over $n!$ terms. It is helpful to analyze the result of $\det(A)$, it is non-zero if and only if when a_1, a_2, \dots, a_n is a permutation of b_1, b_2, \dots, b_n .

Thus, We can construct a polynomial $\mathbf{Q}(x_{11}, x_{12}, \dots, x_{nn}) = \det(\mathbf{A})$. The problem of determination of permutation now transform to determine if $\mathbf{Q}(x_{11}, x_{12}, \dots, x_{nn})$ is identically zero or not.

To solve the problem, we first determine the degree of n^2 multivariate polynomial $\mathbf{Q}(x_{11}, x_{12}, \dots, x_{nn}) \leq n$. We can test its identity on the field \mathbb{Z}_p , where prime $p \geq 2n$.

According to **Schwartz-Zippel Theorem**, the error probability:

$$\Pr[\mathbf{Q}(r_{11}, r_{12}, \dots, r_{nn}) = 0] \leq \frac{n}{p} \leq \frac{n}{2n} = \frac{1}{2}$$

The computation time complexity is dominated by the calculation of determinant, it's $O(n^3)$.

We can independently perform such identity test for $\log n$ times, thus we can reduce the error probability to:

$$\left(\frac{1}{2}\right)^{\log n} = O\left(\frac{1}{n}\right)$$

Combining all of above together, we can solve this problem in $O(n^3 \log n)$ with error probability $O\left(\frac{1}{n}\right)$.

5 Problem 5

The original problem can be transformed to the following form:

$$\begin{aligned}
& \Pr \left[e^{-\epsilon} Z \leq \hat{Z} \leq e^{\epsilon} Z \right] \geq 1 - \delta \\
& \iff \Pr \left[\hat{Z} \leq e^{-\epsilon} Z \text{ or } \hat{Z} \geq e^{\epsilon} Z \right] \leq \delta \\
& \iff \Pr \left[|\ln \hat{Z} - \ln Z| \geq \epsilon \right] \leq \delta
\end{aligned} \tag{8}$$

It's obvious that:

$$\begin{aligned}
\mathbb{E} \left[\ln \hat{Z} \right] &= \mathbb{E} \left[\sum_{i=1}^n \hat{\rho}_i \right] \\
&= \sum_{i=1}^n \mathbb{E} [\hat{\rho}_i] \\
&= \sum_{i=1}^n \mathbb{E} \left[\frac{1}{s} \sum_{j=1}^s X_i^{(j)} \right] \\
&= \sum_{i=1}^n \rho_i \\
&= \ln Z
\end{aligned} \tag{9}$$

Applying Chebyshev's Inequality, we obtain:

$$\Pr \left[|\ln \hat{Z} - \mathbb{E} [\ln \hat{Z}]| \geq \epsilon \right] \leq \frac{\mathbf{Var} [\ln \hat{Z}]}{\epsilon^2}$$

Therefore, we need to calculate the variance of $\ln \hat{Z}$.

$$\begin{aligned}
\mathbf{Var} [\ln \hat{Z}] &= \mathbb{E} \left[\left(\ln \hat{Z} \right)^2 \right] - \mathbb{E}^2 [\ln \hat{Z}] \\
&= \mathbb{E} \left[\left(\sum_{i=1}^n \hat{\rho}_i \right)^2 \right] - \left(\sum_{i=1}^n \rho_i \right)^2
\end{aligned} \tag{10}$$

The first term can be expanded as below:

$$\begin{aligned}
\mathbb{E} \left[\left(\sum_{i=1}^n \hat{\rho}_i \right)^2 \right] &= \mathbb{E} \left[\sum_{i=1}^n \hat{\rho}_i^2 + 2 \sum_{1 \leq i < j \leq n} \hat{\rho}_i \cdot \hat{\rho}_j \right] \\
&= \sum_{i=1}^n \mathbb{E} [\hat{\rho}_i^2] + 2 \sum_{1 \leq i < j \leq n} \mathbb{E} [\hat{\rho}_i \cdot \hat{\rho}_j] \\
&= \sum_{i=1}^n \left(\frac{1}{s} \mathbf{Var} [X_i] + \mathbb{E}^2 [\rho_i] \right) + 2 \sum_{1 \leq i < j \leq n} \mathbb{E} [\hat{\rho}_i] \cdot \mathbb{E} [\hat{\rho}_j] \\
&= \sum_{i=1}^n \left(\frac{1}{s} \rho_i \cdot (1 - \rho_i) + \rho_i^2 \right) + 2 \sum_{1 \leq i < j \leq n} \rho_i \cdot \rho_j
\end{aligned} \tag{11}$$

Combining all of above:

$$\begin{aligned}
\Pr \left[|\ln \hat{Z} - \mathbb{E} [\ln \hat{Z}]| \geq \epsilon \right] &\leq \frac{\mathbf{Var} [\ln \hat{Z}]}{\epsilon^2} \\
&= \frac{\sum_{i=1}^n \left(\frac{1}{s} \rho_i \cdot (1 - \rho_i) + \rho_i^2 \right) + 2 \sum_{1 \leq i < j \leq n} \rho_i \cdot \rho_j - \left(\sum_{i=1}^n \rho_i \right)^2}{\epsilon^2} \\
&= \frac{\frac{1}{s} \sum_{i=1}^n \rho_i \cdot (1 - \rho_i)}{\epsilon^2} \\
&= \delta
\end{aligned} \tag{12}$$

Solve the equation above, we get s :

$$s = \frac{\sum_{i=1}^n \rho_i \cdot (1 - \rho_i)}{\delta \epsilon^2} \geq \frac{\sum_{i=1}^n \frac{1}{4}}{\delta \epsilon^2} = \frac{n}{4 \delta \epsilon^2}$$

6 Problem 6