Problem Set 4 Solutions

学号: MP1933008

姓名: 黄睿

1. Problem1

From the definition of Algorithm $\mathscr{A}$, we have that:

$$\mathbb{E}\left[\hat{Z}\right] = \frac{3}{4}Z$$

Denote $Y_i = \begin{cases} 1, \text{ if } \hat{Z}_i \ success; \\ 0, \ otherwise\,. \end{cases}$, As $\mathbf{X}$ is the median of $\mathbf{s}$ independent trials, $\mathbf{X}\ success \Leftrightarrow \sum_{i=1}^{s} Y_i \geq \frac{1}{2}s\,.$

As each trail is performed independently, $\mathbb{E}\left[Y_i\right] = 3/4, i = 1,2,\cdots, s\,.$

Applying linearity of expectations: $\mathbb{E}\left[\sum_{i=1}^{s} Y_i\right] = \sum_{i=1}^{s} \mathbb{E}\left[Y_i\right] = \frac{3}{4}s.$

Consider $\Pr\left[X\ \ fail\right] < \delta \Leftrightarrow \Pr\left[X\ \ success\right] \geq 1 - \delta$:

$$
\begin{aligned}
\Pr\left[\mathbf{X}\ fail\right] &= \Pr\left[\sum_{i=1}^{s} Y_i < \frac{1}{2}s\right] \\
&= \Pr\left[\sum_{i=1}^{s} Y_i - \frac{3}{4}s < -\frac{1}{4}s\right] \\
&= \Pr\left[\sum_{i=1}^{s} Y_i - \mathbb{E}\left[\sum_{i=1}^{s} Y_i\right] < -\frac{1}{4}s\right] \\
&= \Pr\left[\sum_{i=1}^{s} Y_i - \mathbb{E}\left[\sum_{i=1}^{s} Y_i\right] < -\frac{1}{3}\mathbb{E}\left[\sum_{i=1}^{s} Y_i\right]\right] \\
&< \exp\left(-\frac{E\left[\sum_{i=1}^{s} Y_i\right]\cdot\left(\frac{1}{3}\right)^2}{2}\right) \quad \text{(Chenoff Bound)} \\
&= \exp\left(-\frac{1}{24}s\right) \\
&< \delta
\end{aligned}
$$

Above inequalities hold for $s = \lceil -24\ln\delta \rceil$.

2. Problem2

(1) Q1

Define $|V| = n$ mutually independent random variables :

$X_1, X_2, \cdots, X_n \in \{0,1\}$. Where $X_i = \begin{cases} 1, & \text{if vertex } i \in T; \\ 0, & otherwise. \end{cases}$ .Accordingly, we can define bad events $A_i, (i = 1,2,\cdots,|E| = m)$ as: for edge $e_i \in E$, both vertices in e_i are selected thus e_i will be in the transversal.Then we have:

$$\begin{aligned}\Pr[A_i] &= \Pr[(u,v) \in e_i \text{ is selected}] \\ &= \Pr[u \text{ is selected} \wedge v \text{ is selected}] \\ &= \Pr\left[X_u = 1 \wedge X_v = 1\right] \\ &\le \left(\frac{1}{2e\Delta}\right)^2\end{aligned}$$

Since each vertex v has degree at most Δ, the dependence degree $d \le 2\Delta$. We have:

$$\forall i, \Pr[A_i] \cdot e(d+1) \le \left(\frac{1}{2e\Delta}\right)^2 \cdot e(2\Delta+1) \le 1.$$

The above inequality holds for all $\Delta > 0$.According to **Lovász Local Lemma**(Lovász 1977), it holds that:

$$\Pr\left[\wedge_{i=1}^{m} \overline{A_i}\right] > 0.$$

Thus there must be an independent transversal of $\{S_1, S_2, \cdots, S_r\}$.

(2) Q2

Algorithm can be designed according to **Moser-Tardos** as below:

```
Sample all X_1, X_2, ..., X_n;
While ∃ an occurring bad event A_i:
        resample all X_j ∈ vbl(A_i);
```

From the analysis in class, the algorithm will find an independent transversal in expected polynomial time.

3. Problem3

(1) Q1

For simplicity, we give the LP-relaxed dual program as below,:

$$\begin{aligned} maximize \sum_{C \in \mathscr{C}} y_C & \\ \text{subject to} \sum_{C: v \in C} y_C \le 1, \forall v \in V, & \\ y_C \ge 0, \forall C \in \mathscr{C}. &\end{aligned}$$

(2) Q2

According to the primal-dual schema, we can design the algorithm as below:

> Initially $\mathbf{x} = 0,\ \mathbf{y} = 0$;
> while $\mathscr{C} \neq \varnothing$:
>   pick a cycle $C \in \mathscr{C}$ and raise y_C to 1;
>   set $x_v = 1$for those $v \in C$ and delete all $C : v \in C$ from $\mathscr{C}$;

Each deleted cycle C is incident to a v that $x_v = 1$and all cycles are eventually deleted, thus we get a feasible solution $\mathbf{x}$ satisfying:

$$\forall C \in \mathscr{C},\ \sum_{v \in C} x_v \geq 1$$

For simplicity, we denote the number of vertices in $G(V, E)$ as n, then we obtain:

$\forall C$:either $\sum_{v \in C} x_v \leq n$ or $y_e = 0$

$\forall v$:either $\sum_{C : v \in C} = 1$ or $x_v = 0$

thus, it holds:

$$SOL = \sum_{v \in V} x_v \leq n \cdot OPT$$

The primal-dual schema gets an approximation ratio of n.

4. Problem4

(1) Q1

From the definition of Ising model, we can directly calculate the marginal distribution:

$$
\begin{aligned}
\mu_v\left(+1 \mid \sigma\left(V \backslash \{v\}\right)\right) &= \frac{\mu_v\left(v=+1, \sigma\left(V \backslash \{v\}\right)\right)}{\mu_v\left(\sigma\left(V \backslash \{v\}\right)\right)} \\
&= \frac{\frac{1}{Z}\exp\left(-\sum_{\{u,v\}\in E}\beta\sigma(u)+C\right)}{\frac{1}{Z}\exp\left(-\sum_{\{u,v\}\in E}\beta\sigma(u)\sigma(v)+C\right)},\ C=\sum_{\substack{\{w,u\}\in E\\ w\neq v\\ u\neq v}}\beta\sigma(w)\sigma(u) \\
&= \frac{\exp\left(-\sum_{\{u,v\}\in E}\beta\sigma(u)\right)}{\exp\left(-\sum_{\{u,v\}\in E}\beta\sigma(u)\sigma(v)\right)} \\
&= \frac{\exp\left(-\sum_{\{u,v\}\in E}\beta\sigma(u)\right)}{\exp\left(-\sum_{\{u,v\}\in E}\beta\sigma(u)\right)+\exp\left(\sum_{\{u,v\}\in E}\beta\sigma(u)\right)} \\
&= \frac{1}{1+\exp^2\left(\sum_{\{u,v\}\in E}\beta\sigma(u)\right)}
\end{aligned}
$$

$$
\begin{aligned}
\mu_v\left(-1 \mid \sigma\left(V \backslash \{v\}\right)\right) &= 1-\mu_v\left(+1 \mid \sigma\left(V \backslash \{v\}\right)\right) \\
&= \frac{\exp^2\left(\sum_{\{u,v\}\in E}\beta\sigma(u)\right)}{1+\exp^2\left(\sum_{\{u,v\}\in E}\beta\sigma(u)\right)}
\end{aligned}
$$

(2) Q2

To prove the irreducibility, $\forall \sigma, \tau \in \{-1,+1\}^V$, we need to prove the transition probability $\Pr^t[\tau \mid \sigma] > 0$.Denote $\sigma = (Y_1, Y_2, \cdots, Y_V), \delta = (X_1, X_2, \cdots, X_V)$, where $X_i, Y_i \in \{0,1\}$.

$$
\begin{aligned}
\Pr{}^t[\tau \mid \sigma] &= \Pr\left[X_1 \to Y_1, \cdots, X_V \to Y_V\right] \\
&= \Pr\left[Y_1 \mid X_2, X_3, \cdots, X_V\right] \\
&\cdot \Pr\left[Y_2 \mid Y_1, Y_3, \cdots, X_V\right] \\
&\cdot \Pr\left[Y_3 \mid Y_1, Y_2, \cdots, X_V\right] \\
&\quad \vdots \\
&\cdot \Pr\left[Y_V \mid Y_1, Y_2, \cdots, Y_{V-1}\right]
\end{aligned}
$$

From Q1, we know that all terms of the above equation are positive.Thus, $\Pr[\tau \mid \delta] > 0$, for another word, the Glauber dynamics for Ising model is irreducible.

The aperiodicity holds when $\forall \sigma \in \{-1, +1\}^V, \Pr[\sigma \mid \sigma] > 0$.It's just a special case of irreducibility.

To prove the reversibility of Glauber dynamics, we should prove the detailed balance equation: $\mu(\sigma)\Pr[\tau \mid \sigma] = \mu(\tau)\Pr[\sigma \mid \tau]$.

According to the definition of Glauber dynamics, one step transition can change at most one vertex v.Without loss of generality, let $\sigma(v) = -1$,$\tau(v) = +1$,other cases are similar.

$$
\begin{aligned}
\mu(\sigma)\Pr[\tau \mid \sigma] &= \mu(\sigma)\mu_v(+1 \mid \delta(V \setminus \{v\}) \\
&= \frac{1}{Z}\exp\left(\sum_{\{u,v\} \in E} \beta\sigma(u) + C\right) \cdot \frac{1}{1 + \exp^2\left(\sum_{\{u,v\} \in E} \beta\sigma(u)\right)} \\
&= \frac{\exp(C)}{Z} \cdot \frac{\exp\left(\sum_{\{u,v\} \in E} \beta\sigma(u)\right)}{1 + \exp^2\left(\sum_{\{u,v\} \in E} \beta\sigma(u)\right)}
\end{aligned}
$$

$$
\begin{aligned}
\tau(\sigma)\Pr[\sigma \mid \tau] &= \mu(\tau)\mu_v(-1 \mid \delta(V \setminus \{v\}) \\
&= \frac{1}{Z}\exp\left(-\sum_{\{u,v\}\in E}\beta\sigma(u) + C\right)\cdot\frac{\exp^2\left(\sum_{\{u,v\}\in E}\beta\sigma(u)\right)}{1+\exp^2\left(\sum_{\{u,v\}\in E}\beta\sigma(u)\right)} \\
&= \frac{\exp(C)}{Z}\cdot\frac{\exp\left(\sum_{\{u,v\}\in E}\beta\sigma(u)\right)}{1+\exp^2\left(\sum_{\{u,v\}\in E}\beta\sigma(u)\right)}
\end{aligned}
$$

It’s obvious that $\mu(\sigma)\Pr[\tau \mid \sigma] = \mu(\tau)\Pr[\sigma \mid \tau]$.This proves the reversibility of Glauber dynamics for Ising model.