Notes for Min-Cut and Max-Cut

ritchie huang

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1 Understand Karger's vanilla Min-Cut Algorithm

Let **G** be the graph with **n** nodes, **C** be the min-cut.

Theorem 1

$$Pr\left[\boldsymbol{C} \textit{ is returned}\right] \geq \frac{2}{n(n-1)}$$

Proof 1 *In the* $(i+1)$ *th contraction,* $i = 0, 1, ..., n-3$*. There are only* $(n-i)$ *nodes in the graph* $\boldsymbol{G}$*, so the* ***edges space*** *to contract on is* ***at least*** $\frac{(n-i)|\boldsymbol{C}|}{2}$*. that is, if we denote the size of edges space as* m*, we have* $m \geq \frac{(n-i)|\boldsymbol{C}|}{2}$*.*

Therefore, we obtain:

$$\begin{aligned} Pr\left[\textit{sets of edges contracted in } (i+1) \textit{ th step} \cap \boldsymbol{C} = \emptyset\right] &= 1 - \frac{|\boldsymbol{C}|}{m} \\ &\geq 1 - \frac{|\boldsymbol{C}|}{\frac{(n-i)|\boldsymbol{C}|}{2}} \\ &= 1 - \frac{2}{n-i} \end{aligned} \tag{1}$$

Combining all the $(n-2)$ *steps together:*

$$\begin{aligned} Pr\left[\textit{return} \, \boldsymbol{C}\right] &= Pr\left[\textit{sets of edges contracted in (n - 2) steps} \cap \boldsymbol{C} = \emptyset\right] \\ &\geq \prod_{i=0}^{n-3}(1 - \frac{2}{n-i}) \\ &= \frac{2}{n(n-1)} \end{aligned} \tag{2}$$

If we run the algorithm independently for $\frac{n(n-1)}{2}\log n$ times.

$$\begin{aligned} Pr\left[\text{return } \mathbf{C} \textbf{ at least once}\right] &= 1 - Pr\left[\text{all runs fail to return } \mathbf{C}\right] \\ &\geq 1 - \left(1 - \frac{2}{n(n-1)}\right)^{\frac{n(n-1)}{2}\log n} \\ &= 1 - \left(\frac{1}{e}\right)^{logn} \\ &= 1 - \frac{1}{n} \end{aligned} \tag{3}$$

According to the proof, we are expected to get the correct min-cut with probability **more than** $1 - \frac{1}{n}$.

2 Understand Fast Min-Cut Algorithm

The key mind to get a faster algorithm is to Stop the vanilla algorithm at k-th step in a single run, where k is to be specified. How can we understand this **early stop** ?

If we expand the equation 2:

$$Pr\left[return\mathbf{C}\right] = (1 - \frac{2}{n})(1 - \frac{2}{n-1})(1 - \frac{2}{n-2}) \cdots (1 - \frac{2}{4})(1 - \frac{2}{3}) \quad (4)$$

observe the right-hand side, we can find that with the increase of terms, the probability changes accordingly.