CS2002D Program Design: Monsoon 2021

Asymptotic Notations

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Acknowledgements: Dr. Saleena N, CSED

Rate of Growth or Order of Growth

- Rate/Order of growth – Considering the leading term of a formula
- Ignoring the lower order terms - insignificant for large values of n
- Ignoring leading term’s constant coefficient – constant factors are less significant

Rate of Growth or Order of Growth

- $T(n) = an^2 + bn + c$
- we say $T(n)$ is $\Theta(n^2)$ (“theta of *n*-squared”)
  - simplifying abstraction
  - consider only the leading term of a formula
  - lower order terms- relatively insignificant for large values of n
  - ignore the leading term’s constant coefficients

Asymptotic Efficiency

- input size is large enough, only the order of growth is relevant

- asymptotic efficiency - how the running time increases with the size of the input *in the limit*

- asymptotically more efficient - best choice for all but very small inputs

Asymptotic Notations

- Domain of functions - Set of Natural Numbers
  $\mathrm{N} = 0, 1, 2....$ (as given by CLRS [1])
  - $T(n)$ usually defined only on integer input sizes

- $T(n) = an^2 + bn + c$
  - $T(n)$ is $\Theta(n^2)$ ("theta of n-squared")
  - $T(n)$ is $O(n^2)$ ("Big - Oh of n-squared")
  - $T(n)$ is $\Omega(n^2)$ ("Omega of n-squared")

O notation (big-Oh)

- $T(n) = n^2 + 2n + 1$ for $n > 1$, $\quad T(1) = 4$
- $T(n) \leq 4n^2$, for $n \geq 1$
- $T(n) \leq cn^2$, for $n \geq n_0$ (c=4 and n_0=1)
- we say $T(n)$ is $O(n^2)$

O notation (big-Oh)

- $T(n) = n^2 + 2n + 1$ for $n > 1$, $\quad T(1) = 4$
- How to get c and n_0 such that $T(n) \leq cn^2$, for $n \geq n_0$
  - c should be such that $n^2 + 2n + 1 \leq cn^2$
  - divide by n^2, $\quad 1 + \frac{2}{n} + \frac{1}{n^2} \leq c$
  - for $n \geq 1$, we can choose $c \geq 4$
- $T(n) \leq cn^2$, for $n \geq n_0$ (c=4 and n_0=1)
- we say $T(n)$ is O(n^2)
- $n^2 + 2n + 1$ is O(n^2)

O notation

- $T(n)$ is $O(n^2)$
  - There are positive constants c and n_0 such that $T(n) \le cn^2$ for $n \ge n_0$

Some functions:

$T_1(n) = 5n^2$ $T_2(n) = n^2+2n$ $T_3(n) = n+5$

- $T_1(n) \le 5n^2 \quad for \quad n \ge 1$
- $T_2(n) \le 2n^2 \quad for \quad n \ge 2$
- $T_3(n) \le n^2 \quad for \quad n \ge 3$

Generalizing…..

- *There exists positive constants* $c=5$ *and* $n_0=1$ *such that* $T_1(n) \le cn^2$ *for* $n \ge n_0$
- *There exists positive constants* $c=2$ *and* $n_0=2$ *such that* $T_2(n) \le cn^2$ *for* $n \ge n_0$
- *There exists positive constants* $c=1$ *and* $n_0=3$ *such that* $T_3(n) \le cn^2$ *for* $n \ge n_0$

There exists positive constants c and n_0 ***such that*** $f(n) \le cn^2$ ***for*** $n \ge n_0$

The set $O(n^2)$ *(read "big oh of n^2" or "oh of n^2")*

- *Set of all $f(n)$ such that there exists positive constants c and n_0 such that $f(n) \le cn^2$ for all $n \ge n_0$*
  - Set is denoted by $O(n^2)$
    - $T_1(n) \in O(n^2)$ $T_2(n) \in O(n^2)$ $T_3(n) \in O(n^2)$
- $\boldsymbol{O(n^2)}$ is a set of functions.

$$O(n^2) = \{ f(n): \text{there exists positive constants } c \text{ and } n_0 \text{ such that } 0 \le f(n) \le cn^2 \text{ for all } n \ge n_0 \}$$

The set $O(n^2)$ – contd.

- Give some more functions that belong to the set $O(n^2)$.

  - $f_1(n) = 100n^2 + n + 5$
  - $f_2(n) = 6n + 3$
  - $f_3(n) = 10000n^2$

The set $O(n^3)$

- $O(n^3) = \{ f(n)$: ***there exists positive constants c and*** n_0 ***such that*** $0 \le f(n) \le cn^3$ ***for all*** $n \ge n_0 \}$
- **Some of the elements of** $O(n^3)$
  - $f_4(n) = 100n^3 + 3n^2 + 2$
  - $f_5(n) = 6n + 3$
  - $f_6(n) = 10000n^2$

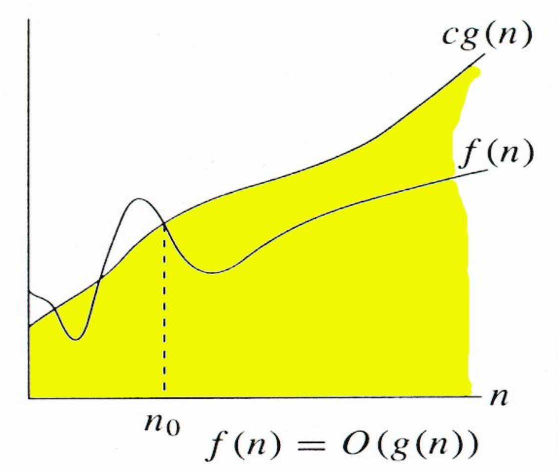
Generalizing….

- $O(n) = \{ f(n)$: *there exists positive constants c and* n_0 *such that* $0 \le f(n) \le c\, n$ *for all* $n \ge n_0 \}$
- $O(n \lg n) = \{ f(n)$: *there exists positive constants c and* n_0 *such that* $0 \le f(n) \le c\, n \lg n$ *for all* $n \ge n_0 \}$

$O(g(n)) = \{ f(n)$: *there exists positive constants c and* n_0 *such that* $0 \le f(n) \le c.\, g(n)$ *for all* $n \ge n_0 \}$

O-notation gives an **upper bound for a function** to within a constant factor.

$f(n)=O(g(n))$, if there are positive constants ***c and*** $\boldsymbol{n_0}$ *such that to the right of* $\boldsymbol{n_0}$*, the value of* ***f(n) always lies on or below cg(n)***.



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Going back ….

- **Insertion Sort**
  - *Worst Case Running time is* $O(n^2)$

- *Worst Case Running time,* $T_1(n) \leq cn^2$ ***for all values of*** $n \geq n_0$ *where c and* n_0 *are positive constants.*

Normally we write f (n) is O(g(n)) or f(n)=O(g(n)) to mean "f (n) is a member of O(g(n))"

Prove $T(n) = n^3+20n+1$ is $O(n^3)$

- by the Big-Oh definition, T(n) is O(n^3) if T(n) ≤ $c \cdot n^3$ for some $n \geq n_0$
- Find out c and n_0

Exercises

1. Is $2n + 10 \in O(n^2)$?

2. Is $n^3 \in O(n^2)$?

The set $\Omega(n)$ *(Read big-omega of n)*

- An example[1]

  $T(n) = 2n + 3$

  $2n \le T(n)$ for $n \ge 1$

  $cn \le T(n)$ for $n \ge 1$ and $c = 2$

  $T(n)$ belongs to $\Omega(n)$

$\Omega(n) = \{ f(n)$: there exists positive constants c and n_0 such that $0 \le c\,n \le f(n)$ for all $n \ge n_0 \}$

Exercises

1. Is $2n + 1 \in \Omega(n)$?
2. Is $2n^2 + 10 \in \Omega(n^2)$?
3. Is $n^3 \in \Omega(n^2)$?

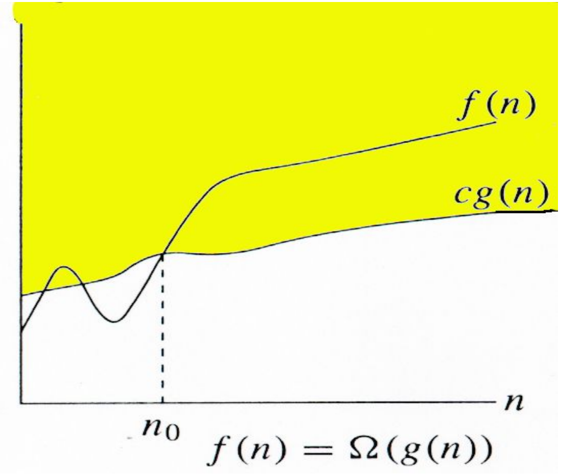
The set $\Omega(g(n))$

$\boldsymbol{\Omega(g(n))}$

$= \{ f(n)$: there exists positive constants c and n_0 such that $0 \leq c\,g(n) \leq f(n)$ for all $n \geq n_0 \}$

Ω-notation gives **a lower bound** for a function to within a constant factor

*$f(n) = \Omega(g(n))$, if there are positive constants **c and n_0** such that to the right of **n_0**, the value of **$f(n)$ always lies on or above $cg(n)$**.*



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The set $\Theta(n)$

- An example[1] - $T(n) = 2n + 3$

$$T(n) \leq 6n \quad \text{for } n \geq 1 \qquad T(n) \text{ is } O(n)$$

$$2n \leq T(n) \quad \text{for } n \geq 1 \qquad T(n) \text{ is } \Omega(n)$$

$T(n)$ belongs to $\Theta(n)$

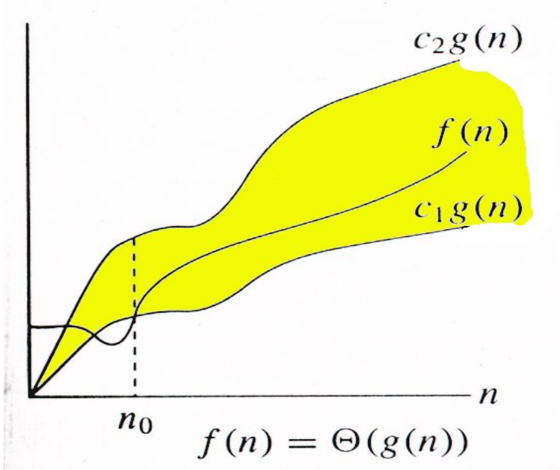
$\Theta(n) = \{ f(n)$: there exists positive constants c_1, c_2 and n_0 such that $0 \leq c_1 n \leq f(n) \leq c_2 n$ for all $n \geq n_0 \}$

The set $\Theta(g(n))$

$\Theta(g(n)) = \{ f(n)$: there exists positive constants c_1, c_2 and n_0 such that $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0 \}$

***Θ*-**notation gives **tight bound** for a function to within constant factors

f(n)=Θ(g(n)), if there exists positive constants c_1, c_2 ***and*** n_0 *such that to the right of* n_0, *the value of f(n) always lies between* $c_1 g(n)$ *and* $c_2 g(n)$ *inclusive.*



Source: http://www.cs.unc.edu/~plaisted/comp122/02-asymp.ppt

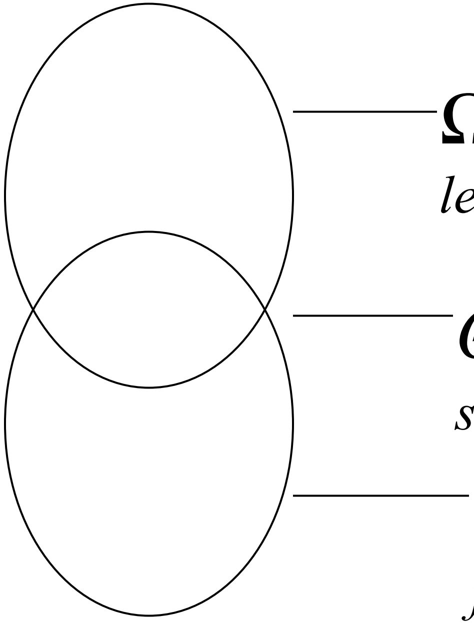
Asymptotic notations – Formal definitions [1]

- $O(g(n)) = \{ f(n)$: there exists positive constants c and n_0 such that $0 \le f(n) \le c\, g(n)$ for all $n \ge n_0 \}$

- $\Omega(g(n)) = \{ f(n)$: there exists positive constants c and n_0 such that $0 \le c\, g(n) \le f(n)$ all $n \ge n_0 \}$

- $\Theta(g(n)) = \{ f(n)$: there exists positive constants c_1, c_2 and n_0 such that $0 \le c_1\, g(n) \le f(n) \le c_2\, g(n)$ for all $n \ge n_0 \}$

To make it more clear [2]



$\Omega(g)$: *functions that grow at least as fast as g*

$\Theta(g)$: *functions that grow at the same rate as g*

$O(g)$: *functions that grow no faster than g*

- $f(n) = \Theta(g(n))$
  - $g(n)$ is an asymptotically tight bound for $f(n)$
- $f(n) = O(g(n))$
  - $g(n)$ is an asymptotic upper bound for $f(n)$
- $f(n) = \Omega(g(n))$
  - $g(n)$ is an asymptotic lower bound for $f(n)$

Theorem [1]

For any two functions $f(n)$ and $g(n)$, we have $f(n)=\Theta(g(n))$ if and only if $f(n)=O(g(n))$ and $f(n)=\Omega(g(n))$.

$$\boxed{\Theta(f(n)) = O(f(n)) \cap \Omega(f(n)).}$$

Examples[1]

- $f(n) = an^2+bn+c$, where a, b, and c are constants and $a>0$
- $f(n) = \Theta(n^2) \rightarrow f(n) = \Omega(n^2)$ *and* $f(n) = O(n^2)$

- **For any polynomial, $p(n)$ of degree k we have $p(n) = \Theta(n^k)$**
- **Any constant function is $\Theta(n^0)$, or $\Theta(1)$.**

Insertion Sort – Running Time

- Best Case running Time is $\Omega(n)$.
  - Implies Running time on any input is $\Omega(n)$.
- Running time is not $\Omega(n^2)$.
- Worst Case running time is $\Omega(n^2)$.
- Is it correct to say best case running Time is $\Theta(n)$?

Is $O(n \lg n)$ algorithm preferred over $O(n^2)$?

- Suppose $T_1(n) \leq 50\, n \lg n$ and $T_2(n) \leq 2n^2$
- Check the values of $T_1(n)$ and $T_2(n)$ when n= 2 and n=1024
- For small input sizes, the $O(n^2)$ algorithm may run faster.
- Once the input size becomes large enough, $O(n \lg n)$ runs faster
  - irrespective of the constant factors
  - irrespective of the implementation.

- Read the corresponding Sections in CLRS.

References

1. Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein *Introduction to Algorithms*, PHI, 2001.
2. Sara Baase and Allen Van Gelder *Computer Algorithms: Introduction to Design & Analysis*, Pearson Education, third edition, 2000.
3. Donald E Knuth. Big omicron and big omega and big theta. *ACM SIGACT News*, 1976.
4. Gilles Brassard, Paul Bratley, *Fundamentals of Algorithmics*, PHI, 1997.