Martingale Layer

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1 Objective

The objective of this document is to determine a function taking as input a vector of positive variables $m_1, m_2, \cdots, m_N$ and returning a vector of probabilities $p_1, p_2, \cdots p_N$ exhibiting a given first order moment μ.

Mathematically, it reads:

$$\forall i \in \{1, \cdots, N\}, \quad p_i \geq 0 \tag{1}$$

$$\sum_{i=1}^{N} p_i = 1 \tag{2}$$

$$\sum_{i=1}^{N} x_i p_i = \mu \tag{3}$$

where $x_1 < x_2 < \cdots < x_N$ is a predefined set of **sorted** real numbers.

To insure that the problem really makes sense, we are going to consider that μ satisfies $x_1 < \mu < x_N$.

2 Solution

Let us split the set of points into two subset depending on their relative positive with respect to μ:

$$\mathcal{I}_+ = \{i \in \{1, \cdots, N\}; \quad x_i > \mu\} \tag{4}$$

$$\mathcal{I}_- = \{i \in \{1, \cdots, N\}; \quad x_i \leq \mu\} \tag{5}$$

By construction, we have:

$$\mu_- \leq \mu < \mu_+ \tag{6}$$

Let us calculate the barycenter $\mu_\pm$ defined by:

$$\mu_\pm = \frac{\chi_\pm}{M_\pm} \tag{7}$$

with

$$\chi_\pm = \sum_{i \in \mathcal{I}_\pm} x_i m_i \tag{8}$$

and

$$M_{\pm} = \sum_{i \in \mathcal{I}_{\pm}} m_i \tag{9}$$

We want to determine $\theta \in [0, 1]$ such that:

$$\mu = (1 - \theta)\mu_- + \theta\mu_+ \tag{10}$$

That yields:

$$\theta = \frac{\mu - \mu_-}{\mu_+ - \mu_-} \tag{11}$$

Expanding Eq;10 using the expressions of $\mu_{\pm}$ written in Eq.7, we get:

$$\mu = \sum_{i \in \mathcal{I}_-} x_i \frac{(1 - \theta)m_i}{M_-} + \sum_{i \in \mathcal{I}_+} x_i \frac{\theta m_i}{M_+} \tag{12}$$

By definition, the first order moment μ relates to the probabilities $p_1, \cdots, p_N$ as follows:

$$\mu = \sum_{i=1}^{N} x_i p_i = \sum_{i \in \mathcal{I}_-} x_i p_i + \sum_{i \in \mathcal{I}_+} x_i p_i \tag{13}$$

Which shows that a legitimate choice for $p_1, \cdots, p_N$ is:

$$p_i = m_i \lambda_i \tag{14}$$

where λ_i is given by:

$$\lambda_i = \begin{cases} \frac{1-\theta}{M_-}; & \text{if } i \in \mathcal{I}_- \\ \frac{\theta}{M_+}; & \text{if } i \in \mathcal{I}_+ \end{cases} \tag{15}$$

3 Conclusions

- We never used the fact that the series $x_1, \cdots, x_N$ was sorted
- The function mapping any positive inputs $\boldsymbol{m} = (m_1, \cdots, m_N)$ to a vector of probabilities $\boldsymbol{p} = (p_1, \cdots, p_N)$ exhibiting some predefined first order moment μ can be written:

  $$p_i(\boldsymbol{m}) = m_i \lambda_i(\boldsymbol{m}) \tag{16}$$

  where

  $$\lambda_i(\boldsymbol{m}) = \begin{cases} \frac{1-\theta(\boldsymbol{m})}{M_-(\boldsymbol{m})}; & \text{if } i \in \mathcal{I}_- \\ \frac{\theta(\boldsymbol{m})}{M_+(\boldsymbol{m})}; & \text{if } i \in \mathcal{I}_+ \end{cases} \tag{17}$$

  with:

  $$M_{\pm}(\boldsymbol{m}) = \sum_{i \in \mathcal{I}_{\pm}} m_i \tag{18}$$

and

$$\theta(\boldsymbol{m}) = \frac{\mu - \mu_-(\boldsymbol{m})}{\mu_+(\boldsymbol{m}) - \mu_-(\boldsymbol{m})} \tag{19}$$

with

$$\mu_\pm(\boldsymbol{m}) = \frac{\chi_\pm(\boldsymbol{m})}{M_\pm(\boldsymbol{m})} \tag{20}$$

$$\chi_\pm(\boldsymbol{m}) = \sum_{i \in \mathcal{I}_\pm} x_i m_i \tag{21}$$