Fokker-Planck using wavelets

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1 Objective

The objective of that document is to make a vague idea less vague regarding the potential usage of wavelets to address the Fokker-Planck equation we want to solve:

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p\nu}{\partial x^2} + \frac{\partial p\nu}{\partial x} \tag{1}$$

$$p(x,0) = \delta(x) \tag{2}$$

where ν is a positive function of (x,t).

2 Functional frame decomposition

One can search for the solution p as a linear combination of some basis functions $g_{i\in\mathcal{I}}$:

$$p(x,t) = \sum_{i\in\mathcal{I}} a_i(t) g_i(x) \tag{3}$$

The coordinates a_i are giving by projections onto the dual frame $g^*_{i\in\mathcal{I}}$:

$$a_i(t) = \int_{\mathbb{R}} g_i^*(x) p(x,t) dx \tag{4}$$

If the basis functions g_i zero at the boundaries together with their derivatives of all order:

$$\begin{aligned}
\dot{a}_i(t) \triangleq \frac{da_i}{dt}(t) &= \int_{\mathbb{R}} g_i^*(x) \frac{\partial p(x,t)}{\partial t} dx && (5)\\
&= \int_{\mathbb{R}} \left(\frac{\partial^2 g_i^*(x)}{\partial x^2} - \frac{\partial g_i^*(x)}{\partial x} \right) p(x,t)\nu(x,t) dx && (6)\\
&= \sum_{j\in\mathcal{I}} G^\nu_{i,j}(t) \dot{a}_j(t) && (7)
\end{aligned}$$

where

$$G^\nu_{i,j}(t) = \int_{\mathbb{R}} \left(\frac{\partial^2 g_i^*(x)}{\partial x^2} - \frac{\partial g_i^*(x)}{\partial x} \right) g_j(x)\nu(x,t) dx \tag{8}$$

Hence, collecting all the coordinates in a vector, that reads:

$$\dot{\boldsymbol{a}}(t) = \boldsymbol{G}^{\boldsymbol{\nu}}(t)\boldsymbol{a}(t) \tag{9}$$

The solution is given by matrix exponentiation:

$$\boldsymbol{a}(t) = e^{\int_0^t \boldsymbol{G}^{\boldsymbol{\nu}}(s)ds}\boldsymbol{a}(0) \tag{10}$$

where

$$a_i(0) = \int_{\mathbb{R}} g_i^*(x)\delta(x)dx = g_i^*(0) \tag{11}$$

One can go even go further decomposing ν onto some well chosen functional basis $\psi_{k\in\mathcal{K}}$:

$$\nu(x,t) = \sum_{k\in\mathcal{K}} v_k(t)\psi_k(x) \tag{12}$$

Giving formally the solution in terms of the matrix exponential of a time varying combination of matrices $\mathbf{\Gamma}_{j\in\mathcal{J}}$ independent of ν:

$$\boldsymbol{a}(t) = e^{\sum_{k\in\mathcal{K}} V_k(t)\mathbf{\Gamma}_k}\boldsymbol{a}(0) \tag{13}$$

where

$$V_k(t) = \int_0^t v_k(s)ds \tag{14}$$

and

$$[\mathbf{\Gamma}_k]_{i,j} = \int_{\mathbb{R}} \left(\frac{\partial^2 g_i^*(x)}{\partial x^2} - \frac{\partial g_i^*(x)}{\partial x} \right) g_j(x)\psi_k(x)dx \tag{15}$$

$$= -\int_{\mathbb{R}} \frac{\partial g_i^*(x)}{\partial x} \left(g_j(x)\psi_k(x) + \frac{\partial g_j(x)\psi_k(x)}{\partial x} \right) dx \tag{16}$$

2.1 Conclusions

We see that:

- The solution to the Fokker-Planck problem somehow all amounts to computing the exponential of a matrix, up to several 1D numerical integrations for the $V_k(t)$.
- That matrix exponential can be made tractable choosing an appropriate frames g and ψ where $\mathbf{\Gamma}_k$ become essentially diagonal
- If g_i are Gaussian mixtures with fixed centroids distributed along a regular grid and with all width of the form $a = 2^p$ then deriving twice with respect to the log-strike x shows that the coordinates a_i are nothing but the coefficients obtained by the wavelet transform of $\frac{\partial^2 p}{\partial x^2}$ using as mother wavelet the Mexican hat.
- The drawback of the expansion onto a functional basis is that the truncated expansion might not lead to a positive worthy of the name density.