Replacement Theory

The Study ~~of~~ of replacement is concerned with situation that ~~are~~ arise when some items such as Machines, men, electric light bulbs etc., need replacement due to their deteriorating efficiency, failure or breakdown (may be either gradual or all of a sudden)

Following are the situations when replacement of certain item needs to be done

(1) An old item has failed and does not work at all, or the old item is expected to fail shortly

(2) The old item has ~~fai~~ deteriorated and work badly or requires expensive maintenance

(3) A better design ~~of~~ equipment has been developed

Replacement problem can be classified into two categories

(a) when the equipment/assets deteriorate with time and the value of money
  (1) Does not change with time
  (2) Changes with time

(b) When the items/units fail completely all of a sudden.

Replacement of Equipment / Asset that Deteriorates Gradually

Replace Policy when Value of money does not Change with time

C: Capital cost of equipment

S: Scrap value of equipment

n: number of years that equipment would be in use

f(t): maintenance cost fuctions

A(n): Average total Annual cost.

Total Cost = TC = Captital cost − Scrap Value + Maintenance cost

Optimal Replacement Policy –

Replace the equipment at the end of n years, if the maintenance cost in the $(n+1)^{th}$ year is more than the average cost in the n^{th} year and the n^{th} year's maintenance cost is less than the previous year's average total cost.

Q A Firm is considering replacement of a machine whose cost price is Rs 12,200 and the scrap value Rs 200. The running (maintenance and operating) cost in rupee are found from experience to be as follows

year :	1	2	3	4	5	6	7	8
Running cost :	200	500	800	1200	1800	2500	3200	4000

when should the machine be replaced

Year of service	Running cost	Cummulative Running cost	Depreciation cost Rs (C−S)	Total cost (Rs) TC		Average cost $A(n) = \frac{TL}{n}$
1	200	200	12,000	12,200		12,200
2	500	700	12,000	12700		6350
3	800	1500	12000	13500		4500
4	1200	2700	12000	14700		3675
5	1800	4500	12000	16500		3300
6	2500	7000	12000	19000	6th	3167
7	3200	10200	12000	22200	7th	3171
8	4000	14200	12000	26200	8th	3275

The average cost at the 7th year is more than the Average cost at the 6th years hence machine should be Replaced at the end of 6th year.

$$= C - S + \int_n^0 f(t)\, dt.$$

Average annual total cost, therefore is

$$A(n) = \frac{1}{n} TC = \frac{C-S}{n} + \frac{1}{n}\int_0^n f(t)\, dt.$$

For minimum cost, we must have $\frac{d}{dn}[A(n)] = 0$

or
$$\frac{-(C-S)}{n^2} - \frac{1}{n^2}\int_0^n f(t)\, dt + \frac{1}{n} f(n) = 0$$

or
$$f(n) = \frac{C-S}{n} + \frac{1}{n}\int_0^n f(t)\, dt \equiv A(n).$$

Clearly,
$$\frac{d^2}{dn^2}[A(n)] > 0 \quad \text{at} \quad f(n) = A(n).$$

This suggests that the *equipment should be replaced when maintenance cost equals the average annual total cost.*

Case 2. When t is a discrete variable. Here, the period of time is considered as fixed and n, t take the values 1, 2, 3, ... Then

$$A(n) = \frac{C-S}{n} + \frac{1}{n}\sum_{t=1}^{n} f(t)$$

Now, $A(n)$ will be a minimum for that value of n, for which

$$A(n+1) \geq A(n) \quad \text{and} \quad A(n-1) \geq A(n).$$

or
$$A(n+1) - A(n) \geq 0 \quad \text{and} \quad A(n) - A(n-1) \leq 0$$

For this, we write

$$A(n+1) = \frac{C-S}{n+1} + \frac{1}{n+1}\sum_{t=1}^{n+1} f(t)$$

$$= \frac{1}{n+1}\left[C - S + \sum_{t=1}^{n} f(t)\right] + \frac{1}{n+1} f(n+1)$$

$$= \frac{1}{n+1}[n A(n) + f(n+1)]$$

$$\therefore \quad A(n+1) - A(n) = \frac{1}{n+1}[f(n+1) - A(n)]$$

Thus
$$A(n+1) - A(n) \geq 0 \Rightarrow f(n+1) \geq A(n).$$

Similarly, it can be shown that

$$A(n) - A(n-1) \leq 0 \Rightarrow f(n) \leq A(n-1).$$

This suggests the optimal replacement policy :

Replace the equipment at the end of n years, if the maintenance cost in the $(n+1)$th year is more than the average total cost in the nth year and the nth year's maintenance cost is less than the previous year's average total cost.

SAMPLE PROBLEMS

1801. *A firm is considering replacement of a machine, whose cost price is Rs. 12,200 and the scrap value, Rs. 200. The running (maintenance and operating) costs in rupees are found from experience to be as follows :*

Year	:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Running cost	:	*200*	*500*	*800*	*1,200*	*1,800*	*2,500*	*3,200*	*4,000*

When should the machine be replaced? **[Meerut M.Sc. (Math.) 2001; Purvanchal M.C.A. 1996]**

Solution. We are given the running cost, $f(n)$, the scrap value $S = $ Rs. 200 and the cost of the machine, $C = $ Rs. 12,200. In order to determine the optimal time n when the machine should be replaced, we calculate an average total cost per year during the life of the machine as shown in table given below :

Year of service n (1)	*Running cost (Rs.)* $f(n)$ (2)	*Cumulative running cost (Rs.)* $\Sigma f(n)$ (3)	*Depreciation cost (Rs.)* $C-S$ (4)	*Total cost (Rs.)* TC (3) + (4) (5)	*Average cost (Rs.)* $A(n)$ (5)/(1) (6)
1	200	200	12,000	12,200	12,200
2	500	700	12,000	12,700	6,350
3	800	1,500	12,000	13,500	4,500
4	1,200	2,700	12,000	14,700	3,675
5	1,800	4,500	12,000	16,500	3,300
6	2,500	7,000	12,000	19,000	3,167
7	3,200	10,200	12,000	22,200	3,171
8	4,000	14,200	12,000	26,200	3,275

From the table it is noted that the average total cost per year, $A(n)$ is minimum in the 6th year (Rs. 3,167). Also the average cost in 7th year (Rs. 3,171) is more than the cost in the 6th year. Hence the machine should be replaced after every 6 years.

1802. (*a*) *Machine A costs Rs. 9,000. Annual operating costs are Rs. 200 for the first year, and then increase by Rs. 2,000 every year. Determine the best age at which to replace the machine. If the optimum replacement policy is followed, what will be the average yearly cost of owning and operating the machine?* ***[Madras B.E. (Comp. Sc.) 1989]***

(*b*) *Machine B costs Rs. 10,000. Annual operating costs are Rs. 400 for the first year, and then increase by Rs. 800 every year. You now have a machine of type A which is one-year old. Should you replace it with B, if so when?* ***[Meerut M.Sc. (Math.) 1989]***

Solution. (*a*) Let the machine have no resale value when replaced. Then, for machine *A*, the average total annual cost $ATC(n)$ is computed as follows :

Year (n)	$f(n)$	$\Sigma f(n)$	$C-S$	TC	$A(n)$
1	200	200	9,000	9,200	9,200
2	2,200	2,400	9,000	11,400	5,700
3	4,200	6,600	9,000	15,600	**5,200**
4	6,200	12,800	9,000	21,800	5,450
5	8,200	21,000	9,000	30,000	6,000

This table shows that the best age for the replacement of machine *A* is 3rd year. The average yearly cost of owning and operating for this period is Rs. 5,200.

(*b*) For machine *B*, the average cost per year can similarly be computed as given in the following table :

Year (n)	$f(n)$	$\Sigma f(n)$	$C-S$	T	$A(n)$
1	400	400	10,000	10,400	10,400
2	1,200	1,600	10,000	11,600	5,800
3	2,000	3,600	10,000	13,600	4,533
4	2,800	6,400	10,000	16,400	4,100
5	3,600	10,000	10,000	20,000	**4,000**
6	4,400	14,400	10,000	24,400	4,066

Since the minimum average cost for machine *B* is lower than that for machine *A*, machine should be replaced by machine *A*.

To decide the time of replacement, we should compare the minimum average cost for *B* (Rs. 4,000) with yearly cost of maintaining and using the machine *A*. Since there is no salvage value of the machine, we shall consider only the maintenance cost. We would keep the machine *A* so long as the yearly maintenance cost is lower than Rs. 4,000 and replace when it exceeds Rs. 4,000.

On the one-year old machine *A*, Rs. 2,200 would be required to be spent in the next year; while Rs. 4,200 would be needed in year following. Thus, we should keep machine *A* for one year and replace it thereafter.