

1 Пересчёт систем координат

1.1 Из декартовых в сферические

$$\rho = \sqrt{x^2 + y^2 + z^2} \quad (1)$$

$$\varphi = \arccos \frac{z}{\rho} \quad (2)$$

$$\theta = \text{atan2}(y, x) \quad (3)$$

1.2 Из сферические в декартовы

$$x = \rho \cdot \sin \varphi \cdot \cos \theta \quad (4)$$

$$y = \rho \cdot \sin \varphi \cdot \sin \theta \quad (5)$$

$$z = \rho \cdot \cos \varphi \quad (6)$$

2 Базовые преобразования

2.1 Перенос точки

$$T^*(\Delta x, \Delta y, \Delta z) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \Delta x & \Delta y & \Delta z & 1 \end{vmatrix} \quad (7)$$

2.2 Масштабирование точки относительно центра координат

$$S^*(Sx, Sy, Sz) = \begin{vmatrix} Sx & 0 & 0 & 0 \\ 0 & Sy & 0 & 0 \\ 0 & 0 & Sz & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \quad (8)$$

2.3 Поворот точки вокруг оси x

$$R_x^*(\alpha) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) & 0 \\ 0 & -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \quad (9)$$

2.4 Поворот точки вокруг оси y

$$R_y^*(\alpha) = \begin{vmatrix} \cos(\alpha) & 0 & -\sin(\alpha) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(\alpha) & 0 & \cos(\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \quad (10)$$

2.5 Поворот точки вокруг оси z

$$R_z^*(\alpha) = \begin{vmatrix} \cos(\alpha) & \sin(\alpha) & 0 & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \quad (11)$$

3 Обратные операции

3.1 Перенос системы координат

$$T^{-1}(\Delta x, \Delta y, \Delta z) = T^*(-\Delta x, -\Delta y, -\Delta z) \quad (12)$$

3.2 Масштабирование осей системы координат

$$S^{-1}(Sx, Sy, Sz) = S^*\left(\frac{1}{Sx}, \frac{1}{Sy}, \frac{1}{Sz}\right) \quad (13)$$

3.3 Поворот системы координат вокруг осей

$$R_i^{-1}(\alpha) = R_i^*(-\alpha) \quad (14)$$

где: i – ось системы координат $(\{x, y, z\})$.

4 Композиция 3D преобразований

4.1 Поворот точки относительно линии, проходящей через начало системы координат на угол α

$$R_z^{-1}(\theta) \cdot R_y^{-1}(\varphi) \cdot \mathbf{R}_z(\alpha) \cdot R_y^{-1}(-\varphi) \cdot R_z^{-1}(-\theta) \quad (15)$$

$$\begin{aligned}
 & \underbrace{\begin{vmatrix} \cos(-\theta) & \sin(-\theta) & 0 & 0 \\ -\sin(-\theta) & \cos(-\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}}_{R_z^{-1}(\theta)} \times \underbrace{\begin{vmatrix} \cos(-\varphi) & 0 & -\sin(-\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(-\varphi) & 0 & \cos(-\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}}_{R_y^{-1}(\varphi)} \\
 & \times \underbrace{\begin{vmatrix} \cos(\alpha) & \sin(\alpha) & 0 & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}}_{\mathbf{R}_z(\alpha)} \\
 & \times \underbrace{\begin{vmatrix} \cos(\varphi) & 0 & -\sin(\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(\varphi) & 0 & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}}_{R_y^{-1}(-\varphi)} \times \underbrace{\begin{vmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}}_{R_z^{-1}(-\theta)} \\
 R_z^{-1}(\theta) \cdot R_y^{-1}(\varphi) &= \begin{vmatrix} \cos(-\theta) & \sin(-\theta) & 0 & 0 \\ -\sin(-\theta) & \cos(-\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} \cos(-\varphi) & 0 & -\sin(-\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(-\varphi) & 0 & \cos(-\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \\
 &= \begin{vmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} \cos(\varphi) & 0 & \sin(\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\varphi) & 0 & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \\
 &= \begin{vmatrix} \cos(\theta) \cos(\varphi) & -\sin(\theta) & \cos(\theta) \sin(\varphi) & 0 \\ \sin(\theta) \cos(\varphi) & \cos(\theta) & \sin(\theta) \sin(\varphi) & 0 \\ -\sin(\varphi) & 0 & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}
 \end{aligned}$$

WolframAlpha

$$\begin{aligned}
 R_y^{-1}(-\varphi) \cdot R_z^{-1}(-\theta) &= \begin{vmatrix} \cos(\varphi) & 0 & -\sin(\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(\varphi) & 0 & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \\
 &= \begin{vmatrix} \cos(\varphi) \cos(\theta) & \cos(\varphi) \sin(\theta) & -\sin(\varphi) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \\ \sin(\varphi) \cos(\theta) & \sin(\varphi) \sin(\theta) & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}
 \end{aligned}$$

WolframAlpha

$$\begin{aligned}
& \left[R_z^{-1}(\theta) \cdot R_y^{-1}(\varphi) \right] \cdot \mathbf{R}_z(\alpha) = \\
& = \begin{vmatrix} \cos(\theta) \cos(\varphi) & -\sin(\theta) & \cos(\theta) \sin(\varphi) & 0 \\ \sin(\theta) \cos(\varphi) & \cos(\theta) & \sin(\theta) \sin(\varphi) & 0 \\ -\sin(\varphi) & 0 & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} \cos(\alpha) & \sin(\alpha) & 0 & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \\
& = \begin{vmatrix} \cos(\theta) \cos(\varphi) \cos(\alpha) + \sin(\theta) \sin(\alpha) & \cos(\theta) \cos(\varphi) \sin(\alpha) - \sin(\theta) \cos(\alpha) & \cos(\theta) \sin(\varphi) & 0 \\ \sin(\theta) \cos(\varphi) \cos(\alpha) - \cos(\theta) \sin(\alpha) & \sin(\theta) \cos(\varphi) \sin(\alpha) + \cos(\theta) \cos(\alpha) & \sin(\theta) \sin(\varphi) & 0 \\ -\sin(\varphi) \cos(\alpha) & -\sin(\varphi) \sin(\alpha) & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}
\end{aligned}$$

WolframAlpha

$$\begin{aligned}
& \left[R_z^{-1}(\theta) \cdot R_y^{-1}(\varphi) \cdot \mathbf{R}_z(\alpha) \right] \cdot \left[R_y^{-1}(-\varphi) \cdot R_z^{-1}(-\theta) \right] = \\
& = \begin{vmatrix} \cos(\theta) \cos(\varphi) \cos(\alpha) + \sin(\theta) \sin(\alpha) & \cos(\theta) \cos(\varphi) \sin(\alpha) - \sin(\theta) \cos(\alpha) & \cos(\theta) \sin(\varphi) & 0 \\ \sin(\theta) \cos(\varphi) \cos(\alpha) - \cos(\theta) \sin(\alpha) & \sin(\theta) \cos(\varphi) \sin(\alpha) + \cos(\theta) \cos(\alpha) & \sin(\theta) \sin(\varphi) & 0 \\ -\sin(\varphi) \cos(\alpha) & -\sin(\varphi) \sin(\alpha) & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \\
& \times \begin{vmatrix} \cos(\varphi) \cos(\theta) & \cos(\varphi) \sin(\theta) & -\sin(\varphi) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \\ \sin(\varphi) \cos(\theta) & \sin(\varphi) \sin(\theta) & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \\
& = \begin{vmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{vmatrix}
\end{aligned}$$

$$\begin{aligned}
r_{11} &= [\cos(\theta) \cos(\varphi) \cos(\alpha) + \sin(\theta) \sin(\alpha)] \cdot [\cos(\varphi) \cos(\theta)] \\
&+ [\cos(\theta) \cos(\varphi) \sin(\alpha) - \sin(\theta) \cos(\alpha)] \cdot [-\sin(\theta)] \\
&+ [\cos(\theta) \sin(\varphi)] \cdot [\sin(\varphi) \cos(\theta)]
\end{aligned}$$

$$\begin{aligned}
r_{12} &= [\cos(\theta) \cos(\varphi) \cos(\alpha) + \sin(\theta) \sin(\alpha)] \cdot [\cos(\varphi) \sin(\theta)] \\
&+ [\cos(\theta) \cos(\varphi) \sin(\alpha) - \sin(\theta) \cos(\alpha)] \cdot [\cos(\theta)] \\
&+ [\cos(\theta) \sin(\varphi)] \cdot [\sin(\varphi) \sin(\theta)]
\end{aligned}$$

$$\begin{aligned}
r_{13} &= [\cos(\theta) \cos(\varphi) \cos(\alpha) + \sin(\theta) \sin(\alpha)] \cdot [-\sin(\varphi)] \\
&+ [\cos(\theta) \sin(\varphi)] \cdot [\cos(\varphi)]
\end{aligned}$$

$$r_{14} = 0$$

$$\begin{aligned}
r_{21} = & [\sin(\theta) \cos(\varphi) \cos(\alpha) - \cos(\theta) \sin(\alpha)] \cdot [\cos(\varphi) \cos(\theta)] \\
& + [\sin(\theta) \cos(\varphi) \sin(\alpha) + \cos(\theta) \cos(\alpha)] \cdot [-\sin(\theta)] \\
& + [\sin(\theta) \sin(\varphi)] \cdot [\sin(\varphi) \cos(\theta)]
\end{aligned}$$

$$\begin{aligned}
r_{22} = & [\sin(\theta) \cos(\varphi) \cos(\alpha) - \cos(\theta) \sin(\alpha)] \cdot [\cos(\varphi) \sin(\theta)] \\
& + [\sin(\theta) \cos(\varphi) \sin(\alpha) + \cos(\theta) \cos(\alpha)] \cdot [\cos(\theta)] \\
& + [\sin(\theta) \sin(\varphi)] \cdot [\sin(\varphi) \sin(\theta)]
\end{aligned}$$

$$\begin{aligned}
r_{23} = & [\sin(\theta) \cos(\varphi) \cos(\alpha) - \cos(\theta) \sin(\alpha)] \cdot [-\sin(\varphi)] \\
& + [\sin(\theta) \sin(\varphi)] \cdot [\cos(\varphi)]
\end{aligned}$$

$$r_{24} = 0$$