

Week 5

Question 8a) should read 4th order.

Question 8b) should read 6th order.

Question 02. a)

Use the method of undetermined coefficients to derive a difference formula for $f''(x_0)$ that uses the points $x_0, x_0 + h, x_0 + 2h$. What is the order of the formula?

$$f''(x_0) \approx Af(x_0) + Bf(x_0 + h) + Cf(x_0 + 2h) \quad (1)$$

The Taylor expansions of $f(x_0 + h)$ and $f(x_0 + 2h)$ are:

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{6}f'''(x_0) + \frac{h^4}{24}f^{(4)}(\xi_1) \quad (2)$$

$$f(x_0 + 2h) = f(x_0) + 2hf'(x_0) + 2h^2f''(x_0) + \frac{4h^3}{3}f'''(x_0) + \frac{2h^4}{3}f^{(4)}(\xi_2) \quad (3)$$

Combining (1), (2), (3):

$$\begin{aligned} f''(x_0) &\approx Af(x_0) + B\left(f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{6}f'''(x_0) + \frac{h^4}{24}f^{(4)}(\xi_1)\right) \\ &\quad + C\left(f(x_0) + 2hf'(x_0) + 2h^2f''(x_0) + \frac{4h^3}{3}f'''(x_0) + \frac{2h^4}{3}f^{(4)}(\xi_2)\right) \\ &= \left(A + B + C\right)f(x_0) + \left(B + 2C\right)hf'(x_0) + \left(\frac{B}{2} + 2C\right)h^2f''(x_0) \\ &\quad + \left(\frac{B}{6} + \frac{4C}{3}\right)h^3f'''(x_0) + \frac{B}{24}h^4f^{(4)}(\xi_1) + \frac{2C}{3}h^4f^{(4)}(\xi_2) \end{aligned} \quad (4)$$

Equating the coefficients of $f(x_0), f'(x_0), f''(x_0)$ on both sides of (4) we obtain a linear system:

$$\begin{cases} 0 = (A + B + C) \\ 0 = (B + 2C)h \\ 1 = \left(\frac{B}{2} + 2C\right)h^2 \end{cases}$$

Solving this system we get

$$A = \frac{-1}{3h^2} \quad B = \frac{2}{3h^2} \quad C = \frac{-1}{3h^2} \quad (5)$$

$$(6)$$

$$f''(x_0) = \frac{-f(x_0) + 2f(x_0 + h) - f(x_0 + 2h)}{3h^2} - \frac{h}{3}f'''(x_0) + \mathcal{O}(h^2) \quad (7)$$

$$f''(x_0) \approx_0 \frac{-f(x_0) + 2f(x_0 + h) - f(x_0 + 2h)}{3h^2} \text{ (First order approximation)} \quad (8)$$

Question 02. b)

$$f''(x_0) \approx \frac{-f(x_0 - 2h) + 16f(x_0 - h) - 30f(x_0) + 16f(x_0 + h) - f(x_0 + 2h)}{12h^2} \text{ (4th order approximation)} \quad (9)$$

Question 03. a)

$$f'''(x_0) \approx \frac{f(x_0) - 4f(x_0 + h) + 6f(x_0 + 2h) - 4f(x_0 + 3h) + f(x_0 + 4h)}{h^2} \quad (\text{First order approximation}) \quad (10)$$

Question 03. b)

$$f'''(x_0) \approx \frac{f(x_0 - 2h) - 4f(x_0 - h) + 6f(x_0) - 4f(x_0 + h) + f(x_0 + 2h)}{h^2} \quad (\text{Second order approximation}) \quad (11)$$