Department of Mathematics and Statistics

Math 391 **Assignment 2** **Fall 2019**

Differentiation and Integration

DUE: MONDAY, NOVEMBER 18TH

01. (10 marks) Let $f(x)$ be a function with as many derivatives as needed at and around $x = x_0$.

a. Derive a difference formula that approximates $f'(x_0)$, and uses the points

$$x_0 - h,\ x_0,\ x_0 + h,\ x_0 + 2\,h$$

b. What is the order of convergence of the formula?

02. (5 marks) Let $f(x)$ be a function with as many derivatives as needed at and around the point x_0. Find the error term for the difference formula

$$f''\big(x_0\big) \approx \frac{-f\big(x_0 + 3\,h\big) + 4\,f\big(x_0 + 2\,h\big) - 5\,f\big(x_0 + h\big) + 2\,f\big(x_0\big)}{h^2}$$

03. (10 marks) If the interval $[a, b]$ is subdivided into n subinterval $[x_i, x_{i+1}]$, $1 \leq i \leq n$ of equal length $h = \dfrac{b-a}{n}$, and if Simpson's rule is successively applied to each of those intervals, one gets the n-interval Composite Simpson's Rule, denoted by $Q_S^{c,n}\big(f\big)$

a. Show that

$$\begin{aligned} Q_S^{c,n}\big(f)\big) = \frac{h}{6}\,\Big(\,f(a) + f(b)\,\Big) + \frac{h}{3}\,\Big(f(a+h) + f(a+2\,h) + \cdot + f\big(a + (n-1)\,h\big)\Big) \\ + \frac{2\,h}{3}\,\Big(f\big(a + \frac{h}{2}\big) + f\big(a + 3\,\frac{h}{2}\big) + \cdots + f\big(a + (2\,n-1)\,\frac{h}{2}\big)\Big) \end{aligned}$$

b. Show that if $f(t)$ has a continuous fourth derivative, then

$$\int_a^b f(t)\,\mathrm{d}t = Q_S^{c,n}\big(f)\big) - \frac{(b-a)^5}{2880\,n^4}\,f^{(4)}(\theta), \qquad \theta \in [a\,,\,b]$$

c. Suppose $-9 \leq f^{(4)}(t) \leq 2$, in the interval $[0\,,\,2]$. Find the smallest number of intervals n, to guarantee that $\left|\displaystyle\int_0^2 f(t)\,\mathrm{d}t - Q_S^{c,n}\big(f\big)\right| \leq 10^{-5}$

04. (10 marks) Write a MATLAB function Composite_Simpson() that implements the composite Simpson's rule. The function should take the following arguments

- fname the handle of the function $f(t)$ whose definite integral is being computed.
- a and b, the end points of the interval of integration.
- n the number of intervals into which $[a, b]$ gets subdivided.

Composite_Simpson() should return the value of the integral.

Test your function for $n = 1, 2, 3$ on the integrals

- $\displaystyle\int_0^1 x^k \,\mathrm{d}x$, $k = 0, 1\,2, 3$, to make sure the degree of precision is 3
- $\displaystyle\int_0^1 \mathrm{e}^{3\,x} \,\mathrm{d}x$, to make sure that $\mathcal{Q}_S^{c,n}\big(f\big)$ is close to the exact value of the integral.

05. (5 marks)
Let $f(x, y)$ be a function of two variables that is defined and continuous in the square region $S = \big[-1\,, 1\big]^2$.

a. Derive an integration rule $Q\big(f\big)$ that approximates $\displaystyle\iint_S f(x, y)\,\mathrm{d}x\,\mathrm{d}y$, by applying Simpson's rule to the iterated integral $\displaystyle\int_{-1}^{1}\left(\int_{-1}^{1} f(x, y)\,\mathrm{d}x\right)\mathrm{d}y$.

b. Use the rule you derived to approximate the double integral $\displaystyle\iint_S \frac{1}{\mathrm{e}^x + \mathrm{e}^y}\,\mathrm{d}x\,\mathrm{d}y$.

Compare to the exact value of the integral $4\,\mathrm{e} - 2\left(\mathrm{e} + \mathrm{e}^{-1}\right)\,\ln\left(\dfrac{\mathrm{e}^2 + 1}{2}\right)$