MATH 391	Worksheet 06	F2019
	Numerical Integration	

01. Recall the 2 and 3-point Gauss quadrature rules for the integral $\displaystyle\int_{-1}^{1} f(x)\,\mathrm{d}x$

$$G_2(f) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \quad \text{and} \quad G_3(f) = \frac{5}{9}\, f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}\, f(0) + \frac{5}{9}\, f\left(\sqrt{\frac{3}{5}}\right)$$

Use $G_2(f)$ and $G_3(f)$ to compute an approximate value of each of the following integrals

a. $\displaystyle\int_0^1 \mathrm{e}^{-t}\,\mathrm{d}t$. Compare with the exact value $1 - \mathrm{e}^{-1}$

b. $\displaystyle\int_0^{\pi/4} \cos^2(t)\,\mathrm{d}x$. Compare with the exact value $\dfrac{\pi - 2}{8}$

c. $\displaystyle\int_0^1 \frac{1}{t^4+1}\,\mathrm{d}t$. Compare with the exact value $\dfrac{\sqrt{2}}{8}\left(\pi + \ln\left(3+\sqrt{2}\right)\right)$

d. $\displaystyle\int_0^1 \frac{\sin(t)}{t}\,\mathrm{d}t$. Compare with the exact value $1.758\,203\,138\cdots$

02. The Fresnel Sine Integral is defined by $\displaystyle S(x) = \int_0^x \sin\left(\frac{\pi\, t^2}{2}\right)\,\mathrm{d}t$.
Use the three-point Gauss integration rule

$$\int_{-1}^{1} f(t)\,\mathrm{d}t \approx \frac{5}{9}\, f\big(-\sqrt{3/5}\big) + \frac{8}{9}\, f\big(0\big) + \frac{5}{9}\, f\big(\sqrt{3/5}\big)$$

to evaluate $S(1)$. In MATLAB, the Fresnel Sine Integral function is implemented as fresnels(x). Compare your value to fresnels(1).

03. Based on the lecture, the nodes of the n-point Gauss quadrature rule are the roots of the degree-n Legendre polynomial, defined by the recurrence formula

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x \\ P_n(x) = \dfrac{2\,n-1}{n}\, P_{n-1}(x) - \dfrac{n-1}{n}\, P_{n-2}(x), \qquad n = 2,\,3,\,4,\,\cdots \end{cases}$$

and the corresponding weights are solutions of the system

$$\begin{cases} \begin{array}{rrcrl} w_1 & +w_2 & +\cdots & +w_n & = \int_{-1}^{1} 1\ \mathrm{d}x \\ x_1\, w_1 & +x_2\, w_2 & +\cdots & +x_n\, w_n & = \int_{-1}^{1} x\ \mathrm{d}x \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^{n-1}\, w_1 & +x_2^{n-1}\, w_2 & +\cdots & +x_n^{n-1}\, w_n & = \int_{-1}^{1} x^{n-1}\ \mathrm{d}x \end{array} \end{cases}$$

Find the 4-point and the 5-point Gauss quadrature rules for the integral $\displaystyle\int_{-1}^{1} f(x)\,\mathrm{d}x$.

04. Derive an integration rule of the form

$$\int_{-1}^{1} x^2\, f(x)\,\mathrm{d}x \approx w_1\, f\big(x_1\big) + w_2\, f\big(x_2\big) = G_{2,x^2}\big(f\big)$$

that will be exact when $f(x)$ is a polynomial of degree up to 3. The integration rule $G_{2,x^2}\big(f\big)$ is called a 2-point **weighted** Gauss quadrature rule, with weight x^2.

05. Find the 3-point and the 4-point weighted Gauss integration rule with weight x^2 for the integral $\int_{-1}^{1} x^2\, f(x)\,\mathrm{d}x$, i.e.,

$$\int_{-1}^{1} x^2\, f(x)\,\mathrm{d}x \approx w_1\, f\big(x_1\big) + w_2\, f\big(x_2\big) + w_3\, f\big(x_3\big)$$

and

$$\int_{-1}^{1} x^2\, f(x)\,\mathrm{d}x \approx w_1\, f\big(x_1\big) + w_2\, f\big(x_2\big) + w_3\, f\big(x_3\big) + w_4\, f\big(x_4\big)$$

What is the degree of precision of each?