MATH 391	Worksheet 08	FALL 2019

Fixed Point Iteration Method

01. Consider the equation $x = 1 + \tan^{-1}(x)$, and let $g(x) = 1 + \tan^{-1}(x)$.

a. Show that the equation has a unique root $r \in \mathbb{R}$.
b. Find an interval $\big[a\,,\,b\big]$ such that $g\big([a\,,\,b]\big) \subset [a\,,\,b]$ and $|g'(x)| < 1, \quad \forall x \in [a\,,\,b]$
c. Show that for any $\alpha \in [a\,,\,b]$, the sequence $\begin{cases} x_1 = \alpha \\ x_n = g\big(x_{n-1}\big), \ n = 2,\, 3,\, \cdots \end{cases}$ converges to the root r.
d. Use the Matlab function myfixedpoint, at the end of this worksheet to generate the first 10 iterates.

Hint: Recall that $-\dfrac{\pi}{2} < \tan^{-1}(x) < \dfrac{\pi}{2}$

02. Repeat problem 01 for each of the following equations.

a. $x = 2^{-x}$ **b.** $x = 3 - 2\,\ln\big(1 - \mathrm{e}^{-x}\big)$

03. For each of the following iterations, determine which will converge to the given fixed point r. Find the order of convergence when the iteration converges. Assume the initial approximation x_1 is close enough to r.

a. $x_n = -16 + 6\,x_{n-1} + \dfrac{12}{x_{n-1}}, \quad r = 2$ **b.** $x_n = \dfrac{2}{3}\,x_{n-1} + \dfrac{1}{x_{n-1}^2}, \quad r = \sqrt[3]{3}$
c. $x_n = \dfrac{12}{1 + x_{n-1}}, \quad r = 3$ **d.** $x_n = \big(2 - x_{n-1}\big)^{1/3}, \quad r = 1$

04. Reformulate the following equations as $x = g(x)$, so that the fixed point iteration algorithm $x_n = g\big(x_{n-1}\big)$ converges.
a. $x^3 - x + 1 = 0$ **b.** $\mathrm{e}^x - \sin(x) = 0$
c. $\ln\big(1 + x\big) - x^2 = 0$ for both zeros **d.** $\mathrm{e}^x - 3\,x^3 = 0$ for both zeros

05. Let r be a fixed point of the function $g(x)$. Suppose $g^{(p)}(x)$ is continuous at and around $x = r$, for some $p \geq 2$, and $g'(r) = \cdots = g^{(p-1)}(r) = 0$. Show that if x_1 is close enough to r, then the sequence $\big\{x_n\big\}_{n\geq 1}$ generated by the fixed point iteration $x_{n+1} = g\big(x_n\big)$ has order of convergence p, more precisely

$$\lim_{n\to+\infty} \frac{r - x_{n+1}}{\big(r - x_n\big)^p} = (-1)^{p-1}\,\frac{g^{(p)}(r)}{p!}$$

Hint: apply Taylor theorem to $g(x)$ at $x = r$.

06. Show that the fixed point iteration algorithm $x_{n+1} = \dfrac{x_n^3 + 3\,a\,x_n}{3\,x_n^2 + a}$, $n = 1, 2, \cdots$ is a third order algorithm for computing $\sqrt{a}$. Hint: Use problem 05
Letting $a = 3$, and starting with $x_1 = 1$, compute x_2, x_3, x_4 and compare each to the exact value $\sqrt{3} = 1.732\,050\,807\,568\,877\,293\,\cdots$

```
 function [r, X, niter, status] = myfixedpoint( xin, myg, tol, maxiter )
%
% Finds an approximate value of the fixed point r of the function g(x),
% i.e. g(r)=r, using the fixed point iteration algorithm:
%
%            || x_1=xin,   x_{n+1}=g(x_n), n=1,2,3,...
%
% Input:
% xin     an initial guess of the fixed point r
% fname   function handle of the function whose fixed point is being
%         approximated.
% tol     is a tolerance to terminate the iterations whenever
%                      |x_{n+1}-x_n|<= tol
% maxiter maximum number of iterations that can be performed.
% Output:
% r       an approximate value of the fixed point r=g(r).
% X       a column matrix whose components are the successive
%         iterates of the fixed point iteration algorithm.
% niter   total number of iterations performed.
% status  a flag. has value 1 if successful and -1 if not.
%
% compute the first iterate
 x1 = xin;
 x2 = myg(x1);
 N = 1;
% distance between the two successive iterates
 delx = abs(x2-x1);
 X = [x1; x2];
 x1 = x2;
% define the stopping criteria
 if nargin == 2
    tol = 10^(-5); maxiter = 100;
 elseif nargin == 3
    maxiter = 100;
 end
%
% generate the successive iterates
 while (delx>tol) & (N<maxiter)
     x2 = myg(x1);
     N = N+1;
     X = [X;x2];
```

```
     delx = abs(x2-x1);
     x1 = x2;
 end
%
% set r equal to the last iterate,  niter to N
 r = x1;
 niter = N;
% determine whether the iterations converged or not
 status = 1;
 if delx>tol
     status = -1;
     disp('=== max number of iterations reached and no convergence ===')
 end
```