MATH 391 **Worksheet 10** **F2019**

Linear Systems - Iterative Methods

01. Consider the systems

(i) $\begin{cases} 3\,x_1 - x_2 + x_3 = 1 \\ 3\,x_1 + 6\,x_2 + 2\,x_3 = 0 \\ 3\,x_1 + 3\,x_2 + 7\,x_3 = 4 \end{cases}$ **(ii)** $\begin{cases} 3\,x_1 + x_2 + x_3 = 5 \\ x_1 + 3\,x_2 - x_3 = 3 \\ 3\,x_1 + x_2 - 5\,x_3 = -1 \end{cases}$

(iii) $\begin{cases} 3\,x_1 + x_2 + x_3 = 6 \\ x_1 + 3\,x_2 + x_3 = 3 \\ x_1 + x_2 + 3\,x_3 = 5 \end{cases}$ **(iv)** $\begin{cases} 2\,x_1 + x_2 - x_3 + x_4 = 4 \\ x_1 - 2\,x_2 + x_3 - 2\,x_4 = 5 \\ 2\,x_1 + 2\,x_2 + 5\,x_3 - x_4 = 6 \\ x_1 - x_2 + x_3 + 4\,x_4 = 7 \end{cases}$

Starting with $\begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, compute the iterates $\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix}$ and $\begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{bmatrix}$, generated by

a. the Jacobi method **b.** the Gauss-Seidel method

02. Show that if $\boldsymbol{A} = \left(a_{ij} \right)_{1\le i,j\le n}$ is an $n\times n$ matrix, then

$$\|\boldsymbol{A}\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}| \qquad \text{and} \qquad \|\boldsymbol{A}\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$$

Hence $\|\boldsymbol{A}\|_1$ is the maximum of the 1-norm of the columns of $\boldsymbol{A}$, while $\|\boldsymbol{A}\|_\infty$ is the maximum of the 1-norm of the rows of $\boldsymbol{A}$.

03. Compute $\|\boldsymbol{A}\|_1$ and $\|\boldsymbol{A}\|_\infty$ for each of the matrices

a. $\boldsymbol{A} = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & -1 & 4 \end{bmatrix}$ **b.** $\boldsymbol{A} = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 3 & -6 & 9 & 3 \\ 2 & 1 & 4 & 1 \\ 1 & -2 & 2 & -2 \end{bmatrix}$

04. Compute $\|\boldsymbol{A}\|_2$ for each of the following matrices

a. $\boldsymbol{A} = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$ **b.** $\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ **c.** $\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$

05. Let $\boldsymbol{I}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$ and $\boldsymbol{D} = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$

a. Show that $|||\boldsymbol{I}_n|||_1 = |||\boldsymbol{I}_n|||_2 = |||\boldsymbol{I}_n|||_\infty = 1$

b. Compute $|||\boldsymbol{D}|||_1$, $|||\boldsymbol{D}|||_2$, and $|||\boldsymbol{D}|||_\infty$

06. For each of the systems in problem (**01**) above, compute

a. the Jacobi iterative matrix $\boldsymbol{T}_J = \boldsymbol{I} - \boldsymbol{D}^{-1}\boldsymbol{A}$ and its infinity norm

b. the Gauss-Seidel iterative matrix $\boldsymbol{T}_{GS} = \boldsymbol{I} - \boldsymbol{R}^{-1}\boldsymbol{A}$ and its infinity norm

Discuss the convergence of each method.

Note: In MATLAB, the command inv(A), computes the inverse of the matrix $\boldsymbol{A}$, while the command norm(A,inf), computes the infinity norm of the matrix $\boldsymbol{A}$.

07. Analyze the convergence of the Jacobi and Gauss-Seidel iterative methods for the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{A} = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}$, $|\alpha| < 1$

08. Suppose the system $\boldsymbol{x} = \boldsymbol{B}\boldsymbol{x} + \boldsymbol{c}$, has a unique solution $\boldsymbol{x} \in \mathbb{R}^n$, and consider the iterative method $\left\| \begin{array}{l} \boldsymbol{x}^{(0)} = \boldsymbol{0} \\ \\ \boldsymbol{x}^{(k)} = \boldsymbol{B}\boldsymbol{x}^{(k-1)} + \boldsymbol{c}, \quad k = 1, 2, 3, \cdots \end{array} \right.$ where $\boldsymbol{B}$ is an $n \times n$ matrix.

a. Show that if $|||\boldsymbol{B}||| < 1$, for some induced matrix norm $|||\cdot|||$, then the sequence $\left\{ \boldsymbol{x}^{(k)} \right\}_{k \geq 0}$ converges to $\boldsymbol{x}$, and $\|\boldsymbol{x} - \boldsymbol{x}^{(k)}\| \leq |||\boldsymbol{B}|||^k \, \|\boldsymbol{x} - \boldsymbol{x}^{(0)}\|$.

b. Show that

$$\|\boldsymbol{x} - \boldsymbol{x}^{(k)}\| \leq \frac{|||\boldsymbol{B}|||^k}{1 - |||\boldsymbol{B}|||} \, \|\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(0)}\|$$

09. Consider the system $\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 \\ -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$

with exact solution $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Use the MATLAB functions myJacobi( ) and myGauss-Seidel( ) from your assignment #3 to solve the system

a. using Jacobi's Method **b.** using Gauss-Seidel Method

Stop the iterations whenever $\dfrac{\|\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}\|_\infty}{\|\boldsymbol{x}^{(k)}\|_\infty} \leq 10^{-3}$

10. Compute the condition number $|||\boldsymbol{A}|||\ |||\boldsymbol{A}^{-1}|||$ of the given matrix $\boldsymbol{A}$. Use the one-norm.

a. $\boldsymbol{A} = \begin{bmatrix} 1/2 & 1/3 \\ 1/3 & 1/4 \end{bmatrix}$ **b.** $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$

Notice that because the matrices are symmetric, the use of the infinity-norm will give the same value of the condition number.

11. Let $\boldsymbol{A}$ be an $n \times n$ invertible matrix, and let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be its eigenvalues. Show

$$\operatorname{cond}(\boldsymbol{A}) \geq \frac{\max\limits_{1 \leq i \leq n} |\lambda_i|}{\min\limits_{1 \leq i \leq n} |\lambda_i|}$$

12. The $n \times n$ Hilbert matrix, is the matrix whose (i, j)-entry is $\dfrac{1}{i+j-1}$. For instance the 2×2 Hilbert matrix is $\begin{bmatrix} \frac{1}{1} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$, and the 3×3 Hilbert matrix is $\begin{bmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$.

Use MATLAB to solve and compare the two solutions of the systems $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{A}\boldsymbol{y} = \boldsymbol{c}$, where $\boldsymbol{A}$ is the 6×6 Hilbert matrix, $\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6.0001 \end{bmatrix}$

Can you explain why despite the fact that $\boldsymbol{b}$ and $\boldsymbol{c}$ are nearly equal, the solutions of the two systems are not.

Note: In MATLAB, the command $\boldsymbol{A} = \text{hilb}(n)$ will generate the $n \times n$ Hilbert matrix and store it in $\boldsymbol{A}$, while the command $\boldsymbol{x} = \boldsymbol{A} \setminus \boldsymbol{b}$, will solve the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, using $\boldsymbol{L}\boldsymbol{U}$ with partial pivoting.