MATH 391	Worksheet 3	F2019

Numerical Differentiation - First Derivative

01. Use the difference formula $f'(x_0) \approx \dfrac{f(x_0+h)-f(x_0-h)}{2\,h}$ and the table below to approximate $f'(1.3)$ with $h=0.1$ and $h=0.01$. Given that the data were generated from $f(x)=3\,x\,\mathrm{e}^x-\cos(x)$, find the actual error.

x	1.20	1.29	1.30	1.31	1.40
$f(x)$	11.59006317	13.78176310	14.04275818	14.30741266	16.86187272

02. From the rounded values of $f(x)=\dfrac{1}{(x+1)^2}$ shown in the table below, determine approximate values of $f'(1.0)$, $f'(1.2)$, and $f'(1.4)$, by use of an appropriate 2-point formula. Compare to the exact values.

x	1.0	1.1	1.2	1.3	1.4
$f(x)$	0.25	0.22675737	0.20661157	0.18903592	0.17361111

03. Analyze the round-off errors as done in the lecture for the centered difference formula

$$f'(x_0)=\frac{f(x_0+h)-f(x_0-h)}{2\,h}-\frac{1}{6}\,f'''(c)\,h^2$$

Assuming that the machine error in evaluating $f(x_0\pm h)$ is $\epsilon=10^{-14}$, estimate the optimal h. Confirm your findings for the function $f(x)=x+\mathrm{e}^x$, with $x_0=0$. Take $h=10^{-1},\cdots,10^{-14}$

Let $\Phi(h)$ be a difference formula that approximates $f'\big(x_0\big)$. if

$$f'\big(x_0\big)-\Phi(h)=O\big(h^p\big),\quad p>0$$

we say that the approximation $f'\big(x_0\big)\approx\Phi(h)$ has order p.

04. Use the degree 2 polynomial in Lagrange form

$$P_2(x)=L_{2,0}(x)\,f\big(x_0\big)+L_{2,1}(x)\,f\big(x_1\big)+L_{2,2}(x)\,f\big(x_2\big)$$

to derive a difference formula based on the values $f(x_0-h)$, $f(x_0)$ and $f(x_0+2\,h)$, that approximates $f'(x_0)$. Use Taylor Theorem

$$f\big(x_0+u\big)=f\big(x_0\big)+\frac{f'\big(x_0\big)}{1!}\,u+\cdots+\frac{f^{(k)}\big(x_0\big)}{k!}\,u^k+\frac{f^{(k+1)}\big(\theta\big)}{(k+1)!}\,u^{k+1},\qquad \theta\in I\big(x_0\,,\,x_0+u\big)$$

to find the order of the difference formula.

05. Use Taylor Theorem to find the order of convergence of each of the following difference formulas.

a. $f'(x_0) \approx \dfrac{-f(x_0 + 2\,h) + 4\,f(x_0 + h) - 3\,f(x_0)}{2\,h}$

b. $f'(x_0) \approx \dfrac{3\,f(x_0) - 4\,f(x_0 - h) + f(x_0 - 2\,h)}{2\,h}$