MATH 391 **Worksheet 4** **F2019**

Numerical Differentiation - Higher Order Formulas

01. Use the difference formula $f''(x_0) \approx \dfrac{f(x_0-h) - 2\,f(x_0) + f(x_0+h)}{h^2}$ and the table below to approximate $f''(1.3)$ with $h = 0.1$ and $h = 0.01$. Given that the data were generated from $f(x) = 3\,x\,e^x - \cos(x)$, find the actual error.

x	1.20	1.29	1.30	1.31	1.40
$f(x)$	11.59006317	13.78176310	14.04275818	14.30741266	16.86187272

02. Use the method of undetermined coefficients to derive a difference formula for $f''\big(x_0\big)$ that uses the points

a. $x_0,\ x_0+h,\ x_0+2\,h$ **b.** $x_0-2\,h,\ x_0-h,\ x_0,\ x_0+h,\ x_0+2\,h$

What is the order of each formula?

03. Use the method of undetermined coefficients to derive a difference formula for $f^{(4)}\big(x_0\big)$ that uses the points

a. $x_0,\ x_0+h,\ x_0+2\,h,\ x_0+3\,h,\ x_0+4\,h$ **b.** $x_0-2\,h,\ x_0-h,\ x_0,\ x_0+h,\ x_0+2\,h$

What is the order of each formula?

04. How well does $\dfrac{2+h}{2\,h^2}\,f(x+h) - \dfrac{2}{h^2}\,f(x) + \dfrac{2-h}{2\,h^2}\,f(x-h)$ approximate $f'(x) + f''(x)$?

05. Find an order two difference formula that approximates $\Big(a(x)\,y'(x)\Big)'$ without using the derivative of $a(x)$.

06. Let $u(x,y)$ be a function of two variables with as many partial derivatives as needed.

a. Find a second order centered difference formula $\Delta_h u(x,y)$ that approximates the Laplacian $\Delta u(x,y) = u_{xx}(x,y) + u_{yy}(x,y)$.

b. What is the error of approximation $E(x,y,h) = \Delta u(x,y) - \Delta_h u(x,y)$

Richardson Extrapolation

07. If $f(x)$ is smooth at and around x_0, then

$$f'(x_0) = \Phi_0(h) - c_0\,h^2 - c_1\,h^4 - c_2\,h^6 - \cdots$$

where $\Phi_0(h) = \dfrac{f(x_0+h) - f(x_0-h)}{2\,h}$, $c_0 = \dfrac{f'''(x_0)}{3!}$, $c_1 = \dfrac{f^{(5)}(x_0)}{5!}$, $c_2 = \dfrac{f^{(7)}(x_0)}{7!}$, $\cdots$

a. Use Richardson extrapolation to find a 2^{nd} order approximation $\Phi_1(h)$ in terms of $\Phi_0(h)$ and $\Phi_0(h/2)$

b. Use Richardson extrapolation a second time to find a 4^{th} order approximation $\Phi_2(h)$ in terms of $\Phi_0(h)$, $\Phi_0(h/2)$ and $\Phi_0(h/4)$

c. Evaluate $\Phi_0(h)$, $\Phi_1(h)$ and $\Phi_2(h)$, if $f(x) = x + \mathrm{e}^x$, $x_0 = 0$, $h = 0.1$, $h = 0.01$, and $h = 0.001$. Compare the computed values with the exact value $f'(0) = 2$

08. Let $f(x)$ be a smooth function at and around x_0, and define

$$\Phi_0(h) = \frac{f(x_0+h) - 2\,f(x_0) + f(x_0-h)}{h^2}$$

a. Show that $\Phi_0(h)$ is an approximation of $f''\big(x_0\big)$, and find its order.

b. Use Richardson's extrapolation to derive a 4^{th} order difference formula for $f''(x_0)$