MATH 391	Worksheet 09	F2019
	Linear Systems - Direct Methods	

01. Given that $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 4 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, solve the system $\boldsymbol{A}\,\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$

02. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n\times n$ matrices. Show the following
- **a.** If $\boldsymbol{A}$ and $\boldsymbol{B}$ are lower triangular, then so is $\boldsymbol{AB}$
- **b.** If $\boldsymbol{A}$ and $\boldsymbol{B}$ are upper triangular, then so is $\boldsymbol{AB}$
- **c.** If $\boldsymbol{A}$ is lower triangular, then so is $\boldsymbol{A}^{-1}$
- **d.** If $\boldsymbol{A}$ is upper triangular, then so is $\boldsymbol{A}^{-1}$

03. Determine how many multiplications/divisions are needed to compute
- **a.** $\boldsymbol{A}\,\boldsymbol{B}$, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n\times n$ matrices.
- **b.** $\boldsymbol{A}\,\boldsymbol{B}$, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n\times n$ triangular matrices.

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04. Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ be three $n\times n$ invertible matrices, and $\boldsymbol{b}, \boldsymbol{c}$ two $n\times 1$ column matrices. Show that $\boldsymbol{x}$ can be computed without having to compute any inverse matrix, but rather by solving systems of equations.
- **a.** $\boldsymbol{x} = \boldsymbol{A}^{-1}\left(\boldsymbol{b} + \boldsymbol{A}^{-1}\boldsymbol{c}\right)$
- **b.** $\boldsymbol{x} = \boldsymbol{B}^{-1}\left(2\,\boldsymbol{A}^{-1} + \boldsymbol{I}\right)\left(\boldsymbol{C}^{-1} + \boldsymbol{A}\right)\boldsymbol{b}$

05. Let $\boldsymbol{A}$ be an $n\times n$ invertible matrix. Suppose $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$, where $\boldsymbol{L}$ is a unit lower triangular matrix, and $\boldsymbol{U}$ is an upper triangular matrix. Show that $\boldsymbol{L}$ and $\boldsymbol{U}$ must be unique.

05. Let $\boldsymbol{A}$ be a tridiagonal $n\times n$ matrix. Suppose $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$ where $\boldsymbol{L}$ is lower triangular and $\boldsymbol{U}$ is upper triangular. Show that both $\boldsymbol{L}$ and $\boldsymbol{U}$ are tridiagonal as well.

06. Find the $\boldsymbol{LU}$ factorization of the following matrices

a. $\begin{bmatrix} 4 & 2 & 0 \\ 4 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix}$ **c.** $\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 3 & 4 & 4 \\ 0 & 2 & 1 & -1 \end{bmatrix}$

07. Use the $\boldsymbol{LU}$ factorizations you have found in the previous problem to solve the systems

a. $\begin{bmatrix} 4 & 2 & 0 \\ 4 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 11 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 14 \\ -7 \end{bmatrix}$

08. Determine which of the following matrices are diagonally dominant and which are symmetric

a. $\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 4 \end{bmatrix}$ **b.** $\boldsymbol{A} = \begin{bmatrix} 4 & 2 & 6 \\ 3 & 0 & 7 \\ -1 & -2 & -3 \end{bmatrix}$

c. $\boldsymbol{A} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 \\ 9 & -11 & 3 & 0 \\ 5 & 4 & 1 & 1 \end{bmatrix}$ **d.** $\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 3 \end{bmatrix}$

09. Find all values of the parameters r and s so that the matrix $\boldsymbol{A}$ is diagonally dominant.

a. $\boldsymbol{A} = \begin{bmatrix} 4 & r & 1 \\ 2s & 5 & 4 \\ s & 2 & r \end{bmatrix}$ **b.** $\boldsymbol{A} = \begin{bmatrix} 3 & 2 & s \\ r & 5 & s \\ 2 & 1 & r \end{bmatrix}$

10. Find the $\boldsymbol{LDL}^t$ factorization of the following matrices

a. $\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ **b.** $\boldsymbol{A} = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}$

c. $\boldsymbol{A} = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 0 & 3 & -1 & 1 \\ 2 & -1 & 6 & 3 \\ 1 & 1 & 3 & 8 \end{bmatrix}$ **d.** $\boldsymbol{A} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

11. Writing a symmetric matrix $\boldsymbol{A}$ in the form $\boldsymbol{A} = \boldsymbol{G}\boldsymbol{G}^t$, where $\boldsymbol{G}$ is a lower triangular matrix, is known as Cholesky factorization. Assuming that Cholesky factorization is possible, write an algorithm that computes $\boldsymbol{G}$. Use your algorithm to find the Cholesky factorization of the matrices

a. $\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 5 & 1 \\ 0 & 2 & 10 \end{bmatrix}$ **b.** $\boldsymbol{A} = \begin{bmatrix} 4 & 6 & -4 & 10 \\ 6 & 10 & -4 & 16 \\ -4 & -4 & 9 & -9 \\ 10 & 16 & -9 & 43 \end{bmatrix}$