11
Exercises

1. Prepare a set of glass or brass grinding plates of good evenness according to the following method: From a stack of 12 approximately quadratic or circular disks with a thickness of ca. 8 mm and an edge length of ca. 100 mm select pairs of plates with about the same camber using a straightedge. One then grinds the plates pairwise against each other so that the respective cambered or concave surfaces are abraded away. A suitable abrasive is water with corundum powder of about 30 μm grain size. Thereafter, the plates are again sorted with a straightedge according to about the same deviation from flatness and repeat the pairwise mutual abrasion of plates of the same curvature. After repeating the procedure again one normally finds that over half the plates are suitable as a grinding base for making plane grinds with a deviation of at most 1 μm per 10,000 μm. Use a finer grain size (for example, 15 μm) in the last steps.

2. Practice grinding plane faces on crystals of various hardness using the grinding plates prepared in Exercise 1. In doing so take special care that grinding is performed under uniform motion in long strokes with little rotation and not too much pressure on the specimen. Otherwise the danger exists that the edges are more strongly abraded. Sometimes it is convenient to use a weakly convex grinding plate to work against this effect. The flatness can be checked with the help of optical interference methods or with a precision straightedge. The next step is to prepare a plane parallel plate. Firstly, plane grind a face and then grind the opposite face under continuous control using a micrometer screw or better using a thickness gauge so that the distances between the faces is constant within the given tolerances over the complete surface. In the last steps, one should use a very fine corundum powder as an abrasive together with a suitable liquid (water, propanol, ethylene glycol, ethylether), whereby it is essential to ensure that no coarse grained abrasives from previous grinding processes are introduced into the fine grinding step. In order to prevent corners and edges from breaking out it is recommended to level the edges by careful grinding.

Physical Properties of Crystals. Siegfried Haussühl.

ISBN: 978-3-527-40543-5

The preparation of plane parallel plates and other geometrically defined specimens is made considerably easier by employing ring-shaped holders, for example, machined from brass. The raw specimens are cemented in the holders such that the area to be abraded away protrudes over the rim of the given holder thus allowing targeted removal during the grinding process.

3. Prove by coordinate transformation applying Neumann's principle that the fundamental polynomials (in the coordinates x_i, homogeneous polynomials of the nth degree) have the following form in the PSG 22:

$$P_0 = a_0, \quad P_1 = 0, \quad P_2 = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2, \quad P_3 = a_4 x_1 x_2 x_3,$$
$$P_4 = a_5 x_1^4 + a_6 x_2^4 + a_7 x_3^4 + a_8 x_1^2 x_2^2 + a_9 x_2^2 x_3^2 + a_{10} x_3^2 x_1^2.$$

Which form has P_5 (polynomial of the 5. degree in x_1, x_2, x_3)?

From these polynomials what can one infer concerning the existence of the corresponding nth rank tensors?

4. In the cylinder groups ∞, ∞m, ∞2 and so on, the tensors up to the fourth rank take on the same form as in the corresponding hexagonal groups PSG 6, 6m, 62 and so on. Prove this by tensor transformation. The condition is that a symmetry equivalent position results from a rotation of an arbitrary angle about the cylinder axis.

5. What is the change in a third- and fourth-rank tensor after a small rotation of the reference system described by the antisymmetric tensor $\{r_{ij}\}$? Calculate

$$\Delta t_{ijk} = t'_{ijk} - t_{ijk} \quad \text{and} \quad \Delta t_{ijkl} = t'_{ijkl} - t_{ijkl} \quad \text{respectively.}$$

6. Prove that symmetric functions $F(x_i)$ under symmetric secondary conditions $G(x_1, x_2, x_3)$ always take on an extreme value along the space diagonals of the Cartesian reference system.

For this purpose, one constructs the auxiliary function

$$H = F(x_1, x_2, x_3) - \lambda G(x_1, x_2, x_3)$$

and substitutes x_i by ru_i, where u_i are the directional cosines of the position vector. According to the method applied in Section 4.3.6, calculate the condition for extreme values of H.

7. The number of independent components of a total symmetric mth-rank tensor in n-dimensional space is

$$Z(m, n) = \frac{(n + m - 1)!}{(n - 1)!m!}.$$

Confirm this formula by a proof of induction. In a triclinic crystal, how large is the number of independent components of an eighth-rank tensor in which the eight positions are divided in pairs of four that are mutually permutable?

8. With the convention introduced in Section 2.2, the position of the basic vectors of the crystal-physical reference system is not uniquely fixed in each case when different equivalent crystallographic arrangements exist. For example, in the PSGs 4/m and 3 both crystallographic reference systems $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ and $\{\boldsymbol{a}'_1 = \boldsymbol{a}_1, \boldsymbol{a}'_2 = -\boldsymbol{a}_2, \boldsymbol{a}'_3 = -\boldsymbol{a}_3\}$ are equivalent (both are right-handed systems and possess the same metric). How do the components c_{1112} and c_{1113}, respectively, of the elasticity tensor transform in a transition from $\boldsymbol{a}_i$ to $\boldsymbol{a}'_i$? Accordingly, one is to check how the components c_{1123} in the PSG $\bar{3}$m transform when an equivalent crystallographic reference system is selected, which results from a 180° rotation about the three-fold axis based initially on the system $\{\boldsymbol{a}_i\}$. In such cases it is necessary to specify the position of the selected reference system via morphological information (indexing certain forms, generated, for example, by spherical growth- or etching methods) or X-ray information (structure factors of certain reflexes) in order to uniquely determine the tensor properties.

9. In which direction appears the largest deviation from a pure longitudinal effect in second-rank tensors (maximum of the angle between inducing and induced vector quantities, for example, in the case of electrical conductivity, between field strength $\boldsymbol{E}$ and current density vector $\boldsymbol{I}$)? The simplest way to carry out the calculation is in the principal axis system. The searched for direction lies in the plane of the smallest and largest principal value.

10. Show that enantiomorphous individuals possess opposite polar effects (polar tensors of odd rank) and that they do not differ in properties described by even-rank tensors when based on the same reference system, respectively. Note that the tensors for right and left individuals in the corresponding right- and left-handed systems have the same tensor components.

11. In double refracting crystals, the maximum ray double refraction appears, to a first approximation, in the direction of the bisector of the largest and smallest semi-axis of the indicatrix when the double refraction is sufficiently small. Prove this assertion.

12. Minimal deflection: when a prism, formed by an optically isotropic medium II or an optically uniaxial crystal, transmits light such that the deflection of the refracted ray is a minimum with respect to the primary ray, then one has for the deflection angle $\overline{\alpha}$ and prism angle φ (only for ordinary rays in optically uniaxial crystals):

$$\frac{\sin(\overline{\alpha}+\varphi)/2}{\sin\varphi/2} = \frac{n_{\mathrm{II}}}{n_{\mathrm{I}}}.$$

Prove this relationship by calculating $d\overline{\alpha}/d\alpha_{\mathrm{I}} = 0$ using the law of refraction. α_{I} is the angle of incidence of the primary ray on the face of a prism. The ray passes symmetrically through the prism in minimal deflection.

13. Calculate from the Lorentz–Lorenz formula

$$\frac{n^2-1}{n^2+2}\cdot\frac{M}{\rho} = R$$

the dependence of the refractive index on hydrostatic pressure under the assumption that the mole refraction R is pressure independent.

14. A fine parallel beam striking a plane parallel plate at perpendicular incidence with normals parallel to an optical axis, experiences a double refraction. Show that the partial rays of all directions of vibration of the incident beam emerge from the crystal on a cylinder with circular cross-section (inner conical refraction). The radius of the cylinder is $r = L\tan\mu$. μ is given by

$$\tan 2\mu = \frac{\sqrt{(n_3^2-n_2^2)(n_2^2-n_1^2)}}{n_1 n_3},$$

where L is the thickness of the crystal plate. Hint: place a Cartesian reference system in the crystal such that the direction of the center principal axis of the indicatrix is parallel $\boldsymbol{e}_2'$ and the direction of the optical axis is parallel $\boldsymbol{e}_1'$. The propagation vector then runs parallel $\boldsymbol{e}_1'$ and an arbitrary $\boldsymbol{D}$-vector of the incident wave has the form

$$\boldsymbol{D} = \boldsymbol{D}_0(\cos\varphi\boldsymbol{e}_2' + \sin\varphi\boldsymbol{e}_3').$$

Now calculate the relation $D_i' = \epsilon_{ij}' E_j'$ by tensor transformation and from this the reversal $E_i' = a_{ij}' D_j'$. This gives the direction of the ray vector $\boldsymbol{s}' \parallel \boldsymbol{E}' \times \boldsymbol{H}'$ with $\boldsymbol{H}' \parallel \boldsymbol{D}' \times \boldsymbol{e}_1'$ as a function of the angle φ and thus a way to calculate the geometrical details.

15. In which direction has a rhombic crystal (calcium formate) a vanishing thermal expansion ($\alpha_{11}' = 0$) when the components of the tensor of thermal expansion have the values $\alpha_{11} = -16.6$, $\alpha_{22} = 68.6$ and $\alpha_{33} = 29.8 \cdot 10^{-6}\mathrm{K}^{-1}$?

16. Derive the conditions existing between the components of the elasticity tensor c_{ijklmn} in isotropic substances. One sets out from the independent components of cubic crystals of the PSG 4/m3 and calculates the effect of an arbitrary n-fold axis, for example, parallel $\boldsymbol{e}_3$, on these components.

17. In the cubic PSG there does not exist a pure converse piezoelectric longitudinal effect ($\hat{d}'_{221}, \hat{d}'_{331} \neq 0$ for any arbitrary electric field $\boldsymbol{E} = E_i \boldsymbol{e}_i \parallel \boldsymbol{e}'_1$). Why?

18. In all crystals possessing the subgroup 22, no change in volume is generated by the first-order electrostrictive effect. Why?

19. The converse piezoelectric effect can be used to construct an electronic position transducer, which experiences a defined change in length proportional to the applied electric voltage. Which voltage must be applied to a 5 mm thick quartz plate, cut perpendicular to a two-fold axis, in order to achieve a change in length of 100 Å(=0.01 μm)? $d_{111} = 2.31 \cdot 10^{-12}$m/V.

20. Why is the deviation from Ohm's law in bismuth (PSG $\bar{3}$m) first observed in the third power of the electric field strength?

21. Under which angle to the six-fold axis in $LiIO_3$ (PSG 6) does phase matching of the fundamental wave and the frequency doubled harmonic appear for the vacuum wavelength $\lambda = 1.06\mu$m? The refractive indices are

$$n_0^{\nu} = 1.860, \quad n_e^{\nu} = 1.719, \quad n_0^{2\nu} = 1.901, \quad n_e^{2\nu} = 1.750.$$

22. With uniaxial tension along $\boldsymbol{e}_i$ one observes, aside from longitudinal dilatation ε_{ii} in the direction of tension, a lateral contraction ε_{jj} perpendicular to the direction of tension. The Poisson relation

$$\nu_{ij} = \frac{\varepsilon_{jj}}{\varepsilon_{ii}} = \frac{s_{iijj}}{s_{iiii}}$$

specifies the magnitude of lateral contraction. Prove that this relation in cubic crystals, for tension along a cube edge, is isotropic and assumes the value $-c_{12}/(c_{11} + c_{12})$ for all directions perpendicular to the edge. Furthermore, derive the lateral contraction relation in rhombic crystals for tension parallel to a rhombic principal direction $\boldsymbol{a}_i$ and for a lateral contraction in the direction $\boldsymbol{a}_j$ perpendicular to this principal direction. It is

$$\nu_{ij} = \frac{-c_{ik}c_{jk} + c_{ij}c_{kk}}{c_{jj}c_{kk} - c_{jk}^2},$$

where $i, j \neq k$.

23. Which form has the elasticity tensor of a cubic crystal when the Cartesian reference system is so selected that $\boldsymbol{e}_3$ is parallel to a three-fold axis, hence

$$\boldsymbol{e}_3 = (\boldsymbol{a}_1 + \boldsymbol{a}_2 + \boldsymbol{a}_3)/a\sqrt{3},$$

and $\boldsymbol{e}_1$ is parallel to a bisector of two cubic principal axes ($\boldsymbol{e}_1 = (\boldsymbol{a}_1 - \boldsymbol{a}_2)/a\sqrt{2}$)?

24. For crystals with subgroup 22, prove, on the basis of the expressions given in Table 4.14 for both ρv^2-values c' and c'' of the waves vibrating in the plane spanned by $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$, that the relationships, $c' - c'' = c_{iijj} + c_{ijij}$ are approximately true, under weak anisotropy conditions, when $\boldsymbol{g} = (\sqrt{2}/2)(\boldsymbol{e}_i \pm \boldsymbol{e}_j)$.

25. In many cases, the physical properties of polycrystalline aggregates can be easily calculated from the properties of single crystals when simple assumptions are made concerning the grain distribution of the aggregate and reasonable boundary conditions are introduced for the transfer of inducing quantities from grain to grain. As an example, we consider the elastic constants of a polycrystalline aggregate. In a first model case, we assume that the deformations are homogeneously distributed over all grains, in a second, that the stress propagates homogeneously through all grains. If the crystals possess anisotropic elastic properties, then both assumptions are only approximately correct. In the first case (Voigt case) we can directly use the relation $\sigma_{ij} = c_{ijkl}\varepsilon_{kl}$ to obtain an average value, in the second case (Reuss case) we must apply the relation $\varepsilon_{ij} = s_{ijkl}\sigma_{kl}$. We calculate the properties of an isotropic medium conveniently by first assuming cubic crystal symmetry and then setting the condition that the anisotropy of the cubic medium is canceled by, for example, letting the material quantities along [100], [110] or [111] and in other directions respectively, take on the same values. Also, the condition of simultaneously fulfilling hexagonal and cubic symmetry leads to isotropy in crystals. The elasticity tensor in cubic crystals has only the three constants $c_{11} = c_{1111}$, $c_{12} = c_{1122}$ and $c_{66} = c_{44} = c_{1212}$.

The condition of isotropy is given when, for example, the longitudinal resistances along [100] and [110] are equal, hence,

$$c_{1111} = (c_{1111} + c_{1122} + 2c_{1212})/2$$

and thus $2c_{44} = c_{11} - c_{12}$. We now demand that the single crystal and the polycrystalline aggregate exhibit the same scalar invariants. In the case of the c-tensor we are dealing with $I_1 = c_{ijkl}\delta_{ij}\delta_{kl}$ and $I_2 = c_{ijkl}\delta_{ik}\delta_{jl}$.

One finds for the single crystal

$$I_1 = c_{1111} + c_{2222} + c_{3333} + 2(c_{1122} + c_{2233} + c_{3311}),$$
$$I_2 = c_{1111} + c_{2222} + c_{3333} + 2(c_{1212} + c_{2323} + c_{3131}).$$

The polycrystalline aggregate gives: $\bar{I}_1 = 3\bar{c}_{1111} + 6\bar{c}_{1122}$ and $\bar{I}_2 = 3\bar{c}_{1111} + 6\bar{c}_{1212} = 6\bar{c}_{1111} - 3\bar{c}_{1122}$ (with $2c_{1212} = c_{1111} - c_{1122}$). This allows the calculation of the elastic constants of the aggregate:

$$\bar{c}_{1111} = \bar{c}_{11} = (I_1 + 2I_2)/15 \quad \text{and} \quad \bar{c}_{1122} = \bar{c}_{12} = (2I_1 - I_2)/15.$$

Completely analogous expressions are valid for the coefficients $\bar{s}_{ijkl}$ and and the corresponding invariants. For cubic crystals, calculate the difference in the values $\bar{c}_{11}$ and $\bar{c}_{12}$ for both cases (homogeneous deformation and homogeneous stress, respectively). The latter is obtained by matrix inversion. In practice, it turns out that the arithmetic mean of both pairs of values comes close to those observed experimentally (see also Kröner, 1958).

26. Show that the number of independent components of the elasticity tensors, in the case of triclinic crystals, is reduced from 21 to 18 when one selects Cartesian axes parallel to the deformation vectors of the three elastic waves propagating in the direction of a dynamic longitudinal effect c'_{1111} (here, a pure longitudinal wave and two pure transverse waves exist; c'_{1111} takes on an extreme value, as shown in Section 4.5.5).

Furthermore, prove that in monoclinic crystals the elastic constant c'_{15} ($= c'_{1113}$) vanishes when one rotates the Cartesian reference system by an angle φ about the axis $\boldsymbol{e}_2$ ($\|2$ and m, respectively) so that the longitudinal effect c'_{1111} takes on an extreme value ($dc'_{1111}/d\varphi = 0$!).

27. Show that in all PSGs with the subgroup 22 no piezoelectric coupling occurs with elastic waves in the propagation direction $\boldsymbol{e}_i$ ($c^E = c^D$).

28. In cubic crystals the energy current in the propagation of elastic transverse waves along the three-fold axis forms a finite angle ζ with the wave normals (elastic internal refraction). Calculate this angle with the help of the relationship discussed at the end of Section 4.5.5. For this purpose, the components s_j of the ray vector, which are proportional to c'_{1j13}, are to be calculated for $j = 1, 2, 3$ in a reference system $\{\boldsymbol{e}'_i\}$, whose basic vector $\boldsymbol{e}'_3$ runs parallel to a three-fold axis. For example, let

$$\boldsymbol{e}'_1 = \frac{1}{\sqrt{2}}(\boldsymbol{e}_1 - \boldsymbol{e}_2), \quad \boldsymbol{e}'_2 = \frac{1}{\sqrt{6}}(\boldsymbol{e}_1 + \boldsymbol{e}_2 - 2\boldsymbol{e}_3), \quad \boldsymbol{e}'_3 = \frac{1}{\sqrt{3}}(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3).$$

In the case of cubic crystals, transformation results in the following components:

$$c'_{1113} = 0, \quad c'_{1213} = (c_{11} - c_{12} - 2c_{44})/3$$

and

$$c'_{1313} = (c_{11} - c_{12} + c_{44})/3.$$

If the deformation vector runs parallel $\boldsymbol{e}'_1$, then one obtains

$$\cos\zeta = \frac{c'_{1313}}{\sqrt{c'^2_{1213} + c'^2_{1313}}}.$$

29. The Grüneisen tensor (after Kitaigorodskii)

$$\gamma_{ij} = \frac{V}{C_V} c^T_{ijkl}\alpha_{kl}$$

is a generalization of the ordinary Grüneisen relation

$$\gamma = \frac{3V}{C_V}\frac{\alpha}{K},$$

where α_{ij} are the components of the tensor of thermal expansion, V is the mole volume, C_V is the specific heat at constant volume (per mole volume) and K is the volume compressibility. Prove the identity of both relations for cubic crystals.

30. Which differences appear between adiabatic and isothermal piezoelectric constants in the following PSGs: 23, 3, mm2?

31. Calculate the form of the magnetoelastic tensor

$$\Delta c_{ijkl} = c_{ijklmn} H_m H_n$$

for the PSG m3. (There exists a total of 13 independent components, as one can immediately calculate from the formula

$$n = \frac{1}{h}\sum_{j=1}^{h} \chi(g_j)!$$

32. Which form has the tensor of first-order magnetostriction in the magnetic PSGs $\underline{\text{m}}3\underline{\text{m}}$, $\underline{\text{m}}3$ and mm$\underline{\text{m}}$ ($\varepsilon_{ij} = m_{ijk}H_k$)?

33. Why does the important scalar product $\boldsymbol{g}\cdot\boldsymbol{G}$ for electrogyration vanish in crystals of the PSG m3 when we have $\boldsymbol{g} \parallel [111]$ and $\boldsymbol{E} \parallel [a\, b . \bar{a} + \bar{b}]$?

34. In the cubic PSGs, which form has the fifth-rank tensor describing the second-order piezoelectric effect according to $D_i = d_{ijklm}\sigma_{jk}\sigma_{lm}$?

35. Calculate the conditions for the direction of the vanishing and maximal longitudinal piezoelectric effect for crystals of the PSG m from the components of the piezoelectric tensor.

36. From the general equation for $\partial v^2/\partial\sigma_{pq}$ derived in Section 4.6.3, derive $\partial c_{1111}/\partial\sigma_{11}$ and $\partial c_{2323}/\partial\sigma_{11}$ for a longitudinal wave and a transverse wave, respectively, which propagate in the direction [100] under uniaxial pressure along [100].

37. The 60 elements of the icosaheder group 235 are distributed over the following five classes: K_1: e, K_2: 2 (15×), K_3: 3^1, 3^2 (each 10×), K_4: 5^1, 5^4 (each 6×) and K_5: 5^2, 5^3 (each 6×). Prove the correctness of the following character table of irreducible representations with the help of the orthogonality relations.

	K_1	K_2	K_3	K_4	K_5
Γ_1	1	1	1	1	1
Γ_2	3	-1	0	$\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{2}(1-\sqrt{5})$
Γ_3	3	-1	0	$\frac{1}{2}(1-\sqrt{5})$	$\frac{1}{2}(1+\sqrt{5})$
Γ_4	4	0	1	-1	-1
Γ_5	5	1	-1	0	0

38. In the cylindrical symmetry, the number of independent tensor components can be calculated from the formula valid for finite groups

$$n = \frac{1}{h}\sum_{j=1}^{h}\chi(g_j),$$

when one sets correspondingly high orders of symmetry of the axes of rotation. Since, however, cylindrical symmetry allows all arbitrary angles of rotation, one can, through a boundary transition, apply the integral

$$n = \frac{1}{2\pi}\int_0^{2\pi}\chi(\varphi)d\varphi$$

instead of the sum.

With the help of this formula calculate the number of independent components for tensors of rank from 1 to 6 for the case of a cylindrical symmetry ∞.

39. As seen in Section 8.3, a second-rank tensor spans a nine-dimensional linear vector space, which can be decomposed into a one-, three- and five-dimensional invariant subspace. In the case of second-rank symmetrical tensors the three-dimensional subspace is empty, and the basic vectors of the remaining subspaces are:

$$V_1 : \ \boldsymbol{e}_1^0 = (\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3)/\sqrt{3}; \quad V_5 : \ \boldsymbol{e}_2^0 = (\boldsymbol{e}_1 + \boldsymbol{e}_2 - 2\boldsymbol{e}_3)/\sqrt{6},$$
$$\boldsymbol{e}_3^0 = (\boldsymbol{e}_1 - \boldsymbol{e}_2)/\sqrt{2}, \quad \boldsymbol{e}_4^0 = \boldsymbol{e}_4, \quad \boldsymbol{e}_5^0 = \boldsymbol{e}_5, \quad \boldsymbol{e}_6^0 = \boldsymbol{e}_6.$$

In highly symmetric crystals, it is often convenient to describe the relationship between two second-rank tensors in the coordinates of these subspaces, the so-called symmetry coordinates. For example, Hooke's law in cubic crystals has the following form:

$$\begin{aligned}
\sigma_1^0 &= \sigma_{11} + \sigma_{22} + \sigma_{33} = (c_{1111} + 2c_{1122})(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \\
&= (c_{11} + 2c_{12})\varepsilon_1^0, \\
\sigma_2^0 &= \sigma_{11} + \sigma_{22} - 2\sigma_{33} = (c_{1111} - c_{1122})(\varepsilon_{11} + \varepsilon_{22} - 2\varepsilon_{33}) \\
&= (c_{11} - c_{12})\varepsilon_2^0, \\
\sigma_3^0 &= \sigma_{11} - \sigma_{22} = (c_{1111} - c_{1122})(\varepsilon_{11} - \varepsilon_{22}) = (c_{11} - c_{12})\varepsilon_3^0, \\
\sigma_4^0 &= \sigma_{23} = 2c_{2323}\varepsilon_{23} = c_{44}\varepsilon_4^0, \\
\sigma_5^0 &= \sigma_{13} = 2c_{3131}\varepsilon_{31} = c_{44}\varepsilon_5^0, \\
\sigma_6^0 &= \sigma_{12} = 2c_{1212}\varepsilon_{12} = c_{44}\varepsilon_6^0,
\end{aligned}$$

Verify these relationships and establish that the quantities ε_i^0 and σ_i^0 connected with the elastic constants

$$c_1^0 = c_{11} + 2c_{12}, \quad c_2^0 = c_3^0 = (c_{11} - c_{12})$$

and

$$c_4^0 = c_5^0 = c_6^0 = c_{44}$$

transform as the spherical harmonic functions

$$Y_0, \quad Y_2^0, \quad Y_2^2 + Y_2^{-2}, \quad Y_2^1 + Y_2^{-1}, \quad Y_2^1 - Y_2^{-1}, \quad Y_2^2 - Y_2^{-2}$$

$$\left(Y_0 \sim \frac{x_1^2 + x_2^2 + x_3^2}{r^2} = 1, \quad Y_2^0 \sim \frac{1}{r^2}(x_1^2 + x_2^2 - 2x_3^2), \right.$$
$$Y_2^2 + Y_2^{-2} \sim \frac{1}{r^2}(x_1^2 - x_2^2), \quad Y_2^1 + Y_2^{-1} \sim \frac{1}{r^2}x_2x_3,$$
$$\left. Y_2^1 - Y_2^{-1} \sim \frac{1}{r^2}x_1x_3, \quad Y_2^2 - Y_2^{-2} \sim \frac{1}{r^2}x_1x_2 \right).$$

That means, that all quantities of a relationship $\sigma_i^0 = c_i^0\varepsilon_i^0$ transform to the same type of symmetry.

40. How many scalar invariants has a fifth-rank tensor of type $t_{3\times3\times3\times(3\times3)_S}$, a sixth-rank tensor of type $t_{(3\times3)_S\times(3\times3)_S\times(3\times3)_S}$ and a third-rank pseudotensor of type $t_{3\times(3\times3)_S}$?