Student name:

1 Prove that each of the statements below is false by finding a counterexample.

a If x and y are any two real numbers, then $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.

b If x and y are any two real numbers, then $\sqrt{x^2 + y^2} = x + y$.

c If x is any real number, then $x^2 > \frac{x}{2}$.

d If m and n are any two natural numbers then $m^n > mn$.

e If m and n are any two natural numbers, then $n^2 - m^2$ is not a prime number.

f If α and β are any two angles, then $\sin(\alpha+\beta) = \sin(\alpha) + \sin(\beta)$.

2 Disprove each of these existence statements by proving that its negation is true.

a There exists some natural number n such that $n^2 + 7n + 12$ is a prime number.

b There exists some natural number n such that $25n^2 - 9$ is a prime number.

c There exists positive integers m and n such that $m^2 + n^2 = 11$.

d There exists real numbers x and y such that such that $x^2 = 2xy - y^2 - \frac{1}{100}$.

3 Suppose that m is an integer.

a Prove that if m is divisible by 4 and n is divisible by 2 then mn is divisible by 8.

b Prove that the converse is not true.

4 Suppose that m and n are integers.

a Prove that if $m - n$ is divisible by 4 then $m^4 - n^4$ is divisible by 4.

b Prove that the converse is not true.

Answers to Chapter 6 Skillsheet 6D

Note: there are many different answers possible for these questions. You need only find one counterexample.

1 **a** Let $x = 9$ and $y = 16$. Then clearly,

$$\sqrt{x+y} = \sqrt{9+16} = \sqrt{25} = 5$$

while

$$\sqrt{9} + \sqrt{16} = 3 + 4 = 7.$$

b Let $x = 3$ and $y = 4$. Then,

$$\sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = 5.$$

while

$$x + y = 3 + 4 = 7.$$

c If $x = \frac{1}{4}$ then $x^2 = \frac{1}{16} < \frac{1}{8} = \frac{x}{2}$.

d If $m = 2$ and $n = 2$ then $m^n = 2^2 = 4 = mn$.

e If $m = 3$ and $n = 4$ then $n^2 - m^2 = 4^2 - 3^2 = 7$ is a prime number.

f If $\alpha = 90°$ and $\beta = 90°$ then

$$\sin(\alpha + \beta) = \sin(180°) = 0$$

while

$$\sin(\alpha) + \sin(\beta) = \sin(90°) + \sin(90°) = 1 + 1 = 2$$

2 **a** The number $n^2 + 7n + 12$ is never a prime number since

$$n^2 + 7n + 12 = (n+3)(n=4)$$

is clearly the product of two numbers, both of which are greater than 1.

b The number $25n^2 - 9$ is never a prime number since

$$25n^2 - 9 = (5n-3)(n+3)$$

is clearly the product of two numbers, both of which are greater than 1.

c We need to prove that there are no positive integers m and n such that $m^2+n^2=11$. To see this, we simply need to check the sums of squares less than 11. We can assume that $m \le n$. Then

$$\begin{aligned}
1^2+2^2&<11\\
1^2+2^2&<11\\
1^2+3^2&<11\\
2^2+2^2&<11\\
2^2+3^2&>11\\
3^2+3^2&>11
\end{aligned}$$

In no instance do we get the required sum. Therefore, there are no positive integers m and n such that $m^2+n^2=11$.

d We need to show that for all real numbers x and y,

$$x^2 \neq 2xy-y^2-\frac{1}{100}$$

To this end, we note that

$$x^2-2xy+y^2=(x-y)^2 \ge 0$$

Therefore,

$$x^2 \ge 2xy-y^2 > 2xy-y^2-\frac{1}{100}.$$

2 **a** If m is divisible by 4 and n is divisible by 2 then $m=4k$ and $n=2j$ for integers k and j. Therefore,

$$mn=(4k)(2j)=8kj$$

is divisible by 8, as required.

b The converse is not true. For example, if $m=8$ and $n=1$ then $mn=8$ is divisible by 8 whereas n is not divisible by 2.

3 **a** Suppose that $m-n$ is divisible by 4 so that $m-n=4k$ for some integer k. Therefore,

$$\begin{aligned}
m^4-n^4&=(m^2-n^2)(m^2+n^2)\\
&=(m-n)(m+n)(m^2+n^2)\\
&=4k(m+n)(m^2+n^2)
\end{aligned}$$

is divisible by 4.

b The converse is not true. For example, if $m = 8$ and $n = 2$ then

$$8^4 - 2^4 = 4080$$

is divisible by 4 whereas $8 - 2 = 6$ is not.