MATHEMATICS
SPECIALIST
UNIT 1

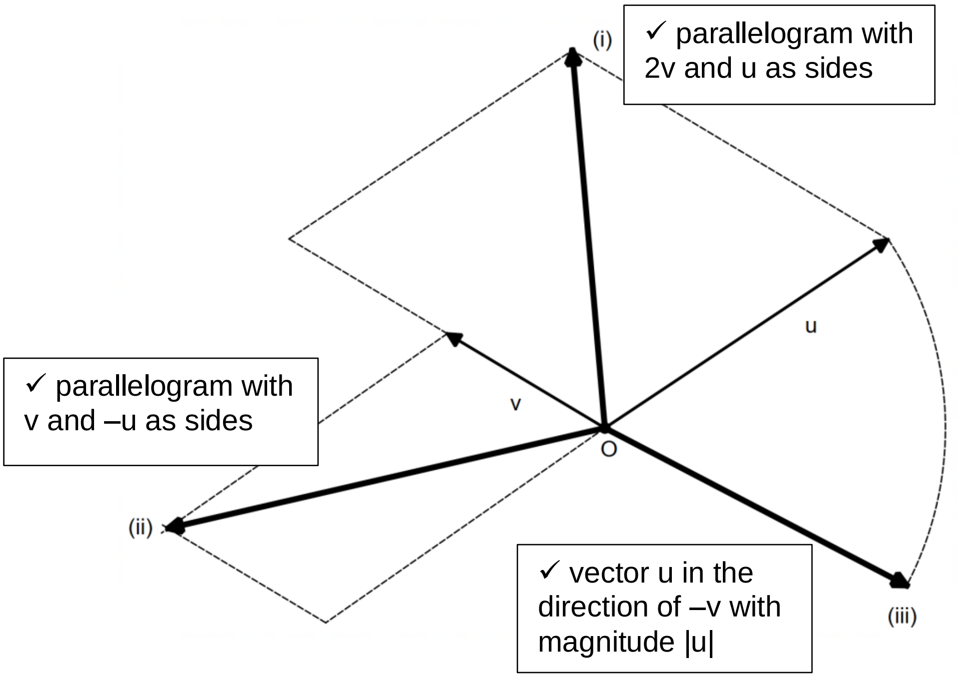
Semester One

2019

SOLUTIONS

Calculator–free Solutions

1.



[6]

2. (a) (i) $\begin{pmatrix} 3 \\ -5 \end{pmatrix} = k\begin{pmatrix} 4 \\ \alpha \end{pmatrix}$ ✓

$\therefore k = \frac{3}{4} \quad \rightarrow \quad \alpha = -\frac{5}{k} = -\frac{20}{3}$ ✓

(ii) $\begin{pmatrix} 4 \\ \alpha \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$ ✓

$\therefore 4 + \alpha = 0 \quad \rightarrow \quad \alpha = -4$ ✓

(iii) $\overrightarrow{PQ} = \begin{pmatrix} 1 \\ \alpha + 5 \end{pmatrix}$

since the *x*–coordinate is already 1 unit in length, then the y–coordinate must be zero. ✓

$\therefore \alpha = -5$ ✓

(iv) PQ as base ⇒ |OP| = |OQ|

$\left|\begin{matrix} 3 \\ -5 \end{matrix}\right| = \left|\begin{matrix} 4 \\ \alpha \end{matrix}\right| \quad \rightarrow \quad \sqrt{34} = \sqrt{16 + \alpha^2}$ ✓

$\therefore \alpha^2 = 18 \quad \rightarrow \quad \alpha = \pm 3\sqrt{2}$ ✓

2. (b) (i) Solving simultaneously (any method, elimination shown below):

$$\begin{array}{lll} \boldsymbol{u} = -6\boldsymbol{i} \ -2\boldsymbol{j} & \times 3 \\ \boldsymbol{v} = \ \ 2\boldsymbol{i} \ +3\boldsymbol{j} & \times 2 \end{array} \rightarrow \begin{array}{l} 3\boldsymbol{u} = -18\boldsymbol{i} \ -6\boldsymbol{j} \\ 2\boldsymbol{v} = \ \ 4\boldsymbol{i} \ +6\boldsymbol{j} \end{array} \downarrow (+)$$

$\therefore \ 3\boldsymbol{u} + 2\boldsymbol{v} = -14\boldsymbol{i} \quad \rightarrow \quad \boldsymbol{i} = -\frac{3}{14}\boldsymbol{u} - \frac{1}{7}\boldsymbol{v}$ ✓✓

similarly (or by substitution):

$$\begin{array}{lll} \boldsymbol{u} = -6\boldsymbol{i} \ -2\boldsymbol{j} & \times 1 \\ \boldsymbol{v} = \ \ 2\boldsymbol{i} \ +3\boldsymbol{j} & \times 3 \end{array} \rightarrow \begin{array}{l} \boldsymbol{u} = -6\boldsymbol{i} \ -2\boldsymbol{j} \\ 3\boldsymbol{v} = \ \ 6\boldsymbol{i} \ +9\boldsymbol{j} \end{array} \downarrow (+)$$

$\therefore \ \boldsymbol{u} + 3\boldsymbol{v} = 7\boldsymbol{j} \quad \rightarrow \quad \boldsymbol{j} = \frac{1}{7}\boldsymbol{u} + \frac{3}{7}\boldsymbol{v}$ ✓✓

(ii) $\boldsymbol{r} = 14\left(-\frac{3}{14}\boldsymbol{u} - \frac{1}{7}\boldsymbol{v}\right) + 7\left(\frac{1}{7}\boldsymbol{u} + \frac{3}{7}\boldsymbol{v}\right)$

$\therefore \boldsymbol{r} = -2\boldsymbol{u} + \boldsymbol{v}$ ✓✓ [14]

3. (a) (i) $20! - 18! = 20 \times 19 \times 18! - 18!$ ✓

$= (20 \times 19 - 1) \times 18!$

$= (380 - 1)k = 379k$ ✓

(ii) $\dfrac{{}^{20}P_3}{{}^{21}C_3} = \dfrac{20!}{3!} \div \dfrac{21!}{3! \times 18!}$ ✓

$= \dfrac{20!}{3!} \times \dfrac{3! \times 18!}{21 \times 20!} = \dfrac{k}{21}$ ✓

(b) RHS $= \binom{n}{n-r} = \dfrac{n!}{(n-r)! \times [n-(n-r)]!}$ ✓

$= \dfrac{n!}{(n-r)![r]!}$ ✓

$= \binom{n}{r} =$ LHS [6]

4. (a) If $m < 1$, then $m > m^2$. ✓

It is NOT always true because it does not work for negatives. ✓

e.g. $m = -2 < 1 \quad \rightarrow \quad m^2 = 4 > m \quad \therefore$ false ✓

The converse is always true for $0 < m < 1$ ✓

(b) If the parallelogram is not a rectangle, then it does not have congruent diagonals. ✓

Yes it is always true as only squares and rectangles have congruent diagonals. ✓

(c) For all rational numbers ✓, there exists two integer numbers a and b ✓

such that p is the quotient of a and b. [8]

5. (a) (i) $2^6 = 1 + 6 + 15 + 20 + 15 + 6 + 1 = 64$ ✓

(ii) $11^5 = (10 + 1)^5$

$= 10^5 + 5 \times 10^4 + 10 \times 10^3 + 10 \times 10^2 + 5 \times 10 + 1^5$ ✓

$= 100\,000 + 50\,000 + 10\,000 + 1\,000 + 50 + 1$

$= 161051$ ✓

(b) (i) $x = 3$ since ${}^6C_3 = 20$ ✓

(ii) $x = 7$ since ${}^7C_5 = 21$ ✓

(iii) $x = 8$ since ${}^8C_2 = {}^8C_6$ ✓

(c) $(2x - y)^5$

$= (2x)^5 + 5(2x)^4(-y)^1 + 10(2x)^3(-y)^2 + 10(2x)^2(-y)^3 + 5(2x)^1(-y)^4 + (-y)^5$ ✓

$= 32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5$ ✓✓

(d) (i) ${}^8C_5 = 56$ ✓

(ii) ${}^2C_2 \times {}^6C_3 = 1 \times 20 = 20$ ✓

(iii) ${}^3C_2 \times {}^5C_3 + {}^3C_3 \times {}^5C_2$ ✓

$= 3 \times 10 + 1 \times 10 = 30 + 10 = 40$ ✓ [13]

6. (a) $\overrightarrow{AD} = \frac{2}{5}\overrightarrow{AB} = \frac{2}{5}(\boldsymbol{b} - \boldsymbol{a})$ ✓

$\overrightarrow{CD} = \overrightarrow{CO} + \overrightarrow{OA} + \overrightarrow{AD} = -\frac{1}{2}\boldsymbol{b} + \boldsymbol{a} + \frac{2}{5}(\boldsymbol{b} - \boldsymbol{a})$ ✓

$\therefore \overrightarrow{CD} = \frac{3}{5}\boldsymbol{a} - \frac{1}{10}\boldsymbol{b}$ ✓

(b) $\overrightarrow{OC} + \overrightarrow{CE} = \overrightarrow{OE} \quad \rightarrow \quad \overrightarrow{OC} + \beta\overrightarrow{CD} = \alpha\overrightarrow{OA}$ given

$\therefore \frac{1}{2}\boldsymbol{b} + \beta\left(\frac{3}{5}\boldsymbol{a} - \frac{1}{10}\boldsymbol{b}\right) = \alpha\boldsymbol{a}$ ✓

$\times 10 \quad \rightarrow \quad 5\boldsymbol{b} + 6\beta\boldsymbol{a} - \beta\boldsymbol{b} = 10\alpha\boldsymbol{a}$

$\rightarrow \quad (6\beta - 10\alpha)\boldsymbol{a} = (\beta - 5)\boldsymbol{b}$

since a and b are non–parallel, then:

$\beta - 5 = 0 \quad \rightarrow \quad \beta = 5$ ✓

$6\beta - 10\alpha = 0 \quad \rightarrow \quad \alpha = \frac{3}{5}\beta = 3$ ✓ [6]

Calculator–assumed Solutions

7. (a) ABC collinear ⇒ AB // BC

$\overrightarrow{AB} = \begin{pmatrix}-2\\4\end{pmatrix} - \begin{pmatrix}4\\-5\end{pmatrix} = \begin{pmatrix}-6\\9\end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix}-6\\10\end{pmatrix} - \begin{pmatrix}-2\\4\end{pmatrix} = \begin{pmatrix}-4\\6\end{pmatrix}$

$\therefore \begin{pmatrix}-6\\9\end{pmatrix} = k\begin{pmatrix}-4\\6\end{pmatrix} \quad \rightarrow \quad \begin{matrix} k = \frac{-6}{-4} = \frac{3}{2} \\ k = \frac{9}{6} = \frac{3}{2} \end{matrix}$ ✓✓

Since k is unique, then AB // BC and hence ABC collinear. ✓

(b) $|AB| = \left|\begin{matrix}-6\\9\end{matrix}\right| = 3\left|\begin{matrix}-2\\3\end{matrix}\right|$ and $|BC| = \left|\begin{matrix}-4\\6\end{matrix}\right| = 2\left|\begin{matrix}-2\\3\end{matrix}\right|$ ✓

$\therefore AB{:}BC = 3{:}2$ ✓ [5]

8. (a) ∠PFO = 35° ✓

Because ΔOFP is isosceles since |OP| = |OF| = radii ✓

(b) ∠FEP = 55° ✓

Since ∠FOP = 110° from ΔOFP, and the angle at the centre is double the size of the angle at the edge. ✓

(c) ∠PQF = ∠FEP = 55° ✓

Angles at the circumference within the same segment are congruent. ✓

(d) ∠CFP = ∠FEP = 55° ✓

The alternate segment theorem ✓

(e) |GC| = |CF| = 11 – |FB| = 11 – 8 = 3 cm ✓

Tangents to a circle from the same external point are congruent. ✓

(f) |AM|×(|AM| + 2×radius) = $|AH|^2$

|AM|×(|AM|+8) = 5^2 ✓

$|AM|^2$+8|AM| – 25 = 0

CAS ⇒ |AM| = $-4 \pm \sqrt{41}$ ✓

∴ |AM| = $\sqrt{41} - 4 \approx 2.40$ cm only solution ✓ [13]

9. (a) (i) Divisible by 3 and 5 = divisible by 15

100 ÷ 15 = 6.6 ⇒ only 6 elements are divisible by 15 ✓

Therefore, assuming every other element is chosen

instead of those 6, we need 100 – 6 +1 = 95 elements ✓

(ii) Divisible by 3 = 100 ÷ 3 = 33.3 ⇒ 33 elements

Divisible by 5 = 100 ÷ 5 = 20 elements ✓

Divisible by 3 or 5 = 33 + 20 – 6 = 47 elements ✓

Assuming the other 53 elements are chosen first,

then 53 + 1 = 54 elements must be chosen ✓

(b) Assuming the highest numbers are chosen first:

100 + 99 + 98 + … + 91 + 90 = 955 ✓

If 89 is chosen next then the sum exceeds 1000. ✓

Therefore, a maximum of 11 elements must be chosen. ✓ [8]

10. (a) $n(M \cup C) = n(M) + n(C) - n(M \cap C)$ ✓

14 334 ✓ = 7 531 + 9 885 – $n(M \cap C)$

∴ $n(M \cap C)$ = 3 082 households ✓

(b) $n(M \cup C \cup B) = n(M) + n(C) + n(B)$

$- n(M \cap C) - n(M \cap B) - n(C \cap B)$

$+ n(M \cap C \cap B)$ ✓

∴ $n(M \cup C \cup B)$ = 7 531 + 9 885 + 4 977 – 3 082 – 2 252 – 4 310 + 1 724

= 14 473 that have all three ✓

Therefore, 16 366 – 14 473 = 1 893 households have neither ✓ [6]

11. (a) (i) ${}^{36}P_4 = 1\,413\,720 \quad OR \quad ({}^{36}C_4 \times 4!)$ ✓

(ii) ${}^{10}C_2 \times {}^{26}C_2 \times 4! = 351\,000$ ✓✓

(iii) ${}^{5}C_1 \times {}^{34}P_2 \times {}^{4}C_1 = 22\,440$ ✓✓

(b) II and III ✓✓

(c) ${}^{x+1}P_3 = {}^{4}C_3 \times {}^{x}P_2$

$\frac{(x+1)!}{(x+1-3)!} = 4 \times \frac{x!}{(x-2)!}$ ✓

$\frac{(x+1) \times x!}{(x-2)!} = 4\frac{x!}{(x-2)!}$ ✓

$(x + 1) = 4 \quad \rightarrow \quad x = 3$ ✓

11. (d) LHS $= \frac{n!}{(n-2)!} + 2n \times \frac{(n-1)!}{(n-1)!}$ ✓

$= \frac{n!}{(n-2)!} \times \frac{(n-1)}{(n-1)} + \frac{2n!}{(n-1)!}$ ✓

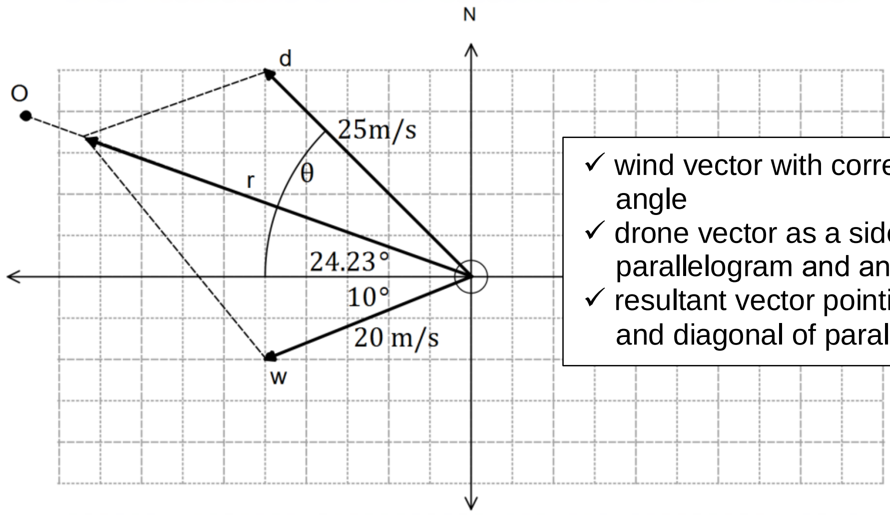
$= \frac{n! \times (n-1) + 2n!}{(n-1)!} = \frac{n!(n-1+2)}{(n-1)!}$ ✓

$= \frac{n! \times (n+1)}{(n+1-2)!} = \frac{(n+1)!}{(n+1-2)} = {}^{n+1}P_2 =$ RHS ✓ [14]

12. (a) $\boldsymbol{w} = \begin{pmatrix} -20\cos 10° \\ -20\sin 10° \end{pmatrix}$

Hovering speed $= -\boldsymbol{w} = \begin{pmatrix} 20\cos 10° \\ 20\sin 10° \end{pmatrix}$ ✓✓

(b) (i)



✓ wind vector with correct x-axis angle
✓ drone vector as a side of parallelogram and angle θ
✓ resultant vector pointing towards O, and diagonal of parallelogram

(ii) $w = \begin{pmatrix} -20\cos 10° \\ -20\sin 10° \end{pmatrix}$ $d = \begin{pmatrix} -25\cos\theta \\ 25\sin\theta \end{pmatrix}$ $r = \begin{pmatrix} -r\cos 24.23° \\ r\sin 24.23 \end{pmatrix}$ ✓✓✓

(iii) $-r\cos 24.23° = -20\cos 10° \quad -25\cos\theta$
$r\sin 24.23° = -20\sin 10° \quad 25\sin\theta$

$\therefore$ $25^2\cos^2\theta = (r\cos 24.23° + 20\cos 10°)^2$
$25^2\sin^2\theta = (r\sin 24.23° + 20\sin 10°)^2$

$\rightarrow 25^2 = (r\cos 24.23° + 20\cos 10°)^2 + (r\sin 24.23° + 20\sin 10°)^2$ ✓✓

CAS $\rightarrow$ $r = 38.8621$ m/s OR $r = -5.7897$ m/s ✓

$\rightarrow$ $\theta = 46.64°$ OR $\theta = 2.94°$ ✓

$\therefore time = \frac{d}{v} = \frac{\sqrt{1200^2+540^2}}{38.8621} =$ 33.86 seconds ✓

bearing $= 270° + \theta = 308.86°T$ ✓ [14]

13. (a) $\sum F_y = 300\sin 62° + 252\sin 56°$ ✓

$= 473.8N$ ✓

Since $473.8N < 500N$ the machinery is not moving upwards ✓

(b) No horizontal component needed ⇒ $\sum F_x = 0$

$\therefore 400\cos 62° = x\cos 56°$ ✓

$\rightarrow x = \frac{400\cos 62°}{\cos 56°} = 335.82N$ ✓

(c) $\sum F_y = 400\sin 62° + 335.82\sin 56°$ ✓

$= 631.59N$ ✓ [7]

14. (a) (i) Let $n \in \mathbb{N}$ with $n = 2k + 1 =$ odd ✓

Then $n^2 + 1 = (2k+1)^2 + 1$

$= 4k^2 + 4k + 2$ ✓

$= 2(2k^2 + 2k + 1)$ ✓

Since 2 is a factor, then $n^2 + 1$ is divisible by 2, and

hence the conjecture is true $\forall\, n \in \mathbb{N}$ ✓

(ii) Contrapositive statement:

"if $n^2 + 1$ is odd, then n is even." ✓

Let $n^2 + 1 = odd = 2k + 1$

$\therefore n^2 = 2k$ ✓

$\Rightarrow n^2 = even \quad \Rightarrow \quad n = even$ ✓

Since the contrapositive statement is true $\forall\, n \in \mathbb{N}$,
then the original conjecture is true $\forall\, n \in \mathbb{N}$ ✓

(b) A ⇒ B:
If the quadrilateral has two diagonals that intersect at right angles, then the quadrilateral is a rhombus, which implies it does have two pairs of parallel sides.

∴ A ⇒ B is true ✓

B ⇒ A:
If the quadrilateral has two pairs of parallel sides then it is a parallelogram, which does not necessarily imply it is a rhombus, and therefore it does not necessarily have diagonals that intersect at right angles.

∴ B ⇒ A is false ✓

Therefore, A ⇔ B is a false statement. ✓

(c) Assume that n is odd and n^2 is even. ✓

Then $\exists\, k \in \mathbb{N}: n = 2k + 1$

$\to n^2 = (2k+1)^2 = 4k^2 + 4k + 1$ ✓

$= 2(2k^2 + 2k) + 1 = 2m + 1 = odd$ ✓

Since n^2 is both even and odd simultaneously, this is a contradiction ✓

and therefore the original conjecture must be true $\forall\, n \in \mathbb{N}, n$ even. [15]

15. (a) (i) $|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB|\cos\theta$ ✓

(ii) $\overrightarrow{AB} \cdot \overrightarrow{AB} = |OA|^2 + |OB|^2 - 2|OA||OB|\cos\theta$

LHS $= \left(\overrightarrow{OB} - \overrightarrow{OA}\right) \cdot \left(\overrightarrow{OB} - \overrightarrow{OA}\right)$ ✓

$= \overrightarrow{OB} \cdot \overrightarrow{OB} - \overrightarrow{OB} \cdot \overrightarrow{OA} - \overrightarrow{OA} \cdot \overrightarrow{OB} + \overrightarrow{OA} \cdot \overrightarrow{OA}$ ✓

$= |OB|^2 + |OA|^2 - 2\,\overrightarrow{OA} \cdot \overrightarrow{OB}$ ✓

$\therefore |OB|^2 + |OA|^2 - 2\,\overrightarrow{OA} \cdot \overrightarrow{OB} = |OA|^2 + |OB|^2 - 2|OA||OB|\cos\theta$

$\to -2\,\overrightarrow{OA} \cdot \overrightarrow{OB} = -2|OA||OB|\cos\theta$ ✓

$\to \overrightarrow{OA} \cdot \overrightarrow{OB} = |OA||OB|\cos\theta$ as required

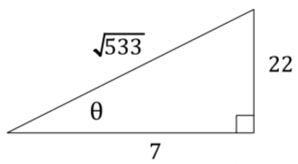
(b) (i) $\begin{pmatrix} 4 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{vmatrix} 4 \\ 5 \end{vmatrix} \times \begin{vmatrix} 2 \\ -3 \end{vmatrix} \cos\theta$

$8 - 15 = \sqrt{41} \times \sqrt{13}\cos\theta$ ✓

$\therefore \cos\theta = -\frac{7}{\sqrt{533}}$ ✓

Since $\cos\theta < 0 \;\Rightarrow\; \theta$ is obtuse ✓

(ii)



$\Rightarrow \sin\theta = \frac{22}{\sqrt{533}}$ ✓

$\therefore$ area ΔOAB $= \frac{1}{2}|OA||OB|\sin\theta$

$= \frac{1}{2}\sqrt{41} \times \sqrt{13} \times \frac{22}{\sqrt{533}} = 11$ units^2 ✓ [10]

16. P, Q, R and S are the midpoints of their respective sides:

$\overrightarrow{OP} = \begin{pmatrix} -0.5 \\ 4 \end{pmatrix} \quad \overrightarrow{OQ} = \begin{pmatrix} 6 \\ 1.5 \end{pmatrix} \quad \overrightarrow{OR} = \begin{pmatrix} 1.5 \\ -4 \end{pmatrix} \quad \overrightarrow{OS} = \begin{pmatrix} -5 \\ -1.5 \end{pmatrix}$ ✓

Therefore:

$\overrightarrow{PQ} = \begin{pmatrix} 6 \\ 1.5 \end{pmatrix} - \begin{pmatrix} -0.5 \\ 4 \end{pmatrix} = \begin{pmatrix} 6.5 \\ -2.5 \end{pmatrix}$

$\overrightarrow{SR} = \begin{pmatrix} 1.5 \\ -4 \end{pmatrix} - \begin{pmatrix} -5 \\ -1.5 \end{pmatrix} = \begin{pmatrix} 6.5 \\ -2.5 \end{pmatrix}$ ✓

$\overrightarrow{PQ} = \overrightarrow{SR} \quad \Rightarrow \therefore \ \overrightarrow{PQ}$ // $\overrightarrow{SR}$ ✓

$\overrightarrow{PS} = \begin{pmatrix} -5 \\ -1.5 \end{pmatrix} - \begin{pmatrix} -0.5 \\ 4 \end{pmatrix} = \begin{pmatrix} -4.5 \\ -5.5 \end{pmatrix}$

$\overrightarrow{QR} = \begin{pmatrix} 1.5 \\ -4 \end{pmatrix} - \begin{pmatrix} 6 \\ 1.5 \end{pmatrix} = \begin{pmatrix} -4.5 \\ -5.5 \end{pmatrix}$ ✓

$\overrightarrow{PS} = \overrightarrow{QR} \quad \Rightarrow \therefore \ \overrightarrow{PS}$ // $\overrightarrow{QR}$ ✓

Since $\overrightarrow{PQ}$ // $\overrightarrow{SR}$ and $\overrightarrow{PS}$ // $\overrightarrow{QR}$ $\Rightarrow$ $PQRS$ is a parallelogram [5]