Student name:

Provide a proof by contradiction for each of the following questions.

1 Prove that every quadrilateral has some interior angle with a magnitude of at least 90°.

2 Suppose that a is an irrational number and b is a non-zero rational number. Prove that then ab is an irrational number.

3 **a** Prove that $\log_2 3$ is an irrational number.

b Hence, prove that $4 + \log_2 3$ is also an irrational number.

4 Prove that there are no integers m and n such that $3m + 21n = 5$.

5 Prove that there are no positive integers m and n such that $m^2 + n^2 = 21$.

6 Prove that the numbers N and $2N$ cannot both be square numbers. Hint: you can use the fact that $\sqrt{2}$ is irrational.

7 **a** Find the smallest natural number N such that $24N$ is a square number.

b Prove by contradiction that there is no natural number N such that $24N$ and $18N$ are both square numbers.

Answers to Chapter 13 Skillsheet 13A

1 Assume the contrary. Then there is some quadrilateral whose angles are all less than 90°. But then the angle sum would be less than 360°, which is a contradiction.

2 Assume the contrary. Then ab is a rational number. This means that

$$ab = \frac{m}{n} \qquad (1)$$

for integers m and n. Since b is rational, this means that

$$b = \frac{p}{q} \qquad (2)$$

for integers p and q. Substitute (2) into (1) to obtain

$$\frac{ap}{q} = \frac{m}{n}$$

$$a = \frac{mq}{np}$$

This mean that a is rational, which is a contradiction.

3 **a** Assume the contrary. Then $\log_2 3$ is a rational number. This means that

$$\log_2 3 = \frac{m}{n} \quad (1)$$

for integers m and n. Therefore

$$2^{\frac{m}{n}} = 3$$

$$\left(2^{\frac{m}{n}}\right)^n = 3^n$$

$$2^m = 3^n$$

The left-hand side of this expression is even and the right-hand side is odd. This is a contradiction. Therefore $\log_2 3$ is not rational.

b Assume the contrary. Then

$$4+\log_2 3=\frac{m}{n}$$

for integers m and n. Therefore

$$\begin{aligned}4+\log_2 3&=\frac{m}{n}\\ \log_2 3&=\frac{m}{n}-4\\ &=\frac{m-4n}{n}\end{aligned}$$

However, this would mean that $\log_2 3$ would be a rational number. This is a contradiction, as we already proved that it is not.

4 If $3m+21n=5$, then

$$\begin{aligned}3m+21n&=5\\ 3(m+7n)&=5\end{aligned}$$

Notice that the left hand side is divisible by 3. However, the right hand side is not divisible by 3, so we obtain a contradiction.

5 Suppose there are integers m and n such that $m^2+n^2=21$. Then

$$m^2=21-n^2.$$

We can then consider cases for n.

$n=1\Rightarrow m^2=20\Rightarrow m=\sqrt{20}$ is not an integer

$n=2\Rightarrow m^2=17\Rightarrow m=\sqrt{17}$ is not an integer.

$n=3\Rightarrow m^2=12\Rightarrow m=\sqrt{12}$ is not an integer.

$n=4\Rightarrow m^2=5\Rightarrow m=\sqrt{5}$ is not an integer.

$n\geq 5\Rightarrow m^2<0\Rightarrow m$ is not a real number.

We obtained a contradiction in each case. Therefore there are no integers m and n such that $m^2+n^2=21$.

6 If N and $2N$ were both squares then $n^2 = N$ and $m^2 = 2N$ for some pair of integers n and m. Eliminating N from these equations gives

$$\frac{m^2}{n^2} = 2$$

$$\frac{m}{n} = \sqrt{2}$$

which would imply that $\sqrt{2}$ is rational. This is clearly a contradiction.

7 **a** Since $24 = 2^3 \times 3$, for $24N$ to be a square we require additional prime factors 2 and 3. Therefore the smallest such N will be $N = 2 \times 3 = 6$.

b Suppose there were a natural number such that $24N$ and $18N$ were both square numbers. Therefore,

$n^2 = 18N$ and $m^2 = 24N$

Since $24 = 2^3 \times 3$, for $24N$ to be a square the number N must contain an odd number of additional factors of 2 and 3.

Since $18 = 2 \times 3^2$, for $18N$ to be a square the number N must contain an odd number of additional factors of 2 and an even number of factors of 3.

Since N cannot have both an even and odd number of factors of 3, we obtain a contradiction.