Student name:

1 Prove that if n is odd, then $n^2 + n$ is even.

2 Prove that if m is a multiple of 2 and n is a multiple of 3 then mn^2 is a multiple of 18.

3 If m and n are any two integers, prove that $(m+n)^2 + (m-n)^2$ is always even.

4 Suppose that n is an odd integer and m is an even integer. Prove that $2n + 3m + 1$ is an odd integer.

5 Suppose that n is an odd integer. Prove that $n^2 + 4n + 7$ is divisible by 4.

6 **a** Prove that for every integer n, the number $(n+1)^2 - n^2$ is an odd number.

b Use your above answer to express 21 as the difference of two square numbers.

7 Prove that $x^2 - x \geq -\frac{1}{4}$ for every real number x.

8 Prove that if a and b are positive real numbers then $\sqrt{ab} \geq \frac{2ab}{a+b}$.

9 Suppose that m and n are natural numbers, where $m > n$. Prove that if $m^2 - n^2$ is a prime number, then $m = n + 1$.

Answers to Chapter 6 Skillsheet 6A

1 If n is odd then $n = 2m+1$ for some integer m. Therefore,

$$\begin{aligned} n^2+n &= (2m+1)^2+2m+1 \\ &= 4m^2+4m+1+2m+1 \\ &= 4m^2+6m+2 \\ &= 2(2m^2+3m+1) \end{aligned}$$

is even.

2 If m is a multiple of 2 and n is a multiple of 3 then $m=2k$ and $n=3j$ for integers k and j. Therefore,

$$\begin{aligned} mn^2 &= (2k)(3j)^2 \\ &= 2k\times 9j^2 \\ &= 18(kj^2) \end{aligned}$$

is a multiple of 18.

3 Expand the brackets to show that

$$\begin{aligned} (m+n)^2+(m-n)^2 &= m^2+2mn+n^2+m^2-2mn+n^2 \\ &= 2m^2+2n^2 \\ &= 2(m^2+n^2) \end{aligned}$$

is even.

4 If n is an odd integer and m is an even integer then $n=2k+1$ and $m=2j$ for integers k and j. Therefore,

$$\begin{aligned} 2n+3m+1 &= 2(2k+1)+3(2j)+1 \\ &= 4k+2+6j+1 \\ &= 2(2k+3j+1)+1 \end{aligned}$$

is odd.

5 If n is an odd integer then $n=2k+1$ for some integer k. Therefore,

$$\begin{aligned} n^2+4n+7 &= (2k+1)^2+4(2k+1)+7 \\ &= 4k^2+4k+1+8k+4+7 \\ &= 4k^2+12k+12 \\ &= 4(k^2+3k+3) \end{aligned}$$

is divisible by 4.

6 **a** Expanding the brackets gives,

$$\begin{aligned}(n+1)^2 - n^2 &= n^2 + 2n + 1 - n^2 \\ &= 2n+1\end{aligned}$$

which is odd.

b Let $n = 10$ in the above equation to give $11^2 - 10^2 = 21$.

7 If we complete the square on the left-hand side we obtain,

$$\begin{aligned}x^2 - x &= x^2 - x + \frac{1}{4} - \frac{1}{4} \\ &= \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} \\ &\geq -\frac{1}{4}\end{aligned}$$

as the term in brackets is non-negative.

8 This can be proved in many ones. Note that the approach shown below *only* works because each step is reversible.

$$\begin{aligned}&\sqrt{ab} \geq \frac{2ab}{a+b} \\ &\Leftrightarrow ab \geq \left(\frac{2ab}{a+b}\right)^2 \\ &\Leftrightarrow ab \geq \frac{4a^2b^2}{(a+b)^2} \\ &\Leftrightarrow 1 \geq \frac{4ab}{(a+b)^2} \\ &\Leftrightarrow (a+b)^2 \geq 4ab \\ &\Leftrightarrow a^2 + 2ab + b^2 \geq 4ab \\ &\Leftrightarrow a^2 - 2ab + b^2 \geq 0 \\ &\Leftrightarrow (a-b)^2 \geq 0\end{aligned}$$

9 Since $m^2 - n^2 = (m-n)(m+n)$ is a prime number and $m + n > 1$, this must mean that $m - n = 1$, or else the number wouldn't be prime. Therefore $m = n + 1$