2.3 – REAL AND COMPLEX NUMBERS (Year 11 Specialist)

Key Symbols / Definitions

Type	Symbol	Explanation	Examples
Real Numbers	$\mathbb{R}$	All numbers, no restrictions	$\frac{3}{7}, -4, \sqrt{3}, -6.7$
Integers	$\mathbb{Z}$	All positive and negative whole numbers (and zero)	-4, -3, -2, -1, 0, 1, 2, 3, 4, 5
Natural Numbers	$\mathbb{N}$	Positive Whole Numbers	1, 2, 3, 4
Rational Numbers	$\mathbb{Q}$	Any number that can be a fraction with two integers $\frac{p}{q}$ where $q \neq 0$ and p and q have no common factors	$3.5 = \frac{7}{2}, -5 = \frac{-5}{1}, 0.\dot{3} = \frac{1}{3}$
Odd number	$2m+1$ for some $m \epsilon Z$	Any integer doubled will be positive, then adding 1 makes it odd, e.g. m = 4, 2 x 4 + 1 = 9	3, 7, 11, -5, -11
Even number	$2m$ for some $m \epsilon Z$	Any integer doubled will be positive, e.g. m = 4, 2 x 4 = 8	2, 10, -18, 26
Divisible by "a"	am for some $m \epsilon Z$	Divisible means it leaves no remainder when divided by "a". In other words the number is a multiple of "a"	21 is divisible by 3 since 21 = 3 x 7 24 is divisible by 2 since 24 = 2 x 12
Perfect square	There exists a^2 for some $a \epsilon Z$	The number is a square number so the square root is a whole number	9, 16, 25, 81
Prime Number	There does not exist ab for some $a,b \in Z$ and $a,b > 1$	The only factors are 1 and itself, for example 7 has no factors except 1 and 7	5, 11, 17, 29
Composite Number	There does exist ab for some $a,b \in Z$ and $a,b > 1$	There is at least one pair of factors which doesn't include one, for example 10 is composite because 2 x 5 = 10	6, 18, 33, 93

FORMULA SHEET

Natural Numbers:	$\mathbb{N} := \{1,2,3, \ldots\}$
Integer Numbers:	$\mathbb{Z} := \{\ldots -2, -1, 0, 1, 2, \ldots\}$
Rational Numbers:	$\mathbb{Q} := \left\{q : q = \frac{a}{b},\ \text{where a and b are integers},\ b \neq 0\right\}$
Irrational Numbers:	Numbers that cannot be expressed as the ratio of two integers
Real Numbers:	The set of all rational and irrational numbers ($\mathbb{R}$)
Complex Numbers:	$\mathbb{C} := \{z : z = ai + b,\ \text{where } a, b \in \mathbb{R},\ i^2 = -1\}$

KEY PARTS OF A PROOF

- Prove only if asked to prove (Prove, clearly show etc.)
- CLEARLY DEFINE ANY VARIABLES YOU ARE USING (use $k \in \mathbb{Z}$ etc.)
- SHOW ALL STEPS
- CLEARLY VALIDATE ANYTHING REQUIRED (i.e. $a^2 \geq 0$ or a^2 will always be positive)
- FINAL PROOF STATEMENT – Make sure you make statements at the end to finish proof (i.e. The contrapositive is true, this implies that the original statement is true or Q.E.D or Justification)
- Never write ……. = ……… to start and manipulate this equation, you have to prove both sides are equal.
- If proved, you can write Q.E.D. = Latin abbreviation for quod erat demonstrandum: "Which was to be demonstrated."
- Only equals, are in a line vertically, meaning this is equal to the line above
- Write a final summary sentence of your findings
- ANY NUMBER SQUARED WILL BE POSITIVE
- Instead of proving a > b, prove a – b > 0 is much easier to justify, usually by square numbers or positive factors

NOTE: When proof starts as an equation, if you manipulate this equation to get to a stage which is true, you must write your proof in the opposite order (You can never start with what you must prove).

KEY SYMBOLS

SYMBOL	$\Rightarrow$	$\Leftrightarrow$	$\not\Leftrightarrow$	$\forall$	$'$	$\exists$
What It means	IMPLICATION This implies	EQUIVALENCE Both imply each other **If and only if**	Don't imply each other	For all	Not	There exists
Example	$x = 2 \Rightarrow 2x = 4$	$x = 2 \Leftrightarrow 2x = 4$	$x = 2 \not\Leftrightarrow x^2 = 4$	$\forall\ x \in \mathbb{R}, x^2 \geq 0$	Rain' = Not Raining	$\exists$ x,y $\in \mathbb{R}$ where x + y = 10

PROVING SOMETHING IS FALSE

It is a lot easier to prove something is false as you need to just show examples or use maths to prove incorrect.

- PROOF BY CONTRADICTION, Assume the statement is true, then prove that it is actually false.
  FOR ALL QUESTIONS $\forall\ P \Rightarrow Q$
- find any COUNTER EXAMPLE which is any situation where the statement is not correct. For example, if someone said all prime numbers are odd, this is false since 2 is a prime number (this is a counter example).
  THERE EXISTS QUESTIONS $\exists\ P \Rightarrow Q$
- Prove the Negation is False (The negation of $\exists\ P \Rightarrow Q$ is $\forall\ P \Rightarrow Q$"

CONTRAPOSITIVE. CONVERSE, INVERSE AND NEGATION

If we start with a TRUE statement like:

- If it is raining, there are clouds (which can also be written rain ⇒ clouds)
- If x = 3, then x^2 = 9 (which can also be written as x = 3 ⇒ x^2 = 9

Type	What it is	Symbol	Rain example	x = 3 example	Is it True
CONTRAPOSITIVE	Opposite things in opposite order	$P \Rightarrow Q$ $Q' \Rightarrow P'$	If there is no clouds, it is not raining	If $x^2 \neq 9$, then $x \neq 3$	YES, if the original was true
CONVERSE	Swapping the order	$P \Rightarrow Q$ $Q \Rightarrow P$	If there is clouds, it is raining	If $x^2 = 9$, then x = 3	Possibly Not, In these cases No $\not\Leftrightarrow$
INVERSE	Opposite things, same order	$P \Rightarrow Q$ $P' \Rightarrow Q'$	If it is not raining, there are no clouds	If $x \neq 3$, then $x^2 \neq 9$	Possibly Not, In these cases No
NEGATION	Swap "for all" and "there exists" and opposite result	$\forall P \Rightarrow Q$ $\exists P \Rightarrow Q'$	For all wet days, there are clouds. There exists some wet days with no clouds	$(\forall n \in \mathbb{N})\ 2n \geq n+1$ $(\exists n \in \mathbb{N})\ 2n < n+1$	If original was true, then No

Complete the following

True Statement	***Contrapositive Statement (True / False)***	***Converse Statement (True / False)***	***Inverse Statement (True / False)***
If today is Monday then tomorrow is Tuesday			
If a number is a multiple of 6, then it is even			
If x = 4, then x^2 = 16			
If a is even, then 2a is even.			

DIRECT PROOFS

Use the information given to you as fact, and then mathematically manipulate this to prove the rest.

Prove that if a is even and b is even, then a + b is even.

Prove that if a is an odd number and b is an odd number, then ab is also odd.

Suppose that n is an odd integer. Prove that $n^2 + 4n + 7$ *is divisible by 4.*

DIRECT PROOFS WITH GREATER THAN OR EQUAL TO

- Whenever you have to prove greater than or equal to, instead of proving a > b, prove a – b > 0 is much easier to justify, <u>usually</u> by justifying that anything squared is positive or positive factors multiply to give a positive answer.
  NOTE: When proof starts as an equation, if you manipulate this equation to get to a stage which is true, you must write your proof in the opposite order (You can never start with what you must prove).

Prove that for $n \in \mathbb{R}$, $n^2 + 1 \geq 2n$

Prove that for $a, b \in \mathbb{R}$ $\frac{a^2+b^2}{2} \geq \left(\frac{a+b}{2}\right)^2$

Let x and y be positive real numbers. Show that for $x > y$ *then* $x^4 > y^4$

PROOF BY CONTRAPOSITIVE

If the Contrapositive is True (opposite things in opposite order), then the original statement is true.
Often needed since the Left part of proof is more complicated than the right part, you must make final statement that since the contrapositive was true, the original statement was true.

Use proof by contraposition to prove that if mn *is even and* $m+n$ *is even, then both* m *and* n *are even*

a) *Use proof by contraposition to prove that if* n^2 *is even, then* n *is even, where* $n \in \mathbb{Z}$.

b) *Hence or otherwise prove that, if* x *and* y *are integers and if* $x^2 + y^2$ *is even, then* $x + y$ *is even.*
(HENCE in Maths usually means use your previous answer to help you)

Prove that if n^2 is divisible by 3, then n is divisible by 3. Hint: Prove the contrapositive by considering two cases.

Hence, prove that $\sqrt{3}$ is irrational.

IF AND ONLY IF (EQUIVALENCE)

If $P \Rightarrow Q$ and the converse $Q \Rightarrow P$, is also true, they are equivalent $Q \Leftrightarrow P$. Therefore, if and only if requires you to prove left to right (original) and right to left (converse).

The number n is even if and only if $n^2 + 2n + 1$ is odd.

PROVE FALSE

It is a lot easier to prove something is false as you need to just show counter examples or use maths to prove incorrect.

FOR ALL QUESTIONS $\forall\ P \Rightarrow Q$

- find any COUNTER EXAMPLE which is any situation where the statement is not correct. For example, if someone said all prime numbers are odd, this is false since 2 is a prime number (this is a counter example).

THERE EXISTS QUESTIONS $\exists\ P \Rightarrow Q$

- Prove the Negation is False (The negation of $\exists\ P \Rightarrow Q$ is $\forall\ P \Rightarrow Q'$

Prove the following are false

For every natural number n, the number $2n^2 - 4n + 7$ is prime

There exists $n \in N$ such that $9n^2 - 1$ is a prime number.

PROOF BY CONTRADICTION

Say the opposite is true, and when you try to prove it you see that mathematically it can't be proved.

Suppose that $5^x = 2$ *, prove that x is irrational*

Suppose that a is rational and b is irrational. Prove that a + b is irrational.

PROVE BY INDUCTION – Covered later in Course in Semester 2

Prove that mathematical statement P(n) works for n=1, then prove that P(k+1) is true if P(k) is true

Prove that $1+3+5+.....(2n-1)=n^2$

REAL NUMBERS

CONVERTING BETWEEN FRACTIONS AND DECIMALS

<u>Converting from Fractions to Decimals</u>

Either convert the fraction to one over 100, since dividing by 10 is easy, or do division.

For example $\frac{16}{25} = \frac{16\times4}{25\times4} = \frac{64}{100} = 64 \div 100 = 0.64$, $\frac{3}{7} \neq \frac{\quad}{100}$ so $\begin{array}{r|l} & 0.\ 4\ \ 2\ \ 8\ \ 5\ \ 7\ \ 1\ \ 4\ldots \\ 7 & 3.\ {}^{3}0\ {}^{2}0\ {}^{6}0\ {}^{4}0\ {}^{5}0\ {}^{1}0\ {}^{3}0\ldots \end{array}$ $= 0.\dot{4}2857\dot{1}$

Convert the following to decimals.

$\frac{9}{20}$ $\frac{3}{11}$ $\frac{4}{15}$ $\frac{11}{27}$

<u>Converting from Decimals to Fractions</u>

<u>If Not Recurring</u>

Multiply decimal by as many tens that converts it to a whole number, and put it at top of fraction, put these 10s as the bottom number of fraction, then simplify the fraction

Eg. $0.45 = \frac{0.45}{1} \times \frac{100}{100} = \frac{45}{100} = \frac{9}{20}$

<u>If Recurring</u>

Use the technique shown

Eg. $n = 0.\dot{5}\dot{4}$

$100n = 54.\dot{5}\dot{4}$ I have completed this step to try and remove the decimal part and leave whole numbers for my fraction

Now if I take these away from each other

$100n - 1n = 54.\dot{5}\dot{4} - 0.\dot{5}\dot{4}$ $\quad 99n = 54$ $\quad n = \frac{54}{99} = \frac{6}{11}$

Convert the following to fractions

$0.\dot{4}\dot{5}$ $\quad 0.\dot{3}0769\dot{2}$ $\quad$ *1.36363636* $\quad$ *0.2666666666*

Prove that $7.\overline{53}$ is a rational number.

A set of real numbers is given by $\{\pi,\ \sqrt{5},\ 0.\overline{36},\ \sqrt[3]{10}\}$. Identify the rational number and clearly show that it satisfies the definition of a rational number.

COMPLEX NUMBERS – Covered later in Course in Semester 2

FORMULA SHEET

For $z = ai + b$, where $a, b \in \mathbb{R}$, $i^2 = -1$

Modulus: $\text{mod } z = |z| = |a+ib| = \sqrt{a^2+b^2}$

Product: $|z_1 z_2| = |z_1||z_2|$

Conjugate: $\bar{z} = a - ib$, $z\bar{z} = |z|^2$, $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$, $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

$\boxed{\sqrt{-1} = i}$ $\quad \sqrt{-25} =$ $\quad \sqrt{-32} =$ $\quad i^3 =$

Basic Form

$z = x + iy$ $\quad \bar{z} = x - iy$ $\quad \text{Re}(z) = x$ $\quad \text{Im}(z) = y$

Conjugate ($\bar{z}$)

Simplify

$(2+3i)+(4-5i)$ $\quad 3(4-2i)-(5-8i)$

$(2+3i)(4-5i)$ $\quad (3-2i)^2$

$\frac{4}{5-i}$ $\quad \frac{2-i}{3+2i}$

Solve

$z^2 + 50 = 0$ $\quad (z-1)^2 + 32 = 0$

$z^2 - 4z + 20 = 0$ $\quad z^2 - 6z + 100 = 0$