Chapter 16 Counting Techniques

If we wish to find the number of elements in a set an obvious way is to list all the elements belonging to the set and then count them. This technique is acceptable for a set with a small number of elements, but not practical for sets with a large number of elements.

For example if we wanted to known the number of 'words' (a word does not have to make sense) which can be made using only two letters of the word SET, the number can be found easily by listing the elements of the set, that is {SE, ES, ST, TS, ET, TE}. The number of words is 6.

However, this method becomes impractical if we wish to perform the task of finding how many three letter words can be made using the letters of the word SECONDARY.

In this chapter we will develop techniques of counting which will enable us to perform tasks such as the one above.

Methods of counting the number of ways of performing a task

Method A Suitable diagram method.
- this method is rather limited.

Method B Tree diagram method.
- this method gives a list of **each** choice in question.

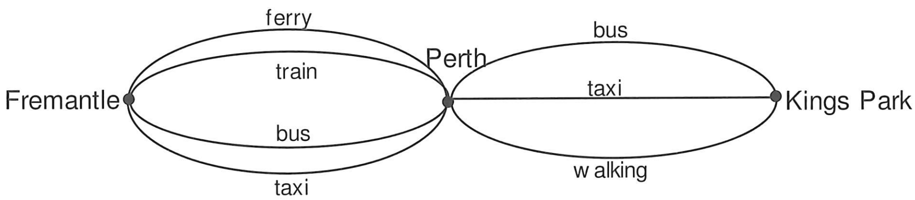
Method C Box diagram method.
- this method gives only the number of choices.

Example 1

A visitor to Western Australia wishes to travel from Fremantle to Kings Park via the city centre of Perth. The visitor has a choice of ferry, train, bus or taxi from Fremantle to Perth and from Perth to Kings Park the visitor has a choice of bus, taxi or walking. In how many ways can the visitor complete the journey from Fremantle to Kings Park via the city centre of Perth.

Solution

Method A Suitable diagram method.



There are 12 ways of completing the journey.

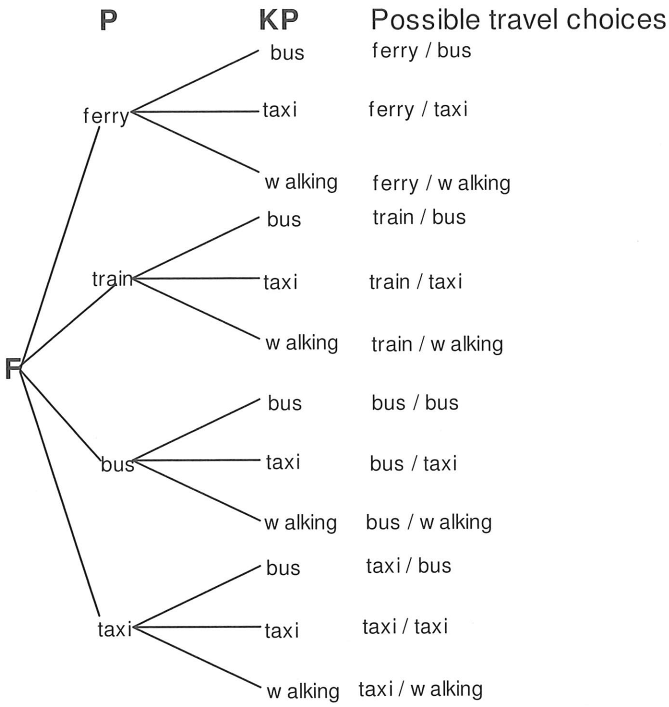
On reflection the method above involved the following steps:

Step 1 Breaking the task of travelling from Fremantle to Kings Park via the city centre of Perth into two stages, that is
Stage 1 Travelling from Fremantle to Perth,
Stage 2 Travelling from Perth to Kings Park.

Step 2 Counting the number of travel choices at each stage, that is
Stage 1 Travelling from Fremantle to Perth there are 4 modes of travel available.
Stage 2 Travelling from Perth to Kings Park there are 3 modes of travel available.

Step 3 Multiplying the number of choices at stage 1 and stage 2 to obtain the number of ways of completing the journey, that is $4 \times 3 = 12$.

Method B Tree diagram method.



This method gives a list of each choice and hence the number of ways the visitor can complete the journey, that is 12 ways.

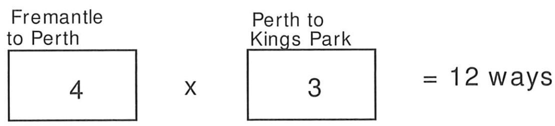
On reflection the tree diagram method above involved the following steps:

Step 1 Breaking the task of travelling from Fremantle to Kings Park via the city centre of Perth into two stages, that is
Stage 1 Travelling from Fremantle to Perth,
Stage 2 Travelling from Perth to Kings Park.

Step 2 Listing on the diagram the number of travel choices at each stage, that is
Stage 1 Travelling from Fremantle to Perth there are 4 modes of travel available.
Stage 2 Travelling from Perth to Kings Park there are 3 modes of travel available.

Step 3 Listing all the available choices and then adding them up. However if we just consider the number of ways only of travelling from Fremantle to Kings Park via Perth, then this is readily found by multiplying the number of choices at each stage, that is 4 x 3 = 12.

Method C Box diagram method.



or more simply $\boxed{4}\boxed{3} = 12$ or $4 \times 3 = 12$

The box diagram method above involved the following steps:

Step 1 Breaking the task of travelling from Fremantle to Kings Park via the city centre of Perth into two stages, that is
Stage 1 Travelling from Fremantle to Perth,
Stage 2 Travelling from Perth to Kings Park.

Step 2 Counting the number of travel choices at each stage, that is
Stage 1 Travelling from Fremantle to Perth there are 4 modes of travel available.
Stage 2 Travelling from Perth to Kings Park there are 3 modes of travel available.

Step 3 Multiplying the number of choices at stage 1 and stage 2 to obtain the number of ways of completing the journey, that is 4 x 3 = 12.

MULTIPLICATION PRINCIPLE

Each of the above methods involved the same set of steps to arrive at the solution to the problem. This set of steps is summarised below and known as the **multiplication principle**.

> **Multiplication Principle**
> To find the number of ways of performing a given task follow the steps below:
> STEP 1 Break the task into stages.
> STEP 2 Count the number of choices at stage one, and after one of these choices has been taken, count the number of choices at stage two, and so on.
> STEP 3 Multiply the number of choices at each stage to determine the number of ways of performing the given task

Example 2

How many motor cycle number plates are possible if each number plate must have two letters, excluding the letters I and O, followed by two digits if:

(a) Repetitions of letters and digits are not allowed.
(b) Repetitions are allowed.

Solution

(a) **Step 1** There are four stages,

Stage 1: selecting the first letter of the number plate,
Stage 2: selecting the second letter of the number plate,
Stage 3: selecting the first digit of the number plate, and
Stage 4: selecting the second letter of the number plate.

Step 2 The number of choices at each stage are,

Stage 1: 24 letters to choose from as I and O must be excluded,
Stage 2: 23 letters to choose from as I and O must be excluded and repetitions are not allowed,
Stage 3: 10 digits to choose from,
Stage 4: 9 digits to choose from as repetitions are not allowed.

Step 3 The number of number plates possible $= 24 \times 23 \times 10 \times 9$
$= 49680$

(b) The number of number plates possible $= 24 \times 24 \times 10 \times 10$
$= 57600$

NOTE: When solving problems choose the method which is usually implied by the question.

EXERCISE 16A

1. There are 5 roads from town A to town B. In how many ways may a person travel by road from A to B and back if the person
   (a) may not use the same road more than once?
   (b) may use the same road more than once?

2. How many different number plates are possible using three letters followed by three digits?

3. A coin is tossed and a die rolled. How many different outcomes are possible?

4. A cafe offers 7 different types of coffee and 4 different kinds of cake. How many different choices has Catherine if she wishes to enjoy a coffee and cake?

5. In how many ways can all the letters of the word ANSWER be arranged?

6. In how many ways may a cricket team of 11 players stand in

   (a) one row for a photograph if the captain must stand in the middle?

   (b) two rows if 6 players are in the back row and 5 in the front row?

7. A PIN (personal identification number) comprises of four digits. How many PINS are possible if

   (a) there are no restrictions?

   (b) a PIN may not contain repeated digits?

   (c) all PINS must be even?

   (d) a PIN cannot start or end with a zero?

8. Using the letters of EDUCATION how many "words" can be formed with

   (a) five letters?

   (b) three letters?

   (c) three letters if the middle letter must be an E?

   (d) five letters if the last letter must be a vowel?

   (Note: A "word" does not have to make sense and each letter can only be used once)

9. A telephone has ten touch pads numbered 0 to 9. In how many ways can a number consisting of four digits be dialled if repetition of the digits is allowed?

10. Three prizes, a first prize, a second prize and a third prize are to be awarded to a class of 18 students. In how many ways can this be done if,

    (a) no one is allowed more than one prize?

    (b) there are no restriction?

11. To enter the Royal Show at the main entrance a married couple have a choice of five turnstiles. In how many ways can they enter the showgrounds if,

    (a) there are no restrictions?

    (b) they both must go through the same turnstile?

    (c) they cannot go through the same turnstile?

12 At the central post office there are six mailboxes next to one another. I have 3 letters to post. In how many ways can I post my letters if,

   (a) each letter must be mailed in a different box?

   (b) the letters are not necessarily mailed in different boxes?

13. Five boys and five girls are to be seated in a row of ten chairs numbered from 1 to 10. In how many ways can they be seated if,

    (a) there are no restrictions?

    (b) a girl must sit in chair 1 and a girl must sit in chair 10?

    (c) a boy must sit in chair 1 and the girls and boys alternate?

    (d) boys and girls must alternate?

    (e) girls must occupy chairs numbered 1, 2, 9 and 10?

    (f) boys must sit together?

14. Five people enter a room in which there are seven chairs. In how many ways may they be seated?

15. Considering only the following digits 2, 3, 6, 7, 9 how many 3 digit numbers can be made if,

(a) there are no restrictions?

(b) they must be even?

(c) digits cannot be repeated?

(d) digits cannot be repeated and the numbers must be even?

16. To gain access to a computer a 5 digit number must be entered. If you knew the password had the following digits 3, 4, 5, 6, 7 in some order with the 3 first and the 7 last and no repetitions, what is the largest number of wrong entries you might enter?

FACTORIAL NOTATION

The product of all counting numbers from n to 1 is called n factorial, and is denoted by n!.

That is $n! = n(n-1)(n-2)(n-3)\ldots 3 \times 2 \times 1.$

Thus we have

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$

NOTE: The number 0! is defined as 1.

Example 3

Evaluate (a) $3! + 3!$ (b) $\frac{9!}{6!}$ (c) $\frac{(n+2)!}{n!}$

Solution

(a)
$$3! + 3! = 3 \times 2 \times 1 + 3 \times 2 \times 1 = 6 + 6 = 12$$

(b)
$$\frac{9!}{6!} = \frac{9 \times 8 \times 7 \times 6!}{6!} = 9 \times 8 \times 7 = 504$$

(c)
$$\frac{(n+2)!}{n!} = \frac{(n+2)(n+1)n!}{n!} = (n+2)(n+1) = n^2 + 3n + 2$$

Your calculator has the ability to compute factorials of positive integers.

5!	
	120
7!	
	5040
3! + 3!	
	12

EXERCISE 16B

1. Evaluate each of the following:

(a) 9! (b) 12! (c) $\frac{12!}{7!}$ (d) $13! - 12!$ (e) $\frac{10!}{7!3!}$

2. Without using the factorial facility on your calculator evaluate each of the following:

(a) $\frac{5!}{2!}$ (b) $\frac{77!}{74!}$ (c) $\frac{100!}{97!3!}$ (d) $5! + 0!$ (e) $\frac{2(10!)}{6!4!}$

3. Simplify (a) $\frac{n!}{(n-1)!}$ (b) $\frac{(n-2)!}{n!}$ (c) $\frac{n!}{5!(n-5)!}$

4. Express each of the following using factorial notation.

(a) $6 \times 5 \times 4 \times 3$ (b) $13 \times 12 \times 11$ (c) $n(n-1)(n-2)$

(d) $n(n+1)(n+2)$ (e) $\frac{6 \times 5}{2 \times 1}$ (f) $\frac{10 \times 9 \times 8}{5 \times 4}$

5. Write each of the following indicated sums as products. (i.e. factorise)

(a) $6! + 5!$ (b) $10! - 8!$ (c) $9! + 3^{2!}8!$

(d) $(n+1)! - n!$ (e) $(n-1)! + (n+1)!$ (f) $n! - (n+1)! + (n-1)!$

6. In how many different orders can the following shopping list be written?

Sugar butter bread milk tea honey

7. In how many ways can a true/false test be answered if there are ten questions in the test and the candidate must answer each question.

PERMUTATIONS (Optional)

A permutation is an arrangement of things or objects in a definite order.
The number of arrangements or permutations may be found using the multiplication principle.

Methods of calculating the number of permutations

Method A Box diagram method.
This method gives the number of permutations.

Method B Tree diagram method.
This method gives a list of each permutation in question.

Method C Permutation notation method.
The notation nPr or ${}^{n}P_{r}$ may be used for the number of permutations or arrangement of r different objects taken from a set which contains n different objects.

That is $\xrightarrow{\text{number of objects to choose from}} {}^{n}P_{r} \xleftarrow{\text{number of objects in each permutation}}$

Hence, ${}^{7}P_{3}$ means the product of 3 successive integers the largest of which is 7.

That is ${}^{7}P_{3} = 7 \times 6 \times 5$
$= 210$

Now ${}^{9}P_{5} = 9 \times 8 \times 7 \times 6 \times 5$
$= 15120$

and ${}^{5}P_{5} = 5 \times 4 \times 3 \times 2 \times 1$
$= 120$ or $5!$

NOTE: If $r = n$, then ${}^{n}P_{r} = {}^{n}P_{n} = n(n-1)(n-2) \ldots 3 \times 2 \times 1 = n!$

Permutations written in factorial form.

We know that ${}^{7}P_{3} = 7 \times 6 \times 5$

Now if we multiply $7 \times 6 \times 5$ by $\frac{4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1}$, that is by 1

we obtain
$$\begin{aligned} {}^{7}P_{3} &= 7 \times 6 \times 5 \times \frac{4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} \\ &= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} \\ &= \frac{7!}{4!} \end{aligned}$$

Now on examining the result above we can see that ${}^{7}P_{3} = \frac{7!}{(7-3)!} = \frac{7!}{4!}$

General case
To find the number of permutations(arrangements) of r objects from a set of n objects we can use the formula

$$^{n}P_{r} = \frac{n!}{(n-r)!} \quad \text{where n and r are non negative integers and } n \geq r$$

The graphics calculator has the ability to compute the number of permutations.

nPr(9,5)	
	15120

Example 4
Consider the set of vowels in the English language, that is, {a, e, i, o, u}.
(a) How many 5 vowel permutations are there?
(b) How many permutations consist of 4 vowels?
(c) How many permutations consist of 2 vowels?
(d) How many 3 vowel arrangements start with the vowel u?
(e) How many 5 vowel permutations have i in the middle?

Solution
(a) The first vowel can be any one of the 5 vowels, the second vowel can be any one of 4 vowels, the third vowel can any one of 3 vowels, the fourth vowel can be any one of 2 vowels and the fifth vowel is the 1 vowel that is left.
Hence the number of permutations = 5 x 4 x 3 x 2 x 1
= 120
There are 120 permutations consisting of the 5 vowels.

(b) The first vowel can be any one of the 5 vowels, the second vowel can be any one of 4 vowels, the third vowel can any one of 3 vowels, the fourth vowel can be any one of the 2 vowels that are left.
Hence the number of permutations = 5 x 4 x 3 x 2
= 120
There are 120 permutations consisting of the 4 vowels.

(c) The first vowel can be any one of the 5 vowels and the second vowel can be any one of the 4 remaining vowels.
Hence the number of permutations = 5 x 4
= 20
There are 20 permutations consisting of the 2 vowels.

(d) The first vowel must be the u and there is just 1 way of selecting this vowel, the second vowel can be one of any 4 vowels and the third vowel can be any one of the remaining 3 vowels.
Hence the number of permutations = 1 x 4 x 3
= 12
There are 12 arrangements starting with u and consisting of 3 vowels.

(e) The first vowel can be any one of 4 vowels as the vowel i must be in the middle, the second vowel can be any one of three vowels, the third vowel must be the vowel i, the fourth vowel can be any one of 2 vowels and the fifth vowel is the 1 vowel that is left.
Hence the number of permutations = 4 x 3 x 1 x 2 x 1
= 24
There are 24 permutations of 5 vowels with the vowel i in the middle.

Example 5
On a shelf above her desk a student has the following subject text books; a Maths text, a Physics text, a History text, a Chemistry text, a Music text and three different English Literature texts.
(a) In how many ways can all these text be arranged on the shelf?
(b) In how many ways can all these texts be permuted if the Maths text must be next to the Music text.
(c) In how many ways can all these texts be arranged on the shelf if the English Literature texts must be together?

Solution
(a) The first position on the shelf may be filled with any of the 8 text books. The next position on the shelf may be filled with any of the remaining 7 text books, the third position by any of the remaining 6 text books and so on.
Hence the number of permutations = 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1
= 8! or 40320
The 8 text books can be arranged in 40320 ways on the shelf.

(b) As the Maths and Music texts have to be next to each other we glue them together and treat them as one text book.
Hence we need to permute 7 text books.
However, when we glue the Maths and Music texts together this can be done in 2! ways, that is Maths-Music or Music-Maths.
Hence the number of permutations = 2! x 7!
= 10080

There are 10080 permutations with the Maths and Music texts next to each other.

(c) As the 3 English Literature texts must be together we glue them together and treat them as one text book and hence we have 6 text books to permute.
Now the 3 English Literature texts can be glued together in 3! ways.
Hence the number of permutations = 3! x 6!
= 4320

There are 4320 permutations with the English Literature together.

Example 6
Answer the following using the letters of the word CHAPTER.
(a) How many permutations of only five letters are possible?
(b) How many of these permutations start with R?
(c) How many "words" of 5 letters start with P and end with T?
(d) How many 5 letter words have C followed immediately by H?
(e) How many 5 letter words have C and H together?
(f) How many 5 letter words have C and H not together?
(g) How many 5 letter words have C, H and A together?

Solution
(a) The first letter can be chosen from 7 different letters, that is it can be chosen in 7 ways, the second letter can be chosen in 6 ways, the third letter can be chosen in 5 ways, the fourth letter can be chosen in 4 ways and the fifth letter can be chosen in 3 ways.
Hence the number of permutations = 7 x 6 x 5 x 4 x 3
= 2520
There are 2520 permutations of 5 letters chosen from the word CHAPTER.

(b) As the permutations must start with R, the first letter can be chosen in 1 way, the second letter can then be chosen in 6 ways, the third letter can be chosen in 5 ways, the fourth letter can be chosen in 4 ways and the fifth letter can be chosen in 3 ways.
Hence the number of permutations = 1 x 6 x 5 x 4 x 3
= 360
There are 360 permutations of 5 letters starting with the letter R.

(c) As the permutations must start with P and end with T the first letter can be chosen in 1 way and the last letter can be chosen in 1 way. The second letter can be chosen in 5 ways, the third letter can be chosen in 4 ways and the fourth letter can be chosen in 3 ways.
Hence the number of permutations = 1 x 5 x 4 x 3 x 1
= 60
There are 60 permutations of 5 letters starting with P and ending in T.

(d) As C is followed by H, that is CH, and CH must be in the five letter word we only have 5 letters available to fill the three remaining places.
Hence the number of permutations starting with CH = 1 x 5 x 4 x 3 = 60
To obtain the total number of 5 letter words under the given condition we must multiply the above result by 4 as CH can take up any of the 4 positions.
Hence the number of permutations = 1 x 5 x 4 x 3 x 4 = 240
There are 240 permutations of 5 letters with C followed by H.

(e) As the permutations must have C and H together, we glue the C and H together we can glue them in 2! ways.
Hence the number of permutations = 1 x 5 x 4 x 3 x 4 x 2! = 480
There are 480 permutations of 5 letters with C and H together.

(f) As the permutations must have C and H not together we make use of the complementary event idea studied in chapter 15.

Number of permutations = Total number – number when together.
= 2520 – 480
= 2040

There are 2040 permutations of 5 letters with C and H not together.

(g) As the permutations must have C, H and A together, we glue the C, H and A together, this can be done in 3! ways.

The number of permutations starting with C, H and A together is = $3! \times 4 \times 3 = 72$

Now the total number of 5 letter words under the given condition is given by multiplying the result above by 3 as the C, H and A arrangement can take up any of 3 positions.

Hence the total number of permutations = $3! \times 4 \times 3 \times 3 = 216$

There are 216 permutations of 5 letters with C, H and A together.

EXERCISE 16C

1. Find the number of different arrangements of the letters of
(a) PERMIT (b) MATHS (c) EDUCATION

2. How many four letter words can be formed using the letters of
(a) RUGBY? (b) SURFING? (c) GOLF?

3. Five people enter a bus which has eight empty seats. In how many ways can these five people be seated?

4. Ten finalists compete in the final of the 100m butterfly at the World Swimming Championships. In how many different ways can
(a) the race finish? (excluding a draw(s)) (b) the medals, gold, silver and bronze be awarded?

5. How many words consisting of seven letters can be formed without vowels and without repetition?

6. Six students have identical school bags. In how many different ways can they pick them up if they have been mixed up?

7. Five extra teachers join the school staff as a result of increased enrolments. They are to be allocated parking bays for their car. In how many ways can the car bays be allocated if
(a) there are exactly five parking bays available? (b) there are exactly nine parking bays available?

8. A van has the following seating plan: 3 seats in the front, 2 in the middle and 3 at the back, that is, it can seat 8 people. A teacher and seven Year 12 students use the van to attend a school function. In how many ways can they be seated, if
(a) the teacher must drive the van? (b) the teacher or one of three students can drive the van?

9. A real estate agency has three display positions on a board in one of its windows and has eight properties to advertise. How many arrangements are possible, if
(a) the most expensive property must be in the middle position on the display board? (b) the two cheapest properties are not to be displayed?

10. Three sets of twins, the Brown twin boys, the Green twin girls and the White twins a boy and a girl, are to be photographed for an article about twins. If each child is of different height in how many ways can they arrange themselves in a row if

(a) there are no restrictions?

(b) the boys must be together and the girls must be together?

(c) the tallest two must be in the middle?

(d) girls and boys must alternate?

(e) the shortest two refuse to stand together?

(f) the family sets must stand next to each other?

(g) they must be in order of height?

11. David has 3 large books, 5 medium sized books and 4 small books. In how many ways can David arrange his books on a shelf

(a) if all the large books are on the left, the small books on the right and the medium sized in between?

(b) if size is not important?

(c) such that books of the same size are together?

(d) if all the large books must be on the right?

(e) such that all the medium sized books are together and no medium sized books are at the ends?

12. The digits 1,2,3,4,5,6,7,8 and 9 are to be used to form a 9 digit code without repetition of digits. How many codes are possible, if

(a) there are no restrictions?

(b) the even digits are on the left and the odd on the right?

(c) the even digits must all be together?

(d) the even and odd digits must alternate?

(e) the three largest digits must be in the middle?

(f) the digits 1 and 9 must be together?

13. In order to choose a music scholarship winner a panel of judges must audition 9 musicians. The musicians are as follows, 4 pianists, 1 oboist and 4 violinists. In how many ways can the auditions be arranged, if

(a) there are no conditions?

(b) the oboist must be first followed by the pianists?

(c) the oboist must be in the middle?

(d) the oboist must be in the middle and the pianists and violinists must alternate?

14. Six different mathematics books, 4 different history books and 3 different science books are to be arranged on a shelf. How many different arrangements are possible, if

(a) there are no restrictions?

(b) the maths books must be on the left, history books in the middle and science books on the right?

(c) the books in each particular subject must be together?

(d) the mathematics books must all be together?

(e) two history books must be at one end and two history books must be at the other end?

COMBINATIONS

A combination is a selection of objects in which the order does not matter.

Methods of calculating the number of combinations.

Method A Listing method.

This method gives a list of all of the possible combinations.
Consider the following:
Given the set of vowels of the English language, that is, {a, e, i, o, u}, how many selections(or combinations or subsets) are there if each combination must contain:
(a) one vowel? (b) two vowels? (c) three vowels? (d) four vowels? (e) five vowels?

(a) If each combination must contain 1 vowel, then the list of such combinations is:

a e i o u

That is, there are 5 combinations such that each combination has 1 vowel taken from 5 vowels.

(b) If each combination must contain 2 vowels, then the list of such combinations is:

ae ai ao au ei eo eu io iu ou

That is, there are 10 combinations such that each combination has 2 vowels taken from 5 vowels.

NOTE:
1. ae and ea are different arrangements of the 2 vowels but are the same combination as the order does not matter. The number of permutations of 2 vowels is given by 2!.
2. The number of permutations of 2 vowels chosen from 5 vowels is given by $^5P_2$. Now $^5P_2 = 5 \times 4 = 20$.
3. The number of combinations of 2 vowels chosen from 5 vowels is 10 and is given by

$$\frac{\text{the number of permutations of two vowels chosen from 5 vowels}}{\text{the number of permutations of 2 vowels}}.$$

Hence the number of combinations is given by $\frac{^5P_2}{2!} = \frac{5\text{x}4}{2\text{x}1} = 10$.

(c) If each combination must contain 3 vowels, then the list of such combinations is:

aei aeo aeu aio aiu aou eio eiu eou iou

That is, there are 10 combinations such that each combination has 3 vowels taken from 5 vowels.

NOTE:
1. aei, aie, eai, eia, iae, and iea are all the different permutations of 3 vowels but are all the same combination as the order does not matter. The number of permutations of 3 vowels is given by 3!.
2. The number of permutations of 3 vowels chosen from 5 vowels is given by $^5P_3 = 5 \times 4 \times 3 = 60$.
3. The number of combinations of 3 vowels chosen from 5 vowels is 10 and is given by

$$\frac{^5P_3}{3!} = \frac{5\text{x}4\text{x}3}{3\text{x}2\text{x}1} = 10.$$

(d) If each combination must contain 4 vowels, then the list of such combinations is:

aeio, aeiu aeou aiou eiou

That is, there are 5 combinations such that each combination has 4 vowels taken from 5 vowels.

NOTE:
1. The number of permutations of 4 vowels is given by 4!.
2. The number of permutations of 4 vowels chosen from 5 vowels is given by $^5P_4 = 5 \times 4 \times 3 \times 2 = 120$.
3. The number of combinations of 4 vowels chosen from 5 vowels is 5 and is given by

$$\frac{^5P_4}{4!} = \frac{5\text{x}4\text{x}3\text{x}2}{4\text{x}3\text{x}2\text{x}1} = 5.$$

(e) If each combination must contain 5 vowels, then there is just one combination: aeiou
That is, there is 1 combination such that has 5 vowels taken from 5 vowels.

NOTE:
1. The number of permutations of 5 vowels is given by 5!.
2. The number of permutations of 5 vowels chosen from 5 vowels is given by $^5P_5 = 5 \times 4 \times 3 \times 2\ 1 = 120$.
3. The number of combinations of 5 vowels chosen from 5 vowels is 1 and is given by

$$\frac{^5P_5}{5!} = \frac{5\text{x}4\text{x}3\text{x}2\text{x}1}{5\text{x}4\text{x}3\text{x}2\text{x}1} = 1.$$

The listing method gives each combination, however it is not a very practical method for finding the number of combinations as it is time consuming and may be confusing for large sets.

Method B Combination notation method.

This method gives the number of combinations without identifying each combination.

Consider selecting r different objects from a set containing n different objects. Using combination notation we can write the above statement as ^nC_r. Hence there are ^nC_r ways of selecting r different objects from a set of n different objects.

$$\underrightarrow{\text{number of objects to choose from}}\ {}^nC_r \ \underleftarrow{\text{number of objects in each selection}}$$

Now from our discussion under method A the number of combinations of r different objects chosen from n different objects is ^nC_r, and is given by $\frac{^nP_r}{r!}$.

$$\text{That is } {}^nC_r = \frac{^nP_r}{r!} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{(n-r)!} \times \frac{1}{r!} = \frac{n!}{(n-r)!r!}$$

The number of combinations ^nC_r, of r different chosen from n different objects is given by: $\mathbf{^nC_r} = \frac{n!}{(n-r)!r!}$

The graphics calculator has the ability to compute the number of permutations.

```
nCr(9,5)
                126
nCr(18,10)
              43758
```

NOTE:

1. ^nC_r may be written as nCr or $\binom{n}{r}$
2. Combinations allow us to find a meaning for 0!

   We know that $^nC_r = \frac{n!}{(n-r)!r!}$

   And if r = n, then $^nC_n = \frac{n!}{(n-n)!n!} = \frac{1}{0!}$

   However we know that $^nC_n = 1$

   Hence $\frac{1}{0!} = 1$

   That is $0! = 1$
3. $^nC_0 = \frac{n!}{(n-0)!0!} = \frac{n!}{n! \times 1} = 1$
4. $^nC_r = {}^nC_{n-r}$

That is choosing to include **r** objects in a combination from a total of n objects is the same as choosing to exclude the other n – r objects.

For example each time I choose 4 students out of 18 students to take to the beach I am at the same time leaving (18 – 4) that is, 14 students behind.

That is $^{18}C_4 = 3060$ and $^{18}C_{(18-4)} = 3060$, hence $^{18}C_4 = {}^{18}C_{18-4} = {}^{18}C_{14}$

Example 7

Evaluate the following (a) $^7C_4$ (b) $\binom{10}{5}$

Solution

(a) $^7C_4 = \frac{7!}{(7-4)!4!} = \frac{7!}{3!4!} = 35$

(b) $\binom{10}{5} = \frac{10!}{(10-5)!5!} = \frac{10!}{5!5!} = 252$

Example 8
In how many ways can a group of 5 students be chosen from a class of 18 students?

Solution
In this example we need to choose 5 from 18

$$\text{Number of combinations} = \binom{18}{5} = \frac{18!}{(18-5)!5!} = \frac{18!}{13!5!} = 8568$$

A group can be chosen 8568 ways.

Example 9
In an examination a candidate must answer 4 questions out of a total of 7. How many combinations of 4 questions are possible?

Solution
In this example we need to choose 4 from 7.

$$\text{Number of combinations} = \binom{7}{4} = \frac{7!}{(7-4)!4!} = \frac{7!}{3!4!} = 35$$

There are 35 combinations of questions.

Example 10
(a) How many selections of 3 chocolates can be made from a tray containing 12 different chocolates?
(b) How many selections of 9 chocolates can be made from a tray containing 12 different chocolates?

Solution
(a) We need to choose 3 from 12.

$$\text{Number of combinations} = \binom{12}{3} = \frac{12!}{(12-3)!3!} = \frac{12!}{9!3!} = 220$$

There are 220 selections.

(b) We need to choose 9 from 12.

$$\text{Number of combinations} = \binom{12}{9} = \frac{12!}{(12-9)!9!} = \frac{12!}{3!9!} = 220$$

There are 220 selections.

Note: The number of combinations of 12 objects taken 3 at a time is the same as the number taken 9 at a time, this is because every time we take 3 objects we leave 9 behind.

Example 11
A committee of 4 is to be chosen from 7 men and 6 women. How many different committees are possible if
(a) there are no restrictions?
(b) there must be an even number of men and women?
(c) the oldest and youngest must be on the committee?

Solution
(a) As there are no restrictions on the composition of the committee we need to choose 4 from 13.

$$\text{Number of combinations} = \binom{13}{4} = \frac{13!}{(13-4)!4!} = 715$$

There are 715 different possible committees.

(b) To answer this question we need to choose 2 men from 7 men and 2 women from 6 women. The word and in such situations implies multiplication.

$$\text{Number of combinations} = \binom{7}{2}\binom{6}{2} = \frac{7!}{(7-2)!2!} \times \frac{6!}{(6-2)!2!} = 315$$

There are 315 committees consisting of and even number of men and women.

NOTE: When we choose 2 men from 7 men and 2 women from 6 women we are choosing 4 people from 13 people and this is verified by $\binom{7}{2}\binom{6}{2}$ as the sum of the top numbers is 13 and the sum of the bottom numbers is 4.

(c) To answer this question we must choose the oldest, the youngest and two others. That is, we must choose 1 from 1 for each of the oldest and youngest and 2 from 11 for the others on the committee.

$$\text{Number of combinations} = \binom{1}{1}\binom{1}{1}\binom{11}{2} = \frac{1!}{(1-1)!1!} \times \frac{1!}{(1-1)!1!} \times \frac{11!}{(11-2)!2!} = 55$$

There are 55 committees with the youngest and oldest.

EXERCISE 16D

1. Evaluate

   (a) ${}^{8}C_4$ (b) $\binom{7}{2}$ (c) $\binom{n}{4}$ (d) $\binom{n}{x}$ (e) $\binom{5}{3}\binom{7}{2}$

2. A cricket team of 11 players is to be chosen from 15 possible players. How many different teams of players can be chosen?

3. A bag contains 5 balls labelled A, B, C, D, and E. In how many ways can 3 balls be chosen? List all the possible 3-subsets.

4. A student must answer any 8 questions out of 10 in a test. How many different choices of 8 questions does he have?

5. An ice-cream stall has 14 flavours of ice-cream. How many different 3-scoop ice-cream cones can you choose from if each scoop must be a different flavour and

   (a) the order of the scoops on the cone doesn't matter?

   (b) the order of the scoops on the cone does matter?

6. A quinella is the first two horses that cross the finishing post. In a race of 13 horses how many quinellas are possible?

7. There are 8 "stand-bys" waiting for seats on the midnight flight of the "Fly By Night" Airlines. If there are 3 seats available on the plane, how many different combinations of "stand-bys" can be chosen to fill them?

8. How many lines can be drawn through 5 points which are arranged on a circle if two points must be used to draw a line?

9. Allan has enough money to buy 3 CD's. In a particular store there are 11 CD's that Allan would like to buy. In how many ways can he choose the CD's?

10. The game of bridge is played with a standard pack of cards, that is 52 cards, and each player is dealt 13 cards. How many bridge hands are possible?

11. To win the first division prize in the game of Lotto a player must select 6 numbers from 45.

    (a) How many possible selections are there?

    (b) Adam chose the numbers 1, 2, 3, 4, 5, 6 and Ben chose the numbers 4, 11, 23, 33, 37, 45. Who has the best chance of winning?

12. A student in Year 10 must choose 6 out of 17 subjects to study in Year 11.

    (a) How many course combinations are available to the student?

    (b) If one of the subjects is compulsory, how many course combinations are possible?

13. In a class of 18 students 8 were born in WA, 7 were born in NSW and 3 were born overseas. In how many ways can 6 students be chosen, if

    (a) there are no restrictions?

    (b) they all had to be born in NSW?

    (c) each birthplace has to have equal representation?

    (d) 3 had to be born in WA, 1 in NSW and the remainder overseas?

14. In a game of poker 5 cards are dealt from a standard pack of 52 cards to each player. How many poker hands
(a) can be dealt?
(b) contain 4 aces and a king?
(c) contain only hearts?
(d) have only picture cards (that is J,Q,K)?
(e) have three fours?
(f) have a pair of aces?

15. A box of computer USBs containing 10 USBs is known to contain 2 defective USBs. If 4 USBs are chosen from the box, how many selections
(a) are possible?
(b) contain 1 defective USB?
(c) contain both defective USBs?
(d) contain only good USB?

16. A collection of 14 CDs consists of 7 classical, 4 pop and 3 country and western. A selection of 7 CDs is made. How many selections
(a) are possible?
(b) have only classical CDs?
(c) have only pop and country and western CDs?
(d) have 3 classical, 2 pop and 2 country and western CDs?
(e) have no pop CDs?
(f) have all of the pop CDs?

17 A school cadet unit consists of 3 Sergeants, 4 Corporals and 28 Privates. How many ways can a squad consisting of
(a) 6 be chosen?
(b) 2 Sergeants, 2 Corporals and 2 privates be chosen?
(c) 6 Privates be chosen?
(d) 3 Sergeants and 3 Corporals be chosen?

18. In how many ways can 6 Physics books and 4 History books be arranged on a shelf if there are 9 Physics books and 8 History books to choose from?

19. A sample of 12 duck eggs is known to contain 9 fertilised eggs and 3 unfertilised eggs. A selection of 3 eggs is made. How many selections of 3 eggs contain
(a) only fertilised eggs?
(b) two fertilised eggs?
(c) only unfertilised eggs?

20. A debating team consists of 4 students.
In how many ways can the team be selected from 8 eligible student if
(a) there are no restrictions?
(b) two of the three year 11 students must be on the team?
(c) two particular girls refuse to be in the same team?

21. The game of bridge is played with a standard pack of cards, that is 52 cards, and each player is dealt 13 cards. Answer the following questions giving your answers in factorial form.
How many bridge hands
(a) consist of red cards only ?
(b) contain the four aces?
(c) contain the 7 of spades?
(d) contain 7 spades?
(e) consist of 5 hearts, 4 diamonds, 3 clubs and 1 spade?
(f) contain the queen of spades, a total of 5 black cards and the remainder red?

ADDITION PRINCIPLE

Mutually Exclusive Events

If two events A, B are such that they are disjoint, that is $A \cap B = \varnothing$, then this tells us that if event A occurs then event B cannot occur.
Events which are disjoint are said to be mutually exclusive events.

Consider the set of vowels of the English language, that is {a, e, i, o, u}.
If A is the event of forming three letter words starting with o, and
B is the event of forming three letter words starting with u.
Then events A and B are mutually exclusive because if a word starts with o it cannot start with u, and if it starts with u it cannot start with o. That is, if A occurs B cannot occur and if B occurs then A cannot occur.

> **Addition Principle**
> If two events A and B are mutually exclusive and event A can be performed in **a** ways and event B can be performed in **b** ways, then **a + b** is the total number of ways of performing either A or B.

Note: This principle can be extended to three or more tasks.
The terms "at least", "or", "at most" are associated with the addition principle.

Example 12

How many different numbers greater than 500 can be formed using the digits 2, 6, 7 and 9 if
(a) repetition of digits is not allowed?
(b) repetition is allowed but the numbers must not consist of more than 4 digits?

Solution

(a) In this situation we must consider 3 digit numbers and 4 digit numbers. Now the events of forming 3 digit numbers and 4 digit numbers are mutually exclusive, hence the addition principle can be applied.

$$\begin{aligned}\text{Number of numbers greater than 500} &= \text{number with 3 digits or with 4 digits}\\ &= 3.3.2 + 4.3.2.1\\ &= 42\end{aligned}$$

There are 42 numbers greater than 500.

(b) $$\begin{aligned}\text{Number of numbers greater than 500} &= 3.4.4 + 4.4.4.4\\ &= 304\end{aligned}$$

There are 304 numbers greater than 500.

Example 13

A four man bobsleigh team is to be selected from 7 Victorians and 5 Tasmanians to compete in the PyeongChang Winter Olympics. How many different teams consist of
(a) no Tasmanians?
(b) at least one Tasmanian?

Solution

(a) $$\begin{aligned}\text{Number of teams with no Tasmanians} &= \text{4 Victorians and no Tasmanians}\\ &= \binom{7}{4}\binom{5}{0}\\ &= 35\end{aligned}$$

There are 35 teams comprising of only Victorians.

(b) The number of teams with at least one Tasmanian means that we require the number of teams with 1 Tasmanian or 2 Tasmanians or 3 Tasmanians or 4 Tasmanians.

$$\begin{aligned}\text{Number of teams with at least 1 Tasmanian} &= \binom{5}{1}\binom{7}{3} + \binom{5}{2}\binom{7}{2} + \binom{5}{3}\binom{7}{1} + \binom{5}{4}\binom{7}{0}\\ &= 175 + 210 + 70 + 5\\ &= 460\end{aligned}$$

There are 460 teams comprising of at least 1 Tasmanian.

Alternatively, the solution to part (b) can be very easily found making use of complementary events.

$$\begin{aligned}\text{Number of teams with at least one Tasmanian} &= \text{Total number of teams} - \text{Number of teams with no Tasmanians}\\ &= \binom{12}{4} - 35\\ &= 460\end{aligned}$$

Example 14
In how many ways can a committee of 3 be chosen from 4 married couples if
(a) there are no restrictions?
(b) the committee must consist of 2 women and 1 man?
(c) a husband and a wife cannot both serve on the same committee?

Solution
(a) As there are no restrictions we need to choose 3 from 8.

Number of committees $= \binom{8}{3} = 56$

The committee can be chosen 56 ways.

(b) Number of committees $= \binom{4}{2}\binom{4}{1} = 6 \times 4 = 24$

The committee can be chosen 24 ways.

(c) Number of committees $= \binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{2}{0} + \binom{2}{1}\binom{2}{1}\binom{2}{0}\binom{2}{1} + \binom{2}{1}\binom{2}{0}\binom{2}{1}\binom{2}{1} + \binom{2}{0}\binom{2}{1}\binom{2}{1}\binom{2}{1} = 8 + 8 + 8 + 8 = 32$

The committees can be chosen 32 ways.

Alternatively part (c) may be argued as follows. We need to choose 3 married couples from 4 and this can be done $\binom{4}{3} = 4$ ways. Now each of these 4 ways can be represented by $\binom{2}{1}\binom{2}{1}\binom{2}{1} = 8$ different committees, hence the committees can be chosen $\binom{4}{3} \times 8 = 32$ways.

Alternatively using the idea of complementary events we need to find the number of committees on which a married couple does serve.
The number of committees on which a married couple does serve can be found by first selecting the married couple that will be on the committee, which is any one of 4 married couples and then selecting one person from the remaining 3 married couples, that is selecting 1 from 6.
Number of committees = Total number of committees – Number of committees on which a married couple serves.

$$= 56 - 4 \times \binom{6}{1}$$
$$= 56 - 24$$
$$= 32$$

EXERCISE 16E

1. How many 3 digit or 4 digit numbers can be formed using only the odd digits if repetition of digits in each is not allowed?

2. From a standard pack of 52 playing cards in how many ways can you choose
(a) an ace or a king? (b) an ace or a spade?

3. Freddie's Food Fair stall offers toasted cheese sandwiches with the following extras, pickle, onion, tomato and chilli. How many different toasted cheese sandwiches can be made if they must have at least one of the given extras?

4. Seven letter words are formed using the letters of the word SUBJECT. How many of these words
(a) start with S? (b) start with S or T?
(c) have S followed immediately by T? (d) have S and T together?
(e) do not have S and T together?

5. How many words of at least five letters can be formed using the letters of the word SUBJECT?

6. How many words of more than four letters can be formed using the letters of the word COMPUTER?

7. A computer software programme for creating a school timetable allows up to 3 capital letters to be used for identifying a teacher, that is for a teacher code. How many teacher codes can be entered if the actual position of the letters does not count and
(a) repetition of letters is not allowed?
(b) repetition of letters is allowed?

8. Pepe's Pizza Place offers cheese, anchovies, mushroom, capsicum and onion as toppings for the plain tomato base of the pizza. How many different pizzas can be made?

9. The game of Powerball involves selecting 5 balls from 45 balls, numbered 1 to 45 from one container and then selecting 1 ball (the powerball) from 45 balls, numbered 1 to 45 from a different container. To win first division a player must correctly identify all six balls. In how many ways can first division
(a) be won?
(b) be won, if the powerball must be a multiple of five?

(c) be won, if all players use only the birthday numbers of their families?

(d) be won, if the number 44 must be in the winning set of numbers?

10. How many 7 digit numbers can be formed using only the digits 1, 2, 3, 4, 5, 6 and 7, if
(a) the number begins with 1 and ends with 7?

(b) there are no more than two digits between the 1 and the 7?

11. Find the number of ways of arranging the letters of the word OBJECTS such that
(a) S is next to T?
(b) S is followed immediately by T?

(c) there are three letters between S and T?
(d) there are at least four letters between S and T?

12. In how many ways can a person make a selection of at least one CD from 8 different CDs?

13. There are 5 vertical dots and 7 horizontal dots as shown.

(a) How many triangles can be formed using the dots as vertices?

(b) How many quadrilaterals can be formed using the dots as vertices?

14. Jim has 13 close friends and he wants to invite 6 of them to dinner at a restaurant. In how many ways can he invite 6 if
(a) two of his friends are married and will only attend together?

(b) two of his friends do not get on with one another and will not attend together?

15. A jeweller has 7 spaces left to fill on a display tray. He decides to fill the spaces with either different men's watches or different women's watches. If he has 8 different men's watches and 9 different women's watches to choose from, in how many ways can he fill the display tray?

16. A student sitting for an examination must answer eight questions. The examination paper is made up of two sections, Section A comprising of 7 questions and Section (B) comprising of 8 questions. The student must answer at least 3 questions from each section. How many different selections are possible?

17. From 8 male and 7 female students, 5 prefects are to be selected. Find the number of ways the selection can be made if,
(a) one particular student must be included and two particular students must be excluded.

(b) the only stipulation is that there must be more males than females.

18. The Fine Food Restaurant offers a menu consisting of 4 soups, 5 entrees, 7 main courses and 6 desserts.
(a) How many different 4 course meals are available to a customer.

(b) If chicken soup and chocolate dessert cake are on the menu, how many of the 4 course meals start with chicken soup and end with chocolate dessert cake.

(c) How many different three course meal can a customer select?

19. A jury of 12 adults is to be selected from 10 men and 8 women. In how many ways can the jury of 12 be selected if
(a) there are no restrictions?
(b) all the women must be selected?

(c) there must be more men then women?
(d) Mrs Green and Mr Brown will not serve on the same jury?

20. The game of bridge is played with a standard pack of cards, that is 52 cards, and each player is dealt 13 cards. Answer the following questions giving your answers in factorial form. How many bridge hands
(a) contain the king of hearts?
(b) contain one king?

(c) contain at least one king?
(d) consist of only red cards and include the king of hearts?

(e) contain at the most three queens?
(f) contain the queen of spades and at most two aces?

Pascal's Triangle

Consider the expansions of the binomial expression $(x+y)^n$ where n is a whole number.

n	$(x+y)^n$	Coefficients of the terms
0	$(x+y)^0 = 1$	1
1	$(x+y)^1 = x + y$	1 1
2	$(x+y)^2 = x^2+2xy+y^2$	1 2 1
3	$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$	1 3 3 1
4	$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$	1 4 6 4 1
5	$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^4$	1 5 10 10 5 1
etc		

The array of coefficients is known as Pascal's Triangle and is named after Blaise Pascal 1623 – 1662, a French mathematician who systemised these results though they were known as early as 1300 AD having been discovered by the Chinese. Pascal did not give a general formula for these binomial coefficients, this was done by an Englishman Sir Isaac Newton 1642 – 1727.

NOTE:
1. Apart from the 1 at each end, any number in a row is the sum of two numbers immediately above it an d on either side of it.
2. The coefficients which are equidistant from the ends of a row are equal in size.
3. Pascal's triangle does not give the general formula for the coefficients of the expansion of $(x+y)^n$.
4. Although Pascal's triangle gives the coefficients of the expansion of $(x+y)^n$ it is time consuming.
5. The expansion of $(x+y)^n$ contains n + 1 terms.
6. In any given term, the sum of the indices associated with x and y is always equal to n.
7. The powers of x decrease from n to 0 and at the same time the powers of y increase from 0 to n.

Example 15

Use Pascal's triangle to write the expansion of $(x+y)^6$.

Solution

To find the coefficients of the terms of $(x+y)^6$ we make use of the sixth row of Pascal's triangle as follows to find the coefficients of the seventh row.

$$\begin{array}{ccccccccccccc} & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \end{array}$$

Now $(x+y)^6 = 1x^6+6x^5y^1+15x^4y^2+20x^3y^3+15x^2y^4+6x^1y^5+1y^6$

$= x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$

Example 16

Use Pascal's triangle to write the expansion of $(x-y)^7$.

Solution

The coefficients of the terms of $(x-y)^7$ will be 1, 7, 21, 35, 35, 21, 7, 1.

Now $(x-y)^7 = [x+(-y)]^7$

$= 1x^7+7x^6(-y)^1+21x^5(-y)^2+35x^4(-y)^3+35x^3(-y)^4+21x^2(-y)^5+7x^1(-y)^6+1(-y)^7$

$= x^7-7x^6y+21x^5y^2-35x^4y^3+35x^3y^4-21x^2y^5+7xy^6-y^7$

Example 17

Use Pascal's triangle to write the expansion of $(2x-3)^6$.

Solution

The coefficients of the terms of $(x+y)^6$.will be 1, 6, 15, 20, 15, 6, 1.

Now $(2x-3)^6 = [2x+(-3)]^6$

$= 1(2x)^6+6(2x)^5(-3)^1+15(2x)^4(-3)^2+20(2x)^3(-3)^3+15(2x)^2(-3)^4+6(2x)^1(-3)^5+(-3)^6$

$= 64x^6-576x^5+2160x^4-4320x^3+4860x^2-2916x+729$

General Form of the Binomial Expansion

Pascal's triangle may be written in $\binom{n}{r}$ form.

$$
\begin{array}{cccccccccccc}
n & & & & & & \text{Coefficients} & & & & & \\
0 & & & & & & \binom{0}{0} & & & & & \\
1 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
2 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
3 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
4 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
5 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

Hence the expansion of $(x+y)^n$ may be written in the form

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{r}x^{n-r}y^r + \ldots + \binom{n}{n-2}x^2y^{n-2} + \binom{n}{n-1}x^1y^{n-1} + \binom{n}{n}x^0y^n$$

That is $(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{r}x^{n-r}y^r + \ldots + \binom{n}{n-2}x^2y^{n-2} + \binom{n}{n-1}xy^{n-1} + y^n$

Now expansions of the form $(x+y)^n$ are called **binomial expansions** because there are two terms in the brackets.

$$\text{Now } (x+y)^n = \underbrace{x^n}_{T_1} + \underbrace{\binom{n}{1}x^{n-1}y}_{T_2} + \underbrace{\binom{n}{2}x^{n-2}y^2}_{T_3} + \ldots + \underbrace{\binom{n}{r}x^{n-r}y^r}_{T_{r+1}} + \ldots + \underbrace{\binom{n}{n-2}x^2y^{n-2}}_{T_{n-1}} + \underbrace{\binom{n}{n-1}xy^{n-1}}_{T_n} + \underbrace{y^n}_{T_{n+1}}$$

Note that the binomial expansion consists of n + 1 terms, that is one more term than the power n. Listed above are the first term, the second term the third term the (r + 1)th term etc.
Now the r + 1)th term or T_{r+1} is called the General Term of the expansion.

That is the General Term of a binomial expansion is given by $T_{r+1} = \binom{n}{r}x^{n-r}y^r$.

Example 18

Expand $(a-2b)^5$

Solution

Using the binomial expansion form, there is no need to determine the coefficients of the terms using Pascal's triangle as shown below.

$$
\begin{aligned}
(a-2b)^5 &= \binom{5}{0}a^5(-2b)^0 + \binom{5}{1}a^4(-2b)^1 + \binom{5}{2}a^3(-2b)^2 + \binom{5}{3}a^2(-2b)^3 + \binom{5}{4}a^1(-2b)^4 + \binom{5}{5}a^0(-2b)^5 \\
&= a^5 - 10a^4b + \frac{5!}{3!2!}a^3 \text{ x } 4b^2 - \frac{5!}{2!3!}a^2 \text{ x } 8b^3 + \frac{5!}{1!4!}a \text{ x } 16b^4 - \frac{5!}{0!5!}32b^5 \\
&= a^5 - 10a^4b + \frac{5\text{x}4}{2\text{x}1}a^3 \text{ x } 4b^2 - \frac{5\text{x}4\text{x}3}{3\text{x}2\text{x}1}a^2 \text{ x } 8b^3 + \frac{5\text{x}4\text{x}3\text{x}2}{4\text{x}3\text{x}2\text{x}1}a \text{ x } 16b^4 - 32b^5 \\
&= a^5 - 10a^4b + 40a^3b^2 - 80a^2b^3 + 80ab^4 - 32b^5
\end{aligned}
$$

Example 19

Find the tenth term of $(a+b)^n$

Solution

To find the tenth term of the binomial expression we could write down all the preceding terms or alternatively we may use the General Term, that is $T_{r+1} = \binom{n}{r}x^{n-r}y^r$.

Hence $\qquad T_{10} = T_{9+1} = \binom{n}{9}a^{n-9}y^9$.

Example 20

Find T_4 in the expansion of $(1-2x)^8$.

Solution

Now $T_4 = T_{3+1} = \binom{8}{3}1^5(-2x)^3 = \frac{8 \times 7 \times 6}{3 \times 2 \times 1}(-8x^3) = -448x^3$

EXERCISE 16F

The questions in this exercise should be completed without the aid of a calculator. On completion answers should then be checked using a calculator.

Expand each of the following:

1. $(a+b)^8$

2. $(a-b)^8$

3. $(a+2b)^6$

4. $(2a-b)^5$

Write down the first five terms of each of the following binomial expressions:

5. $(x+1)^{13}$

6. $(1-3x)^6$

7. $(2x+3y)^7$

8. $(x^2-x)^6$

9. $(2x-1)^5$

10. Find T_4 in the expansion of $(a+b)^{10}$

11. Find T_5 in the expansion of $(x+2y)^6$

12. Find T_6 in the expansion of $(3x-2y)^5$

13. Find middle term in the expansion of $(p-2q)^6$

14. Find the third term in the expansion of $(x+\frac{1}{x})^6$

15. Find the coefficient of x^3 in the expansion of $(2-x)^6$.
-160

16. Find the coefficient of x^4 in the expansion of $(1-2x)^6$.
240

17. Find the coefficient of x^5 in the expansion of $(1-x)^7$.
-21

18. Find the coefficient of x^7 in the expansion of $(3+x)^9$.
324

19. Consider the expansion of the expression $(x+y)^{10}$.
(a) How many terms are in the expansion? (b) What power of x goes with y^7?
(a) 11 (b) x^3 (c) 1024 (d) $252x^5y^5$
(c) What is the sum of the coefficients? (d) Find the middle term of the expression.

Total number of subsets.

Sometimes we need to know to know the total number of subsets that belong to a set.

Using our knowledge of combinations the following table for each $\binom{n}{r}$ may be constructed

n\r	0	1	2	3	4	5	6	7	8	9	10	TOTAL
0	1											1
1	1	1										2
2	1	2	1									4
3	1	3	3	1								8
4	1	4	6	4	1							16
5	1	5	10	10	5	1						32
6	1	6	15	20	15	6	1					64
7	1	7	21	35	35	21	7	1				128
8	1	8	28	56	70	56	28	8	1			256
9	1	9	36	84	126	126	84	36	9	1		512
10	1	10	45	120	210	252	210	120	45	10	1	1024
etc												

If we consider the row where n = 4 we can see that:

There is $\binom{4}{0}$, that is 1 subset containing no elements that can be formed from a 4-set.

There are $\binom{4}{1}$, that is 4 subsets containing 1 element that can be formed from a 4-set.

There are $\binom{4}{2}$, that is 6 subsets containing 2 elements that can be formed from a 4-set.

There are $\binom{4}{3}$, that is 4 subsets containing 3 element that can be formed from a 4-set.

There is $\binom{4}{4}$, that is 1 subset containing 4 elements that can be formed from a 4-set.

Now the total number of of subsets that can be formed from a 4-set is 1 + 4 + 6 + 4 + 1 = 16

If we examine carefully the totals column in the table above we can see that the total for each row is a power of 2, that is $1 = 2^0$, $2 = 2^1$, $4 = 2^2$, $8 = 2^3$, etc.
Thus we can now see that the total number of subsets belonging to a set is given by 2^n where n is the number of elements in the set.

That is, the total number of subsets in an n-set $= 2^n$

We can apply the above in situations **where and only where two outcomes** are possible. This is because when we consider $\binom{n}{r}$ we are in fact dealing with a two outcome situation, that is, when we want to know what $\binom{4}{3}$ is, we are asking the question "How many subsets contain 3 elements and how many do not?" which is a two outcome (binomial) situation.

Consider the set {a, b, c, d}, this set has $2^4 = 16$ subsets made up as follows:

1 subset has no elements, that is { } $\binom{4}{0}$

4 subsets have 1 element, that is {a}, {b}, {c}, {d} $\binom{4}{1}$

6 subsets have 2 element, that is {a,b}, {ac}, {a,d}, {b,c}, {b,d}, {c,d} $\binom{4}{2}$

4 subsets have 3 element, that is {a,b,c}, {a,b,d}, {a,c,d}, {b,c,d} $\binom{4}{3}$

1 subset has 4 elements, that is {a,b,c,d } $\binom{4}{4}$

Example 21
Three coins are tossed. (a) List all the possible outcomes.
(b) Find the total number of outcomes.

Solution

(a)

HHH	HHT	HTT	TTT
	HTH	THT	
	THH	TTH	

(b) From the above we can see that we obtain: 1 outcome of 3 H and 0 T
3 outcomes of 2 H and 1T
3 outcomes of 1 H and 2 T
1 outcome of 0 H and 3 T

That is, the total number of outcomes is given by $1 + 3 + 3 + 1 = 8$

Alternatively the total number of outcomes will be given by 2^3 as each coin has only two outcomes, a head or a tail and hence the Binomial Theorem may be applied.

Note: Tossing three coins once gives exactly the same outcome or **sample space** as tossing one coin three times.

Example 22
Pepe's Pizza Place offers cheese, anchovies, mushroom, capsicum and onion as toppings for the plain tomato base of the pizza. How many different pizzas can be made?

Solution
In this situation we choose either to have a particular topping or not have a particular topping that is, we have a situation in which there are two possible outcomes for each topping in question. Hence we can apply the Binomial Theorem to this situation.

Total number of pizzas $= 2^5 = 32$
Note: This includes a plain pizza with none of the toppings and one with all the toppings.

EXERCISE 16G

1. How many subsets can be formed using each of the following sets?
   (a) {a, e, i, o, u} (b) {0, 1, 2, . . . 9} (c) {(-2,4), (0,-6), (1,5)} (d) {0}

2. Consider the experiment of tossing four one dollar coins.
   (a) List all the possible outcomes.

   (b) How many outcomes are possible?

   (c) In how many of these outcomes will 3 heads be showing?

   (d) In how many of these outcomes will there be the same number of head and tails showing

3. The coins currently in common use in Australia are:
   5c 10c 20c 50c \$1 \$2
   (a) How many sums of money can be formed using only the 50c, \$1 and \$2 coins?

   (b) How many sums of money can be formed using at most three coins?

   (c) How many sums of money can be formed using these coins?

4. If a coin is tossed eight times,
   (a) how many different outcomes are possible?

   (b) How many of these outcomes contain an equal number of heads and tails?

5. The following fillings are available for making a lunch roll: butter, tuna, cheese, tomato, pickle, lettuce, beetroot, onion, cucumber and mayonnaise.
How many different lunch rolls can be made if,
(a) each lunch roll must have 4 fillings.

(b) each lunch roll must have at least 7 fillings.

(c) there are no restrictions.

6. A family has three children.
(a) List how many different families of three children are possible by making use of a tree diagram.

(b) How many of these families have two girls?

(c) In how many of these families is the youngest child a boy, if the family must have children of both sexes?

(d) Which is more likely to occur, a family of two boys and a girl or a family of two girls and a boy? Justify your answer.

7. If a family has 5 children, then one such family may be made up of a boy, girl, girl, boy, girl according to birth order.
(a) How many different 5 children families are possible.

(b) How many of these families have only boys?

(c) How many of these families have more girls than boys?

(d) How many of these families have at least 2 girls?

(e) How many of these families have no more than 3 boys

(f) Do any of these families have an equal number of boys and girls? Why?

CHAPTER SIXTEEN CHECKLIST

You now should be able to:

- display outcome sets using tree diagrams and systematic lists
- use the multiplication principle for counting
- use the addition principle for counting
- use the notation $\binom{n}{r}$ to find the number of subsets of size r in a set of n elements
- determine the total number of subsets which belong to a set
- apply the notation $\binom{n}{r}$ to find the coefficients of binomial expansions
- expand $(x+y)^n$ for n a positive integer
- use Pascal's triangle and its properties

CHAPTER SIXTEEN REVIEW EXERCISE

1. How many 3 digit numbers can be written using the digits 6, 7 and 8 if repetition is allowed?

2. A true-false Science test is made up of 10 questions;
   (a) In how many ways can this test be answered?
   (b) In how many of these ways will all of the questions have been answered correctly?
   (c) In how many of these ways will half the questions be answered correctly and half incorrectly?

3. To win the Times Pool competition a player must insert one letter in each of six boxes in a row. Letters must be entered in alphabetical order and may not be repeated.
   (a) How many possible winning sets of letters are there?
   (b) If a player wanted to make sure that he sent in a winning entry, how many entries must he send in?

4. A car manufacturer produces 7 different models, each available in 6 different colours. In addition a customer can choose one of 6 different sound systems. How many varieties of car can be ordered from the manufacturer?

5. The serial number on a \$50 note consists of two letters followed by eight digits. How many \$50 notes can be printed under these conditions?

6. In long distance direct dialling, the area code consists of three digits and the local number of eight digits. Taking the area code into consideration, how many telephone numbers can be formed if local numbers cannot have digits repeated?

7. A box of chocolates contains 18 different chocolates, 11 of which have soft centres and the remainder have hard centres. Anna selects 5 chocolates from the box.
   (a) How many different selections can Anna make?
   (b) How many different selections can Anna make if the chocolates must all have hard centres?

8. (a) In how many ways can a student choose to answer 5 questions out of 11 if the order of his answers is of no importance?
   (b) The student decides to answer the questions numbered 2, 4, 6, 8 and 11. In how many ways can the student answer these questions?

9. In how many ways can 4 heads and 4 tails be obtained in 8 tosses of a coin?

10. A visitor to Western Australia makes a list of 11 sights he would like to see. He is determined to visit at least 8 of these. How many possible choices does he have?

11. Helen's Hamburger Haven sells hamburgers with the extras cheese, chilli, mustard, ketchup, relish or dill pickle.
    (a) How many hamburgers can be made choosing any three extras?
    (b) How many different hamburgers can be made?

12. The quality control officer at a factory has to inspect a sample of 5 fuses from each box of fuses to make sure that the fuses meet the required standard. If each box contains 100 fuses, how many different samples can the quality control officer chose?

13. A packet of 10 AAA batteries is known to contain 3 batteries which are defective. 4 batteries are selected from this packet to replace the used batteries in a calculator.
    (a) How many selections of 4 batteries may be made?
    (b) How many of these sections will have no no defective batteries.

14. A security firm has 20 security guards. Every night a group of three security guards must be rostered for the midnight to 8am shift, that is, night shift..
    (a) How many consecutive nights can be rostered without two of the groups being identical?
    (b) In how many of these groups would a particular guard be on duty?

15. A school debating club consists of 7 boys and 9 girls, 2 boys and 2 girls are to be selected for a particular debate.
    (a) How many possible speaking orders are there?
    (b) How many speaking orders are there if the first and last speakers must be of the same sex?

16. Marguerite has one of each of the following coins in her pocket, $2. $1, 50c, 20c, 10c. Find the number of ways in which Marguerite can take out a sum of money from her pocket.

17. A school has 8 prefects. In how many ways can at least one prefect be invited by the deputy principal to join the staff for morning tea?

18. A school board consists of 16 members.
(a) In how many ways can three members be chosen?
(b) In how many ways can three members be chosen if the president of the board must be included?

19. n(A) = 10. How many subsets of set A have 7 elements or 8 elements?

20. Twelve points are equally spaced on the circumference of a circle.
(a) Any two points may be joined to form a chord. How many different chords are possible?
(b) Any three points may be joined to form a triangle. How many different triangles are possible?

21. Find the value(s) of n in each of the following:
(a) $\binom{n}{4} = 495$
(b) $\binom{n}{5} = 1287$.

22. The breakfast menu at an Outdoor Education Camp is as follows:

Starter	**Cereal**	**Main Course**
Orange juice	Wheet-Bix	Beans on toast
Tomato juice	Corn flakes	Spaghetti on toast
Apple juice	Coco pops	Scrambled eggs on toast
	Rice bubbles	

A three course breakfast consists of a starter, cereal and a main course.
(a) How many different three course breakfasts are available?
(b) If David is allergic to eggs, how many different breakfasts consisting of three courses are available to him?
(c) How many of these breakfasts have spaghetti on toast as the main course.
(d) How many of these breakfasts have apple juice and coco pops?
(e) If a camper must have at least one course, how many breakfasts are possible.
(f) If the chef allows grated cheese to be sprinkled on the spaghetti on toast main course only, how many different three course breakfast are available?

23. Solve the following equations for x.
(a) $\binom{x}{3} = 84$
(b) $\binom{x}{x-3} = 165$

24. Catherine, Marguerite, David and 12 other students decide to experience a night dive off Rottnest. For safety reasons the diving instructors are permitted to only take 6 students at any one time for a night dive. How many different groups of 6 are possible to go on the first dive if
(a) there are no restrictions?
(b) David must be included?
(c) David must be included and Catherine must be excluded?
(d) David, Catherine and Marguerite will not dive together?

25. A particular game of cards involves a hand of seven cards from a standard pack of 52 cards.
(a) How many different hands can be dealt?
(b) How many hands will contain the four aces?
(c) How many hands will contain four of a kind, that is four cards of the same type?
(d) How many hands contain all cards of the same suit?
(e) How many hands contain an ace, a jack, a queen and the king of hearts?

26 David had 5 tickets to attend the Hopman Cup final. He has seven friends, including Mike and Joe to choose from to go to the concert.
(a) How many ways could he have chosen which of his friends could go to the final if he was going himself.
(b) How many ways could he have chosen his friends if he insisted on going and only one of Mike or Joe attended.
(c) How many ways could he have chosen his friends if he insisted on going and Mike and Joe would not go together?