Chapter 17 Probability

Most of us have some idea of probability from our experiences in everyday living as we all have experienced situations involving the "laws of chance". In this chapter we shall study and develop the laws of probability.

Let us consider the following situation:
The random number generator on a graphics calculator was used to examine whole numbers less than 1000. The random number generator provided the following 182 numbers.

751	501	963	843	679	176	201	529	087	419	656	245	356
615	367	543	384	568	804	228	454	861	684	592	704	288
182	306	133	574	224	533	452	167	184	063	461	844	495
454	705	512	807	009	235	015	828	816	242	386	234	561
981	035	467	874	428	151	625	556	774	243	529	234	998
023	907	175	306	271	388	079	618	276	246	814	286	291
174	143	713	086	046	559	343	589	901	012	386	308	655
617	476	708	495	491	402	034	776	185	122	027	415	697
533	659	924	425	056	414	057	283	442	008	661	733	817
699	389	306	504	869	902	167	939	899	675	707	355	008
009	219	239	059	726	289	275	044	444	933	698	741	196
161	896	206	816	302	736	692	714	859	044	399	242	666
789	198	174	796	967	986	278	993	388	605	368	114	113
142	116	763	197	298	333	218	932	443	638	628	653	253

Now we wish to measure the likelihood or the probability that the next time the random number button is pressed the number will be even?

We can measure probability or assign a numerical value to the likelihood of a certain event happening by using one of three methods:

- **Estimating Probabilities** – that is by gathering data by performing a large number of trials and then using the data in the form of **relative frequencies**.
- **Assigning Probabilities** – that is by assuming that each outcome is **equally likely**.
- **Subjective Probability** – that is by using subjective opinions.

Estimating Probabilities
The chances of the next number generated being even can be determined by considering the relative frequency.

That is, $\text{P(even number on next press)} = \dfrac{\text{number of even numbers}}{\text{total number of presses}}$

If we consider the first 13 trials only

$\text{P(even number on next press of the random number button)} = \dfrac{3}{13} \approx 0.2308$

If we consider the first 39 trials only

$\text{P(even number on next press of the random number button)} = \dfrac{19}{39} \approx 0.4872$

If we consider the first 104 trials only

$\text{P(even number on next press of the random number button)} = \dfrac{51}{104} \approx 0.4904$

If we consider the first 182 trials only

$\text{P(even number on next press of the random number button)} = \dfrac{90}{182} \approx 0.4945$

As can be seen the greater the number of trials considered the closer the result is to 0.5 that is the relative frequency has a limiting value of 0.5.

Assigning Probabilities
In the case under consideration, there are two equally likely possible outcomes, an even number or an odd number.

$$\text{Therefore P(even number on next press)} = \frac{\text{number of even outcomes}}{\text{number of possible outcomes}}$$
$$= \frac{1}{2}$$
$$= 0.5$$

Subjective Probability
In the case under consideration this method does not apply.
We use the subjective probability approach in cases where the two approaches mentioned above do not apply or cannot be applied. For example we use subjective opinions about things such as the chance of rain, the chance of a particular football team winning the next premiership etc. On the basis of past experience and other factors a numerical value may be assigned. The degree of acceptance of this numerical value will depend on the credibility of the author, however it must be noted that not everyone will accept this numerical value.

Before proceeding any further in our study of probability we need to define and explain some of the terminology associated with the study of probability.

Terminology

Experiment: - describes any act that can be repeated under a given set of conditions.

Trial: - is a single performance of an experiment.

Outcome: - is the result of an experiment. Outcomes may be favourable or unfavourable.

Sample Space: - is the set of all possible outcomes.

Sample Point: - is just one outcome of the sample space.

Event: - is any subset of the sample space including the empty set and the sample space itself.

Equally likely outcomes: - the experiment of tossing a fair coin has two possible outcomes, we say that each of these two possible outcomes is equally likely.

We are now in a position to define probability based on equally likely outcomes.

$$P(E) = \frac{\text{number of sample points in E}}{\text{number of sample points in the sample space}}$$
$$= \frac{n(E)}{n(S)}$$
$$= \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}$$

Where, P(E) denotes the probability in favour of a given event E in the sample space S, on condition that the outcomes of the experiment are equally likely.

Range of Probability
The probability of any event E that is, P(E) must satisfy the following probability scale $0 \leq P(E) \leq 1$.
When P(E) = 0 the event E is the empty set, that is, there are no favourable outcomes. This means that the event will never happen, i.e. event E is an impossibility.
When P(E) = 1 the event E is the universal set or the sample space. This means that the event will always happen, i.e. event E is a certainty.
As we move from P(E) = 0 to P(E) = 1 the likelihood of the event E happening increases and may be summarised as follows:

P(E)	**Description**
0	will never happen
between 0 and 0.5	unlikely to happen
0.5	even chance of happening or not happening
between 0.5 and 1	likely to happen
1	will always happen

EXERCISE 17A

1. A survey yielded the following information about the number of children in each family:

Number of children	number of families
0	17
1	35
2	58
3	37
4	12
5	4
	163

A family was chosen at random. Estimate the probability that the next family surveyed will have

(a) no children.

(b) two children.

(c) no more than two children.

(d) at least two children.

A child was selected at random.

(e) Estimate the probability that the child was not the only one in the family.

2. It was found that in 800 rolls of a single die the results were as follows:

Number showing	:	1	2	3	4	5	6
Frequency	:	93	138	134	133	141	161

Estimate the probability that the next roll of the die will result

(a) in a six.

(b) in an odd number.

(c) in a prime number.

(d) in a number less than three.

Comment on the "fairness" of this die.

3. An fair die is rolled and the result showing on the uppermost face is noted. Find the probability that :

(a) it is a six

(b) it is an odd number.

(c) it is a prime number.

(d) it is an even composite number.

(e) it is a number greater than two.

4. From a well shuffled pack of 52 cards a single card is drawn at random. What is the probability that the resulting card is

(a) a nine

(b) the nine of clubs.

(c) a club.

(d) a red card.

(e) not a red card.

(f) not an ace.

5. The sample space of digits S, is given by S = {0, 1, 2, 3, 4, 5, 6, ,7, 8, 9}

A digit is chosen at random from S. Find the probability that it is:

(a) the 8.

(b) less than 8.

(c) a prime digit.

(d) a composite digit.

(e) greater than 9.

(f) neither a prime digit nor a composite digit.

(g) between 2 and 8.

(h) a factor of 12.

6. The letters of the word ACE are permuted.
   (a) How many permutations are possible.

   By first listing all the different arrangements of the letters of the word ACE or otherwise, find the probability that the permutation:
   (b) is EAC.
   (c) ends in A.
   (d) has A in the middle.
   (e) starts with A or ends with A.
   (f) has A next to E.
   (g) starts with A and ends with A.

7. A dreidel is a kind of top with four sides. On each side is a Hebrew letter, the letters are *nun, gimmel, heh* and *shin.* Determine the probability that after the dreidel is spun the uppermost face:
   (a) shows the letter *gimmel.*
   (b) shows the letter *gimmel* or *heh.*
   (c) does not show the letter *shin.*
   (d) shows *nun* or *gimmel* or *heh* or *shin.*

8. The manufacturer of toothpicks decided to survey 100 boxes of toothpicks to determine whether his label of 380 toothpicks per box was justified. The results of the survey were:

Number of toothpicks:	377	378	379	380	381	382	383	384
Number of boxes :	0	2	5	10	44	22	13	4

   Using the results above find the probability that the next box off the production line contained:
   (a) 378 toothpicks.
   (b) 380 toothpicks or more.
   (c) less than the number stated on the box.
   (d) Comment on the manufacturers concern.

9. In a survey 100 adults were asked to state their political affiliation. The results are tabled below.

	Party A	Party B	Party C	Undecided	Total
Male	20	28	2	3	53
Female	14	16	9	8	47
Total	34	44	11	11	100

   A person is chosen at random from this group. Find the probability that the person is:
   (a) male and a supporter of Party B.
   (b) female or a supporter of Party A.
   (c) a supporter of Party A.
   (d) undecided, given that the person is female.

10. The table below shows the order of hands and the number of hands for the game of 5 card stud poker: Stud poker is played with a standard deck of 52 cards.

Order of hands	Number of hands
Royal flush	4
Straight flush	36
Four of a kind	624
Full house	3744
Flush(all cards of the same suit)	5108
Straight	10200
Three of a kind	54912
Two pairs	123552
One pair	**Q**
Highest card	1 302540
TOTAL NUMBER OF HANDS	**P**

   (a) Complete the table by finding the values of P and Q.

   In a game if 5 card poker, find the probability of being dealt
   (b) one pair.
   (c) a full house.
   (d) four of a kind.
   (e) four kings and any other card.

METHODS OF FINDING THE SAMPLE SPACE

In order to be able to calculate the probability of a particular event, knowing the sample space with a listing of all of the sample points or even just the number of sample points is a distinct advantage.

The sample space in most situations may be found by using one or more of the following methods:
(a) Systematic listing of the elements.
(b) Product Table or Grid.
(c) Tree Diagram.
(d) Venn Diagram.
(e) Counting Techniques.
Once the sample space has been found then it is generally obvious what the probability of the event will be.

(a) Systematic listing of the elements.

This method is limited to situations where the number of outcomes is not large. When listing outcomes it is very important to use a system so that all outcomes are listed.

Example 1
A fair coin is tossed and a fair die rolled. Find the probability that:
(a) the coin shows a head and the die a six.
(b) the die shows an even number.
(c) the die shows a number greater than 2

Solution
The outcomes, that is the sample spaces is

H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6.

The sample space contains 12 sample points.

(a) P(coin shows a head and the die a six) $= \dfrac{1}{12}$

(b) P(die shows an even number) $= \dfrac{6}{12} = \dfrac{1}{2}$

(c) P(die shows a number greater than 2) $= \dfrac{8}{12} = \dfrac{2}{3}$

(b) Product Table or Grid.

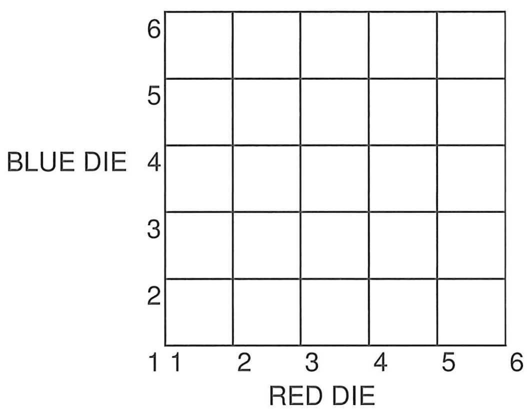
This method is suitable for experiments in which two procedures are involved. The resulting product table or grid shows every sample point in the sample space.

Example 2
Consider the experiment of rolling fair dice, one red and the other blue. Find the probability that the sum of the uppermost faces is:
(a) 6 (b) 5 (c) 12

Solution
Using a product table,

6	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
5	(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
4	(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
3	(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
2	(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
1	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
B / R	**1**	**2**	**3**	**4**	**5**	**6**

or using a grid



There are 36 sample points in the sample space.

(a) P(Sum of 6) $= \dfrac{5}{36}$

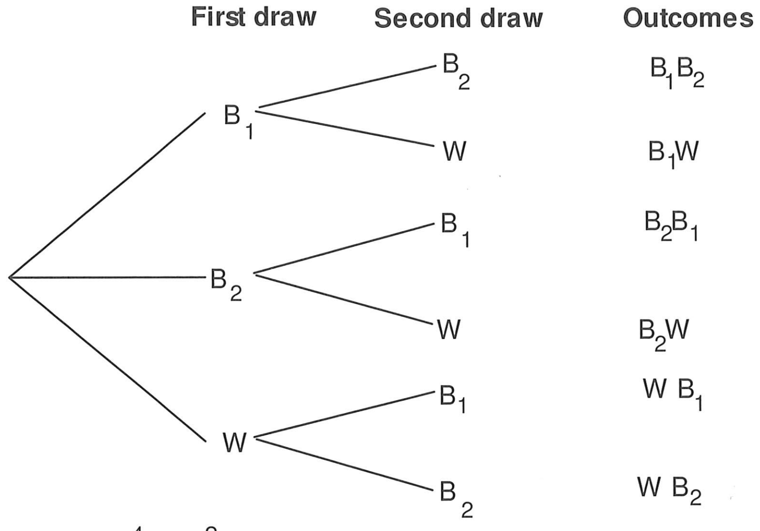
(b) P(Sum of 5) $= \dfrac{4}{36} = \dfrac{1}{9}$

(c) P(Sum of 12) $= \dfrac{1}{36}$

(c) Tree Diagram.
This method gives us an organised way of listing **all** the arrangements in the sample space. Consider the following situation.

Example 3
An box contains 3 balls, 2 blue and 1 white. What is the probability of selecting a blue ball on the second draw if balls are not replaced?

Solution
In this situation with 2 blue balls we must label them B_1 and B_2 so that they can be identified.



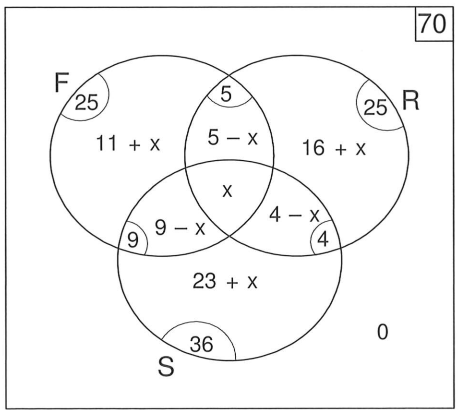
$$P(\text{blue on }2^{nd}\text{ draw}) = \frac{4}{6} = \frac{2}{3}$$

(d) Venn Diagram.
Using a Venn diagram to represent outcomes of an experiment allows us to see the relationships between the outcomes in the sample space or universal set.

Example 4
A survey of 70 students revealed the following information; 25 played football, 25 played rugby, 36 played soccer, 5 played football and rugby, 4 played rugby and soccer, 9 played football and soccer. Every student played at least one of these sports. If a student is selected at random find the probability that the selected student

(a) played all three sports.
(b) played only soccer.
(c) played rugby and soccer only.

Solution
To find the solutions we first need to construct a Venn Diagram will all sections filled in.



Now the number of students in the sample space is 70
Hence, $36 + (11 + x) + (5 - x) + (16 + x) + 0 = 70$
$x = 2$

(a) $P(\text{student played all 3 sports}) = \frac{2}{70} = \frac{1}{35}$

(b) $P(\text{student only played soccer}) = \frac{25}{70} = \frac{5}{14}$

(c) $P(\text{student played rugby and soccer only}) = \frac{2}{70} = \frac{1}{35}$

Probability Venn Diagrams show probabilities in the regions instead of the number of elements. The sum of the probabilities of all the regions must total 1.

Example 5

For two events A and B, it is found that $P(A) = 0.3$, $P(B) = 0.5$ and $P(A \cup B) = 0.6$. Find

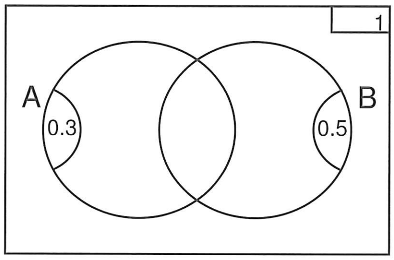
(a) $P(A \cap B)$

(b) $P(A' \cap B)$

(c) $P(A' \cap B')$

Solution

To solve problems involving probability Venn diagrams, we use the same procedure as we used in example 4 above.

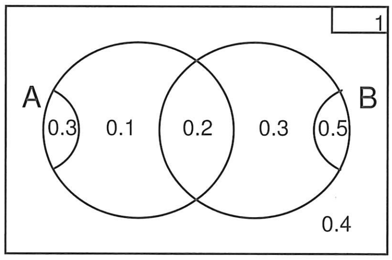


That is we first select the style of Venn diagram to be used and then enter the given information.
To complete the Venn diagram we must make use of the given information that $P(A \cup B) = 0.6$.
We know that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.
Now as areas of the regions show probabilities it then follows that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Hence, $0.6 = 0.3 + 0.5 - P(A \cap B)$.

That is $P(A \cap B) = 0.2$



The Venn diagram can now be completed and is shown on the right.
Once all the regions have an entry the questions can be answered.

(a) $P(A \cap B) = 0.2$

(b) $P(A' \cap B) = 0.3$

(c) $P(A' \cap B') = 0.4$

NOTE: The information contained in the Venn diagram for the example above may be displayed as a two-way table as each of the four regions can be described in terms of the intersection of the sets A, B, $\overline{A}$ and $\overline{B}$.
The two-way table displaying the contents of the Venn diagram above is shown on the right.

	A	$\overline{A}$	
B	0.2 $P(A \cap B)$	0.3 $P(\overline{A} \cap B)$	0.5 $P(B)$
$\overline{B}$	0.1 $P(A \cap \overline{B})$	0.4 $P(\overline{A} \cap \overline{B})$	0.5 $P(\overline{B})$
	0.3 $P(A)$	0.7 $P(\overline{A})$	1 $P(S)$

(e) Counting Techniques.

Using counting techniques we can readily find the number of sample points in the sample space and the number in the event under consideration.

Example 6

Find the probability of a poker hand of 5 cards having

(a) only diamonds.

(b) a pair of kings.

(c) four aces and the king of diamonds.

Solution

To find the number of different 5 card hands, that is, the number of sample points in the sample space we need to choose 5 from 52.

Hence, the number of hands in the sample space $= \binom{52}{5}$

(a) To find the number of hands of 5 cards that contain only diamonds we need to choose 5 from 13 and 0 from 39. That is $\binom{13}{5}\binom{39}{0}$.

Hence, $P(\text{only diamonds}) = \dfrac{\binom{13}{5}\binom{39}{0}}{\binom{52}{5}} = \dfrac{429}{866320}$

(b) The number of favourable sample points is given by choosing 2 from 4 and 3 from 48.

That is $\binom{4}{2}\binom{48}{3}$.

Hence, P(pair of kings) $= \dfrac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}} = \dfrac{2162}{54145}$

(c) The number of favourable sample points is given by choosing 4 from 4, 1 from 1 and 0 from 47.

That is $\binom{4}{4}\binom{1}{1}\binom{47}{0}$.

Hence, P(four aces and the king of diamonds) $= \dfrac{\binom{4}{4}\binom{1}{1}\binom{47}{0}}{\binom{52}{5}} = \dfrac{1}{2598960}$

EXERCISE 17B

Suggested method for Q1 to Q7 systematic listing of elements.

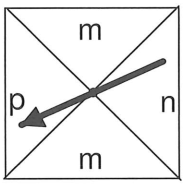
1. A coin and a die are tossed.
(a) List the sample space S.

Find the probability of:
(b) obtaining a head on the coin and a six on the die.
(c) obtaining an even number.
(d) obtaining a number greater than two.
(e) *not* obtaining a six.

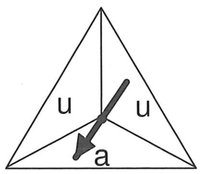
2. A family has three children. Assuming that having a boy or girl are equally likely events:
(a) List all the possible outcomes.

Find the probability that:
(b) the family has all girls.
(c) the family has two boys.
(d) the eldest is a girl.
(e) the only boy is the youngest.
(f) the family has at least one girl.
(g) the middle child is not a boy.

3. Two spinners , A and B as shown are spun simultaneously.



Spinner A



Spinner B

The result of spinner A is recorded, followed by the result of spinner B to form two letter words.
Find the probability of obtaining
(a) the word mu.
(b) a word ending in the letter u.
(c) a word containing a vowel.
(d) the word am.

4. During a power failure the on/off buttons of the TV, computer and radio were repeatedly pressed and it is not certain which of the appliances have been left on. When the power supply is resumed find the probability of
(a) all the appliances being turned on.
(b) only one of them turned on.
(c) at least two of the appliances turned on.
(d) the computer being turned on.

5. Two digits from the digits 5, 7, 8 and 9 are selected. Find the probability that
(a) both digits are odd
(b) the sum of the digits is odd.
(c) the difference between the digits is two.
(d) the difference between the digits is less than three.
(e) the digit 5 was selected first.
(f) the sum of the digits exceeds 13.

6. An examination paper is made up of two sections: Section A and Section B. Section A has two questions numbered Question 1 and Question 2: Section (B) has three questions numbered Question 3, Question 4 and Question 5. Candidates must attempt two questions one from each section. Find the probability that a student chosen at random attempted:
(a) questions 1 and 5.
(b) question 1 first then question 5.
(c) only odd numbered questions.
(d) Q4 if the student had attempted Q2.

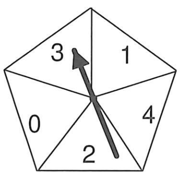
7. Given U = {x:x is an integer, $-2 \le x < 4$}. If a number is chosen at random from U, find the probability:
(a) that it is a whole number.
(b) that it is non negative integer.
(c) that it satisfies the equation $x^2 = 3x + 4$.
(d) that it satisfies the equation $x^2 + 4 = 0$

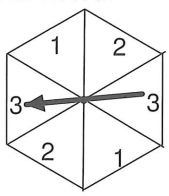
Suggested method for Q8 to Q10 product table or grid.

8. Two fair dice one blue and one red are rolled. Find the probability of
(a) both showing an even number.
(b) both showing the same number.
(c) the red die showing a 3.
(d) both showing a composite number.
(e) the sum of the numbers being less than six.
(f) the number on the blue die being less than the number on the red die.
(g) one die showing the number 4 and the other the number 3.
(h) one die showing a number greater than 4 and the other an even number.

9. The four aces from a pack of cards are selected and the rest of the pack is discarded. The four cards are shuffled, one of the cards is selected and the result noted. The experiment is then repeated using all four cards. Find the probability that:
(a) both aces are of the same suit.
(b) the ace of clubs is selected.
(c) the ace of hearts and the ace of diamonds are selected.
(d) the two aces are of different colour.

10. The spinners shown below are spun and the results noted.





Find the probability of:
(a) both spinners showing a 1.
(b) both spinners showing the same digit.
(c) both spinners showing even digits.
(d) the sum of the digits showing is less than six.
(e) at least one spinner showing a 1.
(f) both spinners showing a prime number.

Suggested method for Q11 to Q16 tree diagram.

11. If the events of having a boy or a girl are equally likely, find the probability that a family of three children has:
    (a) three girls.
    (b) only one boy.
    (c) at least one boy.
    (d) a boy, then two girls.
    (e) the youngest child a boy.
    (f) at most one boy.

12. A bag contains 5 jelly beans, 3 black and 2 green. 2 jelly beans are chosen at random from the bag. Find the probability that:
    (a) they are both black.
    (b) they are both of different colour.
    (c) the second is black.
    (d) the second is black if the first was green.

13. Three fair coins are tossed. Find the probability of the coins showing:
    (a) three heads.
    (b) one head and two tails.
    (c) only one tail.
    (d) the same result.
    (e) at least one head.
    (f) at most two heads.

14. Four letter words are made from the letters of the word JACK. Find the probability that:
    (a) the word ends in K.
    (b) the word starts with J and ends with K.
    (c) the letters C and K are adjacent.
    (d) the letters are in alphabetical order.

15. Three digit numbers are formed using the digits 2, 3, 5 and 7. A number is selected at random. Find the probability that:
    (a) the number ends in 7.
    (b) the number has the digit 5 in the middle.
    (c) the sum of the digits is ten.
    (d) the number is made up of odd digits only.
    (e) the sum of the digits is even.
    (f) the number is divisible by three.

16. Snakes and Ladders is a board game for two or more players. Each player has the same chance of winning a game and a game always results with only one winner. Alice, Boris and Caitlin play two games of Snakes and Ladders.
    By first drawing up a suitable sample space find the probability that
    (a) Boris wins the first game.
    (b) Boris wins both games.
    (c) Boris wins at least one game.
    (d) Boris wins no more than one game.
    (e) Boris wins the second game if Alice wins neither.
    (f) Boris wins the second game if Alice or Caitlin won the first game.
    (g) Boris wins the first game if Alice wins at most one game.

Suggested method for Q17 and Q22 Venn diagram.

17. A study of 80 college students revealed that; 30 studied physics, 25 studied chemistry and 26 studied mathematics. If one of these students studied all three subjects, 6 studied physics and mathematics, 20 studied mathematics only and 10 studied none of these subjects, find the probability that a student chosen at random:
    (a) studied only chemistry.
    (b) studied physics and chemistry.
    (c) studied only physics and mathematics.

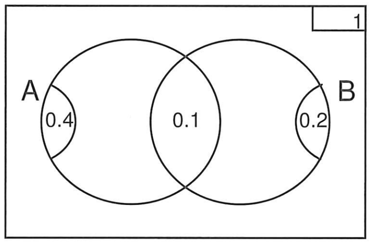
18. At an outdoor education camp it was found 30 students used jam on their toast, 25 used peanut butter, 17 used vegemite, 15 used jam and peanut butter, 5 used jam and vegemite and 8 used peanut butter and vegemite. If all of the 49 students on the camp used at least one of these spreads find the probability that a student chosen at random used:
(a) all three spreads.

(b) peanut butter only.

(a) only jam and peanut butter.

(d) at least 2 of these spreads.

19. Using the information given in the probability Venn diagram, find



(a) $P(A \cap B')$

(b) $P(A \cup B)$

(c) $P(A' \cap B')$

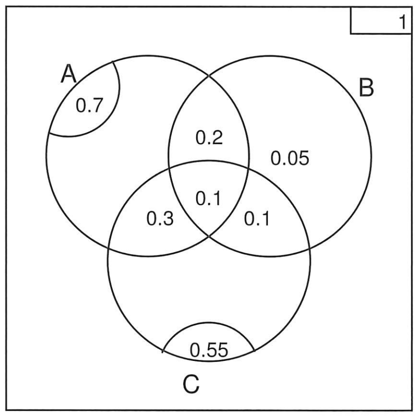
20. If the $P(A) = 0.4$, $P(B) = 0.5$ and $P(A \cap B) = 0.2$, find

(a) $P(A' \cap B)$

(b) $P(A \cap B)'$

(c) $P(A \cup B)'$

21. The diagram shown is a probability Venn diagram. Find each of the following:



(a) $P(B)$

(b) $P(A \cup B)$

(c) $P(A \cup C)$

(d) $P(A')$

(e) $P(A \cup B \cup C)$

22. Given $P(A) = P(B) = 0.6$, $P(C) = 0.5$, $P(A \cap B) = P(B \cap C) = 0.4$, $P(A \cap B \cap C) = 0.3$ and $P(A' \cap B' \cap C) = 0.05$. Find:
(a) $P(A \cap C)$

(b) $P(A \cup B)$

(c) $P(A \cup B \cup C)$

(d) $P(A' \cap B' \cap C')$

Suggested method for Q23 to Q27 Counting Techniques.

23. A box contains ten batteries, three of which are defective. If a battery is chosen at random from the box, determine the probability that:
(a) it is defective. (b) it is not defective.

If two batteries are chosen at random from the box, determine the probability that:
(c) both are defective. (d) one is defective and the other is not.

24. A bridge hand contains 13 cards dealt from a standard pack of 52 cards. Find the probability that a bridge hand has: (Leave answers in factorial form)
(a) only spades. (b) 5 spades and 8 diamonds.

(c) four aces. (d) no picture cards(no J,Q,K).

25. A committee of 5 is to be chosen from 5 men and 7 women. Each committee member is to be selected randomly. Find the probability that the committee contains:
(a) all women. (b) all men.

(c) 2 men and 3 women. (d) a particular man.

26. What is the probability that a hand of 5 cards taken from a standard deck of 52 cards contains (Leave answers in factorial form)
(a) exactly two aces? (b) exactly three aces?

(c) only picture cards?

27. The game of scrabble is a word game and it contains 98 tiles on which a letter is written and two tiles which are blank.
The following list gives the number of tiles which contain a particular letter:

A – 9	B – 2	C – 2	D – 4	E – 12	F – 2	G – 3
H – 2	I – 9	J – 1	K – 1	L – 4	M – 2	N – 6
O – 8	P – 2	Q – 1	R – 6	S – 4	T – 6	U – 4
V – 2	W – 2	X – 1	Y – 2	Z – 1	Blank tiles - 2	

To start the game David selects 7 tiles from a bag containing the 100 tiles.
Find the probability that David (Leave your answers in factorial form)
(a) selects the letters A, C, D, F, K, O and X.

(b) selects three tiles with the letter E, two tiles with the letter S a tile with the letter U and a tile with the letter V.

(c) has the letter Q and one of the tiles with the letter U in his selection of 7 letters.

(d) selects both of the blank tiles.

(e) selects the letters of the word INVOLVE.

Complementary Events Law
Let us consider the following situation.
If two dice are rolled, what is the probability of *not* getting a double?

Examining the product table reveals that it is much easier and quicker to determine the number of sample points favourable to the event "getting a double".
If we define the event A as the event of "getting a double", then

$$P(A) = \frac{6}{36} = \frac{1}{6}$$

Now the probability of not getting a double will be given by

$$P(\overline{A}) = 1 - P(A) = 1 - \frac{1}{6} = \frac{5}{6}$$

6	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	**(6,6)**
5	(1,5)	(2,5)	(3,5)	(4,5)	**(5,5)**	(6,5)
4	(1,4)	(2,4)	(3,4)	**(4,4)**	(5,4)	(6,4)
3	(1,3)	(2,3)	**(3,3)**	(4,3)	(5,3)	(6,3)
2	(1,2)	**(2,2)**	(3,2)	(4,2)	(5,2)	(6,2)
1	**(1,1)**	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
DIE 2 *DIE 1*	*1*	*2*	*3*	*4*	*5*	*6*

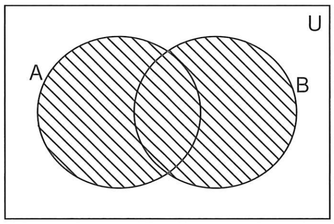
In general,

> If A and $\overline{A}$ are complementary events, then $P(\overline{A}) = 1 - P(A)$

NOTE: 1. $P(A \cup \overline{A}) = 1$
2. $P(A \cap \overline{A}) = P(\varnothing) = 0$
3. The events A and $\overline{A}$ are mutually exclusive.
4. Using the complementary event law simplifies the calculation of probability in some cases.

Addition Law
From our study of set theory we know that the shaded region shown in the Venn diagram is $A \cup B$.
We also know that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
Now if we divide each term in the equation above by the number of sample points in the sample space that is by n(S) we obtain,

$$\frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$$



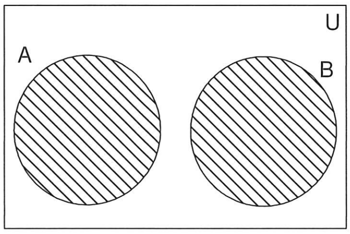
The equation above may now be written as,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Hence the areas of the regions in the Venn diagram may be treated as probabilities.

Mutually Exclusive Events
If two events have no intersection then we say that they are mutually exclusive.
The shaded region represents $A \cup B$ when the events A and B are mutually exclusive.
Now as $A \cap B = \varnothing$ hence $P(A \cap B) = 0$.
Thus if two events *are **mutually exclusive*** the addition law can be stated as,

$$P(A \cup B) = P(A) + P(B).$$



The addition law may be summarised as follows:

> The probability of either events A or B occurring is denoted by $P(A \cup B)$ and is given by:
> $$\mathbf{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$
> However, if events A and B are **mutually exclusive**, then $P(A \cap B) = 0$ and then $P(A \cup B)$ is given by:
> $$\mathbf{P(A \cup B) = P(A) + P(B)}$$

> **The use of *"or"* and *"and"*.**
> When we use the word "or" we mean the union. Thus when we say "A or B" we mean "A or B or both" or"at least one of A or B". Thus the probability of A or B means $P(A \cup B)$.
> When we use the word "and" when referring to events it means intersection. Thus when we say"A and B" we imply that both A and B must occur. Thus when we say the probability of A and B we mean $P(A \cap B)$.

Example 7
A die is rolled and a coin tossed. Find the probability of
(a) a tail. (b) a five (c) a tail and a five. (d) a tail or a five.

Solution
Now the sample space is given by:
S = {(T,1), (T,2), (T,3), (T,4), (T,5), (T,6), (H,1), (H,2), (H,3), (H,4), (H,5), (H,6)}.

(a) $P(\text{tail}) = \frac{6}{12} = \frac{1}{2}$.

(b) $P(\text{five}) = \frac{2}{12} = \frac{1}{6}$

(c) $P(\text{tail and five}) = P(\text{tail}\cap\text{five})$
$= \frac{1}{12}$.

(d) $P(\text{tail or five}) = P(\text{tail}\cup\text{five})$
$= P(\text{tail}) + P(\text{five}) - P(\text{tail}\cap\text{five})$
$= \frac{6}{12} + \frac{2}{12} - \frac{1}{12}$
$= \frac{7}{12}$.

Example 8
Given that $P(A) = \frac{3}{4}$, $P(B)= \frac{1}{2}$ and the $P(A\cup B) = \frac{7}{8}$ determine whether the events A and B are mutually exclusive.

Solution
We know that two events, A and B, are mutually exclusive if $A\cap B = \varnothing$ that is if $P(A\cap B) = 0$. Therefore to establish the events A and B are mutually exclusive we need to show that $P(A\cap B) = 0$.

We know that $P(A\cup B) = P(A) + P(B) - P(A\cap B)$.

Hence $\frac{7}{8} = \frac{3}{4} + \frac{1}{2} - P(A\cap B)$

$P(A\cap B) = \frac{3}{4} + \frac{1}{2} - \frac{7}{8}$
$= \frac{3}{8}$

Now since $P(A\cap B) = \frac{3}{8} \neq 0$, hence the events A and B are not mutually exclusive.

EXERCISE 17C

1. Two fair dice one blue and the other white are rolled.
   (a) Find the number of sample points in the sample space for this experiment using a suitable method.
   Determine:
   (b) P(sum is 6)
   (c) P(sum is 5)
   (d) P(sum is 6 or 5)
   (d) if the events 'sum of six' and 'sum of five' are mutually exclusive? Justify your answer.

2. Two fair dice one blue and the other white are rolled. What is the probability of
   (a) a score of 9?
   (b) a five on the blue die?
   (c) a score of 9 and a five on the blue die?
   (d) a score of 9 or a five on the blue die?

3. From a well shuffled standard pack of 52 cards a single card is drawn. Find the probability that the card is
   (a) a king.
   (b) a red card.
   (c) a king or a red card.
   (d) a king or a heart.
   (e) a king or a queen.
   (f) a six or a ten.
   (g) not a king
   (h) either the king of clubs or a black card

4. If $P(A) = 0.7$, $P(B) = 0.5$ and $P(A\cap B) = 0.4$ find
   (a) $P(\overline{A})$
   (b) $P(\overline{B})$
   (c) $P(\overline{A\cap B})$
   (d) $P(A\cup B)$
   (e) $P(\overline{A\cup B})$
   (f) $P(\overline{A}\cap B)$

5. Given that P(A) =0.36, P(B)= 0.64 and the P(A∪B) =0.8, determine
(a) whether the events A and B are mutually exclusive.

(b) $P(\overline{A \cup B})$ (c) $P(\overline{A} \cap B)$ (d) $P(\overline{A \cap B})$

6. If P(A) = 0.35, P(B) = 0.55 find each of the following if the events A and B are mutually exclusive:
(a) $P(\overline{A})$ (b) $P(\overline{B})$ (c) $P(A \cup B)$

(d) $P(A \cap B)$ (e) $P(\overline{A \cap B})$ (f) $P(A \cap \overline{B})$

7. If $P(\overline{A \cup B}) = 0.4$, P(A) = 0.5 and P(B) = 0.3, find
(a) $P(A \cup B)$ (b) $P(A \cap B)$ (c) $P(A \cap \overline{B})$

8. If $P(A \cap B) = 0.44$, $P(A \cap \overline{B}) = 0.22$ and $P(\overline{A} \cap B) = 0.11$ find
(a) P(A) (b) P(B) (c) $P(\overline{B})$

(d) $P(A \cup B)$ (e) $P(\overline{A \cup B})$ (f) $P(\overline{A \cap B})$

9. Events A, B and C are such that the following probabilities apply:
P(A) = 0.4; P(B) = 0.5; P(C) = 0.6; $P(A \cap B) = P(A \cap C) = 0.3$; $P(B \cap C) = 0.25$; $P(\overline{A \cup B \cup C}) = 0.15$
Determine
(a) $P(A \cap B \cap C)$

(b) $P(A \cap \overline{B} \cap \overline{C})$

(c) $P(\overline{A} \cap B \cap C)$

10. A sample of 200 voters in the recent referendum on the republic issue voted as shown in the table below:

	Yes	No	Vote invalid
Males	40	50	5
Females	30	60	15

If a voter is chosen at random from this sample, find the probability that the voter
(a) is a male (b) voted yes

(c) voted yes and was female (d) was male or voted yes

11. Three unbiased 50c coins are to be tossed. Find the probability of:
(a) three heads. (b) two heads and a tail.

(c) a tail on each of the first two coins and a head on the third coin.

(d) a tail on the first coin and a tail on the third coin.

(e) a tail on the first coin or a tail on the third coin.

(f) at least one tail. (g) at most two tails.

(h) the same result on the first and third coins.

12. Towards the end of a game of scrabble the following four letters are left in the bag C, D, K and M. If Mandy must select three letters find the probability that

(a) she selects the C, D and K.

(b) she selects the C, D and K in alphabetical order.

(c) she selects the C and the D or K.

(d) she selects the C first and the D last.

(e) her selection does not include the C.

13. A sewing box contains 12 buttons of the same size but different colour. Six of the buttons are black, 4 are white and the remainder red. A buttons is selected at random from the box and not replaced, a second button is selected and not replaced and then a third button is selected.
Find the probability that:

(a) the three buttons are black.

(b) the three buttons are not black.

(c) the three buttons are of the same colour.

(d) one of each colour is selected.

(e) at least one of the buttons is black.

14. At a large maternity hospital 6 children were born all at a different times during the first hour on the first of January 2018. If the probability of a child being a male is the same as that of being female, find the probability that,

(a) all the children are male.

(b) all the children are of the same sex.

(c) the oldest 3 are male and the youngest 3 are female.

(d) the oldest is a male and the rest are female.

(e) one is male and the remainder are female.

(f) at least two female.

(g) the birth order of the sexes alternated.

15. At a particular school a survey of the 20 Year Twelve students in form 12A, revealed the following information:
13 students studied Applied Maths
10 students studied Physics
9 students studied Physics and Applied Maths
4 students studied Physics, Applied Maths and Calculus
1 student studied Applied Maths and Calculus only
1 student studied only Physics, and
6 students did not study any of these subjects
Find the probability that a student chosen at random from form 12A:

(a) studied Calculus.

(b) studied Applied Maths only.

(c) studied Calculus or Applied Maths.

(d) did not study Physics.

(e) studied at least two of these subjects.

16. A special set of two digit numbers is to be formed using the digits 1, 2, 3, . . . 9 . The two digit numbers are such that the digits are in ascending order and repetition is not allowed.

(a) How many of these numbers can be formed?

(b) Find the probability that a number chosen at random from this set will be greater than 70.

(c) Find the probability that a number chosen at random from this set has only even digits.

(d) Find the probability that a number chosen at random from this set is not greater than 60.

CONDITIONAL PROBABILITY

Conditional probability is the probability of an event occurring on the condition that another event has occurred. This concept is best introduced by considering a number of examples.

Example 9

Two fair dice one blue and the other white are rolled. Find the probability that,
(a) the sum of the uppermost faces is 6.
(b) the sum of the uppermost faces is 6 given that the white die shows a 4.

Solution

(a) Examination of the sample space reveals that 5 of the sample points in the sample space are favourable to the event sum of 6.

Hence $\quad P(\text{sum of }6) = \dfrac{5}{36}$

6	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
5	**(1,5)**	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
4	(1,4)	**(2,4)**	(3,4)	(4,4)	(5,4)	(6,4)
3	(1,3)	(2,3)	**(3,3)**	(4,3)	(5,3)	(6,3)
2	(1,2)	(2,2)	(3,2)	**(4,2)**	(5,2)	(6,2)
1	(1,1)	(2,1)	(3,1)	(4,1)	**(5,1)**	(6,1)
BLUE *WHITE*	*1*	*2*	*3*	*4*	*5*	*6*

(b) The situation here is that we have been given the extra information or the condition that the white die shows a 4 on the uppermost face.
This means that we only need to consider those sample points in the sample space in which the white die shows a 4. In other words we only need to consider the sample points that have been hi-lighted in the product table.
Hence

$P(\text{sum of 6 given 4 on white}) = \dfrac{1}{6}$

6	(1,6)	(2,6)	(3,6)	**(4,6)**	(5,6)	(6,6)
5	(1,5)	(2,5)	(3,5)	**(4,5)**	(5,5)	(6,5)
4	(1,4)	(2,4)	(3,4)	**(4,4)**	(5,4)	(6,4)
3	(1,3)	(2,3)	(3,3)	**(4,3)**	(5,3)	(6,3)
2	(1,2)	(2,2)	(3,2)	**(4,2)**	(5,2)	(6,2)
1	(1,1)	(2,1)	(3,1)	**(4,1)**	(5,1)	(6,1)
BLUE *WHITE*	*1*	*2*	*3*	*4*	*5*	*6*

Notation for Conditional Probability

We use the symbol $|$ for the word 'given' or the word 'if' to indicate conditional probability.
Thus the probability that some event B occurs given that some event A has occurred is written as $P(B \mid A)$ and read as the 'probability of B given A'.

Example 10

A box contains two blue balls and one white ball. A ball is chosen at random from the box and is not replaced. A second ball is then drawn at random from the box. Find the probability that
(a) the second ball is blue given that the first ball was blue.
(b) the second ball is blue given that the first ball was white.

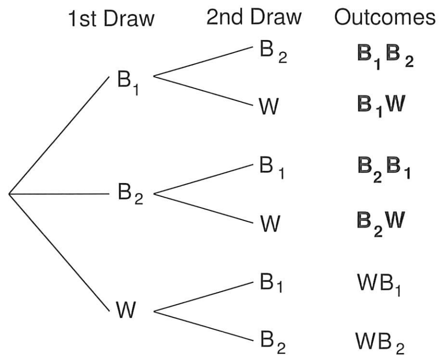
Solution

To solve this problem the sample space must give us the order in which the balls were drawn. A tree diagram will give us all the sample points in the sample space as well as the order in which the balls were drawn.

(a) In this situation we need to restrict our attention to the sample points which have the blue ball as the first ball that is $\{B_1B_2, B_1W, B_2B_1, B_2W\}$ as shown in the tree diagram.
Now there are two sample points $\{B_1B_2, B_2B_1\}$ in this reduced sample space that are favourable to the event the second ball is blue given that the first ball was blue.

Hence $P(\text{2nd blue} \mid 1^{\text{st}} \text{ blue}) = \dfrac{2}{4}$

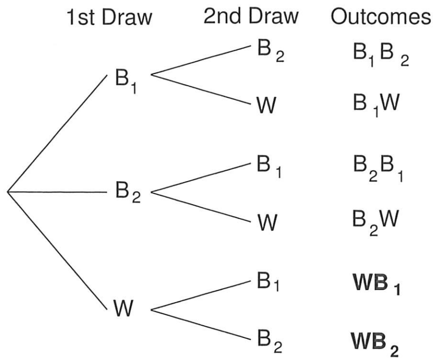
$$= \frac{1}{2}$$



(b) In this case we need to restrict our attention to the sample points which have the white ball as the first ball that is $\{WB_1, WB_2\}$ as shown in the tree diagram.

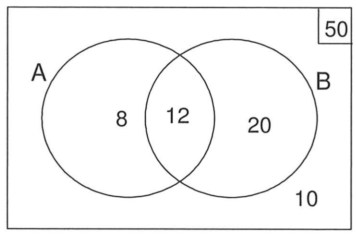
Now there are two sample points $\{WB_1, WB_2\}$ in this reduced sample space that are favourable to the event the second ball is blue given that the first ball was white.

Hence $P(\text{2nd blue} \mid 1^{st} \text{ white}) = \frac{2}{2}$
$= 1$



Example 11

The Venn diagram shows the number of students that study Applied Mathematics(A) and Biology(B) at a particular college.
If a student is selected at random from the universal set, find
(a) P(A).
(b) $P(A \cup B)$.
(c) $P(A \mid B)$.
(d) $P(B \mid A)$.



Solution

(a) The universal set contains 50 students and 20 of these students are in set A.

Hence $P(A) = \frac{20}{50} = \frac{2}{5}$

(b) The number of students in $A \cup B$, that is $n(A \cup B) = 8 + 12 + 20 = 40$

Hence $P(A \cup B) = \frac{40}{50} = \frac{4}{5}$

(c) We are given that the student belongs to set B, that is he is one of the 32 students that belong to set B. Hence we restrict our attention to only those students in set B. Now we wish to find how many of the students which are in set B also are in set A, that is we want to find $n(A \cap B)$. The Venn diagram informs us that $n(A \cap B) = 12$.

Hence $P(A \mid B) = \frac{n(A \cap B)}{n(B)} = \frac{12}{32} = \frac{3}{8}$

(d) Given that the student belongs to set A, that is we restrict our attention to the 20 students which belong to set A. Now the student in question must be one of the 12 of those which are in set B, that is $n(A \cap B) = 12$.

Hence $P(B \mid A) = \frac{n(A \cap B)}{n(A)} = \frac{12}{20} = \frac{3}{5}$

Sometimes it is more convenient to have a ratio of probabilities rather than the number of sample points. This can be easily achieved by dividing the numerator and the denominator by the number of points in the sample space.
Thus we obtain

$$\boxed{P(B \mid A) = \frac{P(A \cap B)}{P(A)}}$$

NOTE: 1. $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

2. $P(A \cap B) = P(A \mid B)\,P(B)$ also $P(A \cap B) = P(B \mid A)\,P(A)$

Example 12

Given $P(A \mid B) = 0.4$, $P(B \mid A) = 0.5$ and $P(A \cap B) = 0.2$. Find (a) P(A) (b) P(B) (c) $P(A \cup B)$.

Solution

(a) $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$

$0.5 = \frac{0.2}{P(A)}$

$P(A) = \frac{0.2}{0.5}$
$= 0.4$

(b) $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

$0.4 = \frac{0.2}{P(B)}$

$P(B) = \frac{0.2}{0.4}$
$= 0.5$

(c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= 0.4 + 0.5 - 0.2$
$= 0.7$

EXERCISE 17D

1. From the set of letters {a, b, c, d, e, f, g, h, i} a letter is selected at random. Find the probability that the letter selected is
   (a) the letter d.
   (b) the letter d given that a consonant is selected.
   (c) the letter e given that a vowel is selected.
   (d) not the letter f given that a consonant is selected.

2. A fair regular die is rolled only once. Find each of the following probabilities.
   (a) P(rolling a 6)
   (b) P(rolling a 6 given that an even number is rolled)
   (c) P(rolling a 6 given that the number is greater than 2)
   (d) P(rolling a 6 given that the number is prime)

3. A number is randomly chosen from the set of integers between –1 and 8. Find each of the following:
   (a) P(3)
   (b) P(2 or 4)
   (c) $P(4 \mid \text{counting number})$
   (d) $P(3 \mid \text{prime number})$
   (e) $P(6 \mid \text{whole number})$
   (f) $P(\text{even} \mid \text{number} > 1)$
   (g) $P(\text{prime} \mid \text{counting number})$
   (h) $P(0 \mid \text{not prime})$

4. Two fair regular dice one black and the other white are both rolled once and the result noted. Find each of the following probabilities.
   (a) P(numbers are identical)
   (b) P(numbers are identical given that the white die shows an even number)
   (c) P(numbers are identical given that sum of the numbers is less than 6)
   (d) P(sum of numbers is greater than 8 given that the numbers are identical)

5. The four aces from a standard pack of cards are selected and the rest of the pack is discarded. The four aces are shuffled, one card is selected and the result noted. The selected card is returned and the experiment is then repeated using all four cards. Find each of the following:
   (a) P(both cards were black)
   (b) P(second card is ace of spades given that the first is black)
   (c) P(second card is red given that the first card is the ace of hearts)
   (d) P(second card is black given that the first card is red)

6. The four aces from a standard pack of cards are selected and the rest of the pack is discarded. The four aces are shuffled, one card is selected and the result noted. The selected card is not returned and the experiment is then repeated using the three remaining cards. Find each of the following:
   (a) P(both cards were black)
   (b) P(second card is ace of spades given that the first is black)
   (c) P(second card is red given that the first card is the ace of hearts)
   (d) P(second card is black given that the first card is red)

7. A box contains 2 blue balls, 1 red ball and 1 white ball. A ball is drawn at random, its colour noted and then it is replaced. A second ball is then drawn at random from this box. Find the following probabilities.
   (a) $P(\text{both blue} \mid \text{first blue})$
   (b) $P(\text{both same colour} \mid \text{first red})$
   (c) $P(\text{both white} \mid \text{both same colour})$
   (d) $P(\text{both different colour} \mid \text{first blue})$
   (e) $P(\text{first red} \mid \text{second white})$

8. A coin is tossed once and the result recorded. It is then tossed again a second time and the result recorded.
   (a) Find the probability that both tosses resulted in a head.

   (b) Given that at least one toss resulted in a head, what is the probability that both tosses resulted in a head.

9. A selection of long term subscribers to the Reader's Digest magazine were each asked which section of the magazine they liked most. The results of the survey are given in the table below.

	Book section	**Humour section**	**Drama section**	**Nature section**
Male	30	142	51	38
Female	52	90	84	43

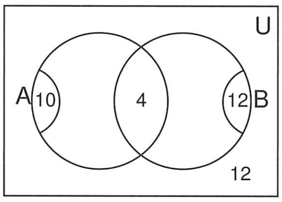
A subscriber is chosen at random from this group. What is the probability that the subscriber
   (a) liked the humour section most?

   (b) was male and liked the humour section the most?

   (c) was male if it was known that the subscriber enjoyed the humour section most?

   (d) enjoyed the drama section the most given that the subscriber was female?

10. The Venn diagram below shows the number of sample points in the events A and B which are contained in the sample space U. Find each of the following:

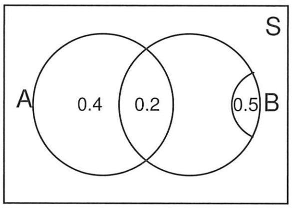


(a) $P(A \cup B)$ = (b) $P(\overline{A})$ =

(c) $P(A \mid B)$ = (d) $P(B \mid A)$ =

(e) $P(A \mid \overline{B})$ = (f) $P(A \cap B \mid A \cup B)$ =

11. The Venn diagram below shows the probabilities associated with A and B in a probability space S. Find each of the following:

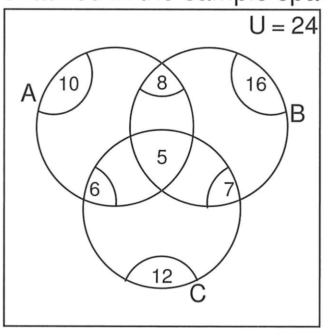


(a) $P(A)$ = (b) $P(A \cup B)$ =

(c) $P(A \mid B)$ = (d) $P(B \mid A)$ =

(e) $P(B \mid \overline{A})$ = (f) $P(\overline{B} \mid A)$ =

12. The Venn diagram below shows the number of sample points in the events A and B which are contained in the sample space U. Find each of the following:

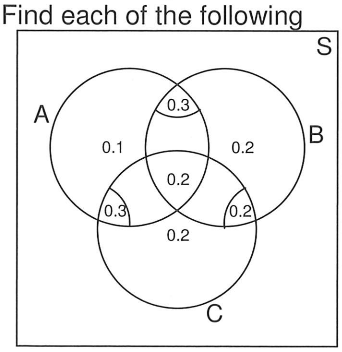


(a) $P(A \cap B \cap C)$ = (b) $P(A \cup B \cup C)$ =

(c) $P(A \mid C)$ = (d) $P(B \mid A)$ =

(e) $P(A \mid \overline{B})$ = (f) $P(A \cap B \mid \overline{C})$ =

13. The Venn diagram below shows the probabilities associated with A, B and C in a probability space S. Find each of the following



(a) $P(C)$ = (b) $P(\overline{A \cup B \cup C})$ =

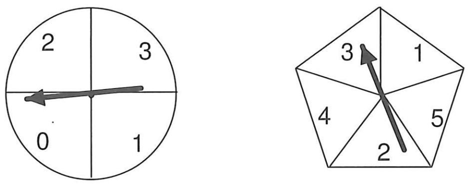
(c) $P(B \mid A)$ = (d) $P(A \mid B)$ =

(e) $P(B \mid \overline{C})$ = (f) $P(B \cap C \mid \overline{A})$ =

14. An ultra-sound procedure revealed that an expectant mother was to give birth to a set of triplets. The ultra-sound procedure could not discern the sex of the children but it is known that the event of having a boy or a girl are equally likely. Determine each of the following:
(a) P(all girls) =
(b) P(2 boys) =
(c) P(first born is a girl) =
(d) P(birth order is boy, girl, girl) =
(e) P(all girls | at least 2 girls) =
(f) P(boy | the first and second babies were male) =
(g) P(girl | the first two of the same sex) =

15. A bag contains 3 green balls and 2 blue balls. Three successive balls are drawn at random from this bag without replacement. Find
(a) P(first ball is green)
(b) P(first two balls are green)
(c) P(second ball is green given that the first ball is green)
(d) P(third ball is green given that the first two balls are green)
(e) P(all the balls are green)
(f) P(the first two balls are of the same colour)

16.



Spinner A Spinner B

Spinner A is spun first and the result noted, then spinner B is spun. Find the following probabilities.
(a) P(both show same number)
(b) P(A shows a 3 and B shows a 4)
(c) P(A shows a 3 | B shows a 4)
(d) P(A odd and B even)
(e) P(A odd if B even)
(f) P(sum is 4 | B < 4)

17. Two events A and B are such that $P(A \cap B) = 0.2$, $P(\overline{A} \cap B) = 0.3$, $P(\overline{A \cup B}) = 0.1$. Find
(a) $P(A)$
(b) $P(\overline{B})$
(c) $P(A \mid B)$
(d) $P(A \mid \overline{B})$
(e) $P(B \mid \overline{A})$
(f) $P(A \mid \overline{A \cap B})$

18. Two events A and B are such that $P(B) = 0.5$, $P(A \mid B) = 0.5$, $P(\overline{A \cup B}) = 0.35$. Find
(a) $P(A \cap B)$
(b) $P(\overline{A})$
(c) $P(B \mid A)$
(d) $P(\overline{B} \mid A)$
(e) $P(\overline{A} \mid \overline{B})$
(f) $P(A \cap B \mid A \cup B)$

19. An examination of the application forms of a group of applicants for a promotional position within a company revealed that 60% of the applicants had an appropriate academic qualification and of these applicants 30% were female. Furthermore 15% of the applicants were male without the appropriate academic qualification.
Find the probability that an applicant chosen at random from this group
(a) is a male.
(b) has the appropriate academic qualification given that the applicant is female.

20. A television company conducting a survey on movie preference asked 400 viewers which of the following movie types they preferred.

comedies musicals thrillers

205 viewers stated that they preferred to watch thriller movies,
194 viewers stated that they preferred to watch musicals,
90 viewers stated that their preference was for musicals and thrillers,
80 viewers stated that they enjoyed all of these movies, and
56 viewers stated that they found all of these movie types too long and boring and hence never watched these movies.
The survey also revealed that the number of viewers that preferred only musicals exceeded the number of viewers that only preferred thrillers by 18 and the number that preferred only comedies by 34.

(a) Find the probability that a viewer chosen at random from this group preferred only comedies.

(b) Find the probability that a viewer chosen at random from those that preferred thriller movies also enjoyed musicals.

(c) Find the probability that a viewer chosen at random from those that stated they preferred musicals and comedies also preferred thrillers.

INDEPENDENT EVENTS

Consider the two events, rolling a die and tossing a coin. These events are said to be independent because the outcome of tossing a coin is not going to be influenced in any way by the result obtained on the roll of the die.
Now consider drawing two black cards from a standard deck of 52 cards without replacement. The probability of drawing the first black card is $\frac{26}{52} = 0.5$. Now by removing one of the black cards the probability of a second black card will be reduced. The probability that the second card is black is $\frac{25}{51} = 0.49$ rounded to two decimal places. In such cases we say that the events are not independent as the occurrence of one of the events influences the other.

Two events are said to be independent of each other if the probability that one of them occurs is unaffected by the occurrence or non-occurrence of the other event.
That is, the events A and B are independent if the probability of B occurring is unaffected by the occurrence or non-occurrence of event A.
This may be stated as follows: $P(B|A) = P(B)$ or $P(A|B) = P(A)$.

$$\boxed{\text{Events A and B are independent if } P(B|A) = P(B) \text{ or if } P(A|B) = P(A)}$$

Example 13

If events A and B are such that $P(A) = 0.5$, $P(B) = 0.8$ and $P(A \cap B) = 0.4$, determine whether the events A and B are independent.

Solution

If the events A and B are independent then the probability of B occurring is unaffected by the occurrence or non-occurrence of A.

That is $P(B|A) = P(B)$

Now $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$ and $P(B) = 0.8$

$$= \frac{0.4}{0.5}$$

$$= 0.8$$

Thus $P(B|A) = P(B)$, hence the events A and B are independent.

NOTE: From the example above $P(A).P(B) = 0.5 \times 0.8 = 0.4 = P(A \cap B)$, hence we can say that if $\mathbf{P(A).P(B) = P(A \cap B)}$ then the event A and the event B are independent.

EXERCISE 17E

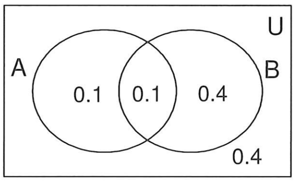
1. For any two events A and B which of the following probability statements define the events A and B as independents events

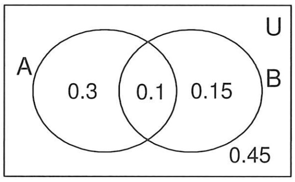
(a) $P(A|B) = P(A)$ (b) $P(B|A) = P(A)$ (c) $\dfrac{P(A \cap B)}{P(A)} = P(B)$

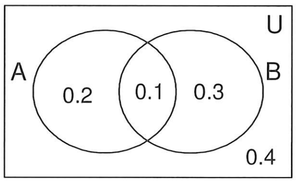
(d) $\dfrac{P(B \cap A)}{P(B)} = P(A)$ (e) $\dfrac{P(B \cap A)}{P(B)} = P(B)$ (f) $P(A).P(B) = P(A \cap B)$

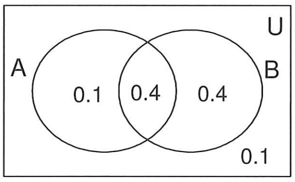
(g) $P(A \cap B) = 0$ (h) $P(A|B) = P(B|A)$

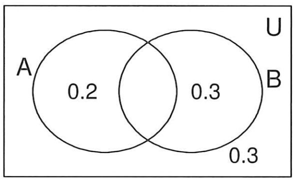
2. The Venn diagrams below show the probability of the events A and B occurring. In each case determine whether the events A and B are independent or dependent. If independent justify your answer with a mathematical statement.

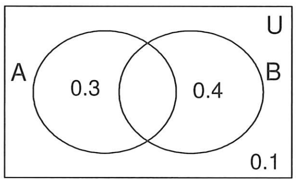
(a)


(b)


(c)


(d)


(e)


(f)


3. Two events A and B are such that $P(A) = 0.4$, $P(B) = 0.6$ and $P(A \cap B) = 0.24$.

(a) Find $P(A|B)$

(b) Show that the events A and B are independent.

(c) Are the events A and $\overline{B}$ dependent? Justify your answer.

4. Two events A and B are such that $P(A) = 0.7$, $P(B) = 0.5$ and $P(A \cup B) = 0.85$. Are the events A and B independent? Justify your answer.

5. Two events A and B are such that $P(A \cap B) = 0.1$, $P(A \cup B) = 0.8$ and $P(A|B) = 0.6$. Are the events A and B independent? Justify your answer.

6. Two events A and B are such that $P(\overline{A}) = 0.6$, $P(B) = 0.5$ and $P(A \cap B) = 0.2$ Are the events A and B independent? Justify your answer.

7. Two events A and B are such that $P(\overline{A}) = \frac{3}{4}$, $P(B) = \frac{4}{5}$ and $P(\overline{A \cup B}) = \frac{3}{20}$ Are the events A and B independent? Justify your answer.

8. Two events A and B are such that $P(A) = 0.5$, $P(A \cap B) = 0.3$ and $P(B|A) = 0.6$. Are the events A and B independent? Justify your answer.

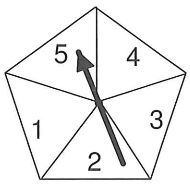
9. Two independent events A and B are such that $P(A) = 0.3$ and $P(B) = 0.4$. Determine
(a) $P(A \cup B)$

(b) $P(A|B)$

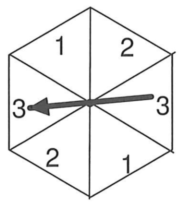
(c) $P(A|\overline{B})$

10. A pair of fair regular dice, one black and the other white, is rolled once. If event A is defined as "a four on the black die," and event B is defined as "a five on the white die." Are the events A and B dependent or independent? Justify your answer using a mathematical statement.

11.



Spinner A



Spinner B

The two spinners shown above are spun simultaneously. Determine

(a) P(1 on spinner B)

(b) P(2 on spinner A and 1 on spinner B)

(c) P(same result on both spinners)

(d) P(sum of 3)

(e) P(sum of 3 and 1 on spinner B)

Given X is the event "sum of 3", Y is the event " 1 on spinner B" and Z is the event "same on both spinners".

(f) Are the events X and Y dependent?

(g) Are the events Y and Z independent?

12. In order to enter a chosen field of study a candidate must sit for a practical test, an aptitude test and a general knowledge test. The tests are totally independent of each other and the probability of passing each test is as follows;
P(practical) = 0.4, P(aptitude) = 0.5 and P(general knowledge) = 0.8.
What is the probability of a candidate;

(a) passing all three test?

(b) failing all three tests?

(c) passing exactly 2 of these tests?

(d) passing at least one of these tests?

13. Records at a maternity hospital reveal that the first three children born on the first day of the new year were all male. Assume that the probability of a male child being born is exactly the same as that of a female child.
(a) What is the probability that the fourth child born on the first day of the new year is a male.

(b) What is the probability that the n^{th} child born on the first day of the new year is a male given that all the preceding births resulted in a male child.

(c) What is the relationship between the events "a boy child being born" and "a girl child being born".

(d) Would this relationship exist if the probability of a girl child being born was greater than that of a male child being born?

14. Two events X and Y are such that $P(X \cup Y) = 0.72$, $P(X \cap Y) = 0.18$ and $P(Y) = 2P(X)$.
(a) Determine P(X) and P(Y).
(b) Are the events X and Y independent. Justify.

15. Events C and D are such that $P(D) = 0.3$, $P(C \cap D) = 0.12$ and $P(C|\overline{D}) = 0.4$. Are the events C and D independent? Justify your response.

16. Two events, A and B in a sample space are such that: $P(A) = \frac{7}{10}$, $P(B) = \frac{2}{5}$, and $P(\overline{A \cup B}) = \frac{1}{10}$. 0.1.
Are the events A and B independent. Justify your answer.

17. Two events, A and B in a sample space are such that: $P(A) = 0.6$, $P(A \cap B) = 0.3$ and $P(A|B) = 0.75$.
Are the events A and B independent. Justify your answer.

MULTIPLICATION LAW

This law enables us to determine the "probability of the event **A and B**" and it is written as $P(A \cap B)$.
The multiplication rule applies to events which are not mutually exclusive, that is to events that can occur together.
As the multiplication law must take into account whether the occurrence or non-occurrence of one event affects the likelihood of the other event occurring we derive it from the conditional probability law.

We know that $\quad P(B|A) = \dfrac{P(A \cap B)}{P(A)}$

Hence $\quad P(A \cap B) = P(A).P(B|A)$

The multiplication law may be stated as

$$\boxed{\mathbf{P(A \cap B) = P(A).P(B|A) = P(B).P(A|B)}}$$

Example 14
A card is drawn from a standard pack of 52 cards and not replaced, then a second card is drawn. Find the probability that both cards are aces.

Solution

The probability of selecting the first ace is given by: $\quad$ P(first card ace) $= \dfrac{4}{52}$.

The probability that the second card is an ace given that the first card was an ace and it was not replaced is given by: $\quad$ P(second card ace | first card ace) $= \dfrac{3}{51}$

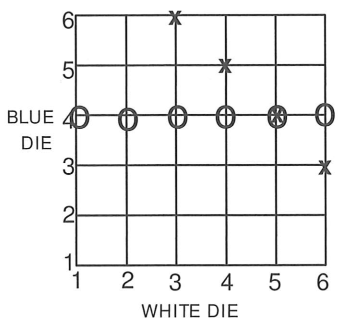
Hence $\quad$ P(both cards are aces) = P(first card is an ace and second card is an ace)
= P(first card is an ace $\cap$ second card is an ace)
= P(first card ace). P(second card ace | first card ace)
$= \dfrac{4}{52} \times \dfrac{3}{51}$
$= \dfrac{1}{221}$

Example 15
A fair white die and a fair blue die are rolled. Find the probability that
(a) the blue die shows a 4.
(b) the sum of the uppermost faces showing is 9.
(c) the blue shows a 4 and the sum is 9.

Solution

(a) P(blue die shows a 4) $= \dfrac{6}{36} = \dfrac{1}{6}$

(b) P(sum is 9) $= \dfrac{4}{36} = \dfrac{1}{9}$



(c) To find P(blue shows a 4 and the sum is 9) we must consider the probability that the sum is 9 given that the blue shows a 4, that is P(sum is 9 | blue shows 4).

P(4 on blue and sum of 9) = P(blue shows 4). P(sum is 9 | blue shows 4)
$= \dfrac{6}{36} \times \dfrac{1}{6}$
$= \dfrac{1}{36}$

Special Multiplication Law
If two events A, B are independent we know that $P(B) = P(B|A)$.
Now we know that $\quad P(A \cap B) = P(A).P(B|A)$
Hence $\quad P(A \cap B) = P(A).P(B)$ $\quad$ if the events A and B are independent.

The multiplication law may now be stated as follows:

If A and B are dependent events then	$\mathbf{P(A \cap B) = P(A).P(B \mid A)}$
If A and B are independent events then	$\mathbf{P(A \cap B) = P(A).P(B)}$ **as** $\mathbf{P(B) = P(B \mid A)}$

If we wish to determine whether two events are independent, the following tests may be applied.

Two events A and B are independent if one of the following statements holds:

$$P(B) = P(B|A)$$
$$P(A) = P(A|B)$$
$$P(A \cap B) = P(A).P(B)$$

Example 16

A card is drawn from a standard pack of 52 cards, it is replaced then a second card is drawn. Find the probability that both cards are aces.

Solution

In this example the drawing of the second card is in no way affected by the outcome of the first draw as the first card is returned to the pack before the second card is drawn that is the outcomes of the two draws are independent of each other.

Hence P(both cards are aces) = P(first card is an ace and second card is an ace)

= P((first card is an ace $\cap$ second card is an ace)

= P(first card ace). P(second card ace)

$$= \frac{4}{52} \times \frac{4}{52}$$

$$= \frac{1}{169}$$

Example 17

A fair white die and a fair blue die are rolled. Consider the following events:

Event A is: the blue die shows a 5.

Event B is: the sum of the uppermost faces is 10.

Event C is: the sum of the uppermost faces is 7.

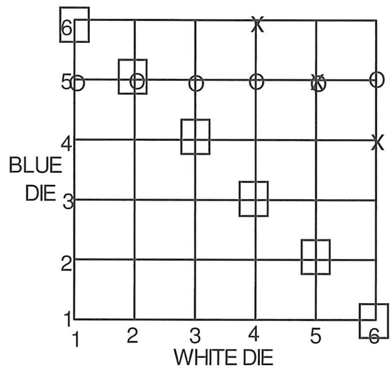
(a) Are the events A and B independent?

(b) Are the events A and C independent?

(c) Which events are mutually exclusive?

Solution

Using the grid on the right the following probabilities can be written



P(A) =6/36 = 1/6 P(B) = 3/36 = 1/12 P(C) = 6/36 = 1/6

$P(A \cap B)$ = 1/36 $P(A \cap C)$ = 1/36

(a) To determine whether the events A and B are independent we need to show that $P(A \cap B) = P(A).P(B)$.
Now P(A).P(B) = 1/6 x 1/12 = 1/72 $\neq P(A \cap B)$
Hence events A and B are not independent.

(b) To determine whether the events A and C are independent we need to show that $P(A \cap C) = P(A).P(C)$.
Now P(A).P(C) = 1/6 x 1/6 = 1/36 = $P(A \cap C)$.
Hence events A and C are independent.

(c) Events B and C are mutually exclusive as $P(B \cap C) = 0$.

EXERCISE 17F

1. A bag containing 4 red and 5 blue glass beads, the glass bead are identical in size and shape. Two beads are chosen at random from the bag one after the other, the first bead being replaced before the second one is chosen.
Calculate the following probabilities.
(a) P(first bead red and second blue)

(b) P(both beads red)

(c) P(both beads blue)

(d) P(both same colour)

(e) P(at least one is blue)

2. A bag containing 4 red and 5 blue glass beads, the glass bead are identical in size and shape. Two beads are chosen at random from the bag one after the other, the first bead chosen is not replaced before the second one is chosen.
Calculate the following probabilities.
(a) P(first bead red and second blue).
(b) P(both beads red).
(c) P(both beads blue).
(d) P(both same colour).
(e) P(at least one is blue).

3. From a well shuffled standard pack of cards, two cards are dealt to a player. Calculate the probability that;
(a) the first is a heart and second is a heart.
(b) the first is a heart and the second is a club.
(c) the first is a heart and second is not a heart.
(d) the first is a heart and the second is the ace of clubs.
(e) both are of the same suit.
(f) one is a king.

4. A survey of Year 11 and 12 students was conducted by the Student Representative Council as to whether the fee for "free dress days" should be increased from \$1 to \$2. The table below summarises the findings of the council.

	Yes	No	Undecided
Year 12	40	15	2
Year 11	15	45	3

The events T and Y are defined as follows:
T defines the event: a student in this survey that is in Year 12
Y defines the event: a student in this survey that responded with a yes.
(a) Find P(T).
(b) Find $P(T \cap Y)$. What information is given by $P(T \cap Y)$?
(c) Find (i) $P(Y | T)$ and (ii) $P(Y | \overline{T})$. . What information is given by these probabilities?

5. Two events A and B are such that P(A) = 0.5, P(B) = 0.4 and $P(A | B) = 0.2$.
(a) Find $P(A \cap B)$.
(b) Find $P(A \cup B)$.
(c) Are the events A and B independent? Justify your answer.

6. Two events A and B are such that $P(A) = P(B) = P(B | A) = 0.4$.
(a) Find $P(A \cap B)$.
(b) Find $P(\overline{A \cup B})$.
(c) Are the events A and B independent? Justify your answer.

7. Three events A, B and C are such that $P(A) = 0.5$, $P(B) = P(C) = 0.6$, $P(A \cap B \cap C) = 0.2$, $P(A \cap C) = 0.3$, $P(\overline{A \cup B \cup C}) = 0.1$ and $P(B \mid A) = 0.6$.
(a) Find $P(A \cap B)$.

(b) Find $P(B \cap C)$

(c) Which of the events A, B and C are dependent?

8. Two letters of the word CAPERS are chosen at random and used to form "two letter words".
What is the probability that:
(a) they form the word RS?
(b) the letters R and S are selected?
(c) they are both consonants?
(d) they are both vowels?
(e) the first will be a consonant and the second a vowel?
(f) one will be a consonant and the other a vowel?

9. Three letters of the word CENTRIFUGAL are randomly chosen to form "three letter words".
What is the probability that
(a) they form the word TEN?
(b) the letters T, E and N are selected?
(c) they are all consonants?
(d) they are all vowels?
(e) the first will be a consonant, the second a vowel and the third a consonant?
(f) 1 will be a vowel and the other 2 consonants?

10. The results of a poll of residents in a cross section of a suburb is shown in the table below:

	Labor	Liberal	Others	Total
Male	60	45	35	140
Female	50	70	40	160
Total	110	115	75	300

What is the probability that a resident randomly selected from this group
(a) voted for Labor?
(b) was male and voted for Labor?
(c) voted for Labor and was female?
(d) voted for Labor given they were female?
(e) was male if they voted Liberal?

11. Two events A and B are such that $P(A) = \frac{3}{7}$ and $P(A \cup B) = \frac{3}{4}$.
Determine P(B), under each of the following conditions:
(a) If $P(\overline{B}) = \frac{1}{2}$.

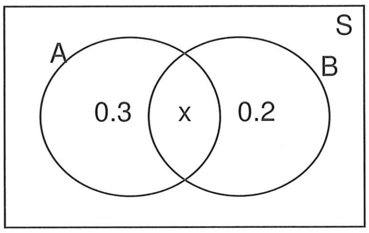
(b) If the events A and B are independent events.
(c) If the events A and B are mutually exclusive.

12. David and his sister Marguerite audition for different parts in the forthcoming school production of the musical Oaklahoma. The probability that David gets a part in the school production is 0.4 and the probability that his sister gets a part is 0.5. The probability that David gets a part given that his sister gets a part is 0.7.
(a) What is the probability that both David and Marguerite get a part in the production?

(b) What is the probability that Marguerite gets a part if her brother has already been given a part?

(c) Explain whether the events of David and his sister being successful in getting a part in the production are independent.

13. The Venn diagram shows the probabilities of the events A and B in the probability space S.
Determine the value of x if
(a) the events A and B are mutually exclusive.



(b) the events A and B are independent. (c) $P(A|B) = 0.25$

14. A particular brand of beach hat is available in sizes, small medium and large and colours, white, cream and blue. The following table summarises the stock of this beach hat at a particular beach kiosk at the start of summer.

	small	medium	large
white	30	50	15
cream	25	60	20
blue	10	30	10

(a) Calculate the probability that a customer purchases a medium sized hat.

(b) Calculate the probability that a customer purchases a medium sized blue hat.

A customer with three children purchases three small hats. What is the probability that
(c) all the hats will be cream?

(d) at most one will be blue?

(e) at least one will be white?

15. A student of both Physics and Chemistry estimates that she has a 50% chance of passing the Semester 1 Physics examination, 60% chance of passing the Semester 1 Chemistry examination and a 30% chance of passing both examinations.
On the basis of the above information;
(a) Are the events of passing these exams by this student dependent or independent? Justify your answer.

(b) What are her chances of failing both examinations?

(c) What are her chances of passing Physics but not Chemistry?

(d) What are her chances of passing one of the examinations but not the other?

16. Two events A and B are such that $P(A|B) = 0.25$ and $P(B|A) = 0.2$.
(a) If $P(A \cup B) = 0.8$, find $P(A \cap B)$.
(b) If A and B are independent find $P(A \cap B)$.

17. Before the onset of winter, two elderly people decide to get immunised against influenza. One of the randomly selected people is immunised with vaccine A and the other with vaccine B.
The probability that vaccine A will protect a person from influenza is 0.65 and the probability that vaccine B will protect against influenza is 0.55.
(a) Determine the probability that both people develop influenza.

(b) What is the probability that exactly one of these two people develop influenza?

(c) What is the probability that if exactly one of the two people develop influenza it is the person that was immunised with vaccine B?

Probability Tree Diagrams

We have in the past made use of tree diagrams for equally likely events to find the outcomes of an experiment and then by examining the sample points in the sample space determined the required probabilities.

Tree diagrams may be used in cases where the outcomes are not equally likely as well as equally likely by assigning probabilities to the branches of the tree diagram. Tree diagrams with probabilities assigned to each branch of the tree are known as ***probability tree diagrams***.

When we use a probability tree diagram we can find the probability of a particular sequence of outcomes by using the multiplication laws to find the probabilities along each branch of the tree. When using a probability tree diagram we must make certain that the sum of each set of branches is one and the sum of the ends of all the branches is one, if not an error has been made.

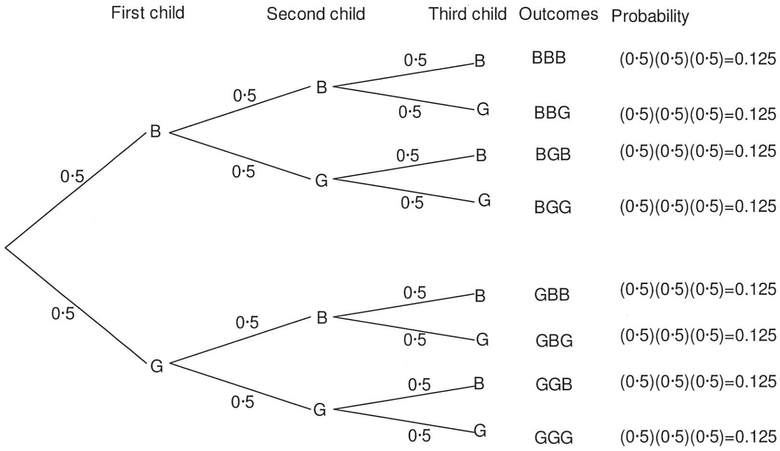
Example 18

If the event of having a boy or a girl is equally likely, find the probability that a family of three children has:

(a) three boys. (b) only one boy. (c) the youngest child a boy.

Solution

To solve this problem the sample space must give us the order of births, hence a tree diagram should be used.



Note that in this case since each outcome is equally likely the probabilities at the end of each branch have the same value and the sum of the probabilities is equal to one.

(a) P(three boys) = P(1st child boy and 2nd child boy and 3rd child boy)
= 0.5 x 0.5 x 0.5
= 0.125.

(b) P(only 1 boy) = P(only 1st child boy or only 2nd child boy or only 3rd child boy)
= (0.5x 0.5 x 0.5) + (0.5 x 0.5x 0.5) + (0.5 x 0.5x 0.5)
= 0.375

(c) P(youngest a boy) = P($B_1B_2B_3$ or $B_1G_2B_3$ or $G_1B_2B_3$ or $G_1G_2B_3$)
= (0.5 x 0.5 x 0.5) + (0.5 x 0.5x 0.5) + (0.5 x 0.5 x 0.5) + (0.5 x 0.5 x 0.5)
= 0.5

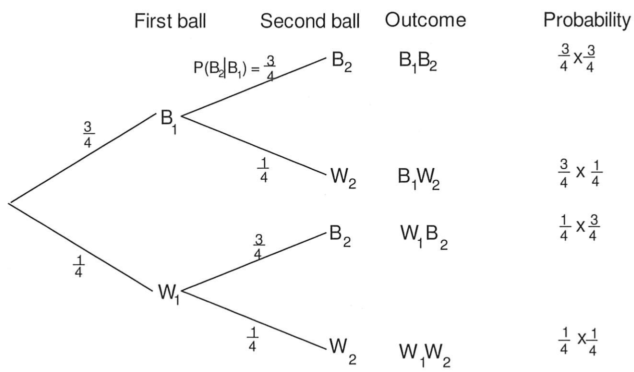
Example 19

Two balls are selected at random from a box containing 3 blue balls and 1 white ball.

(a) If the balls are replaced after each selection find the probability:
  (i) that both balls are blue.
  (ii) the first ball is blue and the second is white.

Solution

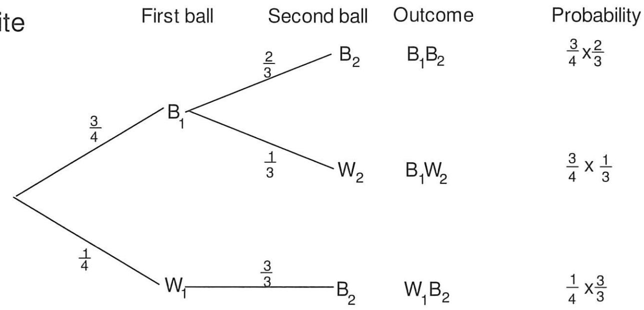
(a) (i) P(both blue) = P(B_1 and B_2)
= P($B_1 \cap B_2$)
= P(B_1) P(B_2)
= $\frac{3}{4} \times \frac{3}{4}$
= $\frac{9}{16}$



(ii) P(1st ball blue and 2nd ball white)
= P(B_1 and W_2)
= P($B_1 \cap W_2$)
= P(B_1)P(W_2)
= $\frac{3}{4} \times \frac{1}{4}$
= $\frac{3}{16}$

Note that in this case the **trials are independent** because the first ball was replaced before the second one was drawn.

(b) If the balls are not replaced after each selection find the probability:
  (i) that both balls are blue.
  (ii) the first ball is blue and the second is white

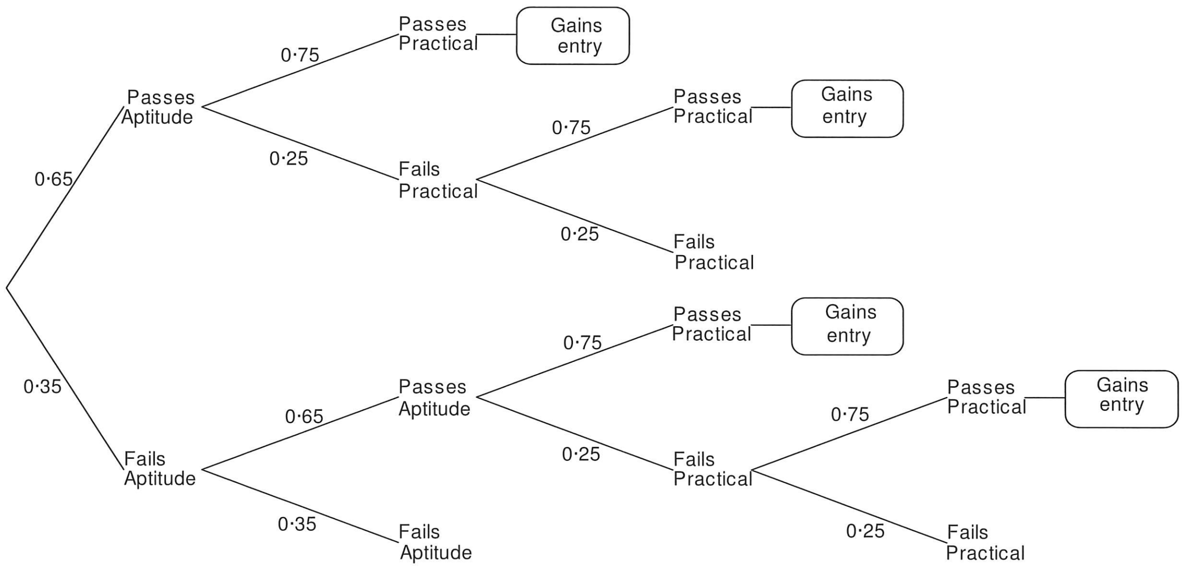


(b) (i) P(both blue) $= P(B_1 \text{ and } B_2)$
$= P(B_1)\,P(B_2|B_1)$
$= \frac{3}{4} \times \frac{2}{3}$
$= \frac{1}{2}$

(ii) P(1st ball blue and 2nd ball white)
$= P(B_1 \text{ and } W_2)$
$= P(B_1)\,P(W_2|B_1)$
$= \frac{3}{4} \times \frac{1}{3}$
$= \frac{1}{4}$

Note that that in this case **the trials are dependent** because the first ball is not replaced before the second ball is chosen. Hence the number available for the second selection is not the same as for the first selection, that is the result of the second selection depends upon what ball was selected first.

Example 20
In order to enter a special field of study a student must first pass an aptitude test and then a practical test. The student has a 65% chance of passing the aptitude test and a 75% chance of passing the practical test. If the student fails either test he may sit for it a second time, the probability of passing the second time remains the same. Find the probability the student enters this special field of study.



P(Gains entry) $= (0.65)(0.75) + (0.65)(0.25)(0.75) + (0.35)(0.65)(0.75) + (0.35)(0.65)(0.25)(0.75)$
$= 0.4875 + 0.121875 + 0.170625 + 0.04265625$
$= 0.82265625$

EXERCISE 17G

1. Two beads are selected at random from a bag containing 5 beads, 4 blue beads and 1 red bead. The first bead is replaced before the second bead is selected.
Find the probability that
(a) both beads are blue.

(b) the first bead is blue and the second is red.

(c) the first bead is red and the second is blue.

(d) the beads are of different colour.

(e) the beads are of the same colour.

(f) the beads are of different colour given that the first bead was red.

2. Two beads are selected at random from a bag containing 5 beads, 4 blue beads and 1 red bead. The first bead is not replaced before the second bead is selected.
Find the probability
(a) that both beads are blue.

(b) first bead is blue and the second is red.

(c) first bead is red and the second is blue.

(d) that the beads are of different colour.

(e) that the beads are of the same colour.

(f) that the beads are of different colour given that the first bead was red.

3. A bag contains 5 red beads and 4 blue beads. Two beads are randomly selected from this bag, the first bead not being replaced before the second bead is selected.
Determine
(a) P(both red)

(b) P(same colour)

(c) P(both red given same colour)

4. A bag contains 10 glass beads, 6 of these bead are coloured red. Two beads are randomly selected from this bag the first bead being replaced before the second one is selected.
Find the probability that
(a) both beads are red.

(b) both beads are not red.

(c) the first bead is red and the other is not red.

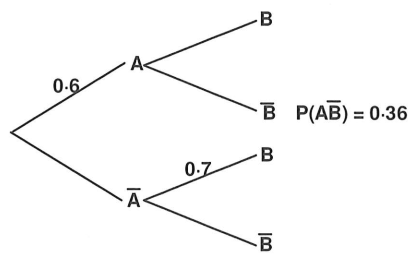
(d) the first bead is red given the second is not red.

5. The probability tree diagram shown on the right gives the probabilities of the two events A and B occurring.
(a) Complete the tree diagram by entering the missing probabilities.
(b) Explain what information is given by the 0.6.

(c) Explain what information is given by the 0.7.

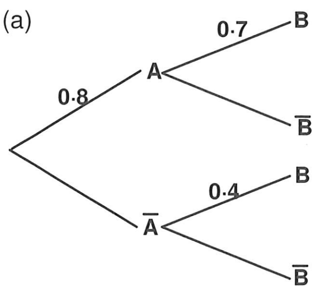
(d) Explain what information is given by the 0.36.

(e) Show the relationship between the events A and B in a Venn diagram.

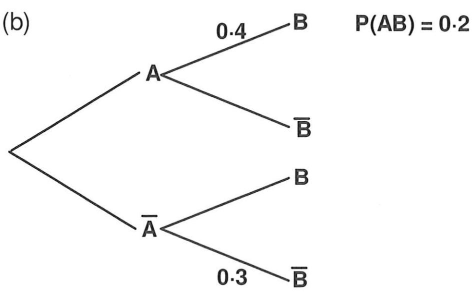


6. For each of the following probability tree diagrams draw a Venn diagram.

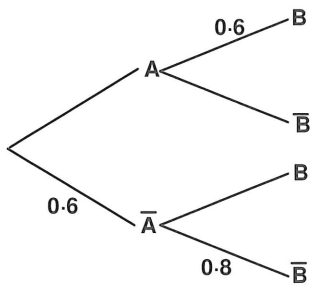
(a)



(b)



7. Consider the probability tree diagram shown below. Determine each of the following:



(a) $P(A)$

(b) $P(\overline{B} \mid A)$

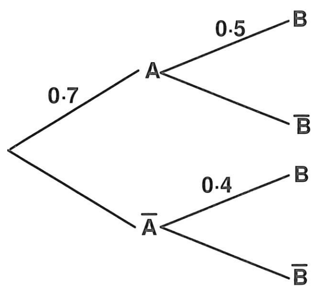
(c) $P(A \cap \overline{B})$

(d) $P(B)$

(e) $P(A \mid B)$

(f) $P(A \mid \overline{B})$

8. Consider the probability tree diagram shown below. Determine each of the following:



(a) $P(A \cap B)$

(b) $P(\overline{A} \cap B)$

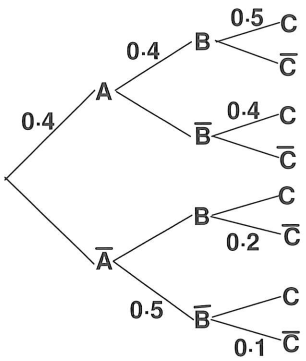
(c) $P(\overline{B} \mid A)$

(d) $P(B \mid \overline{A})$

(e) $P(\overline{B})$

(f) $P(A \mid \overline{B})$

9. Consider the probability tree diagram shown below. Determine each of the following:



(a) $P(B \mid A)$

(b) $P(\overline{B} \mid \overline{A})$

(c) $P(C \mid A \cap B)$

(d) $P(\overline{C} \mid \overline{A \cup B})$

(e) $P(A \cap B \cap C)$

(f) $P(B)$

(g) $P(C)$

(h) $P(A \mid \overline{B})$

10. In Darwin if it is cloudy in August, the probability of it being very humid is 0.775. If it is not cloudy then the probability of it not being very humid is 0.845. On average 14 days out of the 31 days of August are cloudy.
What is the probability that
(a) on August 6th this year it will be cloudy and very humid?

(b) it will very humid on August 6th this year?

(c) August 6th this year will be cloudless if it is very humid?

11. A test for a particular disease gives a positive result 95% of the time for a person with the disease and a positive result 4% of the time for a person not suffering from this disease. It is known that only 6% of the population have this disease.
A person is randomly selected from this population and tested for this disease.
What is the probability that this person:
(a) tests positive?

(b) has the disease and tests negative?

(c) does not have the disease and tests positive?

(d) does not have the disease given they tested positive.

12. During winter a householder finds that the water from the solar hot water system is hot enough 45% of the time on clear days and only 5% of the time on overcast days. The probability that any winters day is overcast is 0.7. If the water is not hot enough the householder needs to switch on the electric booster to heat the water to the desired temperature. What is the probability that on any given winters day the householder needs to switch on the electric booster?

13. A domestic water tap has two parts, the O-ring and the washer, one or both of which needs to be replaced if defective as the tap will leak.
The company manufacturing these taps finds that the probability of the O-ring being defective is 0.001 and the probability that the washer is defective is 0.01. It is also known that whether the O-ring is defective is independent of whether the washer is defective.
A randomly selected tap is tested by a quality control engineer. What is the probability that
(a) the tap leaks?

(b) the tap leaks because one of the two parts is defective?

(c) the washer is defective given that the tap leaks?

14. The probability that bad weather will necessitate postponing the athletics carnival scheduled for Friday of this week is 0.1. If the athletics carnival is postponed because of the weather the probability that it will be held the following Monday is 0.4. If the weather permits the athletics carnival to be held on Friday then the probability that the weather is bad on Monday is 0.4.
Find the probability that
(a) the athletics carnival is held on Friday of this week.

(b) the athletics carnival is held on the Monday.

(c) the carnival had to be postponed a second time.

(d) the weather was fine on Monday.

15. The regular goalkeeper of a soccer team injures his back during training a week before the final match of the season. The teams doctor and physiotherapist give the injured goalkeeper a 55% chance of playing the final match of the season.
If the regular goalkeeper does not play the final game then the team has a 40% chance of winning. However, if the goalkeeper recovers and plays, then the team has a 75% chance of winning.
(a) Find the probability that the team won the final game of the season.

(b) If the regular goalkeeper played, what is the probability that the team wins the final game?

(c) If the team lost the final, what is the probability that the regular goalkeeper did not play?

16. In order to be eligible to enter a special field of study a student must first pass an aptitude test and then a practical test. The student has a 70% chance of passing the aptitude test and a 80% chance of passing the practical test. If the student fails either test he may sit for it a second time, the probability of passing the second time remains the same.
(a) Find the probability the student is eligible to enter this special field of study.
(b) Given that the student failed the practical test twice, what is the probability that he also failed the aptitude test?

CHAPTER SEVENTEEN CHECKLIST

You now should be able to:

- identify and understand the probability scale: $0 \le P(A) \le 1$ and if $P(A) = 0$ is then the occurrence of A is an impossibility and if $P(A) = 1$ then the occurrence of A is a certainty
- determine the probabilities of events using
  - systematic lists
  - product tables
  - tree diagrams
  - Venn diagrams
  - counting techniques
- use the relationship $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ to calculate probabilities
- use the relationship $P(\bar{A}) = 1 - P(A)$
- use the formulas $P(A|B) = \frac{P(A \cap B)}{P(B)}$ and $P(B|A) = \frac{P(A \cap B)}{P(A)}$
- identify conditional probability situations and calculate conditional probabilities
- use the formulas $P(A \cap B) = P(A).P(B|A) = P(B).P(A|B)$ to calculate probabilities
- identify independent events using the formulas: $P(A \cap B) = P(A).P(B)$;
  $P(A|B) = P(A)$
  $P(B|A) = P(B)$
- identify mutually exclusive events using the relationship $P(A \cap B) = 0$

CHAPTER SEVENTEEN REVIEW EXERCISE

1. Bridge is a card game for two teams, each team consisting of two players. On Monday evenings Mr and Mrs Brown and Mr and Mrs Green meet to play bridge. Each Monday they select the makeup of the teams for the evening by drawing their names from a hat.
(a) How many different teams can be made up?
(b) On how many of these teams is Mrs Brown?
Find the probability that
(c) both teams have members of the same sex.
(d) each of the teams has a female member.
(e) each of the teams consists of husband and wife.

2. In probability sample space S the events A and B have probabilities $\frac{2}{3}$ and $\frac{1}{2}$ respectively.
   If $P(B \mid A) = \frac{1}{2}$, find
   (a) $P(A \cup B)$.
   (b) $P(\overline{A} \mid B)$.
   (c) whether the events $\overline{A}$ and B are mutually exclusive. Justify your answer.
   (d) whether the events $\overline{A}$ and B are independent. Justify your answer.

3. Tom and Jack are finalists in a diving competition. The probability that Tom executes a perfect dive on his first attempt is 0.7 while the probability that Jack executes a perfect dive on his first attempt is 0.6. The probability that Tom executes a perfect dive on both his first and second attempts is 0.65. What is the probability that
   (a) both divers execute a perfect dive on their first attempt?
   (b) either diver executes a perfect dive on their first attempt?
   (c) Tom executes a perfect dive on his second attempt given that he executed a perfect dive on his first attempt.
   (d) Tom executes a perfect dive on his first attempt but not on his second attempt.

4. A group of investors at a conference for shareholders of a particular company were surveyed as to the type of shares they had in their portfolios out of communications shares, mining shares and banking shares.
   The survey found that:
   61% of the investors had banking shares
   60% of the investors had communications shares
   53% of the investors had mining shares
   24% of the investors had all three types
   36% of the investors had mining and banking shares
   and half of the investors had only one share type
   The number of investors that held only communications shares was double the number that held only mining shares.
   An investor from this group is chosen at random. What is the probability that this investor
   (a) had only banking shares?
   (b) had communication and banking shares?
   (c) had only mining and communication shares given he had communications shares?

5. The Dean of Studies at a particular school is investigating the relationship between regular study habits by students and their performance in assessment tasks.
   The Dean of Studies found that
   - 60% of the students had regular study habits
   - if a student studied regularly there is a 80% chance that their performance in assessment tasks improved during the year
   - if a student studied irregularly there is 15% chance that their performance in assessment tasks improved during the year

   What is the probability that a student chosen at random from this school
   (a) studied regularly and the student's performance in assessment tasks improved during the year?
   (b) studied irregularly and the student's performance in assessment tasks improved during the year?
   (c) studied irregularly and the student's performance in assessment tasks did not improved during the year?
   (d) studied regularly if the student's performance in assessment tasks improved during the year?
   (e) studied irregularly if the student's performance in assessment tasks improved during the year?

6. Using the fact that for any two events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
   (a) Show that $P(A \mid B) = 1 - P(\overline{A} \mid B)$
   (b) What restriction(s), if any, must be placed on the events A and B in proving the above relationship?

7. Records kept by the teacher in charge of late arrivals for a school on any given day reveal the following probability distribution.

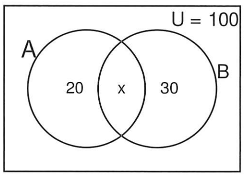
Number of students that are late	0	1	2	3	4
Probability	0.35	0.25	0.2	0.15	0.05

   (a) What is the probability that on any given day only one student is late for school.
   (b) What is the probability that at least one student is late for school on any given day.
   (c) What is the probability that at most two students are late for school on any given day.

8. During the subject selection process at a particular school the Year 10 students are required to select six or seven subjects for their eleventh year of study. The Mathematics Department offers the following four mathematics courses; Specialist Maths, Methods, Applications and Essentials. The selection of the mathematics courses was of interest to the Head of Mathematics and the responses of the 82 Year 10 students are summarised below:
   - 2 students selected none of the mathematics courses on offer,
   - 4 students selected only Essentials and no other mathematics course,
   - 39 students selected Specialist Maths,
   - 44 students selected Applications,
   - one-third of the Specialist Maths students also selected Methods,
   - all of the students selecting Methods also selected Specialist Maths, and
   - none of the students selected Applications and Methods.

   (a) Find the number of students that selected two mathematics courses to study in Year 11.
   (b) What relationship exists between the set of students selecting Specialist Maths and the set of students selecting Methods,
   What is the probability that a student selected at random
   (c) selected to study only one of the mathematics courses?
   (d) selected at most one of these subjects?
   (e) selected Applications given that they selected at most one of the mathematics courses.
   (f) selected exactly two of the mathematics courses given that they selected Specialist Maths?

9. The Venn diagram on the right shows the number of sample points in the events A and B which are contained in sample space U.
   Determine the value of x if
   (a) the events A and B are mutually exclusive.
   (b) the events A and B are independent. (c) $P(B|A) = \frac{5}{9}$

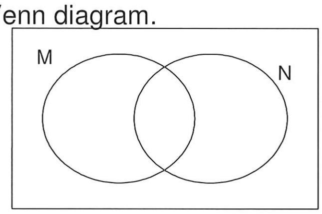


10. Complete the following statements, given that the two events M and N are such that $P(M) > 0$.
   (a) The probability of the events M and N occurring is given by
   (b) If events M and N are mutually exclusive, then the probability of M and N occurring is
   (c) If the events M and N are independent, then the probability of the events M and N occurring is given by

11. Blood tests are performed on blood samples taken from vegetarians to determine the presence of iron radicals in the blood. It was found that for 5% of blood samples with a deficiency in iron radicals the blood test indicated that there was no deficiency, and for 8% of blood samples without an iron deficiency the tests indicated an iron deficiency.
   Previous medical research has found that 60% of all vegetarians have a deficiency of iron radicals in their blood.
   A vegetarian is selected at random and a blood sample is taken for iron radical testing.
   What is the probability that the vegetarian in question
   (a) was correctly tested as not having an iron deficiency?
   (b) tested as having a deficiency in iron radicals?

12. Jonno is about to sit for his driving test and a mathematics test on the same day. He has a 56% chance of passing both tests and only a 60% chance of passing his mathematics test if he does not pass his driving test. If Jonno has a 70% chance of passing his driving test, find the probability that
   (a) he passes his mathematics test if he passed his driving test.
   (b) he passes at least one of the tests.
   (c) he passes his mathematics test.

13. Two events M and N are such that: $P(M) = 0.65$, $P(M \cap N) = 0.3$ and $P(\overline{M} \cap N) = 0.2$.
   Make use of the given information about M and N to complete the Venn diagram.
   (a) Determine $P(\overline{M \cup N})$. (b) Determine $P(N|M)$.
   (c) Are the events M and N mutually exclusive?
   Justify your answer.
   (d) Are the events M and N independent? Justify your answer.



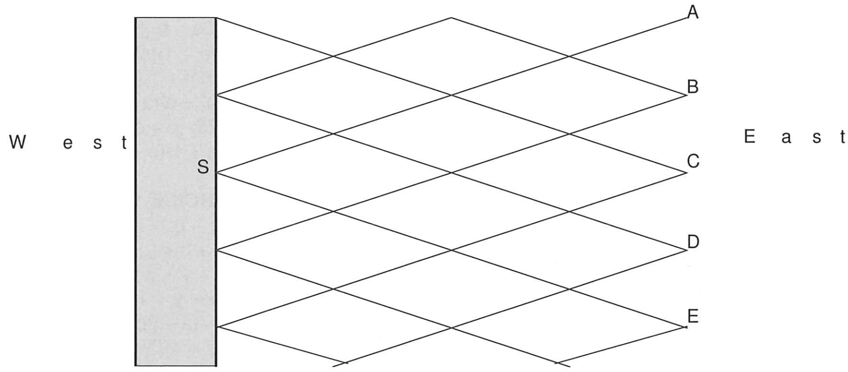
14. A carton of one dozen eggs is known to contain 3 bad eggs. A sample of 4 eggs is randomly chosen from this carton.
   (a) What is the probability that this sample contains no bad eggs?
   (b) What is the probability that this sample contains t lease one bad egg?

15. Extensive medical records show that 1 in 50 adults in a certain country suffer from diabetes. Of these, 25% are insulin dependent and the remainder are not. In a blood test designed to detect diabetes it was found that 90% of those who are insulin dependent will give a positive result and 80% of those who are not insulin dependent will give a positive result.
Also, 5% of people without diabetes give a positive blood test result.
(a) Show the sample space using an appropriate diagram.
(b) Determine the probability that a randomly chosen adult from this country will have diabetes and show a positive blood test result.
(c) Determine the probability that a randomly chosen adult from this country will have diabetes given that the blood test shows a negative result.

16. On any given day Gloria drinks either tea or coffee. The probability that Gloria drinks a cup of coffee after drinking a cup of tea is 0.7, whereas the probability that she drinks a cup of tea after drinking a cup of coffee is 0.8. This morning Gloria started her day with a cup of tea. What is the probability that her third cup will be coffee?

17. A farmer travelling from his farm into the city passes by four service stations each offering food and fuel at discounted prices. Because of the distance between his farm and the city the farmer must stop at least once to purchase fuel.
The probability that the farmer stops at the first service station on his way into the city is 0.3.
If the farmer stops at the first service station, then the probability of him stopping at the second service station is 0.1. If on the other hand he did not stop at the first service station, then the probability of him stopping at the second service station is 0.6.
If the farmer had not stopped prior to reaching the third service station, then the probability he stops at this service station is 0.8, otherwise the probability of the farmer stopping at this service station is 0.4.
If the farmer had made only one stop prior to reaching the fourth service station, then the probability of him stopping at this service station is 0.5 and if he had made more than one stop then the probability of him stopping again is 0.3.
(a) What is the probability that the farmer stops at all four service stations?
(b) What is the probability that the farmer stops at only two of the four service stations?
(c) What is the probability that the farmer only stops once in travelling from his farm into the city?
(d) If the farmer only stopped once on his way into the city, what is the probability that he stopped at the second service station?

18. An ant climbs a fence post to position S and then from S always proceeds eastwards along the wire strands of the diamond patterned fencing mesh as shown below.



At junction S, there is a 50% chance that the ant takes the top wire strand. On coming to the next and subsequent junctions of the wire mesh the probability that the ant takes the top wire strand remains unchanged. If the ant starts at junction S,
(a) find the probability that the ant passes through point A.
(b) find the probability that the ant passes through point B.
(c) through which point is the ant most likely to pass through. Find the probability of the ant passing through this point.

Observation revealed that at S the ant took the bottom wire strand.
(d) Through which point is the ant most likely to pass through? Justify your answer.
A second ant also starts from S. The probability that this ant always takes the top wire strand is 0.2.
(e) Through which point is this ant most likely to pass through? Justify your answer.