8 Relativistic Landau

Starting from Fokker-Planck-Landau equation has the following structure (we keep the notation consistent with Strain and Taskovic [ST19])

$$\partial_t f = \nabla_p \cdot \left(\int_{\mathbb{R}^3} \Phi(p,q) \cdot (f(q)\nabla_p f(p) - f(p)\nabla_q f(q)) \; dq \right). \tag{1}$$

where Φ is a collision kernel. Specifically, if we let the single particle kinetic energy as $\mathcal{E}(p)$, this collision kernel can be written as the product of a scalar field Λ and a tensor field $\mathbb{S}$:

$$\Phi(p,q) = \Lambda\,(p,q)\,\mathbb{S}\,(\nabla_p \mathcal{E}(p), \nabla_q \mathcal{E}(q))\,.$$

The tensor field $\mathbb{S}(\mathbf{v},\mathbf{w})$ depends on velocities $\mathbf{v},\mathbf{w} \in \mathbb{R}^3$. Denoting the relative velocity as $\mathbf{u} := \mathbf{v} - \mathbf{w}$, and the mean velocity as $\mathbf{z} := \frac{\mathbf{v}+\mathbf{w}}{2}$, then we consider tensors $\mathbb{S}$ of the form,

$$\mathbb{S}(\mathbf{v},\mathbf{w}) := \begin{cases} (\mathbf{u}\cdot\mathbf{u})I - \mathbf{u}\otimes\mathbf{u}, & \text{non-relativistic,} \\ (\mathbf{u}\cdot\mathbf{u})I - \mathbf{u}\otimes\mathbf{u} - (\mathbf{z}\times\mathbf{u})\otimes(\mathbf{z}\times\mathbf{u})\,, & \text{relativistic.} \end{cases} \tag{2}$$

where I is the 3×3 identity matrix. Moreover, the scalar field is given by

$$\Lambda(p,q) := \frac{(\rho+1)^2}{\mathcal{E}(p)\mathcal{E}(q)}(\tau\rho)^{-3/2} \tag{3}$$

Where

$$\rho(p,q) := \mathcal{E}(p)\mathcal{E}(q) - p\cdot q - 1$$
$$\tau(p,q) := \mathcal{E}(p)\mathcal{E}(q) - p\cdot q + 1$$

so that $\tau = \rho + 2$. Finally, the energy in the **relativistic case** is given by

$$\mathcal{E}(x) = \sqrt{1+|x|^2}$$

where $x \in \mathbb{R}^3$. I will abbreviate with $\mathcal{E}_x = \mathcal{E}(x)$. The velocity is then given by

$$\nabla\mathcal{E}(x) = \frac{x}{\sqrt{1+|x|^2}} = \frac{x}{\mathcal{E}_x}$$

8.1 Goal

As seen in the proof of theorem 5, the key to rotational equivariance (invariance, covariance) is to prove that the collision kernel has the property

$$\Phi(\mathbf{R}p,q) = \mathbf{R}\Phi(p,\mathbf{R}^T q)\mathbf{R}^T$$

where $\mathbf{R}$ is a 3×3 rotation matrix. (It should satisfy $\mathbf{R}^{-1} = \mathbf{R}^T$, $\det(\mathbf{R}) = 1$.) There are two parts for this. First, the scalar field must satisfy

$$\Lambda(\mathbf{R}p,q) = \Lambda(p,\mathbf{R}^T q) \tag{4}$$

and the tensor field must satisfy

$$= \mathbf{R}\mathbb{S}(p,\mathbf{R}^T q)\mathbf{R}^T \tag{5}$$

8.2 Key Property

Here I highlight what it seems to me to be the key property. In both the relativistic and non-relativistic case we have that

$$\mathcal{E}(x) = F(r(x))$$

That is, given a vector $x \in \mathbb{R}^3$, the energy only depends on the radial part of the vector, $r(x) = \sqrt{x \cdot x}$. Then we have that this energy function is invariant with respect of rotations $\mathbf{R}$,

$$\mathcal{E}(\mathbf{R}x) = F(r(\mathbf{R}x)) = F(\sqrt{\mathbf{R}x \cdot \mathbf{R}x}) = F(\sqrt{x \cdot x}) = F(r(x)) = \mathcal{E}(x)$$

Moreover, the gradient of this energy is rotational covariant (it commutes with rotation matrices). First, compute the gradient

$$\begin{aligned}
\nabla\mathcal{E}(x) &= \nabla F(r(x)) \\
&= \partial_r F(r)\nabla r(x) \\
&= \partial_r F(r)x/r(x) \\
&= G(r(x))x
\end{aligned}$$

where $G(r(x)) = \partial_r F(r)/r$ is another scalar. Then it is easy to see

$$\begin{aligned}
\nabla\mathcal{E}(\mathbf{R}x) &= G(r(\mathbf{R}x))\mathbf{R}x \\
&= \mathbf{R}(G(r(x))x) \\
&= \mathbf{R}\nabla\mathcal{E}(x)
\end{aligned}$$

We can see that is the case for the relativistic energy (for the non-relativistic it is easier)

$$\mathcal{E}(\mathbf{R}x) = \sqrt{1 + |\mathbf{R}x|^2} = \sqrt{1 + |x|^2} = \mathcal{E}(x)$$

$$\nabla\mathcal{E}(\mathbf{R}x) = \frac{\mathbf{R}x}{\mathcal{E}_{\mathbf{R}x}} = \mathbf{R}\frac{x}{\mathcal{E}_x} = \mathbf{R}\nabla\mathcal{E}(x)$$

8.3 The Scalar Field

Note

$$\begin{aligned}
\rho(\mathbf{R}p, q) &= \mathcal{E}(\mathbf{R}p)\mathcal{E}(q) - (\mathbf{R}p) \cdot q - 1 \\
&= \mathcal{E}_p\mathcal{E}_{\mathbf{R}^T q} - (\mathbf{R}p) \cdot (\mathbf{R}\mathbf{R}^T q) - 1 \\
&= \mathcal{E}_p\mathcal{E}_{\mathbf{R}^T q} - p \cdot \mathbf{R}^T q - 1 \\
&= \rho(p, \mathbf{R}^T q)
\end{aligned}$$

Similarly, $\tau(\mathbf{R}p, q) = \tau(p, \mathbf{R}^T q)$. We then check

$$\begin{aligned}
\Lambda(\mathbf{R}p, q) &= \frac{(\rho(\mathbf{R}p, q) + 1)^2}{\mathcal{E}(\mathbf{R}p)\mathcal{E}(q)}(\tau(\mathbf{R}p, q)\rho(\mathbf{R}p, q))^{-3/2} \\
&= \frac{(\rho(p, \mathbf{R}^T q) + 1)^2}{\mathcal{E}(p)\mathcal{E}(\mathbf{R}^T q)}(\tau(p, \mathbf{R}^T q)\rho(p, \mathbf{R}^T q))^{-3/2} \\
&= \Lambda(p, \mathbf{R}^T q)
\end{aligned}$$

8.4 The Tensor Field

The tensor is given by

$$
\begin{aligned}
\mathbb{S}(p,q) = &(\nabla\mathcal{E}(p) - \nabla\mathcal{E}(q))^2 \\
&- (\nabla\mathcal{E}(p) - \nabla\mathcal{E}(q))^{\otimes} \\
&- \big((\nabla\mathcal{E}(p) + \nabla\mathcal{E}(q))/2 \times (\nabla\mathcal{E}(p) - \nabla\mathcal{E}(q))\big)^{\otimes}
\end{aligned}
$$

Where for simplicity I use the notation $x^2 = (x \cdot x)I$ and $x^{\otimes} = x \otimes x$ (or equivalently $x^{\otimes} = xx^T$), $x \in \mathbb{R}^3$ and I is the 3×3 identity matrix. The goal is to compute $\mathbb{S}(\mathbf{R}p, q)$. We check term by term. For the first one,

$$
\begin{aligned}
\nabla\mathcal{E}(\mathbf{R}p) - \nabla\mathcal{E}(q) &= \nabla\mathcal{E}(\mathbf{R}p) - \nabla\mathcal{E}(\mathbf{R}\mathbf{R}^T q) \\
&= \mathbf{R}(\nabla\mathcal{E}(p) - \nabla\mathcal{E}(\mathbf{R}^T q)) \\
&\Longrightarrow \\
(\nabla\mathcal{E}(\mathbf{R}p) - \nabla\mathcal{E}(q))^2 &= (\nabla\mathcal{E}(p) - \nabla\mathcal{E}(\mathbf{R}^T q))^2
\end{aligned}
$$

Where we used the fact that $\mathbf{R}x \cdot \mathbf{R}x = x \cdot x$. Using the previous, it is easy to see that for the second one

$$
\begin{aligned}
(\nabla\mathcal{E}(\mathbf{R}p) - \nabla\mathcal{E}(q))^{\otimes} &= (\mathbf{R}(\nabla\mathcal{E}(p) - \nabla\mathcal{E}(\mathbf{R}^T q)))^{\otimes} \\
&= \mathbf{R}\big\{\nabla\mathcal{E}(p) - \nabla\mathcal{E}(\mathbf{R}^T q)\big\}^{\otimes}\mathbf{R}^T
\end{aligned}
$$

Here I used a small calculation:

$$
(\mathbf{R}x)^{\otimes} = (\mathbf{R}x)(\mathbf{R}x)^T = \mathbf{R}xx^T\mathbf{R}^T = \mathbf{R}(x^{\otimes})\mathbf{R}^T
$$

Remark. The point of the previous was to use the fact that if $u(p,q) = \nabla\mathcal{E}(p) - \nabla\mathcal{E}(q)$, then

$$
u(\mathbf{R}p, q) = \mathbf{R}u(p, \mathbf{R}^T q)
$$

And apply to the other compute both u^2 and $u^{\otimes}$.

We can prove something similar for $z(p,q) = (\nabla\mathcal{E}(p) + \nabla\mathcal{E}(q))/2$. Note

$$
\begin{aligned}
z(\mathbf{R}p, q) &= (\nabla\mathcal{E}(\mathbf{R}p) + \nabla\mathcal{E}(q))/2 \\
&= (\nabla\mathcal{E}(\mathbf{R}p) + \neq \mathbf{R}\mathbf{R}^T q))/2 \\
&= \mathbf{R}(\nabla\mathcal{E}(p) + \neq \mathbf{R}^T q))/2 \\
&= \mathbf{R}z(p, \mathbf{R}^T q)
\end{aligned}
$$

Using these two we can easily compute the last term. First consider the cross product...

8.5 The Tensor Field (again)

Consider the relative and mean velocity

$$
\begin{aligned}
u(p,q) &= \nabla\mathcal{E}(p) - \nabla\mathcal{E}(q) \\
z(p,q) &= (\nabla\mathcal{E}(p) + \nabla\mathcal{E}(q))/2
\end{aligned}
$$

Then, consider

$$
\begin{aligned}
u(\mathbf{R}p,q) &= \nabla\mathcal{E}(\mathbf{R}p) - \nabla\mathcal{E}(q) \\
&= \mathbf{R}(\nabla\mathcal{E}(p) - \nabla\mathcal{E}(\mathbf{R}^T q)) \\
&= \mathbf{R}u(p,\mathbf{R}^T q) \\
z(\mathbf{R}p,q) &= (\nabla\mathcal{E}(\mathbf{R}p) + \nabla\mathcal{E}(q))/2 \\
&= \mathbf{R}(\nabla\mathcal{E}(p) + \nabla\mathcal{E}(\mathbf{R}^T q))/2 \\
&= \mathbf{R}z(p,\mathbf{R}^T q)
\end{aligned}
$$

Then, consider the first term, using the fact that rotations preserve inner products

$$
\begin{aligned}
u(\mathbf{R}p,q)\cdot u(\mathbf{R}p,q) &= (\mathbf{R}u(p,\mathbf{R}^T q))\cdot(\mathbf{R}u(p,\mathbf{R}^T q)) \\
&= u(p,\mathbf{R}^T q)\cdot u(p,\mathbf{R}^T q)
\end{aligned}
$$

This is a scalar, but we care about the tensor:

$$
\begin{aligned}
u(\mathbf{R}p,q)\cdot u(\mathbf{R}p,q)I &= u(p,\mathbf{R}^T q)\cdot u(p,\mathbf{R}^T q)I \\
&= u(p,\mathbf{R}^T q)\cdot u(p,\mathbf{R}^T q)\mathbf{R}\mathbf{R}^T \\
&= \mathbf{R}(u(p,\mathbf{R}^T q)\cdot u(p,\mathbf{R}^T q))\mathbf{R}^T
\end{aligned}
$$

As for the second term

$$
\begin{aligned}
u(\mathbf{R}p,q)\otimes u(\mathbf{R}p,q) &= (\mathbf{R}u(p,\mathbf{R}^T q))\otimes(\mathbf{R}u(p,\mathbf{R}^T q)) \\
&= (\mathbf{R}u(p,\mathbf{R}^T q))(\mathbf{R}u(p,\mathbf{R}^T q))^T \\
&= \mathbf{R}(u(p,\mathbf{R}^T q)u(p,\mathbf{R}^T q)^T)\mathbf{R}^T \\
&= \mathbf{R}(u(p,\mathbf{R}^T q)\otimes u(p,\mathbf{R}^T q))\mathbf{R}^T
\end{aligned}
$$

As for the third term, we first consider the cross product, and use the fact that $(\mathbf{R}a\times\mathbf{R}b) = \mathbf{R}(a\times b)$ for rotation matrices.

$$
\begin{aligned}
z(\mathbf{R}p,q)\times u(\mathbf{R}p,q) &= (\mathbf{R}z(p,\mathbf{R}^T q))\times(\mathbf{R}u(p,\mathbf{R}^T q)) \\
&= \mathbf{R}(z(p,\mathbf{R}^T q)\times u(p,\mathbf{R}^T q))
\end{aligned}
$$

Therefore, the last term is

$$
\begin{aligned}
&(z(\mathbf{R}p,q)\times u(\mathbf{R}p,q))\otimes(z(\mathbf{R}p,q)\times u(\mathbf{R}p,q)) \\
=&(\mathbf{R}(z(p,\mathbf{R}^T q)\times u(p,\mathbf{R}^T q)))\otimes(\mathbf{R}(z(p,\mathbf{R}^T q)\times u(p,\mathbf{R}^T q))) \\
=&\mathbf{R}((z(p,\mathbf{R}^T q)\times u(p,\mathbf{R}^T q))\otimes(z(p,\mathbf{R}^T q)\times u(p,\mathbf{R}^T q))\mathbf{R}^T
\end{aligned}
$$

Now we can put everything together. Again, the tensor part of the relativistic kernel is given by

$$
\begin{aligned}
\mathbb{S}(p,q) = &(u(p,q)\cdot u(p,q))I \\
&- (u(p,q)\otimes u(p,q)) \\
&- (z(p,q)\times u(p,q))\otimes(z(p,q)\times u(p,q))
\end{aligned}
$$

Then we consider

$$
\begin{aligned}
\mathbb{S}(\mathbf{R}p,q) &= (u(\mathbf{R}p,q)\cdot u(\mathbf{R}p,q))I \\
&- (u(\mathbf{R}p,q)\otimes u(\mathbf{R}p,q)) \\
&- (z(\mathbf{R}p,q)\times u(\mathbf{R}p,q))\otimes (z(\mathbf{R}p,q)\times u(\mathbf{R}p,q)) \\
&= \mathbf{R}(u(p,\mathbf{R}^Tq)\cdot u(p,\mathbf{R}^Tq))\mathbf{R}^T \\
&- \mathbf{R}(u(p,\mathbf{R}^Tq)\otimes u(p,\mathbf{R}^Tq))\mathbf{R}^T \\
&- \mathbf{R}((z(p,\mathbf{R}^Tq)\times u(p,\mathbf{R}^Tq))\otimes (z(p,\mathbf{R}^Tq)\times u(p,\mathbf{R}^Tq))\mathbf{R}^T \\
&= \mathbf{R}\mathbb{S}(p,\mathbf{R}^Tq)\mathbf{R}^T
\end{aligned}
$$

8.6 Rotational Covariance of Relativistic Landau

Now, we put the previous together

Theorem 4. *For a rotation matrix $\boldsymbol{R}$ and the relativistic kernel $\Phi(p,q)$, we have*

$$\Phi(\boldsymbol{R}p,q) = \boldsymbol{R}\Phi(p,\boldsymbol{R}^Tq)\boldsymbol{R}^T$$

Proof.

$$\Phi(\mathbf{R}p,q) = \Lambda(\mathbf{R}p,q)\mathbb{S}(\mathbf{R}p,q) = \Lambda(p,\mathbf{R}^Tq)\mathbf{R}\mathbb{S}(p,\mathbf{R}^Tq)\mathbf{R}^T = \mathbf{R}\Phi(p,\mathbf{R}^Tq)\mathbf{R}^T$$

Where we use the two relations derived for the scalar field Λ and the tensor $\mathbb{S}$ before. □

Now that this is established, we may repeat the previous argument to show that the Landau Equation is rotational covariant.

Theorem 5 (Landau's rotational covariant)**.** *The Landau collision operator is rotational covariant.*

Proof. Let $x = \mathbf{R}^Tq$ so that $\mathbf{R}x = q$ and $\mathrm{d}x = \mathrm{d}q$. Then, by the previous, we have

$$\Phi(\mathbf{R}p,q) = \mathbf{R}\Phi(p,x)\mathbf{R}^T$$

Then, we check,

$$
\begin{aligned}
Q(f,f)(\mathbf{R}p) &= \nabla_{\mathbf{R}p}\cdot\left(\int_q \Phi(\mathbf{R}p,q)\Big(g(q)\nabla_{\mathbf{R}p}f(\mathbf{R}p) - f(\mathbf{R}p)\nabla_q g(q)\Big)\mathrm{d}q\right) \\
&= \nabla_{\mathbf{R}p}\cdot\left(\int_x \mathbf{R}\Phi(p,x)\mathbf{R}^T\Big(g(q)\nabla_{\mathbf{R}p}f(\mathbf{R}p) - f(\mathbf{R}p)\nabla_q g(q)\Big)\mathrm{d}q\right) \\
&= \nabla_{\mathbf{R}p}\cdot\left(\int_x \mathbf{R}\Phi(p,x)\mathbf{R}^T\Big(g(\mathbf{R}x)\nabla_{\mathbf{R}p}f(\mathbf{R}p) - f(\mathbf{R}p)\nabla_{\mathbf{R}x} g(\mathbf{R}x)\Big)\mathrm{d}x\right) \\
&= \mathbf{R}\nabla_p\cdot\mathbf{R}\left(\int_x \Phi(p,x)\mathbf{R}^T\Big(g(\mathbf{R}x)\mathbf{R}\nabla_p f(\mathbf{R}p) - f(\mathbf{R}p)\mathbf{R}\nabla_x g(\mathbf{R}x)\Big)\mathrm{d}x\right) \\
&= \mathbf{R}\nabla_p\cdot\mathbf{R}\left(\int_x \Phi(p,x)\mathbf{R}^T\mathbf{R}\Big(g(\mathbf{R}x)\nabla_p f(\mathbf{R}p) - f(\mathbf{R}p)\nabla_x g(\mathbf{R}x)\Big)\mathrm{d}x\right) \\
&= \nabla_p\cdot\left(\int_x \Phi(p,x)\Big(g(\mathbf{R}x)\nabla_p f(\mathbf{R}p) - f(\mathbf{R}p)\nabla_x g(\mathbf{R}x)\Big)\mathrm{d}x\right) \\
&= Q(Rf,Rg)(p)
\end{aligned}
$$

Where we use that $\nabla_{\mathbf{R}p} = \mathbf{R}\nabla_p$ and similarly for x. □

8.7 A Conservative Numerical Scheme

The previous shows that the relativistic Landau equation is rotational covariant (equivariant). As we have learned from [HT23], this implies that the discrete collision tensor will be sparse if we use the same basis functions described before. As far as I know, this will lead the most efficient solver for the **relativistic** Landau equation, or at least a very efficient one, worth of studying.

There is one issue that needs to be discussed: conservation laws. The collisional invariants for the relativistic equation are given by the set

$$\left\{1, p_x, p_y, p_z, \mathcal{E}(p) = \sqrt{1+|p|^2}\right\}$$

The first 4 are in our test space, and therefore the scheme will preserve these moments without need for an extra strategy. However, the energy (which went from being a polynomial $\mathcal{E}(p) = |p|^2$, to not being one) is no longer in the test space. How do we solve this issue? Here we use the strategy from the work with **Kun Huang**. First, let us review the weak form of the Landau equation.

$$\begin{aligned}
\int_p Q(f,f)(p)\phi(p)\mathrm{d}p &= -\int_p\int_q \Phi(p,q)(f(q)\nabla_p f(p) - f(p)\nabla_q f(q))\cdot\nabla_p\phi(p)dqdp \\
&= -\int_{p,q} \Phi f(q)\nabla_p f(p)\cdot\nabla_p\phi(p) + \int_{p,q}\Phi f(p)\nabla_q f(q)\cdot\nabla_p\phi(p) \\
&= -\int_{p,q} \Phi f(p)\nabla_q f(q)\cdot\nabla_q\phi(q) + \int_{p,q}\Phi f(p)\nabla_q f(q)\cdot\nabla_p\phi(p) \\
\int_p \partial_t f\phi(p)\mathrm{d}p &= -\int_{p,q} f(p)\nabla_q f(q)\cdot\Phi(p,q)(\nabla_q\phi(q) - \nabla_p\phi(p))\mathrm{d}p\mathrm{d}q
\end{aligned}$$

Thus, we see that the conservation of mass and momentum are independent of the kernel Φ, but the conservation of energy comes from the fact that

$$\Phi(p,q)(\nabla\mathcal{E}(p) - \nabla\mathcal{E}(q)) = 0$$

Or more specifically,

$$\mathbb{S}(\nabla\mathcal{E}(p), \nabla\mathcal{E}(q))\cdot(\nabla\mathcal{E}(p) - \nabla\mathcal{E}(q)) = 0$$

(I am using $\cdot$ and no dot as equivalent, I used it in the second line because it is easier to read, but you know what I mean.) The discrete conservation will come from

$$\mathbb{S}(\nabla\mathcal{E}_h(p), \nabla\mathcal{E}_h(q))\cdot(\nabla\mathcal{E}_h(p) - \nabla\mathcal{E}_h(p)) = 0$$

That is, we replace the energy $\mathcal{E}$ for a discrete energy $\mathcal{E}_h$. This energy will be simply a projection of the energy into a subspace of our test functions - the subspace conformed by radially symmetric functions. Specifically,

$$\mathcal{E}_h = \Pi_h^r[\mathcal{E}]$$

Where the projection operator maps into the subspace of radially symmetric functions. Recall that our test space V_h was defined as containing linear combinations of functions of the form

$$\phi_i(x) = \phi_i(r,e) = L_k^{l+1/2}(r^2)r^l Y_{l,m}(e)$$

where $r = ||x||$ and $e = x/r$, L_k^α is an associated Laguerre polynomial, and $Y_{l,m}$ is a real spherical harmonic. The index i is a shorthand for the triplet (k,l,m). Then, set $l,m = 0$ so that the spherical harmonic is a constant κ and obtain radial test functions.

$$\phi_i(x) = \phi_i(r) = \kappa L_k^{l+1/2}(\rho^2)\rho^l$$

Again, the span of these test functions conform a well-defined *subspace* V_h^r of our discrete space (closed under addition and scaling, contain the 0, etc.) **AND** they are a complete space in the sense that we can approximate any function arbitrary well by taking the space big enough. In other words, the projector Π_h^r that maps into V_h^r will satisfy

$$||\Pi_h^r[\mathcal{E}] - \mathcal{E}|| \to 0$$

in some appropriate norm, as $h \to 0$ (which means that we take the test space large enough.). The main difference with Kun's work is that this projector does not arbitrarily map into the whole space V_h, but rather only onto the radially symmetric functions V_h^r. This is reasonable, because the energy $\mathcal{E}$ is radially symmetric to begin with, and this subspace is well-defined within out discrete space.

The reason for this restriction is to preserve rotational covariance (equivariance) of the discrete collision operator: $\mathcal{E}_h$ will be radially symmetric by definition. Therefore, as seen in subsection 8.2, $\mathcal{E}_h$ shares the key property (rotational invariance), and therefore the whole argument follows. This is why I insisted in proving rotational covariance using an abstract $\mathcal{E}$ as opposed to $\sqrt{1+|p|^2}$

All in all, this suggests that it should be possible to design a fast and conservative scheme for the relativistic Landau equation.

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