THE LOGIC OF COMPLEXITY ZOOLOGY

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Propositions in Complexity Zoology are inclusions of the form $\mathsf{C}_1 \subseteq \mathsf{C}_2$, where C_1 and C_2 are complexity classes. Each such inclusion is true or false in a particular *world* in the sense of modal logic: for a world W, we write $\mathsf{C}_1 \subseteq_W \mathsf{C}_2$ to indicate that the inclusion is true in W. Each world W has a transitive dual W^* with respect to which the following inference rules are true:

$$\mathsf{C}_1 \subseteq_W \mathsf{C}_2 \,\&\, \mathsf{C}_2 \subseteq_{W^*} \mathsf{C}_3 \Longrightarrow \mathsf{C}_1 \subseteq_W \mathsf{C}_3,$$
$$\mathsf{C}_1 \subseteq_{W^*} \mathsf{C}_2 \,\&\, \mathsf{C}_2 \subseteq_W \mathsf{C}_3 \Longrightarrow \mathsf{C}_1 \subseteq_W \mathsf{C}_3.$$

Additionally, there is a partial ordering $\rightarrow$ on the set of worlds such that if $W_1 \rightarrow W_2$, then the following inference rule is true:

$$\mathsf{C}_1 \subseteq_{W_1} \mathsf{C}_2 \Longrightarrow \mathsf{C}_1 \subseteq_{W_2} \mathsf{C}_2.$$

In the current version of Complexity Zoology, there are six worlds:

$$\begin{aligned}
E &\longleftrightarrow \text{every oracle},\\
A &\longleftrightarrow \text{every algebraic oracle},\\
X &\longleftrightarrow \text{some algebraic oracle},\\
R &\longleftrightarrow \text{the random oracle},\\
T &\longleftrightarrow \text{the trivial oracle},\\
O &\longleftrightarrow \text{some oracle}.
\end{aligned}$$

For example, we write $\mathsf{C}_1 \subseteq_X \mathsf{C}_2$ if $\mathsf{C}_1^A \subseteq \mathsf{C}_2^A$ for some algebraic oracle A. The worlds E, A, R, and T are all *transitive* worlds in the sense that they are their own transitive duals: $E^* = E$, $A^* = A$, $R^* = R$, and $T^* = T$. On the other hand $X^* = A$ and $O^* = E$.

The remaining inference rules pertain to complexity class operators. For each operator op and world W,

$$\mathsf{C}_1 \subseteq_W \mathsf{C}_2 \Longrightarrow \mathsf{op} \cdot \mathsf{C}_1 \subseteq_W \mathsf{op} \cdot \mathsf{C}_2$$

is an inference rule. There is also a special pair of inference rules involving the co and cocap operators: for a complexity class C, $\mathsf{cocap} \cdot \mathsf{C}$ is the *meet* of C and $\mathsf{co} \cdot \mathsf{C}$. For each world W, we have

$$\mathsf{C}_1 \subseteq_W \mathsf{C}_2 \,\&\, \mathsf{C}_1 \subseteq_{W^*} \mathsf{C}_2 \Longleftrightarrow \mathsf{C}_1 \subseteq_W \mathsf{cocap} \cdot \mathsf{C}_2,$$
$$\mathsf{C}_1 \subseteq_{W^*} \mathsf{C}_2 \,\&\, \mathsf{C}_1 \subseteq_W \mathsf{C}_2 \Longleftrightarrow \mathsf{C}_1 \subseteq_W \mathsf{cocap} \cdot \mathsf{C}_2.$$

This logical framework – the propositions that specify inclusions, the worlds, and the inference rules – represents the basic knowledge and reasoning of which Complexity Zoology is capable. However, since the purpose of Complexity Zoology is to partially automate the process of surveying complexity theory, it requires a formal system that can describe not merely whether a statement is true or false, but whether it is regarded as proven, unproven, or open.

Each of the statements that we have described so far, along with the inference rules, can be formally expressed in the language of set theory. We assume that there is a set P of inclusion statements and negations thereof that are regarded as *proven*. We assume that P is closed under the logical system of Complexity Zoology. In other words, if CZ is the set of formalized inference rules that Complexity Zoology uses, and if $CZ \cup P \vdash \varphi$ (φ is a logical consequence of $CZ \cup P$), where φ is a formalized inclusion of complexity classes, then $\varphi \in P$. Thus, we regard CZ as a system that is simple enough so that any of its implications from proven facts should also be regarded as proven.

A formal inclusion φ is regarded as *proven* if it lies in P, *disproven* if its negation $\neg\varphi$ lies in P, and *open* if it is neither proven nor disproven. A statement is also sometimes called *provable* if it is not disproven and *disprovable* if it is not proven. We use the following notation:

$$\begin{aligned}
P(\varphi) &\longleftrightarrow \varphi \text{ is proven,}\\
D(\varphi) &\longleftrightarrow \varphi \text{ is disproven,}\\
p(\varphi) &\longleftrightarrow \varphi \text{ is provable,}\\
d(\varphi) &\longleftrightarrow \varphi \text{ is disprovable,}\\
O(\varphi) &\longleftrightarrow \varphi \text{ is open.}
\end{aligned}$$

The data input is a partial description of the set P in the form of a list of statements that are proven, disproven, or open. Complexity Zoology then attempts to deduce as much as it can about which inclusions are proven, disproven, or open. Complexity Zoology has no understanding of complexity theory as such; its strength is in organizing the state of knowledge in the field – that is, Complexity Zoology is an expert at surveying complexity theory, not in complexity theory itself.

Internally, Complexity Zoology represents propositions as a quadruple

$$(\text{status}, \text{world}, \mathsf{C}_1, \mathsf{C}_2),$$

where the status is either proven, disproven, provable, or disprovable; the quadruple is interpreted as "the inclusion $\mathsf{C}_1 \subseteq \mathsf{C}_2$ has the specified status in the specified world." Then, from the base inference rules of the system CZ, we generate the full set of inference rules as follows:

(1) For any formalized inclusion φ, $P(\varphi) \Longrightarrow p(\varphi)$ and $D(\varphi) \Longrightarrow d(\varphi)$.
(2) If $\varphi_1 \,\&\, \cdots \,\&\, \varphi_n \Longrightarrow \psi$ is an inference rule, then
$$P(\varphi_1) \,\&\, \cdots \,\&\, P(\varphi_n) \Longrightarrow P(\psi).$$
(3) The partial involutions $P \mapsto D \mapsto P$ and $P \mapsto d \mapsto P, D \mapsto p \mapsto D$ are implication-reversing. For example, if $P(\varphi) \Longrightarrow P(\psi)$, then $D(\psi) \Longrightarrow D(\varphi)$, and if
$$P(\varphi_1) \,\&\, \cdots \,\&\, P(\varphi_n) \Longrightarrow P(\psi),$$
then
$$P(\varphi_1) \,\&\, \cdots \,\&\, d(\psi) \Longrightarrow d(\varphi_n).$$

For instance, if $\mathsf{C}_1^A \subseteq \mathsf{C}_2^A$ for every oracle A and $\mathsf{C}_1^A \not\subseteq \mathsf{C}_3^A$ for some oracle A, then $\mathsf{C}_2^A \not\subseteq \mathsf{C}_3^A$ for some oracle A. In our notation,

$$P(\mathsf{C}_1 \subseteq_E \mathsf{C}_2) \,\&\, D(\mathsf{C}_1 \subseteq_E \mathsf{C}_3) \Longrightarrow D(\mathsf{C}_2 \subseteq_E \mathsf{C}_3).$$