Deriving fractional moments using the Moment Generating Function

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Dr. yyyy (co-reader)

Abstract

The abstract should summarize the contents of the thesis. It should be clear, descriptive, self-explanatory and not longer than a third of a page. Please avoid using mathematical formulas as much as possible. Keywords might be given.

Keywords: Fractional moments, Moment Generating Function.

1 Introduction

Statistical moments such as the mean and the variance are essential tools to characterize data and its distribution. Moments of even higher order are useful regarding the shape of the distribution. Less known moments, however, are the fractional moments. While these moments may not be that significant in comparison with the first for integer moments of a distribution, they can be very useful in certain applications. While these moments can be obtained by ordinary integration, the field of fractional calculus, which finds many application in physics, provides a more general framework to compute these moments. In this paper, we will combine the aforementioned technique with the Moment Generating Function to obtain new expressions for the fractional moments of a distribution.

2 General Guidelines

3 Fractional Calculus

In order to obtain the expression as mentioned in section 1, some advanced tools are required. We can find this in the field of fractional calculus. We define the following:

Definition 3.1. Let D be the differential operator, such that $Df(x) = \frac{d}{dx}f(x)$. Then the fractional derivative of order α is defined as $D^\alpha f(x) = \frac{d^\alpha}{dx^\alpha}f(x)$.

In this definition, α can be any real number. When taking regular derivatives, $\alpha \in \mathbb{N}$. In our case, we are interested in instances where $\alpha \in \mathbb{Q}, \alpha \geq 0$.

It is also possible to study derivatives of negative order, which can be used to obtain moments of negative order of a function. A derivative of negative order is simply an integral of positive order. This is defined as follows:

Definition 3.2. Let I be the integral operator, such that $If(x) = \int f(x)dx$. Then the fractional integral of order α is defined as $(I^\alpha f)(x) = \frac{1}{(n-1)!}\int(x-t)^{n-1}f(t)dt$.

Combining the previous two definition, we can obtain the following definition.

Definition 3.3. The differintegral operator is defined as

$$R^\alpha f(x) = \begin{cases} I^{|\alpha|}f(x) & \text{if } \alpha < 0 \\ D^\alpha f(x) & \text{if } \alpha > 0 \\ f(x) & \text{if } \alpha = 0 \end{cases} \tag{1}$$

A lot of different definition have been used to compute a fractional derivative. In this paper, we will focus on the following fractional derivatives:

Definition 3.4. The Riemann-Liouville fractional derivative of order α is defined as:

$$D^\alpha f(x) = \frac{d^n}{dx^n}D_x^{-(n-\alpha)}f(x) = \frac{d^n}{dx^n}I_x^{n-\alpha}f(x) = \frac{d^n}{dx^n}\frac{1}{\Gamma(n-\alpha)}\int_0^x(x-t)^{n-\alpha-1}f(t)dt \tag{2}$$

Where $n = \lceil\alpha\rceil$, the ceiling function and $\Gamma(.)$ is the Gamma function, see section A.

Remark 3.1. For values $\alpha \in \mathbb{N}, n = \alpha$, so $\Gamma(n-\alpha) = \Gamma(0)$, which is undefined. Thus for $\alpha \in \mathbb{N}$, including the value 0, we simply define: $D^\alpha f(x) = \frac{d^\alpha}{dx^\alpha}f(x)$

A modification of the Riemann-Liouville derivative is the Caputo-Fabrizio derivative, which is defined as follows:

Definition 3.5. The Caputo-Fabrizio fractional derivative of order $\alpha, \alpha \in [0, 1)$ is defined as:

$$D^\alpha f(x) = \frac{1}{1-\alpha}\int_0^x \exp(\frac{-\alpha}{1-\alpha}(x-t))f'(t)dt \tag{3}$$

Lastly, we will define the Grünwald-Letnikov derivative, which is defined as follows:

Definition 3.6. The Grünwald-Letnikov fractional derivative of order α is defined as:

$$D^\alpha f(x) = \lim_{h\to 0}\frac{1}{h^\alpha}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}f(x-kh) \tag{4}$$

Where $\binom{\alpha}{k}$ is the binomial coefficient, see section A.

Proposition 3.1. *The fractional derivatives above adhere to the following properties:*

(i) Linearity: Let $f(x), g(x)$ be functions and $a, b, x \in \mathbb{R}$. Then we have that $D^\alpha(af(x) + bg(x)) = aD^\alpha f(x) + bD^\alpha g(x)$.

(ii) $D^\alpha f(x)$ for $\alpha = 0, = f(x)$

(iii) for sufficiently smooth functions f, we have that $D^{\alpha+\beta} f(x) = D^\alpha(D^\beta f(x)) = D^\beta(D^\alpha f(x))$, with $\alpha, \beta \in \mathbb{R}$. Note that, for 3.5, this only holds for $\beta \in \mathbb{N}, \alpha \in [0, 1)$.

Proof. (i) We will proof for the Riemann-Liouville derivative, the proof for the Caputo-Fabrizio derivative is very similar and the Grünwald-Letnikov derivative is a direct consequence of the linearity of the sum.

$$D^\alpha(af(x) + bg(x)) = \frac{d^n}{dx^n} \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1}(af(t) + bg(t))dt$$

$$= \frac{d^n}{dx^n}\left(\frac{a}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f(t)dt + \frac{b}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} g(t)dt\right)$$

Where we simply split the integral and put the constants in front.

$$= \frac{d^n}{dx^n} \frac{a}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f(t)dt + \frac{d^n}{dx^n} \frac{b}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} g(t)dt$$

As the regular derivative operator is just linear.

$$= aD^\alpha f(x) + bD^\alpha g(x)$$

(ii) Intuitively, this makes perfect sense, as the 0-th derivative is just no derivative, so just the function $f(x)$. But for these derivatives, a little bit more effort is needed to prove this rather obvious fact.
For the Grünwald-Letnikov derivative we get:

$$D^0 f(x) = \lim_{h \to 0} \frac{1}{h^0} \sum_{k=0}^{\infty} (-1)^k \binom{0}{k} f(x-kh) = \lim_{h \to 0} \frac{1}{1} \sum_{k=0}^{\infty} (-1)^k \frac{0!}{k!(0-k)!} f(x-kh).$$

The factorial Identity of the binomial coefficient only holds for $0 \leq k \leq \alpha$. Since $\alpha = 0$ and k is always a positive integer lesser or equal to $\alpha, k = 0$. Thus, we get:

$$= \lim_{h \to 0} \sum_{k=0}^{\infty} (-1)^0 \frac{0!}{0!(0-0)!} f(x-0h) = \lim_{h \to 0} f(x-0h) = f(x).$$

For the Caputo-Fabrizio derivative, we get the following:

$$D^0 f(x) = \frac{1}{1-0} \int_0^x \exp\left(\frac{0}{1-0}(x-t)\right) f'(t)dt = \int_0^x f'(t)dt = f(x).$$

Finally, for the Riemann-Liouville derivative, we can simply make use of 3.3 and 3.1 to note that in this context $\alpha = 0$ is included in the natural integers. So $D^\alpha = \frac{d^\alpha}{dx^\alpha} f(x) = d^0 dx^0 f(x) = f(x)$.

(iii) The proof for the Riemann-Liouville derivative is given by Koning (2015). And the proof for the Caputo-Fabrizio derivative is given by Losada and Nieto (2015).For the Grünwald-Letnikov derivative, we get:

$$D^{\alpha}(D^{\beta}f(x)) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}(\frac{1}{h^{\beta}}\sum_{l=0}^{\infty}(-1)^l\binom{\beta}{l}f(x-lh-kh))$$

$$= \lim_{h\to 0}\frac{1}{h^{\alpha+\beta}}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}\sum_{l=0}^{\infty}(-1)^l\binom{\beta}{l}f(x-(k+l)h).$$

We substitute $m = k + l$ to deal with the dubble sums:

$$\lim_{h\to 0}\frac{1}{h^{\alpha+\beta}}\sum_{m=0}^{\infty}f(x-mh)\sum_{k=0}^{m}(-1)^k(-1)^{m-k}\binom{\alpha}{k}\binom{\beta}{m-k}$$

Now we make use of an identify from A to obtain:

$$= \lim_{h\to 0}\frac{1}{h^{\alpha+\beta}}\sum_{m=0}^{\infty}(-1)^m\binom{\alpha+\beta}{m}f(x-mh) = D^{\alpha+\beta}f(x).$$

It can be shown in an exactly similar way that the latter expression is equal to $D^{\beta}(D^{\alpha}f(x))$. □

We will now provide a few numerical examples of these fractional derivatives. For simplicity, we will let $a = 0$:

Example 3.1. (i)

$$D^{\frac{3}{2}}_{RL}(c) = \frac{d^2}{dx^2}\frac{1}{\Gamma(2-\frac{3}{2})}\int_0^x (x-t)^{2-\frac{3}{2}-1}c dt$$

$$= \frac{d^2}{dx^2}\frac{c}{\sqrt{\pi}}\int_0^x (x-t)^{-\frac{1}{2}}dt = \frac{d^2}{dx^2}\frac{-2c}{\sqrt{\pi}}\sqrt{x-t}\Big|_0^x$$

$$= \frac{d^2}{dx^2}\frac{2c\sqrt{x}}{\sqrt{\pi}} = \frac{-c}{2\pi(x)^{\frac{3}{2}}} \neq 0.$$

As stated, the fractional of a constant is not equal to zero when using the Riemann-Liouville derivative, this is also the case for the Grünwald-Letnikov derivative, but not for the Caputo-Fabrizio derivative.

(ii)

$$D^{\frac{1}{2}}_{CF}(\frac{x}{2}) = \frac{1}{1-\frac{1}{2}}\int_0^x \exp(\frac{-\frac{1}{2}}{1-\frac{1}{2}}(x-t))\frac{1}{2}dt = \int_0^x \exp(t-x)dt$$

$$= \exp(t-x)\Big|_0^x = 1-\exp(-x).$$

For $x \geq 0$, this expression is equal to the CDF of the exponential distribution with $\lambda = 1$. A remarkble result.

(iii) We will compute another famous result with the Riemann-Liouville derivative:

$$D^{\frac{1}{2}}_{RL}(x) = \frac{d}{dx}\frac{1}{\Gamma(1-\frac{1}{2})}\int_0^x (x-t)^{1-\frac{1}{2}-1}t dt$$

Let $u = x - t$, such that $\frac{du}{dt} = -1, dt = -du$:

$$= \frac{d}{dx}\frac{1}{\sqrt{\pi}}\int_0^x (u)^{-\frac{1}{2}}(u-x)du = \frac{d}{dx}\frac{1}{\sqrt{\pi}}(\int_0^x \sqrt{u}du - x\int_0^x \frac{1}{\sqrt{u}}du)$$

$$= \frac{d}{dx}\frac{1}{\sqrt{\pi}}(\frac{2u^{\frac{3}{2}}}{3} - 2xu^{\frac{1}{2}}\Big|_0^x) = \frac{d}{dx}\frac{1}{\sqrt{\pi}}(\frac{2(x-t)^{\frac{3}{2}}}{3} - 2x(x-t)^{\frac{1}{2}}\Big|_0^x)$$

$$= \frac{d}{dx}\frac{-1}{\sqrt{\pi}}(\frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{3}{2}} = \frac{d}{dx}\frac{\frac{4}{3}x^{\frac{2}{3}}}{\sqrt{\pi}})$$

$$= \frac{2\sqrt{x}}{\sqrt{\pi}}.$$

3.1 Formatting the Front Page

The front page just contains

- Title of the thesis.
- Author (your name and student number).
- Date.
- References or logo(s) of
  - VU University.
  - School of Business and Economics.
  - Department of Econometrics and Operations Research.
- Master Thesis.
- Thesis Committee and its members.

4 Mathematics

4.1 Mathematical Expressions in Text and in Displays

Display only the most important equations, and number only the displayed equations that are explicitly referenced in the text. To conserve space, simple mathematical expressions such as $\bar{Y} = n^{-1}\sum_{i=1}^{n} Y_i$ may be incorporated into the text. Mathematical expressions that are more complicated or that must be referenced later should be displayed, as in

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2.$$

If a display is referenced in the text, then enclose the equation number in parentheses and place it flush with the right-hand margin of the column. This is automatically obtained by the `equation` environment accompanied by the `\label` command. For example, the quadratic equation has the general form

$$ax^2 + bx + c = 0, \text{ where } a \neq 0. \tag{5}$$

In the text, each reference to an equation number should also be enclosed in parentheses by using `\eqref{<labelgiven>}`. For example, the solution to (5) is given in (6) in Appendix D.

If the equation is at the end of a sentence, then you should end the equation with a period. If the sentence in question continues beyond the equation, then you should end the equation with the appropriate punctuation—that is, a comma, semicolon, or no punctuation mark.

4.2 Symbols, Commands, Environments, Etc

See
`https://en.wikibooks.org/wiki/LaTeX/Mathematics`
and
`https://en.wikibooks.org/wiki/LaTeX/Advanced_Mathematics`.

4.3 Definitions, Theorems, Etc.

Definitions, theorems, propositions, etc. should be formatted using the `amsthm` package. Number these items separately and sequentially. Examples are given below

Definition 4.1. In colloquial New Zealand English, the term *dopey mongrel* is used to refer to someone who has exhibited less than stellar intelligence.

Theorem 4.1. *If a proceedings editor from New Zealand accidentally deletes his draft of the author kit shortly after completing it, he would be considered to be a dopey mongrel.*

Proof. The proof follows by the principle of contradiction. Suppose the editor is not a dopey mongrel, then he is smart enough to save the author kit. □

Corollary 4.1. *One of the proceedings editors is a dopey mongrel.*

Proof. This follows immediately from Theorem 4.1. □

5 Figures and Tables

Figures and tables should be centered within the text and should not extend beyond the right and left margins of the paper. Figures and tables can make use of color. However, try to select colors that can be differentiated when printing in black and white in consideration of vast majority of people using such printers. Figures and tables are numbered sequentially, but separately, using arabic numerals.

5.1 Tables

Each table should appear in the document after the paragraph in which the table is first referenced. However, if the table is getting split across pages, it is okay to include it after a few paragraphs from its first reference. One-line captions are centered, while multiline captions are left justified. The captions appear *above* the table. See Tables 1 and 2 for examples.

Table 1: Table captions appear above the table, and if they are longer than one line they are left justified. Captions are written using normal sentences with full punctuation. It is fine to have multiple-sentence captions that help to explain the table.

Creature	IQ	Diet
dog	70	anything
cat	75	almost nothing
human	60	ice cream
dolphin	120	fish fillet

Table 2: Counting in Maori.

English	Maori
one	tahi
two	rua
three	toru
four	wha

Table 3: Alternative table aligning columns

	2012	2013
Mean	1.23	1.23456
Variance	13454.4	3435.456

Note: Alternatively, place a longer explanation as a note at the bottom of the table, allowing the caption to be more concise. This table uses explicit alignment of the numbers in the columns, using either 3 or 5 decimals.

5.2 Figures

Each figure should appear in the document after the paragraph in which the figure is first referenced. One-line captions are centered, while multiline captions are left justified. Figure captions appear below the figure. To include figures you use package `graphicx`, and in the document the command

`\includegraphics{<graphs/graphicfilename>}`.

The graphicfilename usually does not have to include an extension; PdfLaTeX would search for extensions it recognises. In general, it is advisable to keep your graphs in a separate directory, to avoid clogging up your LaTeX directory.

If the graph is scaled correctly, one should be able to use a

`\includegraphics[width=\textwidth]{<graphs/graphicfilename>}`

to ensure that the graph fills the width of the paper.

5.3 References to Tables and Figures

References to tables and figures identified by number are capitalized. For example, "We see in Table 2 that..." and "We see in the previous table that..." are both correct. Be sure to use the `\label` command within the figure or table environment and refer to the associated figure or table using `Table \ref{<labelgiven>}`. Please do not use hard coded figure/table numbers. This is error prone.

5.4 Graphics Formats

As graphics files in your document you use `.jpg`, `.png`, `.pdf`, or `.eps` files. But there are tools to convert these formats into one another. The main difference between the formats is how they store the images and how well suited they are for specific graphics. In general we can choose between bitmap and vector graphics. Bitmap graphics are well suited for photographies (jpg is very common here) or for screenshots (png is a lossless encoding (in contrast to jpg), and is thus better suited for all those cases where you have sharp edges in your graphics). Vector graphics are the encoding to be chosen for all kinds of drawings (diagrams, charts, ...). In contrast to bitmap formats, they can be scaled to any size without any loss of sharpness. This makes it possible to read such graphics even if two pages are printed on one sheet of paper, or if the documents are read electronically.

So what to choose for your Latex document? As a rule of thumb you should always prefer pdf and eps. In general these two encodings can contain both, bitmap and vector graphics. But there is no need (and no use) to convert your bitmaps to any of these.

You include figures via the `\includegraphics` command. You must use the `pdflatex` or `xelatex` command to generate your pdf file, as was done with this file.

6 Algorithms

Typeset algorithms by using the `algorithmic` environment of the `algorithm` package. The command `\begin{algorithmic}` can be given the optional argument of a positive integer, which if given will cause line numbering to occur at multiples of that integer. E.g. `\begin{algorithmic}[5]` will enter the algorithmic environment and number every fifth line. Below is an example of typesetting a basic algorithm (remember to add the

`\usepackage{algorithm,algorithmic}`

statement to your document preamble). The pseudocode is centered by using the `minipage` environment.

```
\begin{center}
\begin{minipage}{10cm}
\begin{algorithm}[H]
\caption{Polar Method for Normal Random Numbers}
\begin{algorithmic}[1]
\REPEAT
   \STATE $U_1\sim {\sf U}(0,1)$ \COMMENT{Generate uniform on $(0,1)$}
   \STATE $U_2\sim {\sf U}(0,1)$
   \STATE $V_1 \gets 2U_1 -1$ \COMMENT{Uniform on $(-1,1)$}
   \STATE $V_2 \gets 2U_2 -1$
   \STATE $W \gets V_1^2 + V_2^2$
\UNTIL {$W < 1$}
\STATE \textbf{return} $V_1\,\sqrt{(-2\,\ln(W)/W}$
\end{algorithmic}
\end{algorithm}
\end{minipage}
\end{center}
```

This produces

Algorithm 1 Polar Method for Normal Random Numbers

1: **repeat**
2: $U_1 \sim \mathsf{U}(0,1)$ {Generate uniform on $(0,1)$}
3: $U_2 \sim \mathsf{U}(0,1)$
4: $V_1 \leftarrow 2U_1 - 1$ {Uniform on $(-1,1)$}
5: $V_2 \leftarrow 2U_2 - 1$
6: $W \leftarrow V_1^2 + V_2^2$
7: **until** $W < 1$
8: **return** $V_1 \sqrt{(-2 \ln(W)/W}$

7 Bibliography Management

Managing your bibliography requires two specifications: the citation style in your document, and the style of the reference list at the end of the thesis.

7.1 Citing a Reference

To cite a reference in the text, use the author-date method. Thus, Ross (2006) could also be cited parenthetically (Ross, 2006). For a work with three or more authors, use an abbreviated form. For example, a work by Evans, Keith and Kroese would be cited in one of the following ways: Evans et al. (2007) or (Evans et al., 2007).

Parenthetical citations are enclosed in parentheses ( ), not square brackets [ ]. The items in a series of such citations are usually separated by commas. If an item in the series of parenthetical citations contains punctuation because (for example) it refers to a work with three or more coauthors, then all items should be separated by semicolons.

The following is a list of correct forms of citations:

- Brown and Edwards (1993),
- (Brown and Edwards 1993),
- (Brown and Edwards, 1993),
- Brown and Edwards (1993), Smith (1997),
- (Brown and Edwards 1993; Smith 1997; Brown et al. 1997).
- (Brown and Edwards, 1993; Smith, 1997; Brown et al., 1997).

7.2 List of References

Place the list of references after the appendices. The section heading is **References**, and it is not numbered. List only references that are cited in the text. Arrange the references in alphabetical order (chronologically for a particular author or group of authors); do not number the references. Give complete references without abbreviations. To identify multiple references by the same authors and year, append a lower case letter to the year of publication; for example, 1984a and 1984b.

Use hanging indentation to distinguish individual entries. Do not insert additional space between references.

The bibliographic style for a journal article is:
<Surname of first author>, <First author initials>, <Initials and surnames of other authors> (<year>). <Article title>. <*Journal Name in Headline Italics*> <Volume number>, <page numbers>.

The article title may come between quotes but this is not required. The format for other types of reference can be inferred from the examples in the references, which include:

- a technical report (Chien, 1989),
- a proceedings article (Evans et al., 2007),
- a journal article (Alon et al., 2005),
- a book by 2 authors (Asmussen and Glynn, 2007),
- a chapter in a book (Asmussen and Rubinstein, 1995),
- an unpublished thesis or dissertation (Garvels, 2000),
- a document available on the web (Felgenhauer and Javis, 2005).

7.3 BibTeX

These formats are obtained by creating a BibTeX database for all your references. A BibTeX database is stored as a `.bib` file. It is a plain text file, and so can be viewed and edited easily. One benefit of using BibTeX is that bibliography formatting and referencing can be greatly simplified: the correct citation and reference list style is automatically created. How the BibTeX items are entered in the `bib` file will be explained in section 7.6.

Once you have created the `bib` file with your references it needs to be included in your document.

7.4 Biblatex Package

Advise is to use the `biblatex` package. It is a reimplementation of the classic BibTeX providing modern bibliographic facilities such as advanced name disambiguation, smart crossref data inheritance, configurable sorting schemes, dynamic datasource modification, etc. The prerequisite is that you need to use Biber to process your `bib` files. Older TeX distributions do not have Biber included, you need to check this or you find out when you cannot run your `bib` files using the `biblatex` package. In that case, see next section for the classic BibTeX implementation.

- In the preamble you put

```
\usepackage[natbib,
            style=authoryear,
            maxcitenames=2,
            maxbibnames=99,
            hyperref,
            backend=biber]{biblatex}
\addbibresource{bib/<bibfilename>.bib}
```

- At the end of your document, where the reference list is supposed to come you put

  `\printbibliography`

- In your document, wherever you need to cite a reference you put `\citet{<bibentrykey>}` (for the author name(s) followed by the year in parentheses), or `\citep{<bibentrykey>}` to get the entire citation in parentheses.

7.5 Classic BibTeX

In case Biber does not work in your TeX distribution. Use the `natbib` bibliography style with `chicago` citation style.

- In the preamble you put

  `\usepackage{natbib}`.

- At the end of your document, where the reference list is supposed to come you put

```
\bibliographystyle{chicago}
\bibliography{bib/<bibfilename>.bib}
```

- Citations are `\citet{<bibentrykey>}` (for the author name(s) followed by the year in parentheses), or `\citep{<bibentrykey>}` to get the entire citation in parentheses.

7.6 BibTeX Entries

See the `EORthesis.bib` file available in Canvas, or use the templates that you can find, e.g.,

`https://en.wikibooks.org/wiki/LaTeX/Bibliography_Management`

and

`https://en.wikipedia.org/wiki/BibTeX`.

7.7 Compilation

You need to obtain the final pdf version of document with correct citations and reference list.

(a). In case you use Biblatex, you run consecutively `pdflatex, biber, pdflatex`. In stead of `pdflatex`, you might also run `xelatex`.

(b). In case you use classic BibTeX, you run consecutively `pdflatex, bibtex, pdflatex, pdflatex`. In stead of `pdflatex`, you might also run `xelatex`.

7.8 Example

In Asmussen and Glynn (2007) we see that they mention (Blitzstein and Diaconis, 2010), and after some more comments of Asmussen and Rubinstein (1995), we know that Evans et al. (2007) have published the same results as we can find in (Alon et al., 2005; Garvels, 2000; Ross, 2006), however in a more general context.

8 Conclusion

The conclusion summarizes the results of your research. Also it gives possible directions of further research.

Acknowledgements

Place the acknowledgments section, if needed, after the main text, but before any appendices and the references. The section heading is not numbered.

A Relevant functions and Identities

We define the (Euler-)Gamma function as follows:

Definition A.1. for $\Re(z) > 0$, we have the following: $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$

The Gamma function can be seen as an extension of the factorial function, for non-integers. This function is defined for complex numbers and all there subsets (so also real numbers), as long as the condition above holds. For positive integers values z, we have the following identity: $\Gamma(z) = (z-1)!$ Other important identities are:

- $\Gamma(z+1) = z\Gamma(z)$
- $\Gamma(2) = \Gamma(1) = 1$
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Definition A.2. The falling factorial is defined as follows: $(x)_n = \prod_{k=0}^{n-1}(x-k)$, which is a polynomial

Definition A.3. For $0 \leq k \leq n$, the Binomial Coefficient is defined as follows: $\binom{n}{k}$, where $n, k \in \mathbb{N}$.

We can derive the following factorial identity, which is convenient to work with analytically: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. For numerically computing expressions containing the Binomial Coefficient, the following identity is computationally more efficient: $\binom{n}{k} = \frac{(n)_k}{k!}$. With $(n)_k$ as in A.2. Since we have established in A.1 that $\Gamma(z) = (z-1)!$, we can rewrite our factorial identify to:

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1) \cdot \Gamma(n-k+1)} = \frac{n}{k} \frac{\Gamma(n)}{\Gamma(k) \cdot \Gamma(n-k+1)}$$

Definition A.4. Vandermonde's identity: for non-negative integers, k, l, m, n, we have that

$$\sum_{k=0}^{l} \binom{m}{k} \cdot \binom{n}{l-k} = \binom{m+n}{l}$$

.

A modification on the latter identity has been called the Chu-Vandermonde identity. This is the same identity, but it his been proven that the identities still hold for complex values m, n as long as l is a positive integer (Askey (1975)).

B Appendix

Place any appendices after the acknowledgments, starting on a new page. The appendices are numbered **A**, **B**, **C**, and so forth. The appendices contain material that you would like to share with the reader but that would hinder the flow of reading. For instance long proofs of theorems, code of algorithms, data, etc.

C Appendix

You might give more appendices. For instance Appendix B for proofs, Appendix C for data.

D Appendix

The solution to (5) has the form

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{if } \ a \neq 0. \tag{6}$$

References

Alon, G., D. Kroese, T. Raviv, and R. Rubinstein (2005). Application of the cross-entropy method to the buffer allocation problem in a simulation-based environment. *Annals of Operations Research 134*(1), 137–151.

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