Deriving fractional moments using the Moment Generating Function

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Abstract

The abstract should summarize the contents of the thesis. It should be clear, descriptive, self-explanatory and not longer than a third of a page. Please avoid using mathematical formulas as much as possible. Keywords might be given.

Keywords: Fractional moments, Moment Generating Function, Fractional Calculus.

1 Introduction

Statistical moments, defined as the n-th moment of a random variable X, with probability density function (PDF) $f_X(x)$

$$\mathbb{E}[X^n] = \begin{cases} \int_{-\infty}^{\infty} (x-c)^n f_X(x)\, dx & \text{if } f_X(x) \text{ is continuous,} \\ \sum_i (x_i - c_i)^n f_X(x_i) & \text{if } f_X(x) \text{ is discrete.} \end{cases}, n \in \mathbb{N}$$

are essential tools to characterize data and its distribution. Moments of the first and second order, the mean and variance respectively, provide one with essential information about the average and measure of dispersion of a random variable. Moments of even higher order are useful regarding the shape and symmetry of the distribution. Less known moments, however, are the fractional moments. The latter are defined in precisely the same manner as the integer moments, but now with $n \in \mathbb{R}$, or even $n \in \mathbb{C}$. From this point on, when fractional moments are considered, we denote $\alpha\mathbb{C}$, instead of n, to be its moment's order. While these moments may not find as much usage in comparison with the integer moments of a distribution, they can be very useful in certain applications.

Fractional moments play a significant role in a variety fields, including finance, economics, and statistics An example of the latter is its application in approximating integer moments as described by Novi Inverardi and Tagliani (2024). This is especially useful when (for $\alpha \in \mathbb{R}$) the $\lceil \alpha \rceil$-th central integer moment might not exist, while its fractional moment does. Finding an existing fractional moment close to a non-existing integer moment, could still provide one with information about this integer moment. For example, the student-t distribution with $\nu = 2$ degrees of freedom only has central and raw moments of order k, where $0 < k < \nu$. This implies that its second central moment ($k = 2$) does not exist. One could however consider the $k-th$ central moment where $k = 1.95$ and interpret its value as the variance of the distribution. Fractional moments are also used in financial modelling, particularly in the context of Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models. These models are commonly applied to time series data such as financial returns and capture the dynamic volatility that changes over time. The GARCH model achieves this by modeling the volatility based on the returns and variances of previous time periods. Hansen and Tong (2024) have obtained a method of finding fractional absolute moments of the cumulative return, which would have been impossible when using any other method. Gzyl et al. (2013) have also introduced the usage of Fractional moments in risk-models, specifically insurance models. In such models, often the probability density function of total ruin, the event where an insurance company's capital becomes negative, is unknown. The so-called "Method of maximum entropy" has been developed to find these densities. This method takes fractional moments as its input, as they have been proven to be able to characterize its distribution Lin (1992). This method has proven to be a useful alternative to existing methods, such as inverse Laplace transformations. This is the case as this new method takes less values than the latter as input, making it computationally more efficient. Beyond finance and risk modelling, fractional moments also have important applications in engineering. Examples include optimizing signal processing and control systems as well as studying the response characteristics of random vibration systems. Wang et al. (2025) has shown that when using the concept of fractional moments for the latter, accuracy and stability is higher compared to traditional methods, such as Taylor expansions. Furthermore, in terms of simplicity and efficiency, the method of fractional moments is advantageous, as its computation steps are straightforward and avoid convergence issues, significantly reducing the resources required for computation. Working with fractional moments has allowed Wang et al. (2025) to obtain both analytical, as well as numerical solutions to problems within their research field, which again proves its viability. Another application within the field of engineering, can be found in the identification of distributions of non-linear systems. Di Matteo et al. (2014) have shown that complex fractional moments allow one to solve equations such as the Kolmogorov or Fokker-Planck equation, a characterization of continuous-time Markov processes. After performing a Mellin transformation on this system of equations, the resulting system is a linear system in terms of complex fractional moments. The latter can now be solved rather easily and taking the Inverse Mellin transformation on these solutions immediately provides one with the solutions of the non-linear system. Advantages of using complex fractional moments instead of integer moments is that, when applying the Mellin transformation, the relevant PDF is restored on its entire support. This is not the case for the integer moments. This method of using complex fractional moments has been proven to have a rather high accuracy and is applicable to any stable kind of non-linear system of equations Di Matteo et al. (2014).

2 Methodology

2.1 Existing methods of obtaining fractional moments

The traditional method of computing fractional moments is rather straightforward. Similarly to integer moments, one simply computes the summation (in the discrete case) or integral (in the continuous case) of $x^{\alpha} \cdot f_x(x)$, where $f_x(x)$ denotes the probability density function of the random variable X. In the context of fractional moments, $\alpha \in \mathbb{R}$ instead of $\mathbb{Z}$ (assuming that negative moments exist). Hansen and Tong (2024) introduce an alternative approach to computing fractional moments using the complex moment generating function (CMGF), which they apply in the context of the aforementioned GARCH models. One of their key expressions is given by:

$$\mathbb{E}\,|X-\xi|^{\alpha} = \frac{\Gamma(\alpha+1)}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-\xi z}M_X(z)+e^{\xi z}M_X(-z)}{z^{\alpha+1}}dt$$

where $z = s + it, s \in \mathbb{N}_+, \xi \in \mathbb{R}$ and α of course the order of the moment.

This formulation extends upon the traditional moment generating function (MGF) but avoids the process of taking derivatives, making it computationally efficient. The inclusion of the Gamma function is logical, as it extends the factorial function to real values, aligning well with the computation of fractional moments. Since this method relies on integration, rather than differentiation, it avoids numerical issues that might arise when computing derivatives, such as obtaining rather great approximation errors.

2.2 Obtaining fractional moments by using the moment generating function

While the CMGF method provides an efficient and elegant alternative to the traditional method, this thesis explores a different approach: computing fractional moments directly by applying fractional derivatives to the MGF. The MGF is widely used for computing integer moments by differentiation around zero. Extending this approach to fractional orders requires us to take fractional derivatives. Thus, we need to define such fractional derivative operators. These fractional derivatives have a long history and often make use of the aforementioned Gamma function in combination with some integral. This means that, for continuous random variables, where we integrate the MGF, we will have to do double integration. A consequence might be that obtaining analytical expressions of these moments may not always be possible. A lot of alternative expressions of these fractional derivatives have been created, mostly based on different interpretations of the latter in the field of physics. In this thesis, we will focus on computing the MGF using the Riemann-Liouville derivative, the Caputo-Fabrizio derivative and the Grünwald-Letnikov fractional derivative. Each of these fractional derivatives in combination with the MGF might lead to different moments expressions for the same distribution and same fractional order of the moment. Thus, it is essential to compare each of these definitions with the traditional way of computing fractional moments, to derive their accuracy and conclude which approach is most suitable for fractional moment computation. Similar to the expression of the moments of a random variable, their 'biases' may also be hard to derive analytically, depending on its distribution.

2.3 Order and methods of research

To realize the differences between the aforementioned fractional derivatives, the order of research will be as follows. First, a mathematical groundwork for the fractional derivatives will be laid, in which a number of their respective properties will be discussed. Next, we will revise some basic definitions of statistical moments and the moment generating function and see how some of these properties change when considering fractional moments in combination with the moment generating function. If possible, analytical biases for each fractional derivative in combination with the MGF will be computed, which will conclude the theoretical research of this thesis. We will conduct a simulation using the programming language Julia, to now obtain numerical biases instead of analytical biases. Finally, we will consider a practical case (WIP: TO BE DECIDED) in which all three derivatives will once again be compared.

3 Fractional Calculus

3.1 Overview and applications of fractional derivatives

Although not the main topic of interest of this thesis, it is useful to have some knowledge about the history and applications of fractional derivatives. The study of fractional derivatives has been relevant as early as the year 1695 when the concept of such a derivative was implicitly discussed by Leibniz and Bernoulli Katugampola (2014). Since then, numerous definitions of fractional derivatives have been developed. The best-known definition of the fractional derivative is the Riemann-Liouville derivative, its upper derivative of a function $f(x)$ of order α is denoted as

$$\frac{d^n}{dx^n}\frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}f(t)dt, \text{ where } n = \lceil\alpha\rceil$$

Kilbas et al. (2006). Michele Caputo (1967) defined a variation on this derivative, where instead of $\frac{d^n}{dx^n}$ in front of the integral, we have $\frac{d^n}{dt^n}$ inside the integral. Due to this adjustment, it is possible to have initial value conditions expressed as the traditional derivatives of integer-order, which made these fractional differential equation problems more intuitive. Other fractional derivatives, such as the Hadamard (1892) and Riesz (1949) derivative, have been defined to take advantage of particular beneficial properties. For example, each of the latter derivatives can be written as a Fourier transformation. As a consequence, the analytical expressions, can often be simplified. A rather unique derivative is the Grünwald-Letnikov derivative which, in contrast to all the aforementioned derivatives, is not based on integral. Instead, it generalizes the difference quotient, $\frac{f(x+h)-f(x)}{h}$, to fractional orders using binomial coefficients Atici et al. (2021). This variety of definitions emphasises how dependent fractional derivatives are on different physical interpretations and practical applications. Beyond their theoretical significance, fractional derivatives have been of significant importance in various scientific fields since the 19th century. Examples include fractional Fourier transformations, a generalization of the regular Fourier transformations Missbauer (2012), fractional diffusion equation models, describing the motion of particles in liquids as a consequence of thermal molecular motions Einstein (1905) and the fractional Schrödinger equation, a generalization of the Schrödinger equation, often used in quantum mechanics Laskin (2002). Their applications are less common in the fields of finance or economics, as fractional derivatives are mainly used to describe natural phenomena Boulaaras et al. (2023). Yet they still offer some great potential. (Symmetric) Levy flights make use of fractional derivatives in order to solve partial differential equations which describe random walk processes in time series Scalas et al. (2000). The development of fractional derivatives also led to the notion of fractional Brownian motions, a generalization of the Brownian motion Mandelbrot and Van Ness (1968). The latter is a continuous-time stochastic process which, similar to Levy flights, may be used to model random walk processes.

3.2 Formal definitions of fractional derivatives

In order to obtain the expressionS mentioned in section 2, some advanced tools are required. We can find these in the field of fractional calculus. We define the following:

Definition 3.2.1. Let D be the differential operator, such that $Df(x) = \frac{d}{dx}f(x)$. Then the fractional derivative of order α is defined as

$$D^\alpha f(x) = \frac{d^\alpha}{dx^\alpha}f(x)$$

.

In this definition, α can be any real number. When taking regular derivatives, $\alpha \in \mathbb{N}$. In most of our cases, we are interested in the instance where $\alpha \in \mathbb{R}_+$.

It is also possible to study derivatives of negative order, which can be used to obtain moments of negative order of a function, provided that such an order exists. A derivative of negative order is simply an integral of positive order. This is defined as follows:

Definition 3.2.2. Let I be the integral operator, such that $If(x) = \int f(x)dx$. Then the fractional integral of order α is defined as

$$(I^\alpha f)(x) = \frac{1}{(\alpha-1)!}\int (x-t)^{\alpha-1} f(t)dt$$

Cauchy (1823).

Combining the previous two definitions, we obtain the following, more general, definition.

Definition 3.2.3. The differintegral operator is defined as

$$R^\alpha f(x) = \begin{cases} I^{|\alpha|} f(x) & \text{if } \alpha < 0 \\ D^\alpha f(x) & \text{if } \alpha > 0 \\ f(x) & \text{if } \alpha = 0 \end{cases}, \text{ with } \alpha \in \mathbb{R}. \tag{1}$$

As mentioned in section 3.1, quite a number of different definitions have been proposed to compute a fractional derivative. Some of these definitions are rather similar, thus, in this paper, we will focus on some of the more well known fractional derivatives. We will start with the most famous fractional derivative, which laid the foundation of the study of fractional derivatives as early as in 1832.

Definition 3.2.4. The left-side Riemann-Liouville fractional derivative of order α is defined as:

$$D^\alpha_{a+} f(x) = \frac{d^n}{dx^n} D_x^{-(n-\alpha)} f(x) = \frac{d^n}{dx^n} I_x^{n-\alpha} f(x) = \frac{d^n}{dx^n}\frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1} f(t)dt \tag{2}$$

Joseph Liouville (1832).

Here, $n = \lceil \alpha \rceil$, the ceiling function and $\Gamma(.)$ is the Gamma function, see section Appendix A.

Note that we just defined the left-side Riemann-Liouville fractional derivative, suggesting that there also exists a right side derivative. In the case of the latter, we would evaluate the associated integral the other way around. Namely, $D^\alpha_{b-} f(x) = \frac{d^n}{dx^n}\frac{1}{\Gamma(n-\alpha)}\int_x^b (x-t)^{n-\alpha-1} f(t)dt$. We intend on using the left-side derivative, as is supported by Tarasov (2023). The reason is, due to the fact that many functions in probability theory, most importantly the cumulative distribution function, are defined as an integral from some constant to x.

Remark 3.2.1. For values $\alpha \in \mathbb{N}_+, n = \lceil \alpha \rceil = \alpha$, so $\Gamma(n-\alpha) = \Gamma(0)$, which is undefined. Thus for $\alpha \in \mathbb{N}_+$, we define: $D^\alpha f(x) = \frac{d^\alpha}{dx^\alpha} f(x)$, which is simply the regular expression for derivatives of integer order.

A modification of the Riemann-Liouville derivative is the Caputo-Fabrizio derivative, which is defined as follows:

Definition 3.2.5. The Caputo-Fabrizio fractional derivative of order $\alpha, \alpha \in [0, 1)$ is defined as:

$$D^\alpha f(x) = \frac{1}{1-\alpha}\int_a^x \exp\left(\frac{-\alpha}{1-\alpha}(x-t)\right) f'(t)dt \tag{3}$$

Caputo and Fabrizio (2015). With $a \in [-\infty, x)$. The Caputo-Fabrizio is always defined as the integral from some constant to the variable x. This is another reason for choosing to work with the left-side Riemann-Liouville integral. In this way, it will be more straightforward to compare the two integrals. What is more, note that for the Caputo-Fabrizio derivative, the order $\alpha \in [0, 1)$. This does not mean, however, that one can only compute fractional derivatives of order 1 or lower. There exists a rather convenient property of the differintegral operator which allows one to combine orders of derivatives, which will be discussed in a moment.

Lastly, we will define the Grünwald-Letnikov derivative, which is defined as follows:

Definition 3.2.6. The Grünwald-Letnikov fractional derivative of order α is defined as:

$$D^\alpha f(x) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x - kh) \tag{4}$$

Where $\binom{\alpha}{k}$ is the binomial coefficient, with $0 \leq k \leq \alpha$, see section Appendix A.

Zhmakin (2022). It is immediately clear, observing the summmation symbol instead of the integral, that this derivative behaves quite differently from the two derivatives defined above. The Grünwald-Letnikov derivative is an extension on derivatives based of the concept of finite differences Flajolet and Sedgewick (1995).

We will consider a number of properties which come in useful when working with fractional derivatives.

Proposition 3.2.1. *The fractional derivatives above adhere to the following properties:*

(i) Linearity: Let $f(x), g(x)$ be functions and $a, b, x \in \mathbb{R}$. Then we have that $D^\alpha(af(x) + bg(x)) = aD^\alpha f(x) + bD^\alpha g(x)$.

(ii) $D^\alpha f(x) = f(x)$, for $\alpha = 0$

(iii) for sufficiently smooth functions f, we have that $D^{\alpha+\beta} f(x) = (D^\alpha(D^\beta) f(x)) = D^\beta(D^\alpha f(x))$, with $\alpha, \beta \in \mathbb{R}$. Note that, for definition 3.2.5, this property only holds for $\beta \in \mathbb{N}, \alpha \in [0, 1)$.

The proofs of these of the properties stated in this proposition can be found in Appendix B. Most of these proofs have been provided by myself, while some other proofs, which are outside the scope of thesis are based on other papers. Note that the third property is especially useful for the Caputo-Fabrizio derivative. This property allows one to take fractional derivatives of order greater than 1 by comparing fractional derivatives and regular integer derivatives.

We will now provide a few numerical examples of these fractional derivatives. For simplicity, we will let $a = 0$:

Example 3.2.1. (i) We consider the Riemann-Liouville derivative of order $\frac{3}{2}$ for some constant $c \in \mathbb{R}$:

$$\begin{aligned} D^{\frac{3}{2}}_{RL}(c) &= \frac{d^2}{dx^2} \frac{1}{\Gamma(2 - \frac{3}{2})} \int_0^x (x - t)^{2 - \frac{3}{2} - 1} c\,dt \\ &= \frac{d^2}{dx^2} \frac{c}{\sqrt{\pi}} \int_0^x (x - t)^{-\frac{1}{2}} dt = \frac{d^2}{dx^2} \frac{-2c}{\sqrt{\pi}} \sqrt{x - t}\Big|_0^x \\ &= \frac{d^2}{dx^2} \frac{2c\sqrt{x}}{\sqrt{\pi}} = \frac{-c}{2x\sqrt{\pi x}} \neq 0. \end{aligned}$$

As stated, the fractional derivative of a constant is not equal to zero when using the Riemann-Liouville derivative. This is also the case for the Grünwald-Letnikov derivative, but not for the Caputo-Fabrizio derivative.

(ii) We compute the semi-derivative of $\frac{1}{2}$ using the Caputo-Fabrizio derivative:

$$D^{\frac{1}{2}}_{CF}(\frac{1}{2}) = \frac{1}{1-\frac{1}{2}} \int_0^x \exp\left(\frac{-\frac{1}{2}}{1-\frac{1}{2}}(x-t)\right)\frac{1}{2}dt = \int_0^x \exp(t-x)dt$$

$$= \exp(t-x)\Big|_0^x = 1 - \exp(-x).$$

For $x \geq 0$, this expression is equal to the CDF of the exponential distribution with $\lambda = 1$. A remarkble result.

(iii) We will compute another famous result using the Riemann-Liouville derivative, namely, the semi-derivative of x:

$$D^{\frac{1}{2}}_{RL}(x) = \frac{d}{dx}\frac{1}{\Gamma(1-\frac{1}{2})}\int_0^x (x-t)^{1-\frac{1}{2}-1} t dt$$

U-substitution: $u = x - t$, such that $\frac{du}{dt} = -1, dt = -du$:

$$= \frac{d}{dx}\frac{1}{\sqrt{\pi}}\int_0^x (u)^{-\frac{1}{2}}(u-x)du = \frac{d}{dx}\frac{1}{\sqrt{\pi}}\left(\int_0^x \sqrt{u}du - x\int_0^x \frac{1}{\sqrt{u}}du\right)$$

$$= \frac{d}{dx}\frac{1}{\sqrt{\pi}}\left(\frac{2u^{\frac{3}{2}}}{3} - 2xu^{\frac{1}{2}}\right)\Big|_0^x = \frac{d}{dx}\frac{1}{\sqrt{\pi}}\left(\frac{2(x-t)^{\frac{3}{2}}}{3} - 2x(x-t)^{\frac{1}{2}}\right)\Big|_0^x$$

$$= \frac{d}{dx}\frac{-1}{\sqrt{\pi}}\left(\frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{3}{2}}\right) = \frac{d}{dx}\frac{4x^{\frac{2}{3}}}{3\sqrt{\pi}}$$

$$= \frac{2\sqrt{x}}{\sqrt{\pi}}.$$

The observant reader might notice that no explicit examples of the Grünwald-Letnikov derivative have been provided. The latter is due to the fact that is rather difficult to obtain analytical expressions for this derivative. Thus, later on in this thesis, when computing fractional moments and their assiocated biases, the main focus for the Grünwald-Letnikov derivative will be on its numerical computations.

Remark 3.2.2. As shortly mentioned in the introduction, it is possible to generalize the order of derivatives even further, extending α to be in $\mathbb{C}$ instead of $\alpha \in \mathbb{R}$. This means that, when combining such derivatives with the moment generating function, we will obtain complex moments. Since statisical moments of complex order do almost not find any usage in applications, as they lack interpretability, they are not the main focus of this research. For the interested reader, Love (1971) has done some impressive research on the fundamentals of derivatives of complex order. The obtained expressions are somewhat similar to those of the fractional derivatives which have been discussed above.

4 Conclusion

The conclusion summarizes the results of your research. Also it gives possible directions of further research.

Acknowledgements

Place the acknowledgments section, if needed, after the main text, but before any appendices and the references. The section heading is not numbered.

A (Appendix): Relevant functions and Identities

We define the (Euler-)Gamma function as follows:

Definition A.0.1. for $\Re(z) > 0$, we have the following: $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$

The Gamma function can be seen as an extension of the factorial function, for non-integers. This function is defined for complex numbers and all there subsets (so also real numbers), as long as the condition above holds. For positive integers values z, we have the following identity: $\Gamma(z) = (z-1)!$ Other important identities, not necessarily for z an integer, are:

- $\Gamma(z+1) = z\Gamma(z)$
- $\Gamma(2) = \Gamma(1) = 1$
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Definition A.0.2. The falling factorial is defined as follows: $(x)_n = \prod_{k=0}^{n-1}(x-k)$, which is a polynomial

Definition A.0.3. For $0 \leq k \leq n$, the Binomial Coefficient is defined as follows: $\binom{n}{k}$, where $n, k \in \mathbb{N}$.

We can derive the following factorial identity, which is convenient to work with analytically: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. For numerically computing expressions containing the Binomial Coefficient, the following identity is computationally more efficient: $\binom{n}{k} = \frac{(n)_k}{k!}$. With $(n)_k$ as in A.0.2. Since we have established in A.0.1 that $\Gamma(z) = (z-1)!$, we can rewrite our factorial identify to:

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\cdot\Gamma(n-k+1)} = \frac{n}{k}\cdot\frac{\Gamma(n)}{\Gamma(k)\cdot\Gamma(n-k+1)}$$

Definition A.0.4. Vandermonde's identity: for non-negative integers, k, l, m, n, we have that

$$\sum_{k=0}^{l}\binom{m}{k}\cdot\binom{n}{l-k} = \binom{m+n}{l}$$

.

A modification on the latter identity has been called the Chu-Vandermonde identity. This is the same identity, but it his been proven that the identities still hold for complex values m, n as long as l is a positive integer (Askey (1975)).

For the interchange-ability of derivatives and integrals and sums, we can apply the following two theorems:

B (Appendix): Proofs

B.1 Proofs section 3

Proof. Proof of 3.2.1

(i) We will proof for the Riemann-Liouville derivative, the proof for the Caputo-Fabrizio derivative is very similar and the Grünwald-Letnikov derivative is a direct consequence of the linearity of the sum.

$$D^\alpha(af(x) + bg(x)) = \frac{d^n}{dx^n}\frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}(af(t) + bg(t))dt$$

$$= \frac{d^n}{dx^n}\left(\frac{a}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1} f(t)dt + \frac{b}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1} g(t)dt\right)$$

Where we simply split the integral and put the constants in front.

$$= \frac{d^n}{dx^n}\frac{a}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1} f(t)dt + \frac{d^n}{dx^n}\frac{b}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1} g(t)dt$$

As the regular derivative operator is linear.

$$= aD^\alpha f(x) + bD^\alpha g(x)$$

(ii) Intuitively, this makes perfect sense, as the 0-th derivative is just no derivative, so just the function $f(x)$. But for these derivatives, a little bit more effort is needed to prove this rather obvious fact.

For the Grünwald-Letnikov derivative we get:

$$D^0 f(x) = \lim_{h\to 0}\frac{1}{h^0}\sum_{k=0}^{\infty}(-1)^k\binom{0}{k}f(x-kh) = \lim_{h\to 0}\frac{1}{1}\sum_{k=0}^{\infty}(-1)^k\frac{0!}{k!(0-k)!}f(x-kh).$$

The factorial Identity of the binomial coefficient only holds for $0 \leq k \leq \alpha$. Since $\alpha = 0$ and k is always a positive integer lesser or equal to $\alpha, k = 0$. Thus, we get:

$$= \lim_{h\to 0}\sum_{k=0}^{\infty}(-1)^0\frac{0!}{0!(0-0)!}f(x-0h) = \lim_{h\to 0} f(x-0h) = f(x).$$

For the Caputo-Fabrizio derivative, we obtain the following:

$$D^0 f(x) = \frac{1}{1-0}\int_0^x \exp\left(\frac{0}{1-0}(x-t)\right) f'(t)dt = \int_0^x f'(t)dt = f(x).$$

Finally, for the Riemann-Liouville derivative, we can simply make use of 3.2.3 and 3.2.1 to note that in this context $\alpha = 0$ is included in the natural integers. So $D^\alpha = \frac{d^\alpha}{dx^\alpha}f(x) = d^0 dx^0 f(x) = f(x)$ by the first fundamental theorem of calculus.

(iii) The proof for the Riemann-Liouville derivative is given by Koning (2015). And the proof for the Caputo-Fabrizio derivative is given by Losada and Nieto (2015).For the Grünwald-Letnikov derivative, we get:

$$D^\alpha(D^\beta f(x)) = \lim_{h\to 0}\frac{1}{h^\alpha}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}(\frac{1}{h^\beta}\sum_{l=0}^{\infty}(-1)^l\binom{\beta}{l}f(x-lh-kh))$$

$$= \lim_{h\to 0}\frac{1}{h^{\alpha+\beta}}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}\sum_{l=0}^{\infty}(-1)^l\binom{\beta}{l}f(x-(k+l)h).$$

We substitute $m = k + l$ to deal with the dubble sums:

$$\lim_{h\to 0}\frac{1}{h^{\alpha+\beta}}\sum_{m=0}^{\infty}f(x-mh)\sum_{k=0}^{m}(-1)^k(-1)^{m-k}\binom{\alpha}{k}\binom{\beta}{m-k}$$

Now we make use of an identify from Appendix A to obtain:

$$= \lim_{h\to 0}\frac{1}{h^{\alpha+\beta}}\sum_{m=0}^{\infty}(-1)^m\binom{\alpha+\beta}{m}f(x-mh) = D^{\alpha+\beta}f(x).$$

It can be shown in an exactly similar way that the latter expression is equal to $D^\beta(D^\alpha f(x))$.

□

C Appendix C

TODO: ADD TABLE OF COMMON DISTRIBUTIONS + SUPPORT, MGF, ETC.

D Appendix

Place any appendices after the acknowledgments, starting on a new page. The appendices are numbered **A**, **B**, **C**, and so forth. The appendices contain material that you would like to share with the reader but that would hinder the flow of reading. For instance long proofs of theorems, code of algorithms, data, etc.

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