Deriving fractional moments using the Moment Generating Function

Jelle Reisinger

(2780350)

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Thesis commission:

Prof. dr. xxxx (supervisor)

Dr. yyyy (co-reader)

Abstract

The abstract should summarize the contents of the thesis. It should be clear, descriptive, self-explanatory and not longer than a third of a page. Please avoid using mathematical formulas as much as possible. Keywords might be given.

Keywords: Fractional moments, Moment Generating Function.

1 Introduction

Statistical moments such as the mean and the variance are essential tools to characterize data and its distribution. Moments of even higher order are useful regarding the shape of the distribution. Less known moments, however, are the fractional moments. While these moments may not be that significant in comparison with the first for integer moments of a distribution, they can be very useful in certain applications. While these moments can be obtained by ordinary integration, the field of fractional calculus, which finds many application in physics, provides a more general framework to compute these moments. In this paper, we will combine the aforementioned technique with the Moment Generating Function to obtain new expressions for the fractional moments of a distribution.

2 Fractional Calculus

2.1 Fractional derivatives

In order to obtain the expression as mentioned in section section 1, some advanced tools are required. We can find this in the field of fractional calculus. We define the following:

Definition 2.1.1. Let D be the differential operator, such that $Df(x) = \frac{d}{dx}f(x)$. Then the fractional derivative of order α is defined as $D^\alpha f(x) = \frac{d^\alpha}{dx^\alpha}f(x)$.

In this definition, α can be any real number. When taking regular derivatives, $\alpha \in \mathbb{N}$. In our case, we are interested in instances where $\alpha \in \mathbb{Q}, \alpha \geq 0$.

It is also possible to study derivatives of negative order, which can be used to obtain moments of negative order of a function. A derivative of negative order is simply an integral of positive order. This is defined as follows:

Definition 2.1.2. Let I be the integral operator, such that $If(x) = \int f(x)dx$. Then the fractional integral of order α is defined as $(I^\alpha f)(x) = \frac{1}{(n-1)!}\int (x-t)^{n-1}f(t)dt$.

Combining the previous two definition, we can obtain the following definition.

Definition 2.1.3. The differintegral operator is defined as

$$R^\alpha f(x) = \begin{cases} I^{|\alpha|}f(x) & \text{if } \alpha < 0 \\ D^\alpha f(x) & \text{if } \alpha > 0 \\ f(x) & \text{if } \alpha = 0 \end{cases} \tag{1}$$

A lot of different definition have been used to compute a fractional derivative. In this paper, we will focus on the following fractional derivatives:

Definition 2.1.4. The Riemann-Liouville fractional derivative of order α is defined as:

$$D^\alpha f(x) = \frac{d^n}{dx^n}D_x^{-(n-\alpha)}f(x) = \frac{d^n}{dx^n}I_x^{n-\alpha}f(x) = \frac{d^n}{dx^n}\frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}f(t)dt \tag{2}$$

Where $n = \lceil \alpha \rceil$, the ceiling function and $\Gamma(.)$ is the Gamma function, see section Appendix A.

Remark 2.1.1. For values $\alpha \in \mathbb{N}, n = \alpha$, so $\Gamma(n-\alpha) = \Gamma(0)$, which is undefined. Thus for $\alpha \in \mathbb{N}$, including the value 0, we simply define: $D^\alpha f(x) = \frac{d^\alpha}{dx^\alpha}f(x)$

A modification of the Riemann-Liouville derivative is the Caputo-Fabrizio derivative, which is defined as follows:

Definition 2.1.5. The Caputo-Fabrizio fractional derivative of order $\alpha, \alpha \in [0,1)$ is defined as:

$$D^\alpha f(x) = \frac{1}{1-\alpha}\int_0^x \exp(\frac{-\alpha}{1-\alpha}(x-t))f'(t)dt \tag{3}$$

Lastly, we will define the Grünwald-Letnikov derivative, which is defined as follows:

Definition 2.1.6. The Grünwald-Letnikov fractional derivative of order α is defined as:

$$D^\alpha f(x) = \lim_{h\to 0}\frac{1}{h^\alpha}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}f(x-kh) \tag{4}$$

Where $\binom{\alpha}{k}$ is the binomial coefficient, see section Appendix A.

Proposition 2.1.1. *The fractional derivatives above adhere to the following properties:*

(i) Linearity: Let $f(x), g(x)$ be functions and $a, b, x \in \mathbb{R}$. Then we have that $D^\alpha(af(x) + bg(x)) = aD^\alpha f(x) + bD^\alpha g(x)$.

(ii) $D^\alpha f(x)$ for $\alpha = 0, = f(x)$

(iii) for sufficiently smooth functions f, we have that $D^{\alpha+\beta} f(x) = D^\alpha(D^\beta f(x)) = D^\beta(D^\alpha f(x))$, with $\alpha, \beta \in \mathbb{R}$. Note that, for 2.1.5, this only holds for $\beta \in \mathbb{N}, \alpha \in [0, 1)$.

Proof. (i) We will proof for the Riemann-Liouville derivative, the proof for the Caputo-Fabrizio derivative is very similar and the Grünwald-Letnikov derivative is a direct consequence of the linearity of the sum.

$$D^\alpha(af(x) + bg(x)) = \frac{d^n}{dx^n}\frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}(af(t) + bg(t))dt$$

$$= \frac{d^n}{dx^n}(\frac{a}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1} f(t)dt + \frac{b}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1} g(t)dt)$$

Where we simply split the integral and put the constants in front.

$$= \frac{d^n}{dx^n}\frac{a}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1} f(t)dt + \frac{d^n}{dx^n}\frac{b}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1} g(t)dt$$

As the regular derivative operator is just linear.

$$= aD^\alpha f(x) + bD^\alpha g(x)$$

(ii) Intuitively, this makes perfect sense, as the 0-th derivative is just no derivative, so just the function $f(x)$. But for these derivatives, a little bit more effort is needed to prove this rather obvious fact.
For the Grünwald-Letnikov derivative we get:

$$D^0 f(x) = \lim_{h\to 0}\frac{1}{h^0}\sum_{k=0}^{\infty}(-1)^k\binom{0}{k} f(x-kh) = \lim_{h\to 0}\frac{1}{1}\sum_{k=0}^{\infty}(-1)^k\frac{0!}{k!(0-k)!} f(x-kh).$$

The factorial Identity of the binomial coefficient only holds for $0 \leq k \leq \alpha$. Since $\alpha = 0$ and k is always a positive integer lesser or equal to $\alpha, k = 0$. Thus, we get:

$$= \lim_{h\to 0}\sum_{k=0}^{\infty}(-1)^0\frac{0!}{0!(0-0)!} f(x-0h) = \lim_{h\to 0} f(x-0h) = f(x).$$

For the Caputo-Fabrizio derivative, we get the following:

$$D^0 f(x) = \frac{1}{1-0}\int_0^x \exp(\frac{0}{1-0}(x-t))f'(t)dt = \int_0^x f'(t)dt = f(x).$$

Finally, for the Riemann-Liouville derivative, we can simply make use of 2.1.3 and 2.1.1 to note that in this context $\alpha = 0$ is included in the natural integers. So $D^\alpha = \frac{d^\alpha}{dx^\alpha} f(x) = d^0 dx^0 f(x) = f(x)$.

(iii) The proof for the Riemann-Liouville derivative is given by Koning (2015). And the proof for the Caputo-Fabrizio derivative is given by Losada and Nieto (2015).For the Grünwald-Letnikov derivative, we get:

$$D^{\alpha}(D^{\beta}f(x)) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}(\frac{1}{h^{\beta}}\sum_{l=0}^{\infty}(-1)^l\binom{\beta}{l}f(x-lh-kh))$$

$$= \lim_{h\to 0}\frac{1}{h^{\alpha+\beta}}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}\sum_{l=0}^{\infty}(-1)^l\binom{\beta}{l}f(x-(k+l)h).$$

We substitute $m = k + l$ to deal with the dubble sums:

$$\lim_{h\to 0}\frac{1}{h^{\alpha+\beta}}\sum_{m=0}^{\infty}f(x-mh)\sum_{k=0}^{m}(-1)^k(-1)^{m-k}\binom{\alpha}{k}\binom{\beta}{m-k}$$

Now we make use of an identify from Appendix A to obtain:

$$= \lim_{h\to 0}\frac{1}{h^{\alpha+\beta}}\sum_{m=0}^{\infty}(-1)^m\binom{\alpha+\beta}{m}f(x-mh) = D^{\alpha+\beta}f(x).$$

It can be shown in an exactly similar way that the latter expression is equal to $D^{\beta}(D^{\alpha}f(x))$.
□

We will now provide a few numerical examples of these fractional derivatives. For simplicity, we will let $a = 0$:

Example 2.1.1. (i)

$$D^{\frac{3}{2}}_{RL}(c) = \frac{d^2}{dx^2}\frac{1}{\Gamma(2-\frac{3}{2})}\int_0^x (x-t)^{2-\frac{3}{2}-1}c\,dt$$

$$= \frac{d^2}{dx^2}\frac{c}{\sqrt{\pi}}\int_0^x (x-t)^{-\frac{1}{2}}dt = \frac{d^2}{dx^2}\frac{-2c}{\sqrt{\pi}}\sqrt{x-t}\Big|_0^x$$

$$= \frac{d^2}{dx^2}\frac{2c\sqrt{x}}{\sqrt{\pi}} = \frac{-c}{2\pi(x)^{\frac{3}{2}}} \neq 0.$$

As stated, the fractional of a constant is not equal to zero when using the Riemann-Liouville derivative, this is also the case for the Grünwald-Letnikov derivative, but not for the Caputo-Fabrizio derivative.

(ii)

$$D^{\frac{1}{2}}_{CF}(\frac{x}{2}) = \frac{1}{1-\frac{1}{2}}\int_0^x \exp(\frac{-\frac{1}{2}}{1-\frac{1}{2}}(x-t))\frac{1}{2}dt = \int_0^x \exp(t-x)dt$$

$$= \exp(t-x)\Big|_0^x = 1-\exp(-x).$$

For $x \geq 0$, this expression is equal to the CDF of the exponential distribution with $\lambda = 1$. A remarkble result.

(iii) We will compute another famous result with the Riemann-Liouville derivative:

$$D^{\frac{1}{2}}_{RL}(x) = \frac{d}{dx}\frac{1}{\Gamma(1-\frac{1}{2})}\int_0^x (x-t)^{1-\frac{1}{2}-1}t\,dt$$

Let $u = x - t$, such that $\frac{du}{dt} = -1, dt = -du$:

$$= \frac{d}{dx}\frac{1}{\sqrt{\pi}}\int_0^x (u)^{-\frac{1}{2}}(u-x)du = \frac{d}{dx}\frac{1}{\sqrt{\pi}}(\int_0^x \sqrt{u}du - x\int_0^x \frac{1}{\sqrt{u}}du)$$

$$= \frac{d}{dx}\frac{1}{\sqrt{\pi}}(\frac{2u^{\frac{3}{2}}}{3} - 2xu^{\frac{1}{2}}\Big|_0^x) = \frac{d}{dx}\frac{1}{\sqrt{\pi}}(\frac{2(x-t)^{\frac{3}{2}}}{3} - 2x(x-t)^{\frac{1}{2}}\Big|_0^x)$$

$$= \frac{d}{dx}\frac{-1}{\sqrt{\pi}}(\frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{3}{2}} = \frac{d}{dx}\frac{\frac{4}{3}x^{\frac{2}{3}}}{\sqrt{\pi}})$$

$$= \frac{2\sqrt{x}}{\sqrt{\pi}}.$$

2.2 Complex derivatives

WIP

3 Moments

Now that we have defined the required techniques to compute fractional derivatives, we can apply them to the Moment Generating Function. First, however, we need to define some concepts relevant to computing moments.

3.1 Moments

Definition 3.1.1. The n-th moment of a PDF $f_X(x)$ is defined as:

$$E[X^n] = \begin{cases} \int_{-\infty}^{\infty} (x-c)^n f_X(x)\,dx & \text{if } f_X(x) \text{ is continuous,} \\ \sum_i (x_i - c_i)^n f_X(x_i) & \text{if } f_X(x) \text{ is discrete.} \end{cases}$$

If $c = \mu_x$, the first moment of $f(x)$, then our higher moments are called central moments. In our situation, we will focus on the case $c = 0$, which is called a raw moment. We do this as the Moment Generating Function, which we will soon define, only computes raw moments of higher order. A moment of order $n$is said to exist, if $E[X^n] < \infty$.

Proposition 3.1.1. *We can derive the following simple properties:*

(i) If $E[X^n]$ does not exist, then $E[X^k]$ also does not exist, for $k \geq n$.

(ii) If $E[X^k]$ does exist, then all its lower moments, i.e. $E[X^n]$, for $n \leq k$. Also exist.

Proof. (i) If $f(x)$ is a discrete distribution, we get:

$$E[X^n] = \sum_i x_i^n f(x_i) = \infty, E[X^k] = \sum_i x_i^k f(x_i)$$

$$= \sum_i x_i^n \cdot x^{k-n} f(x_i) \geq \sum_i x_i^n f(x_i) = \infty, \text{ as } k \geq n$$

We can prove this for continuous functions in a similar way.

(ii) This simply follows from the previous proposition, as the current proposition is just the contrapositive statement of the previous proposition.

□

Example 3.1.1. *Let $f_x(x) \sim C(0,1)$ where $C(x_0, y_0)$*
is the cauchy distribution funtion , does the second moment exist?
We use 3.1.1 and begin with verifying that the first moment exists:

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)} dx.$$

For large values of x, the inside is equal to $\frac{1}{x}$, splitting the integral:

$$= \int_1^{\infty} \frac{1}{x} dx - \int_{-\infty}^{1} \frac{1}{x} dx$$

Where both integrals diverge logarithmically.

3.2 Moments of negative order

We take a quick look at (raw) moments of negative order, and if we can define these in the same way as above, let us look at the continuous case first:

$$E[X^-n] = \int_{-\infty}^{\infty} x^{-n} f_x(x)dx = \int_{-\infty}^{\infty} (\frac{1}{x})^n f_x(x)dx.$$

We can immediately observe a rather obvious problem. This integral is not defined at $x = 0$ and diverges for values of x in a neighbourhood of 0. Khuri and Casella (2002) have stated the following corollary for the existence of the first negative moment:

Corollary 3.2.1. *If $f(x)$ is a continious pdf defined on $(-\infty, \infty)$, and if*

$$\lim_{x \to 0} \frac{f(x)}{|x|^\alpha} < \infty$$

, for $\alpha > 0$, then

$$E[X^{-1}]\textit{exists}.$$

Not a lot of common distribution functions adhere to this corollary, however, the Gamma function does:

Example 3.2.1. Let $f_X(x) \sim \Gamma(\alpha, \lambda) =$

$$\frac{x^{\alpha-1}e^{-\lambda x}\lambda^\alpha}{\Gamma(\alpha)}.$$

This PDF is defined on $(0, \infty)$. So the function is not defined on $\mathbb{R}$. This, however, is not a problem, as we can just evaluate the right limit. Since the Gamma function already uses α as a parameter, we will look at $\frac{f(x)}{x^\beta}, \beta > 0$:

$$\lim_{x \to 0_+} \frac{f(x)}{x^\beta} = \lim_{x \to 0_+} \frac{x^{\alpha-1-\beta}e^{-\lambda x}\lambda^\alpha}{\Gamma(\alpha)} = \lim_{x \to 0_+} \frac{x^{\alpha-(1+\beta)}e^{-\lambda x}\lambda^\alpha}{\Gamma(\alpha)}.$$

So for $\alpha \geq \beta + 1, \lim_{x \to 0_+} \frac{f(x)}{x^\beta} < \infty$. So the first negative moment of the Gamma distribution should exist.

We will compute the first negative moment:

$$E[X^{-1}] = \int_0^\infty x^{-1} \frac{x^{\alpha-1}e^{-\lambda x}\lambda^\alpha}{\Gamma(\alpha)} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-2}e^{-\lambda x}dx$$

Using the substitution $u = \lambda x, \frac{du}{dx} = \lambda, dx = \frac{du}{\lambda}$, we get:

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty (\frac{u}{\lambda})^{\alpha-2}e^{-u}du = \frac{\lambda^\alpha}{\Gamma(\alpha)\lambda^{\alpha-1}} \int_0^\infty (\frac{u}{\lambda})^{\alpha-2}e^{-u}du.$$

This integral is equal to $\Gamma(\alpha - 1)$ (See A). So we get:

$$E[X^{-1}] = \frac{\lambda^\alpha \Gamma(\alpha - 1)}{\Gamma(\alpha)\lambda^{\alpha-1}} = \frac{\lambda\Gamma(\alpha - 1)}{(\alpha - 1)\Gamma(\alpha - 1)} = \frac{\lambda}{(\alpha - 1)}.$$

So, for $\alpha \neq 1, E[X^{-1}] = \frac{\lambda}{(\alpha-1)}$. Fortunately, this is always the case, since we had just derived that the integral is only finite when $\alpha \geq \beta + 1,$ with $\beta > 0$. In other words, $\alpha > 1$. So this holds.

3.3 The Moment Generating Function

We will finally define the Moment Generating Function, one of the most significant subjects of this thesis.

Definition 3.3.1. The Moment Generating Function (MGF) of a variable X, is defined as

$$M_X(t) = E[e^{tX}]$$

provided that $E[e^t X] < \infty$ on some interval $(-h, h)$, which contains 0, for some $h > 0$.

Remark 3.3.1. Obtaining the expression $M_X(t) = E[e^{tX}]$ is pretty trivial. Generally, it just requires quite a few steps of analytic evalution. This is not that interesting nor relevant to this research. We will give one explicit example on how to compute the Moment Generating Function for a specific distribution. And for later cases, when we make use of an expression of the Moment Generating Function, we will simply refer to figure 2.1.5.

Example 3.3.1. Let $f_X(x) \sim \Gamma(\alpha, \lambda)$ be the Gamma distribution with PDF:

$$f_X(x) = \frac{x^{\alpha-1}e^{-\lambda x}\lambda^\alpha}{\Gamma(\alpha)}.$$

Let $t < \lambda$ (in any other case, the integral diverges).The moment generating function $M_X(t)$ is given by:

$$M_X(t) = \int_0^\infty e^{tx} f_X(x)\,dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{(t-\lambda)x}\,dx.$$

We make use of the substitution $u = -(t-\lambda)x$, $\frac{du}{dx} = (\lambda - t)$, $dx = \frac{du}{(\lambda-t)}$, so:

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{u}{(\lambda-t)}^{\alpha-1} e^{-u} \frac{du}{\lambda - t} = \frac{\lambda^\alpha}{\Gamma(\alpha)(\lambda-t)^\alpha} \int_0^\infty u^{\alpha-1}e^{-u}\,du.$$

This integral is just the definition of the Gamma Function, $\Gamma(\alpha)$, so we obtain:

$$= \frac{\lambda^\alpha \Gamma(\alpha)}{\Gamma(\alpha)(\lambda-t)^\alpha} = (\frac{\lambda}{\lambda - t})^\alpha = M_X(t)$$

We will state the theorem which makes the MGF so useful.

Theorem 3.3.1. *If $M_X(t)$ exists on some interval $-h, h$, as defined before, we have that:*

$$E[X^n] = M_X^{(n)}(0), \textit{ for } n \in \mathbb{N}$$

Proof. The proof is fairly straightforward:

$$M_X^{(n)}(t) = \frac{d^n}{dt^n} \int_{-\infty}^\infty e^{tx} f_X(x)dx = \int_{-\infty}^\infty \frac{d^n}{dt^n} e^{tx} f_X(x)dx$$

(We can interchange differentiation and integration since all partial derivatives of $e^{tx}f(x)$ are continuous and the absolute value of the integral converges, as we assume the n-th moment exists, see A).

$$= \int_{-\infty}^\infty x^n e^{tx} f_X(x)dx, \text{evaluate at } t = 0 := \int_{-\infty}^\infty x^n e^{0x} f_X(x)dx$$

$$= \int_{-\infty}^\infty x^n f_X(x)dx = E[x^n]$$

The proof for the case that $f_X(x)$ is discrete is very similar. In that case, one would have to change the order of the derivative and summation, which has also been justified in A. □

Remark 3.3.2. Since the validity of interchanging the order of differentiation and integration is based on characteristics of the integral and its inside, we can extend Theorem 3.3.1 to real values α.

We introduce the following properties for the Moment Generating Function $M_X(t)$:

Proposition 3.3.1. *For X, Y random variables, we have that:*

(i) $M_X^{(0)}(t) = E[e^{0X}] = 1$. *This property can be used to check if a given is really a PDF.*

(ii) Location scale-transform. Assuming $M_X(t)$ exists, for constant μ, σ, we have that:

$$M_{\mu+\sigma X}(t) = e^{\mu t} \cdot M_X(\sigma t)$$

(iii) If $X \perp Y$, then $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$.

Proof. (i) This is trivial. For continuous functions of x, we get:

$$M_X^{(0)}(t) = \int_{-\infty}^{\infty} x^0 e^{tx} f_X(x)dx = \int_{-\infty}^{\infty} 1e^{0x} f_X(x)dx = \int_{-\infty}^{\infty} f_X(x)dx.$$

Assuming that $f(x)$ is a PDF, this integrates to 1 by definition. If this integral is not equal to 1, this implies that $f(x)$ is not a PDF. The proof for the discrete case is the exact but with a summation instead of an integral sign.

(ii)

$$M_{\mu+\sigma X}(t) = E[e^{(\mu+\sigma x)t}] = E[e^{\mu t} \cdot e^{\sigma x t}].$$

Since this is the expectation of x, every term that is not dependent on x can be taken out of the summation:

$$= e^{\mu t} \cdot E[e^{\sigma x t}] = e^{\mu t} \cdot M_x(\sigma t)$$

(iii)

$$M_{X+Y}(t) = E[e^{(x+y)t}] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{(x+y)t} f_{X,Y}(x,y)dxdy$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{xt} \cdot e^{yt} f_{X,Y}(x,y)dxdy.$$

$f_{X,Y}(x,y)$ is the joint pdf for X, Y. But we know that the latter is equal to $f_X(x) \cdot f_Y(y)$, if X, Y are independent. Thus we get:

$$= (\int_{-\infty}^{\infty} e^{xt} f_X(x)dx)(\int_{-\infty}^{\infty} e^{yt} f_Y(y)dy) = M_X(t) \cdot M_Y(t).$$

□

To illustrate the usage of fractional derivatives with the MGF, we will give the following numerical examples. We let $f(x) = \lambda \cdot e^{-\lambda x}$ on $x \in [0, \infty)$. We will calculate its semi-moment using the Riemann-Liouville derivative, the Caputo-Fabrizio derivative and just the regular integral to compute moments.

Example 3.3.2. (i)

3.3.1 Computing negative moments using the Moment Generating Function

We have seen that under specific conditions, it is possible to compute negative, or so called inverse moments of distribution functions. This technique can be extended to the moment generating function. In the 20-th century, Cressie et al. (1981) have published the following remarkable theorem:

Theorem 3.3.2. *Assuming the negative n-th raw moment exists, the negative n-th raw moment can be computed as follows:*

$$E[X^{-n}] = \frac{1}{\Gamma(n)} \int_0^\infty t^{n-1} M_X(-t) dt$$

, where n is a positive integer.

The proof of this Theorem can be found in B.

Let us again compute the first inverse moment of the Gamma distribution, but now by making use of the latter theorem for Moment Generating Functions!

Example 3.3.3. Let $f_x(x) \sim \Gamma(\alpha, \lambda) =$

$$\frac{x^{\alpha-1} e^{-\lambda x} \lambda^\alpha}{\Gamma(\alpha)}, M_X(t) = (\frac{\lambda}{\lambda - t})^\alpha$$

$$E[X^{-1}] = \frac{1}{\Gamma(1)} \int_0^\infty t^{1-1} (\frac{\lambda}{\lambda - (-t)})^\alpha dt = \int_0^\infty (\frac{\lambda}{\lambda + t})^\alpha dt$$

$$= \lambda^\alpha \int_0^\infty (\lambda + t)^{-\alpha} dt, \text{ Let } u = \lambda + t, \frac{du}{dt} = 1, dt = du : \lambda^\alpha \int_0^\infty u^{-\alpha} du$$

$$= \lambda^\alpha \frac{u^{1-\alpha}}{1-\alpha}\Big|_0^\infty = \lambda^\alpha \frac{(\lambda + t)^{1-\alpha}}{1-\alpha}\Big|_0^\infty$$

$$= \lambda^\alpha (0 - \frac{\lambda^{1-\alpha}}{1-\alpha}) = \frac{-\lambda}{1-\alpha} = \frac{\lambda}{\alpha - 1}.$$

Which corresponds with our result from Example 3.2.1.

3.3.2 Laplace Transformation

WIP

3.3.3 Characteristic function and Fourier Transformations

WIP

4 Conclusion

The conclusion summarizes the results of your research. Also it gives possible directions of further research.

Acknowledgements

Place the acknowledgments section, if needed, after the main text, but before any appendices and the references. The section heading is not numbered.

A (Appendix): Relevant functions and Identities

We define the (Euler-)Gamma function as follows:

Definition A.0.1. for $\Re(z) > 0$, we have the following: $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$

The Gamma function can be seen as an extension of the factorial function, for non-integers. This function is defined for complex numbers and all there subsets (so also real numbers), as long as the condition above holds. For positive integers values z, we have the following identity: $\Gamma(z) = (z-1)!$ Other important identities, not necessarily for z an integer, are:

- $\Gamma(z+1) = z\Gamma(z)$
- $\Gamma(2) = \Gamma(1) = 1$
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Definition A.0.2. The falling factorial is defined as follows: $(x)_n = \prod_{k=0}^{n-1}(x-k)$, which is a polynomial

Definition A.0.3. For $0 \leq k \leq n$, the Binomial Coefficient is defined as follows: $\binom{n}{k}$, where $n, k \in \mathbb{N}$.

We can derive the following factorial identity, which is convenient to work with analytically: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. For numerically computing expressions containing the Binomial Coefficient, the following identity is computationally more efficient: $\binom{n}{k} = \frac{(n)_k}{k!}$. With $(n)_k$ as in A.0.2. Since we have established in A.0.1 that $\Gamma(z) = (z-1)!$, we can rewrite our factorial identify to:

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\cdot\Gamma(n-k+1)} = \frac{n}{k}\frac{\Gamma(n)}{\Gamma(k)\cdot\Gamma(n-k+1)}$$

Definition A.0.4. Vandermonde's identity: for non-negative integers, k, l, m, n, we have that

$$\sum_{k=0}^{l}\binom{m}{k}\cdot\binom{n}{l-k} = \binom{m+n}{l}$$

.

A modification on the latter identity has been called the Chu-Vandermonde identity. This is the same identity, but it his been proven that the identities still hold for complex values m, n as long as l is a positive integer (Askey (1975)).

For the interchange-ability of derivatives and integrals and sums, we can apply the following two theorems:

Theorem A.0.1. *Leibnitz's Rule: Let $f(x,\theta), a(\theta), b(\theta)$ be differentiable with respect to θ, then we have that:*

$$\frac{d}{d\theta}\int_{a(\theta)}^{b(\theta)} f(x,\theta)dx = f(b(\theta),\theta)\frac{db(\theta)}{d\theta} - f(a(\theta),\theta)\frac{da(\theta)}{d\theta} + \int_{a(\theta)}^{b(\theta)}\frac{\partial f(x,\theta)}{\partial\theta}dx.$$

For the special case, where $a(\theta), b(\theta)$ are constant we have that:

$$\frac{d}{d\theta}\int_a^b f(x,\theta)dx = \int_a^b \frac{\partial f(x,\theta)}{\partial d\theta}.$$

For the interchange-ability of derivatives and summations, the following theorem has been given by Casella and Berger (2002):

Theorem A.0.2. *Suppose that the series $\sum_{x=0}^{\infty} h(\theta, x)$ converges for all θ in an interval (a, b) of real numbers and*

(i) $\frac{\partial h(\theta,x)}{\partial \theta}$ *is continuous for all* x

(ii) $\sum_{x=0}^{\infty} \frac{\partial h(\theta,x)}{\partial \theta}$ *converges uniformly on every closed bounded subinterval of* (a, b)

Then:

$$\frac{d}{d\theta}\left(\sum_{x=0}^{\infty} h(\theta, x)\right) = \sum_{x=0}^{\infty} \frac{\partial h(\theta, x)}{\partial \theta}$$

B (Appendix): Proofs

Proof. Theorem 3.3.2: Suppose for the moment that X is a positive random variable. Since $xf(x)$ is integrable for $x > 0$, we have:

$$E(X) = \int_0^\infty x\,dF(x) = \int_0^\infty \int_0^\infty e^{tx}\,dt\,dF(x).$$

We can interchange the order of integration as follows:

$$E(X) = \int_0^\infty e^{tx}\,dF(x)\,dt = \int_0^\infty M_X(-t)\,dt.$$

The interchange of the order of integration is subject to $E(e^{-tX})$ being integrable from $t = 0$ to $t = \infty$.

Finally, by substituting X^{-1} for X, we find:

$$E(X^{-1}) = \int_0^\infty M_X^{-1}(-t)\,dt.$$

There are two natural ways to generalize (1) to $E(X^{-1})$; one way gives:

$$E(X^{-n}) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty M_X(-t_n)\,dt_n \cdots dt_2 dt_1, \tag{2}$$

while the second way gives:

$$E(X^{-n}) = \frac{1}{\Gamma(n)} \int_0^\infty t^{n-1} M_X(-t)\,dt.$$

Cressie et al. (1981) □

C Appendix C

TODO: ADD TABLE OF COMMON DISTRIBUTIONS + SUPPORT, MGF, ETC.

D Appendix

Place any appendices after the acknowledgments, starting on a new page. The appendices are numbered **A**, **B**, **C**, and so forth. The appendices contain material that you would like to share with the reader but that would hinder the flow of reading. For instance long proofs of theorems, code of algorithms, data, etc.

References

Askey, R. (1975). *Orthogonal Polynomials and Special Functions*. Society for Industrial and Applied Mathematics.

Casella, G. and R. L. Berger (2002). *Statistical inference*, Volume 2. Duxbury Pacific Grove, CA.

Cressie, N., A. S. Davis, J. L. Folks, and G. E. Policello (1981). The moment-generating function and negative integer moments. *The American Statistician 35*(3), 148–150.

Khuri, A. and G. Casella (2002). The existence of the first negative moment revisited. *The American Statistician 56*(1), 44–47.

Koning, D. (2015). *Fractional Calculus*. Ph. D. thesis, University of Groningen.

Losada, J. and J. Nieto (2015, 04). Properties of a new fractional derivative without singular kernel. *Prog Fract Differ Appl 1*, 87–92.