Deriving fractional moments using the Moment Generating Function

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Abstract

The abstract should summarize the contents of the thesis. It should be clear, descriptive, self-explanatory and not longer than a third of a page. Please avoid using mathematical formulas as much as possible. Keywords might be given.

Keywords: Fractional moments, Moment Generating Function, fractional calculus.

1 Introduction

Statistical moments such as the mean and the variance are essential tools to characterize data and its distribution. Moments of even higher order are useful regarding the shape of the distribution. Less known moments, however, are the fractional moments. While these moments may not be that significant in comparison with the first for integer moments of a distribution, they can be very useful in certain applications. While these moments can be obtained by ordinary integration, the field of fractional calculus, which finds many application in physics, provides a more general framework to compute these moments. In this paper, we will combine the aforementioned technique with the Moment Generating Function to obtain new expressions for the fractional moments of a distribution.

2 Literature Review

2.1 Application of fractional moments

Fractional moments play a significant role in a variety fields, including finance, economics, and statistics An example of the latter is its application in approximating integer moments as described by Novi Inverardi and Tagliani (2024). This is especially useful when (for $\alpha \in \mathbb{R}$) the $\lceil \alpha \rceil$-th central integer moment might not exist, while its fractional moment does. Finding an existing fractional moment close to a non-existing integer moment, could still provide one with information about this integer moment. For example, the student-t distribution with $\nu = 2$ degrees of freedom only has central and raw moments of order k, where $0 < k < \nu$. This implies that its second central moment ($k = 2$) does not exist. One could however consider the $k-th$ central moment where $k = 1.95$ and interpret its value as the variance of the distribution. Fractional moments are also used in financial modelling, particularly in the context of Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models. These models are commonly applied to time series data such as financial returns and capture the dynamic volatility that changes over time. The GARCH model achieves this by modeling the volatility based on the returns and variances of previous time periods. Hansen and Tong (2024) have obtained a method of finding fractional absolute moments of the cumulative return, which would have been impossible when using any other method. Gzyl et al. (2013) have also introduced the usage of Fractional moments in risk-models, specifically insurance models. In such models, often the probability density function of total ruin, the event where an insurance company's capital becomes negative, is unknown. The so-called "Method of maximum entropy" has been developed to find these densities. This method takes fractional moments as its input, as they have been proven to be able to characterize its distribution Lin (1992). This method has proven to be a useful alternative to existing methods, such as inverse Laplace transformations. This is the case as this new method takes less values than the latter as input, making it computationally more efficient. Beyond finance and risk modelling, fractional moments also have important applications in engineering. Examples include optimizing signal processing and control systems as well as studying the response characteristics of random vibration systems. Wang et al. (2025) has shown that when using the concept of fractional moments for the latter, accuracy and stability is higher compared to traditional methods, such as Taylor expansions. Furthermore, in terms of simplicity and efficiency, the method of fractional moments is advantageous, as its computation steps are straightforward and avoid convergence issues, significantly reducing the resources required for computation. Working with fractional moments has allowed Wang et al. (2025) to obtain both analytical, as well as numerical solutions to problems within their research field, which again proves its viability. Another application within the field of engineering, can be found in the identification of distributions of non-linear systems. Di Matteo et al. (2014) have shown that complex fractional moments allow one to solve equations such as the Kolmogorov or Fokker-Planck equation, a characterization of continuous-time Markov processes. After performing a Mellin transformation on this system of equations, the resulting system is a linear system in terms of complex fractional moments. The latter can now be solved rather easily and taking the Inverse Mellin transformation on these solutions immediately provides one with the solutions of the non-linear system. Advantages of using complex fractional moments instead of integer moments is that, when applying the Mellin transformation, the relevant PDF is restored on its entire support. This is not the case for the integer moments. This method of using complex fractional moments has been proven to have a rather high accuracy and is applicable to any stable kind of non-linear system of equations Di Matteo et al. (2014).

2.2 Obtaining fractional moments

The traditional method of computing fractional moments is rather straightforward. Similarly to integer moments, one simply computes the summation (in the discrete case) or integral (in the continuous case) of $x^\alpha \cdot f_x(x)$, where $f_x(x)$ denotes the probability density function of the random variable X. In the context of fractional moments, $\alpha \in \mathbb{R}$ instead of $\mathbb{Z}$ (assuming that negative moments exist). Hansen and Tong (2024) introduce an alternative approach to computing fractional moments using the complex moment generating function (CMGF), which they apply in the context of the aforementioned GARCH models. One of their key expressions is given by:

$$\mathbb{E}\,|X-\xi|^r = \frac{\Gamma(r+1)}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-\xi z}M_X(z)+e^{\xi z}M_X(-z)}{z^{r+1}}dt$$

where $z = s + it, s \in \mathbb{N}_+, \xi \in \mathbb{R}$ and r of course the order of the moment.

This formulation extends upon the traditional moment generating function (MGF) but avoids the process of taking derivatives, making it computationally efficient. The inclusion of the Gamma function is logical, as it extends the factorial function to real values, aligning well with the computation of fractional moments. Since this method relies on integration, rather than differentiation, it avoids numerical issues that might arise when computing derivatives, such as obtaining rather great approximation errors. While the CMGF method provides an efficient and elegant alternative, this thesis explores a different approach: computing fractional moments directly by applying fractional derivatives to the MGF. The MGF is widely used for computing integer moments by differentiation around zero. To extend this approach to fractional orders requires us to take fractional derivatives. Thus, we need to define such fractional derivative operators. These fractional derivatives have a long history and often make use of the aforementioned Gamma function in combination with some integral. This means that, for continuous random variables, where we integrate the MGF, we will have to do double integration. A consequence might be that obtaining analytical expressions of these moments may not always be possible. A lot of alternative expressions of these fractional derivatives have been created, mostly based on different interpretations of the latter in the field of physics. This implies that different expressions of fractional derivatives in combination with the MGF might obtain different moments expressions for the same distribution and same fractional order of the moment. Thus, it is essential to compare each of these definitions with the traditional way of computing fractional moments, to derive their accuracy and conclude which approach is most suitable for fractional moment computation. Similar to the expression of the moments of a random variable, their 'Biases' may also be hard to derive analytically, depending on its distribution.

2.3 Fractional derivatives

Although not the main topic of interest of this thesis, it is useful to have some knowledge about the history and applications of fractional derivatives. The study of fractional derivatives has been relevant as early as the year 1695 when the concept of such a derivative was implicitly discussed by Leibniz and Bernoulli Katugampola (2014). Since then, numerous definitions of fractional derivatives have been developed. The best-known definition of the fractional derivative is the Riemann-Liouville derivative, its upper derivative of a function $f(x)$ of order α is denoted as

$$\frac{d^n}{dx^n}\frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}f(t)dt, \text{ where } n = \lceil\alpha\rceil$$

Kilbas et al. (2006). Michele Caputo (1967) defined a variation on this derivative, where instead of $\frac{d^n}{dx^n}$ in front of the integral, we have $\frac{d^n}{dt^n}$ inside the integral. Due to this adjustment, it is possible to have initial value conditions expressed as the traditional derivatives of integer-order, which made these fractional differential equation problems more intuitive. Other fractional derivatives, such as the Hadamard (1892) and Riesz (1949) derivative, have been defined to take advantage of particular beneficial properties. For example, each of the latter derivatives can be written as a Fourier transformation. As a consequence, the analytical expressions, can often be simplified. A rather unique derivative is the Grünwald-Letnikov derivative which, in contrast to all the aforementioned derivatives, is not based on integral. Instead, it generalizes the difference quotient, $\frac{f(x+h)-f(x)}{h}$, to fractional orders using binomial coefficients Atici et al. (2021). This variety of definitions emphasises how dependent fractional derivatives are on different physical interpretations and practical applications. Beyond their theoretical significance, fractional derivatives have been of significant importance in various scientific fields since the 19th century. Examples include fractional Fourier transformations, a generalization of the regular Fourier transformations Missbauer (2012), fractional diffusion equation models, describing the motion of particles in liquids as a consequence of thermal molecular motions Einstein (1905) and the fractional Schrödinger equation, a generalization of the Schrödinger equation, often used in quantum mechanics Laskin (2002). Their applications are less common in the fields of finance or economics, as fractional derivatives are mainly used to describe natural phenomena Boulaaras et al. (2023). Yet they still offer some great potential. (Symmetric) Levy flights make use of fractional derivatives in order to solve partial differential equations which describe random walk processes in time series Scalas et al. (2000). The development of fractional derivatives also led to the notion of fractional Brownian motions, a generalization of the Brownian motion Mandelbrot and Van Ness (1968). The latter is a continuous-time stochastic process which, similar to Levy flights, may be used to model random walk processes.

3 Fractional Calculus

3.1 Fractional derivatives

In order to obtain the expression as mentioned in section section 1, some advanced tools are required. We can find this in the field of fractional calculus. We define the following:

Definition 3.1.1. Let D be the differential operator, such that $Df(x) = \frac{d}{dx}f(x)$. Then the fractional derivative of order α is defined as $D^{\alpha}f(x) = \frac{d^{\alpha}}{dx^{\alpha}}f(x)$.

In this definition, α can be any real number. When taking regular derivatives, $\alpha \in \mathbb{N}$. In our case, we are interested in instances where $\alpha \in \mathbb{Q}, \alpha \geq 0$.

It is also possible to study derivatives of negative order, which can be used to obtain moments of negative order of a function. A derivative of negative order is simply an integral of positive order. This is defined as follows:

Definition 3.1.2. Let I be the integral operator, such that $If(x) = \int f(x)dx$. Then the fractional integral of order α is defined as $(I^{\alpha}f)(x) = \frac{1}{(n-1)!}\int (x-t)^{n-1}f(t)dt$.

Combining the previous two definition, we can obtain the following definition.

Definition 3.1.3. The differintegral operator is defined as

$$R^{\alpha}f(x) = \begin{cases} I^{|\alpha|}f(x) & \text{if } \alpha < 0 \\ D^{\alpha}f(x) & \text{if } \alpha > 0 \\ f(x) & \text{if } \alpha = 0 \end{cases} \tag{1}$$

A lot of different definition have been used to compute a fractional derivative. In this paper, we will focus on the following fractional derivatives:

Definition 3.1.4. The Riemann-Liouville fractional derivative of order α is defined as:

$$D^{\alpha}f(x) = \frac{d^n}{dx^n}D_x^{-(n-\alpha)}f(x) = \frac{d^n}{dx^n}I_x^{n-\alpha}f(x) = \frac{d^n}{dx^n}\frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}f(t)dt \tag{2}$$

Where $n = \lceil \alpha \rceil$, the ceiling function and $\Gamma(.)$ is the Gamma function, see section Appendix A.

Remark 3.1.1. For values $\alpha \in \mathbb{N}, n = \alpha$, so $\Gamma(n-\alpha) = \Gamma(0)$, which is undefined. Thus for $\alpha \in \mathbb{N}$, including the value 0, we simply define: $D^{\alpha}f(x) = \frac{d^{\alpha}}{dx^{\alpha}}f(x)$

A modification of the Riemann-Liouville derivative is the Caputo-Fabrizio derivative, which is defined as follows:

Definition 3.1.5. The Caputo-Fabrizio fractional derivative of order $\alpha, \alpha \in [0,1)$ is defined as:

$$D^{\alpha}f(x) = \frac{1}{1-\alpha}\int_0^x \exp(\frac{-\alpha}{1-\alpha}(x-t))f'(t)dt \tag{3}$$

Lastly, we will define the Grünwald-Letnikov derivative, which is defined as follows:

Definition 3.1.6. The Grünwald-Letnikov fractional derivative of order α is defined as:

$$D^{\alpha}f(x) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}f(x-kh) \tag{4}$$

Where $\binom{\alpha}{k}$ is the binomial coefficient, see section Appendix A.

Proposition 3.1.1. *The fractional derivatives above adhere to the following properties:*

(i) Linearity: Let $f(x), g(x)$ be functions and $a, b, x \in \mathbb{R}$. Then we have that $D^\alpha(af(x) + bg(x)) = aD^\alpha f(x) + bD^\alpha g(x)$.

(ii) $D^\alpha f(x)$ for $\alpha = 0, = f(x)$

(iii) for sufficiently smooth functions f, we have that $D^{\alpha+\beta} f(x) = D^\alpha(D^\beta f(x)) = D^\beta(D^\alpha f(x))$, with $\alpha, \beta \in \mathbb{R}$. Note that, for 3.1.5, this only holds for $\beta \in \mathbb{N}, \alpha \in [0, 1)$.

Proof. (i) We will proof for the Riemann-Liouville derivative, the proof for the Caputo-Fabrizio derivative is very similar and the Grünwald-Letnikov derivative is a direct consequence of the linearity of the sum.

$$D^\alpha(af(x) + bg(x)) = \frac{d^n}{dx^n} \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1}(af(t) + bg(t))dt$$

$$= \frac{d^n}{dx^n}\left(\frac{a}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f(t)dt + \frac{b}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} g(t)dt\right)$$

Where we simply split the integral and put the constants in front.

$$= \frac{d^n}{dx^n} \frac{a}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f(t)dt + \frac{d^n}{dx^n} \frac{b}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} g(t)dt$$

As the regular derivative operator is just linear.

$$= aD^\alpha f(x) + bD^\alpha g(x)$$

(ii) Intuitively, this makes perfect sense, as the 0-th derivative is just no derivative, so just the function $f(x)$. But for these derivatives, a little bit more effort is needed to prove this rather obvious fact.
For the Grünwald-Letnikov derivative we get:

$$D^0 f(x) = \lim_{h\to 0} \frac{1}{h^0} \sum_{k=0}^{\infty} (-1)^k \binom{0}{k} f(x-kh) = \lim_{h\to 0} \frac{1}{1} \sum_{k=0}^{\infty} (-1)^k \frac{0!}{k!(0-k)!} f(x-kh).$$

The factorial Identity of the binomial coefficient only holds for $0 \leq k \leq \alpha$. Since $\alpha = 0$ and k is always a positive integer lesser or equal to $\alpha, k = 0$. Thus, we get:

$$= \lim_{h\to 0} \sum_{k=0}^{\infty} (-1)^0 \frac{0!}{0!(0-0)!} f(x-0h) = \lim_{h\to 0} f(x-0h) = f(x).$$

For the Caputo-Fabrizio derivative, we get the following:

$$D^0 f(x) = \frac{1}{1-0} \int_0^x \exp\left(\frac{0}{1-0}(x-t)\right) f'(t)dt = \int_0^x f'(t)dt = f(x).$$

Finally, for the Riemann-Liouville derivative, we can simply make use of 3.1.3 and 3.1.1 to note that in this context $\alpha = 0$ is included in the natural integers. So $D^\alpha = \frac{d^\alpha}{dx^\alpha} f(x) = d^0 dx^0 f(x) = f(x)$.

(iii) The proof for the Riemann-Liouville derivative is given by Koning (2015). And the proof for the Caputo-Fabrizio derivative is given by Losada and Nieto (2015).For the Grünwald-Letnikov derivative, we get:

$$D^{\alpha}(D^{\beta}f(x)) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}(\frac{1}{h^{\beta}}\sum_{l=0}^{\infty}(-1)^l\binom{\beta}{l}f(x-lh-kh))$$

$$= \lim_{h\to 0}\frac{1}{h^{\alpha+\beta}}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}\sum_{l=0}^{\infty}(-1)^l\binom{\beta}{l}f(x-(k+l)h).$$

We substitute $m = k + l$ to deal with the dubble sums:

$$\lim_{h\to 0}\frac{1}{h^{\alpha+\beta}}\sum_{m=0}^{\infty}f(x-mh)\sum_{k=0}^{m}(-1)^k(-1)^{m-k}\binom{\alpha}{k}\binom{\beta}{m-k}$$

Now we make use of an identify from Appendix A to obtain:

$$= \lim_{h\to 0}\frac{1}{h^{\alpha+\beta}}\sum_{m=0}^{\infty}(-1)^m\binom{\alpha+\beta}{m}f(x-mh) = D^{\alpha+\beta}f(x).$$

It can be shown in an exactly similar way that the latter expression is equal to $D^{\beta}(D^{\alpha}f(x))$. □

We will now provide a few numerical examples of these fractional derivatives. For simplicity, we will let $a = 0$:

Example 3.1.1. (i)

$$D^{\frac{3}{2}}_{RL}(c) = \frac{d^2}{dx^2}\frac{1}{\Gamma(2-\frac{3}{2})}\int_0^x (x-t)^{2-\frac{3}{2}-1}c dt$$

$$= \frac{d^2}{dx^2}\frac{c}{\sqrt{\pi}}\int_0^x (x-t)^{-\frac{1}{2}}dt = \frac{d^2}{dx^2}\frac{-2c}{\sqrt{\pi}}\sqrt{x-t}\Big|_0^x$$

$$= \frac{d^2}{dx^2}\frac{2c\sqrt{x}}{\sqrt{\pi}} = \frac{-c}{2\pi(x)^{\frac{3}{2}}} \neq 0.$$

As stated, the fractional of a constant is not equal to zero when using the Riemann-Liouville derivative, this is also the case for the Grünwald-Letnikov derivative, but not for the Caputo-Fabrizio derivative.

(ii)

$$D^{\frac{1}{2}}_{CF}(\frac{x}{2}) = \frac{1}{1-\frac{1}{2}}\int_0^x \exp(\frac{-\frac{1}{2}}{1-\frac{1}{2}}(x-t))\frac{1}{2}dt = \int_0^x \exp(t-x)dt$$

$$= \exp(t-x)\Big|_0^x = 1-\exp(-x).$$

For $x \geq 0$, this expression is equal to the CDF of the exponential distribution with $\lambda = 1$. A remarkble result.

(iii) We will compute another famous result with the Riemann-Liouville derivative:

$$D^{\frac{1}{2}}_{RL}(x) = \frac{d}{dx}\frac{1}{\Gamma(1-\frac{1}{2})}\int_0^x (x-t)^{1-\frac{1}{2}-1}t dt$$

Let $u = x - t$, such that $\frac{du}{dt} = -1, dt = -du$:

$$= \frac{d}{dx}\frac{1}{\sqrt{\pi}}\int_0^x (u)^{-\frac{1}{2}}(u-x)du = \frac{d}{dx}\frac{1}{\sqrt{\pi}}(\int_0^x \sqrt{u}du - x\int_0^x \frac{1}{\sqrt{u}}du)$$

$$= \frac{d}{dx}\frac{1}{\sqrt{\pi}}(\frac{2u^{\frac{3}{2}}}{3} - 2xu^{\frac{1}{2}}\Big|_0^x) = \frac{d}{dx}\frac{1}{\sqrt{\pi}}(\frac{2(x-t)^{\frac{3}{2}}}{3} - 2x(x-t)^{\frac{1}{2}}\Big|_0^x)$$

$$= \frac{d}{dx}\frac{-1}{\sqrt{\pi}}(\frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{3}{2}} = \frac{d}{dx}\frac{\frac{4}{3}x^{\frac{2}{3}}}{\sqrt{\pi}})$$

$$= \frac{2\sqrt{x}}{\sqrt{\pi}}.$$

3.2 Complex derivatives

WIP

4 Moments

Now that we have defined the required techniques to compute fractional derivatives, we can apply them to the Moment Generating Function. First, however, we need to define some concepts relevant to computing moments.

4.1 Moments

Definition 4.1.1. The n-th moment of a PDF $f_X(x)$ is defined as:

$$E[X^n] = \begin{cases} \int_{-\infty}^{\infty} (x-c)^n f_X(x)\, dx & \text{if } f_X(x) \text{ is continuous,} \\ \sum_i (x_i - c_i)^n f_X(x_i) & \text{if } f_X(x) \text{ is discrete.} \end{cases}$$

If $c = \mu_x$, the first moment of $f(x)$, then our higher moments are called central moments. In our situation, we will focus on the case $c = 0$, which is called a raw moment. We do this as the Moment Generating Function, which we will soon define, only computes raw moments of higher order. A moment of order $n$is said to exist, if $E[X^n] < \infty$.

Proposition 4.1.1. *We can derive the following simple properties:*

(i) If $E[X^n]$ does not exist, then $E[X^k]$ also does not exist, for $k \geq n$.

(ii) If $E[X^k]$ does exist, then all its lower moments, i.e. $E[X^n]$, for $n \leq k$. Also exist.

Proof. (i) If $f(x)$ is a discrete distribution, we get:

$$E[X^n] = \sum_i x_i^n f(x_i) = \infty, E[X^k] = \sum_i x_i^k f(x_i)$$

$$= \sum_i x_i^n \cdot x^{k-n} f(x_i) \geq \sum_i x_i^n f(x_i) = \infty, \text{ as } k \geq n$$

We can prove this for continuous functions in a similar way.

(ii) This simply follows from the previous proposition, as the current proposition is just the contrapositive statement of the previous proposition.

□

Example 4.1.1. *Let $f_x(x) \sim C(0,1)$ where $C(x_0, y_0)$*
is the cauchy distribution funtion , does the second moment exist?
We use 4.1.1 and begin with verifying that the first moment exists:

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)} dx.$$

For large values of x, the inside is equal to $\frac{1}{x}$, splitting the integral:

$$= \int_1^{\infty} \frac{1}{x} dx - \int_{-\infty}^{1} \frac{1}{x} dx$$

Where both integrals diverge logarithmically.

4.2 Moments of negative order

We take a quick look at (raw) moments of negative order, and if we can define these in the same way as above, let us look at the continuous case first:

$$E[X^-n] = \int_{-\infty}^{\infty} x^{-n} f_x(x)dx = \int_{-\infty}^{\infty} (\frac{1}{x})^n f_x(x)dx.$$

We can immediately observe a rather obvious problem. This integral is not defined at $x = 0$ and diverges for values of x in a neighbourhood of 0. Khuri and Casella (2002) have stated the following corollary for the existence of the first negative moment:

Corollary 4.2.1. *If $f(x)$ is a continious pdf defined on $(-\infty, \infty)$, and if*

$$\lim_{x\to 0} \frac{f(x)}{|x|^\alpha} < \infty$$

, for $\alpha > 0$, then

$$E[X^{-1}]\textit{exists}.$$

Not a lot of common distribution functions adhere to this corollary, however, the Gamma function does:

Example 4.2.1. Let $f_X(x) \sim \Gamma(\alpha, \lambda) =$

$$\frac{x^{\alpha-1}e^{-\lambda x}\lambda^\alpha}{\Gamma(\alpha)}.$$

This PDF is defined on $(0, \infty)$. So the function is not defined on $\mathbb{R}$. This, however, is not a problem, as we can just evaluate the right limit. Since the Gamma function already uses α as a parameter, we will look at $\frac{f(x)}{x^\beta}, \beta > 0$:

$$\lim_{x\to 0_+} \frac{f(x)}{x^\beta} = \lim_{x\to 0_+} \frac{x^{\alpha-1-\beta}e^{-\lambda x}\lambda^\alpha}{\Gamma(\alpha)} = \lim_{x\to 0_+} \frac{x^{\alpha-(1+\beta)}e^{-\lambda x}\lambda^\alpha}{\Gamma(\alpha)}.$$

So for $\alpha \geq \beta + 1, \lim_{x\to 0_+} \frac{f(x)}{x^\beta} < \infty$. So the first negative moment of the Gamma distribution should exist.

We will compute the first negative moment:

$$E[X^{-1}] = \int_0^\infty x^{-1} \frac{x^{\alpha-1}e^{-\lambda x}\lambda^\alpha}{\Gamma(\alpha)} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-2}e^{-\lambda x}dx$$

Using the substitution $u = \lambda x, \frac{du}{dx} = \lambda, dx = \frac{du}{\lambda}$, we get:

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty (\frac{u}{\lambda})^{\alpha-2} e^{-u} du = \frac{\lambda^\alpha}{\Gamma(\alpha)\lambda^{\alpha-1}} \int_0^\infty (\frac{u}{\lambda})^{\alpha-2} e^{-u} du.$$

This integral is equal to $\Gamma(\alpha - 1)$ (See A). So we get:

$$E[X^{-1}] = \frac{\lambda^\alpha \Gamma(\alpha-1)}{\Gamma(\alpha)\lambda^{\alpha-1}} = \frac{\lambda\Gamma(\alpha-1)}{(\alpha-1)\Gamma(\alpha-1)} = \frac{\lambda}{(\alpha-1)}.$$

So, for $\alpha \neq 1, E[X^{-1}] = \frac{\lambda}{(\alpha-1)}$. Fortunately, this is always the case, since we had just derived that the integral is only finite when $\alpha \geq \beta + 1,$ with $\beta > 0$. In other words, $\alpha > 1$. So this holds.

4.3 The Moment Generating Function

We will finally define the Moment Generating Function, one of the most significant subjects of this thesis.

Definition 4.3.1. The Moment Generating Function (MGF) of a variable X, is defined as

$$M_X(t) = E[e^{tX}]$$

provided that $E[e^t X] < \infty$ on some interval $(-h, h)$, which contains 0, for some $h > 0$.

Remark 4.3.1. Obtaining the expression $M_X(t) = E[e^{tX}]$ is pretty trivial. Generally, it just requires quite a few steps of analytic evalution. This is not that interesting nor relevant to this research. We will give one explicit example on how to compute the Moment Generating Function for a specific distribution. And for later cases, when we make use of an expression of the Moment Generating Function, we will simply refer to figure 3.1.5.

Example 4.3.1. Let $f_X(x) \sim \Gamma(\alpha, \lambda)$ be the Gamma distribution with PDF:

$$f_X(x) = \frac{x^{\alpha-1}e^{-\lambda x}\lambda^\alpha}{\Gamma(\alpha)}.$$

Let $t < \lambda$ (in any other case, the integral diverges).The moment generating function $M_X(t)$ is given by:

$$M_X(t) = \int_0^\infty e^{tx} f_X(x)\, dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{(t-\lambda)x}\, dx.$$

We make use of the substitution $u = -(t-\lambda)x$, $\frac{du}{dx} = (\lambda - t)$, $dx = \frac{du}{(\lambda-t)}$, so:

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{u}{(\lambda-t)}^{\alpha-1} e^{-u} \frac{du}{\lambda - t} = \frac{\lambda^\alpha}{\Gamma(\alpha)(\lambda-t)^\alpha} \int_0^\infty u^{\alpha-1} e^{-u}\, du.$$

This integral is just the definition of the Gamma Function, $\Gamma(\alpha)$, so we obtain:

$$= \frac{\lambda^\alpha \Gamma(\alpha)}{\Gamma(\alpha)(\lambda-t)^\alpha} = (\frac{\lambda}{\lambda-t})^\alpha = M_X(t)$$

We will state the theorem which makes the MGF so useful.

Theorem 4.3.1. *If $M_X(t)$ exists on some interval $-h, h$, as defined before, we have that:*

$$E[X^n] = M_X^{(n)}(0), \textit{ for } n \in \mathbb{N}$$

Proof. The proof is fairly straightforward:

$$M_X^{(n)}(t) = \frac{d^n}{dt^n} \int_{-\infty}^\infty e^{tx} f_X(x) dx = \int_{-\infty}^\infty \frac{d^n}{dt^n} e^{tx} f_X(x) dx$$

(We can interchange differentiation and integration since all partial derivatives of $e^{tx} f(x)$ are continuous and the absolute value of the integral converges, as we assume the n-th moment exists, see A).

$$= \int_{-\infty}^\infty x^n e^{tx} f_X(x) dx, \text{evaluate at } t = 0 := \int_{-\infty}^\infty x^n e^{0x} f_X(x) dx$$

$$= \int_{-\infty}^\infty x^n f_X(x) dx = E[x^n]$$

The proof for the case that $f_X(x)$ is discrete is very similar. In that case, one would have to change the order of the derivative and summation, which has also been justified in A. □

Remark 4.3.2. Since the validity of interchanging the order of differentiation and integration is based on characteristics of the integral and its inside, we can extend Theorem 4.3.1 to real values α.

We introduce the following properties for the Moment Generating Function $M_X(t)$:

Proposition 4.3.1. *For X, Y random variables, we have that:*

(i) $M_X^{(0)}(t) = E[e^{0X}] = 1$. *This property can be used to check if a given is really a PDF.*

(ii) Location scale-transform. Assuming $M_X(t)$ exists, for constant μ, σ, we have that:

$$M_{\mu+\sigma X}(t) = e^{\mu t} \cdot M_X(\sigma t)$$

(iii) If $X \perp Y$, then $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$.

Proof. (i) This is trivial. For continuous functions of x, we get:

$$M_X^{(0)}(t) = \int_{-\infty}^{\infty} x^0 e^{tx} f_X(x)dx = \int_{-\infty}^{\infty} 1e^{0x} f_X(x)dx = \int_{-\infty}^{\infty} f_X(x)dx.$$

Assuming that $f(x)$ is a PDF, this integrates to 1 by definition. If this integral is not equal to 1, this implies that $f(x)$ is not a PDF. The proof for the discrete case is the exact but with a summation instead of an integral sign.

(ii)

$$M_{\mu+\sigma X}(t) = E[e^{(\mu+\sigma x)t}] = E[e^{\mu t} \cdot e^{\sigma x t}].$$

Since this is the expectation of x, every term that is not dependent on x can be taken out of the summation:

$$= e^{\mu t} \cdot E[e^{\sigma x t}] = e^{\mu t} \cdot M_x(\sigma t)$$

(iii)

$$M_{X+Y}(t) = E[e^{(x+y)t}] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{(x+y)t} f_{X,Y}(x,y)dxdy$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{xt} \cdot e^{yt} f_{X,Y}(x,y)dxdy.$$

$f_{X,Y}(x,y)$ is the joint pdf for X, Y. But we know that the latter is equal to $f_X(x) \cdot f_Y(y)$, if X, Y are independent. Thus we get:

$$= (\int_{-\infty}^{\infty} e^{xt} f_X(x)dx)(\int_{-\infty}^{\infty} e^{yt} f_Y(y)dy) = M_X(t) \cdot M_Y(t).$$

□

To illustrate the usage of fractional derivatives with the MGF, we will give the following numerical examples. We let $f(x) = \lambda \cdot e^{-\lambda x}$ on $x \in [0, \infty)$. We will calculate its semi-moment using the Riemann-Liouville derivative, the Caputo-Fabrizio derivative and just the regular integral to compute moments.

Example 4.3.2. (i)

4.3.1 Computing negative moments using the Moment Generating Function

We have seen that under specific conditions, it is possible to compute negative, or so called inverse moments of distribution functions. This technique can be extended to the moment generating function. In the 20-th century, Cressie et al. (1981) have published the following remarkable theorem:

Theorem 4.3.2. *Assuming the negative n-th raw moment exists, the negative n-th raw moment can be computed as follows:*

$$E[X^{-n}] = \frac{1}{\Gamma(n)} \int_0^{\infty} t^{n-1} M_X(-t) dt$$

, where n is a positive integer.

The proof of this Theorem can be found in B.

Let us again compute the first inverse moment of the Gamma distribution, but now by making use of the latter theorem for Moment Generating Functions!

Example 4.3.3. Let $f_x(x) \sim \Gamma(\alpha, \lambda) =$

$$\frac{x^{\alpha-1} e^{-\lambda x} \lambda^{\alpha}}{\Gamma(\alpha)}, M_X(t) = (\frac{\lambda}{\lambda - t})^{\alpha}$$

$$E[X^{-1}] = \frac{1}{\Gamma(1)} \int_0^{\infty} t^{1-1} (\frac{\lambda}{\lambda - (-t)})^{\alpha} dt = \int_0^{\infty} (\frac{\lambda}{\lambda + t})^{\alpha} dt$$

$$= \lambda^{\alpha} \int_0^{\infty} (\lambda + t)^{-\alpha} dt, \text{ Let } u = \lambda + t, \frac{du}{dt} = 1, dt = du : \lambda^{\alpha} \int_0^{\infty} u^{-\alpha} du$$

$$= \lambda^{\alpha} \frac{u^{1-\alpha}}{1-\alpha}\Big|_0^{\infty} = \lambda^{\alpha} \frac{(\lambda + t)^{1-\alpha}}{1-\alpha}\Big|_0^{\infty}$$

$$= \lambda^{\alpha}(0 - \frac{\lambda^{1-\alpha}}{1-\alpha}) = \frac{-\lambda}{1-\alpha} = \frac{\lambda}{\alpha - 1}.$$

Which corresponds with our result from Example 4.2.1.

4.3.2 Laplace Transformation

WIP

4.3.3 Characteristic function and Fourier Transformations

WIP

5 Conclusion

The conclusion summarizes the results of your research. Also it gives possible directions of further research.

Acknowledgements

Place the acknowledgments section, if needed, after the main text, but before any appendices and the references. The section heading is not numbered.

A (Appendix): Relevant functions and Identities

We define the (Euler-)Gamma function as follows:

Definition A.0.1. for $\Re(z) > 0$, we have the following: $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$

The Gamma function can be seen as an extension of the factorial function, for non-integers. This function is defined for complex numbers and all there subsets (so also real numbers), as long as the condition above holds. For positive integers values z, we have the following identity: $\Gamma(z) = (z-1)!$ Other important identities, not necessarily for z an integer, are:

- $\Gamma(z+1) = z\Gamma(z)$
- $\Gamma(2) = \Gamma(1) = 1$
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Definition A.0.2. The falling factorial is defined as follows: $(x)_n = \prod_{k=0}^{n-1}(x-k)$, which is a polynomial

Definition A.0.3. For $0 \leq k \leq n$, the Binomial Coefficient is defined as follows: $\binom{n}{k}$, where $n, k \in \mathbb{N}$.

We can derive the following factorial identity, which is convenient to work with analytically: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. For numerically computing expressions containing the Binomial Coefficient, the following identity is computationally more efficient: $\binom{n}{k} = \frac{(n)_k}{k!}$. With $(n)_k$ as in A.0.2. Since we have established in A.0.1 that $\Gamma(z) = (z-1)!$, we can rewrite our factorial identify to:

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1) \cdot \Gamma(n-k+1)} = \frac{n}{k} \frac{\Gamma(n)}{\Gamma(k) \cdot \Gamma(n-k+1)}$$

Definition A.0.4. Vandermonde's identity: for non-negative integers, k, l, m, n, we have that

$$\sum_{k=0}^{l} \binom{m}{k} \cdot \binom{n}{l-k} = \binom{m+n}{l}$$

.

A modification on the latter identity has been called the Chu-Vandermonde identity. This is the same identity, but it his been proven that the identities still hold for complex values m, n as long as l is a positive integer (Askey (1975)).

For the interchange-ability of derivatives and integrals and sums, we can apply the following two theorems:

Theorem A.0.1. *Leibnitz's Rule: Let $f(x,\theta), a(\theta), b(\theta)$ be differentiable with respect to θ, then we have that:*

$$\frac{d}{d\theta} \int_{a(\theta)}^{b(\theta)} f(x,\theta)dx = f(b(\theta),\theta)\frac{db(\theta)}{d\theta} - f(a(\theta),\theta)\frac{da(\theta)}{d\theta} + \int_{a(\theta)}^{b(\theta)} \frac{\partial f(x,\theta)}{\partial\theta}dx.$$

For the special case, where $a(\theta), b(\theta)$ are constant we have that:

$$\frac{d}{d\theta} \int_{a}^{b} f(x,\theta)dx = \int_{a}^{b} \frac{\partial f(x,\theta)}{\partial d\theta}.$$

For the interchange-ability of derivatives and summations, the following theorem has been given by Casella and Berger (2002):

Theorem A.0.2. *Suppose that the series $\sum_{x=0}^{\infty} h(\theta, x)$ converges for all θ in an interval (a, b) of real numbers and*

(i) $\frac{\partial h(\theta,x)}{\partial \theta}$ *is continuous for all* x

(ii) $\sum_{x=0}^{\infty} \frac{\partial h(\theta,x)}{\partial \theta}$ *converges uniformly on every closed bounded subinterval of* (a, b)

Then:

$$\frac{d}{d\theta}\left(\sum_{x=0}^{\infty} h(\theta, x)\right) = \sum_{x=0}^{\infty} \frac{\partial h(\theta, x)}{\partial \theta}$$

B (Appendix): Proofs

Proof. Theorem 4.3.2: Suppose for the moment that X is a positive random variable. Since $xf(x)$ is integrable for $x > 0$, we have:

$$E(X) = \int_0^\infty x\,dF(x) = \int_0^\infty \int_0^\infty e^{tx}\,dt\,dF(x).$$

We can interchange the order of integration as follows:

$$E(X) = \int_0^\infty e^{tx}\,dF(x)\,dt = \int_0^\infty M_X(-t)\,dt.$$

The interchange of the order of integration is subject to $E(e^{-tX})$ being integrable from $t = 0$ to $t = \infty$.

Finally, by substituting X^{-1} for X, we find:

$$E(X^{-1}) = \int_0^\infty M_X^{-1}(-t)\,dt.$$

There are two natural ways to generalize (1) to $E(X^{-1})$; one way gives:

$$E(X^{-n}) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty M_X(-t_n)\,dt_n \cdots dt_2 dt_1, \tag{2}$$

while the second way gives:

$$E(X^{-n}) = \frac{1}{\Gamma(n)} \int_0^\infty t^{n-1} M_X(-t)\,dt.$$

Cressie et al. (1981) □

C Appendix C

TODO: ADD TABLE OF COMMON DISTRIBUTIONS + SUPPORT, MGF, ETC.

D Appendix

Place any appendices after the acknowledgments, starting on a new page. The appendices are numbered **A**, **B**, **C**, and so forth. The appendices contain material that you would like to share with the reader but that would hinder the flow of reading. For instance long proofs of theorems, code of algorithms, data, etc.

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