CSCI 338: Assignment 1 (7 points)

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Problem 1

Prove that $1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$.

Proof. Let $f(n)$ represent, as described above, the summation of the series of natural numbers to the power of 4. Denoted as follows:

$$\begin{aligned} f(n) &= \sum_{i=1}^{n} i^4 \\ &= 1^4 + 2^4 + 3^4 + \cdots + n^4 \\ &= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \end{aligned}$$

Given $f(n)$, when $n = 1$,

$$f(1) = \sum_{i=1}^{1} i^4 = 1^4 = 1,$$

$$\frac{1(1+1)(2(1)+1)(3(1)^2+3(1)-1)}{30} = \frac{1(2)(3)(5)}{30} = \frac{30}{30} = 1$$

So, when $n = 1$, it is true such that $f(n) = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$. Now, let us assume that this is also true for $n \leq k$. Then, by definition,

$$
\begin{aligned}
f(k+1) &= \underline{1^4+2^4+3^4+\cdots+k^4}+(k+1)^4 \\
&= \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}+(k+1)^4 \\
&= \frac{(k+1)}{30}[k(2k+1)(3k^2+3k-1)+30(k+1)^3] \\
&= \frac{(k+1)}{30}[(2k^2+k)(3k^2+3k-1)+30(k^3+3k^2+3k+1)] \\
&= \frac{(k+1)}{30}[6k^4+39k^3+91k^2+89k+30] \\
&= \frac{1}{30}[6k^5+45k^4+130k^3+180k^2+119k+1] \\
&= \frac{1}{30}[(2k^3+9k^2+13k+6)(3k^2+9k+5)] \\
&= \frac{1}{30}[(k^2+3k+2)(2k+3)(3k^2+9k+5)] \\
&= \frac{1}{30}[(k^2+3k+2)(2k+3)(3k^2+9k+5)] \\
&= \frac{1}{30}[(k+1)(k+2)(2k+3)(3k^2+6k+3+3k+3-1)] \\
&= \frac{1}{30}[(k+1)((k+1)+1)(2(k+1)+1)(3(k+1)^2+3(k+1)-1)] \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)(3(k+1)^2+3(k+1)-1)}{30}
\end{aligned}
$$

By substituting n back in for $k+1$, we will get back $f(n)$;

$$
\begin{aligned}
f(k+1) &= \frac{(k+1)((k+1)+1)(2(k+1)+1)(3(k+1)^2+3(k+1)-1)}{30} \\
&= \frac{1}{30}[(\underline{k+1})((\underline{k+1})+1)(2(\underline{k+1})+1)(3(\underline{k+1})^2+3(\underline{k+1})-1)] \\
&= \frac{1}{30}[(\underline{n})((\underline{n})+1)(2(\underline{n})+1)(3(\underline{n})^2+3(\underline{n})-1)] \\
&= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \\
&= f(n)
\end{aligned}
$$

$\therefore f(n) = \sum_{i=1}^{n} i^4 = 1^4+2^4+3^4+\cdots+n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

$\square$

Problem 2

Given a planar graph $P = (V, E)$, we have Euler's formula: $|V| + |F| - |E| = 2$, where F is the set of faces of P and E is the set of edges of P. Let $|V| = n$, where V is the set of vertices of P. Prove that $|F|$ is at most $2n$.

Proof. For a planar graph P with v vertices and f faces, it must be true that $|F|$ is at most $2n$.

If P is a forest or a tree, then there exists only one face. Such that,

$$|F| = 1 \leq 2$$

Considering all other planar graphs that are NOT either a forest or a tree, from the perspective of each individual face, the sum of the number of edges in P will be a total of $2|E|$. Each face in P is also required to have at least 3 edges, denoted as $3|F|$. Since each face must have at least 3 edges and the total number of edges is at most 2 times the number of faces, this can be described as such,

$$3|F| \leq 2|E| = \frac{3}{2}|F| \leq |E|$$

Given Euler's formula,

$$\begin{aligned}
|V| + |F| - |E| = 2 &\iff n + |F| - |E| = 2 \\
&\iff n + |F| - \frac{3}{2}|F| \geq 2 \\
&\iff -\frac{1}{2}|F| \geq 2 - n \\
&\iff |F| \leq 2n - 4 \\
|V| + |F| - |E| = 2 &\iff |F| \leq 2n
\end{aligned}$$

$\therefore$ If Euler's formula is true, then it must be true that $|F| \leq 2$. $\square$

Problem 3

Prove that in any simple graph there is a path from any vertex of odd degree to some other vertex of odd degree.

Proof. A simple graph is a graph such that has no pair of vertices a, b has more than one single edge connecting a and b. Given a simple graph $G = (V, E)$, there are two possible scenarios; either G is connected or is not connected. Provided that G is a non-connected graph, it is the union of a connected graph. Thus, it is only necessary to provide evidence of proof for the connected case.

Given G, a simple graph that is connected, suppose there exists at least one vertex v such that $v \in V$ and $degree(v)$ is odd. Recall that in a connected graph, there is a path p from any vertex v, u such that $v \in V$ and $u \in V$, and that the total sum of the degree of G must be even.

$\therefore$ Since such a path p must exist between two vertices v, u in a connected graph G, if vertex v has an odd degree there must be another vertex of odd degree because the total sum of the degree of G must be even. $\square$

Problem 4

A fully binary tree T is a tree such that all internal nodes have two children. Prove that a fully binary tree with n internal nodes in total has $n + 1$ leaves.

Proof. Provided a binary tree T, such that T_n represents the number of nodes in the tree. Consider tree T_0, a binary tree with no internal nodes. Thus, $T_0 = 1$ and the base case holds true such as described as follows:

$$\begin{aligned} T_n &= 2n + 1 \\ T_0 &= 2(0) + 1 \\ &= 1 \end{aligned}$$

Now, let us assume that for some number of internal nodes k such that, $k \in \{0\} \cup Z^+$, and $T_n = 2n + 1$ for all $n \leq k$.

$$\begin{aligned} T_n &= 2n + 1 \\ 2\underline{n} + 1 &= 2(\underline{k + 1}) + 1 \\ &= 2k + 2 + 1 \\ &= 2(k + 1) + 1 \\ T_{k+1} &= 2k + 2 \end{aligned}$$

$\therefore$ Since T_{k+1} holds, it must be true that a fully binary tree T with n internal nodes has a total of $2n + 1$. $\square$

Problem 5

Given an undirected graph $G = (V, E)$, the breadth-first-search starting at $v \in V$ ($bfs(v)$ for short) is to generate a shortest path tree starting at vertex $v \in V$. The diameter of G is the longest of all shortest paths $\delta(u, v), u, v \in V$.

When G is a tree, the following algorithm is proposed to compute the diameter of G.

1. Run $bfs(w), w \in V$, and compute the vertex $x \in V$ furthest from w.
2. Run $bfs(x)$ and compute the vertex $y \in V$ furthest from x.
3. Return $\delta(x, y)$ as the diameter of G.

Prove that this algorithm is correct; i.e., $\delta(x, y)$ is in fact the longest among all the shortest paths between $u, v \in V$.

Proof. Suppose vertices a, b have the longest shortest path such that $bfs(b) = n$ and $n = a$. Assume that for some vertex m such that $m \in V$ and $bfs(m) = n$ and $n \neq a$. Given $\delta(m, a) \geq \delta(m, b)$ and $bfs(m) = n$ where $n \neq a$, then $\delta(m, n) \geq \delta(m, a)$ must be true. Recalling that $\delta(a, b)$ is the diameter, but a longer diameter is found by m and a. Thus, we have a contradiction.

$\therefore$ Since $bfs(a)$ must provide endpoint b on the longest shortest path, the algorithm above must compute the diameter. $\square$