Jan 15

Sequences and tuples

Cartesian product and/or cross product of A and B:
set of all pairs (x, y) s.t. $x \in A$ and $y \in B$.

Ex $A = \{a, b\}$. $B = \{1, 2, 3\}$

$$A \times B = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}.$$

Sam $\rightarrow$ Tom

$\langle \text{Sam}, \text{Tom} \rangle$

Relations and functions

A function (mapping) maps some inputs to outputs.

$$f : D \rightarrow R$$

D — Domain, set of all possible inputs
R — Range, set of all possible outputs
- If all elements in R could be used, the function f is said to be onto R.

Ex. abs, $|\cdot| : \mathbb{Z} \rightarrow \mathbb{Z}$ (is not onto $\mathbb{Z}$, as -5 is never used in the range)
$\text{abs}(-5) = 5$

add : $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ (is onto $\mathbb{Z}$).

square: $\mathbb{Z} \rightarrow \mathbb{Z}$ (is not onto $\mathbb{Z}$, as 3 is never used in the range).
$3^2 \rightarrow 9$, $4^2 \rightarrow 16$

A predicate / property is a function whose range is {True, False}.

Ex. Goldbach's conjecture:
For every even integer ≥ 4, it can be written as the summation of 2 prime numbers.

$24 = 11 + 13$, $50 = 3 + 47$

A property whose domain is a set of k-tuples $A_1 \times A_2 \times \cdots \times A_k$ is called a relation (k-ary relation).
In practice, $k = 2$ typically.

Equivalence relation:
A special relation R which is
① refexive, i.e., xRx for all x.
② symmetric, i.e., for every x and y, if xRy then yRx.
③ transitive, i.e., for every x, y, z, if xRy, yRz then xRz.

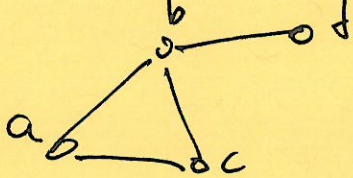
Ex $\equiv_5$, for $i, j \in N$, $i \equiv_5 j$ if $i-j$ is a multiple of 5.

① $i \equiv_5 i$, as $i - i = 0 = 5 \times 0$.
② $i \equiv_5 j$, then $i - j = 5 \times \Delta$; of course, $j - i = 5 * (-\Delta)$
$\therefore j \equiv_5 i$
③ $i \equiv_5 j$, $j \equiv_5 k \Rightarrow i \equiv_5 k$.

$i - j = 5 \times \Delta_1$
$j - k = 5 \times \Delta_2$
$i - k = 5 \times (\Delta_1 + \Delta_2)$

Graphs (cont.)

$G = (V, E)$



- **Subgraph**: A graph G is a subgraph of H if the nodes/edges of G are subsets that of H.
- Enumerating all subgraphs would be costly.
- Connectness: A **path** is a sequence of nodes connected by edges.

$\langle a, c, b, d \rangle$ ✓
$\langle a, c, d \rangle$ ✗

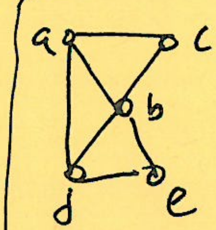
A **simple** path is a path that has no repeated nodes.

$\langle b, c, a, b, d \rangle$ is not simple

A **path** is a **cycle** if it starts and ends at the same node.

A Simple cycle is one that doesn't repeat any node (except at the end).

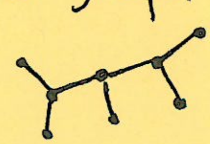
Ex: $\langle a, b, c \rangle$ or $\langle a, b, c, a \rangle$



$a \rightarrow b$

Hamiltonian cycle

A tree is a connected graph with no cycle.



Proofs:

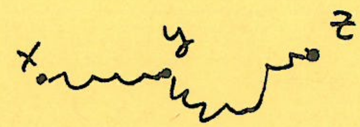
① Direct method.

Ex 3. Let xRy be the relation that there is a path between x and y in G. Then R is an equivalence relation.

proof: - xRx, obvious.

- If xRy, i.e., there is a path ~~between~~ from x to y, then there is a path from y to x,

$\therefore$ yRx.



- If xRy, yRz,

by def, ~~there~~ a path from x to z (through y), therefore xRz. $\square$

② By contradiction.

Ex 1. $\sqrt{2}$ is irrational (rational: $\frac{x}{y}$, $x, y \in N$).

$\sqrt{2} \approx 1.414..$

proof. Assume $\sqrt{2}$ is rational, i.e.,

$\sqrt{2} = \frac{m}{n}$, where m, n can only be divisible by 1, i.e.

$\frac{9}{6}$ would be $\frac{3}{2}$.

Take square func, we have

$2 = \frac{m^2}{n^2}$, or $2n^2 = m^2$, which implies m is even.

If $m = 2k$, $2n^2 = (2k)^2$, which is $n^2 = 2k^2$, this implies n is also even. Then m, n are divisible by 2, a contradiction to the assumption. $\square$

$\therefore$ $\sqrt{2}$ is irrational.