CSCI 338: Assignment 3 (7 points)

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Problem 1

Design context-free grammars for the following languages

(1.1) $A = \{a^n b^m | n \neq 2m\}$.

$S \to aaSb \mid A \mid B \mid ab$ miss "aab"
$A \to aA \mid a$
$B \to bB \mid b$

(1.2) $B = \{a^i b^j c^k | i, j, k \geq 0$ and either $i = j$ or $j = k\}$.

$S \to AC \mid BD$
$A \to aAb \mid \epsilon$
$C \to cC \mid \epsilon$
$B \to aB \mid \epsilon$
$D \to bDc \mid \epsilon$

(1.3) $C = \{a^n b^m | n = 3m\}$.

$S \to aaaSb \mid \epsilon$

(1.4) $D = \{a^n b^m | n \leq m + 3\}$.

$S \to X \mid aX \mid aaX \mid aaaX$
$X \to aXb \mid \epsilon$
$Y \to bY \mid b$

Problem 2

Decide whether the following grammar is ambiguous.

$S \to AB \mid aaB$
$A \to a \mid Aa$
$B \to b$

Proof. By definition, a string w is derived ***ambiguously*** in a context-free grammar if it has two or more different leftmost derivations. A context-free grammar is ***ambiguous*** if it generates some string ambiguously.

To show that the above grammar is ambiguous, we must show that there are two or more different leftmost derivations for some string.

Derivation 1	Derivation 2
S	S
AB	aaB
AaB	aab
aaB	
aab	

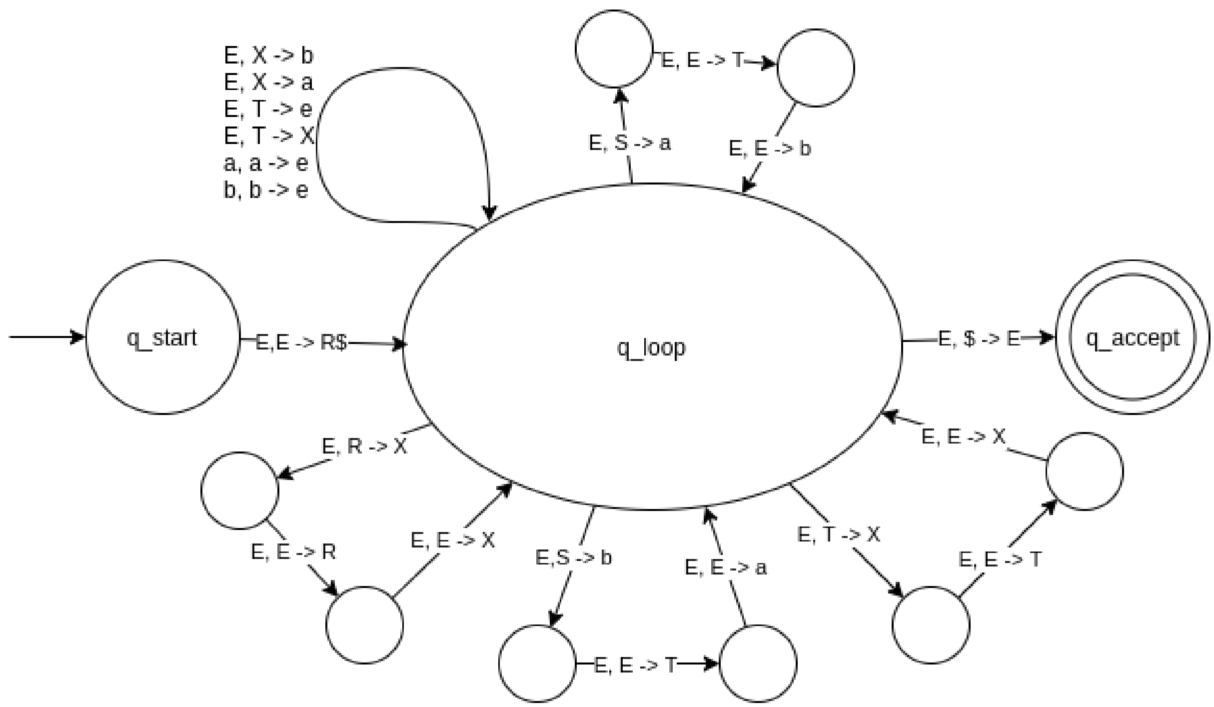
Each derivation above, using the leftmost rule, generates its own unique and distinct parse tree, each resulting in the same output.

$\therefore$ The given grammar must be ambiguous. $\square$

Problem 3

Convert the following CFG G to an equivalent PDA.

$R \to XRX|S$
$S \to aTb|bTa$
$T \to XTX|X|\epsilon$
$X \to a|b$



Problem 4

Let $G = (V, \Sigma, R, S)$ be the following grammar. $V = \{S, T, U\}$; $\Sigma = \{0, \#\}$; and R is the set of rules:

$S \to TT|U$
$T \to 0T|T0|\#$
$U \to 0U00|\#$

(**4.1**) Describe $L(G)$ in English.

The language of grammar G, or $L(G)$, may take the form of two possibilities. The two options are described as follows:

1. The first possibility is a string which contains 0's and #'s with at least two # symbols and any number of 0's on either side and or in between the # symbols.

2. The second possibility is a string beginning with n zeros, the a # symbol, and the $2n$ zeros.

(**4.2**) Prove that $L(G)$ is not regular.

Proof. Assume that $L(G)$ is regular. Since $L(G)$ is regular, we can apply the pumping lemma. Let p be the pumping length, and let s be a string such that $s \in L(G)$ where $|s| \geq p$. By definition of the pumping lemma, the following must be true:

1. $xy^iz \in A$ for $i \geq 0$

2. $|y| > 0$

3. $|xy| \leq p$

Now, we must choose $s = 0^p\#0^{2p} \mid s \in L(G)$.
By (2), $i = |y|$, and by (3) $|xy| \leq p$, so y must consist of only zeros.
We pump up $xy^2z = 0^{p+i}\#0^{2p} \notin L(G)$ because the increased zeros before the # symbol without changing those following it, so $2(p + i) \neq 2p$.
$\therefore$ A contradiction of the pumping lemma, so $L(G)$ must not be regular. $\square$

Problem 5

Convert the following CFG into an equivalent CFG in Chomsky Normal Form

$A \to BAB \mid B \mid \epsilon$
$B \to 00 \mid \epsilon$

Step 1 First, we add a new start variable S and the rule $S \to A$ where A was the original start variable.

$S \to A$
$A \to BAB \mid B \mid \epsilon$
$B \to 00 \mid \epsilon$

Step 2 Take care and remove all ϵ-rules.

2a Remove $B \to \epsilon$

$S \to A$
$A \to BAB \mid BA \mid AB \mid A \mid B \mid \epsilon$
$B \to 00$

2b Remove $A \to \epsilon$

$S \to A \mid \epsilon$
$A \to BAB \mid BA \mid AB \mid A \mid B \mid BB$
$B \to 00$

We do not need to remove the ϵ-rule $S \to \epsilon$ since S is the start variable.

Step 3 Handle all unit-rules.

3a Remove $A \to A$

$S \to A \mid \epsilon$
$A \to BAB \mid BA \mid AB \mid B \mid BB$
$B \to 00$

3b Remove $A \to B$

$S \to A \mid \epsilon$
$A \to BAB \mid BA \mid AB \mid 00 \mid BB$
$B \to 00$

3c Remove $S \to A$

$S \to BAB \mid BA \mid AB \mid 00 \mid BB \mid \epsilon$
$A \to BAB \mid BA \mid AB \mid 00 \mid BB$
$B \to 00$

3d Replace ill-placed terminals by new variable C

$S \to BAB \mid BA \mid AB \mid CC \mid BB \mid \epsilon$
$A \to BAB \mid BA \mid AB \mid CC \mid BB$
$B \to CC$
$C \to 0$

Step 4 Finally, convert all remaining rules into the proper form. (i.e. by replacing each rule $A \to u_1 u_2 \dots u_k$, where $k \geq 3$)

$S \to BD_1 \mid BA \mid AB \mid CC \mid BB \mid \epsilon$
$A \to BD_2 \mid BA \mid AB \mid CC \mid BB$
$B \to CC$
$C \to 0$
$D_1 \to AB$
$D_2 \to AB$

Problem 6

Using pumping lemma to prove that the following languages are not context-free.

(**6.1**) $L = \{a^n b^j c^k | k = nj\}$.

Proof. Assume L is regular, then choose $s = a^p b^p c^{p^2}$ where p is the pumping length. By the pumping lemma, s may be decomposed into $uvxyz$, such that:

(1) $uv^i xy^i z \in L$ for $i \geq 0$
(2) $|vy| > 0$
(3) $|vxy| \leq p$

There are three possible cases, let's examine each:

1. vxy is made up of b's and c's.

2. vxy does not have an c's.

3. vxy contains only c's.

In case 1, pumping uv^2xy^2z take the form $a^p b^{p+i} c^{p^2+j}$. Then, $p(p+i) = p^2+ip > p^2 + j$ since $i > 0$, and by (3) $j > p$. Therefore $uv^2xy^2z \notin L$.

In case 2, pumping uv^2xy^2z violates (2) because there would be more a's or b's without changing the number of c's.

Lastly, in case 3, when pumping uv^2xy^2z also violates (2) because there is more c's without changing the number of a's or b's.

$\therefore$ A contradiction in all cases, so L must not be regular. $\square$

(6.2) $L = \{a^n b^j | n \geq (j-1)^3\}$.

Proof. Assume L is regular, then choose $s = a^{(p-1)^3} b^p$ where p is the pumping length. By the pumping lemma, s may be decomposed into $uvxyz$, such that:

(1) $uv^i xy^i z \in L$ for $i \geq 0$
(2) $|vy| > 0$
(3) $|vxy| \leq p$

There are three possible cases, let's examine each:

1. vxy is made up of only a's.

2. vxy contains both a's and b's.

3. vxy contains only b's.

In case 1, by pumping down such that, $uv^0 xy^0 z = uxz$. By rule (2) $i = |vy| > 0$, so uxz has the form $a^{(p-1)^3 - i} b^p$. But, $(p-1)^3 - i < (p-1)^3$. Therefore $uxz \notin L$.

In case 2, pumping up $uv^2 xy^2 z$ takes the form $a^{(p-1)^3+i} b^{p+j}$, and we know that $j \neq 0$ so $(p-1+j)^3 \geq (p)^3$. Then, $(p-1)^3 + i < (p-1)^3 + 3p^2 - 3p + 1 = p^3$ for $p > 1$. Therefore $uxz \notin L$.

In case 3, pumping up $uv^2 xy^2 z$ takes the form $a^{(p-1)^3} b^{p+i}$. Thus, increasing the number of b's without altering the number of a's. By (2), $i = |vy| > 0$, then $(p-1)^3 < (p+i-1)^3$. Therefore $uxz \notin L$.

$\therefore$ A contradiction in all cases, so L must not be regular. $\square$