Sketch of solutions for Assignment #4

1. <u>proof</u>: Assume that B is countable, by def, there is a correspondence f between B and N. In other words, the elements in B can be listed as $f(1), f(2), \cdots$, which are listed in the table

i	$f(i)$
1	a b a b a b ⋯
2	b a b a b a ⋯
3	a a b a b b ⋯
⋮	

Now, construct an infinite binary sequence x such that its ith bit is the complement of the ith bit of $f(i)$, i.e., in the above example,

$$x = b\,b\,a \cdots$$

Clearly, $x \in B$, but $x \neq f(i)$ for any i.

$\therefore$ f is not a correspondence between N and B.

$\therefore$ B is uncountable. $\square$

2. <u>Answer</u>: (a) order $\langle i,j,k \rangle$ according to their sum $i+j+k$.

A tie is broken by the lexicographical order of $\langle i,j,k \rangle$'s.

or (b) Order $2^i 3^j 5^k$.

3. As PDA's and CFG's are equivalent, it suffices to study the problem

$$INFINITE_{CFG} = \{ \langle G \rangle \mid G \text{ is a CFG and } L(G) \text{ is infinite} \}$$

Construct a dependency graph for the variables in G such that there is an edge (A, B) whenever there is a rule (production) in G, e.g.,

$$A \to xBy.$$

Then, G has a repeating variable <u>if and only if</u> the dependency graph has <u>a cycle</u> (this can be found using depth-first-search, covered in 232).

Finally, L(G) is infinite if S can reach a repeating variable which could terminate (i.e., this variable is able to generate some strings composed of terminals).

// repeating variables are used in the proof of pumping lemma for CFL's.

4. <u>proof</u>: Constmct a TM M' on input x

- If $|x| =$ odd, then run M on w and <u>accept</u> x if M accepts w

// M accepts $w \Rightarrow M'$ accepts any x of odd length

M doesn't accept $w \Rightarrow M'$ accepts nothing.

Assume that R decides ODD_{TM}, constmct TM S for A_{TM}:

S: on $\langle M, w \rangle$

① Constmct M'.

② Run R on $\langle M' \rangle$.

③ If R accepts, <u>accept</u>; if R rejects, <u>reject</u>.

$\therefore$ S is a decider for A_{TM}, a contradiction.

$\therefore$ ODD_{TM} is undecidable. $\square$

5. Question 5.1.

Def. $EQ_{CFG} = \{\langle G, H\rangle \mid G, H$ are CFGs & $L(G) = L(H)\}$

We'll try to reduce ALL_{CFG} to EQ_{CFG}.

First, it is easy to construct a grammar to generate Σ^*: $S \to 1S \mid 0S \mid \varepsilon$.

Assume that EQ_{CFG} is decidable and D decides EQ_{CFG}.

We construct TM S_1 to decide ALL_{CFG}:

S_1: on input $\langle G\rangle$

1) Let H be the grammar $S \to 1S \mid 0S \mid \varepsilon$.
2) Run $D(\langle G, H\rangle)$.
3) If D accepts, accept; if D rejects, reject.

So ALL_{CFG} is decidable, a contradiction. $\square$

6. Question 5.2

We need to show that $\overline{EQ_{CFG}}$ is Turing-recognizable.

We construct a TM M as follows.

M: On input $\langle G_1, G_2\rangle$: on each $w \in \Sigma^*$

1). Run S, which decides A_{CFG}, on $\langle G_1, w\rangle$;
   Run S on $\langle G_2, w\rangle$.
2) If S returns one 'accept' and one 'reject', accept; otherwise, goto step 1).

7. Question 5.3

$$\left[\frac{aa}{a}\right]\left[\frac{aa}{a}\right]\left[\frac{b}{a}\right]\left[\frac{ab}{abab}\right]$$