CSCI 338 Computer Science Theory

Self-Evaluation Test (30 minutes)

Question 1

Given a planar graph $P = (V, E)$, we have Euler's formula: $|V| + |F| - |E| = 2$, where F (resp. E) is the set of faces (resp. edges) of P and $|F|$ (resp. $|E|$) is the size of F (resp. E). Let $|V| = n$. Prove that the number of edges in E is less than $3n$.

Proof:

- When P has only one face, or, P is a tree or forest, the claim is obviously true.
- Counting the edges in P face by face, we have a total of $2|E|$ edges, and as each face has at least 3 edges, we've

$$2|E| \geq 3|F|, \text{ or } |F| \leq \frac{2}{3}|E|.$$

Then, $|V| + \frac{2}{3}|E| - |E| \geq 2$, or $\frac{1}{3}|E| \leq |V| - 2$,

hence $|E| \leq 3|V| - 6 = 3n - 6 < 3n$. $\square$

Question 2

Peter makes a claim "If I have a ball absolutely round in my hand, then within 30 seconds I can raise the temperature in Bozeman by 20 degrees." How do you proceed to find a counterexample for this claim?

- This claim is in the form of $A \rightarrow B$ (i.e., if A is true then B is true).
- To disprove such a claim you should show an instance $A \rightarrow \neg B$ (i.e., you are given a ball absolutely round, but you can't raise Bozeman's temperature by 20 degrees).
- Try to show A is always false won't give you a counterexample, as $F \rightarrow T$, $F \rightarrow F$ are always considered true.

Question 3

Proof:

Prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$.

Let $f(n) = 1^3 + 2^3 + \cdots + n^3$, we will show $f(n) = \frac{1}{4}n^2(n+1)^2$ by induction on n.

① Basis: $f(1) = 1^3 = 1 = \frac{1}{4}1^2(1+1)^2$

② IH: Assume that $f(n) = \frac{1}{4}n^2(n+1)^2$ for $n \le K$.

③ IS: $f(K+1) = 1^3 + 2^3 + \cdots + K^3 + (K+1)^3$ // by def

$= (1^3 + 2^3 + \cdots + K^3) + (K+1)^3$

$= \frac{1}{4}K^2(K+1)^2 + (K+1)^3$ // by IH

$= \frac{1}{4}(K+1)^2[K^2 + 4(K+1)]$

$= \frac{1}{4}(K+1)^2(K+2)^2$

$= \frac{1}{4}(K+1)^2 * [(K+1)+1]^2.$ $\square$