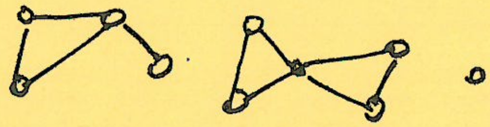
Jan 20

<u>Ex2</u>. A simple graph with n vertices and n edges must contain a cycle.



proof. (By contradiction).

Assume G has n vertices and n edges, but G contains no cycle, then G is either a tree or a forest.

 tree

If G is a tree or forest, then G has at most $\underline{n-1}$ edges.

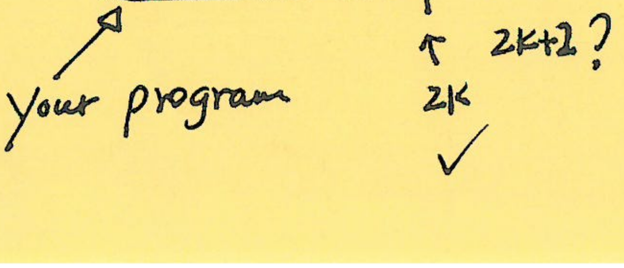
 forest

This contradicts with the assumption that G has n edges. Hence G must contain a cycle. $\square$

<u>Induction</u> (usually on integers)

Goldbach Conjecture: Any ~~(positive)~~ even integer $\geq 4$ can be written as the sum of 2 prime numbers.

<u>Ex</u>. $48 = 11 + 37$



Ex 1. $f(n) = \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \dots + n^2 = \frac{1}{6} n(n+1)(2n+1)$.

proof. (By induction)

Basis. when $n=1$, $f(n) = 1^2 = 1$.

$\frac{1}{6} n(n+1)(2n+1) = \frac{1}{6} \cdot 1 \cdot 2 \cdot 3 = 1$

So, $f(n) = \frac{1}{6} n(n+1)(2n+1)$.

Inductive Hypothesis (IH): Assume $f(n) = \frac{1}{6} n(n+1)(2n+1)$ for $n \le K$.

Inductive step (IS):

By def $f(K+1) = \underline{1^2 + 2^2 + \dots + K^2} + (K+1)^2$,

then for the first K items, we can use IH.

$$\therefore f(K+1) = \left[\frac{1}{6} K(K+1)(2K+1)\right] + (K+1)^2 \quad // \text{by IH}$$

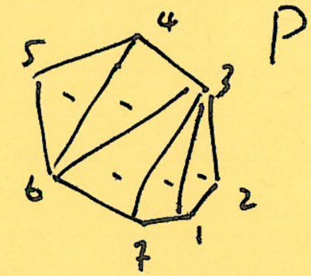
$$= \frac{1}{6}(K+1)\{K(2K+1) + 6(K+1)\}$$

$$= \frac{1}{6}(K+1)\{2K^2 + 7K + 6\}$$

$$= \frac{1}{6}(K+1)(K+2)(2K+3)$$

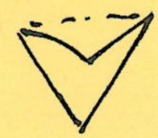
$$= \frac{1}{6}(\underline{K+1})[(\underline{K+1})+1] \cdot [2(\underline{K+1})+1]$$

**Ex2**. Any convex polygon with $n$ vertices can be triangulated into $n-2$ triangles.



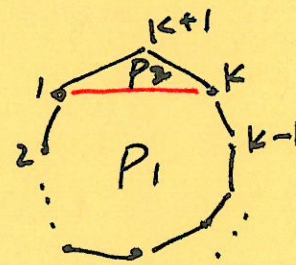

**Proof**: (By induction).

**Basis** when $n=3$, this convex polygon $P$ has $1 = n-2$ triangle in it.



**IH**: Assume a convex polygon $P$ with $n$ vertices can be triangulated into $n-2$ triangles for $n \le K$.

**IS**: Let $P$ be a convex polygon with $K+1$ vertices: $1, 2, \cdots, K+1$.




we connect 1 with $K$, this diagonal $\overline{1K}$ (or $(1,K)$) separates $P$ into 2 parts: $P_1$, $P_2$. Certainly the claim is true for both $P_1$ and $P_2$ as they have at most $K$ vertices by the IH.

$$\begin{aligned}\#\text{ of triangles in } P &= \#\text{ triangles in } P_1 + \#\text{ triangles in } P_2 \\ &= (K-2) + 1 \\ &= K-1 \ (= (K+1)-2).\end{aligned}$$

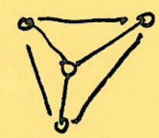
$\square$

# proof by constuction ($\neq$ by ~~example~~)

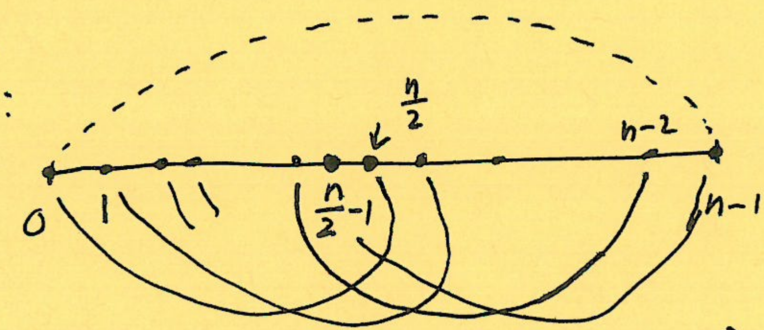
(only for existential claims)

**Ex.** For every even $n \geq 2$, there exists a 3-regular graph with $n$ nodes.

By example:



**proof**:




Constuct $V = \{0, 1, \cdots, n-1\}$.

$E = \{(i, i+1) \mid \text{for } 0 \leq i \leq n-2\}$ // horizontal edges

$\cup \{(i, i+\frac{n}{2}) \mid \text{for } 0 \leq i \leq \frac{n}{2}-1\}$ // arcs below the horizontal edges

$\cup \{(0, n-1)\}$ // dashed edge.

$\square$

## Directed graph

A graph where all edges are directed.

$G = (V, E)$, $E = \{\langle u, v \rangle \mid u, v \in V\}$

(in general $\langle u, v \rangle \neq \langle v, u \rangle$ $\quad u \rightarrow v$

## Strings and languages

alphabet: any finite set, each element is also called a <u>symbol</u>.

<u>Ex</u> $\Sigma = \{a, c, g, t\}$.

$S = aacttagtac$, $|S| = 10$

string: a finite sequence of symbols.

language: a set of strings.