

CSCI 338: Assignment 3 (7 points)

River Kelly

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Problem 1

Design context-free grammars for the following languages

(1.1) $A = \{a^n b^m \mid n \neq 2m\}.$

$$S \rightarrow aaSb \mid A \mid B \mid ab$$

$$A \rightarrow aA \mid a$$

$$B \rightarrow bB \mid b$$

(1.2) $B = \{a^i b^j c^k \mid i, j, k \geq 0 \text{ and either } i = j \text{ or } j = k\}.$

$$S \rightarrow AC \mid BD$$

$$A \rightarrow aAb \mid \epsilon$$

$$C \rightarrow cC \mid \epsilon$$

$$B \rightarrow aB \mid \epsilon$$

$$D \rightarrow bDc \mid \epsilon$$

(1.3) $C = \{a^n b^m \mid n = 3m\}.$

$$S \rightarrow aaaSb \mid \epsilon$$

(1.4) $D = \{a^n b^m \mid n \leq m + 3\}.$

$$S \rightarrow X \mid aX \mid aaX \mid aaaX$$

$$X \rightarrow aXb \mid \epsilon$$

$$Y \rightarrow bY \mid b$$

Problem 2

Decide whether the following grammar is ambiguous.

$$\begin{aligned} S &\rightarrow AB \mid aaB \\ A &\rightarrow a \mid Aa \\ B &\rightarrow b \end{aligned}$$

Proof. By definition, a string w is derived **ambiguously** in a context-free grammar if it has two or more different leftmost derivations. A context-free grammar is **ambiguous** if it generates some string ambiguously.

To show that the above grammar is ambiguous, we must show that there are two or more different leftmost derivations for some string.

Derivation 1	Derivation 2
S	S
AB	aaB
AaB	aab
aaB	
aab	

Each derivation above, using the leftmost rule, generates its own unique and distinct parse tree, each resulting in the same output.

\therefore The given grammar must be ambiguous.

□

Problem 3

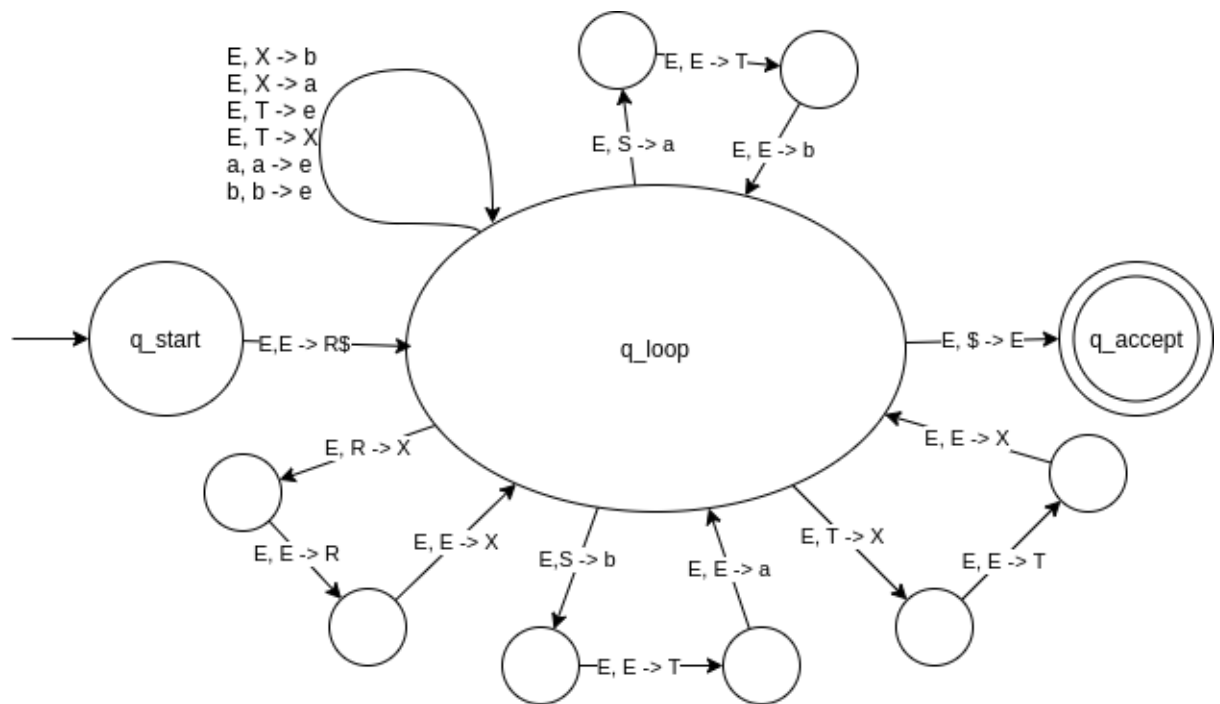
Convert the following CFG G to an equivalent PDA.

$R \rightarrow XRX|S$

$S \rightarrow aTb|bTa$

$T \rightarrow XTX|X|\epsilon$

$X \rightarrow a|b$



Problem 4

Let $G = (V, \Sigma, R, S)$ be the following grammar. $V = \{S, T, U\}$; $\Sigma = \{0, \#\}$; and R is the set of rules:

$$\begin{aligned} S &\rightarrow TT|U \\ T &\rightarrow 0T|T0|\# \\ U &\rightarrow 0U00|\# \end{aligned}$$

(4.1) Describe $L(G)$ in English.

The language of grammar G , or $L(G)$, may take the form of two possibilities. The two options are described as follows:

1. The first possibility is a string which contains 0's and #'s with at least two # symbols and any number of 0's on either side and or in between the # symbols.
2. The second possibility is a string beginning with n zeros, then a # symbol, and then $2n$ zeros.

(4.2) Prove that $L(G)$ is not regular.

Proof. Assume that $L(G)$ is regular. Since $L(G)$ is regular, we can apply the pumping lemma. Let p be the pumping length, and let s be a string such that $s \in L(G)$ where $|s| \geq p$. By definition of the pumping lemma, the following must be true:

1. $xy^iz \in A$ for $i \geq 0$
2. $|y| > 0$
3. $|xy| \leq p$

Now, we must choose $s = 0^p \# 0^{2p} \mid s \in L(G)$.

By (2), $i = |y|$, and by (3) $|xy| \leq p$, so y must consist of only zeros.

We pump up $xy^2z = 0^{p+i} \# 0^{2p} \notin L(G)$ because the increased zeros before the # symbol without changing those following it, so $2(p+i) \neq 2p$.

\therefore A contradiction of the pumping lemma, so $L(G)$ must not be regular. \square

Problem 5

Convert the following CFG into an equivalent CFG in Chomsky Normal Form

$$\begin{aligned} A &\rightarrow BAB \mid B \mid \epsilon \\ B &\rightarrow 00 \mid \epsilon \end{aligned}$$

Step 1 First, we add a new start variable S and the rule $S \rightarrow A$ where A was the original start variable.

$$\begin{aligned} S &\rightarrow A \\ A &\rightarrow BAB \mid B \mid \epsilon \\ B &\rightarrow 00 \mid \epsilon \end{aligned}$$

Step 2 Take care and remove all ϵ -rules.

2a Remove $B \rightarrow \epsilon$

$$\begin{aligned} S &\rightarrow A \\ A &\rightarrow BAB \mid BA \mid AB \mid A \mid B \mid \epsilon \\ B &\rightarrow 00 \end{aligned}$$

2b Remove $A \rightarrow \epsilon$

$$\begin{aligned} S &\rightarrow A \mid \epsilon \\ A &\rightarrow BAB \mid BA \mid AB \mid A \mid B \mid BB \\ B &\rightarrow 00 \end{aligned}$$

We do not need to remove the ϵ -rule $S \rightarrow \epsilon$ since S is the start variable.

Step 3 Handle all unit-rules.

3a Remove $A \rightarrow A$

$$\begin{aligned} S &\rightarrow A \mid \epsilon \\ A &\rightarrow BAB \mid BA \mid AB \mid B \mid BB \\ B &\rightarrow 00 \end{aligned}$$

3b Remove $A \rightarrow B$

$$\begin{aligned} S &\rightarrow A \mid \epsilon \\ A &\rightarrow BAB \mid BA \mid AB \mid 00 \mid BB \\ B &\rightarrow 00 \end{aligned}$$

3c Remove $S \rightarrow A$

$$\begin{aligned} S &\rightarrow BAB \mid BA \mid AB \mid 00 \mid BB \mid \epsilon \\ A &\rightarrow BAB \mid BA \mid AB \mid 00 \mid BB \\ B &\rightarrow 00 \end{aligned}$$

3d Replace ill-placed terminals by new variable C

$$\begin{aligned} S &\rightarrow BAB \mid BA \mid AB \mid CC \mid BB \mid \epsilon \\ A &\rightarrow BAB \mid BA \mid AB \mid CC \mid BB \\ B &\rightarrow CC \\ C &\rightarrow 0 \end{aligned}$$

Step 4 Finally, convert all remaining rules into the proper form. (i.e. by replacing each rule $A \rightarrow u_1 u_2 \dots u_k$, where $k \geq 3$)

$$\begin{aligned} S &\rightarrow BD_1 \mid BA \mid AB \mid CC \mid BB \mid \epsilon \\ A &\rightarrow BD_2 \mid BA \mid AB \mid CC \mid BB \\ B &\rightarrow CC \\ C &\rightarrow 0 \\ D_1 &\rightarrow AB \\ D_2 &\rightarrow AB \end{aligned}$$

Problem 6

Using pumping lemma to prove that the following languages are not context-free.

$$(6.1) L = \{a^n b^j c^k \mid k = nj\}.$$

Proof. Assume L is regular, then choose $s = a^p b^p c^{p^2}$ where p is the pumping length. By the pumping lemma, s may be decomposed into $uvxyz$, such that:

- (1) $uv^i xy^i z \in L$ for $i \geq 0$
- (2) $|vy| > 0$
- (3) $|vxy| \leq p$

There are three possible cases, let's examine each:

- 1. vxy is made up of b 's and c 's.
- 2. vxy does not have an a 's.
- 3. vxy contains only c 's.

In case 1, pumping $uv^2 xy^2 z$ take the form $a^p b^{p+i} c^{p^2+j}$. Then, $p(p+i) = p^2 + ip > p^2 + j$ since $i > 0$, and by (3) $j > p$. Therefore $uv^2 xy^2 z \notin L$.

In case 2, pumping $uv^2 xy^2 z$ violates (2) because there would be more a 's or b 's without changing the number of c 's.

Lastly, in case 3, when pumping $uv^2 xy^2 z$ also violates (2) because there is more c 's without changing the number of a 's or b 's.

\therefore A contradiction in all cases, so L must not be regular. □

(6.2) $L = \{a^n b^j \mid n \geq (j-1)^3\}$.

Proof. Assume L is regular, then choose $s = a^{(p-1)^3} b^p$ where p is the pumping length. By the pumping lemma, s may be decomposed into $uvxyz$, such that:

- (1) $uv^i xy^i z \in L$ for $i \geq 0$
- (2) $|vy| > 0$
- (3) $|vxy| \leq p$

There are three possible cases, let's examine each:

- 1. vxy is made up of only a 's.
- 2. vxy contains both a 's and b 's.
- 3. vxy contains only b 's.

In case 1, by pumping down such that, $uv^0 xy^0 z = uxz$. By rule (2) $i = |vy| > 0$, so uxz has the form $a^{(p-1)^3-i} b^p$. But, $(p-1)^3 - i < (p-1)^3$. Therefore $uxz \notin L$.

In case 2, pumping up $uv^2 xy^2 z$ takes the form $a^{(p-1)^3+i} b^{p+j}$, and we know that $j \neq 0$ so $(p-1+j)^3 \geq (p)^3$. Then, $(p-1)^3 + i < (p-1)^3 + 3p^2 - 3p + 1 = p^3$ for $p > 1$. Therefore $uxz \notin L$.

In case 3, pumping up $uv^2 xy^2 z$ takes the form $a^{(p-1)^3} b^{p+i}$. Thus, increasing the number of b 's without altering the number of a 's. By (2), $i = |vy| > 0$, then $(p-1)^3 < (p+i-1)^3$. Therefore $uxz \notin L$.

\therefore A contradiction in all cases, so L must not be regular. □