CSCI 338: Assignment 4

River Kelly

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Problem 1

Let $\mathcal{B}$ be the set of all infinite sequences over $\{a, b\}$. Show that $\mathcal{B}$ is uncountable, using a proof by diagonalization.

Proof. To prove that $\mathcal{B}$ is uncountable using diagonalization, we must show, by contradiction, that no correspondence exists between $\mathcal{N}$ and $\mathcal{B}$. Suppose that a correspondence f existed between $\mathcal{N}$ and $\mathcal{B}$. For such correspondence to exist, f must pair all the members of $\mathcal{N}$ with all the members of $\mathcal{B}$. We must show that f fails to work as it should by finding an x in $\mathcal{B}$ that is not paired with anything in $\mathcal{N}$, a contradiction.

Assuming that $\mathcal{B}$ is countable, the elements of $\mathcal{B}$ may be ordered as $b_1, b_2, b_3, \ldots b_n$, and a correspondence f exists between $\mathcal{N}$ and $\mathcal{B}$, the following table shows a few values of a hypothetical correspondence.

n	$f(n)$
1	$\underline{\text{a}}$bbab...
2	b$\underline{\text{b}}$aab...
3	ab$\underline{\text{a}}$ba...
4	aaa$\underline{\text{b}}$a...
$\vdots$	$\vdots$

We can now construct the desired x by taking the elements of the diagonal and complementing them so that to ensure that $x \neq f(n)$ for any n. Continuing in this way down the diagonal of the table for f, we obtain all the digits of x. Note that underlined values in the table above represent the construction for the inputs of x. Furthermore, the value of x would be

$$x = \text{baba} \ldots$$

But $x \in \mathcal{B}$, so then some $b_i = x$. But by the construction of x is different from b_i at the i^{th} spot. A contradiction.

$\therefore \mathcal{B}$ is not countable. $\square$

Problem 2

Let $T = \{(i, j, k) | i, j, k \in \mathcal{N}\}$. Show that T is countable.

Proof. To prove that T is countable, we must show that there exists some correspondence f between $\mathcal{N}$ and T. To do this, we will construct a list which shows a few values of a hypothetical correspondence f between $\mathcal{N}$ and T.

n	$f(n)$
1	$(0, 0, 0)$
2	$(1, 0, 0)$
3	$(0, 1, 0)$
4	$(0, 0, 1)$
5	$(2, 0, 0)$
6	$(0, 2, 0)$
$\vdots$	$\vdots$

For each tuple (i, j, k), let s be the sum such that $s = i + j + k$ and $s \in \mathcal{N}$. This implies that we can enumerate all tuples in T because there are a finite number of tuples whose sum is equal to s. This shows the existence of a one-to-one correspondence between $\mathcal{N}$ and $\mathcal{T}$.

$\therefore T$ must be countable. $\square$

Problem 3

Let $INFINITE_{PDA} = \{< M > | M$ is a PDA and $L(M)$ is an infinite language$\}$. Show that $INFINITE_{PDA}$ is decidable.

Proof. To prove that $INFINITE_{PDA}$ is decidable, we will construct a Turning Machine that decides it. Let M denote such a Turning Machine, we can the convert M into a Context Free Grammar (CFG) - lets call N. The CFG N can be converted into Chomsky Normal Form, called N', and we can check if the there exists a derivation $A \Rightarrow uAv$ where $u, v \in \Sigma$.

There is an infinite language $L(M)$ if $A \Rightarrow uAv$ is a derivation in N', and there is not an infinite language $L(M)$ is there is not a derivation $A \Rightarrow uAv$ in N'. The Turning Machine M, which decides $INFINITE_{PDA}$ as follows:

$M =$ "On input $< M >$ a PDA:
1. Convert M into an equivalent CFG N.
2. Convert N into an equivalent CFG N' in Chomsky Normal Form.
3. If the derivation $A \Rightarrow uAv$ is included in N', accept, else, reject."

$\therefore INFINITE_{PDA}$ is decidable. $\square$

Problem 4

Let $\Sigma = \{a, b\}$. Define the following language *ODD*$_{TM}$:

ODD$_{TM}$ $= \{< M > | M$ is a TM and $L(M)$ contains only strings of odd length $\}$.

Prove that *ODD*$_{TM}$ is undecidable.

Proof. To prove *ODD*$_{TM}$ is undecidable, we will show that A_{TM} reduces to *ODD*$_{TM}$. Assume that *ODD*$_{TM}$ is decidable with Turning Machine R. Let $< M, w >$ be the input into A_{TM}.

First, we will construct Turning Machine S on input $< M, w >$:

$S =$ "On input x:
1. If $|x|$ is odd, accept.
2. If $|x|$ is even, run M on w. If M accepts, accept. Otherwise, reject."

Now we can construct a TM to run R on $< M, w >$ which decides A_{TM};

$H =$ "On input $< M, w >$:
1. Run R on $< M, w >$.
2. If R accepts, accept. If R rejects, reject."

But A_{TM} is undecidable, a contradiction.

$\therefore$ *ODD*$_{TM}$ is undecidable. $\square$

Problem 5

Show that EQ_{CFG} is undecidable.

Proof. To prove that EQ_{CFG} is undecidable, we will construct a TM S to decide ALL_{CFG}.

Recall that,

$$EQ_{CFG} = \{\langle G_1, G_2 \rangle \mid G_1 \text{ and } G_2 \text{ are CFGs and } L(G_1) = L(G_2)\}$$

and the ALL_{CFG} will reduce to EQ_{CGF} where,

$$ALL_{CGF} = \{\langle G \rangle \mid G \text{ is a CFG and } L(G) = \Sigma^*\}$$

Let TM R decide EQ_{CFG} and the details of TM S be:

$S =$ "On input G a CFG:
1. Let G_0 be the CFG such that $L(G_0) = \Sigma^*$
2. Run R on input $\langle G, G_0 \rangle$.
3. If R accepts, accept. Otherwise reject."

ALL_{CFG} is undecidable, and TM S decides ALL_{CFG}, a contradiction.

$\therefore$ EQ_{CGF} must also be undecidable. $\square$

Problem 6

Show that EQ_{CFG} is co-Turing-recognizable.

Proof. To prove that EQ_{CFG} is co-Turing-recognizable we will construct a TM S that recognizes $\overline{EQ_{CFG}}$.

Recall that a language is co-Turing-recognizable if, and only if, its complement is a Turing-recognizable language.

$$\overline{EQ_{CFG}} = \{\langle G_1, G_2\rangle \mid G_1 \text{ and } G_2 \text{ are CFGs and } L(G_1) \neq L(G_2)\}$$

The details of TM S that recognizes $\overline{EQ_{CFG}}$:

$S =$ "On input $\langle G_1, G_2\rangle$, where G_1 and G_2 are CFGs:
1. If at least one CFG, either G_1 and G_2, is invalid, accept.
2. Convert the equivalent Chomsky Normal Form CFGs of G_1 and G_2, G_1' and G_2' respectively.
3. Repeat step 4 for $i = 1, 2, 3 \ldots$
4. Test G_1' and G_2' to generate a unique string $s \in \Sigma^*$
If exactly one is valid and one is invalid, accept."

$\therefore$ TM S recognizes $\overline{EQ_{CFG}}$, thus EQ_{CFG} is co-Turing-recognizable. $\square$

Problem 7

Find a match in the following instance of the Post Correspondence Problem.

$$\left\{ \left[\frac{ab}{abab}\right], \left[\frac{b}{a}\right], \left[\frac{aba}{b}\right], \left[\frac{aa}{a}\right] \right\}$$

Proof. A matching result can be found/made/created by a combination of all the upper strings and all of the lower strings such that reading both of the top and bottom are the same.

Consider the collection as a list of dominos numbered 1-4. There is a match given the sequence 4,4,2 and 1. The match is show as:

$$\left[\frac{aa}{a}\right], \left[\frac{aa}{a}\right], \left[\frac{b}{a}\right], \left[\frac{ab}{abab}\right] = \frac{aaaabab}{aaaabab}$$

$\square$