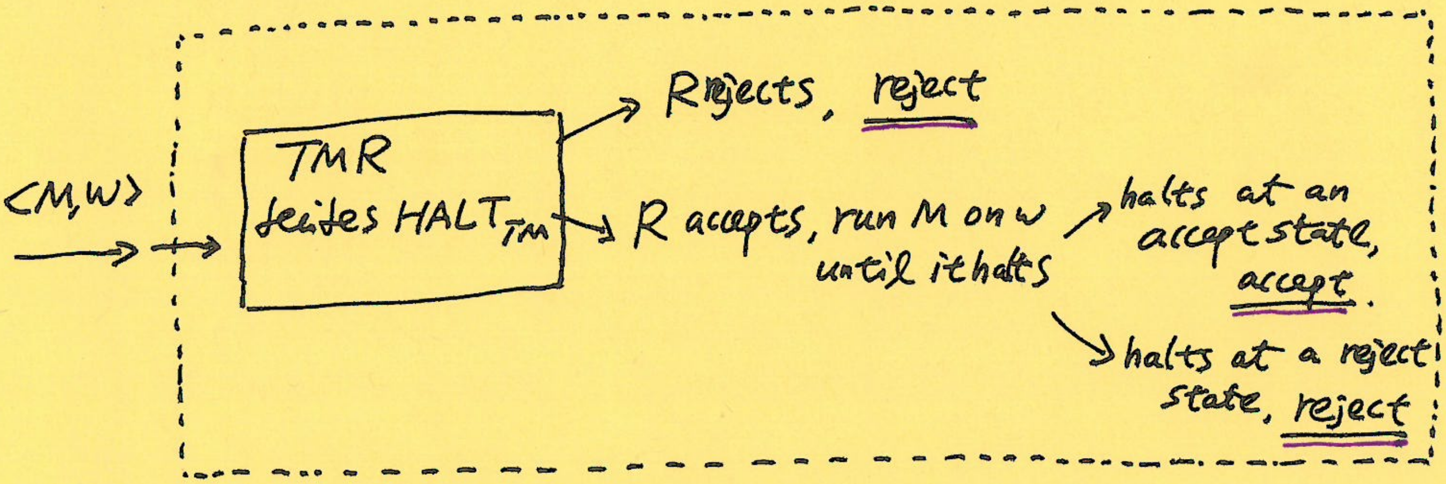
Chapter 5 Reducibility

- Reduction: Convert A to B s.t. the solution to B can be used to solve A ($A \propto B$, $A \leq B$)
- Implication: If we know A is hard, then B is at least as hard as A —— provided that the transformation cost (from A to B) is small. $\boxed{\text{Sorting} \leq_{O(n)} \text{2D CH}}$

$A_{TM} = \{\langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$

$HALT_{TM} = \{\langle M, w \rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$



a TM S for A_{TM}, assuming R decides $HALT_{TM}$ (notice the order)

Thm 5.1 $HALT_{TM}$ is undecidable

proof. Assume that TM R decides $HALT_{TM}$.

we construct a TM S for A_{TM}:

1. Run R on $\langle M, w \rangle$
2. If R rejects, reject.
3. If R accepts, simulate M on w until it halts
   - 3.1 If M accepts w, accept;
   - 3.2 If M rejects w, reject.

Therefore, A_{TM} is decidable and this is a contradiction with Thm 4.11 (that A_{TM} is undecidable).

$\therefore$ $HALT_{TM}$ is undecidable. $\square$

The most important thing here is to relate whatever problem you need to show to be undecidable with a known undecidable problem (usually A_{TM}).

$FINITE_{TM} = \{\langle M \rangle \mid M$ is a TM and $L(M)$ is finite$\}$.

$$A_{TM} \propto FINITE_{TM}$$

input $\langle M, w \rangle$? input $\langle M' \rangle$

You have to build the logical connection (bridge)!

Here is what we do:

- Construct TM M' on input $\langle x \rangle$

  1. Run M on w and accept x if M accepts w.

Here $\langle M, w \rangle$ is the input for A_{TM}.

this is not written openly in a book

If M accepts w $\Rightarrow$ M' accepts any input x
$\Rightarrow$ $L(M')$ is infinite

If M doesn't accept w $\Rightarrow$ M' doesn't accept any input x
$\Rightarrow$ $L(M')$ is finite

$L(M') = \phi$, but we need to relate it to $FINITE_{TM}$

with this logical relation, the formal proof becomes easy.

<u>proof</u>. Suppose R decides $FINITE_{TM}$, we construct a TM S for A_{TM}.

1. Construct M' as above.
2. Run R on $\langle M' \rangle$
3. If R accepts, <u>reject</u>,
   If R rejects, <u>accept</u>.

$\therefore$ S is a decider for A_{TM}, a contradiction.

$\therefore$ $FINITE_{TM}$ is undecidable. $\square$

$$E_{TM} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \phi\}$$

$$A_{TM} \leq E_{TM}$$

$$\langle M, w\rangle \xrightarrow{?} \langle M_1\rangle$$

Thm 5.2 E_{TM} is undecidable.

- First construct TM M_1 on input x:
  1. If $x \neq w$, reject
  2. If $x = w$, run M on w and accept if M accepts w // $\langle M, w\rangle$ is the input for A_{TM}.

// M accepts $w \Rightarrow M_1$ accepts $w \Rightarrow L(M_1) \neq \phi$

M doesn't accept $w \Rightarrow M_1$ accepts nothing $\Rightarrow L(M_1) = \phi$

<u>proof</u>: Assume that E_{TM} is decidable and R is the decider, we'll construct TM S for A_{TM}.

S: on $\langle M, w\rangle$
1. Construct M_1.
2. Run R on $\langle M_1\rangle$.
3. If R accepts, <u>reject</u>,
   if R rejects, <u>accept</u>.

Therefore, S is a decider for A_{TM}, a contradiction.

$\therefore$ E_{TM} is undecidable. $\square$

Exercise. Two-diff-TM = $\{\langle M \rangle \mid$ M is a TM and L(M) contains two different strings of the same length$\}$.

Yes instance: $L(M) = \{a, b\}$ or $\{aa, bb, ba, bab\}$

No instance: $L(M) = \{a, bb, aaa, bbba\}$ or ϕ

IDEA: A_{TM} ? Two-diff-TM

$\langle M, w \rangle$ $\langle M' \rangle$

- Try to spend about 30 minutes
- If you can get it, you already have no problem with reductions.
- If not, don't be upset, people learn things at different paces. Just keep up, more examples to come.