Student No.	Name

CSCI 338 Computer Science Theory

Test 2 — 70 minutes (10 points)

Note: If you don't have a printer, you should write the answers on white papers. After you finish, scan a .pdf file to D2L (under Assignments/Test 2). *Note also that this is an open book test, while all physical resources are allowed, resorting for external human help constitutes a* **plagiarism.**

Question 1

Let R_1 be the set of all the positive real numbers less than 1, i.e.,

$$R_1 = \{x | 0 < x < 1\}.$$

Prove that R_1 is uncountable.

proof: Suppose that R_1 is countable, i.e., there is a correspondence f between R_1 and N (as R_1 is obviously infinite).

Then, we can list elements in R_1 as $f(1), f(2), f(3), \ldots$, i.e.

i	$f(i)$
1	0.353535 ···
2	0.978654 ···
3	0.141596 ···
⋮	···

we then construct x by flipping the i-th digit of $f(i)$ to some different value, e.g.,

$$x = 0.546\cdots$$

Clearly, $x \in R_1$, but $x \neq f(i)$ for any i.

$\therefore f$ is not a correspondence between R_1 and N.

$\therefore R_1$ is uncountable. $\square$

Question 2

Determine whether the following grammar is ambiguous. Explain your reason.

$$S \to aSbS|bSaS|\epsilon.$$

Ambiguous.

The reason is that we can derive ab in 2 different ways

- $S \Rightarrow a\underline{S}bS \Rightarrow a\varepsilon b\underline{S} \Rightarrow a\varepsilon b\varepsilon = ab$
- $S \Rightarrow aSb\underline{S} \Rightarrow a\underline{S}b\varepsilon \Rightarrow a\varepsilon b\varepsilon = ab.$

The solution might not be unique.

Question 3

Is the language $L = \{a^i b^j c^k | i < j < k\}$ context-free? You must give enough details to justify your answer.

It is not context-free.

We give a sketch of proof using the pumping lemma.

- Suppose L is context free. select $s = a^p b^{p+1} c^{p+2}$, where p is the pumping length, then, by the pumping lemma, s can be decomposed into $s = uvxyz$. s.t.

① $uv^ixy^iz \in L$, for $i \geq 0$,

② $|vxy| \leq p$,

③ $|vy| > 0$.

Case 1. If vxy only contain one type of symbols then

1.1) vxy contain a's, v^2xy^2 would increase the # of a's.

1.2) vxy contain b's, v^0xy^0 would decrease the # of b's.

1.3) vxy contain c's, v^0xy^0 would decrease the # of c's.

In all cases, pumping up/down would result $uv^ixy^iz \notin L$ for $i = 2$ or 0.

Case 2. If vxy contain 2 types of symbols, then uv^2xy^2z would contain a subsequence a··b··a··b or b··c··b··c, hence can't be in L.

∴ We have a contradiction to the pumping lemma.

Therefore, L is not context-free. □

Question 4

In class we mentioned that the general Post Correspondence Problem (PCP), i.e., when $|\Sigma| \geq 2$, is undecidable. Show that if $\Sigma = \{a\}$ then the restricted problem PCP-1 is in fact decidable.

Proof: As a domino card only uses letter a's, we model each card C_i as an integer I_i, i.e.,

I_i = # of a's above the mid-line on C_i
– # of a's below the mid-line on C_i.

Ex. C_1 $\left[\frac{aaa}{a}\right] \rightarrow I_1 = 2$

C_2 $\left[\frac{a}{aa}\right] \rightarrow I_2 = -1$

Now we run the following algorithm:

① If some $I_j = 0$, return C_j and accept.

② If all I_j's > 0, reject.

③ If all I_j's < 0, reject.

④ If $I_j > 0$, $I_K < 0$, then the PCP solution is $|I_K|$ copies of C_j followed with I_j copies of C_K, accept.

Ex. With only C_1, C_2, the solution is

$\left[\frac{aaa}{a}\right]$ $\left[\frac{a}{aa}\right]$ $\left[\frac{a}{aa}\right]$

C_1 C_2 C_2.

Question 5

Define $ALL_{TM} = \{< M > | M \text{ is a Turing machine with } L(M) = \Sigma^*\}$. Prove that ALL_{TM} is undecidable by a reduction from A_{TM}.

Proof: Assume that ALL_{TM} is decided by a TM R, we'll construct a TM S for A_{TM}.

- construct a TM M':

M': on input x

- Run M on w and accept x if M accepts w.

// M accepts $w \Rightarrow M'$ accepts all $x \Rightarrow L(M') = \Sigma^*$

// M doesn't accept $w \Rightarrow M'$ accepts nothing $\Rightarrow L(M') \neq \Sigma^*$

Now construct S for A_{TM}:

S: on $<M, w>$

① construct M'

② Run R on $<M'>$

③ If R accepts, accept; if R rejects, reject.

$\therefore$ S is a decider for A_{TM}, a contradiction.

$\therefore$ ALL_{TM} is undecidable. $\square$