CSCI 338: Assignment 1 (7 points)

This assignment is due on **Thursday, Jan 28, 8:00pm**. You will need to use Latex to generate a single pdf file and upload it under *Assignment 1* on D2L. There will be a penalty for not using Latex (to finish the assignment). This is **not** a group-assignment, so you must finish the assignment by yourself.

Problem 1

Prove that $1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$.

Proof: Let the left-hand side be $f(n)$. We use induction on n to show that $f(n) = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$.

Basis: When $n = 1$, $f(1) = 1^4 = 1$, the right-hand side evaluates to $\frac{1(1+1)(2+1)(3+3-1)}{30} = 1$. So $f(n) = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$, when $n = 1$.

Inductive Hypothesis: Assume that the claim is true for $n \leq k$, i.e., $f(n) = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$, when $n \leq k$.

Inductive Step: By definition

$$f(k+1) = 1^4 + 2^4 + 3^4 + \cdots + (k-1)^4 + k^4 + (k+1)^4.$$

Therefore,

$$f(k+1) = \{1^4 + 2^4 + 3^4 + \cdots + (k-1)^4 + k^4\} + (k+1)^4 = f(k) + (k+1)^4.$$

Following the inductive hypothesis,

$$f(k+1) = \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} + (k+1)^4.$$

Therefore, $f(k+1) = \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} + (k+1)^4 = (k+1) \cdot \frac{k(2k+1)(3k^2+3k-1)}{30} + (k+1) \cdot \frac{30(k+1)^3}{30} = (k+1) \cdot \frac{k(2k+1)(3k^2+3k-4)+30(k+1)^3}{30} = (k+1) \cdot \frac{(6k^4+39k^3+91k^2+89k+30)}{30} =$

$(k+1)(k+2) \cdot \frac{(6k^3+27k^2+37k+15)}{30} = (k+1)(k+2)(2k+3) \cdot \frac{(3k^2+9k+5)}{30} = \frac{(k+1)(k+2)(2k+3)(3(k+1)^2+3(k+1)-1)}{30}$.

□

Problem 2

Given a planar graph $P = (V, E)$, we have Euler's formula: $|V| + |F| - |E| = 2$, where F is the set of faces of P and E is the set of edges of P. Let $|V| = n$, where V is the set of vertices of P. Prove that $|F|$ is at most $2n$.

Proof: We will use a direct (argument) method.

From $|V| + |F| - |E| = 2$, or $n + |F| - |E| = 2$, we can eliminate $|E|$ as follows. We count the number of edges as they appear in each face, so we have a total count of $2|E|$ (as each one is counted exactly twice). On the other hand, each face has at least 3 edges, so

$$3|F| \leq 2|E|.$$

Now, put this into Euler's formula, we have

$$n + |F| - \frac{3}{2}|F| \geq 2,$$

which means

$$n - 2 \geq \frac{1}{2}|F|,$$

or

$$|F| \leq 2n - 4 < 2n.$$

□

Problem 3

Prove that in any simple graph there is a path from any vertex of odd degree to some other vertex of odd degree.

Proof: We will use proof by contradiction.

Without loss of generality (WLOG), assume that the graph G is connected (otherwise, we could argue on any connected component). Assume that the claim is false, i.e., G only has a vertex v with $deg(v)$ odd. (If G had at least two such nodes, since G is connected, the claim would have already been proven.)

Then,
$$\sum_{u \in V(G)} deg(u) = deg(v) + \sum_{u \in V(G)-\{v\}} deg(u),$$
which is odd. But this contradicts with the fact, which we proved in class, that $\sum_{u \in V(G)} deg(u)$ is even.

□

Problem 4

A fully binary tree T is a tree such that all internal nodes have two children. Prove that a fully binary tree with n internal nodes in total has $n+1$ leaves.

Proof: Let $f(n)$ be the number of leaves in a fully binary tree with n internal nodes. We prove $f(n) = n + 1$ by induction.

Basis: When $n = 1$, a fully binary tree with $n = 1$ internal nodes has 2 leaves, so $f(1) = 2 = n + 1$.

Inductive Hypothesis (IH): Assume that the claim is true for $n \leq k$, i.e., $f(n) = n + 1$ when $n \leq k$.

Inductive Step: Consider a tree T with $k + 1$ internals nodes (Figure 1). Pick any internal node x with 2 leaf children, x_1 and x_2.

We construct a new fully binary tree T' by deleting x, x_1 and x_2 and then add back a (black) leaf node y at the position of x. By IH, T' has k internal nodes and hence has $k + 1$ leaves in total. We can then build T back by deleting y and add x, x_1 and x_2 back to T'. The total number of leaves in T is then

$$\text{\# of leaves in } T' - 1 + 2 = (k + 1) + 1.$$

Therefore, we have $f(k + 1) = (k + 1) + 1$.

□

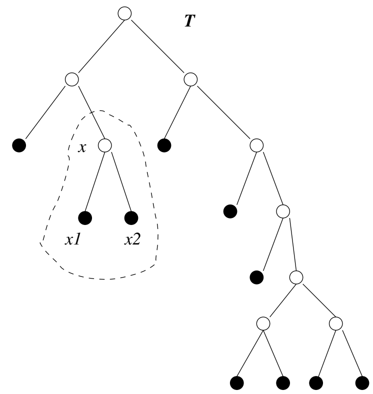

Figure 1: **Illustration for Problem 4.**

Problem 5

Given an undirected graph $G = (V, E)$, the breadth-first-search starting at $v \in V$ ($bfs(v)$ for short) is to generate a shortest path tree starting at vertex $v \in V$. The diameter of G is the longest of all shortest paths $\delta(u, v), u, v \in V$.

When G is a tree, the following algorithm is proposed to compute the diameter of G.

1. Run $bfs(w), w \in V$, and compute the vertex $x \in V$ furthest from w.
2. Run $bfs(x)$ and compute the vertex $y \in V$ furthest from x.
3. Return $\delta(x, y)$ as the diameter of G.

Prove that this algorithm is correct; i.e., $\delta(x, y)$ is in fact the longest among all the shortest paths between $u, v \in V$.

Proof: We prove this claim by contradiction.

Assume that $\delta(x, y)$ is not the diameter of tree G and instead $\delta(a, b)$ is the diameter of G, i.e., $\delta(x, y) < \delta(a, b)$. WLOG, assume that $\delta(a, b)$ and $\delta(w, x)$ have an intersection o.

Since x is the farthest from w, we have

$$\delta(w, b) \leq \delta(w, x).$$

In other words,

$$\delta(b, o) \le \delta(o, x). \tag{1}$$

Similarly, as y is the farthest from x, we have

$$\delta(x, a) \le \delta(x, y).$$

In other words, $\delta(a, p) \le \delta(p, y)$, or equivalently,

$$\delta(a, o) + \delta(o, p) \le \delta(p, y). \tag{2}$$

We add (1) and (2) to have

$$\delta(a, b) + \delta(o, p) \le \delta(x, y) + \delta(o, p),$$

that is

$$\delta(a, b) \le \delta(x, y).$$

This is a contradiction to the assumption that $\delta(a, b) > \delta(x, y)$.

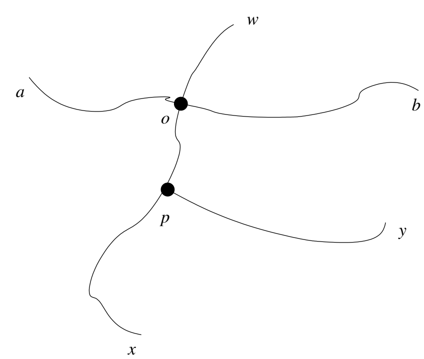

Figure 2: **Tree G, illustration for Problem 5.**

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