Exercise Two_diff_TM = { <M> | M is a TM and L(M) contains two different strings of the same length}.

Prove that Two_diff_TM is undecidable.

Proof. We will reduce A_{TM} to Two_diff_TM.

First, let $\Sigma = \{a, b\}$. Construct M' as follows:

- M', on input x

  If $x \neq aaa$ and $x \neq bbb$ then reject.

  If $x = aaa$ or $x = bbb$ then run M on w and accept x if M accepts w.

M accepts w $\Rightarrow$ M' accepts aaa, bbb $\Rightarrow$ L(M') contains two different strings of the same length}

M doesn't accept w $\Rightarrow$ M' accepts nothing $\Rightarrow$ L(M') doesn't contain two different strings of the same length

Now, let R be a decider for Two_diff_TM. We construct TM S for A_{TM}:

① Construct M'.

② Run R on <M'>.

③ If R accepts <M'>, accept, if R rejects <M'>, reject.

$\therefore$ S decides A_{TM}, a contradiction.

$\therefore$ Two_diff_TM is undecidable. □

Map reducibility:

$\leq_m$ or $\leq$ or $\propto$

$A \leq_m B$ if there is a computable function $f: \Sigma^* \to \Sigma^*$, such that for every w,

$$w \in A \text{ iff } f(w) \in B$$

f is also called a **reduction**.

// Intuitively, you solve A by calling a solution for B once. In the future (graduate school, say), you could encounter more complex reductions.

$$REGULAR_{TM} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is regular}\}$$

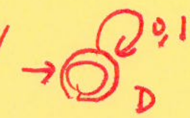
$$\underset{\langle M, w\rangle}{A_{TM}} \xrightarrow{?} \underset{\langle M_2 \rangle}{REGULAR_{TM}}$$

Construct M_2, on input x // $\Sigma = \{0,1\}$

1. If $x = 0^n1^n$, accept.
2. If $x \neq 0^n1^n$, run M on w and accept if M accepts w

// M doesn't accept $w \implies M_2$ accepts $\{0^n1^n\}$
$\implies L(M_2)$ is not regular

M accepts $w \implies M_2$ accepts Σ^*
$\implies L(M_2)$ is regular

// 

$L(D) = \Sigma^*$

Exercise write a proof using the previous logical reasoning that $REGULAR_{TM}$ is undecidable.

Thm 5.3 $REGULAR_{TM}$ is undecidable.

proof: Let R decide $REGULAR_{TM}$, we construct TM S for A_{TM}.

1. Construct TM M_2.
2. Run R on $\langle M_2 \rangle$.
3. If R accepts, accept, if R rejects, reject.

$\therefore$ S is a decider for A_{TM}, a contradiction.

$\therefore$ $REGULAR_{TM}$ is undecidable. $\square$

$EQ_{TM} = \{\langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TM's and } L(M_1) = L(M_2)\}$

proof: Let TM R decide EQ_{TM}, we construct TM S for E_{TM}.

S: on input $\langle M \rangle$

1. Run R on $\langle M, M_1 \rangle$, where M_1 is a TM that rejects all inputs // $\rightarrow \emptyset$
2. If R accepts, accept, if R rejects, reject.

Then S is a decider for E_{TM}, a contradiction to Thm 5.2

$\therefore$ EQ_{TM} is undecidable. $\square$

A Turing-unrecognizable language (chapter 4)

<u>Def</u>. A language is co-Turing-recognizable if it is the complement of a Turing-recognizable language.

<u>Thm</u> 4.22. A language is <u>decidable</u> if and only if it is both Turing-recognizable and co-Turing-recognizable.

$\boxed{\text{Proof}}$: "$\Rightarrow$" only if part:

If A is decidable, $\bar{A}$ is also decidable (just switch Y/N).
Certainly, both A and $\bar{A}$ are Turing-recognizable.

"$\Leftarrow$" if part:

If both $A, \bar{A}$ are Turing-recognizable, let M_1, M_2 be the corresponding recognizer for $A, \bar{A}$.
We then construct TM M for A:

M, on input x:

1. Run M_1, M_2 on x at the same time.
2. If M_1 accepts, <u>accept</u>; if M_2 accepts, <u>reject</u>. $\square$

Since x is either in A or $\bar{A}$, M is a decider.

<u>Corollary</u> 4.23.

$\overline{A_{TM}}$ is not Turing-recognizable // what is $\overline{A_{TM}}$?

<u>Proof</u>: If $\overline{A_{TM}}$ is Turing-recognizable, then as A_{TM} is also Turing-recognizable, by Thm 4.22, A_{TM} is decidable. Then we have a contradiction. $\square$