Definition: Conditional Probability

The **conditional probability** of an Event A given event B is the probability that A occurs, given that B has occurred. It is denoted $P(A|B)$.

Formal Definition

Given two events A and B in a Probability Space, with $P(B) > 0$, the conditional probability of A given B is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Intuition

- $P(A|B)$ represents the "updated" probability of A after learning that B occurred
- We restrict our attention to the subset of outcomes where B occurs
- We rescale probabilities so that $P(B|B) = 1$

Properties

1. **Range**: $0 \leq P(A|B) \leq 1$
2. **Certainty**: $P(B|B) = 1$ when $P(B) > 0$
3. **Impossibility**: $P(\emptyset|B) = 0$
4. **Subset**: If $A \subseteq B$, then $P(A|B) = \frac{P(A)}{P(B)}$

Multiplication Rule

Rearranging the definition gives:

$$P(A \cap B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A)$$

For multiple events:

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap \cdots \cap A_{n-1})$$

Independence

Events A and B are **independent** if and only if:

$$P(A|B) = P(A)$$

This means knowing B occurred doesn't change the probability of A.

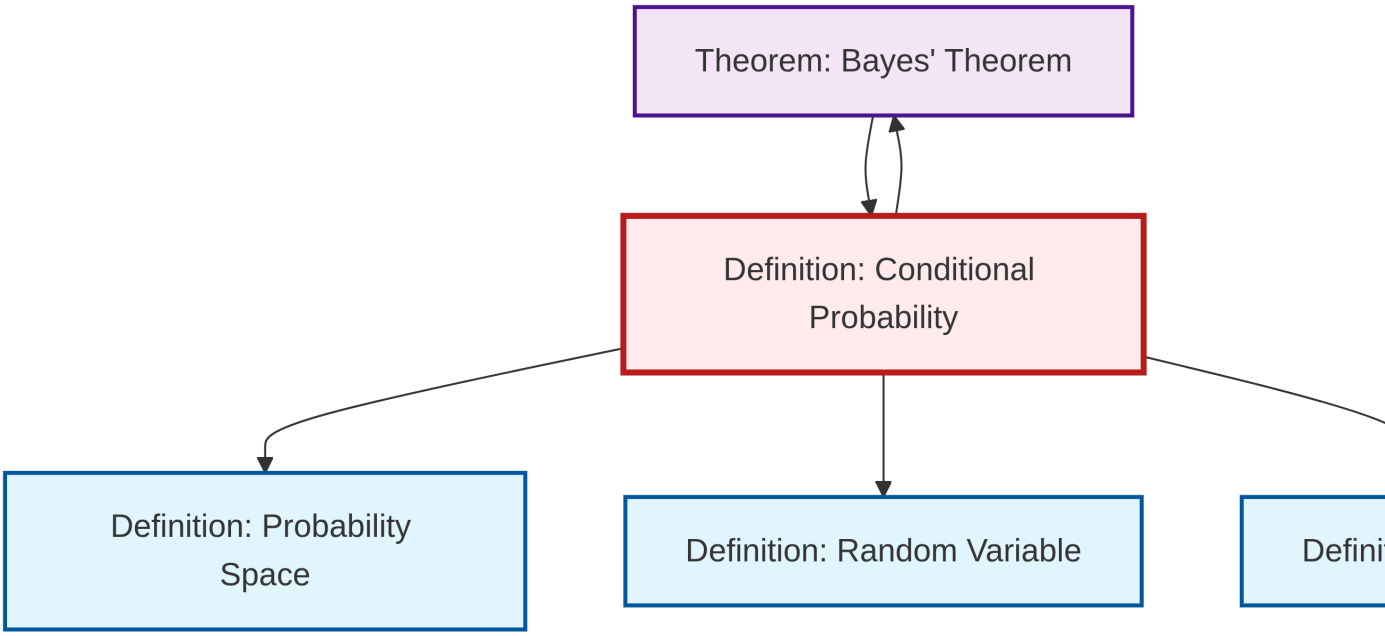
Examples

1. **Card Drawing**: Drawing an ace from a standard deck
   - $P(\text{Ace}) = \frac{4}{52} = \frac{1}{13}$
   - $P(\text{Ace}|\text{Red card}) = \frac{2}{26} = \frac{1}{13}$ (independent)
   - $P(\text{Ace}|\text{Face card}) = 0$ (mutually exclusive)
2. **Medical Testing**: Disease prevalence and test accuracy
   - If $P(\text{Disease}) = 0.01$ and test is 95% accurate
   - $P(\text{Positive}|\text{Disease})$ represents sensitivity

Related Concepts

- Bayes' Theorem: Relates $P(A|B)$ and $P(B|A)$
- Law of Total Probability: Uses conditional probabilities for calculation
- Conditional expectation extends this concept to Random Variables

Dependency Graph



Local dependency graph