Example: Polynomial Continuity

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Polynomial functions are continuous everywhere on $\mathbb{R}$, providing fundamental examples of continuous functions.

Statement

Every polynomial function $p : \mathbb{R} \to \mathbb{R}$ defined by:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_i \in \mathbb{R}$, is continuous at every point $c \in \mathbb{R}$.

Proof Strategy

We build up from simple cases:

1. **Constant function**: $f(x) = a_0$ is continuous
   - For any $\varepsilon > 0$, choose any $\delta > 0$
   - Then $|f(x) - f(c)| = |a_0 - a_0| = 0 < \varepsilon$
2. **Identity function**: $f(x) = x$ is continuous
   - For any $\varepsilon > 0$, choose $\delta = \varepsilon$
   - If $|x - c| < \delta$, then $|f(x) - f(c)| = |x - c| < \delta = \varepsilon$
3. **Products and sums**: Using the arithmetic properties of continuous functions:
   - If f and g are continuous at c, then $f + g$ and $f \cdot g$ are continuous at c
   - By induction, x^n is continuous for all $n \in \mathbb{N}$
   - Therefore, $a_i x^i$ is continuous for each term
   - The sum $p(x) = \sum_{i=0}^{n} a_i x^i$ is continuous

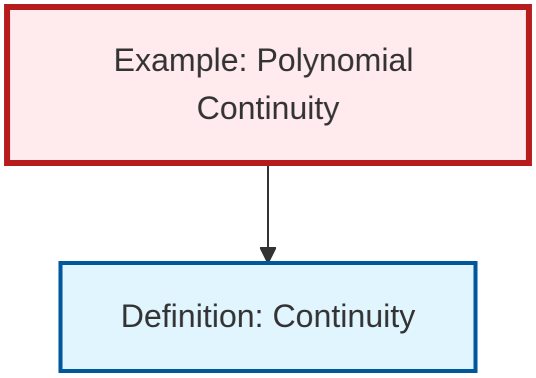
Specific Example

Consider $p(x) = x^3 - 2x + 1$. To verify continuity at $c = 1$: - $p(1) = 1 - 2 + 1 = 0$ - For sequences $x_n \to 1$: $p(x_n) = x_n^3 - 2x_n + 1 \to 1^3 - 2(1) + 1 = 0 = p(1)$

Importance

Polynomial continuity is crucial because: - Polynomials approximate other functions (Taylor series) - They form a dense subset of continuous functions (Weierstrass approximation) - They provide concrete examples for testing theorems about continuity

Dependency Graph



Local dependency graph