

Definition: Convergence

Convergence describes the behavior of a [Sequence](#) whose terms approach a specific value as the index increases. A convergent sequence gets arbitrarily close to its limit.

Formal Definition

A sequence (a_n) in a metric space (X, d) **converges** to a [Limit of a Sequence](#) $L \in X$ if:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n > N, \quad d(a_n, L) < \varepsilon$$

We write: $\lim_{n \rightarrow \infty} a_n = L$ - $a_n \rightarrow L$ as $n \rightarrow \infty$ - (a_n) converges to L

In Real Numbers

For a real sequence (a_n) converging to $L \in \mathbb{R}$:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n > N, \quad |a_n - L| < \varepsilon$$

Properties of Convergent Sequences

1. **Uniqueness:** A sequence can have at most one limit
2. **Boundedness:** Every convergent sequence is bounded
3. **Preservation under arithmetic:**
 - If $a_n \rightarrow A$ and $b_n \rightarrow B$, then:
 - $a_n + b_n \rightarrow A + B$
 - $a_n \cdot b_n \rightarrow A \cdot B$
 - $a_n/b_n \rightarrow A/B$ (if $B \neq 0$ and $b_n \neq 0$)

Types of Convergence

Pointwise Convergence

For function sequences $f_n : X \rightarrow Y$:

$$f_n \rightarrow f \text{ pointwise if } \forall x \in X, f_n(x) \rightarrow f(x)$$

Uniform Convergence

$$f_n \rightarrow f \text{ uniformly if } \sup_{x \in X} |f_n(x) - f(x)| \rightarrow 0$$

Absolute Convergence

For series: $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges

Examples

1. **Convergent:** $a_n = \frac{1}{n} \rightarrow 0$
2. **Convergent:** $a_n = \frac{n+1}{n} = 1 + \frac{1}{n} \rightarrow 1$
3. **Divergent:** $a_n = (-1)^n$ oscillates between -1 and 1
4. **Divergent:** $a_n = n$ grows without bound

Cauchy Criterion

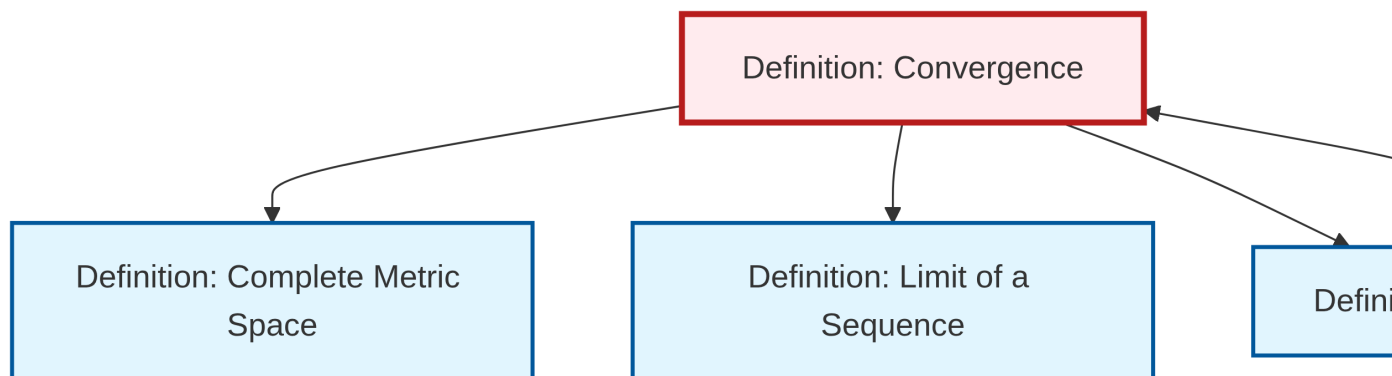
In [Complete Metric Spaces](#), a sequence converges if and only if it is a Cauchy sequence:

$$\forall \varepsilon > 0, \exists N : \forall m, n > N, \quad d(a_m, a_n) < \varepsilon$$

Applications

- Foundation of calculus and analysis
- Numerical methods and approximation
- Probability (law of large numbers)
- Functional analysis (operator convergence)

Dependency Graph



Local dependency graph