

## Definition: Vector Space

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A **vector space** (or linear space) over a field  $F$  is a **Set**  $V$  together with two **Binary Operations**:

1. **Vector addition:**  $+: V \times V \rightarrow V$
2. **Scalar multiplication:**  $\cdot: F \times V \rightarrow V$

such that the following axioms are satisfied:

#### Addition Axioms

For all  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ :

1. **Associativity:**  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
2. **Commutativity:**  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. **Identity element:** There exists  $\mathbf{0} \in V$  such that  $\mathbf{v} + \mathbf{0} = \mathbf{v}$  for all  $\mathbf{v} \in V$
4. **Inverse elements:** For each  $\mathbf{v} \in V$ , there exists  $-\mathbf{v} \in V$  such that  $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$

#### Scalar Multiplication Axioms

For all  $a, b \in F$  and  $\mathbf{u}, \mathbf{v} \in V$ :

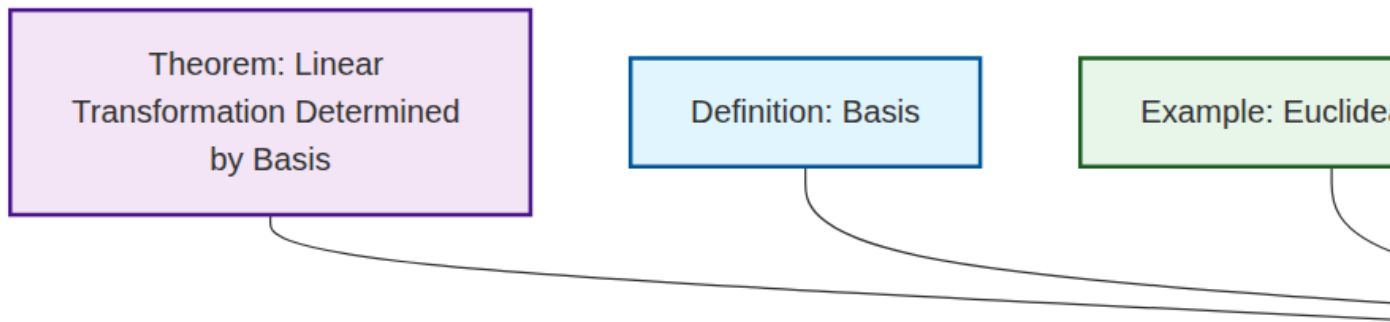
5. **Associativity:**  $a(b\mathbf{v}) = (ab)\mathbf{v}$
6. **Identity:**  $1\mathbf{v} = \mathbf{v}$ , where 1 is the multiplicative identity in  $F$
7. **Distributivity over vector addition:**  $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
8. **Distributivity over scalar addition:**  $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

Elements of  $V$  are called **vectors**, and elements of  $F$  are called **scalars**.

#### Remarks

- The vector space axioms ensure that  $(V, +)$  forms an abelian **Group**
- The field  $F$  determines the “type” of vector space (e.g., real vector space when  $F = \mathbb{R}$ )
- Vector spaces are fundamental structures in linear algebra and appear throughout mathematics

## Dependency Graph



Local dependency graph