

## Definition: Conditional Probability

The **conditional probability** of an **Event**  $A$  given event  $B$  is the probability that  $A$  occurs, given that  $B$  has occurred. It is denoted  $P(A|B)$ .

### Formal Definition

Given two events  $A$  and  $B$  in a **Probability Space**, with  $P(B) > 0$ , the conditional probability of  $A$  given  $B$  is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

### Intuition

- $P(A|B)$  represents the “updated” probability of  $A$  after learning that  $B$  occurred
- We restrict our attention to the subset of outcomes where  $B$  occurs
- We rescale probabilities so that  $P(B|B) = 1$

### Properties

1. **Range:**  $0 \leq P(A|B) \leq 1$
2. **Certainty:**  $P(B|B) = 1$  when  $P(B) > 0$
3. **Impossibility:**  $P(\emptyset|B) = 0$
4. **Subset:** If  $A \subseteq B$ , then  $P(A|B) = \frac{P(A)}{P(B)}$

### Multiplication Rule

Rearranging the definition gives:

$$P(A \cap B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A)$$

For multiple events:

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \dots P(A_n|A_1 \cap \dots \cap A_{n-1})$$

### Independence

Events  $A$  and  $B$  are **independent** if and only if:

$$P(A|B) = P(A)$$

This means knowing  $B$  occurred doesn't change the probability of  $A$ .

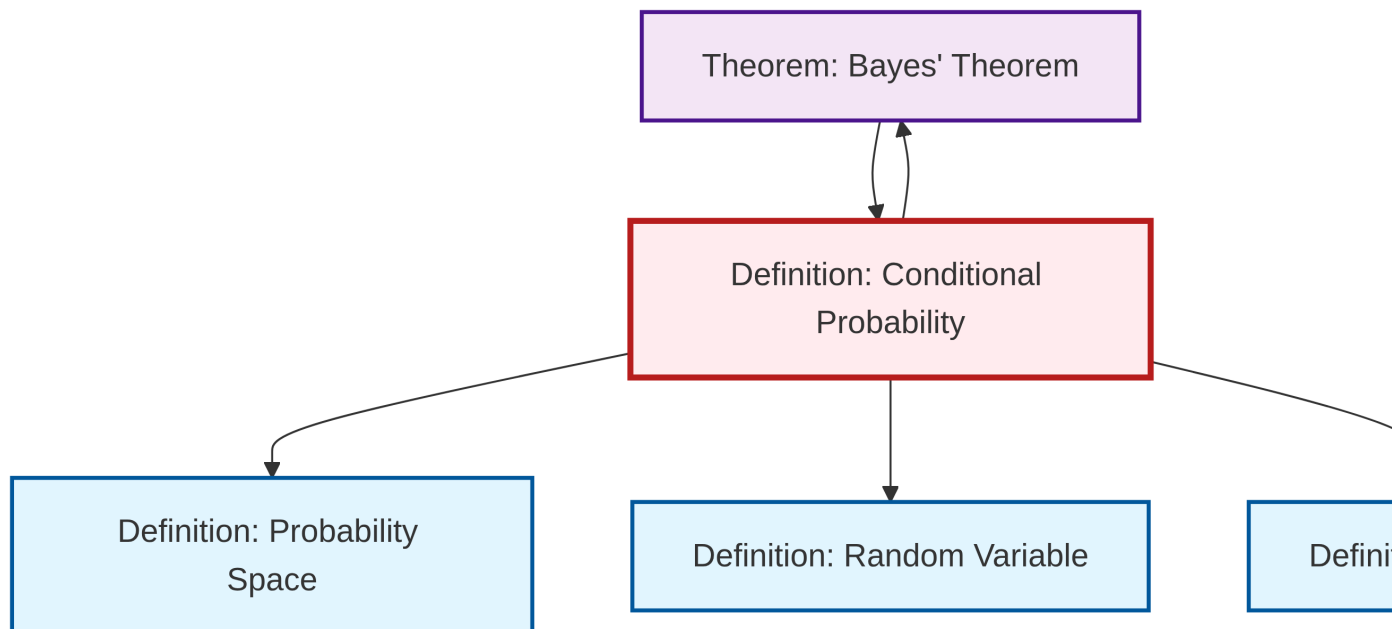
## Examples

1. **Card Drawing:** Drawing an ace from a standard deck
  - $P(\text{Ace}) = \frac{4}{52} = \frac{1}{13}$
  - $P(\text{Ace}|\text{Red card}) = \frac{2}{26} = \frac{1}{13}$  (independent)
  - $P(\text{Ace}|\text{Face card}) = 0$  (mutually exclusive)
2. **Medical Testing:** Disease prevalence and test accuracy
  - If  $P(\text{Disease}) = 0.01$  and test is 95% accurate
  - $P(\text{Positive}|\text{Disease})$  represents sensitivity

## Related Concepts

- **Bayes' Theorem:** Relates  $P(A|B)$  and  $P(B|A)$
- **Law of Total Probability:** Uses conditional probabilities for calculation
- Conditional expectation extends this concept to **Random Variables**

## Dependency Graph



Local dependency graph