

## Definition: Integral

The **integral** of a function represents the signed area under its curve, accumulation of quantities, or the inverse operation to differentiation. We focus on the Riemann integral for real-valued functions.

### Riemann Integral

#### Partition and Riemann Sum

Given  $f : [a, b] \rightarrow \mathbb{R}$  and a partition  $P = \{a = x_0 < x_1 < \dots < x_n = b\}$ :

A **Riemann sum** is:

$$S(f, P) = \sum_{i=1}^n f(c_i)(x_i - x_{i-1})$$

where  $c_i \in [x_{i-1}, x_i]$  is a sample point.

#### Definition of the Integral

The function  $f$  is **Riemann integrable** on  $[a, b]$  if the [Limit of a Sequence](#):

$$\int_a^b f(x) dx = \lim_{\|P\| \rightarrow 0} S(f, P)$$

exists and is independent of the choice of sample points, where  $\|P\| = \max_i(x_i - x_{i-1})$ .

### Fundamental Theorem of Calculus

#### Part I

If  $f$  is continuous on  $[a, b]$  and  $F(x) = \int_a^x f(t) dt$ , then:

$$F'(x) = f(x)$$

#### Part II

If  $f$  is continuous on  $[a, b]$  and  $F$  is an antiderivative of  $f$ :

$$\int_a^b f(x) dx = F(b) - F(a)$$

### Properties

#### 1. Linearity:

$$\bullet \int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

- $\int_a^b c f(x) dx = c \int_a^b f(x) dx$
2. **Additivity:**  $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$
  3. **Monotonicity:** If  $f(x) \leq g(x)$  on  $[a, b]$ , then  $\int_a^b f(x) dx \leq \int_a^b g(x) dx$
  4. **Reverse limits:**  $\int_a^b f(x) dx = - \int_b^a f(x) dx$

### Classes of Integrable Functions

- All **continuous** functions on  $[a, b]$
- All monotonic functions on  $[a, b]$
- All piecewise continuous functions with finitely many discontinuities

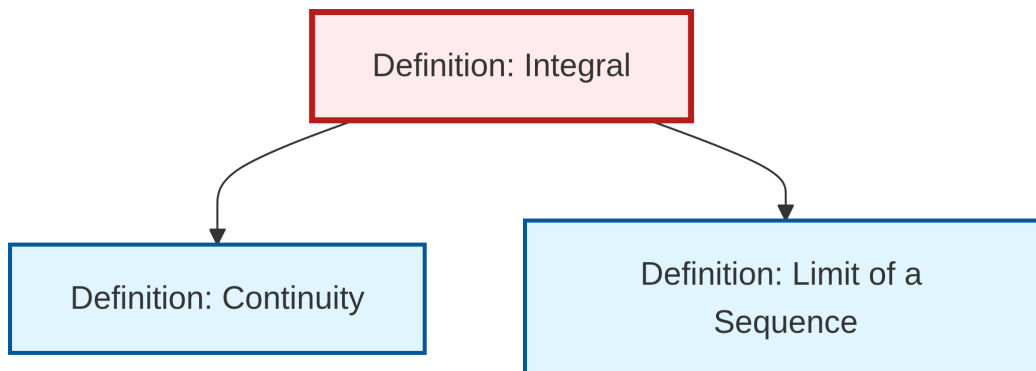
### Examples

1. **Power functions:**  $\int x^n dx = \frac{x^{n+1}}{n+1} + C$  (for  $n \neq -1$ )
2. **Exponential:**  $\int e^x dx = e^x + C$
3. **Trigonometric:**  $\int \sin x dx = -\cos x + C$
4. **Definite integral:**  $\int_0^1 x^2 dx = \frac{1}{3}$

### Applications

- Area and volume calculations
- Physics: work, center of mass, moments
- Probability: expected values, distributions
- Differential equations: solution methods

### Dependency Graph



Local dependency graph