Definition: Composition

Composition is the fundamental operation in a Category that combines two compatible Morphisms to produce a third morphism.

Formal Definition

In a category $\mathcal{C}$, given morphisms: - $f : A \to B$ - $g : B \to C$

Their **composition** is a morphism:

$$g \circ f : A \to C$$

The composition is defined whenever the codomain of f equals the domain of g.

Axioms

Composition must satisfy:

1. Associativity

For morphisms $f : A \to B$, $g : B \to C$, $h : C \to D$:

$$(h \circ g) \circ f = h \circ (g \circ f)$$

2. Identity Laws

For any morphism $f : A \to B$ and identity morphisms id_A, id_B: - **Left identity**: $\mathrm{id}_B \circ f = f$ - **Right identity**: $f \circ \mathrm{id}_A = f$

Notation

Various notations for composition: - $g \circ f$ (standard, read "g after f") - gf (shortened form) - $f; g$ (diagrammatic order, "f then g") - fg (sometimes used, but can be confusing)

Examples

Set Category

In **Set**, morphisms are functions: - If $f : X \to Y$ and $g : Y \to Z$ - Then $(g \circ f)(x) = g(f(x))$ for all $x \in X$

Linear Transformations

In the category of vector spaces: - Morphisms are linear maps - Composition is matrix multiplication (in finite dimensions)

Group Homomorphisms

In **Grp** (category of groups): - If $\phi : G \to H$ and $\psi : H \to K$ are homomorphisms - $(\psi \circ \phi)(g) = \psi(\phi(g))$ is also a homomorphism

Properties

1. **Non-commutativity**: Generally $g \circ f \neq f \circ g$
2. **Partial operation**: Not all pairs of morphisms can be composed
3. **Preservation of structure**: Composition preserves the categorical structure

Commutative Diagrams

Composition is often visualized using commutative diagrams:

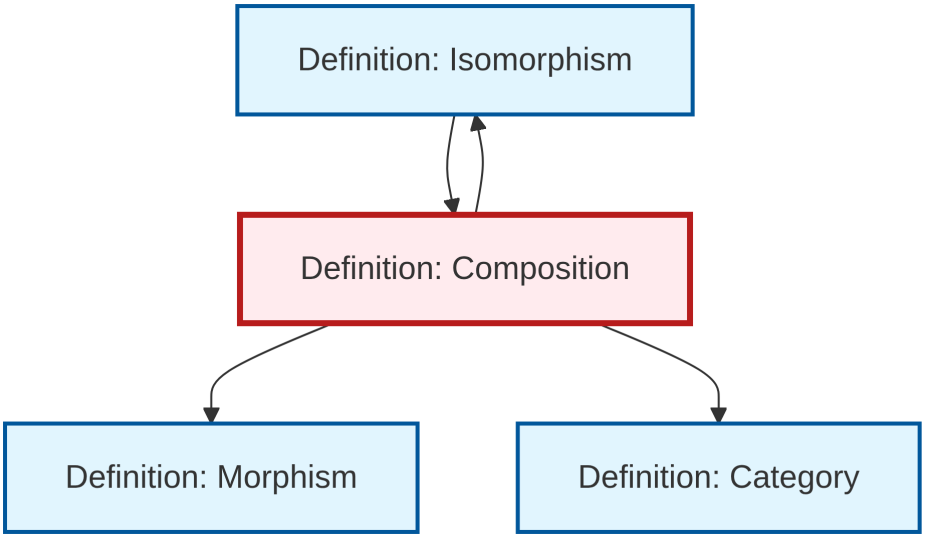
```
A ---f---> B
|          |
h          g
|          |
v          v
C ---k---> D
```

This diagram commutes if $k \circ h = g \circ f$

Related Concepts

- Isomorphism: Morphisms with two-sided inverses under composition
- Endomorphisms: Morphisms that can be composed with themselves
- Monoids: Single-object categories where all morphisms compose

Dependency Graph



Local dependency graph