Theorem: Field Characteristic

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Every Field has characteristic either 0 or a Prime Number number.

Definitions

The **characteristic** of a field F, denoted $\mathrm{char}(F)$, is the smallest positive integer n such that:

$$\underbrace{1 + 1 + \ldots + 1}_{n \text{ times}} = 0$$

If no such n exists, we say $\mathrm{char}(F) = 0$.

Statement

For any field F: 1. If $\mathrm{char}(F) \neq 0$, then $\mathrm{char}(F) = p$ for some prime p 2. If $\mathrm{char}(F) = 0$, then F contains a copy of $\mathbb{Q}$ 3. If $\mathrm{char}(F) = p$, then F contains a copy of $\mathbb{F}_p$

Proof Sketch

Suppose $\mathrm{char}(F) = n > 0$. If $n = ab$ with $1 < a, b < n$, then:

$$(a \cdot 1)(b \cdot 1) = (ab) \cdot 1 = n \cdot 1 = 0$$

Since fields have no zero divisors, either $a \cdot 1 = 0$ or $b \cdot 1 = 0$, contradicting the minimality of n. Therefore n must be prime.

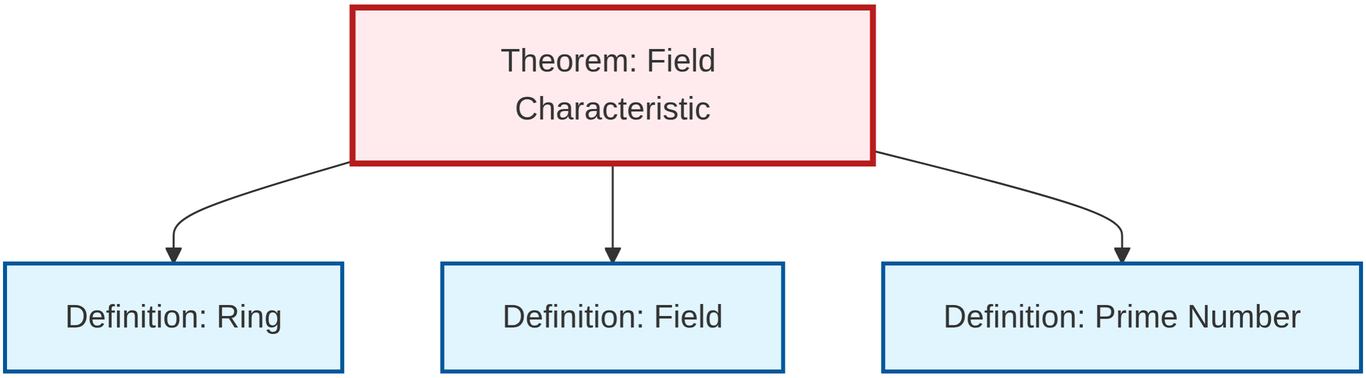
Examples

- $\mathrm{char}(\mathbb{Q}) = 0$, $\mathrm{char}(\mathbb{R}) = 0$, $\mathrm{char}(\mathbb{C}) = 0$
- $\mathrm{char}(\mathbb{F}_p) = p$ for any prime p
- $\mathrm{char}(GF(p^n)) = p$ for any prime power p^n

Consequences

- In characteristic p: $(a + b)^p = a^p + b^p$ (Freshman's Dream)
- The Frobenius map $x \mapsto x^p$ is a field homomorphism in characteristic p

Dependency Graph



Local dependency graph