Definition: Isomorphism

An **isomorphism** in a Category is a Morphism that has a two-sided inverse. Isomorphic objects are considered "the same" from the categorical perspective.

Formal Definition

A morphism $f : A \to B$ in a category $\mathcal{C}$ is an **isomorphism** if there exists a morphism $g : B \to A$ such that: 1. $g \circ f = \mathrm{id}_A$ 2. $f \circ g = \mathrm{id}_B$

The morphism g is called the **inverse** of f, denoted f^{-1}.

Properties

1. **Uniqueness of Inverse**: If f has an inverse, it is unique
2. **Symmetry**: If $f : A \to B$ is an isomorphism, then $f^{-1} : B \to A$ is also an isomorphism
3. **Composition**: The Composition of isomorphisms is an isomorphism:
   - If $f : A \to B$ and $g : B \to C$ are isomorphisms
   - Then $g \circ f : A \to C$ is an isomorphism with $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Isomorphic Objects

Objects A and B are **isomorphic**, written $A \cong B$, if there exists an isomorphism $f : A \to B$.

Isomorphism is an equivalence relation: - **Reflexive**: $A \cong A$ via id_A - **Symmetric**: $A \cong B$ implies $B \cong A$ - **Transitive**: $A \cong B$ and $B \cong C$ implies $A \cong C$

Examples in Different Categories

Set

- Isomorphisms are bijective functions
- Two sets are isomorphic iff they have the same cardinality

Group

- Isomorphisms are bijective group homomorphisms
- Example: $(\mathbb{Z}, +) \cong (2\mathbb{Z}, +)$ via $n \mapsto 2n$

Top (Topological Spaces)

- Isomorphisms are homeomorphisms (continuous bijections with continuous inverse)
- Example: Open interval $(0, 1)$ is homeomorphic to $\mathbb{R}$

Vect (Vector Spaces)

- Isomorphisms are invertible linear transformations
- Finite-dimensional spaces are isomorphic iff they have the same dimension

Special Types

Automorphism

An isomorphism from an object to itself: $f : A \to A$

Natural Isomorphism

An isomorphism in the functor category (see Natural Transformation)

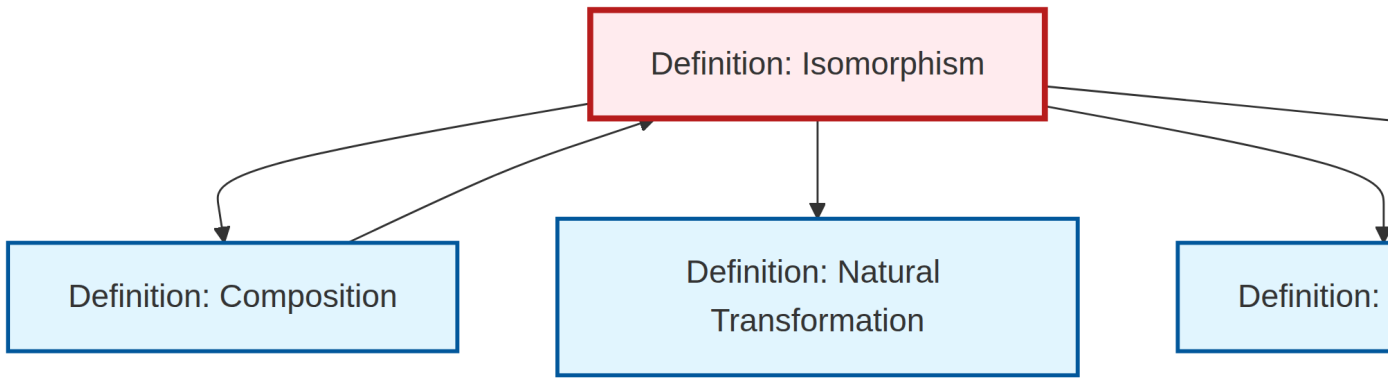
Categorical Perspective

- Isomorphism captures the idea of "sameness" in a category
- Category theory studies properties invariant under isomorphism
- The principle of equivalence: isomorphic objects are indistinguishable within the category

Non-Examples

- In **Set**: Injective but non-surjective functions
- In **Group**: Non-bijective homomorphisms
- Continuous functions that aren't homeomorphisms

Dependency Graph



Local dependency graph