Definition: Kernel

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Let $T : V \to W$ be a Linear Transformation between s. The **kernel** (or **null space**) of T is the set of all vectors in V that map to the zero vector in W:

$$\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}_W\}$$

Alternative Names

The kernel is also known as: - Null space (denoted $\text{null}(T)$ or $N(T)$) - The pre-image of zero: $T^{-1}(\{\mathbf{0}_W\})$

Properties

1. **Subspace**: $\ker(T)$ is always a subspace of V
   - Contains $\mathbf{0}_V$ since $T(\mathbf{0}_V) = \mathbf{0}_W$
   - Closed under addition: if $\mathbf{u}, \mathbf{v} \in \ker(T)$, then $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{0} + \mathbf{0} = \mathbf{0}$
   - Closed under scalar multiplication: if $\mathbf{v} \in \ker(T)$ and $a \in F$, then $T(a\mathbf{v}) = aT(\mathbf{v}) = a\mathbf{0} = \mathbf{0}$
2. **Injectivity criterion**: T is injective (one-to-one) if and only if $\ker(T) = \{\mathbf{0}_V\}$
3. **Dimension**: The dimension of $\ker(T)$ is called the **nullity** of T

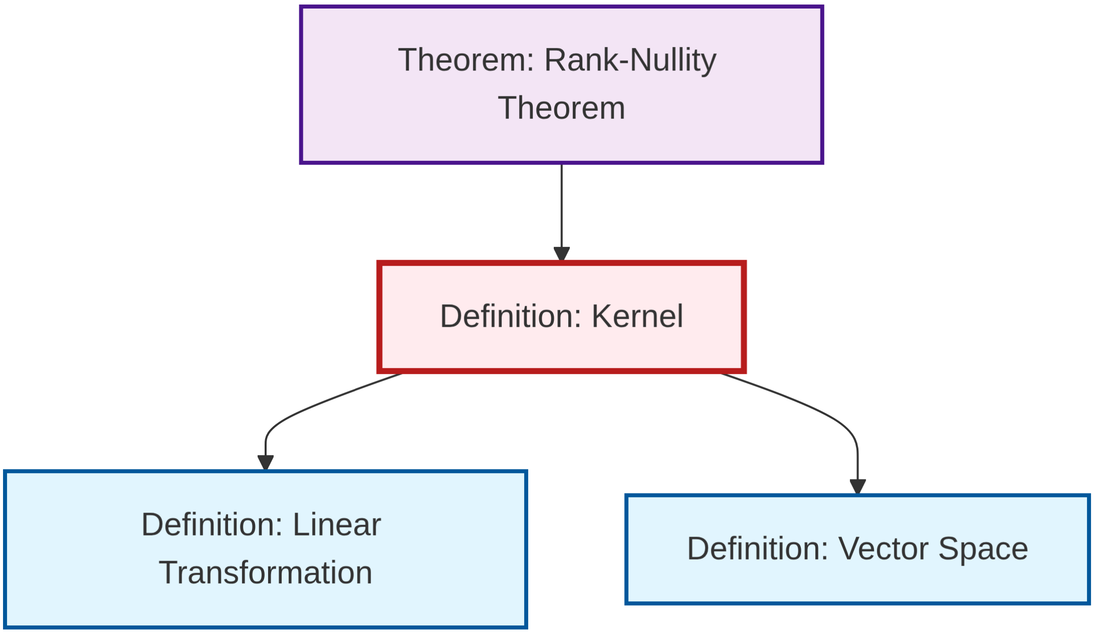
Example

For a matrix transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T_A(\mathbf{x}) = A\mathbf{x}$:

$$\ker(T_A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$$

This is precisely the solution set of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$.

Dependency Graph



Local dependency graph