

# Axiom: Mathematical Induction

## Axiom: Mathematical Induction

The principle of mathematical induction is a fundamental axiom for the natural numbers that allows us to prove statements about all natural numbers by verifying a base case and an inductive step.

### Statement

Let  $P(n)$  be a property defined for natural numbers. If: 1. **Base case:**  $P(1)$  is true 2. **Inductive step:** For all  $k \in \mathbb{N}$ , if  $P(k)$  is true, then  $P(k+1)$  is true

Then  $P(n)$  is true for all  $n \in \mathbb{N}$ .

### Formal Statement

$$[P(1) \wedge \forall k \in \mathbb{N}(P(k) \implies P(k+1))] \implies \forall n \in \mathbb{N} P(n)$$

### Variants

#### Strong Induction

If: 1.  $P(1)$  is true 2. For all  $k \in \mathbb{N}$ , if  $P(1), P(2), \dots, P(k)$  are all true, then  $P(k+1)$  is true

Then  $P(n)$  is true for all  $n \in \mathbb{N}$ .

### Well-Ordering Principle

Every non-empty subset of  $\mathbb{N}$  has a least element. This is equivalent to the induction axiom.

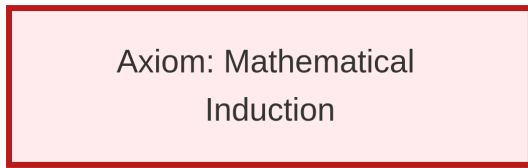
### Intuition

Induction works like climbing an infinite ladder: - We can reach the first rung (base case) - From any rung, we can reach the next rung (inductive step) - Therefore, we can reach any rung

### Role in Mathematics

Mathematical induction is: - Part of the Peano axioms that define natural numbers - Essential for proving statements about infinite sets - The foundation for recursive definitions and algorithms - Generalizable to other well-ordered sets (transfinite induction)

## Dependency Graph



Local dependency graph