Definition: Field

Field

A **field** is a Ring in which every non-zero element has a multiplicative inverse.

Formal Definition

A field $(F, +, \cdot)$ is a commutative ring with unity such that $(F \setminus \{0\}, \cdot)$ forms an abelian Group.

Explicitly, a field satisfies:

1. $(F, +)$ is an abelian group with identity 0
2. $(F \setminus \{0\}, \cdot)$ is an abelian group with identity 1
3. Distributivity: $a \cdot (b + c) = a \cdot b + a \cdot c$
4. $0 \neq 1$ (non-triviality)

Properties

- Every field is an integral domain
- Every finite integral domain is a field
- Fields have no zero divisors: if $ab = 0$, then $a = 0$ or $b = 0$
- Every non-zero element a has a unique inverse a^{-1} such that $a \cdot a^{-1} = 1$

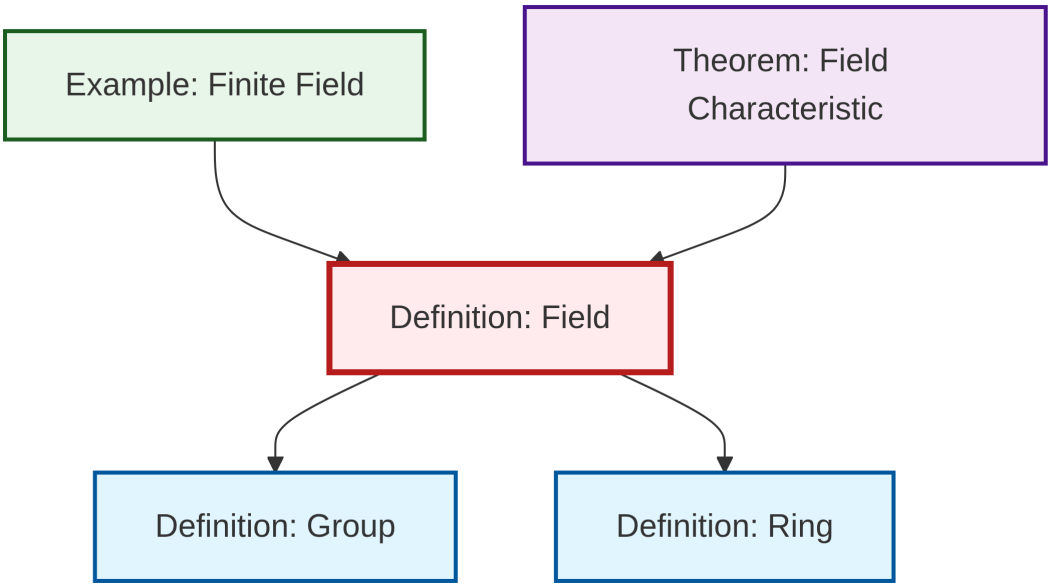
Examples

- $\mathbb{Q}$ - rational numbers
- $\mathbb{R}$ - real numbers
- $\mathbb{C}$ - complex numbers
- $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ - integers modulo prime p
- $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ - field extension

Non-Examples

- $\mathbb{Z}$ is not a field (no multiplicative inverse for 2)
- $\mathbb{Z}/6\mathbb{Z}$ is not a field (has zero divisors)

Dependency Graph



Local dependency graph