

## Theorem: Bayes' Theorem

**Bayes' Theorem** provides a way to calculate **Conditional Probability** in the “reverse” direction. It relates  $P(A|B)$  to  $P(B|A)$ , allowing us to update beliefs based on new evidence.

### Statement

For **Events**  $A$  and  $B$  with  $P(A) > 0$  and  $P(B) > 0$ :

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

### Alternative Forms

#### Using Law of Total Probability

If  $\{A_1, A_2, \dots, A_n\}$  form a partition of the sample space:

$$P(A_i|B) = \frac{P(B|A_i) \cdot P(A_i)}{\sum_{j=1}^n P(B|A_j) \cdot P(A_j)}$$

### Binary Case

For event  $A$  and its complement  $A^c$ :

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B|A) \cdot P(A) + P(B|A^c) \cdot P(A^c)}$$

### Proof

From the definition of conditional probability: 1.  $P(A|B) = \frac{P(A \cap B)}{P(B)}$  2.  $P(B|A) = \frac{P(A \cap B)}{P(A)}$

From (2):  $P(A \cap B) = P(B|A) \cdot P(A)$

Substituting into (1):

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

### Terminology

- $P(A)$ : **Prior probability** of  $A$
- $P(A|B)$ : **Posterior probability** of  $A$  given evidence  $B$
- $P(B|A)$ : **Likelihood** of observing  $B$  given  $A$
- $P(B)$ : **Marginal probability** or normalizing constant

## Applications

### Medical Diagnosis

- $A$ : Patient has disease
- $B$ : Test is positive
- Calculate  $P(\text{Disease}|\text{Positive test})$  from known test accuracy

### Machine Learning

- Naive Bayes classifiers
- Bayesian inference and parameter estimation
- Updating model beliefs with new data

### Example: Medical Test

Given: - Disease prevalence:  $P(D) = 0.001$  (0.1%) - Test sensitivity:  $P(+|D) = 0.99$  (99%) - Test specificity:  $P(-|D^c) = 0.95$ , so  $P(+|D^c) = 0.05$

Find  $P(D|+)$ :

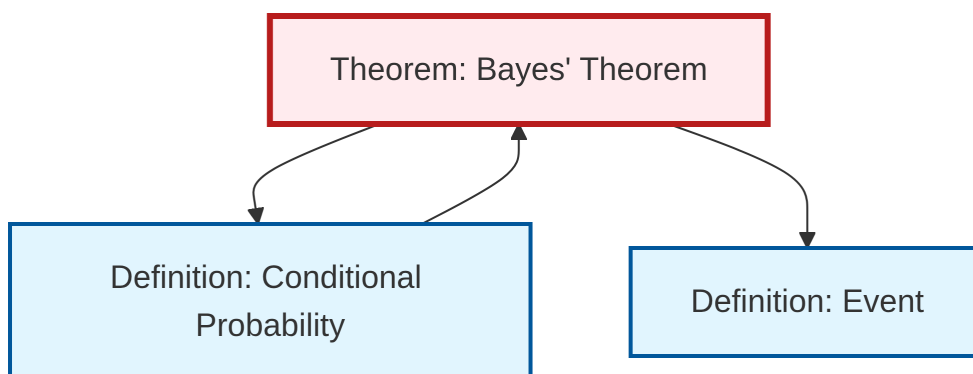
$$P(D|+) = \frac{0.99 \times 0.001}{0.99 \times 0.001 + 0.05 \times 0.999} = \frac{0.00099}{0.05094} \approx 0.0194$$

Despite a positive test, the probability of having the disease is only about 1.94%!

### Bayesian Updating

Bayes' theorem enables iterative belief updating: 1. Start with prior  $P(A)$  2. Observe evidence  $B$  3. Update to posterior  $P(A|B)$  4. Use posterior as new prior for next observation

### Dependency Graph



Local dependency graph