Definition: Natural Transformation

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A **natural transformation** provides a way to transform one Functor into another while respecting the internal structure of the categories involved.

Formal Definition

Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors between categories $\mathcal{C}$ and $\mathcal{D}$. A natural transformation $\eta : F \Rightarrow G$ consists of:

- For each object X in $\mathcal{C}$, a Morphism $\eta_X : F(X) \to G(X)$ in $\mathcal{D}$

Such that for every morphism $f : X \to Y$ in $\mathcal{C}$, the following diagram commutes:

```
F(X) --F(f)--> F(Y)
 |              |
_X             _Y
 ↓              ↓
G(X) --G(f)--> G(Y)
```

That is, $\eta_Y \circ F(f) = G(f) \circ \eta_X$.

Components and Naturality

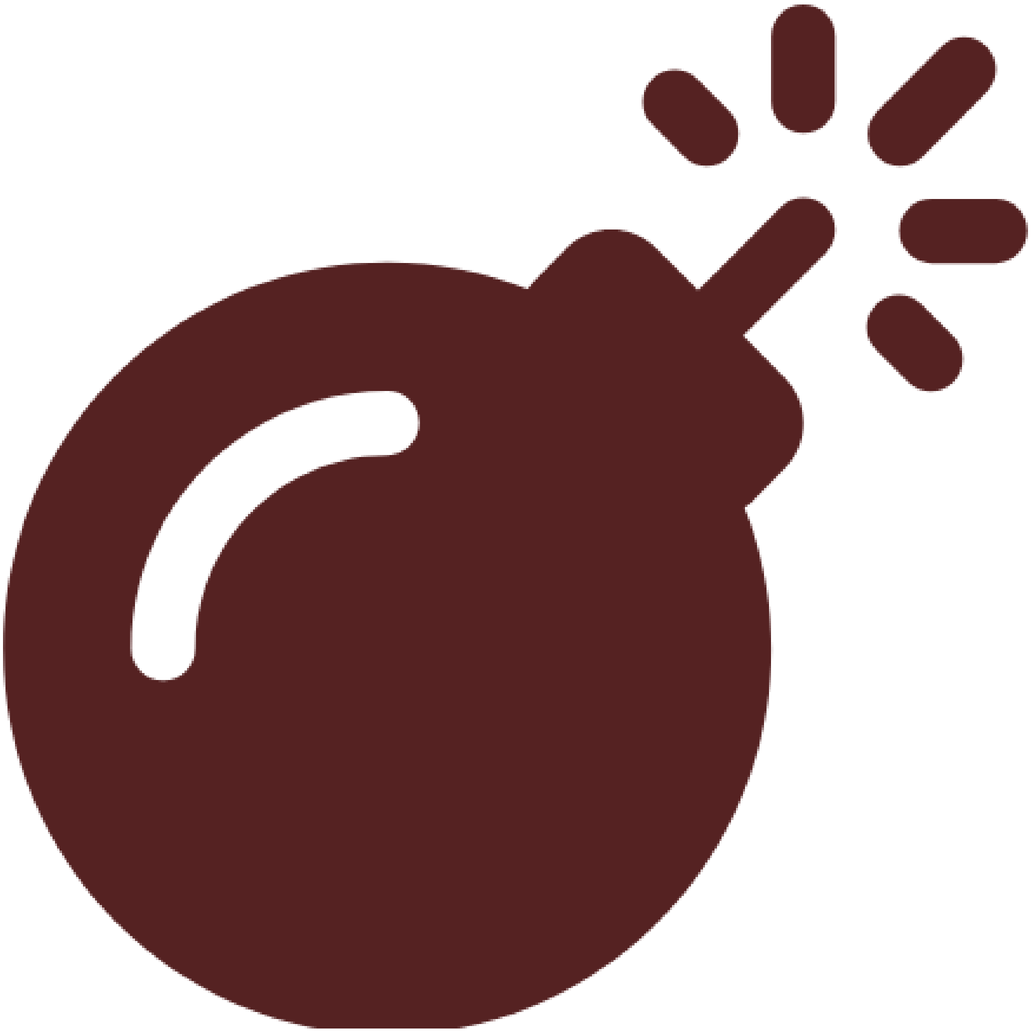
- Each morphism η_X is called a **component** of the natural transformation
- The commutativity condition is called the **naturality condition**
- When this condition holds, we say that η is **natural in X**

Special Cases

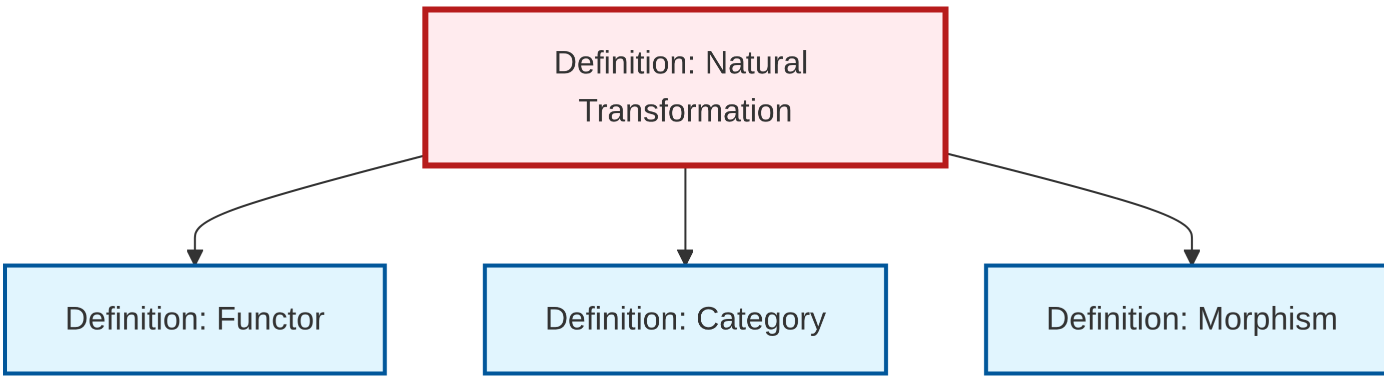
1. **Natural Isomorphism**: A natural transformation where every component η_X is an isomorphism
2. **Identity Natural Transformation**: For any functor F, the identity transformation id_F has components $(\mathrm{id}_F)_X = \mathrm{id}_{F(X)}$

Composition

Natural transformations can be composed: - **Vertical Composition**: If $\eta : F \Rightarrow G$ and $\mu : G \Rightarrow H$, then $\mu \circ \eta : F \Rightarrow H$ - **Horizontal Composition**: Natural transformations can also be composed with functors



Dependency Graph



Local dependency graph