

# Theorem: Cantor's Theorem

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For any set  $A$ , there is no surjection from  $A$  to its power set  $\mathcal{P}(A)$ . In particular,  $|A| < |\mathcal{P}(A)|$ .

### Statement

Let  $A$  be any set. Then there exists no function  $f : A \rightarrow \mathcal{P}(A)$  that is surjective (onto).

### Proof

We prove this by contradiction using Cantor's diagonal argument.

Suppose  $f : A \rightarrow \mathcal{P}(A)$  is surjective. Define the set:

$$B = \{x \in A : x \notin f(x)\}$$

Note that  $B \subseteq A$ , so  $B \in \mathcal{P}(A)$ .

Since  $f$  is assumed to be surjective, there must exist some  $a \in A$  such that  $f(a) = B$ .

Now we ask: Is  $a \in B$ ?

- If  $a \in B$ , then by definition of  $B$ , we have  $a \notin f(a) = B$ . Contradiction!
- If  $a \notin B$ , then  $a \in f(a)$ , which by definition of  $B$  means  $a \in B$ . Contradiction!

Both cases lead to a contradiction. Therefore, no such surjection  $f$  can exist.

### Consequences

1. **Infinite hierarchy:** Starting with any infinite set, we can construct an infinite sequence of sets with strictly increasing cardinalities
2. **No largest cardinal:** There is no set of all sets
3. **Uncountability:**  $\mathcal{P}(\mathbb{N})$  is uncountable (since  $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$ )

### Special Case

For finite sets: If  $|A| = n$ , then  $|\mathcal{P}(A)| = 2^n > n$  for all  $n \geq 0$ .

### Mermaid Diagram

```
graph TD
    A[Cantor's Theorem] --> B[No surjection  $A \rightarrow \mathcal{P}(A)$ ]
    B --> C[Diagonal Argument]
    C --> D[Construct  $B = \{x : x \notin f(x)\}$ ]
    D --> E[Contradiction]
```

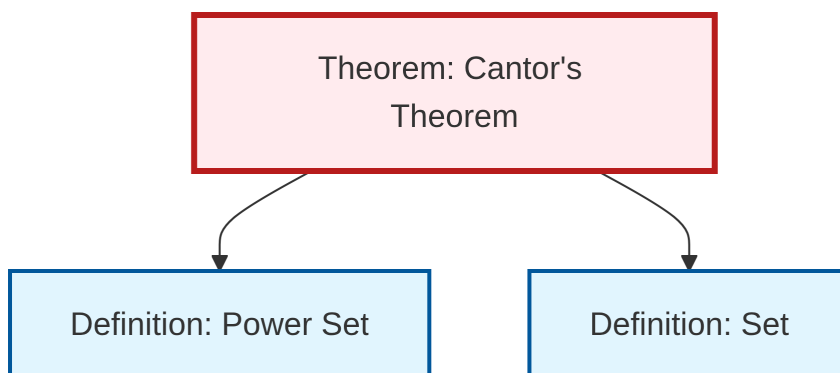
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A --> F[|A| < |P(A)|]
F --> G[Infinite Hierarchy]
F --> H[P( ) Uncountable]

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style E fill:#fbb,stroke:#333,stroke-width:2px
style F fill:#bfb,stroke:#333,stroke-width:2px

```

## Dependency Graph



Local dependency graph