Example: Integers under Addition

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The integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ form a Group under the Binary Operation of addition.

Verification of Group Axioms

We denote this group as $(\mathbb{Z}, +)$ and verify each group axiom:

1. Closure

For any $a, b \in \mathbb{Z}$, the sum $a + b$ is also an integer. This follows from the definition of integer addition.

2. Associativity

For all $a, b, c \in \mathbb{Z}$:

$$(a + b) + c = a + (b + c)$$

This is a fundamental property of integer addition.

3. Identity Element

The integer 0 serves as the identity element: - For any $a \in \mathbb{Z}$: $a + 0 = 0 + a = a$

By Uniqueness of Identity, this identity element is unique.

4. Inverse Elements

For each $a \in \mathbb{Z}$, the integer $-a$ is its inverse: - $a + (-a) = (-a) + a = 0$

Additional Properties

The group $(\mathbb{Z}, +)$ has several important additional properties:

1. **Commutativity**: $a + b = b + a$ for all $a, b \in \mathbb{Z}$
   - This makes $(\mathbb{Z}, +)$ an **abelian group**
2. **Infinite order**: The group has infinitely many elements
3. **Cyclic**: The group is generated by the single element 1:
   - Every integer can be written as $n \cdot 1$ for some $n \in \mathbb{Z}$

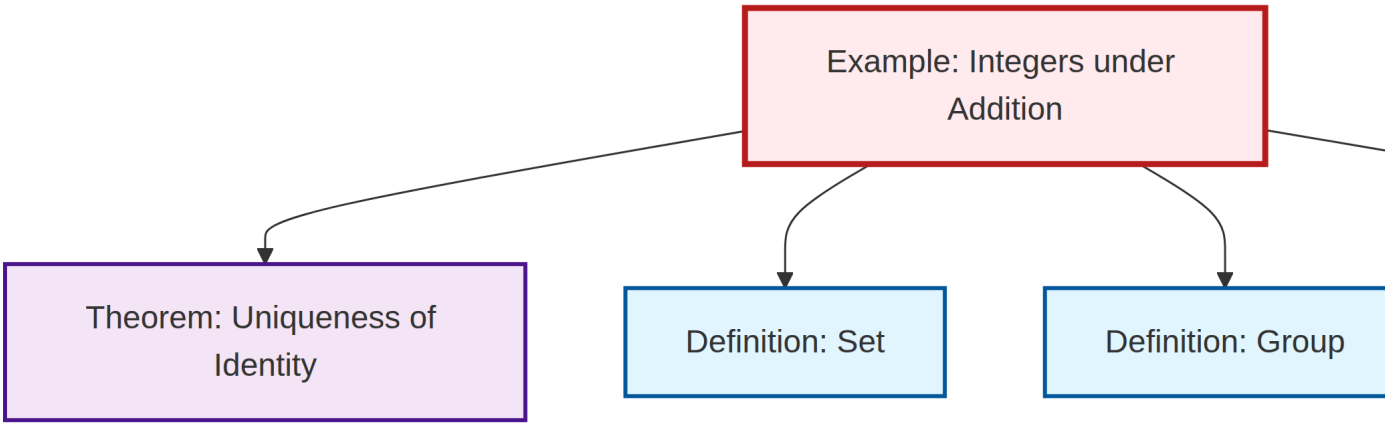
Subgroups

Notable subgroups of $(\mathbb{Z}, +)$ include: - $n\mathbb{Z} = \{nk : k \in \mathbb{Z}\}$ for any $n \in \mathbb{N}$ - For example, $2\mathbb{Z}$ is the subgroup of even integers

Related Examples

- The rationals $(\mathbb{Q}, +)$ form a group containing $(\mathbb{Z}, +)$ as a subgroup
- The integers modulo n, denoted $\mathbb{Z}/n\mathbb{Z}$, form finite groups

Dependency Graph



Local dependency graph