Theorem: Cantor's Theorem

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For any set A, there is no surjection from A to its power set $\mathcal{P}(A)$. In particular, $|A| < |\mathcal{P}(A)|$.

Statement

Let A be any set. Then there exists no function $f : A \to \mathcal{P}(A)$ that is surjective (onto).

Proof

We prove this by contradiction using Cantor's diagonal argument.

Suppose $f : A \to \mathcal{P}(A)$ is surjective. Define the set:

$$B = \{x \in A : x \notin f(x)\}$$

Note that $B \subseteq A$, so $B \in \mathcal{P}(A)$.

Since f is assumed to be surjective, there must exist some $a \in A$ such that $f(a) = B$.

Now we ask: Is $a \in B$?

- If $a \in B$, then by definition of B, we have $a \notin f(a) = B$. Contradiction!
- If $a \notin B$, then $a \notin f(a)$, which by definition of B means $a \in B$. Contradiction!

Both cases lead to a contradiction. Therefore, no such surjection f can exist.

Consequences

1. **Infinite hierarchy**: Starting with any infinite set, we can construct an infinite sequence of sets with strictly increasing cardinalities
2. **No largest cardinal**: There is no set of all sets
3. **Uncountability**: $\mathcal{P}(\mathbb{N})$ is uncountable (since $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$)

Special Case

For finite sets: If $|A| = n$, then $|\mathcal{P}(A)| = 2^n > n$ for all $n \geq 0$.

Mermaid Diagram

```
graph TD
    A[Cantor's Theorem] --> B[No surjection A → P(A)]
    B --> C[Diagonal Argument]
    C --> D[Construct B = {x: x   f(x)}]
    D --> E[Contradiction]
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    A --> F[|A| < |P(A)|]
    F --> G[Infinite Hierarchy]
    F --> H[P( ) Uncountable]

    style A fill:#f9f,stroke:#333,stroke-width:2px
    style B fill:#bbf,stroke:#333,stroke-width:2px
    style E fill:#fbb,stroke:#333,stroke-width:2px
    style F fill:#bfb,stroke:#333,stroke-width:2px
```
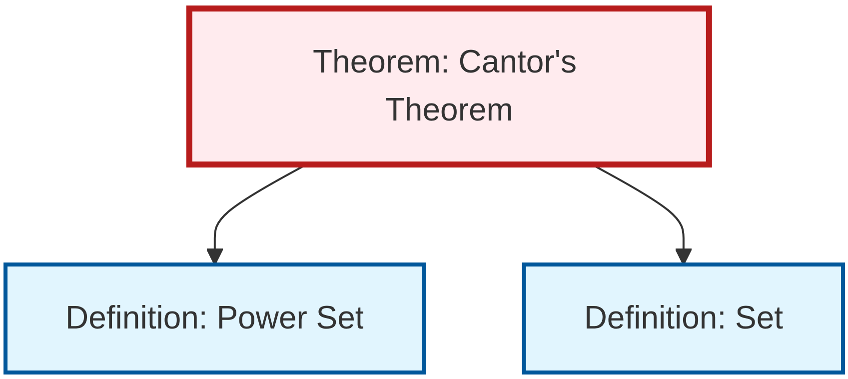

Dependency Graph



Local dependency graph