Example: Pascal's Triangle

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Pascal's Triangle is a triangular array of Binomial Coefficient s that provides a visual representation of many combinatorial properties.

Construction

The triangle is constructed with the following rules: 1. The top row (row 0) contains just the number 1 2. Each subsequent row starts and ends with 1 3. Each interior number is the sum of the two numbers above it

```
Row 0:          1
Row 1:        1   1
Row 2:      1   2   1
Row 3:    1   3   3   1
Row 4:  1   4   6   4   1
Row 5: 1   5  10  10   5   1
```

Connection to Binomial Coefficients

The entry in row n and position k (counting from 0) is exactly $\binom{n}{k}$:

- Row 0: $\binom{0}{0} = 1$
- Row 1: $\binom{1}{0} = 1$, $\binom{1}{1} = 1$
- Row 2: $\binom{2}{0} = 1$, $\binom{2}{1} = 2$, $\binom{2}{2} = 1$
- Row 3: $\binom{3}{0} = 1$, $\binom{3}{1} = 3$, $\binom{3}{2} = 3$, $\binom{3}{3} = 1$

Key Properties

1. **Symmetry**: Each row is symmetric, reflecting $\binom{n}{k} = \binom{n}{n-k}$
2. **Row Sums**: The sum of row n is 2^n:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

3. **Hockey Stick Pattern**: The sum along any diagonal equals the entry below and to the side:

$$\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$$

4. **Fibonacci Numbers**: The sums of the shallow diagonals give Fibonacci numbers

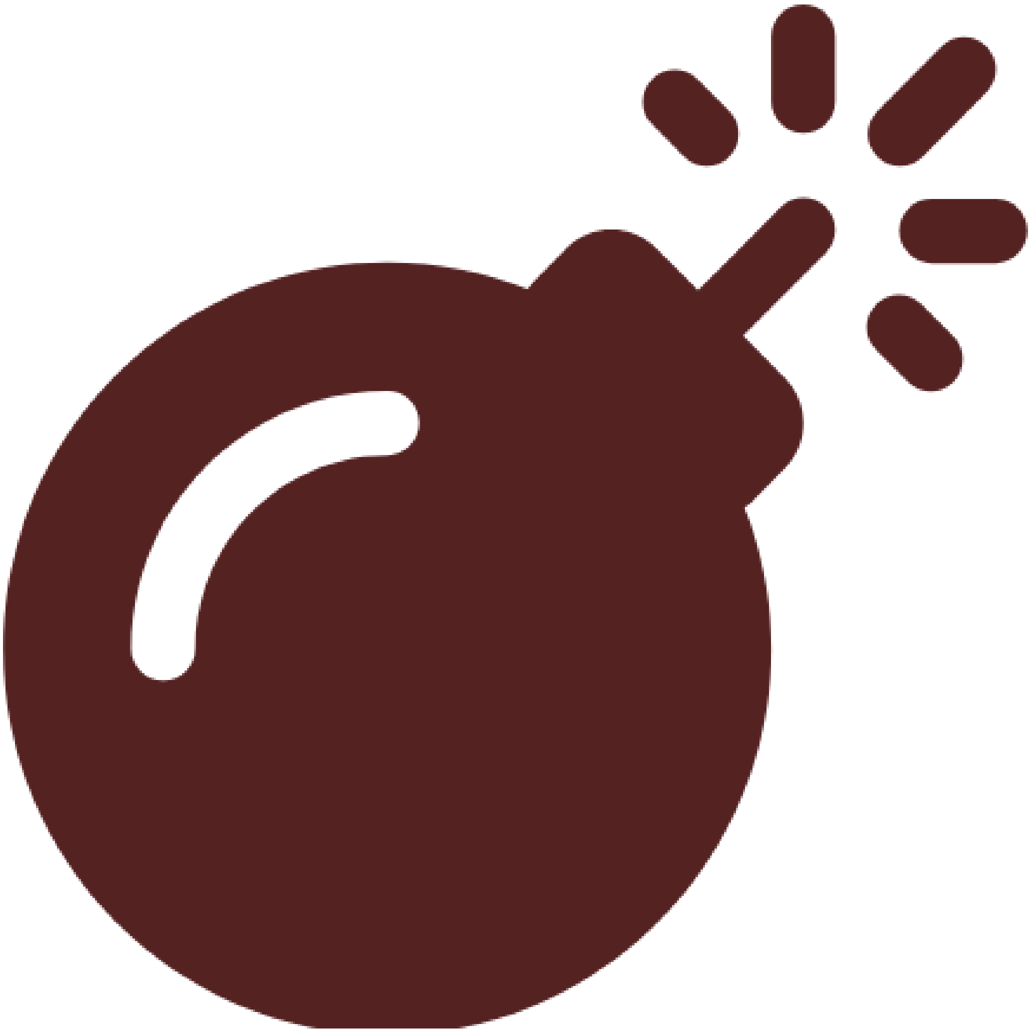
Applications

1. **Binomial Expansion**: Row n gives the coefficients of $(x + y)^n$
2. **Probability**: Used in calculating binomial probabilities
3. **Combinatorial Identities**: Visual proofs of many identities
4. **Number Theory**: Connections to divisibility properties

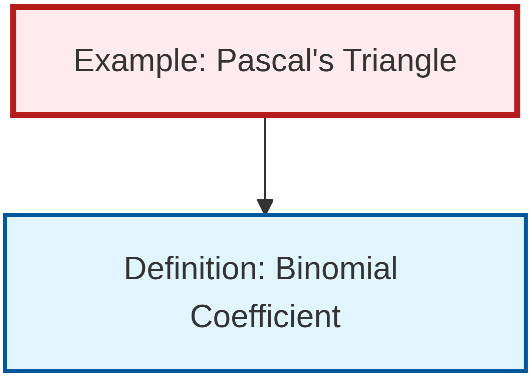
Generating Function

The entries in Pascal's Triangle can be generated using:

$$\frac{1}{(1-x)^{n+1}} = \sum_{k=0}^{\infty} \binom{n+k}{k} x^k$$



Dependency Graph



Local dependency graph