

Theorem: Pythagorean Theorem

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In a [Euclidean space](#), two vectors are orthogonal if and only if the square of the norm of their sum equals the sum of the squares of their norms.

Statement

Let $\mathbf{x}, \mathbf{y} \in \mathbb{E}^n$. Then:

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0 \quad \text{if and only if} \quad \|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

Proof

(\Rightarrow) Suppose $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. Then:

$$\|\mathbf{x} + \mathbf{y}\|^2 = \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle \tag{1}$$

$$= \langle \mathbf{x}, \mathbf{x} \rangle + 2\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle \tag{2}$$

$$= \|\mathbf{x}\|^2 + 2 \cdot 0 + \|\mathbf{y}\|^2 \tag{3}$$

$$= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \tag{4}$$

(\Leftarrow) Suppose $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$. Expanding the left side:

$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2 \tag{5}$$

Comparing with the given equality:

$$\|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

Therefore, $2\langle \mathbf{x}, \mathbf{y} \rangle = 0$, which implies $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

Classical Form

In \mathbb{E}^2 , for a right triangle with legs of length a and b and hypotenuse of length c :

$$a^2 + b^2 = c^2$$

Mermaid Diagram

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graph TD
  A[Pythagorean Theorem] --> B[Orthogonality]
  A --> C[Norm Properties]
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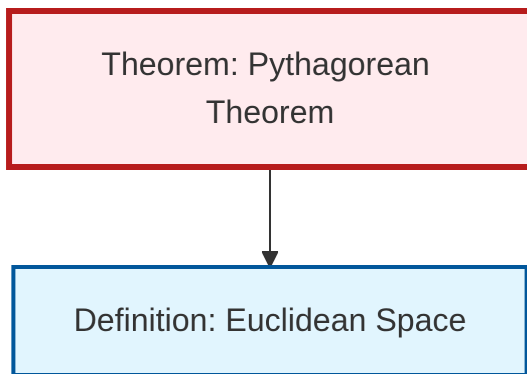
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B --> D[Inner Product = 0]
C --> E[ $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ ]
D <--> E
A --> F[Euclidean Space]

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style B fill:#bbf,stroke:#333,stroke-width:2px
style C fill:#bbf,stroke:#333,stroke-width:2px
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style E fill:#bfb,stroke:#333,stroke-width:2px

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Dependency Graph



Local dependency graph