# Definition: Distributivity

**Distributivity** is a property that relates two Binary Operations, describing how one operation "distributes" over the other.

#### Formal Definition

Given a set S with two binary operations \* and +:

### Left Distributivity

Operation \* is **left distributive** over + if for all  $a, b, c \in S$ :

$$a * (b + c) = (a * b) + (a * c)$$

## Right Distributivity

Operation \* is **right distributive** over + if for all  $a, b, c \in S$ :

$$(a+b)*c = (a*c) + (b*c)$$

### Full Distributivity

Operation \* is **distributive** over + if it is both left and right distributive.

### Classic Examples

#### Arithmetic

- Multiplication over addition:
  - -a(b+c) = ab + ac (left distributive)
  - -(a+b)c = ac + bc (right distributive)
- Multiplication over subtraction: a(b-c) = ab ac

## Set Theory

- Intersection over union:  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- Union over intersection:  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

## Logic

- AND over OR:  $p \land (q \lor r) = (p \land q) \lor (p \land r)$
- **OR over AND**:  $p \lor (q \land r) = (p \lor q) \land (p \lor r)$

# Linear Algebra

- Matrix multiplication over addition: A(B+C) = AB + AC
- Scalar multiplication over vector addition:  $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$

# Non-Distributive Operations

#### Arithmetic

- Division over addition: a + a / c + a / c
  Exponentiation over multiplication: a + c / c (but a + c + a / c)

# Other Operations

- Square root over addition:  $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$
- Composition over addition (when defined):  $(f+g) \circ h \neq (f \circ h) + (g \circ h)$  in general

# Algebraic Structures

Distributivity is a key requirement for:

- 1. Rings: Multiplication must distribute over addition
- 2. **Fields**: Extension of rings with additional properties
- 3. Vector spaces: Scalar multiplication distributes over vector addition
- 4. Lattices: Meet and join operations distribute over each other

# **Special Forms**

## Infinite Distributivity

For infinite operations:

$$a\cdot \left(\sum_{i\in I}b_i\right) = \sum_{i\in I}(a\cdot b_i)$$

## Conditional Distributivity

Some operations distribute only under certain conditions: - GCD over LCM: Sometimes but not always distributive - Min/Max operations: Distribute over each other

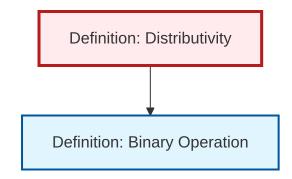
# **Applications**

- 1. **Algebra**: Expanding expressions like  $(x+2)(x+3) = x^2 + 5x + 6$
- 2. Computer Science: Optimizing expressions in compilers
- 3. Circuit Design: Boolean algebra for logic gates
- 4. Category Theory: Distributive categories and monoidal structures

## Related Concepts

- FOIL method: Application of distributivity for binomial multiplication
- Modular distributivity: Weaker form in lattice theory
- Near-rings: Structures with one-sided distributivity

# Dependency Graph



Local dependency graph