Definition: Associativity

Associativity is a fundamental property of a Binary Operation that allows us to perform multiple operations without needing to specify the order of evaluation using parentheses.

Formal Definition

A binary operation $*$ on a set S is **associative** if for all $a, b, c \in S$:

$$(a * b) * c = a * (b * c)$$

When this property holds, we can write $a * b * c$ without ambiguity.

Extended Associativity

By induction, if $*$ is associative, then for any elements $a_1, a_2, \ldots, a_n$: - All possible ways of parenthesizing $a_1 * a_2 * \cdots * a_n$ yield the same result - We can write the expression without parentheses

Examples of Associative Operations

Arithmetic

- **Addition**: $(a + b) + c = a + (b + c)$
- **Multiplication**: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Set Theory

- **Union**: $(A \cup B) \cup C = A \cup (B \cup C)$
- **Intersection**: $(A \cap B) \cap C = A \cap (B \cap C)$

Logic

- **AND**: $(p \land q) \land r = p \land (q \land r)$
- **OR**: $(p \lor q) \lor r = p \lor (q \lor r)$

Functions

- **Composition**: $(f \circ g) \circ h = f \circ (g \circ h)$

Strings

- **Concatenation**: $(AB)C = A(BC)$ for strings A, B, C

Non-Associative Operations

Arithmetic

- **Subtraction**: $(a-b)-c \neq a-(b-c)$
  - Example: $(5-3)-1=1$ but $5-(3-1)=3$
- **Division**: $(a \div b) \div c \neq a \div (b \div c)$
  - Example: $(8 \div 4) \div 2 = 1$ but $8 \div (4 \div 2) = 4$
- **Exponentiation**: $(a^b)^c \neq a^{(b^c)}$
  - Example: $(2^3)^2 = 64$ but $2^{(3^2)} = 512$

Vector Operations

- **Cross product**: $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

Importance

1. **Algebraic Structures**: Associativity is required for:

   - Semigroups
   - Monoids
   - Groups
   - Rings (for both operations)

2. **Computation**: Allows flexible evaluation order for optimization

3. **Generalization**: Enables definition of products/sums over arbitrary index sets

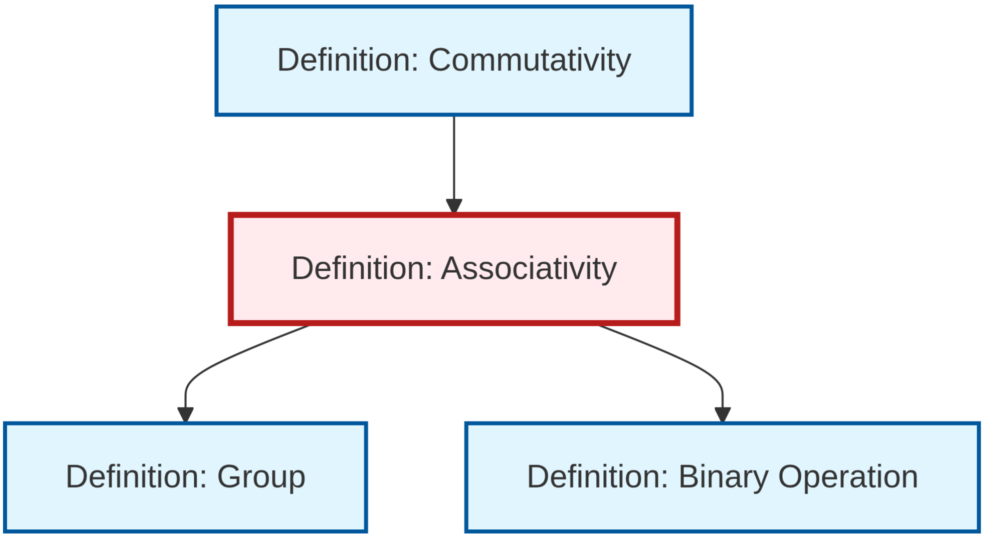
Related Properties

- **Commutativity**: $a * b = b * a$ (independent of associativity)
- **Power associativity**: $(x^m)^n = x^{mn}$ (weaker than full associativity)
- **Alternative property**: $(aa)b = a(ab)$ and $a(bb) = (ab)b$ (weaker)

Consequences

In structures with associativity: - Can define powers: $a^n = a * a * \cdots * a$ (n times) - Can extend to infinite products (with appropriate convergence) - Matrix multiplication is associative, enabling efficient computation

Dependency Graph



Local dependency graph