Axiom: Mathematical Induction

Axiom: Mathematical Induction

The principle of mathematical induction is a fundamental axiom for the natural numbers that allows us to prove statements about all natural numbers by verifying a base case and an inductive step.

Statement

Let $P(n)$ be a property defined for natural numbers. If: 1. **Base case**: $P(1)$ is true 2. **Inductive step**: For all $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Formal Statement

$$[P(1) \land \forall k \in \mathbb{N}(P(k) \implies P(k+1))] \implies \forall n \in \mathbb{N}\, P(n)$$

Variants

Strong Induction

If: 1. $P(1)$ is true 2. For all $k \in \mathbb{N}$, if $P(1), P(2), \ldots, P(k)$ are all true, then $P(k+1)$ is true

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Well-Ordering Principle

Every non-empty subset of $\mathbb{N}$ has a least element. This is equivalent to the induction axiom.

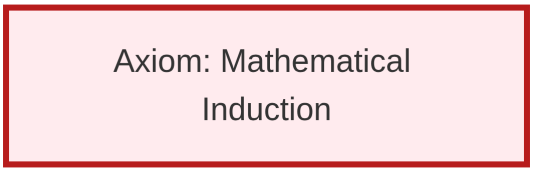
Intuition

Induction works like climbing an infinite ladder: - We can reach the first rung (base case) - From any rung, we can reach the next rung (inductive step) - Therefore, we can reach any rung

Role in Mathematics

Mathematical induction is: - Part of the Peano axioms that define natural numbers - Essential for proving statements about infinite sets - The foundation for recursive definitions and algorithms - Generalizable to other well-ordered sets (transfinite induction)

Dependency Graph



Local dependency graph