Theorem: Pigeonhole Principle

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The **Pigeonhole Principle** is a fundamental counting principle stating that if you distribute more items than containers, at least one container must contain more than one item.

Statement

Basic Form: If n items are placed into k containers where $n > k$, then at least one container must contain more than one item.

General Form: If n items are placed into k containers, then at least one container must contain at least $\lceil n/k \rceil$ items.

Formal Statement

Let $f : A \to B$ be a function where A and B are finite Set s with $|A| > |B|$. Then f is not injective; that is, there exist distinct elements $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$.

Proof

By contradiction. Suppose f is injective. Then each element of A maps to a distinct element of B, implying $|A| \leq |B|$. This contradicts our assumption that $|A| > |B|$.

Generalizations

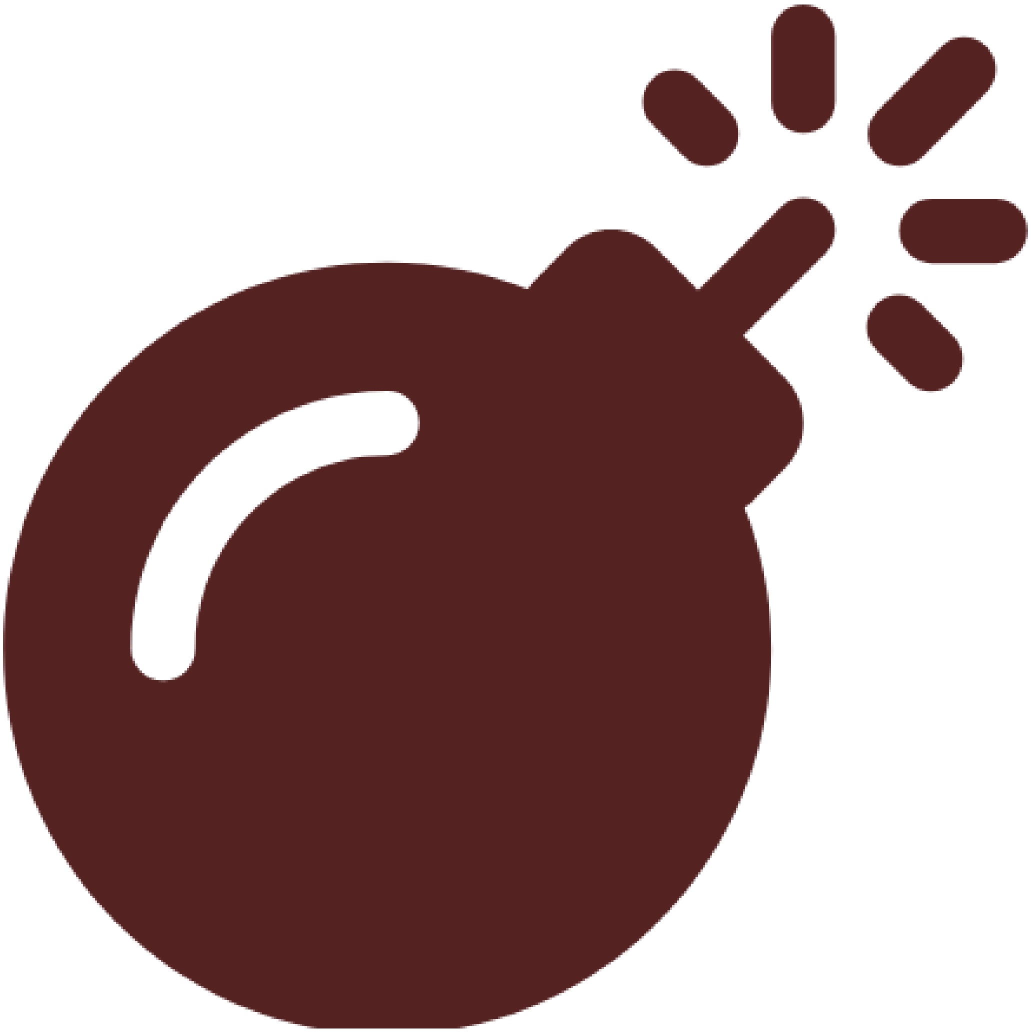
1. **Infinite Version**: An infinite set cannot be mapped injectively into a finite set.
2. **Probabilistic Version**: If items are distributed randomly, the expected number of items in each container provides bounds on the maximum load.
3. **Multidimensional Version**: Extends to partitioning higher-dimensional spaces.

Applications

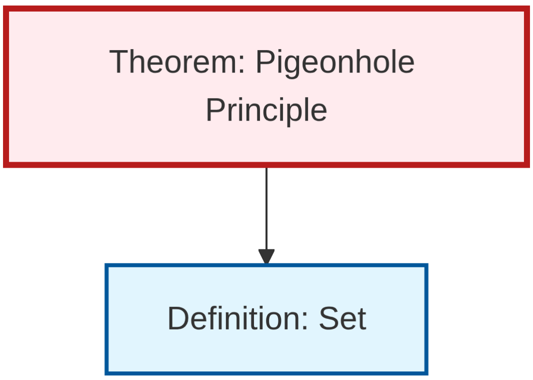
1. **Existence Proofs**: Often used to prove that certain configurations must exist
2. **Computer Science**: Hash table collision analysis, load balancing
3. **Number Theory**: Proving properties about integer sequences
4. **Graph Theory**: Showing certain substructures must exist in large graphs

Classic Examples

- In any group of 13 people, at least two must have birthdays in the same month
- Among any 6 people, either 3 are mutual friends or 3 are mutual strangers
- In any sequence of $n^2 + 1$ distinct real numbers, there is either an increasing subsequence or a decreasing subsequence of length $n + 1$



Dependency Graph



Local dependency graph