Definition: Functor

Functor

A **functor** is a structure-preserving mapping between Category structures.

Formal Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ consists of:

1. An **object mapping**: For each object $A \in \mathrm{Ob}(\mathcal{C})$, an object $F(A) \in \mathrm{Ob}(\mathcal{D})$
2. A **morphism mapping**: For each morphism $f : A \to B$ in $\mathcal{C}$, a morphism $F(f) : F(A) \to F(B)$ in $\mathcal{D}$

satisfying:

Functor Laws

1. **Identity preservation**: $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for all objects A
2. **Composition preservation**: $F(g \circ f) = F(g) \circ F(f)$ for all composable morphisms f and g

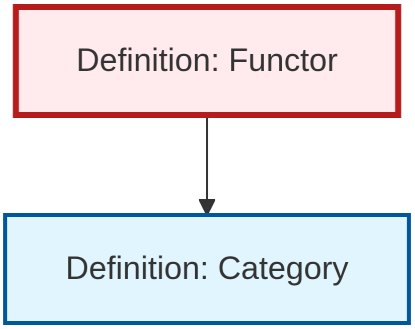
Types of Functors

- **Covariant functor**: As defined above
- **Contravariant functor**: Reverses the direction of morphisms
- **Faithful functor**: Injective on morphism sets
- **Full functor**: Surjective on morphism sets
- **Fully faithful functor**: Both faithful and full
- **Essentially surjective**: Every object in $\mathcal{D}$ is isomorphic to $F(A)$ for some A

Examples

- The forgetful functor from **Grp** to **Set**
- The free group functor from **Set** to **Grp**
- The fundamental group functor from **Top** to **Grp**

Dependency Graph



Local dependency graph