Theorem: Intermediate Value Theorem

Intermediate Value Theorem

A Continuity function from a Connected Space space to the real numbers attains all intermediate values.

Statement

Let $f : X \to \mathbb{R}$ be a continuous function where X is a connected topological space. If $a, b \in f(X)$ with $a < b$, then for every $c \in (a, b)$, there exists $x \in X$ such that $f(x) = c$.

Classical Version

For the special case where $X = [a, b] \subseteq \mathbb{R}$ is a closed interval:

If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < c < f(b)$ (or $f(b) < c < f(a)$), then there exists $x_0 \in (a, b)$ such that $f(x_0) = c$.

Proof Idea

The proof relies on the fact that: 1. The continuous image of a connected space is connected 2. Connected subsets of $\mathbb{R}$ are intervals 3. Intervals contain all intermediate values

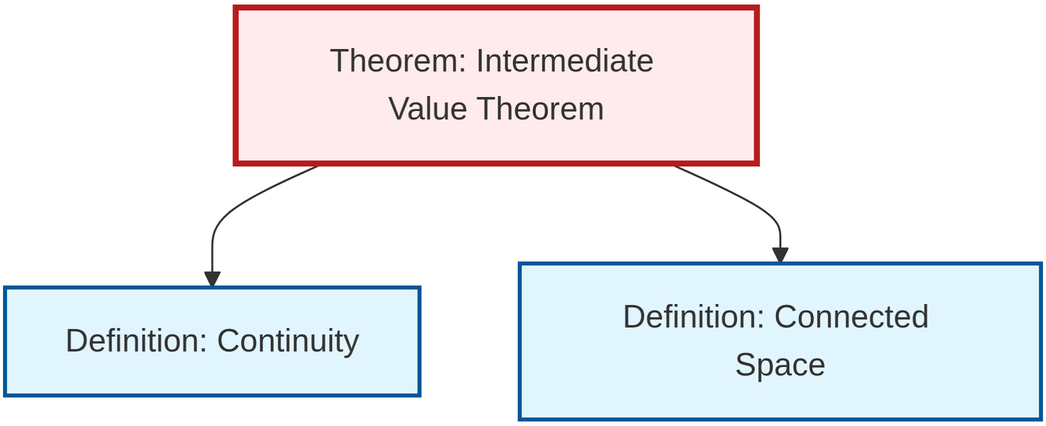
Applications

- Finding roots of continuous functions
- Proving existence of fixed points
- Establishing properties of continuous functions on intervals
- Analysis of differential equations

Example

Any polynomial of odd degree has at least one real root, since: - $\lim_{x\to\infty} P(x) = \infty$ and $\lim_{x\to-\infty} P(x) = -\infty$ (or vice versa) - By IVT, $P(x) = 0$ for some x

Dependency Graph



Local dependency graph