Example: Closed Interval is Compact

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The closed interval [a, b] in \mathbb{R} (with the standard topology) is a Compact Space space.

Statement

For any $a, b \in \mathbb{R}$ with $a \leq b$, the closed interval $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ is compact.

Intuition

Compactness of [a, b] means that from any collection of open sets that cover [a, b], we can always select a finite number of them that still cover [a, b]. This captures the idea that [a, b] is "finite in extent" despite containing infinitely many points.

Proof Sketch

By the Heine-Borel Theorem, a subset of \mathbb{R}^n is compact if and only if it is closed and bounded. We verify:

- 1. Closed: [a, b] is closed because its complement $(-\infty, a) \cup (b, \infty)$ is open.
- 2. **Bounded**: [a, b] is bounded since all its elements lie between a and b.

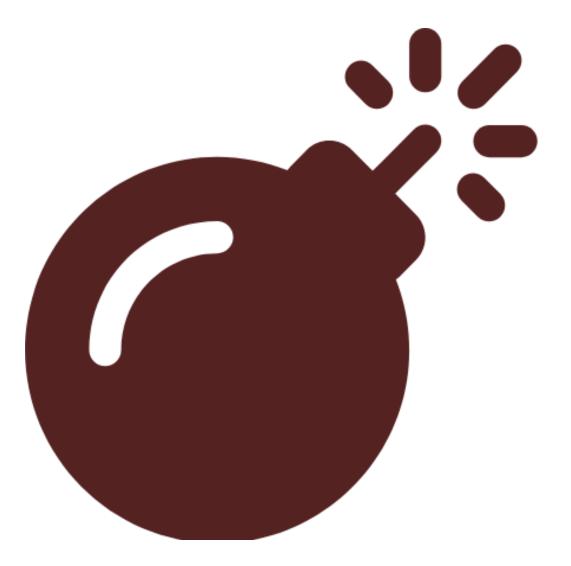
Therefore, [a, b] is compact.

Contrast with Non-Examples

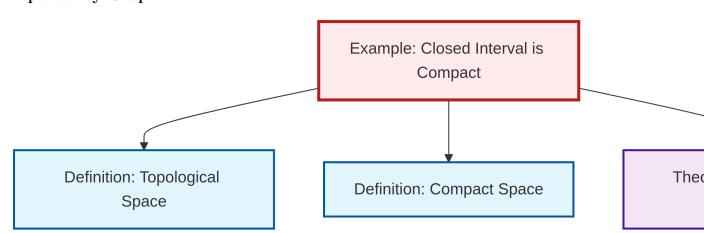
- The open interval (a, b) is **not compact** (it's bounded but not closed)
- The ray $[a, \infty)$ is **not compact** (it's closed but not bounded)
- The entire real line \mathbb{R} is **not compact** (it's closed but not bounded)

Applications

The compactness of closed intervals is fundamental in analysis: - It ensures that continuous functions on [a,b] attain their maximum and minimum - It's key to proving the uniform continuity of continuous functions on [a,b] - It's essential for the Riemann integrability of continuous functions



Dependency Graph



Local dependency graph