Definition: Distributivity

Distributivity is a property that relates two Binary Operations, describing how one operation "distributes" over the other.

Formal Definition

Given a set S with two binary operations $*$ and $+$:

Left Distributivity

Operation $*$ is **left distributive** over $+$ if for all $a, b, c \in S$:

$$a * (b + c) = (a * b) + (a * c)$$

Right Distributivity

Operation $*$ is **right distributive** over $+$ if for all $a, b, c \in S$:

$$(a + b) * c = (a * c) + (b * c)$$

Full Distributivity

Operation $*$ is **distributive** over $+$ if it is both left and right distributive.

Classic Examples

Arithmetic

- **Multiplication over addition**:
  - $a(b + c) = ab + ac$ (left distributive)
  - $(a + b)c = ac + bc$ (right distributive)
- **Multiplication over subtraction**: $a(b - c) = ab - ac$

Set Theory

- **Intersection over union**: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- **Union over intersection**: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Logic

- **AND over OR**: $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$
- **OR over AND**: $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

Linear Algebra

- **Matrix multiplication over addition**: $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
- **Scalar multiplication over vector addition**: $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$

Non-Distributive Operations

Arithmetic

- **Division over addition**: $\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$
- **Exponentiation over multiplication**: $a^{bc} \neq a^b \cdot a^c$ (but $a^{b+c} = a^b \cdot a^c$)

Other Operations

- **Square root over addition**: $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$
- **Composition over addition** (when defined): $(f+g) \circ h \neq (f \circ h) + (g \circ h)$ in general

Algebraic Structures

Distributivity is a key requirement for:

1. **Rings**: Multiplication must distribute over addition
2. **Fields**: Extension of rings with additional properties
3. **Vector spaces**: Scalar multiplication distributes over vector addition
4. **Lattices**: Meet and join operations distribute over each other

Special Forms

Infinite Distributivity

For infinite operations:

$$a \cdot \left(\sum_{i \in I} b_i \right) = \sum_{i \in I} (a \cdot b_i)$$

Conditional Distributivity

Some operations distribute only under certain conditions: - **GCD over LCM**: Sometimes but not always distributive - **Min/Max operations**: Distribute over each other

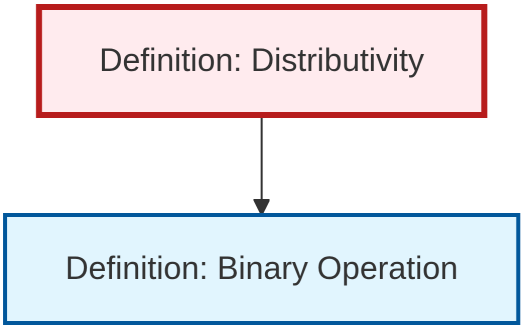
Applications

1. **Algebra**: Expanding expressions like $(x+2)(x+3) = x^2 + 5x + 6$
2. **Computer Science**: Optimizing expressions in compilers
3. **Circuit Design**: Boolean algebra for logic gates
4. **Category Theory**: Distributive categories and monoidal structures

Related Concepts

- **FOIL method**: Application of distributivity for binomial multiplication
- **Modular distributivity**: Weaker form in lattice theory
- **Near-rings**: Structures with one-sided distributivity

Dependency Graph



Local dependency graph