Definition: Linear Transformation

Definition: Linear Transformation

Let V and W be s over the same field F. A function $T : V \to W$ is called a **linear transformation** (or **linear map**) if it satisfies the following two properties:

Linearity Conditions

For all vectors $\mathbf{u}, \mathbf{v} \in V$ and all scalars $a, b \in F$:

1. **Additivity**: $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. **Homogeneity**: $T(a\mathbf{u}) = aT(\mathbf{u})$

These two conditions can be combined into a single condition:

$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$$

Important Properties

If $T : V \to W$ is a linear transformation, then:

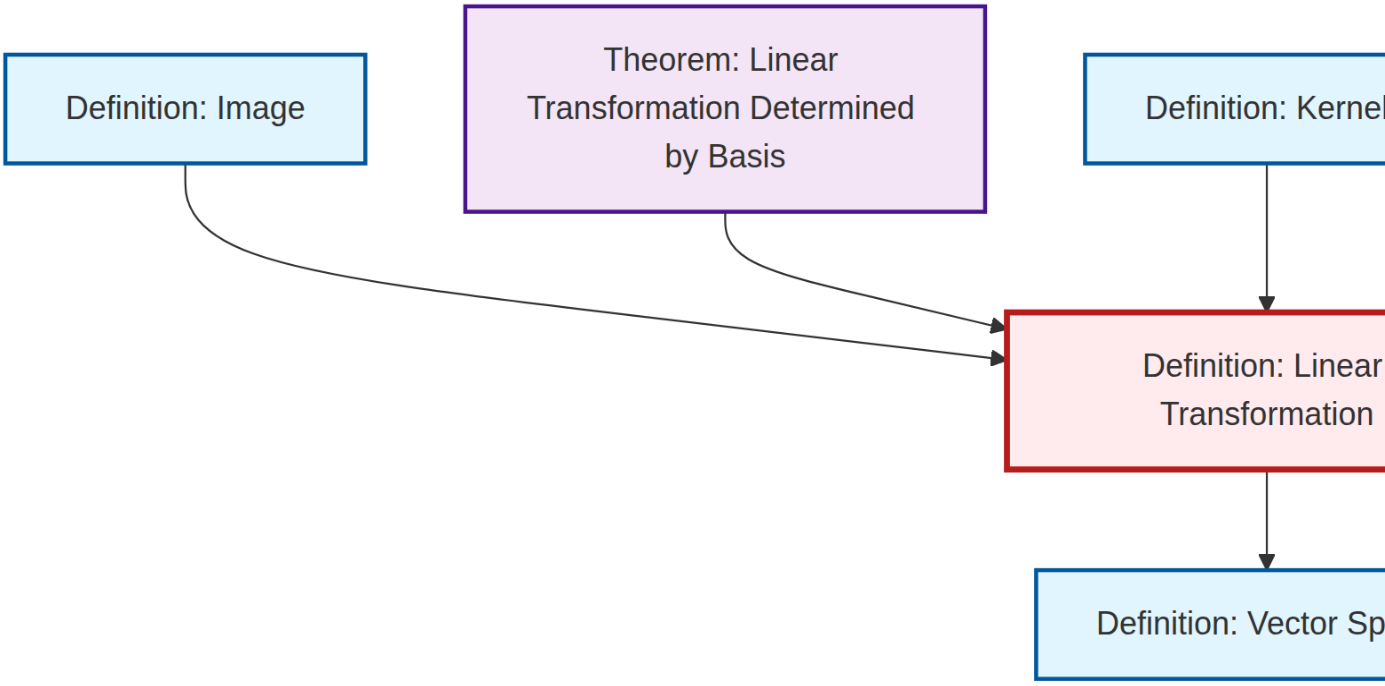
- $T(\mathbf{0}_V) = \mathbf{0}_W$ (maps zero to zero)
- $T(-\mathbf{v}) = -T(\mathbf{v})$ for all $\mathbf{v} \in V$
- $T\left(\sum_{i=1}^{n} a_i \mathbf{v}_i\right) = \sum_{i=1}^{n} a_i T(\mathbf{v}_i)$ for any linear combination

Special Types

- **Linear functional**: When $W = F$ (the field of scalars)
- **Linear operator**: When $V = W$ (transformation from a space to itself)
- **Isomorphism**: When T is bijective (one-to-one and onto)

Linear transformations preserve the vector space structure and are the morphisms in the category of vector spaces.

Dependency Graph



Local dependency graph