Definition: Topological Space

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A **topological space** is a Set X together with a collection τ of subsets of X satisfying the following axioms:

Axioms for a Topology

The collection τ must satisfy:

1. **Empty set and whole space**: $\emptyset \in \tau$ and $X \in \tau$
2. **Arbitrary unions**: If $\{U_i\}_{i \in I}$ is any collection of sets in τ, then
$$\bigcup_{i \in I} U_i \in \tau$$
3. **Finite intersections**: If $U_1, U_2, \dots, U_n \in \tau$, then
$$\bigcap_{i=1}^{n} U_i \in \tau$$

Terminology

- The collection τ is called a **topology** on X
- The sets in τ are called **open sets**
- The pair (X, τ) is called a **topological space**
- When the topology is clear from context, we may simply refer to "the topological space X"

Examples

1. **Discrete topology**: $\tau = \mathcal{P}(X)$ (all subsets are open)
2. **Indiscrete topology**: $\tau = \{\emptyset, X\}$ (only the empty set and X are open)
3. **Standard topology on** $\mathbb{R}$: Open sets are unions of open intervals

Properties

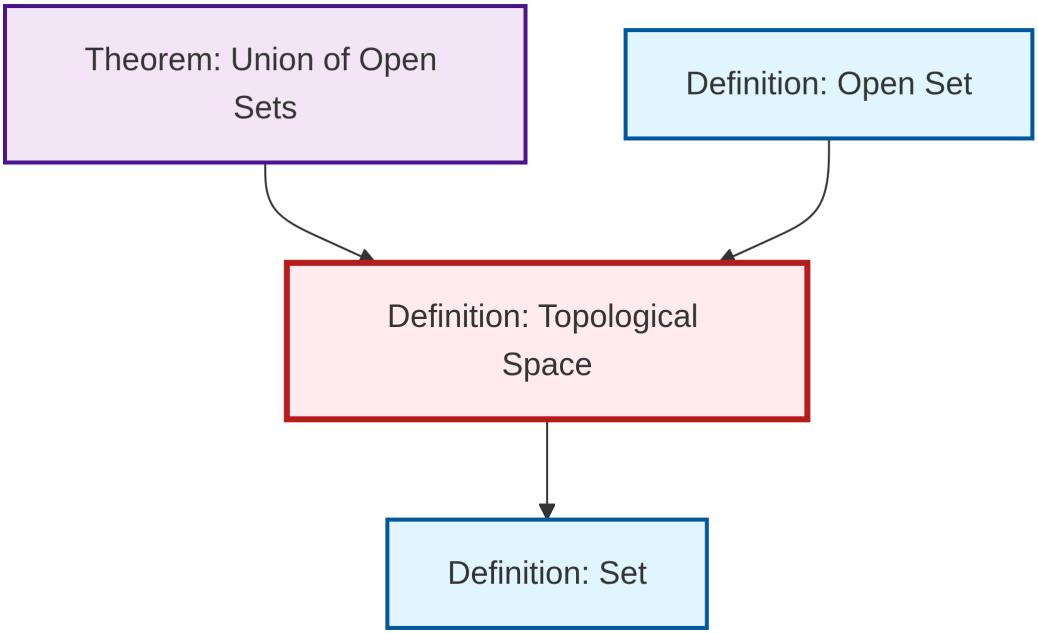
Key properties that follow from these axioms:

- The union of any collection of open sets is open
- The intersection of finitely many open sets is open
- Complements of open sets are called **closed sets**

Related Concepts

- Definition: Open Set
- Definition: Continuous Function (coming soon)
- Definition: Homeomorphism (coming soon)
- Definition: Basis for a Topology (coming soon)

Dependency Graph



Local dependency graph