

## Theorem: Law of Large Numbers

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The **Law of Large Numbers** states that the sample average of independent and identically distributed **random variables** converges to their **expected value** as the sample size increases.

#### Weak Law of Large Numbers

Let  $X_1, X_2, \dots$  be independent and identically distributed random variables with finite expected value  $\mu = E[X_i]$  and finite variance  $\sigma^2 = \text{Var}(X_i)$ .

Define the sample average:

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$$

Then for any  $\epsilon > 0$ :

$$\lim_{n \rightarrow \infty} P(|\bar{X}_n - \mu| > \epsilon) = 0$$

This is convergence in probability:  $\bar{X}_n \xrightarrow{P} \mu$ .

#### Strong Law of Large Numbers

Under the same conditions, we have almost sure convergence:

$$P\left(\lim_{n \rightarrow \infty} \bar{X}_n = \mu\right) = 1$$

This is written as:  $\bar{X}_n \xrightarrow{a.s.} \mu$ .

#### Proof Sketch (Weak Law)

Using Chebyshev's inequality:

$$P(|\bar{X}_n - \mu| > \epsilon) \leq \frac{\text{Var}(\bar{X}_n)}{\epsilon^2}$$

Since the  $X_i$  are independent:

$$\text{Var}(\bar{X}_n) = \text{Var}\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i) = \frac{\sigma^2}{n}$$

Therefore:

$$P(|\bar{X}_n - \mu| > \epsilon) \leq \frac{\sigma^2}{n\epsilon^2} \rightarrow 0 \text{ as } n \rightarrow \infty$$

## Applications

1. Empirical frequencies converge to probabilities
2. Sample means approximate population means
3. Foundation for statistical inference
4. Monte Carlo methods

## Example

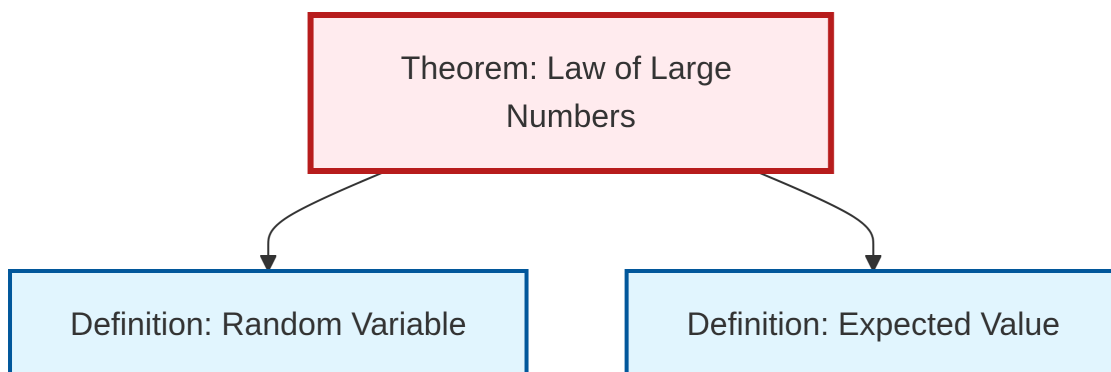
For fair coin flips (Bernoulli( $1/2$ )), the proportion of heads converges to  $1/2$  as the number of flips increases.

## Mermaid Diagram

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graph TD
    A[Law of Large Numbers] --> B[Sample Average]
    B --> C[" $\bar{X} = (1/n)\sum X$ "]
    A --> D[Weak Law]
    A --> E[Strong Law]
    D --> F["Convergence in Probability"]
    E --> G["Almost Sure Convergence"]
    F --> H[" $P(|\bar{X} - \mu| > \epsilon) \rightarrow 0$ "]
    G --> I[" $P(\lim \bar{X} = \mu) = 1$ "]

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    style D fill:#bbf,stroke:#333,stroke-width:2px
    style E fill:#bbf,stroke:#333,stroke-width:2px
    style H fill:#bfb,stroke:#333,stroke-width:2px
    style I fill:#bfb,stroke:#333,stroke-width:2px
```

## Dependency Graph



Local dependency graph