

## Definition: Convergence

**Convergence** describes the behavior of a [Sequence](#) whose terms approach a specific value as the index increases. A convergent sequence gets arbitrarily close to its limit.

### Formal Definition

A sequence  $(a_n)$  in a metric space  $(X, d)$  **converges** to a [Limit of a Sequence](#)  $L \in X$  if:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n > N, \quad d(a_n, L) < \varepsilon$$

We write:  $\lim_{n \rightarrow \infty} a_n = L$  -  $a_n \rightarrow L$  as  $n \rightarrow \infty$  -  $(a_n)$  converges to  $L$

### In Real Numbers

For a real sequence  $(a_n)$  converging to  $L \in \mathbb{R}$ :

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n > N, \quad |a_n - L| < \varepsilon$$

### Properties of Convergent Sequences

1. **Uniqueness:** A sequence can have at most one limit
2. **Boundedness:** Every convergent sequence is bounded
3. **Preservation under arithmetic:**
  - If  $a_n \rightarrow A$  and  $b_n \rightarrow B$ , then:
    - $a_n + b_n \rightarrow A + B$
    - $a_n \cdot b_n \rightarrow A \cdot B$
    - $a_n/b_n \rightarrow A/B$  (if  $B \neq 0$  and  $b_n \neq 0$ )

### Types of Convergence

#### Pointwise Convergence

For function sequences  $f_n : X \rightarrow Y$ :

$$f_n \rightarrow f \text{ pointwise if } \forall x \in X, f_n(x) \rightarrow f(x)$$

#### Uniform Convergence

$$f_n \rightarrow f \text{ uniformly if } \sup_{x \in X} |f_n(x) - f(x)| \rightarrow 0$$

#### Absolute Convergence

For series:  $\sum_{n=1}^{\infty} a_n$  converges absolutely if  $\sum_{n=1}^{\infty} |a_n|$  converges

## Examples

1. **Convergent:**  $a_n = \frac{1}{n} \rightarrow 0$
2. **Convergent:**  $a_n = \frac{n+1}{n} = 1 + \frac{1}{n} \rightarrow 1$
3. **Divergent:**  $a_n = (-1)^n$  oscillates between -1 and 1
4. **Divergent:**  $a_n = n$  grows without bound

## Cauchy Criterion

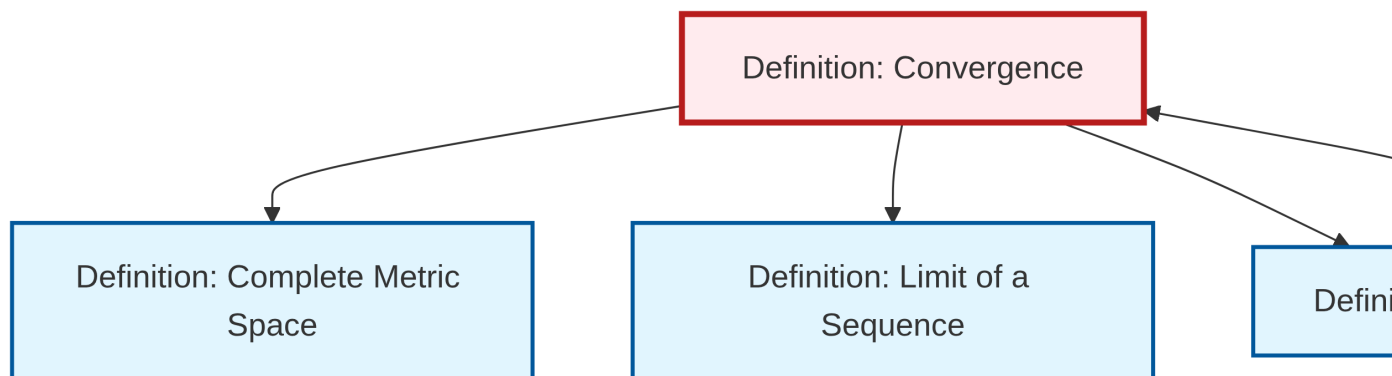
In [Complete Metric Spaces](#), a sequence converges if and only if it is a Cauchy sequence:

$$\forall \varepsilon > 0, \exists N : \forall m, n > N, \quad d(a_m, a_n) < \varepsilon$$

## Applications

- Foundation of calculus and analysis
- Numerical methods and approximation
- Probability (law of large numbers)
- Functional analysis (operator convergence)

## Dependency Graph



Local dependency graph