# Theorem: Every Metric Space is Hausdorff

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**Statement**: Every Metric Space $(X, d)$ with the metric topology is a Hausdorff Space space.

### Proof

Let $(X, d)$ be a metric space and let $x, y \in X$ be two distinct points with $x \neq y$.

Since $x \neq y$, by the identity of indiscernibles property of metrics, we have $d(x, y) > 0$.

Let $r = \frac{d(x,y)}{2} > 0$.

Consider the open balls: - $U = B(x, r) = \{z \in X : d(x, z) < r\}$ - $V = B(y, r) = \{z \in X : d(y, z) < r\}$

We claim that $U \cap V = \emptyset$.

**Proof by contradiction**: Suppose there exists $z \in U \cap V$. Then: - $z \in U$ implies $d(x, z) < r = \frac{d(x,y)}{2}$ - $z \in V$ implies $d(y, z) < r = \frac{d(x,y)}{2}$

By the triangle inequality:

$$d(x, y) \leq d(x, z) + d(z, y) = d(x, z) + d(y, z) < \frac{d(x, y)}{2} + \frac{d(x, y)}{2} = d(x, y)$$

This gives us $d(x, y) < d(x, y)$, which is a contradiction.

Therefore $U \cap V = \emptyset$, and we have found disjoint open neighborhoods separating $x$ and $y$.

Since $x$ and $y$ were arbitrary distinct points, $(X, d)$ is Hausdorff. $\square$

### Consequences

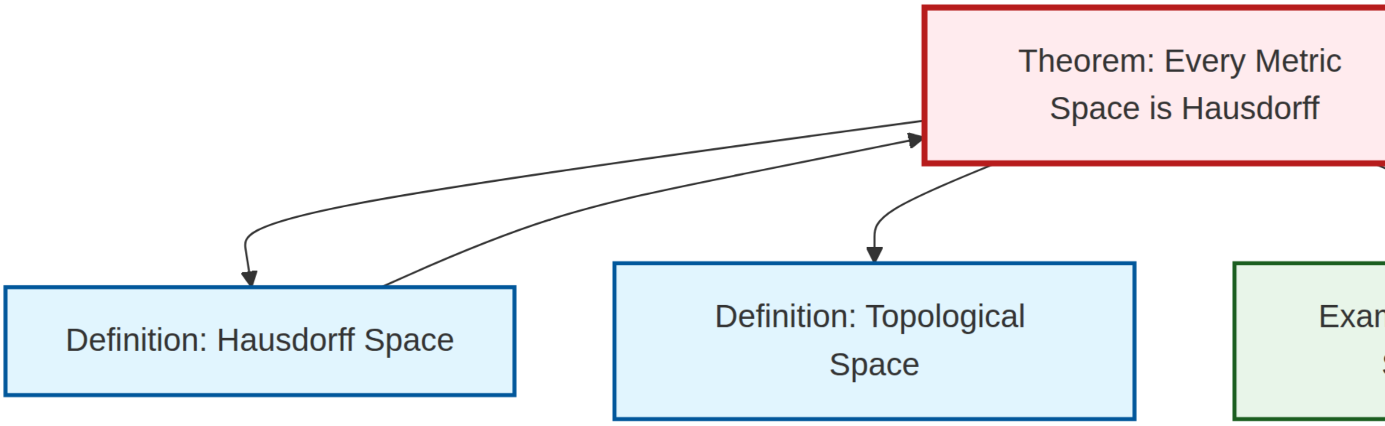
This theorem has several important implications:

1. **Uniqueness of limits**: In any metric space, convergent sequences have unique limits
2. **Closed singletons**: Every point $\{x\}$ in a metric space forms a closed set
3. **Nice separation properties**: Metric spaces automatically satisfy all the lower separation axioms ($T_0$, $T_1$, $T_2$)

### See Also

- Metric Space - The structure that guarantees the Hausdorff property
- Hausdorff Space - The separation property satisfied by all metric spaces
- Real Line with Standard Metric - A concrete example of a metric space that is Hausdorff

### Dependency Graph




Local dependency graph