

## Theorem: Linear Transformation Determined by Basis

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A [Linear Transformation](#) between finite-dimensional [s](#) is uniquely determined by its action on a [Basis](#).

#### Statement

Let  $V$  and  $W$  be vector spaces over a field  $F$ , and let  $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  be a basis for  $V$ .

For any choice of vectors  $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n \in W$ , there exists a unique linear transformation  $T : V \rightarrow W$  such that:

$$T(\mathbf{v}_i) = \mathbf{w}_i \text{ for } i = 1, 2, \dots, n$$

#### Proof Sketch

**Existence:** Define  $T$  on an arbitrary vector  $\mathbf{v} \in V$  by: 1. Express  $\mathbf{v}$  uniquely as a linear combination:  $\mathbf{v} = \sum_{i=1}^n a_i \mathbf{v}_i$  2. Define  $T(\mathbf{v}) = \sum_{i=1}^n a_i \mathbf{w}_i$  3. Verify that  $T$  is linear by checking additivity and homogeneity

**Uniqueness:** If  $S : V \rightarrow W$  is another linear transformation with  $S(\mathbf{v}_i) = \mathbf{w}_i$ , then by linearity:

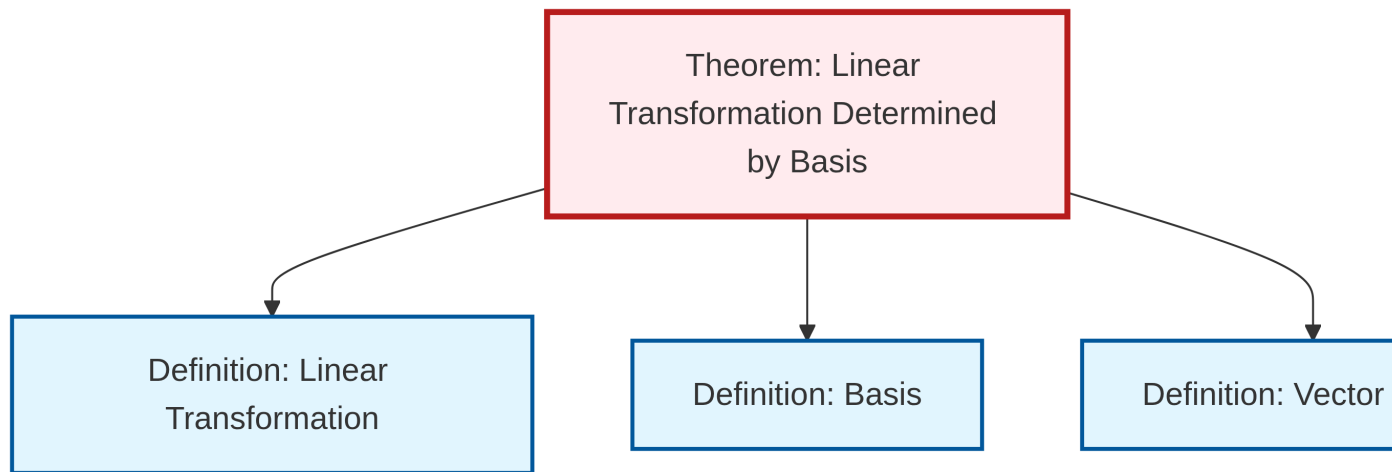
$$S(\mathbf{v}) = S\left(\sum_{i=1}^n a_i \mathbf{v}_i\right) = \sum_{i=1}^n a_i S(\mathbf{v}_i) = \sum_{i=1}^n a_i \mathbf{w}_i = T(\mathbf{v})$$

Therefore,  $S = T$ .

#### Consequences

This theorem shows that: - Linear transformations are completely determined by finitely many values - The space of linear transformations  $\text{Hom}(V, W)$  has dimension  $\dim(V) \times \dim(W)$  - Matrix representations of linear transformations arise naturally from this principle

## Dependency Graph



Local dependency graph