Definition: Continuity

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A function $f : A \to \mathbb{R}$ (where $A \subseteq \mathbb{R}$) is **continuous** at a point $c \in A$ if the function value equals the limit at that point.

Epsilon-Delta Definition

f is continuous at c if:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in A$:

$$|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$$

Sequential Definition

Equivalently, f is continuous at c if:

For every sequence (x_n) in A with $x_n \to c$, we have $f(x_n) \to f(c)$.

Limit Definition

When c is a limit point of A, f is continuous at c if and only if:

$$\lim_{x \to c} f(x) = f(c)$$

This requires three conditions: 1. $f(c)$ is defined 2. $\lim_{x \to c} f(x)$ exists 3. The limit equals the function value

Types of Continuity

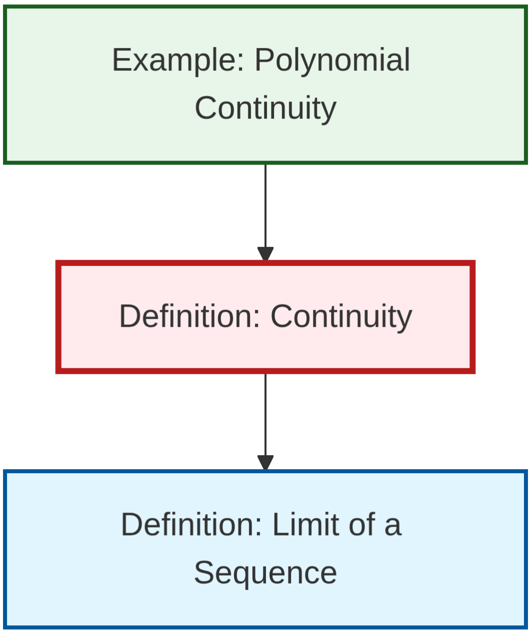
- **Continuous on a set**: f is continuous on A if it is continuous at every point in A
- **Uniformly continuous**: A stronger condition where δ depends only on ε, not on the point
- **Lipschitz continuous**: An even stronger condition with a linear bound on the rate of change

Discontinuities

If f is not continuous at c, we say f has a **discontinuity** at c. Types include: - **Removable**: The limit exists but doesn't equal $f(c)$ - **Jump**: Left and right limits exist but are different - **Essential**: The limit doesn't exist

Continuity captures the intuitive notion of a function having "no breaks or jumps."

Dependency Graph



Local dependency graph