Theorem: Law of Large Numbers

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The **Law of Large Numbers** states that the sample average of independent and identically distributed random variables converges to their expected value as the sample size increases.

Weak Law of Large Numbers

Let $X_1, X_2, ...$ be independent and identically distributed random variables with finite expected value $\mu = E[X_i]$ and finite variance $\sigma^2 = \mathrm{Var}(X_i)$.

Define the sample average:

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Then for any $\epsilon > 0$:

$$\lim_{n\to\infty} P(|\bar{X}_n - \mu| > \epsilon) = 0$$

This is convergence in probability: $\bar{X}_n \xrightarrow{P} \mu$.

Strong Law of Large Numbers

Under the same conditions, we have almost sure convergence:

$$P\left(\lim_{n\to\infty} \bar{X}_n = \mu\right) = 1$$

This is written as: $\bar{X}_n \xrightarrow{a.s.} \mu$.

Proof Sketch (Weak Law)

Using Chebyshev's inequality:

$$P(|\bar{X}_n - \mu| > \epsilon) \leq \frac{\mathrm{Var}(\bar{X}_n)}{\epsilon^2}$$

Since the X_i are independent:

$$\mathrm{Var}(\bar{X}_n) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}(X_i) = \frac{\sigma^2}{n}$$

Therefore:

$$P(|\bar{X}_n - \mu| > \epsilon) \leq \frac{\sigma^2}{n\epsilon^2} \to 0 \text{ as } n \to \infty$$

Applications

1. **Empirical frequencies converge to probabilities**
2. **Sample means approximate population means**
3. **Foundation for statistical inference**
4. **Monte Carlo methods**

Example

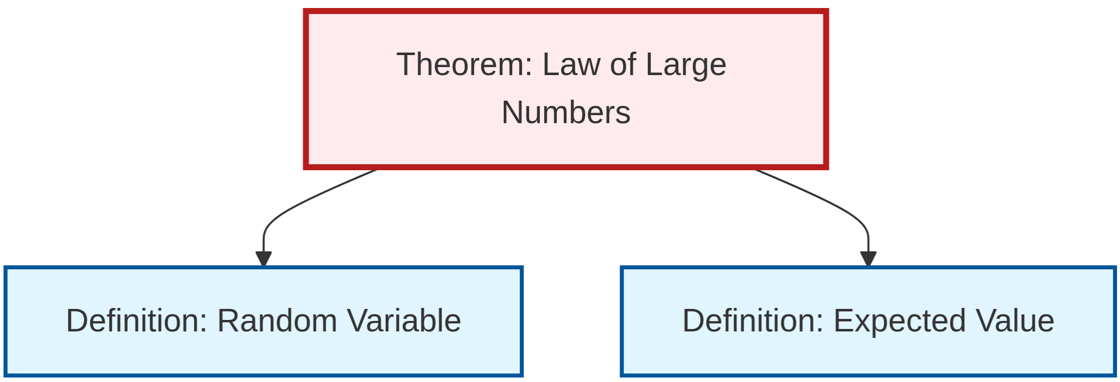
For fair coin flips (Bernoulli(1/2)), the proportion of heads converges to 1/2 as the number of flips increases.

Mermaid Diagram

```
graph TD
    A[Law of Large Numbers] --> B[Sample Average]
    B --> C[X̄  = (1/n)Σ X ]
    A --> D[Weak Law]
    A --> E[Strong Law]
    D --> F[Convergence in Probability]
    E --> G[Almost Sure Convergence]
    F --> H[P(|X̄  -  | >  ) → 0]
    G --> I[P(lim X̄  =  ) = 1]

    style A fill:#f9f,stroke:#333,stroke-width:2px
    style D fill:#bbf,stroke:#333,stroke-width:2px
    style E fill:#bbf,stroke:#333,stroke-width:2px
    style H fill:#bfb,stroke:#333,stroke-width:2px
    style I fill:#bfb,stroke:#333,stroke-width:2px
```

Dependency Graph



Local dependency graph