Example: Matrix Transformation

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Every Matrix defines a Linear Transformation, providing the most concrete and computational example of linear maps.

Construction

Let A be an $m \times n$ matrix with entries in a field F. Define $T_A : F^n \to F^m$ by:

$$T_A(\mathbf{x}) = A\mathbf{x}$$

where $\mathbf{x}$ is viewed as a column vector and $A\mathbf{x}$ denotes matrix-vector multiplication.

Verification of Linearity

For vectors $\mathbf{x}, \mathbf{y} \in F^n$ and scalars $a, b \in F$:

$$T_A(a\mathbf{x} + b\mathbf{y}) = A(a\mathbf{x} + b\mathbf{y}) = aA\mathbf{x} + bA\mathbf{y} = aT_A(\mathbf{x}) + bT_A(\mathbf{y})$$

This follows from the distributive and scalar multiplication properties of matrices.

Concrete Example

Consider the 2×3 matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

The linear transformation $T_A : \mathbb{R}^3 \to \mathbb{R}^2$ maps:

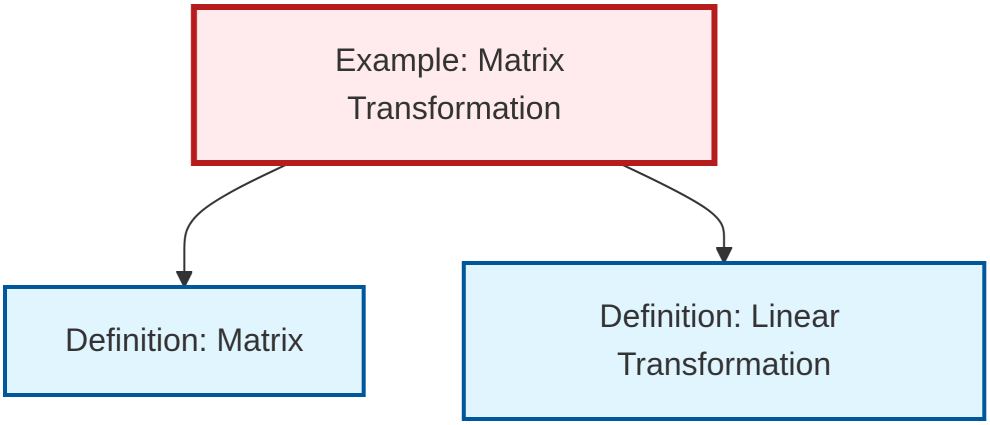
$$T_A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ 4x + 5y + 6z \end{pmatrix}$$

Important Facts

- Every linear transformation between finite-dimensional vector spaces can be represented by a matrix
- The columns of A are the images of the standard basis vectors: $T_A(\mathbf{e}_i) = \mathbf{a}_i$ (the i-th column of A)
- Composition of linear transformations corresponds to matrix multiplication
- The rank of A equals the dimension of the image of T_A

Matrix transformations provide the computational foundation for linear algebra.

Dependency Graph



Local dependency graph