

# Theorem: Yoneda Lemma

The **Yoneda Lemma** is a fundamental result in **Category** theory that relates **Functors** to representable functors. It states that an object is completely determined by its relationships to all other objects.

## Statement

Let  $\mathcal{C}$  be a category and  $F : \mathcal{C} \rightarrow \mathbf{Set}$  be a functor. For any object  $A \in \mathcal{C}$ , there is a natural bijection:

$$\text{Nat}(\text{Hom}(A, -), F) \cong F(A)$$

where: -  $\text{Hom}(A, -) : \mathcal{C} \rightarrow \mathbf{Set}$  is the representable functor -  $\text{Nat}(\text{Hom}(A, -), F)$  is the set of **Natural Transformations**

## The Bijection

### Forward Direction

Given a natural transformation  $\alpha : \text{Hom}(A, -) \Rightarrow F$ , we get an element of  $F(A)$  by:

$$\alpha \mapsto \alpha_A(\text{id}_A) \in F(A)$$

### Reverse Direction

Given an element  $x \in F(A)$ , we define a natural transformation  $\alpha$  by:

$$\alpha_B(f) = F(f)(x) \text{ for } f : A \rightarrow B$$

## Yoneda Embedding

The Yoneda Lemma gives rise to the **Yoneda embedding**:

$$\mathcal{Y} : \mathcal{C} \rightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$$

$$A \mapsto \text{Hom}(-, A)$$

This embedding is: - **Full**: Every natural transformation between representables comes from a morphism - **Faithful**: Different morphisms give different natural transformations

## Corollaries

### 1. Yoneda Principle

Objects  $A$  and  $B$  are isomorphic if and only if  $\text{Hom}(-, A) \cong \text{Hom}(-, B)$  as functors.

## 2. Representability

A functor  $F$  is representable if and only if it is naturally isomorphic to some  $\text{Hom}(A, -)$ .

### Proof Sketch

1. Show that the assignment  $\alpha \mapsto \alpha_A(\text{id}_A)$  is well-defined
2. Verify that the reverse construction gives a natural transformation
3. Check that these constructions are mutual inverses
4. Prove naturality in both  $A$  and  $F$

## Examples and Applications

### Universal Properties

The Yoneda Lemma provides a systematic way to understand universal properties: an object with a universal property represents a particular functor.

### Limits and Colimits

The limit of a functor  $F$  represents the functor  $\text{Hom}(-, \lim F)$ .

### Algebraic Topology

In the category of topological spaces, the functor  $\pi_n(-)$  (n-th homotopy group) is represented by the n-sphere  $S^n$ .

### Philosophical Significance

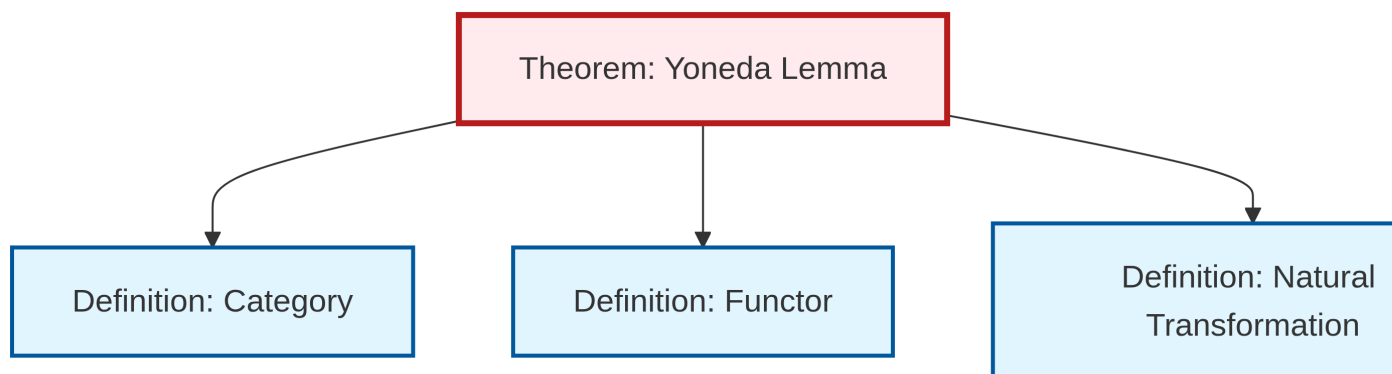
The Yoneda Lemma embodies the idea that: > “An object is completely determined by its relationships to all other objects”

This perspective shifts focus from internal structure to external relationships, a key insight of category theory.

### Variations

- **Contravariant Yoneda:** For contravariant functors  $F : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$
- **Enriched Yoneda:** Generalizes to enriched categories
- **2-Yoneda:** Version for 2-categories

### Dependency Graph



Local dependency graph