Theorem: Linear Transformation Determined by Basis

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A Linear Transformation between finite-dimensional s is uniquely determined by its action on a Basis.

Statement

Let V and W be vector spaces over a field F, and let $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis for V.

For any choice of vectors $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n \in W$, there exists a unique linear transformation $T : V \to W$ such that:

$$T(\mathbf{v}_i) = \mathbf{w}_i \text{ for } i = 1, 2, \dots, n$$

Proof Sketch

Existence: Define T on an arbitrary vector $\mathbf{v} \in V$ by: 1. Express $\mathbf{v}$ uniquely as a linear combination: $\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{v}_i$ 2. Define $T(\mathbf{v}) = \sum_{i=1}^{n} a_i \mathbf{w}_i$ 3. Verify that T is linear by checking additivity and homogeneity

Uniqueness: If $S : V \to W$ is another linear transformation with $S(\mathbf{v}_i) = \mathbf{w}_i$, then by linearity:

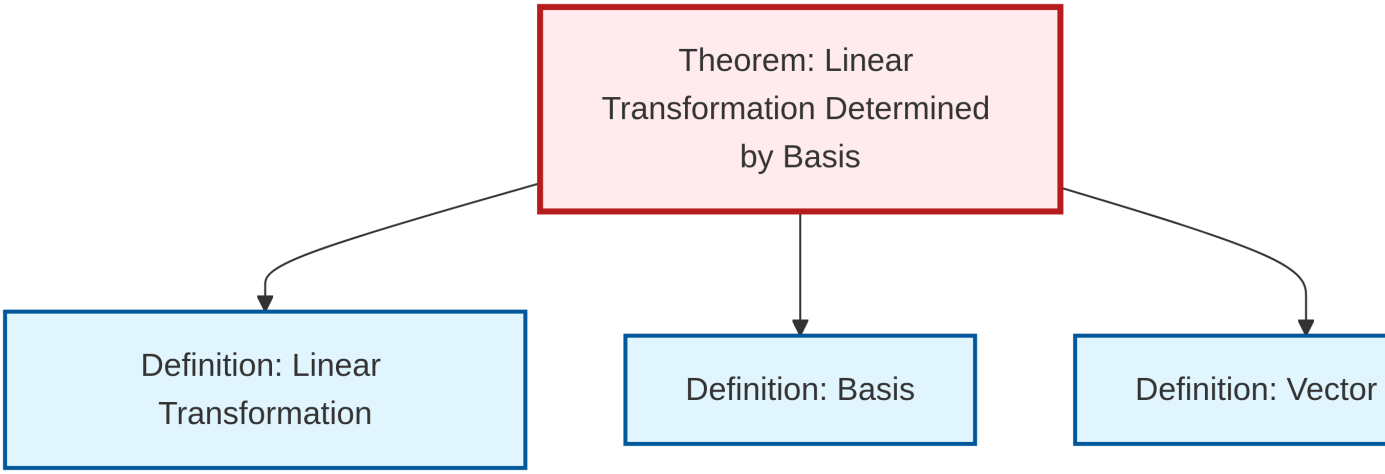
$$S(\mathbf{v}) = S\left(\sum_{i=1}^{n} a_i \mathbf{v}_i\right) = \sum_{i=1}^{n} a_i S(\mathbf{v}_i) = \sum_{i=1}^{n} a_i \mathbf{w}_i = T(\mathbf{v})$$

Therefore, $S = T$.

Consequences

This theorem shows that: - Linear transformations are completely determined by finitely many values - The space of linear transformations $\mathrm{Hom}(V, W)$ has dimension $\dim(V) \times \dim(W)$ - Matrix representations of linear transformations arise naturally from this principle

Dependency Graph



Local dependency graph