

## Example: Closed Interval is Compact

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The closed interval  $[a, b]$  in  $\mathbb{R}$  (with the standard topology) is a [Compact Space](#) space.

#### Statement

For any  $a, b \in \mathbb{R}$  with  $a \leq b$ , the closed interval  $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$  is compact.

#### Intuition

Compactness of  $[a, b]$  means that from any collection of open sets that cover  $[a, b]$ , we can always select a finite number of them that still cover  $[a, b]$ . This captures the idea that  $[a, b]$  is “finite in extent” despite containing infinitely many points.

#### Proof Sketch

By the [Heine-Borel Theorem](#), a subset of  $\mathbb{R}^n$  is compact if and only if it is closed and bounded. We verify:

1. **Closed:**  $[a, b]$  is closed because its complement  $(-\infty, a) \cup (b, \infty)$  is open.
2. **Bounded:**  $[a, b]$  is bounded since all its elements lie between  $a$  and  $b$ .

Therefore,  $[a, b]$  is compact.

#### Contrast with Non-Examples

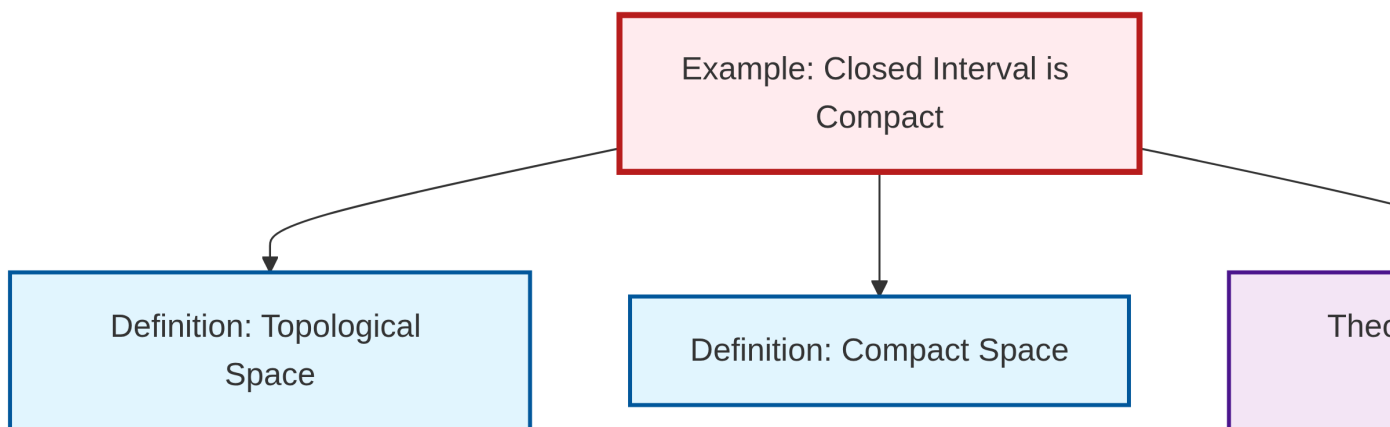
- The open interval  $(a, b)$  is **not compact** (it's bounded but not closed)
- The ray  $[a, \infty)$  is **not compact** (it's closed but not bounded)
- The entire real line  $\mathbb{R}$  is **not compact** (it's closed but not bounded)

#### Applications

The compactness of closed intervals is fundamental in analysis: - It ensures that continuous functions on  $[a, b]$  attain their maximum and minimum - It's key to proving the uniform continuity of continuous functions on  $[a, b]$  - It's essential for the Riemann integrability of continuous functions



Dependency Graph



Local dependency graph