# Theorem: Heine-Borel Theorem

## Heine-Borel Theorem

A subset of Euclidean space $\mathbb{R}^n$ is Compact Space if and only if it is Closed Set and bounded.

### Statement

For a subset $K \subseteq \mathbb{R}^n$ (with the standard Metric Space topology), the following are equivalent:

1. $K$ is compact
2. $K$ is closed and bounded

where bounded means there exists $M > 0$ such that $\|x\| \leq M$ for all $x \in K$.

### Implications

#### Forward Direction

If $K$ is compact in $\mathbb{R}^n$, then: - $K$ is closed (since $\mathbb{R}^n$ is Hausdorff) - $K$ is bounded (by considering the open cover $\{B(0, n)\}_{n=1}^{\infty}$)

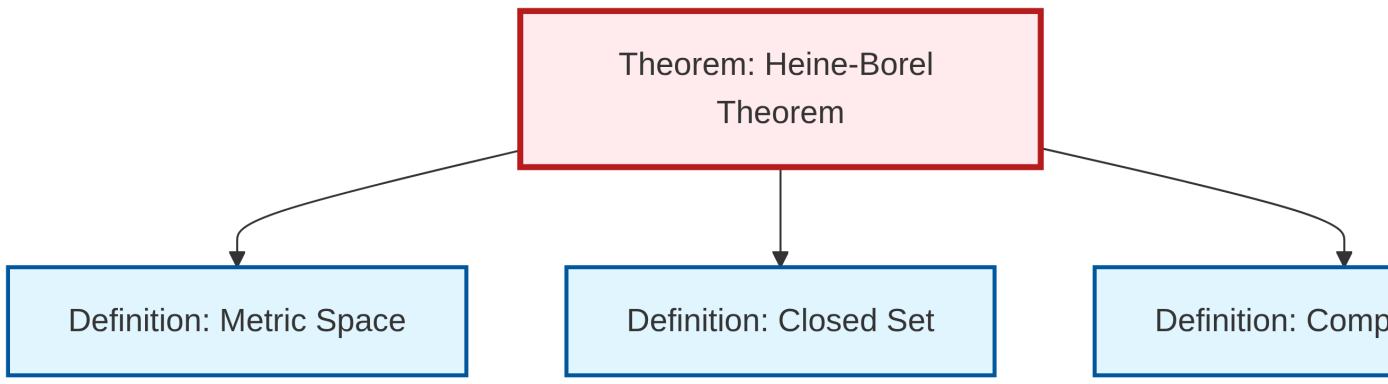
#### Reverse Direction

If $K$ is closed and bounded in $\mathbb{R}^n$, then $K$ is compact.

### Applications

- Characterizing compact sets in finite-dimensional normed spaces
- Proving existence of extrema for continuous functions
- Establishing convergence of sequences in $\mathbb{R}^n$

### Dependency Graph




Local dependency graph