Definition: Vector Space

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A **vector space** (or linear space) over a field F is a Set V together with two Binary Operations:

1. **Vector addition**: $+ : V \times V \to V$
2. **Scalar multiplication**: $\cdot : F \times V \to V$

such that the following axioms are satisfied:

Addition Axioms

For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$:

1. **Associativity**: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
2. **Commutativity**: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. **Identity element**: There exists $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$
4. **Inverse elements**: For each $\mathbf{v} \in V$, there exists $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$

Scalar Multiplication Axioms

For all $a, b \in F$ and $\mathbf{u}, \mathbf{v} \in V$:

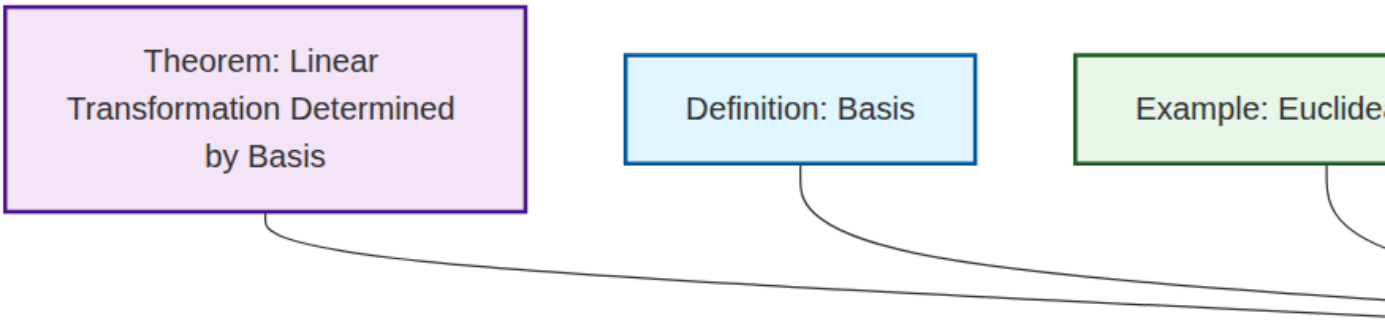
5. **Associativity**: $a(b\mathbf{v}) = (ab)\mathbf{v}$
6. **Identity**: $1\mathbf{v} = \mathbf{v}$, where 1 is the multiplicative identity in F
7. **Distributivity over vector addition**: $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
8. **Distributivity over scalar addition**: $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

Elements of V are called **vectors**, and elements of F are called **scalars**.

Remarks

- The vector space axioms ensure that $(V, +)$ forms an abelian Group
- The field F determines the "type" of vector space (e.g., real vector space when $F = \mathbb{R}$)
- Vector spaces are fundamental structures in linear algebra and appear throughout mathematics

Dependency Graph



Local dependency graph