Definition: Span

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Let V be a Vector Space over a field F, and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} \subseteq V$ be a set of vectors. The **span** of S, denoted $\mathrm{span}(S)$, is the set of all linear combinations of vectors in S:

$$\mathrm{span}(S) = \left\{ \sum_{i=1}^{k} a_i \mathbf{v}_i : a_i \in F \right\}$$

Alternative Notation

The span is also commonly denoted as: - $\langle S \rangle$ or $\langle \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \rangle$ - $L(S)$ (for "linear span") - $\mathrm{Lin}(S)$

Properties

1. **Subspace**: $\mathrm{span}(S)$ is always a subspace of V
2. **Smallest containing subspace**: $\mathrm{span}(S)$ is the smallest subspace of V that contains S
3. **Closure under linear combinations**: If $\mathbf{u}, \mathbf{w} \in \mathrm{span}(S)$ and $a, b \in F$, then $a\mathbf{u} + b\mathbf{w} \in \mathrm{span}(S)$

Special Cases

- $\mathrm{span}(\emptyset) = \{\mathbf{0}\}$ (the zero subspace)
- $\mathrm{span}(\{\mathbf{v}\}) = \{a\mathbf{v} : a \in F\}$ (the line through $\mathbf{v}$)
- If S spans V, we say S is a **spanning set** or **generating set** for V

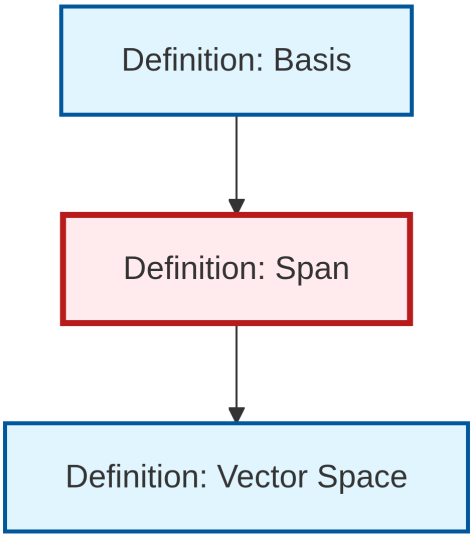
Infinite Sets

For an infinite set $S \subseteq V$, the span consists of all *finite* linear combinations:

$$\mathrm{span}(S) = \left\{ \sum_{i=1}^{n} a_i \mathbf{v}_i : n \in \mathbb{N}, \mathbf{v}_i \in S, a_i \in F \right\}$$

The concept of span connects individual vectors to the subspaces they generate.

Dependency Graph



Local dependency graph