Theorem: Union of Open Sets

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Let (X, τ) be a Topological Space. Then the union of any collection of open sets is open.

Statement

If $\{U_i\}_{i \in I}$ is any collection of open sets (where I is an arbitrary index set), then

$$\bigcup_{i \in I} U_i \in \tau$$

Proof

This follows directly from the definition of a topology.

Let $\{U_i\}_{i \in I}$ be an arbitrary collection of sets where each $U_i \in \tau$.

By the second axiom in the definition of a Topological Space, the arbitrary union of sets in τ must also be in τ.

Therefore, $\bigcup_{i \in I} U_i \in \tau$, which means $\bigcup_{i \in I} U_i$ is open. $\square$

Important Note

This theorem holds for **arbitrary** unions, including: - Finite unions - Countably infinite unions - Uncountably infinite unions

However, the analogous statement for intersections is **not** true in general. Only **finite** intersections of open sets are guaranteed to be open.

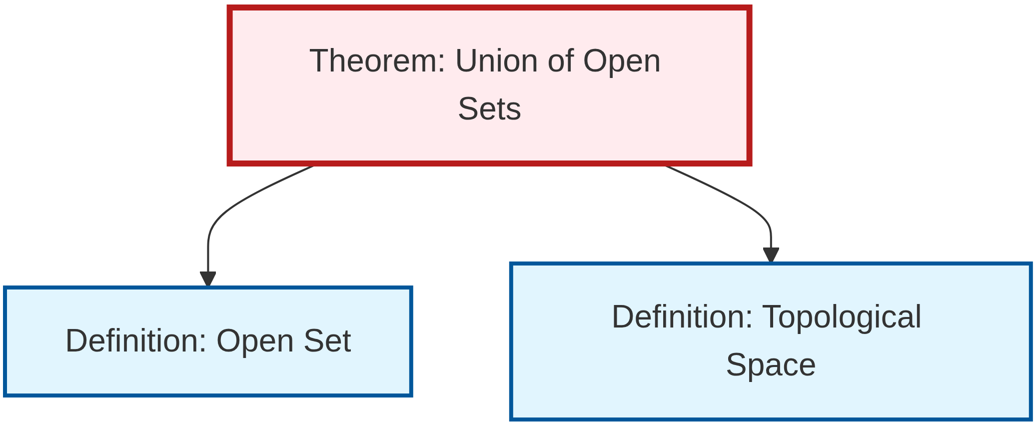
Examples

1. In $\mathbb{R}$ with the standard topology:
   - $\bigcup_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = (-1, 1)$ is open
   - $\bigcup_{x \in (0,1)} (x - \frac{1}{2}, x + \frac{1}{2}) = (-\frac{1}{2}, \frac{3}{2})$ is open
2. Counter-example for infinite intersections:
   - $\bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$ is not open in $\mathbb{R}$

Applications

This property is fundamental in: - Defining basis and subbasis for topologies - Constructing new topological spaces - Proving continuity of functions

Dependency Graph



Local dependency graph