Definition: Linear Independence

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Let V be a Vector Space over a field F. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} \subseteq V$ is called **linearly independent** if the only solution to the equation:

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$$

is $a_1 = a_2 = \cdots = a_k = 0$, where $a_i \in F$ are scalars.

Equivalent Conditions

The following are equivalent:

1. $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is linearly independent
2. No vector in the set can be expressed as a linear combination of the others
3. If $\sum_{i=1}^{k} a_i\mathbf{v}_i = \sum_{i=1}^{k} b_i\mathbf{v}_i$, then $a_i = b_i$ for all i

Linear Dependence

A set of vectors that is not linearly independent is called **linearly dependent**. This means there exist scalars $a_1, a_2, \dots, a_k$, not all zero, such that:

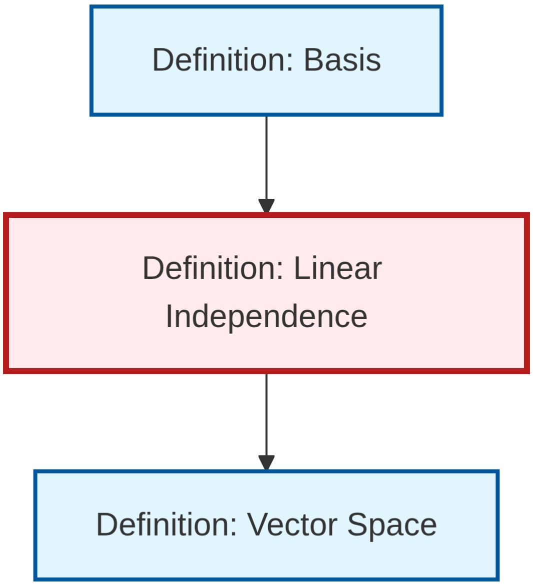
$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$$

Important Properties

- The empty set $\emptyset$ is linearly independent by convention
- A set containing the zero vector is always linearly dependent
- Any subset of a linearly independent set is linearly independent
- If a set is linearly dependent, any superset is also linearly dependent

Linear independence is a fundamental concept that determines when vectors provide "independent directions" in a vector space.

Dependency Graph



Local dependency graph