

## Definition: Complete Metric Space

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A [Metric Space](#)  $(X, d)$  is called **complete** if every Cauchy sequence in  $X$  converges to a [Limit of a Sequence](#) in  $X$ .

### Cauchy Sequences

A sequence  $(x_n)_{n=1}^{\infty}$  in a metric space  $(X, d)$  is called a **Cauchy sequence** if for every  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $m, n \geq N$ :

$$d(x_m, x_n) < \varepsilon$$

Intuitively, the terms of a Cauchy sequence get arbitrarily close to each other as the sequence progresses.

### Relationship to Convergence

- Every convergent sequence is Cauchy
- In a complete metric space, every Cauchy sequence converges
- A metric space is complete if and only if the converse holds: every Cauchy sequence converges

### Examples

**Complete metric spaces:** -  $\mathbb{R}$  with the standard metric (see [Real Line with Standard Metric](#))  
-  $\mathbb{C}$  with the standard metric - Any closed subset of a complete metric space -  $C[a, b]$  (continuous functions on  $[a, b]$ ) with the supremum metric

**Incomplete metric spaces:** -  $\mathbb{Q}$  with the standard metric (e.g., the sequence  $(x_n)$  where  $x_n$  is the  $n$ -th decimal approximation of  $\sqrt{2}$  is Cauchy but doesn't converge in  $\mathbb{Q}$ ) -  $(0, 1)$  with the standard metric -  $C[a, b]$  with the  $L^1$  metric

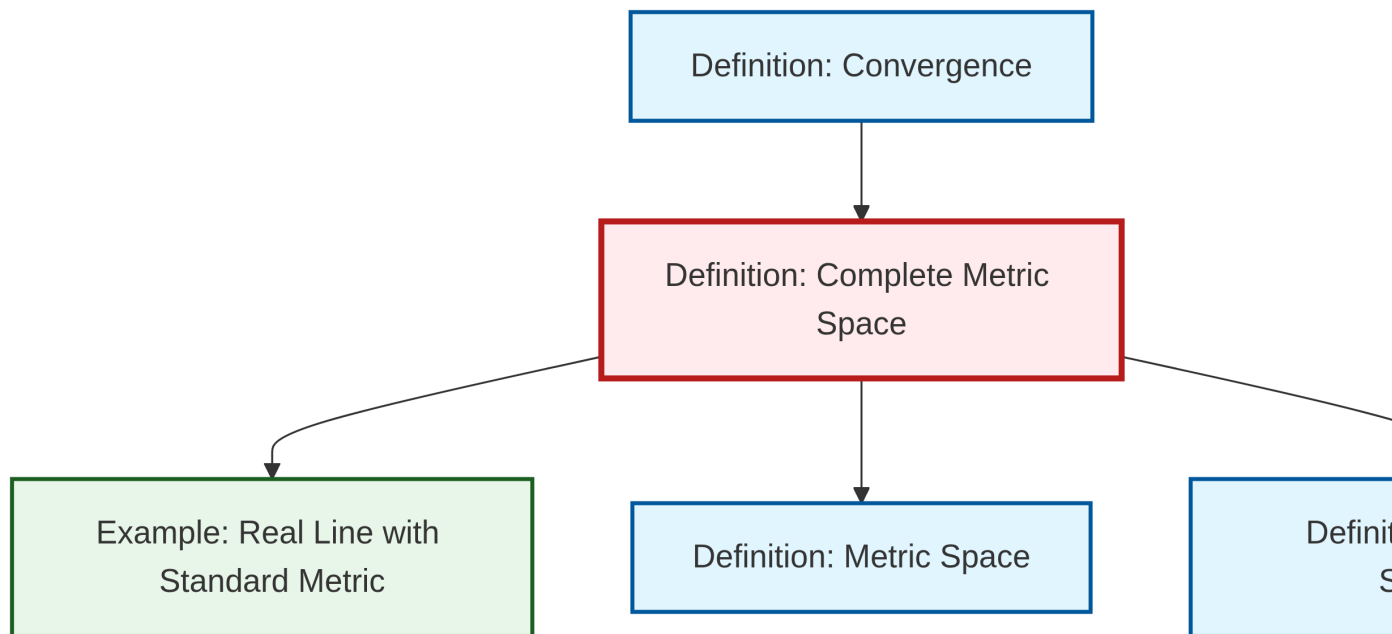
### Importance

Completeness is crucial for: - **Fixed point theorems:** The Banach fixed-point theorem requires completeness - **Existence theorems:** Many existence results in analysis rely on completeness - **Construction of  $\mathbb{R}$ :** The real numbers can be constructed as the completion of  $\mathbb{Q}$

### See Also

- [?@thm-banach-fixed-point](#) - A fundamental theorem requiring completeness
- [?@def-banach-space](#) - A complete normed vector space
- [?@thm-baire-category](#) - An important theorem about complete metric spaces

## Dependency Graph



Local dependency graph