

# Theorem: Rank-Nullity Theorem

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For any **Linear Transformation**  $T : V \rightarrow W$  between finite-dimensional **s**, the dimensions of the **Kernel** and **Image** satisfy a fundamental relationship.

### Statement

Let  $T : V \rightarrow W$  be a linear transformation where  $V$  is finite-dimensional. Then:

$$\dim(V) = \dim(\ker(T)) + \dim(\operatorname{im}(T))$$

where: -  $\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}$  is the kernel (null space) of  $T$  -  $\operatorname{im}(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\}$  is the image (range) of  $T$

### Alternative Formulation

Using standard terminology:

$$\operatorname{nullity}(T) + \operatorname{rank}(T) = \dim(V)$$

where: -  $\operatorname{nullity}(T) = \dim(\ker(T))$  -  $\operatorname{rank}(T) = \dim(\operatorname{im}(T))$

### Proof Outline

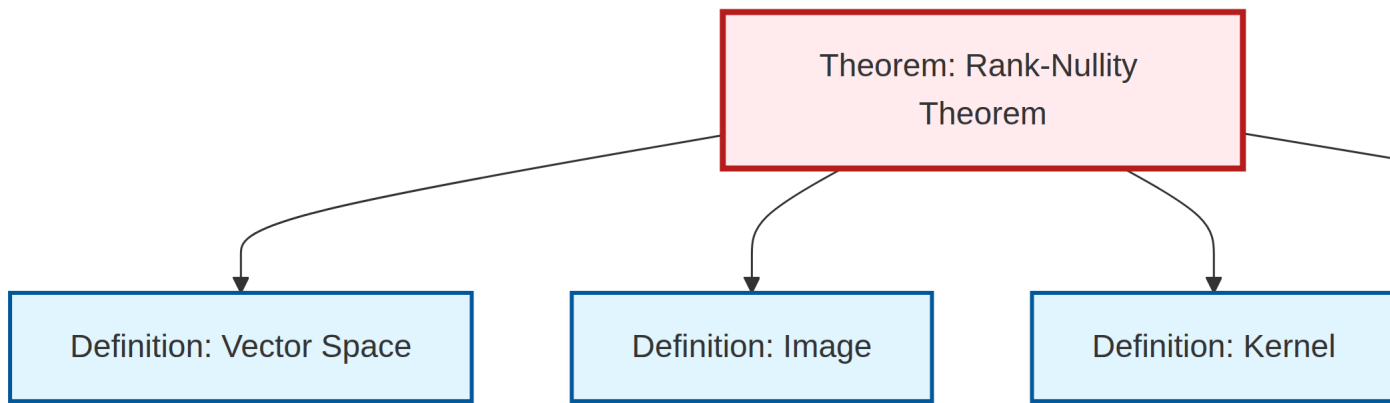
1. Let  $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$  be a basis for  $\ker(T)$
2. Extend to a basis  $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}_1, \dots, \mathbf{v}_r\}$  for  $V$
3. Show that  $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_r)\}$  is a basis for  $\operatorname{im}(T)$
4. Therefore:  $k + r = \dim(V)$ , where  $k = \dim(\ker(T))$  and  $r = \dim(\operatorname{im}(T))$

### Consequences

- **Injectivity criterion:**  $T$  is injective if and only if  $\dim(\ker(T)) = 0$
- **Surjectivity criterion:**  $T$  is surjective if and only if  $\dim(\operatorname{im}(T)) = \dim(W)$
- **Isomorphism criterion:** For  $\dim(V) = \dim(W)$ ,  $T$  is an isomorphism if and only if  $T$  is injective (or surjective)

This theorem is fundamental to understanding the structure of linear transformations and solving systems of linear equations.

## Dependency Graph



## Local dependency graph