Definition: Complete Metric Space

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A Metric Space (X, d) is called **complete** if every Cauchy sequence in X converges to a Limit of a Sequence in X.

Cauchy Sequences

A sequence $(x_n)_{n=1}^{\infty}$ in a metric space (X, d) is called a **Cauchy sequence** if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$:

$$d(x_m, x_n) < \varepsilon$$

Intuitively, the terms of a Cauchy sequence get arbitrarily close to each other as the sequence progresses.

Relationship to Convergence

- Every convergent sequence is Cauchy
- In a complete metric space, every Cauchy sequence converges
- A metric space is complete if and only if the converse holds: every Cauchy sequence converges

Examples

Complete metric spaces: - $\mathbb{R}$ with the standard metric (see Real Line with Standard Metric) - $\mathbb{C}$ with the standard metric - Any closed subset of a complete metric space - $C[a, b]$ (continuous functions on $[a, b]$) with the supremum metric

Incomplete metric spaces: - $\mathbb{Q}$ with the standard metric (e.g., the sequence (x_n) where x_n is the n-th decimal approximation of $\sqrt{2}$ is Cauchy but doesn't converge in $\mathbb{Q}$) - $(0, 1)$ with the standard metric - $C[a, b]$ with the L^1 metric

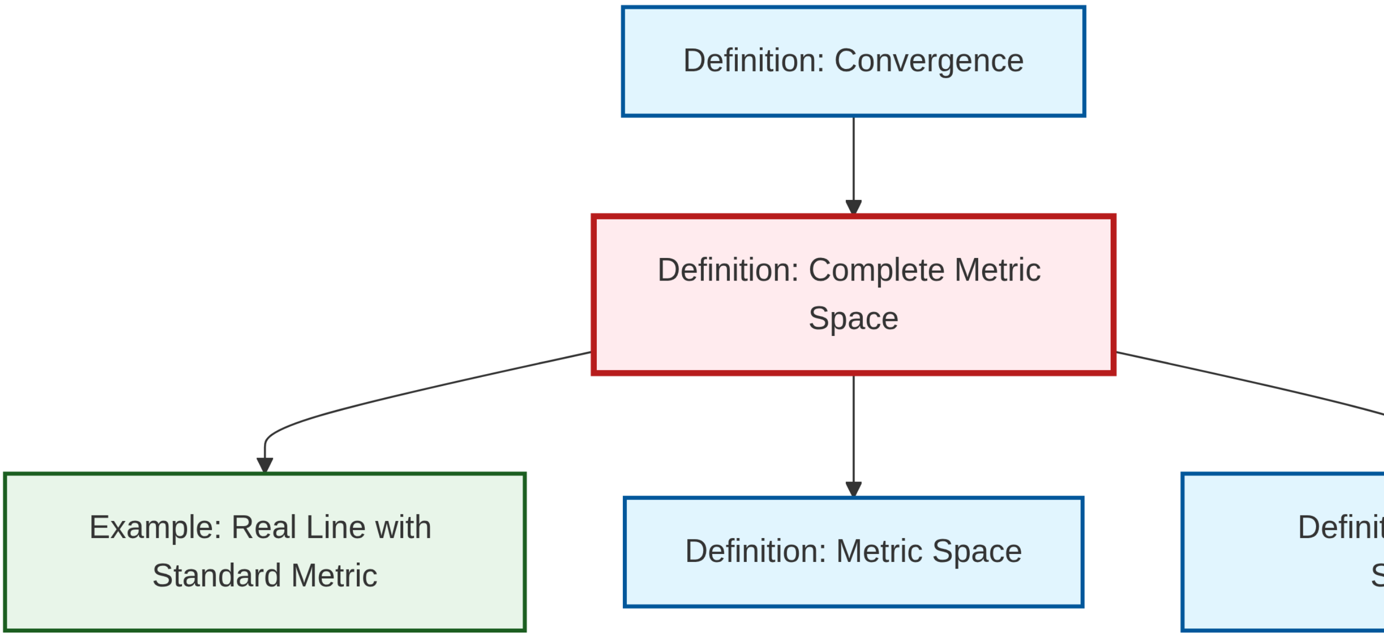
Importance

Completeness is crucial for: - **Fixed point theorems**: The Banach fixed-point theorem requires completeness - **Existence theorems**: Many existence results in analysis rely on completeness - **Construction of $\mathbb{R}$**: The real numbers can be constructed as the completion of $\mathbb{Q}$

See Also

- **?@thm-banach-fixed-point** - A fundamental theorem requiring completeness
- **?@def-banach-space** - A complete normed vector space
- **?@thm-baire-category** - An important theorem about complete metric spaces

Dependency Graph



Local dependency graph