Definition: Commutativity

Commutativity is a property of a Binary Operation where the order of the operands does not affect the result.

Formal Definition

A binary operation $*$ on a set S is **commutative** (or **abelian**) if for all $a, b \in S$:

$$a * b = b * a$$

Examples of Commutative Operations

Arithmetic

- **Addition**: $a + b = b + a$
- **Multiplication**: $a \cdot b = b \cdot a$

Set Theory

- **Union**: $A \cup B = B \cup A$
- **Intersection**: $A \cap B = B \cap A$
- **Symmetric difference**: $A \triangle B = B \triangle A$

Logic

- **AND**: $p \wedge q = q \wedge p$
- **OR**: $p \vee q = q \vee p$
- **XOR**: $p \oplus q = q \oplus p$

Number Theory

- **GCD**: $\gcd(a, b) = \gcd(b, a)$
- **LCM**: $\text{lcm}(a, b) = \text{lcm}(b, a)$

Non-Commutative Operations

Arithmetic

- **Subtraction**: $a - b \neq b - a$ (unless $a = b$)
  - Example: $5 - 3 = 2$ but $3 - 5 = -2$
- **Division**: $a \div b \neq b \div a$ (unless $a = b$ or both equal 1)
  - Example: $6 \div 2 = 3$ but $2 \div 6 = \frac{1}{3}$

Linear Algebra

- **Matrix multiplication**: $\mathbf{AB} \neq \mathbf{BA}$ in general
- **Cross product**: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ (anti-commutative)

Other

- **Function composition**: $f \circ g \neq g \circ f$ in general
- **String concatenation**: "ab" $\neq$ "ba"

Importance

1. **Abelian Groups**: Groups with commutative operation
   - Examples: $(\mathbb{Z}, +)$, $(\mathbb{R}^*, \cdot)$
2. **Simplification**: Allows reordering of terms
   - In expressions like $a + b + c + d$, can rearrange freely
3. **Parallel Computation**: Commutative operations can be parallelized more easily

Relationship with Other Properties

- **Independent of Associativity**: An operation can be:
  - Commutative but not associative
  - Associative but not commutative
  - Both (e.g., addition)
  - Neither (e.g., subtraction)

Special Cases

Commutators

For non-commutative operations, the **commutator** measures failure of commutativity: - In groups: $[a, b] = a * b * a^{-1} * b^{-1}$ - In rings: $[a, b] = ab - ba$

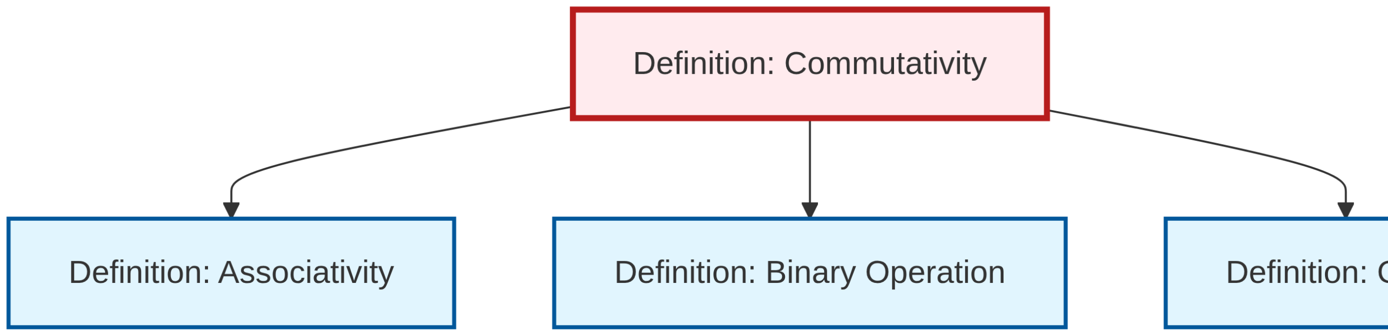
Graded Commutativity

In graded algebras: $ab = (-1)^{|a||b|}ba$ where $|a|$ is the degree of a

Applications

- **Cryptography**: Commutative encryption allows flexible ordering
- **Database queries**: Commutative operations enable query optimization
- **Physics**: Commuting observables can be measured simultaneously

Dependency Graph



Local dependency graph