Theorem: Rank-Nullity Theorem

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For any Linear Transformation $T : V \to W$ between finite-dimensional s, the dimensions of the Kernel and Image satisfy a fundamental relationship.

Statement

Let $T : V \to W$ be a linear transformation where V is finite-dimensional. Then:

$$\dim(V) = \dim(\ker(T)) + \dim(\mathrm{im}(T))$$

where: - $\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}$ is the kernel (null space) of T - $\mathrm{im}(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\}$ is the image (range) of T

Alternative Formulation

Using standard terminology:

$$\mathrm{nullity}(T) + \mathrm{rank}(T) = \dim(V)$$

where: - $\mathrm{nullity}(T) = \dim(\ker(T))$ - $\mathrm{rank}(T) = \dim(\mathrm{im}(T))$

Proof Outline

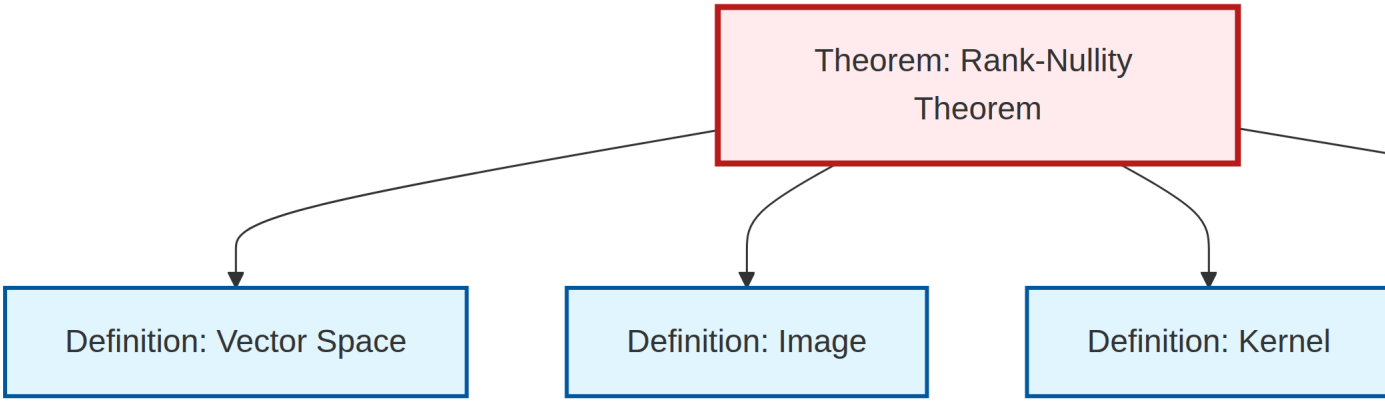
1. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be a basis for $\ker(T)$
2. Extend to a basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}_1, \dots, \mathbf{v}_r\}$ for V
3. Show that $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_r)\}$ is a basis for $\mathrm{im}(T)$
4. Therefore: $k + r = \dim(V)$, where $k = \dim(\ker(T))$ and $r = \dim(\mathrm{im}(T))$

Consequences

- **Injectivity criterion**: T is injective if and only if $\dim(\ker(T)) = 0$
- **Surjectivity criterion**: T is surjective if and only if $\dim(\mathrm{im}(T)) = \dim(W)$
- **Isomorphism criterion**: For $\dim(V) = \dim(W)$, T is an isomorphism if and only if T is injective (or surjective)

This theorem is fundamental to understanding the structure of linear transformations and solving systems of linear equations.

Dependency Graph



Local dependency graph