Theorem: Bayes' Theorem

Bayes' Theorem provides a way to calculate Conditional Probability in the "reverse" direction. It relates $P(A|B)$ to $P(B|A)$, allowing us to update beliefs based on new evidence.

Statement

For Events A and B with $P(A) > 0$ and $P(B) > 0$:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

Alternative Forms

Using Law of Total Probability

If $\{A_1, A_2, ..., A_n\}$ form a partition of the sample space:

$$P(A_i|B) = \frac{P(B|A_i) \cdot P(A_i)}{\sum_{j=1}^{n} P(B|A_j) \cdot P(A_j)}$$

Binary Case

For event A and its complement A^c:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B|A) \cdot P(A) + P(B|A^c) \cdot P(A^c)}$$

Proof

From the definition of conditional probability: 1. $P(A|B) = \frac{P(A \cap B)}{P(B)}$ 2. $P(B|A) = \frac{P(A \cap B)}{P(A)}$

From (2): $P(A \cap B) = P(B|A) \cdot P(A)$

Substituting into (1):

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

Terminology

- $P(A)$: **Prior probability** of A
- $P(A|B)$: **Posterior probability** of A given evidence B
- $P(B|A)$: **Likelihood** of observing B given A
- $P(B)$: **Marginal probability** or normalizing constant

Applications

Medical Diagnosis

- A: Patient has disease
- B: Test is positive
- Calculate $P(\text{Disease}|\text{Positive test})$ from known test accuracy

Machine Learning

- Naive Bayes classifiers
- Bayesian inference and parameter estimation
- Updating model beliefs with new data

Example: Medical Test

Given: - Disease prevalence: $P(D) = 0.001$ (0.1%) - Test sensitivity: $P(+|D) = 0.99$ (99%) - Test specificity: $P(-|D^c) = 0.95$, so $P(+|D^c) = 0.05$

Find $P(D|+)$:

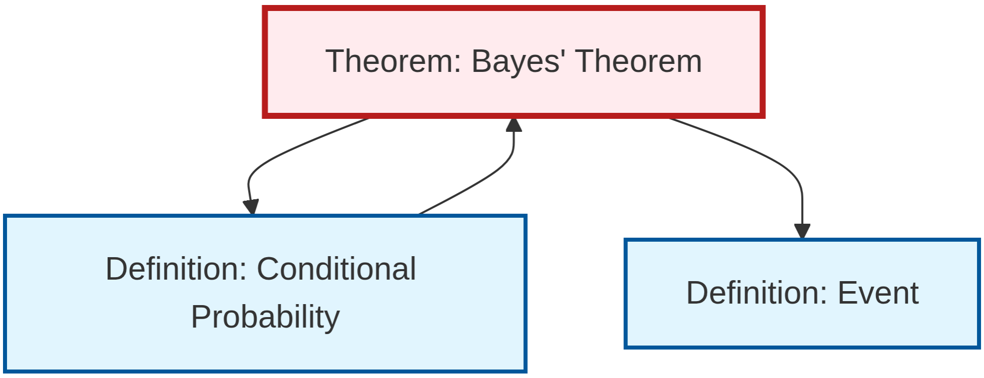
$$P(D|+) = \frac{0.99 \times 0.001}{0.99 \times 0.001 + 0.05 \times 0.999} = \frac{0.00099}{0.05094} \approx 0.0194$$

Despite a positive test, the probability of having the disease is only about 1.94%!

Bayesian Updating

Bayes' theorem enables iterative belief updating: 1. Start with prior $P(A)$ 2. Observe evidence B 3. Update to posterior $P(A|B)$ 4. Use posterior as new prior for next observation

Dependency Graph



Local dependency graph