Example: Real Line with Standard Metric

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The real numbers $\mathbb{R}$ with the standard metric form one of the most fundamental examples of a Metric Space.

Definition

The **standard metric** on $\mathbb{R}$ is defined by:

$$d(x, y) = |x - y|$$

for all $x, y \in \mathbb{R}$, where $|x - y|$ denotes the absolute value of $x - y$.

Verification of Metric Axioms

We verify that d satisfies all four metric axioms:

1. **Non-negativity**: Since absolute value is always non-negative, $d(x, y) = |x - y| \geq 0$ for all $x, y \in \mathbb{R}$.
2. **Identity of indiscernibles**:
   - If $x = y$, then $d(x, y) = |x - y| = |0| = 0$
   - If $d(x, y) = 0$, then $|x - y| = 0$, which implies $x - y = 0$, so $x = y$
3. **Symmetry**: $d(x, y) = |x - y| = |-(y - x)| = |y - x| = d(y, x)$
4. **Triangle inequality**: For any $x, y, z \in \mathbb{R}$:

$$d(x, z) = |x - z| = |(x - y) + (y - z)| \leq |x - y| + |y - z| = d(x, y) + d(y, z)$$

Open Balls

In this metric space, the open ball $B(a, r)$ centered at a with radius $r > 0$ is:

$$B(a, r) = \{x \in \mathbb{R} : |x - a| < r\} = (a - r, a + r)$$

This is simply the open interval of length $2r$ centered at a.

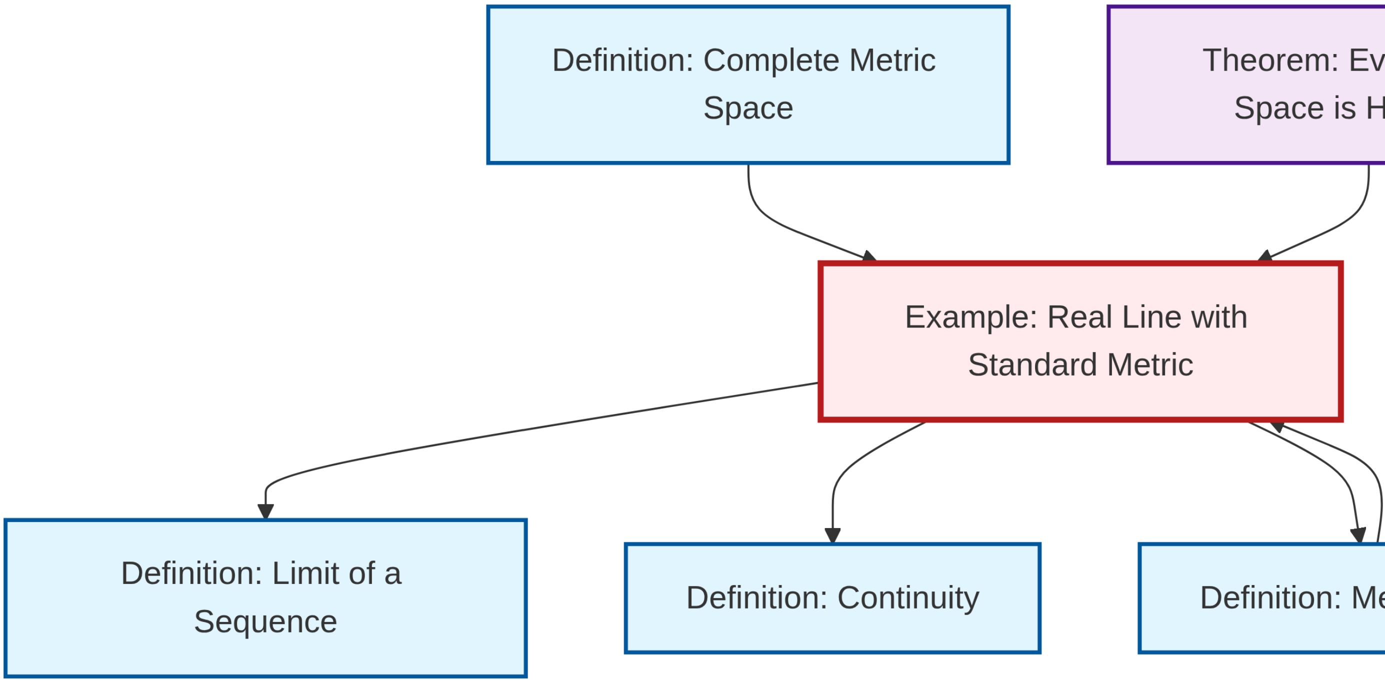
Properties

- The metric topology induced by d is the standard topology on $\mathbb{R}$
- $(\mathbb{R}, d)$ is a **complete metric space**: every Cauchy sequence converges
- The space is **separable**: the rationals $\mathbb{Q}$ form a countable dense subset
- It is **connected** but not **compact**

See Also

- Limit of a Sequence - Limits in calculus are defined using this metric
- Continuity - Continuous functions preserve this metric structure

Dependency Graph



Local dependency graph