Definition: Metric Space

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A **metric space** is a Set X together with a function $d : X \times X \to \mathbb{R}$ called a **metric** (or **distance function**) that satisfies certain axioms capturing the intuitive properties of distance.

Metric Axioms

For all $x, y, z \in X$, the metric d must satisfy:

1. **Non-negativity**: $d(x, y) \geq 0$
2. **Identity of indiscernibles**: $d(x, y) = 0$ if and only if $x = y$
3. **Symmetry**: $d(x, y) = d(y, x)$
4. **Triangle inequality**: $d(x, z) \leq d(x, y) + d(y, z)$

The pair (X, d) is called a metric space.

Examples

1. **Euclidean metric** on $\mathbb{R}^n$:

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

2. **Discrete metric** on any set X:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

3. **Manhattan metric** on $\mathbb{R}^n$:

$$d(x, y) = \sum_{i=1}^{n} |x_i - y_i|$$

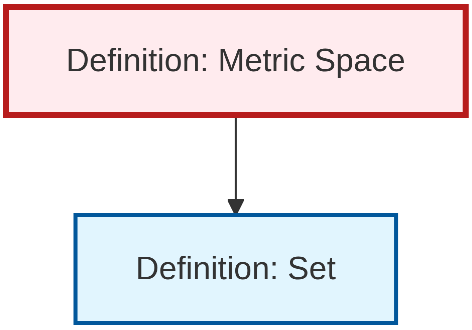
Key Concepts

- **Open ball**: $B(x, r) = \{y \in X : d(x, y) < r\}$
- **Closed ball**: $\bar{B}(x, r) = \{y \in X : d(x, y) \leq r\}$
- **Bounded set**: $A \subseteq X$ is bounded if $\exists M > 0, x_0 \in X$ such that $A \subseteq B(x_0, M)$

Importance

Metric spaces: - Generalize the notion of distance beyond Euclidean space - Provide a framework for studying continuity and convergence - Form the foundation for analysis and topology - Connect geometry with analysis

Dependency Graph



Local dependency graph