Definition: Image

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Let $T : V \to W$ be a Linear Transformation between s. The **image** (or **range**) of T is the set of all vectors in W that are outputs of T:

$$\text{im}(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\} = T(V)$$

Alternative Names

The image is also known as: - Range (denoted range(T) or $R(T)$) - The image of the domain under T

Properties

1. **Subspace**: $\text{im}(T)$ is always a subspace of W
   - Contains $\mathbf{0}_W$ since $T(\mathbf{0}_V) = \mathbf{0}_W$
   - Closed under addition: if $\mathbf{w}_1 = T(\mathbf{v}_1)$ and $\mathbf{w}_2 = T(\mathbf{v}_2)$, then $\mathbf{w}_1 + \mathbf{w}_2 = T(\mathbf{v}_1) + T(\mathbf{v}_2) = T(\mathbf{v}_1 + \mathbf{v}_2) \in \text{im}(T)$
   - Closed under scalar multiplication: if $\mathbf{w} = T(\mathbf{v})$ and $a \in F$, then $a\mathbf{w} = aT(\mathbf{v}) = T(a\mathbf{v}) \in \text{im}(T)$
2. **Surjectivity criterion**: T is surjective (onto) if and only if $\text{im}(T) = W$
3. **Dimension**: The dimension of $\text{im}(T)$ is called the **rank** of T

Relationship to Basis

If $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for V, then:

$$\text{im}(T) = \text{span}\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$$

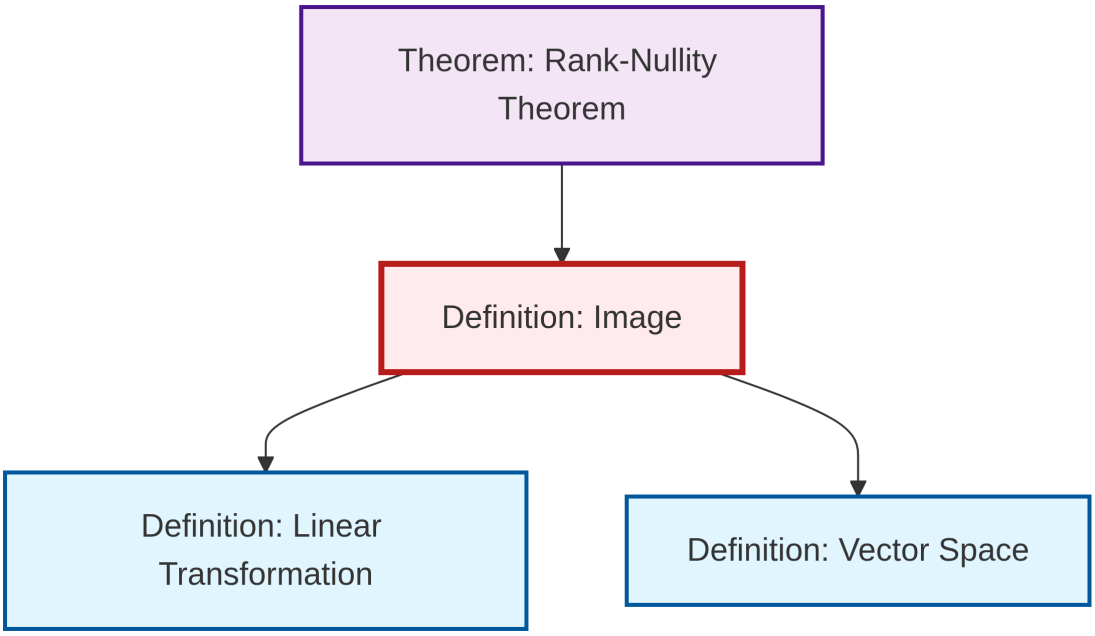
Example

For a matrix transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T_A(\mathbf{x}) = A\mathbf{x}$:

$$\text{im}(T_A) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}$$

This is the column space of A, i.e., the span of the columns of A.

Dependency Graph



Local dependency graph