

## Definition: Image

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Let  $T : V \rightarrow W$  be a [Linear Transformation](#) between [s](#). The **image** (or **range**) of  $T$  is the set of all vectors in  $W$  that are outputs of  $T$ :

$$\text{im}(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\} = T(V)$$

### Alternative Names

The image is also known as: - Range (denoted  $\text{range}(T)$  or  $R(T)$ ) - The image of the domain under  $T$

### Properties

1. **Subspace:**  $\text{im}(T)$  is always a subspace of  $W$ 
  - Contains  $\mathbf{0}_W$  since  $T(\mathbf{0}_V) = \mathbf{0}_W$
  - Closed under addition: if  $\mathbf{w}_1 = T(\mathbf{v}_1)$  and  $\mathbf{w}_2 = T(\mathbf{v}_2)$ , then  $\mathbf{w}_1 + \mathbf{w}_2 = T(\mathbf{v}_1) + T(\mathbf{v}_2) = T(\mathbf{v}_1 + \mathbf{v}_2) \in \text{im}(T)$
  - Closed under scalar multiplication: if  $\mathbf{w} = T(\mathbf{v})$  and  $a \in F$ , then  $a\mathbf{w} = aT(\mathbf{v}) = T(a\mathbf{v}) \in \text{im}(T)$
2. **Surjectivity criterion:**  $T$  is surjective (onto) if and only if  $\text{im}(T) = W$
3. **Dimension:** The dimension of  $\text{im}(T)$  is called the **rank** of  $T$

### Relationship to Basis

If  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is a basis for  $V$ , then:

$$\text{im}(T) = \text{span}\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$$

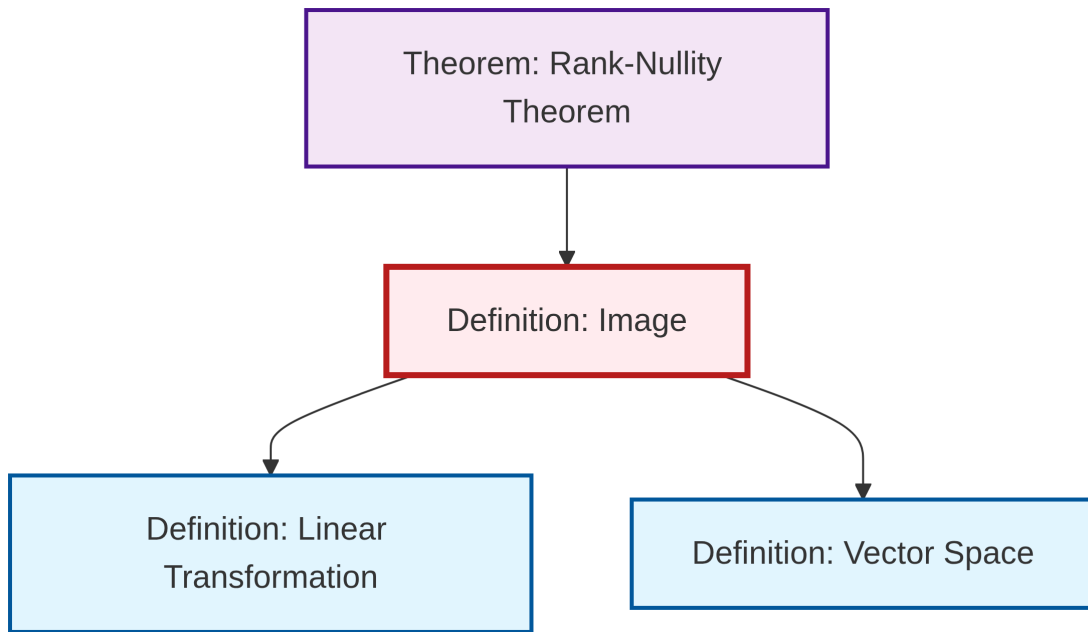
### Example

For a matrix transformation  $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  defined by  $T_A(\mathbf{x}) = A\mathbf{x}$ :

$$\text{im}(T_A) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}$$

This is the column space of  $A$ , i.e., the span of the columns of  $A$ .

## Dependency Graph



Local dependency graph