Theorem: Mean Value Theorem

The **Mean Value Theorem (MVT)** states that for a differentiable function on an interval, there exists at least one point where the instantaneous rate of change equals the average rate of change over the interval.

Statement

If $f : [a, b] \to \mathbb{R}$ satisfies: 1. f is continuous on the closed interval $[a, b]$ 2. f is differentiable on the open interval (a, b)

Then there exists at least one $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometric Interpretation

The theorem guarantees that somewhere between a and b, the tangent line to the curve is parallel to the secant line connecting $(a, f(a))$ and $(b, f(b))$.

Special Cases

Rolle's Theorem

If additionally $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

This is MVT with horizontal secant line.

Cauchy Mean Value Theorem

For functions f, g both continuous on $[a, b]$ and differentiable on (a, b), with $g'(x) \neq 0$ on (a, b):

There exists $c \in (a, b)$ such that:

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof Sketch

1. Define auxiliary function: $h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x - a)$
2. Note that $h(a) = h(b) = 0$
3. Apply Rolle's theorem to h
4. Find c where $h'(c) = 0$, which gives the result

Consequences

1. **Monotonicity**: If $f' > 0$ on (a, b), then f is strictly increasing
2. **Constant functions**: If $f' = 0$ on (a, b), then f is constant
3. **Lipschitz continuity**: If $|f'| \leq M$ on (a, b), then $|f(x) - f(y)| \leq M|x - y|$

Applications

L'Hôpital's Rule

Used in the proof for evaluating limits of indeterminate forms

Taylor's Theorem

MVT generalizes to higher derivatives, leading to Taylor approximations

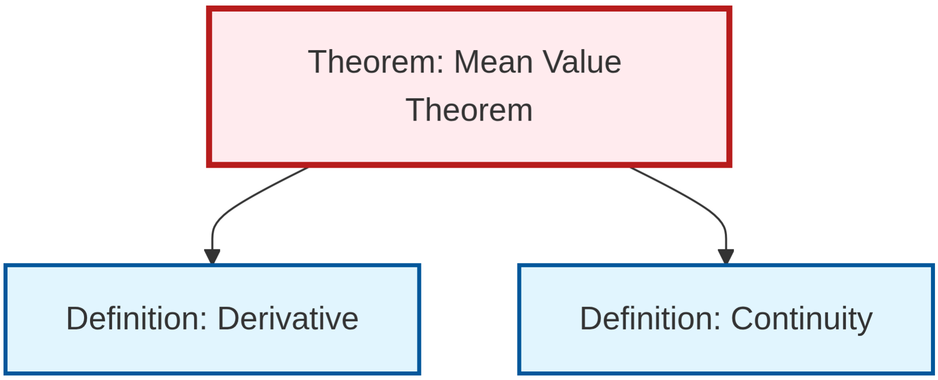
Numerical Analysis

- Error bounds for numerical differentiation
- Convergence analysis of iterative methods

Examples

1. **Verify MVT**: For $f(x) = x^2$ on $[0, 2]$:
   - Average slope: $\frac{4-0}{2-0} = 2$
   - $f'(x) = 2x$, so $f'(c) = 2$ gives $c = 1 \in (0, 2)$
2. **Application**: Prove that $|\sin x - \sin y| \leq |x - y|$:
   - By MVT: $\sin x - \sin y = \cos c \cdot (x - y)$ for some c
   - Since $|\cos c| \leq 1$, the result follows

Dependency Graph



Local dependency graph