

## Definition: Isomorphism

An **isomorphism** in a **Category** is a **Morphism** that has a two-sided inverse. Isomorphic objects are considered “the same” from the categorical perspective.

### Formal Definition

A morphism  $f : A \rightarrow B$  in a category  $\mathcal{C}$  is an **isomorphism** if there exists a morphism  $g : B \rightarrow A$  such that: 1.  $g \circ f = \text{id}_A$  2.  $f \circ g = \text{id}_B$

The morphism  $g$  is called the **inverse** of  $f$ , denoted  $f^{-1}$ .

### Properties

1. **Uniqueness of Inverse:** If  $f$  has an inverse, it is unique
2. **Symmetry:** If  $f : A \rightarrow B$  is an isomorphism, then  $f^{-1} : B \rightarrow A$  is also an isomorphism
3. **Composition:** The **Composition** of isomorphisms is an isomorphism:
  - If  $f : A \rightarrow B$  and  $g : B \rightarrow C$  are isomorphisms
  - Then  $g \circ f : A \rightarrow C$  is an isomorphism with  $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

### Isomorphic Objects

Objects  $A$  and  $B$  are **isomorphic**, written  $A \cong B$ , if there exists an isomorphism  $f : A \rightarrow B$ .

Isomorphism is an equivalence relation: - **Reflexive:**  $A \cong A$  via  $\text{id}_A$  - **Symmetric:**  $A \cong B$  implies  $B \cong A$  - **Transitive:**  $A \cong B$  and  $B \cong C$  implies  $A \cong C$

### Examples in Different Categories

#### Set

- Isomorphisms are bijective functions
- Two sets are isomorphic iff they have the same cardinality

#### Group

- Isomorphisms are bijective group homomorphisms
- Example:  $(\mathbb{Z}, +) \cong (2\mathbb{Z}, +)$  via  $n \mapsto 2n$

#### Top (Topological Spaces)

- Isomorphisms are homeomorphisms (continuous bijections with continuous inverse)
- Example: Open interval  $(0, 1)$  is homeomorphic to  $\mathbb{R}$

#### Vect (Vector Spaces)

- Isomorphisms are invertible linear transformations
- Finite-dimensional spaces are isomorphic iff they have the same dimension

## Special Types

### Automorphism

An isomorphism from an object to itself:  $f : A \rightarrow A$

### Natural Isomorphism

An isomorphism in the functor category (see [Natural Transformation](#))

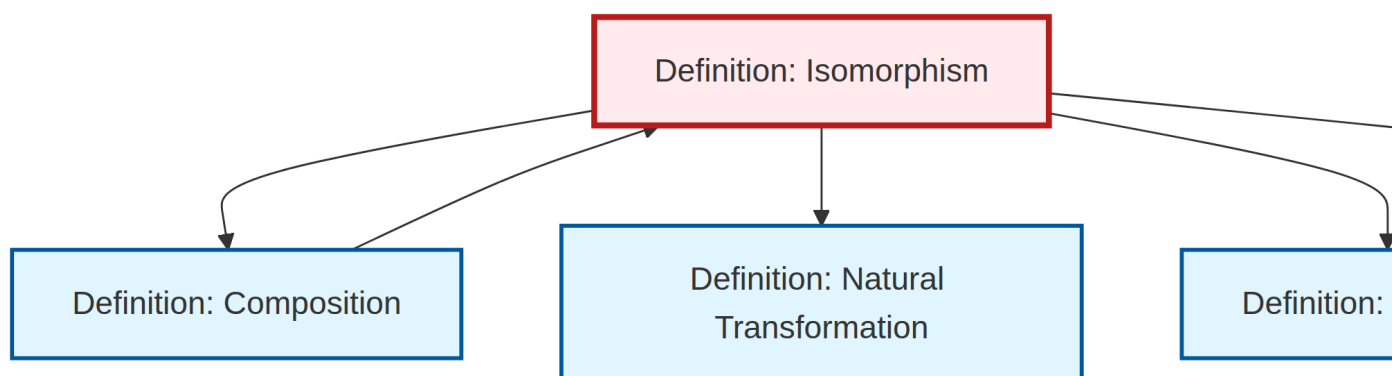
## Categorical Perspective

- Isomorphism captures the idea of “sameness” in a category
- Category theory studies properties invariant under isomorphism
- The principle of equivalence: isomorphic objects are indistinguishable within the category

## Non-Examples

- In **Set**: Injective but non-surjective functions
- In **Group**: Non-bijective homomorphisms
- Continuous functions that aren't homeomorphisms

## Dependency Graph



Local dependency graph