

## Theorem: Mean Value Theorem

The **Mean Value Theorem (MVT)** states that for a differentiable function on an interval, there exists at least one point where the instantaneous rate of change equals the average rate of change over the interval.

### Statement

If  $f : [a, b] \rightarrow \mathbb{R}$  satisfies: 1.  $f$  is **continuous** on the closed interval  $[a, b]$  2.  $f$  is differentiable on the open interval  $(a, b)$

Then there exists at least one  $c \in (a, b)$  such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

### Geometric Interpretation

The theorem guarantees that somewhere between  $a$  and  $b$ , the tangent line to the curve is parallel to the secant line connecting  $(a, f(a))$  and  $(b, f(b))$ .

### Special Cases

#### Rolle's Theorem

If additionally  $f(a) = f(b)$ , then there exists  $c \in (a, b)$  such that  $f'(c) = 0$ .

This is MVT with horizontal secant line.

#### Cauchy Mean Value Theorem

For functions  $f, g$  both continuous on  $[a, b]$  and differentiable on  $(a, b)$ , with  $g'(x) \neq 0$  on  $(a, b)$ :

There exists  $c \in (a, b)$  such that:

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

### Proof Sketch

1. Define auxiliary function:  $h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$
2. Note that  $h(a) = h(b) = 0$
3. Apply Rolle's theorem to  $h$
4. Find  $c$  where  $h'(c) = 0$ , which gives the result

### Consequences

1. **Monotonicity:** If  $f' > 0$  on  $(a, b)$ , then  $f$  is strictly increasing
2. **Constant functions:** If  $f' = 0$  on  $(a, b)$ , then  $f$  is constant
3. **Lipschitz continuity:** If  $|f'| \leq M$  on  $(a, b)$ , then  $|f(x) - f(y)| \leq M|x - y|$

## Applications

### L'Hôpital's Rule

Used in the proof for evaluating limits of indeterminate forms

### Taylor's Theorem

MVT generalizes to higher derivatives, leading to Taylor approximations

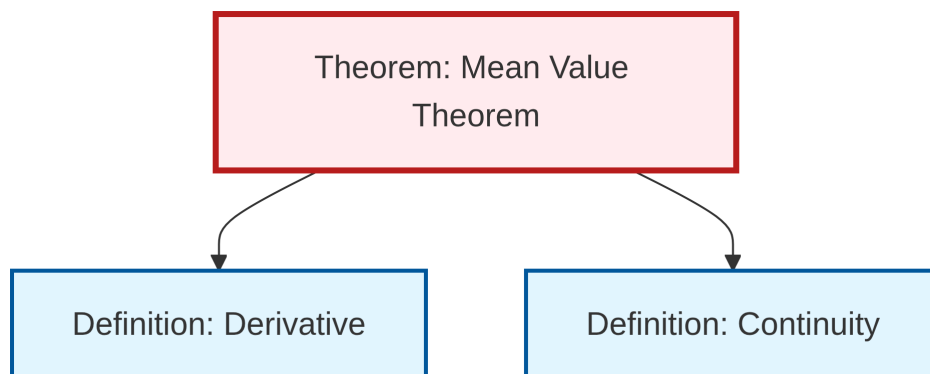
### Numerical Analysis

- Error bounds for numerical differentiation
- Convergence analysis of iterative methods

### Examples

1. **Verify MVT:** For  $f(x) = x^2$  on  $[0, 2]$ :
  - Average slope:  $\frac{4-0}{2-0} = 2$
  - $f'(x) = 2x$ , so  $f'(c) = 2$  gives  $c = 1 \in (0, 2)$
2. **Application:** Prove that  $|\sin x - \sin y| \leq |x - y|$ :
  - By MVT:  $\sin x - \sin y = \cos c \cdot (x - y)$  for some  $c$
  - Since  $|\cos c| \leq 1$ , the result follows

### Dependency Graph



Local dependency graph