

## Definition: Kernel

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Let  $T : V \rightarrow W$  be a [Linear Transformation](#) between [s](#). The **kernel** (or **null space**) of  $T$  is the set of all vectors in  $V$  that map to the zero vector in  $W$ :

$$\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}_W\}$$

### Alternative Names

The kernel is also known as: - Null space (denoted  $\text{null}(T)$  or  $N(T)$ ) - The pre-image of zero:  $T^{-1}(\{\mathbf{0}_W\})$

### Properties

1. **Subspace:**  $\ker(T)$  is always a subspace of  $V$ 
  - Contains  $\mathbf{0}_V$  since  $T(\mathbf{0}_V) = \mathbf{0}_W$
  - Closed under addition: if  $\mathbf{u}, \mathbf{v} \in \ker(T)$ , then  $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{0} + \mathbf{0} = \mathbf{0}$
  - Closed under scalar multiplication: if  $\mathbf{v} \in \ker(T)$  and  $a \in F$ , then  $T(a\mathbf{v}) = aT(\mathbf{v}) = a\mathbf{0} = \mathbf{0}$
2. **Injectivity criterion:**  $T$  is injective (one-to-one) if and only if  $\ker(T) = \{\mathbf{0}_V\}$
3. **Dimension:** The dimension of  $\ker(T)$  is called the **nullity** of  $T$

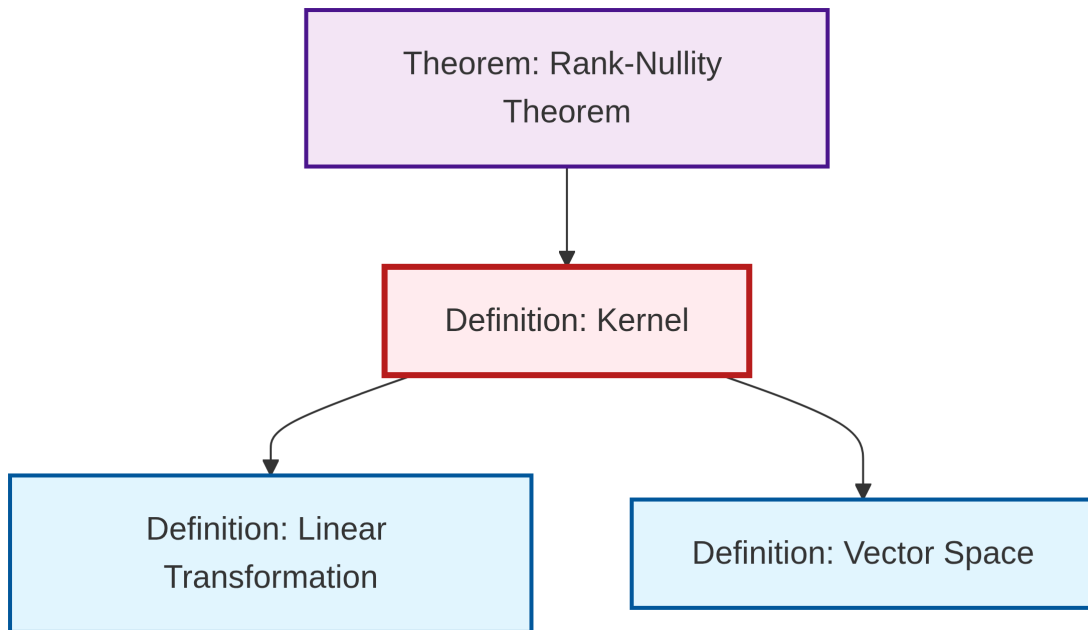
### Example

For a matrix transformation  $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  defined by  $T_A(\mathbf{x}) = A\mathbf{x}$ :

$$\ker(T_A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$$

This is precisely the solution set of the homogeneous linear system  $A\mathbf{x} = \mathbf{0}$ .

## Dependency Graph



Local dependency graph