Definition: Integral

The **integral** of a function represents the signed area under its curve, accumulation of quantities, or the inverse operation to differentiation. We focus on the Riemann integral for real-valued functions.

Riemann Integral

Partition and Riemann Sum

Given $f : [a, b] \to \mathbb{R}$ and a partition $P = \{a = x_0 < x_1 < ... < x_n = b\}$:

A **Riemann sum** is:

$$S(f, P) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$$

where $c_i \in [x_{i-1}, x_i]$ is a sample point.

Definition of the Integral

The function f is **Riemann integrable** on $[a, b]$ if the Limit of a Sequence:

$$\int_a^b f(x)\, dx = \lim_{||P|| \to 0} S(f, P)$$

exists and is independent of the choice of sample points, where $||P|| = \max_i(x_i - x_{i-1})$.

Fundamental Theorem of Calculus

Part I

If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\, dt$, then:

$$F'(x) = f(x)$$

Part II

If f is continuous on $[a, b]$ and F is an antiderivative of f:

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Properties

1. **Linearity**:
   - $\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$

- $\int_a^b c f(x)\,dx = c \int_a^b f(x)\,dx$

2. **Additivity**: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
3. **Monotonicity**: If $f(x) \leq g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$
4. **Reverse limits**: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Classes of Integrable Functions

- All continuous functions on $[a, b]$
- All monotonic functions on $[a, b]$
- All piecewise continuous functions with finitely many discontinuities

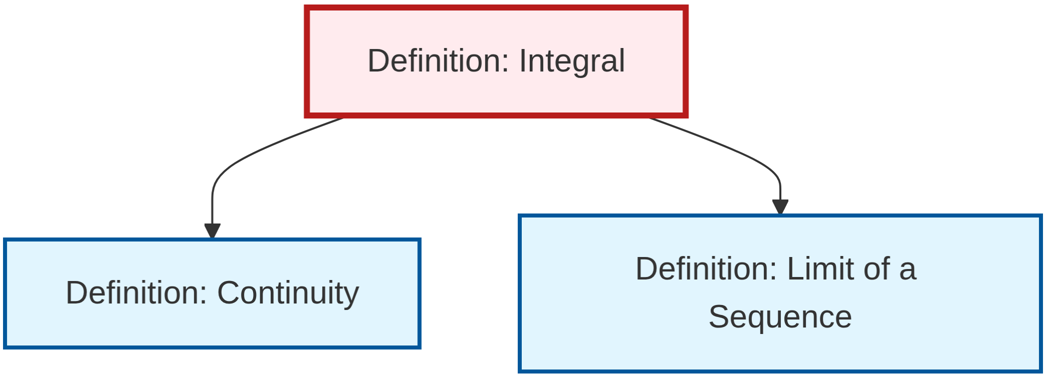
Examples

1. **Power functions**: $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ (for $n \neq -1$)
2. **Exponential**: $\int e^x\,dx = e^x + C$
3. **Trigonometric**: $\int \sin x\,dx = -\cos x + C$
4. **Definite integral**: $\int_0^1 x^2\,dx = \frac{1}{3}$

Applications

- Area and volume calculations
- Physics: work, center of mass, moments
- Probability: expected values, distributions
- Differential equations: solution methods

Dependency Graph



Local dependency graph