# Definition: Continuity

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A function $f : A \to \mathbb{R}$ (where $A \subseteq \mathbb{R}$) is **continuous** at a point $c \in A$ if the function value equals the limit at that point.

### Epsilon-Delta Definition

$f$ is continuous at $c$ if:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in A$:

$$|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$$

### Sequential Definition

Equivalently, $f$ is continuous at $c$ if:

For every sequence $(x_n)$ in $A$ with $x_n \to c$, we have $f(x_n) \to f(c)$.

### Limit Definition

When $c$ is a limit point of $A$, $f$ is continuous at $c$ if and only if:

$$\lim_{x \to c} f(x) = f(c)$$

This requires three conditions: 1. $f(c)$ is defined 2. $\lim_{x \to c} f(x)$ exists 3. The limit equals the function value

### Types of Continuity

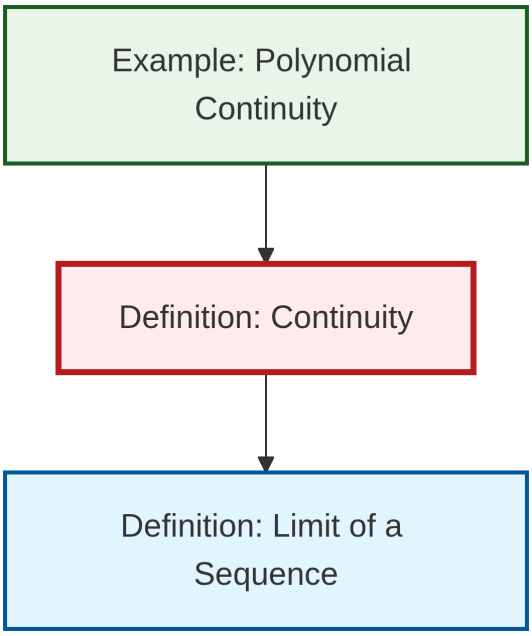
- **Continuous on a set**: $f$ is continuous on $A$ if it is continuous at every point in $A$
- **Uniformly continuous**: A stronger condition where $\delta$ depends only on $\varepsilon$, not on the point
- **Lipschitz continuous**: An even stronger condition with a linear bound on the rate of change

### Discontinuities

If $f$ is not continuous at $c$, we say $f$ has a **discontinuity** at $c$. Types include: - **Removable**: The limit exists but doesn't equal $f(c)$ - **Jump**: Left and right limits exist but are different - **Essential**: The limit doesn't exist

Continuity captures the intuitive notion of a function having "no breaks or jumps."

### Dependency Graph




Local dependency graph