

## Theorem: Chain Rule

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The chain rule provides a formula for computing the [derivative](#) of a composition of functions.

#### Statement

Let  $f : (a, b) \rightarrow \mathbb{R}$  and  $g : (c, d) \rightarrow \mathbb{R}$  where  $f((a, b)) \subseteq (c, d)$ .

If  $f$  is differentiable at  $x_0 \in (a, b)$  and  $g$  is differentiable at  $f(x_0)$ , then the composition  $g \circ f$  is differentiable at  $x_0$  and:

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

#### Leibniz Notation

If  $y = f(x)$  and  $z = g(y)$ , then:

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

#### Proof Outline

Let  $h = g \circ f$ . We need to show:

$$\lim_{x \rightarrow x_0} \frac{h(x) - h(x_0)}{x - x_0} = g'(f(x_0)) \cdot f'(x_0)$$

Define  $y = f(x)$  and  $y_0 = f(x_0)$ . For  $x \neq x_0$ :

$$\frac{h(x) - h(x_0)}{x - x_0} = \frac{g(f(x)) - g(f(x_0))}{x - x_0}$$

When  $f(x) \neq f(x_0)$ :

$$= \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0}$$

Since  $f$  is differentiable at  $x_0$ , it is [continuous](#) there, so  $f(x) \rightarrow f(x_0)$  as  $x \rightarrow x_0$ .

Taking the limit as  $x \rightarrow x_0$ , we get the desired result.

## Extended Chain Rule

For a composition of  $n$  functions  $f_1 \circ f_2 \circ \dots \circ f_n$ :

$$(f_1 \circ f_2 \circ \dots \circ f_n)' = f_1'(f_2 \circ \dots \circ f_n) \cdot f_2'(f_3 \circ \dots \circ f_n) \dots f_n'$$

## Examples

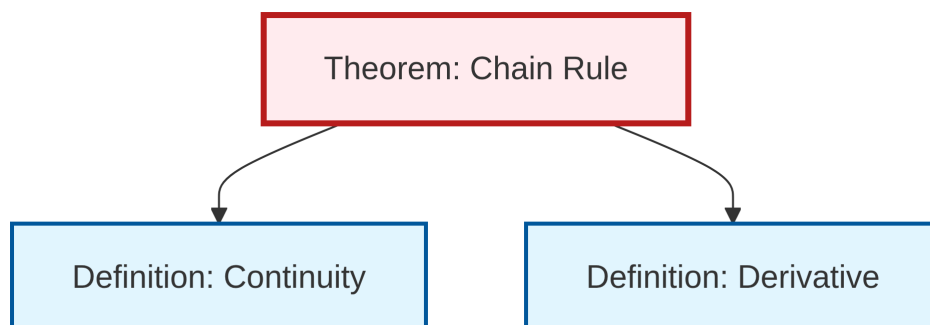
1. If  $h(x) = \sin(x^2)$ , then  $h'(x) = \cos(x^2) \cdot 2x$
2. If  $h(x) = e^{\sqrt{x}}$ , then  $h'(x) = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$
3. If  $h(x) = \ln(\sin(x))$ , then  $h'(x) = \frac{1}{\sin(x)} \cdot \cos(x) = \cot(x)$

## Mermaid Diagram

```
graph TD
    A[Chain Rule] --> B["(g f)' = g'(f(x)) · f'(x)"]
    A --> C[Composition of Functions]
    C --> D["f: x → y"]
    C --> E["g: y → z"]
    B --> F["Leibniz: dz/dx = dz/dy · dy/dx"]
    A --> G[Requires]
    G --> H["f differentiable at x"]
    G --> I["g differentiable at f(x)"]

    style A fill:#f9f,stroke:#333,stroke-width:2px
    style B fill:#bfb,stroke:#333,stroke-width:2px
    style F fill:#bfb,stroke:#333,stroke-width:2px
```

## Dependency Graph



Local dependency graph