

# Theorem: Intermediate Value Theorem

## Intermediate Value Theorem

A [Continuity](#) function from a [Connected Space](#) space to the real numbers attains all intermediate values.

### Statement

Let  $f : X \rightarrow \mathbb{R}$  be a continuous function where  $X$  is a connected topological space. If  $a, b \in f(X)$  with  $a < b$ , then for every  $c \in (a, b)$ , there exists  $x \in X$  such that  $f(x) = c$ .

### Classical Version

For the special case where  $X = [a, b] \subseteq \mathbb{R}$  is a closed interval:

If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous and  $f(a) < c < f(b)$  (or  $f(b) < c < f(a)$ ), then there exists  $x_0 \in (a, b)$  such that  $f(x_0) = c$ .

### Proof Idea

The proof relies on the fact that: 1. The continuous image of a connected space is connected 2. Connected subsets of  $\mathbb{R}$  are intervals 3. Intervals contain all intermediate values

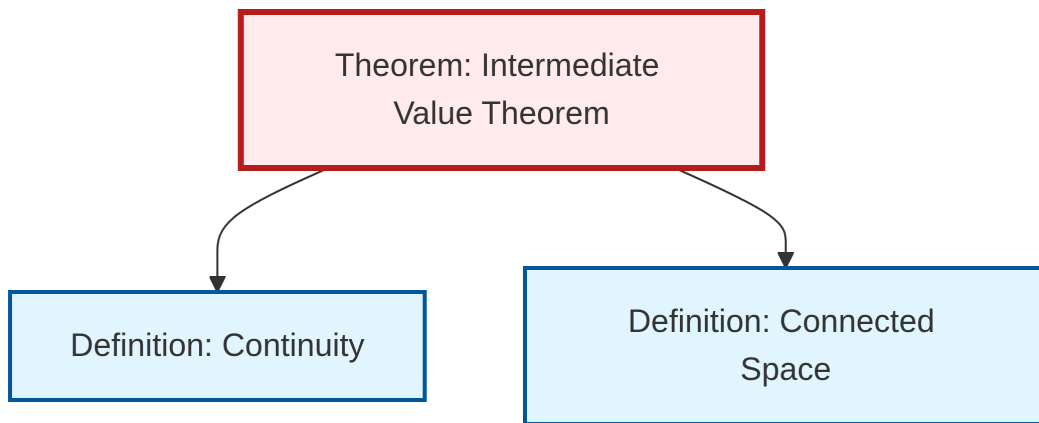
### Applications

- Finding roots of continuous functions
- Proving existence of fixed points
- Establishing properties of continuous functions on intervals
- Analysis of differential equations

### Example

Any polynomial of odd degree has at least one real root, since: -  $\lim_{x \rightarrow \infty} P(x) = \infty$  and  $\lim_{x \rightarrow -\infty} P(x) = -\infty$  (or vice versa) - By IVT,  $P(x) = 0$  for some  $x$

## Dependency Graph



## Local dependency graph