## Definition: Field

### Field

A field is a Ring in which every non-zero element has a multiplicative inverse.

#### Formal Definition

A field  $(F, +, \cdot)$  is a commutative ring with unity such that  $(F \setminus \{0\}, \cdot)$  forms an abelian Group.

Explicitly, a field satisfies:

- 1. (F, +) is an abelian group with identity 0
- 2.  $(F \setminus \{0\}, \cdot)$  is an abelian group with identity 1
- 3. Distributivity:  $a \cdot (b+c) = a \cdot b + a \cdot c$
- 4.  $0 \neq 1$  (non-triviality)

#### **Properties**

- Every field is an integral domain
- Every finite integral domain is a field
- Fields have no zero divisors: if ab = 0, then a = 0 or b = 0
- Every non-zero element a has a unique inverse  $a^{-1}$  such that  $a \cdot a^{-1} = 1$

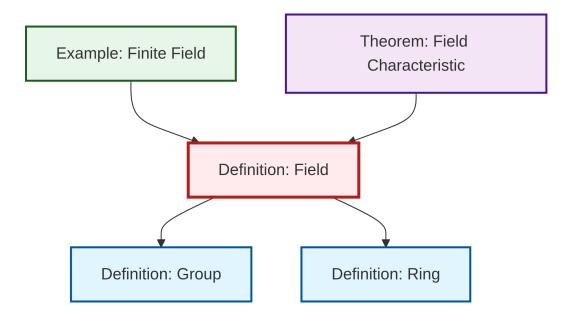
#### Examples

- $\mathbb{Q}$  rational numbers
- $\mathbb{R}$  real numbers
- $\mathbb C$  complex numbers
- $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$  field extension

#### Non-Examples

- $\mathbb{Z}$  is not a field (no multiplicative inverse for 2)
- $\mathbb{Z}/6\mathbb{Z}$  is not a field (has zero divisors)

# Dependency Graph



Local dependency graph