Theorem: Yoneda Lemma

The **Yoneda Lemma** is a fundamental result in Category theory that relates Functors to representable functors. It states that an object is completely determined by its relationships to all other objects.

Statement

Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to \mathbf{Set}$ be a functor. For any object $A \in \mathcal{C}$, there is a natural bijection:

$$\mathrm{Nat}(\mathrm{Hom}(A, -), F) \cong F(A)$$

where: - $\mathrm{Hom}(A, -) : \mathcal{C} \to \mathbf{Set}$ is the representable functor - $\mathrm{Nat}(\mathrm{Hom}(A, -), F)$ is the set of Natural Transformations

The Bijection

Forward Direction

Given a natural transformation $\alpha : \mathrm{Hom}(A, -) \Rightarrow F$, we get an element of $F(A)$ by:

$$\alpha \mapsto \alpha_A(\mathrm{id}_A) \in F(A)$$

Reverse Direction

Given an element $x \in F(A)$, we define a natural transformation α by:

$$\alpha_B(f) = F(f)(x) \text{ for } f : A \to B$$

Yoneda Embedding

The Yoneda Lemma gives rise to the **Yoneda embedding**:

$$\mathcal{Y} : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$$

$$A \mapsto \mathrm{Hom}(-, A)$$

This embedding is: - **Full**: Every natural transformation between representables comes from a morphism - **Faithful**: Different morphisms give different natural transformations

Corollaries

1. Yoneda Principle

Objects A and B are isomorphic if and only if $\mathrm{Hom}(-, A) \cong \mathrm{Hom}(-, B)$ as functors.

2. Representability

A functor F is representable if and only if it is naturally isomorphic to some $\mathrm{Hom}(A, -)$.

Proof Sketch

1. Show that the assignment $\alpha \mapsto \alpha_A(\mathrm{id}_A)$ is well-defined
2. Verify that the reverse construction gives a natural transformation
3. Check that these constructions are mutual inverses
4. Prove naturality in both A and F

Examples and Applications

Universal Properties

The Yoneda Lemma provides a systematic way to understand universal properties: an object with a universal property represents a particular functor.

Limits and Colimits

The limit of a functor F represents the functor $\mathrm{Hom}(-, \lim F)$.

Algebraic Topology

In the category of topological spaces, the functor $\pi_n(-)$ (n-th homotopy group) is represented by the n-sphere S^n.

Philosophical Significance

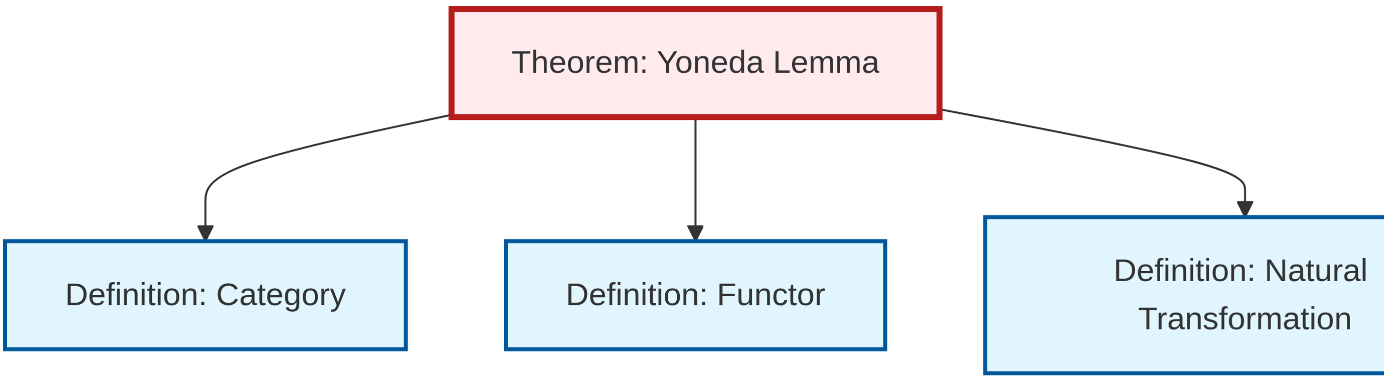
The Yoneda Lemma embodies the idea that: > "An object is completely determined by its relationships to all other objects"

This perspective shifts focus from internal structure to external relationships, a key insight of category theory.

Variations

- **Contravariant Yoneda**: For contravariant functors $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$
- **Enriched Yoneda**: Generalizes to enriched categories
- **2-Yoneda**: Version for 2-categories

Dependency Graph



Local dependency graph