Theorem: De Morgan's Laws

De Morgan's Laws describe the relationship between Union, Intersection, and set complement. They state that the complement of a union equals the intersection of complements, and the complement of an intersection equals the union of complements.

Statement

For any Sets A and B in a universal set U:

1. $(A \cup B)^c = A^c \cap B^c$
2. $(A \cap B)^c = A^c \cup B^c$

Where X^c denotes the complement of set X in U.

Proof

First Law: $(A \cup B)^c = A^c \cap B^c$

We prove by showing mutual subset inclusion.

Part 1: $(A \cup B)^c \subseteq A^c \cap B^c$

Let $x \in (A \cup B)^c$. Then: - $x \notin A \cup B$ - $x \notin A$ and $x \notin B$ (by definition of union) - $x \in A^c$ and $x \in B^c$ (by definition of complement) - $x \in A^c \cap B^c$ (by definition of intersection)

Part 2: $A^c \cap B^c \subseteq (A \cup B)^c$

Let $x \in A^c \cap B^c$. Then: - $x \in A^c$ and $x \in B^c$ - $x \notin A$ and $x \notin B$ - $x \notin A \cup B$ - $x \in (A \cup B)^c$

Therefore, $(A \cup B)^c = A^c \cap B^c$.

Second Law: $(A \cap B)^c = A^c \cup B^c$

The proof follows a similar structure using the logical equivalence: $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

Generalized De Morgan's Laws

For any collection of sets $\{A_i : i \in I\}$:

1. $\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c$
2. $\left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c$

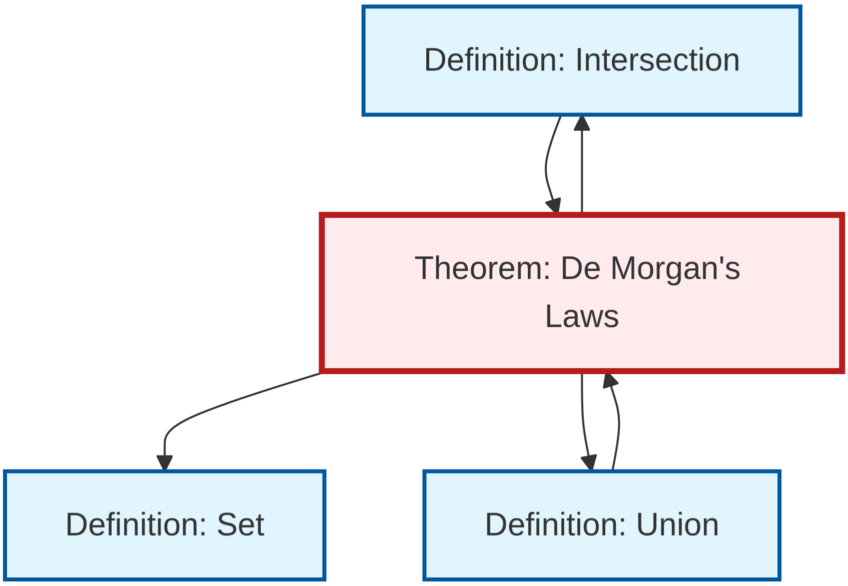
Applications

1. **Logic**: Corresponding laws in Boolean algebra and propositional logic
2. **Probability**: $P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B)$
3. **Computer Science**: Circuit design and Boolean function simplification
4. **Set Theory**: Simplifying complex set expressions

Examples

- If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$ in $U = \{1, 2, 3, 4, 5, 6\}$:
  - $A \cup B = \{1, 2, 3, 4, 5\}$, so $(A \cup B)^c = \{6\}$
  - $A^c = \{4, 5, 6\}$ and $B^c = \{1, 2, 6\}$
  - $A^c \cap B^c = \{6\}$

Dependency Graph



Local dependency graph