

Definition: Natural Transformation

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A **natural transformation** provides a way to transform one **Functor** into another while respecting the internal structure of the categories involved.

Formal Definition

Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be two functors between categories \mathcal{C} and \mathcal{D} . A natural transformation $\eta : F \Rightarrow G$ consists of:

- For each object X in \mathcal{C} , a **Morphism** $\eta_X : F(X) \rightarrow G(X)$ in \mathcal{D}

Such that for every morphism $f : X \rightarrow Y$ in \mathcal{C} , the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ | & & | \\ X & & Y \\ \downarrow & & \downarrow \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

That is, $\eta_Y \circ F(f) = G(f) \circ \eta_X$.

Components and Naturality

- Each morphism η_X is called a **component** of the natural transformation
- The commutativity condition is called the **naturality condition**
- When this condition holds, we say that η is **natural in X**

Special Cases

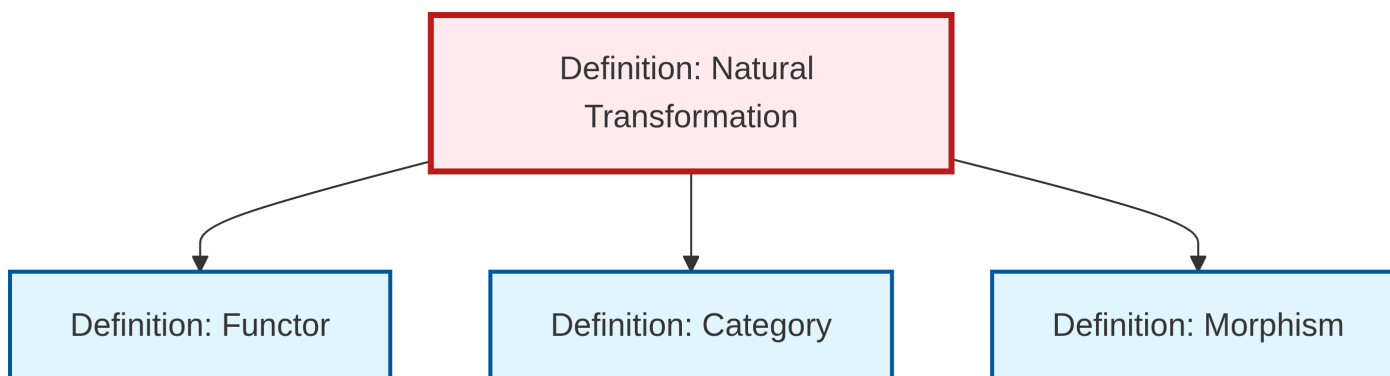
1. **Natural Isomorphism:** A natural transformation where every component η_X is an isomorphism
2. **Identity Natural Transformation:** For any functor F , the identity transformation id_F has components $(\text{id}_F)_X = \text{id}_{F(X)}$

Composition

Natural transformations can be composed: - **Vertical Composition:** If $\eta : F \Rightarrow G$ and $\mu : G \Rightarrow H$, then $\mu \circ \eta : F \Rightarrow H$ - **Horizontal Composition:** Natural transformations can also be composed with functors



Dependency Graph



Local dependency graph