Theorem: Arithmetic of Limits

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If sequences (a_n) and (b_n) converge, then their sum, difference, product, and (under conditions) quotient also converge, with limits given by the corresponding arithmetic operations.

Statement

Let (a_n) and (b_n) be sequences of real numbers with s: - $\lim_{n\to\infty} a_n = A$ - $\lim_{n\to\infty} b_n = B$

Then:

1. **Sum Rule**: $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. **Difference Rule**: $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. **Product Rule**: $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
4. **Constant Multiple Rule**: For any $c \in \mathbb{R}$, $\lim_{n\to\infty}(c \cdot a_n) = c \cdot A$
5. **Quotient Rule**: If $B \neq 0$ and $b_n \neq 0$ for all n, then:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$$

Proof Sketch (Sum Rule)

Given $\varepsilon > 0$: - Since $a_n \to A$, there exists N_1 such that $|a_n - A| < \varepsilon/2$ for all $n > N_1$ - Since $b_n \to B$, there exists N_2 such that $|b_n - B| < \varepsilon/2$ for all $n > N_2$ - Let $N = \max(N_1, N_2)$

For $n > N$:

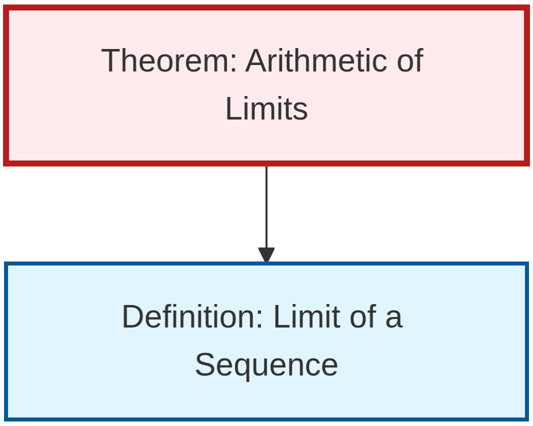
$$|(a_n + b_n) - (A + B)| = |(a_n - A) + (b_n - B)| \leq |a_n - A| + |b_n - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Applications

These rules allow us to: - Compute limits of polynomial sequences - Analyze rational sequences - Build complex limits from simpler ones

The theorem shows that the limit operation preserves algebraic structure.

Dependency Graph



Local dependency graph