

## Definition: Identity Element

An **identity element** for a **Binary Operation** is an element that leaves any other element unchanged when combined with it.

### Formal Definition

Let  $(S, *)$  be a set  $S$  with a binary operation  $*$ . An element  $e \in S$  is called:

#### Left Identity

If for all  $a \in S$ :

$$e * a = a$$

#### Right Identity

If for all  $a \in S$ :

$$a * e = a$$

#### Two-Sided Identity (or simply Identity)

If  $e$  is both a left identity and a right identity.

#### Uniqueness

If an identity element exists, it is unique:

**Proof:** Suppose  $e$  and  $e'$  are both identities. Then: -  $e = e * e'$  (since  $e'$  is a right identity) -  $e * e' = e'$  (since  $e$  is a left identity) - Therefore  $e = e'$

### Examples

#### Arithmetic Operations

- **Addition on  $\mathbb{R}$ :** Identity is 0, since  $a + 0 = 0 + a = a$
- **Multiplication on  $\mathbb{R}$ :** Identity is 1, since  $a \cdot 1 = 1 \cdot a = a$

#### Matrix Operations

- **Matrix addition:** The zero matrix **0**
- **Matrix multiplication:** The identity matrix **I** with 1s on diagonal

#### Set Operations

- **Union:** The empty set  $\emptyset$ , since  $A \cup \emptyset = A$
- **Intersection:** The universal set  $U$ , since  $A \cap U = A$

## Function Composition

- In the set of functions  $f : X \rightarrow X$ , the identity function  $\text{id}_X(x) = x$

## Non-Examples

- **Subtraction on  $\mathbb{R}$ :** No identity element exists
  - No right identity:  $a - e = a$  implies  $e = 0$
  - But 0 is not a left identity:  $0 - a = -a \neq a$

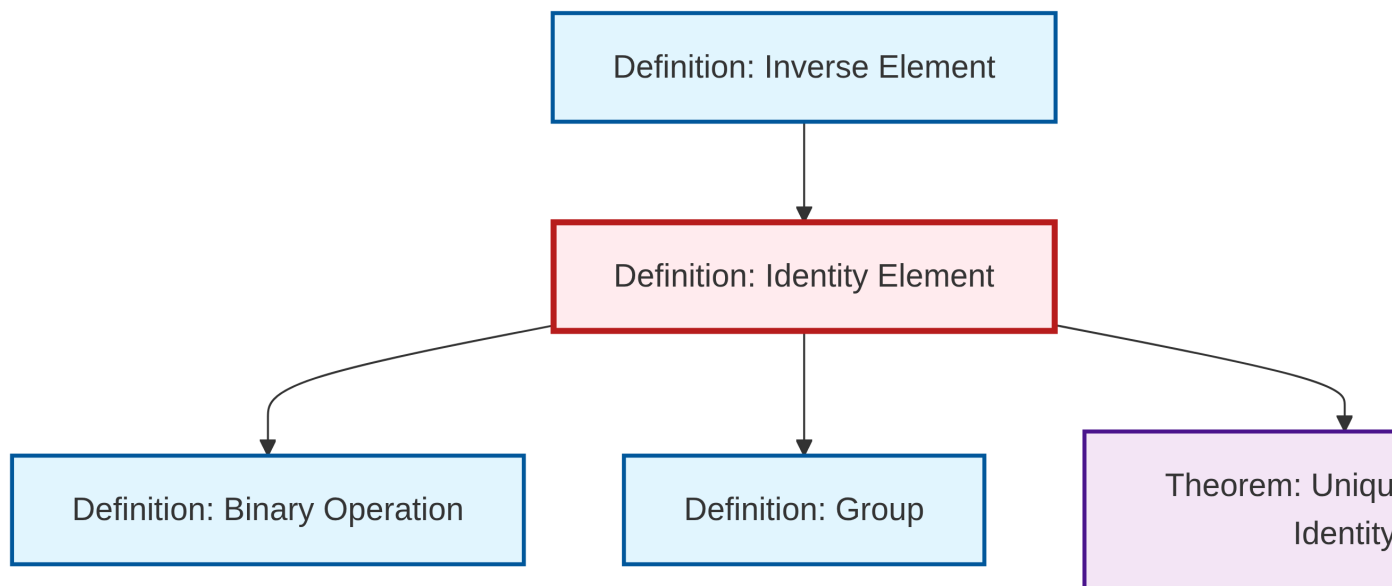
## Related Concepts

- **Group:** Requires an identity element
- **Monoid:** A set with an associative operation and identity
- **Inverse elements:** Defined relative to an identity element
- **Uniqueness of Identity:** Proves uniqueness in group context

## Properties

1. In a **Group**, every element has an inverse with respect to the identity
2. The identity element is its own inverse:  $e * e = e$
3. Identity elements are preserved by homomorphisms between algebraic structures

## Dependency Graph



Local dependency graph