Theorem: Pythagorean Theorem

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In a Euclidean space, two vectors are orthogonal if and only if the square of the norm of their sum equals the sum of the squares of their norms.

Statement

Let $\mathbf{x}, \mathbf{y} \in \mathbb{E}^n$. Then:

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0 \quad \text{if and only if} \quad \|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

Proof

($\Rightarrow$) Suppose $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. Then:

$$\begin{aligned}
\|\mathbf{x} + \mathbf{y}\|^2 &= \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle && (1)\\
&= \langle \mathbf{x}, \mathbf{x} \rangle + 2\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle && (2)\\
&= \|\mathbf{x}\|^2 + 2 \cdot 0 + \|\mathbf{y}\|^2 && (3)\\
&= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 && (4)
\end{aligned}$$

($\Leftarrow$) Suppose $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$. Expanding the left side:

$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2 \quad (5)$$

Comparing with the given equality:

$$\|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

Therefore, $2\langle \mathbf{x}, \mathbf{y} \rangle = 0$, which implies $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

Classical Form

In $\mathbb{E}^2$, for a right triangle with legs of length a and b and hypotenuse of length c:

$$a^2 + b^2 = c^2$$

Mermaid Diagram

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graph TD
    A[Pythagorean Theorem] --> B[Orthogonality]
    A --> C[Norm Properties]
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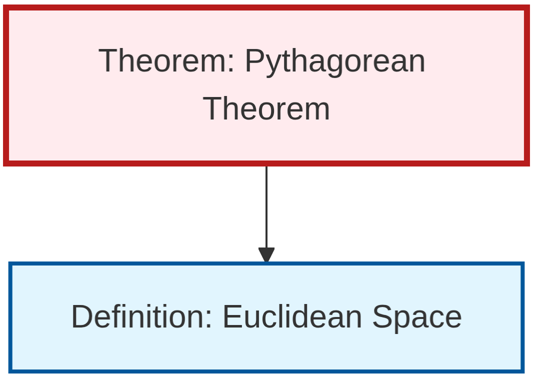
    B --> D[Inner Product = 0]
    C --> E[||x + y||² = ||x||² + ||y||²]
    D <--> E
    A --> F[Euclidean Space]

    style A fill:#f9f,stroke:#333,stroke-width:2px
    style B fill:#bbf,stroke:#333,stroke-width:2px
    style C fill:#bbf,stroke:#333,stroke-width:2px
    style D fill:#bfb,stroke:#333,stroke-width:2px
    style E fill:#bfb,stroke:#333,stroke-width:2px
```

Dependency Graph



Local dependency graph