Definition: Vector Space

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A vector space (or linear space) over a field F is a Set V together with two Binary Operations:

- 1. Vector addition: $+: V \times V \to V$
- 2. Scalar multiplication: $\cdot: F \times V \to V$

such that the following axioms are satisfied:

Addition Axioms

For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$:

- 1. Associativity: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- 2. Commutativity: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- 3. Identity element: There exists $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$
- 4. Inverse elements: For each $\mathbf{v} \in V$, there exists $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$

Scalar Multiplication Axioms

For all $a, b \in F$ and $\mathbf{u}, \mathbf{v} \in V$:

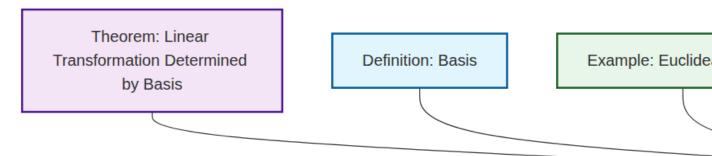
- 5. Associativity: $a(b\mathbf{v}) = (ab)\mathbf{v}$
- 6. **Identity**: $1\mathbf{v} = \mathbf{v}$, where 1 is the multiplicative identity in F
- 7. Distributivity over vector addition: $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
- 8. Distributivity over scalar addition: $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

Elements of V are called **vectors**, and elements of F are called **scalars**.

Remarks

- The vector space axioms ensure that (V, +) forms an abelian Group
- The field F determines the "type" of vector space (e.g., real vector space when $F = \mathbb{R}$)
- Vector spaces are fundamental structures in linear algebra and appear throughout mathematics

Dependency Graph



Local dependency graph