Example: Closed Interval is Compact

Example: Closed Interval is Compact

The closed interval $[a, b]$ in $\mathbb{R}$ (with the standard topology) is a Compact Space space.

Statement

For any $a, b \in \mathbb{R}$ with $a \leq b$, the closed interval $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ is compact.

Intuition

Compactness of $[a, b]$ means that from any collection of open sets that cover $[a, b]$, we can always select a finite number of them that still cover $[a, b]$. This captures the idea that $[a, b]$ is "finite in extent" despite containing infinitely many points.

Proof Sketch

By the Heine-Borel Theorem, a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. We verify:

1. **Closed**: $[a, b]$ is closed because its complement $(-\infty, a) \cup (b, \infty)$ is open.
2. **Bounded**: $[a, b]$ is bounded since all its elements lie between a and b.

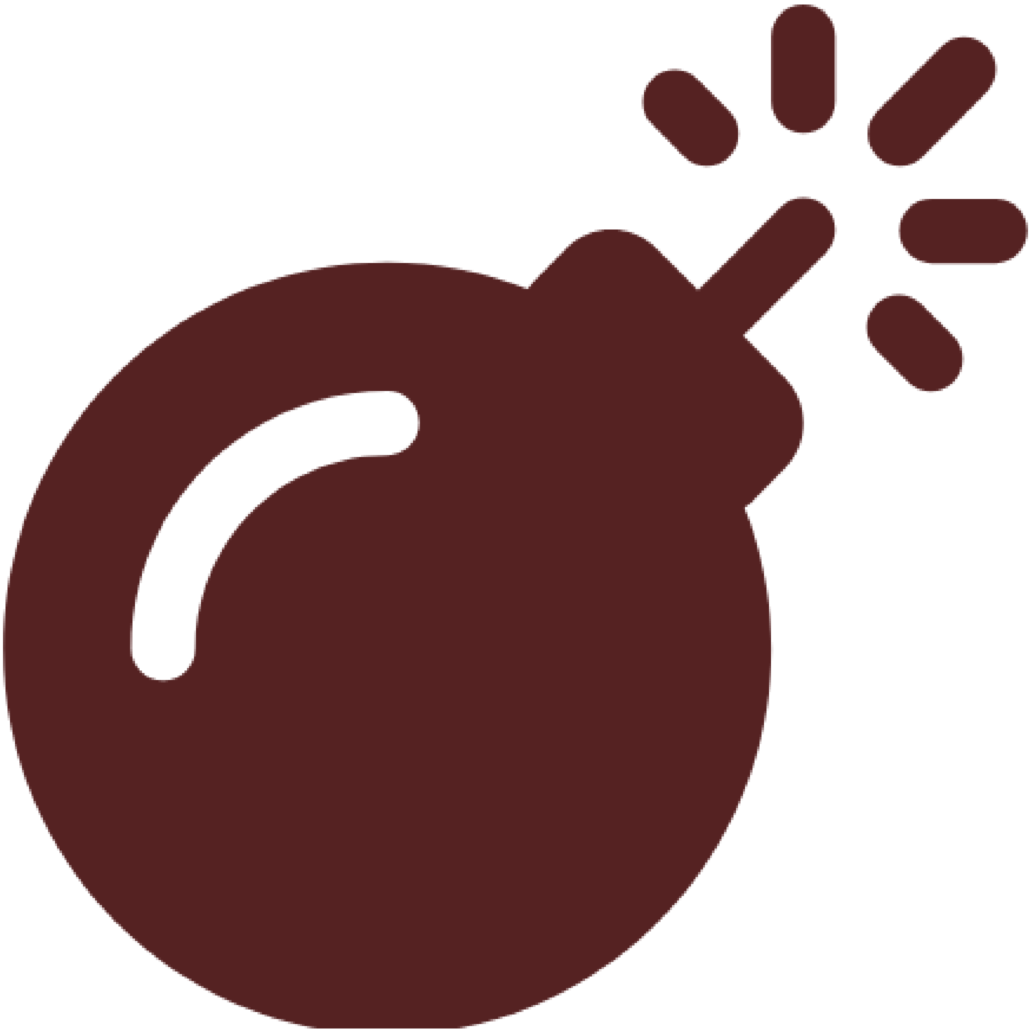
Therefore, $[a, b]$ is compact.

Contrast with Non-Examples

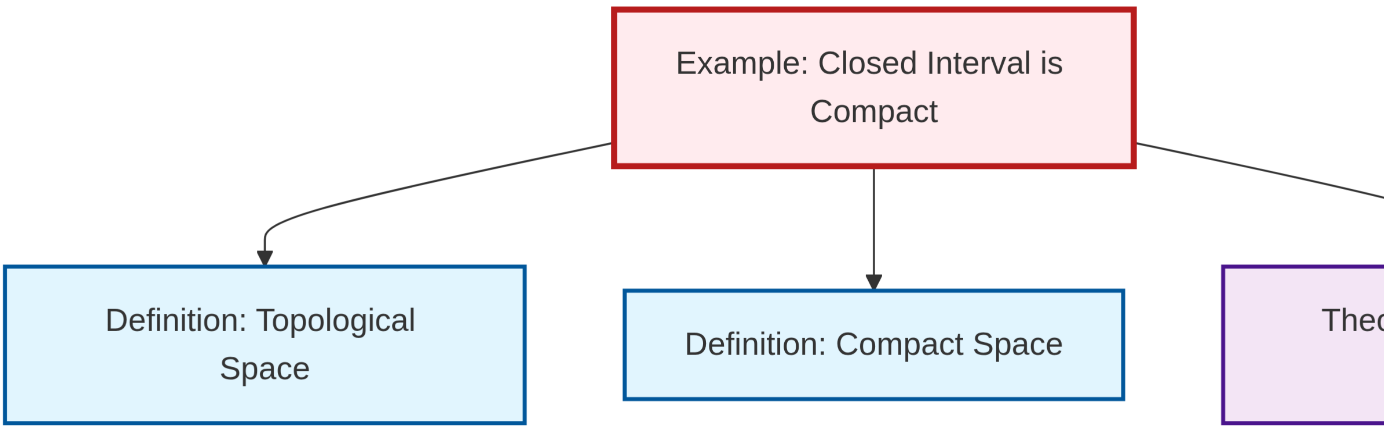
- The open interval (a, b) is **not compact** (it's bounded but not closed)
- The ray $[a, \infty)$ is **not compact** (it's closed but not bounded)
- The entire real line $\mathbb{R}$ is **not compact** (it's closed but not bounded)

Applications

The compactness of closed intervals is fundamental in analysis: - It ensures that continuous functions on $[a, b]$ attain their maximum and minimum - It's key to proving the uniform continuity of continuous functions on $[a, b]$ - It's essential for the Riemann integrability of continuous functions



Dependency Graph



Local dependency graph