Definition: Identity Element

An **identity element** for a Binary Operation is an element that leaves any other element unchanged when combined with it.

Formal Definition

Let $(S, *)$ be a set S with a binary operation $*$. An element $e \in S$ is called:

Left Identity

If for all $a \in S$:

$$e * a = a$$

Right Identity

If for all $a \in S$:

$$a * e = a$$

Two-Sided Identity (or simply Identity)

If e is both a left identity and a right identity.

Uniqueness

If an identity element exists, it is unique:

Proof: Suppose e and e' are both identities. Then: - $e = e * e'$ (since e' is a right identity) - $e * e' = e'$ (since e is a left identity) - Therefore $e = e'$

Examples

Arithmetic Operations

- **Addition on** $\mathbb{R}$: Identity is 0, since $a + 0 = 0 + a = a$
- **Multiplication on** $\mathbb{R}$: Identity is 1, since $a \cdot 1 = 1 \cdot a = a$

Matrix Operations

- **Matrix addition**: The zero matrix $\mathbf{0}$
- **Matrix multiplication**: The identity matrix $\mathbf{I}$ with 1s on diagonal

Set Operations

- **Union**: The empty set $\emptyset$, since $A \cup \emptyset = A$
- **Intersection**: The universal set U, since $A \cap U = A$

Function Composition

- In the set of functions $f : X \to X$, the identity function $\mathrm{id}_X(x) = x$

Non-Examples

- **Subtraction on** $\mathbb{R}$: No identity element exists
  - No right identity: $a - e = a$ implies $e = 0$
  - But 0 is not a left identity: $0 - a = -a \neq a$

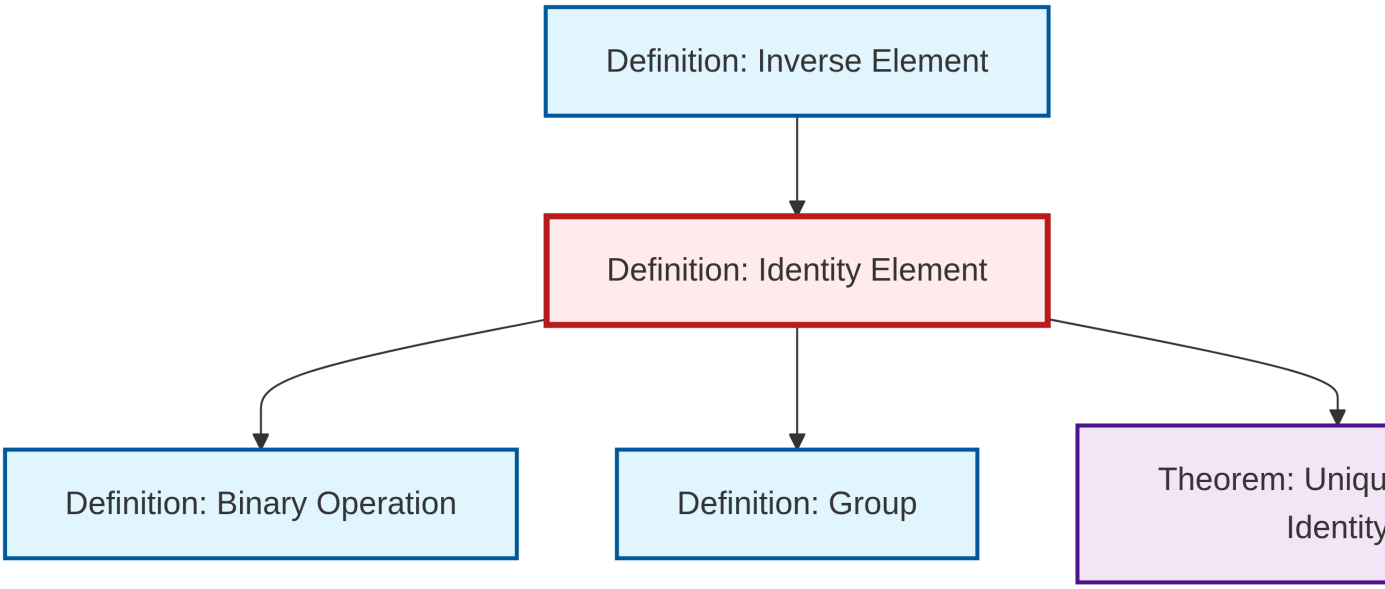
Related Concepts

- Group: Requires an identity element
- **Monoid**: A set with an associative operation and identity
- **Inverse elements**: Defined relative to an identity element
- Uniqueness of Identity: Proves uniqueness in group context

Properties

1. In a Group, every element has an inverse with respect to the identity
2. The identity element is its own inverse: $e * e = e$
3. Identity elements are preserved by homomorphisms between algebraic structures

Dependency Graph



Local dependency graph