Theorem: Heine-Borel Theorem

Heine-Borel Theorem

A subset of Euclidean space $\mathbb{R}^n$ is Compact Space if and only if it is Closed Set and bounded.

Statement

For a subset $K \subseteq \mathbb{R}^n$ (with the standard Metric Space topology), the following are equivalent:

1. K is compact
2. K is closed and bounded

where bounded means there exists $M > 0$ such that $\|x\| \leq M$ for all $x \in K$.

Implications

Forward Direction

If K is compact in $\mathbb{R}^n$, then: - K is closed (since $\mathbb{R}^n$ is Hausdorff) - K is bounded (by considering the open cover $\{B(0, n)\}_{n=1}^{\infty}$)

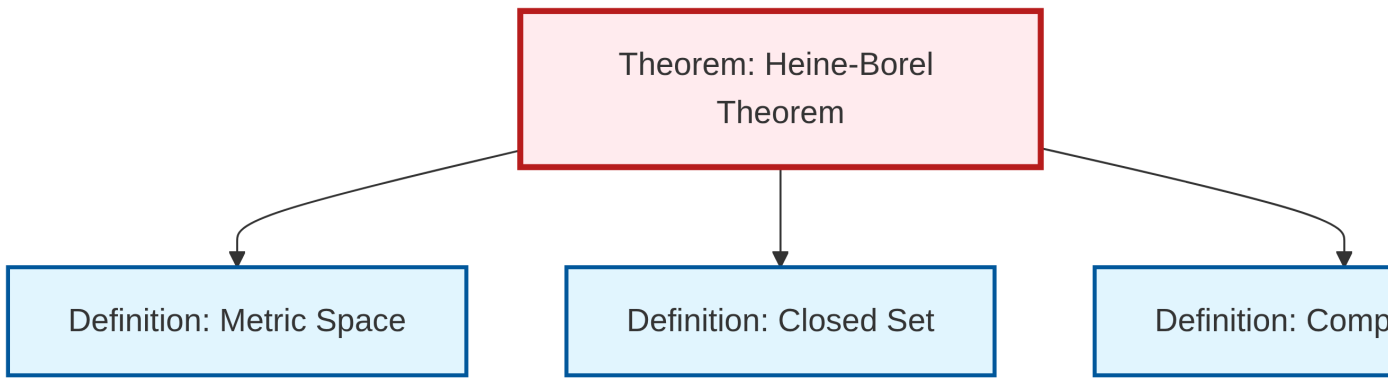
Reverse Direction

If K is closed and bounded in $\mathbb{R}^n$, then K is compact.

Applications

- Characterizing compact sets in finite-dimensional normed spaces
- Proving existence of extrema for continuous functions
- Establishing convergence of sequences in $\mathbb{R}^n$

Dependency Graph



Local dependency graph