## Definition: Natural Transformation

### **Definition:** Natural Transformation

A natural transformation provides a way to transform one Functor into another while respecting the internal structure of the categories involved.

#### Formal Definition

Let  $F,G:\mathcal{C}\to\mathcal{D}$  be two functors between categories  $\mathcal{C}$  and  $\mathcal{D}$ . A natural transformation  $\eta:F\Rightarrow G$  consists of:

• For each object X in  $\mathcal{C}$ , a Morphism  $\eta_X : F(X) \to G(X)$  in  $\mathcal{D}$ 

Such that for every morphism  $f: X \to Y$  in  $\mathcal{C}$ , the following diagram commutes:

That is,  $\eta_Y \circ F(f) = G(f) \circ \eta_X$ .

#### Components and Naturality

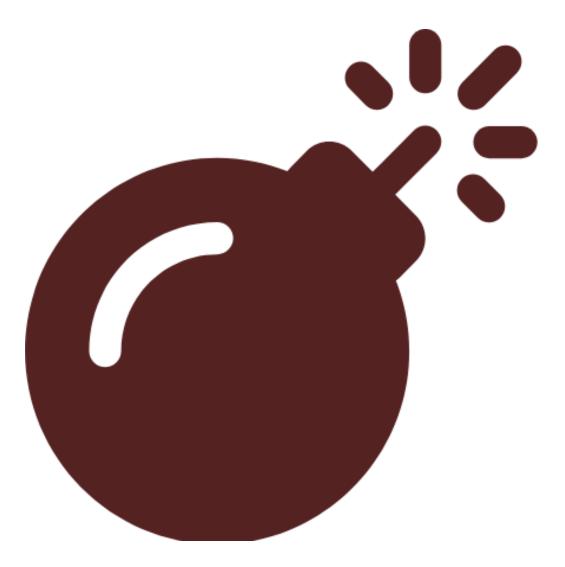
- Each morphism  $\eta_X$  is called a **component** of the natural transformation
- The commutativity condition is called the **naturality condition**
- When this condition holds, we say that  $\eta$  is **natural in X**

#### **Special Cases**

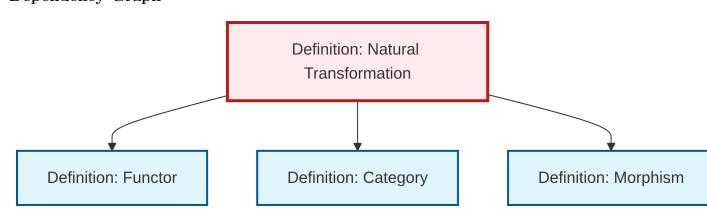
- 1. Natural Isomorphism: A natural transformation where every component  $\eta_X$  is an isomorphism
- 2. **Identity Natural Transformation**: For any functor F, the identity transformation  $\mathrm{id}_F$  has components  $(\mathrm{id}_F)_X = \mathrm{id}_{F(X)}$

### Composition

Natural transformations can be composed: - Vertical Composition: If  $\eta: F \Rightarrow G$  and  $\mu: G \Rightarrow H$ , then  $\mu \circ \eta: F \Rightarrow H$  - Horizontal Composition: Natural transformations can also be composed with functors



# Dependency Graph



Local dependency graph