2.1 Posterior Distribution

$$\text{Posterior} = \text{likelihood} \times \text{prior}$$

$$= \prod_{i=1}^{N} \sqrt{\frac{\tau}{2\pi}} \exp\left(-\frac{\tau}{2}(x_i-\mu)^2\right) \times \text{prior}$$

$$= \left(\frac{\tau}{2\pi}\right)^{\frac{N}{2}} \exp \sum_{i=1}^{N}\left(-\frac{\tau}{2}(x_i-\mu)^2\right) \times \text{prior}$$

$$= \left(\frac{\tau}{2\pi}\right)^{N/2} \exp\left(-\frac{\tau}{2}\left(N(\mu-\bar{x})^2 + \sum_{i=1}^{N}(x_i-\bar{x})^2\right)\right) \times \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \cdot \left(\frac{\nu_0}{2\pi}\right)^{\frac{1}{2}} \tau^{\alpha_0-\frac{1}{2}} \exp\left(-\frac{\tau}{2}\left[\nu_0(\mu-\mu_0)^2 + 2\beta_0\right]\right)$$

$$= \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \cdot \left(\frac{1}{2\pi}\right)^{\frac{N}{2}} \cdot \left(\frac{\nu_0}{2\pi}\right)^{\frac{1}{2}} \tau^{\frac{N}{2}+\alpha_0-\frac{1}{2}} \cdot \exp\left[-\frac{\tau}{2}\left(N(\mu-\bar{x})^2 + \sum_{i=1}^{N}(x_i-\bar{x})^2 - \frac{\tau}{2}\left[\nu_0(\mu-\mu_0)^2 + 2\beta_0\right]\right)\right]$$

$$= \text{updated NG}$$

consider the coefficient of μ inside "exp":

μ^2 : $-\frac{\tau}{2}N - \frac{\tau}{2}\nu_0 = -\frac{\tau}{2}\nu_N \Rightarrow \nu_N = \nu_0 + N$

μ : $-\frac{\tau}{2}\cdot(-2N\bar{x}) + \frac{\tau}{2}\nu_0 \cdot 2\mu_0 = -\frac{\tau}{2}\nu_N \cdot (-2\mu_N) \Rightarrow \mu_N = (N\bar{x} + \nu_0\mu_0)/(\nu_0 + N)$

const : $-\frac{\tau}{2}\cdot N\bar{x}^2 - \frac{\tau}{2}\cdot\sum_{i=1}^{N}(x_i-\bar{x})^2 - \frac{\tau}{2}\nu_0\cdot\mu_0^2 - \frac{\tau}{2}\cdot 2\beta_0 = -\frac{\tau}{2}\left[\nu_N \cdot \mu_N^2 + 2\beta_N\right]$

$$\Rightarrow N\bar{x}^2 + \sum_{i=1}^{N}(x_i-\bar{x})^2 + \nu_0\mu_0^2 + 2\beta_0 = \nu_N\mu_N^2 + 2\beta_N$$

$$\Rightarrow \beta_N = \beta_0 + \frac{1}{2}\sum_{i=1}^{N}(x_i-\bar{x})^2 + \frac{1}{2}N\bar{x}^2 + \frac{1}{2}\nu_0\mu_0^2 - \frac{(N\bar{x}+\nu_0\mu_0)^2}{2(\nu_0+N)}$$

consider the power of τ:

$$\frac{N}{2} + \alpha_0 - \frac{1}{2} = \tau^{\alpha_N - \frac{1}{2}} \Rightarrow \alpha_N = \alpha_0 + \frac{N}{2}$$

2.2.1

When $P(y=c|x) = P(y=c'|x) \Rightarrow$ decision boundary

$$\pi_c |\Sigma_c|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x-\mu_c)^T \Sigma_c^{-1} (x-\mu_c)\right) = \pi_{c'} |\Sigma_{c'}|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x-\mu_{c'})^T \Sigma_{c'}^{-1} (x-\mu_{c'})\right)$$

$$\Rightarrow \frac{\pi_c}{\pi_{c'}} \times \exp\left(-\frac{1}{2}(x-\mu_c)^T \Sigma_c^{-1}(x-\mu_c) + \frac{1}{2}(x-\mu_{c'})^T \Sigma_{c'}^{-1}(x-\mu_{c'})\right) = 1.$$

$\because e^0 = 1$

$$\therefore -\frac{1}{2}(x-\mu_c)^T \Sigma_c^{-1}(x-\mu_c) + \frac{1}{2}(x-\mu_{c'})^T \Sigma_{c'}^{-1}(x-\mu_{c'}) = 0$$

$$\Rightarrow \frac{-1}{2}x^2 + \mu_c x - \frac{1}{2}\mu_c^2 + \frac{1}{2}x^2 - \mu_{c'} x + \frac{1}{2}\mu_{c'}^2 = 0$$

$$\Rightarrow (\mu_c - \mu_{c'})x = \frac{1}{2}\mu_c^2 - \frac{1}{2}\mu_{c'}^2$$

$$\Rightarrow \quad x = \frac{\frac{1}{2}\mu_c^2 - \frac{1}{2}\mu_{c'}^2}{\mu_c - \mu_{c'}} = \frac{1}{2} \times (\mu_c + \mu_{c'})$$

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Conclusion: if 2 class share the same covariance, the decision boundary will lie in the middle of two means.

2.2.2

if covariance are different, then the x^2 term in the calculation cannot be eliminated, which leads to non-linear decision boundary.