1 Least Square Fitting: A Straight Line

$$\text{Data}: \quad \{x_i,\, y_i \pm \sigma_i\}_{i=1,2,\ldots,N} \tag{1.1}$$

$$\text{Fit}: \quad y = a_1 x + a_0 \tag{1.2}$$

Objective(Cost) function / Residual Square:

$$\chi^2 = \sum_i^N \frac{[y_i - (a_1 x_i + a_0)]^2}{\sigma_i^2} \tag{1.3}$$

$$\frac{\partial \chi^2}{\partial a_0} = 2 \sum_i^N \frac{[y_i - (a_1 x_i + a_0)]}{\sigma_i^2}(-1) \tag{1.4}$$

$$\frac{\partial \chi^2}{\partial a_1} = 2 \sum_i^N \frac{[y_i - (a_1 x_i + a_0)]}{\sigma_i^2}(-x_i) \tag{1.5}$$

$$\frac{\partial \chi^2}{\partial a_0} = 0 \quad \Rightarrow 2 \sum_i^N \frac{[y_i - (a_1 x_i + a_0)]}{\sigma_i^2}(-1) = 0 \tag{1.6}$$

or,

$$\sum_i^N \frac{y_i}{\sigma_i^2} - a_1 \sum_i^N \frac{x_i}{\sigma_i^2} - a_0 \sum_i^N \frac{1}{\sigma_i^2} = 0 \tag{1.7}$$

or,

$$\left(\sum_i^N \frac{1}{\sigma_i^2} \right) a_0 + \left(\sum_i^N \frac{x_i}{\sigma_i^2} \right) a_1 = \sum_i^N \frac{y_i}{\sigma_i^2} \tag{1.I}$$

$$\frac{\partial \chi^2}{\partial a_1} = 0 \quad \Rightarrow 2 \sum_i^N \frac{[y_i - (a_1 x_i + a_0)]}{\sigma_i^2}(-x_i) = 0 \tag{1.8}$$

or,

$$\sum_i^N \frac{x_i\, y_i}{\sigma_i^2} - a_1 \sum_i^N \frac{x_i^2}{\sigma_i^2} - a_0 \sum_i^N \frac{x_i}{\sigma_i^2} = 0 \tag{1.9}$$

or,

$$\left(\sum_i^N \frac{x_i}{\sigma_i^2} \right) a_0 + \left(\sum_i^N \frac{x_i^2}{\sigma_i^2} \right) a_1 = \sum_i^N \frac{x_i\, y_i}{\sigma_i^2} \tag{1.II}$$

So, we get a system of two linear equations:

$$\left(\sum_i^N \frac{1}{\sigma_i^2}\right) a_0 + \left(\sum_i^N \frac{x_i}{\sigma_i^2}\right) a_1 = \sum_i^N \frac{y_i}{\sigma_i^2} \tag{1.I}$$

$$\left(\sum_i^N \frac{x_i}{\sigma_i^2}\right) a_0 + \left(\sum_i^N \frac{x_i^2}{\sigma_i^2}\right) a_1 = \sum_i^N \frac{x_i\, y_i}{\sigma_i^2} \tag{1.II}$$

Eqs. (1.I) and (1.II) in the matrix form

$$\begin{bmatrix} \left(\sum_i^N \frac{1}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x_i}{\sigma_i^2}\right) \\ \left(\sum_i^N \frac{x_i}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x_i^2}{\sigma_i^2}\right) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \left(\sum_i^N \frac{y_i}{\sigma_i^2}\right) \\ \left(\sum_i^N \frac{x_i\, y_i}{\sigma_i^2}\right) \end{bmatrix} \tag{1.III}$$

Let's define

$$X = \begin{bmatrix} \frac{1}{\sigma_1} & \frac{x_1}{\sigma_1} \\ \frac{1}{\sigma_2} & \frac{x_2}{\sigma_2} \\ \vdots & \vdots \\ \frac{1}{\sigma_N} & \frac{x_N}{\sigma_N} \end{bmatrix} \qquad Y = \begin{bmatrix} \frac{y_1}{\sigma_1} \\ \frac{y_2}{\sigma_2} \\ \vdots \\ \frac{y_N}{\sigma_N} \end{bmatrix} \qquad a = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \tag{1.10}$$

then

$$X^T X = \begin{bmatrix} \frac{1}{\sigma_1} & \frac{1}{\sigma_2} & \cdots & \frac{1}{\sigma_N} \\ \frac{x_1}{\sigma_2} & \frac{x_2}{\sigma_2} & \cdots & \frac{x_N}{\sigma_N} \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma_1} & \frac{x_1}{\sigma_1} \\ \frac{1}{\sigma_2} & \frac{x_2}{\sigma_2} \\ \vdots & \vdots \\ \frac{1}{\sigma_N} & \frac{x_N}{\sigma_N} \end{bmatrix} = \begin{bmatrix} \left(\sum_i^N \frac{1}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x_i}{\sigma_i^2}\right) \\ \left(\sum_i^N \frac{x_i}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x_i^2}{\sigma_i^2}\right) \end{bmatrix} \tag{1.11}$$

and

$$X^T Y = \begin{bmatrix} \frac{1}{\sigma_1} & \frac{1}{\sigma_2} & \cdots & \frac{1}{\sigma_N} \\ \frac{x_1}{\sigma_2} & \frac{x_2}{\sigma_2} & \cdots & \frac{x_N}{\sigma_N} \end{bmatrix} \begin{bmatrix} \frac{y_1}{\sigma_1} \\ \frac{y_2}{\sigma_2} \\ \vdots \\ \frac{y_N}{\sigma_N} \end{bmatrix} = \begin{bmatrix} \left(\sum_i^N \frac{y_i}{\sigma_i^2}\right) \\ \left(\sum_i^N \frac{x_i\, y_i}{\sigma_i^2}\right) \end{bmatrix} \tag{1.12}$$

i.e. the matrix Eq. (1.III) is

$$\boxed{\left(X^T X\right) a = X^T Y} \tag{1.A}$$

e.g. Least_Square_st_line_fitting_01.R

2 Least Square Fitting: A Quadratic

$$\text{Data}: \quad \{x_i,\, y_i \pm \sigma_i\}_{i=1,2,\ldots,N} \tag{2.13}$$

$$\text{Fit}: \quad y = a_2 x^2 + a_1 x + a_0 \tag{2.14}$$

Objective(Cost) function / Residual Square:

$$\chi^2 = \sum_i^N \frac{[y_i - (a_2 x_i^2 + a_1 x_i + a_0)]^2}{\sigma_i^2} \tag{2.15}$$

$$\frac{\partial \chi^2}{\partial a_0} = 2\sum_i^N \frac{[y_i - (a_2 x_i^2 + a_1 x_i + a_0)]}{\sigma_i^2}(-1) \tag{2.16}$$

$$\frac{\partial \chi^2}{\partial a_1} = 2\sum_i^N \frac{[y_i - (a_2 x_i^2 + a_1 x_i + a_0)]}{\sigma_i^2}(-x_i) \tag{2.17}$$

$$\frac{\partial \chi^2}{\partial a_2} = 2\sum_i^N \frac{[y_i - (a_2 x_i^2 + a_1 x_i + a_0)]}{\sigma_i^2}(-x_i^2) \tag{2.18}$$

$$\frac{\partial \chi^2}{\partial a_0} = 0 \tag{2.19}$$

$$\Downarrow \tag{2.20}$$

$$\sum_i^N \frac{y_i}{\sigma_i^2} - a_2 \sum_i^N \frac{x_i^2}{\sigma_i^2} - a_1 \sum_i^N \frac{x_i}{\sigma_i^2} - a_0 \sum_i^N \frac{1}{\sigma_i^2} = 0 \tag{2.21}$$

or,

$$\left(\sum_i^N \frac{1}{\sigma_i^2}\right) a_0 + \left(\sum_i^N \frac{x_i}{\sigma_i^2}\right) a_1 + \left(\sum_i^N \frac{x_i^2}{\sigma_i^2}\right) a_2 = \sum_i^N \frac{y_i}{\sigma_i^2} \tag{2.I}$$

$$\frac{\partial \chi^2}{\partial a_1} = 0 \tag{2.22}$$

$$\Downarrow \tag{2.23}$$

$$\sum_i^N \frac{x_i\, y_i}{\sigma_i^2} - a_2 \sum_i^N \frac{x_i^3}{\sigma_i^2} - a_1 \sum_i^N \frac{x_i^2}{\sigma_i^2} - a_0 \sum_i^N \frac{x_i}{\sigma_i^2} = 0 \tag{2.24}$$

or,

$$\left(\sum_i^N \frac{x_i}{\sigma_i^2}\right) a_0 + \left(\sum_i^N \frac{x_i^2}{\sigma_i^2}\right) a_1 + \left(\sum_i^N \frac{x_i^3}{\sigma_i^2}\right) a_2 = \sum_i^N \frac{x_i\, y_i}{\sigma_i^2} \tag{2.II}$$

$$\frac{\partial \chi^2}{\partial a_2} = 2\sum_i^N \frac{[y_i - (a_2 x_i^2 + a_1 x_i + a_0)]}{\sigma_i^2}(-x_i^2) \tag{2.25}$$

$$\frac{\partial \chi^2}{\partial a_2} = 0 \tag{2.26}$$

$$\Downarrow \tag{2.27}$$

$$\sum_i^N \frac{x_i^2\, y_i}{\sigma_i^2} - a_2 \sum_i^N \frac{x_i^4}{\sigma_i^2} - a_1 \sum_i^N \frac{x_i^3}{\sigma_i^2} - a_0 \sum_i^N \frac{x_i^2}{\sigma_i^2} = 0 \tag{2.28}$$

or,

$$\left(\sum_i^N \frac{x_i^2}{\sigma_i^2}\right) a_0 + \left(\sum_i^N \frac{x_i^3}{\sigma_i^2}\right) a_1 + \left(\sum_i^N \frac{x_i^4}{\sigma_i^2}\right) a_2 = \sum_i^N \frac{x_i^2\, y_i}{\sigma_i^2} \tag{2.III}$$

So, we get a system of three linear equations:

$$\left(\sum_i^N \frac{1}{\sigma_i^2}\right) a_0 + \left(\sum_i^N \frac{x_i}{\sigma_i^2}\right) a_1 + \left(\sum_i^N \frac{x_i^2}{\sigma_i^2}\right) a_2 = \sum_i^N \frac{y_i}{\sigma_i^2} \tag{2.I}$$

$$\left(\sum_i^N \frac{x_i}{\sigma_i^2}\right) a_0 + \left(\sum_i^N \frac{x_i^2}{\sigma_i^2}\right) a_1 + \left(\sum_i^N \frac{x_i^3}{\sigma_i^2}\right) a_2 = \sum_i^N \frac{x_i\, y_i}{\sigma_i^2} \tag{2.II}$$

$$\left(\sum_i^N \frac{x_i^2}{\sigma_i^2}\right) a_0 + \left(\sum_i^N \frac{x_i^3}{\sigma_i^2}\right) a_1 + \left(\sum_i^N \frac{x_i^4}{\sigma_i^2}\right) a_2 = \sum_i^N \frac{x_i^2\, y_i}{\sigma_i^2} \tag{2.III}$$

Eqs. (2.I), (2.II) and (2.III) in the matrix form

$$\begin{bmatrix} \left(\sum_i^N \frac{1}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x_i}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x_i^2}{\sigma_i^2}\right) \\ \left(\sum_i^N \frac{x_i}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x_i^2}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x_i^3}{\sigma_i^2}\right) \\ \left(\sum_i^N \frac{x_i^2}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x_i^3}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x_i^4}{\sigma_i^2}\right) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \left(\sum_i^N \frac{y_i}{\sigma_i^2}\right) \\ \left(\sum_i^N \frac{x_i\, y_i}{\sigma_i^2}\right) \\ \left(\sum_i^N \frac{x_i^2\, y_i}{\sigma_i^2}\right) \end{bmatrix} \tag{2.IV}$$

Let's define

$$X = \begin{bmatrix} \frac{1}{\sigma_1} & \frac{x_1}{\sigma_1} & \frac{x_1^2}{\sigma_1} \\ \frac{1}{\sigma_2} & \frac{x_2}{\sigma_2} & \frac{x_2^2}{\sigma_2} \\ \vdots & \vdots & \vdots \\ \frac{1}{\sigma_N} & \frac{x_N}{\sigma_N} & \frac{x_N^2}{\sigma_N} \end{bmatrix} \qquad Y = \begin{bmatrix} \frac{y_1}{\sigma_1} \\ \frac{y_2}{\sigma_2} \\ \vdots \\ \frac{y_N}{\sigma_N} \end{bmatrix} \qquad a = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \tag{2.29}$$

then

$$X^T X = \begin{bmatrix} \frac{1}{\sigma_1} & \frac{1}{\sigma_2} & \cdots & \frac{1}{\sigma_N} \\ \frac{x_1}{\sigma_2} & \frac{x_2}{\sigma_2} & \cdots & \frac{x_N}{\sigma_N} \\ \frac{x_1^2}{\sigma_2} & \frac{x_2^2}{\sigma_2} & \cdots & \frac{x_N^2}{\sigma_N} \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma_1} & \frac{x_1}{\sigma_1} & \frac{x_1^2}{\sigma_1} \\ \frac{1}{\sigma_2} & \frac{x_2}{\sigma_2} & \frac{x_2^2}{\sigma_2} \\ \vdots & \vdots & \vdots \\ \frac{1}{\sigma_N} & \frac{x_N}{\sigma_N} & \frac{x_N^2}{\sigma_N} \end{bmatrix} = \begin{bmatrix} \left(\sum_i^N \frac{1}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x_i}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x_i^2}{\sigma_i^2}\right) \\ \left(\sum_i^N \frac{x_i}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x_i^2}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x_i^3}{\sigma_i^2}\right) \\ \left(\sum_i^N \frac{x_i^2}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x_i^3}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x_i^4}{\sigma_i^2}\right) \end{bmatrix} \tag{2.30}$$

and

$$X^T Y = \begin{bmatrix} \frac{1}{\sigma_1} & \frac{1}{\sigma_2} & \cdots & \frac{1}{\sigma_N} \\ \frac{x_1}{\sigma_2} & \frac{x_2}{\sigma_2} & \cdots & \frac{x_N}{\sigma_N} \\ \frac{x_1^2}{\sigma_2} & \frac{x_2^2}{\sigma_2} & \cdots & \frac{x_N^2}{\sigma_N} \end{bmatrix} \begin{bmatrix} \frac{y_1}{\sigma_1} \\ \frac{y_2}{\sigma_2} \\ \vdots \\ \frac{y_N}{\sigma_N} \end{bmatrix} = \begin{bmatrix} \left(\sum_i^N \frac{y_i}{\sigma_i^2}\right) \\ \left(\sum_i^N \frac{x_i\, y_i}{\sigma_i^2}\right) \\ \left(\sum_i^N \frac{x_i^2\, y_i}{\sigma_i^2}\right) \end{bmatrix} \tag{2.31}$$

i.e. the matrix Eq. (2.III) is

$$\boxed{\left(X^T X\right) a = X^T Y} \tag{2.A}$$

e.g. Least_Square_quadratic_curve_fitting_01.R

3 Least Square Fitting: A Plain

$$\text{Data}: \quad \{x_i,\, y_i,\, z_i \pm \sigma_i\}_{i=1,2,\ldots,N} \tag{3.32}$$

$$\text{Fit}: \quad z = a_2 y + a_1 x + a_0 \tag{3.33}$$

Objective(Cost) function / Residual Square:

$$\chi^2 = \sum_i^N \frac{[z_i - (a_2 y_i + a_1 x_i + a_0)]^2}{\sigma_i^2} \tag{3.34}$$

$$\frac{\partial \chi^2}{\partial a_0} = 2 \sum_i^N \frac{[z_i - (a_2 y_i + a_1 x_i + a_0)]}{\sigma_i^2}(-1) \tag{3.35}$$

$$\frac{\partial \chi^2}{\partial a_1} = 2 \sum_i^N \frac{[z_i - (a_2 y_i + a_1 x_i + a_0)]}{\sigma_i^2}(-x_i) \tag{3.36}$$

$$\frac{\partial \chi^2}{\partial a_2} = 2 \sum_i^N \frac{[z_i - (a_2 y_i + a_1 x_i + a_0)]}{\sigma_i^2}(-y_i) \tag{3.37}$$

$$\frac{\partial \chi^2}{\partial a_0} = 0 \tag{3.38}$$

$$\Downarrow \tag{3.39}$$

$$\sum_i^N \frac{z_i}{\sigma_i^2} - a_2 \sum_i^N \frac{y_i}{\sigma_i^2} - a_1 \sum_i^N \frac{x_i}{\sigma_i^2} - a_0 \sum_i^N \frac{1}{\sigma_i^2} = 0 \tag{3.40}$$

or,

$$\left(\sum_i^N \frac{1}{\sigma_i^2}\right) a_0 + \left(\sum_i^N \frac{x_i}{\sigma_i^2}\right) a_1 + \left(\sum_i^N \frac{y_i}{\sigma_i^2}\right) a_2 = \sum_i^N \frac{z_i}{\sigma_i^2} \tag{3.I}$$

$$\frac{\partial \chi^2}{\partial a_1} = 0 \tag{3.41}$$

$$\Downarrow \tag{3.42}$$

$$\sum_i^N \frac{x_i\, z_i}{\sigma_i^2} - a_2 \sum_i^N \frac{x_i\, y_i}{\sigma_i^2} - a_1 \sum_i^N \frac{x_i^2}{\sigma_i^2} - a_0 \sum_i^N \frac{x_i}{\sigma_i^2} = 0 \tag{3.43}$$

or,

$$\left(\sum_i^N \frac{x_i}{\sigma_i^2}\right) a_0 + \left(\sum_i^N \frac{x_i^2}{\sigma_i^2}\right) a_1 + \left(\sum_i^N \frac{x_i\, y_i}{\sigma_i^2}\right) a_2 = \sum_i^N \frac{x_i\, z_i}{\sigma_i^2} \tag{3.II}$$

$$\frac{\partial \chi^2}{\partial a_2} = 0 \tag{3.44}$$

$$\Downarrow \tag{3.45}$$

$$\sum_i^N \frac{y_i\, z_i}{\sigma_i^2} - a_2 \sum_i^N \frac{y_i^2}{\sigma_i^2} - a_1 \sum_i^N \frac{x_i\, y_i}{\sigma_i^2} - a_0 \sum_i^N \frac{y_i}{\sigma_i^2} = 0 \tag{3.46}$$

or,

$$\left(\sum_i^N \frac{y_i}{\sigma_i^2}\right) a_0 + \left(\sum_i^N \frac{x_i\, y_i}{\sigma_i^2}\right) a_1 + \left(\sum_i^N \frac{y_i^2}{\sigma_i^2}\right) a_2 = \sum_i^N \frac{y_i\, z_i}{\sigma_i^2} \tag{3.III}$$

So, we get a system of three linear equations:

$$\left(\sum_i^N \frac{1}{\sigma_i^2}\right) a_0 + \left(\sum_i^N \frac{x_i}{\sigma_i^2}\right) a_1 + \left(\sum_i^N \frac{y_i}{\sigma_i^2}\right) a_2 = \sum_i^N \frac{z_i}{\sigma_i^2} \tag{3.I}$$

$$\left(\sum_i^N \frac{x_i}{\sigma_i^2}\right) a_0 + \left(\sum_i^N \frac{x_i^2}{\sigma_i^2}\right) a_1 + \left(\sum_i^N \frac{x_i\, y_i}{\sigma_i^2}\right) a_2 = \sum_i^N \frac{x_i\, z_i}{\sigma_i^2} \tag{3.II}$$

$$\left(\sum_i^N \frac{y_i}{\sigma_i^2}\right) a_0 + \left(\sum_i^N \frac{x_i\, y_i}{\sigma_i^2}\right) a_1 + \left(\sum_i^N \frac{y_i^2}{\sigma_i^2}\right) a_2 = \sum_i^N \frac{y_i\, z_i}{\sigma_i^2} \tag{3.III}$$

Eqs. (I), (II) and (III) in the matrix form

$$\begin{bmatrix} \left(\sum_i^N \frac{1}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x_i}{\sigma_i^2}\right) & \left(\sum_i^N \frac{y_i}{\sigma_i^2}\right) \\ \left(\sum_i^N \frac{x_i}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x_i^2}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x_i\, y_i}{\sigma_i^2}\right) \\ \left(\sum_i^N \frac{y_i}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x_i\, y_i}{\sigma_i^2}\right) & \left(\sum_i^N \frac{y_i^2}{\sigma_i^2}\right) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \left(\sum_i^N \frac{z_i}{\sigma_i^2}\right) \\ \left(\sum_i^N \frac{x_i\, z_i}{\sigma_i^2}\right) \\ \left(\sum_i^N \frac{y_i\, z_i}{\sigma_i^2}\right) \end{bmatrix} \tag{3.IV}$$

Let's define

$$X = \begin{bmatrix} \frac{1}{\sigma_1} & \frac{x_1}{\sigma_1} & \frac{y_1}{\sigma_1} \\ \frac{1}{\sigma_2} & \frac{x_2}{\sigma_2} & \frac{y_2}{\sigma_2} \\ \vdots & \vdots & \vdots \\ \frac{1}{\sigma_N} & \frac{x_N}{\sigma_N} & \frac{y_N}{\sigma_N} \end{bmatrix} \qquad Y = \begin{bmatrix} \frac{z_1}{\sigma_1} \\ \frac{z_2}{\sigma_2} \\ \vdots \\ \frac{z_N}{\sigma_N} \end{bmatrix} \qquad a = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \tag{3.47}$$

then

$$X^T X = \begin{bmatrix} \frac{1}{\sigma_1} & \frac{1}{\sigma_2} & \cdots & \frac{1}{\sigma_N} \\ \frac{x_1}{\sigma_2} & \frac{x_2}{\sigma_2} & \cdots & \frac{x_N}{\sigma_N} \\ \frac{y_1}{\sigma_2} & \frac{y_2}{\sigma_2} & \cdots & \frac{y_N}{\sigma_N} \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma_1} & \frac{x_1}{\sigma_1} & \frac{y_1}{\sigma_1} \\ \frac{1}{\sigma_2} & \frac{x_2}{\sigma_2} & \frac{y_2}{\sigma_2} \\ \vdots & \vdots & \vdots \\ \frac{1}{\sigma_N} & \frac{x_N}{\sigma_N} & \frac{y_1}{\sigma_N} \end{bmatrix} = \begin{bmatrix} \left(\sum_i^N \frac{1}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x_i}{\sigma_i^2}\right) & \left(\sum_i^N \frac{y_i}{\sigma_i^2}\right) \\ \left(\sum_i^N \frac{x_i}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x^2}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x_i\, y_i}{\sigma_i^2}\right) \\ \left(\sum_i^N \frac{y_i}{\sigma_i^2}\right) & \left(\sum_i^N \frac{x_i\, y_i}{\sigma_i^2}\right) & \left(\sum_i^N \frac{y_i^2}{\sigma_i^2}\right) \end{bmatrix} \tag{3.48}$$

and

$$X^T Y = \begin{bmatrix} \frac{1}{\sigma_1} & \frac{1}{\sigma_2} & \cdots & \frac{1}{\sigma_N} \\ \frac{x_1}{\sigma_2} & \frac{x_2}{\sigma_2} & \cdots & \frac{x_N}{\sigma_N} \\ \frac{y_1}{\sigma_2} & \frac{y_2}{\sigma_2} & \cdots & \frac{y_N}{\sigma_N} \end{bmatrix} \begin{bmatrix} \frac{z_1}{\sigma_1} \\ \frac{z_2}{\sigma_2} \\ \vdots \\ \frac{z_N}{\sigma_N} \end{bmatrix} = \begin{bmatrix} \left(\sum_i^N \frac{y_i}{\sigma_i^2}\right) \\ \left(\sum_i^N \frac{x_i\, z_i}{\sigma_i^2}\right) \\ \left(\sum_i^N \frac{y_i\, z_i}{\sigma_i^2}\right) \end{bmatrix} \tag{3.49}$$

i.e. the matrix Eq. (3.IV) is

$$\boxed{\left(X^T X\right) a = X^T Y} \tag{3.A}$$

e.g. Least_Square_multilinear_fitting_01.R