Artificial Intelligence

Propositional Logic

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Propositional logic …

Logic:

The word logic refers to the use and study of valid reasoning. Logic contains rules and techniques to formalize statements, to make them precise.

In other words, accepted rules for making precise statements represents logic.

- Logic for computer science: programming, artificial intelligence, logic circuits, database.
- Logic

– Represents knowledge precisely

– Helps to extract information (inference)

Reasoning in Daily Life

Logic can be seen in action all around us:

In a restaurant, your Father has ordered Fish, your Mother ordered vegetarian, and you ordered Meat. Out of the kitchen comes some new person carrying the three plates. What will happen?

Symbolic Logic (Logical Operators)

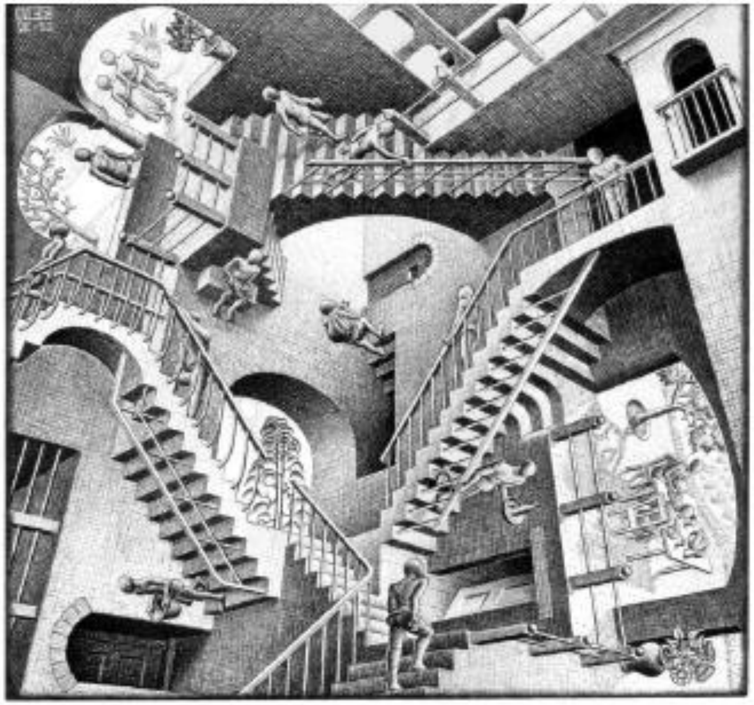
Symbol	In natural language	Technical name
$\neg$	not	negation
$\wedge$	and	conjunction
$\vee$	or	disjunction
$\rightarrow$	if ... then	implication
$\leftrightarrow$	if and only if	equivalence

Paradox

A paradox is a statement that *cannot be assigned a truth value*.

- A paradox is not a proposition.
- Example: the liar paradox

  "This statement is false"



Art Work by Escher ("Relativity")

Negation ¬

- Negation (not) of p: ¬p (~ p is always also used)

p	¬p
T	F
F	T

Truth Table

- p: You shall pass

¬p: You shall not pass

Disjunction V

- Disjunction (or) of p with q: pVq

p	q	pVq	qVp
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

True when
`at least one'
of them is
true

- p V q ≡ q V p, i.e. operator V commutes

means "equivalent"

Conjunction ∧

- Conjunction (and) of p with q: $p \wedge q$

p	q	p∧q
T	T	T
T	F	F
F	T	F
F	F	F

True only when `both' of them are true

- ∧ is also commutative: $p \wedge q \equiv q \wedge p$

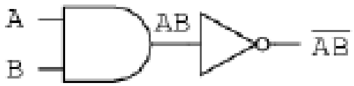
De Morgan’s Law

$$\neg (p \land q) \equiv \neg p \lor \neg q$$

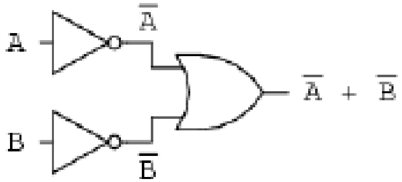
$$\neg (p \lor q) \equiv \neg p \land \neg q$$

p q	¬p ¬q p ∧ q	¬(p ∧ q)	¬p ∨¬q
T T	F F T	F	F
T F	F T F	T	T
F T	T F F	T	T
F F	T T F	T	T

Augustus De Morgan
(1806-1871)



. . . is equivalent to . . .



$$\overline{AB} = \overline{A} + \overline{B}$$

Contradiction

A statement that is always false is called a contradiction.

Example: This course is easy
`and' this course is not easy

$$p \wedge (\neg p) \equiv F$$

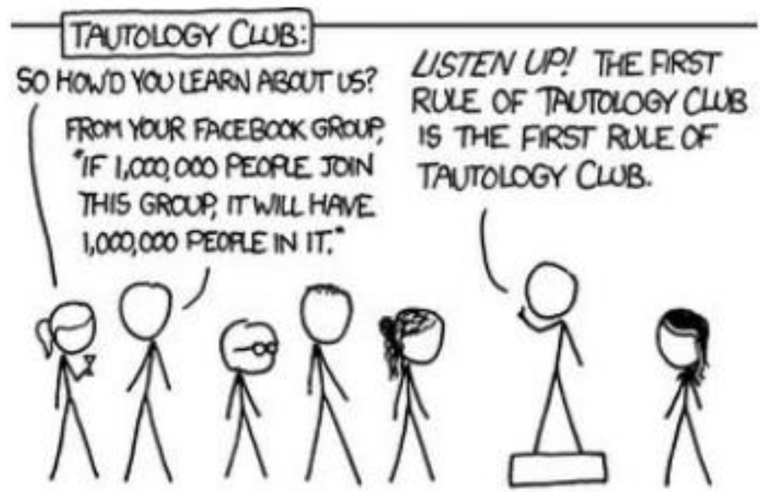
Tautology

An expression that always gives a true value is called a tautology .

Example: $p \vee (\neg p) \equiv T$

p	¬p	p∨¬p
T	F	T
F	T	T

Always true!



Somewhat similar to “Head I win, Tail you lose“

Equivalent Expressions

Consider the following three statements

- Alice is not married but Bob is not single

  $\neg h \wedge \neg b$

- Bob is not single and Alice is not married

  $\neg b \wedge \neg h$

- Neither Bob is single nor Alice is married

  $\neg(b \vee h)$

- These three statements are equivalent

  $\neg h \wedge \neg b \equiv \neg b \wedge \neg h \equiv \neg(b \vee h)$

Propositional logic …

Proposition:

A proposition is a statement either true or false, but not both.

- The statement "1+1 > 3" is false
- "5 > 3" is true

Both the above statements are propositions.

- The statement "What a great book!" is not a proposition because someone is expressing an opinion.

Propositional logic …

Interpretation of a logical expression

Assignment of a specific set of truth values to the individual statements of a given logical expression is said to be an interpretation of the expression.

Example

Logical expression : $F(A,B) = (A \vee B) \vee (A \vee \neg B)$

An interpretation : $A := T$, $B := F$, $F(A,B) = T$

Propositional logic …

Equivalence

Two logical expressions are said to be equivalent if they attain the same truth value for all possible interpretations.

Example

1 $A \longrightarrow B \equiv \neg A \vee B$

2 $(A \longrightarrow B) \longrightarrow C \equiv A \longrightarrow (B \longrightarrow C)$

Propositional logic …

Equivalence …

A	B	C	$A \rightarrow B$	$(A \rightarrow B) \rightarrow C$	$B \rightarrow C$	$A \rightarrow (B \rightarrow C)$
F	F	F	T	F	T	T
F	F	T	T	T	T	T
F	T	F	T	F	F	T
F	T	T	T	T	T	T
T	F	F	F	T	T	T
T	F	T	F	T	T	T
T	T	F	T	F	F	F
T	T	T	T	T	T	T

Propositional logic …

Tautology

Statements that are TRUE under all possible interpretations.

Contradiction

Statements that are FALSE under all possible interpretations.

Consistency

A collection of statements is consistent if the statements can all be true simultaneously.

Propositional logic … consistency

Is the following statements consistent?

Sales of houses fall off interest rates rise. Auctioneers are not happy if sales of houses fall off. Interest rates are rising. Auctioneers are happy.

S : Sales of houses fall off.

R : Interest rates rise.

H : Auctioneers are happy.

Propositional logic … consistency

S : Sales of houses fall off.

R : Interest rates rise.

H : Auctioneers are happy.

Sales of houses fall off if interest rates rise. $(R \longrightarrow S)$
Auctioneers are not happy if sales of houses fall off.

$(S \longrightarrow \neg H)$

Interest rates are rising. (R)

Auctioneers are happy. (H)

Propositional logic … consistency

S	R	H	$R \rightarrow S$	$S \rightarrow \neg H$	$(R \rightarrow S) \wedge (S \rightarrow \neg H) \wedge R \wedge H$
F	F	F	T	T	F
F	F	T	T	T	F
F	T	F	F	T	F
F	T	T	F	T	F
T	F	F	T	T	F
T	F	T	T	F	F
T	T	F	T	T	F
T	T	T	T	F	F

Propositional logic …

Validity of an Argument

An argument is valid if the conclusion is true whenever the premises are true.

Argument 1

If the violinist plays the concerto, then crowds will come if the prices are not too high. If the violinist plays the concerto, the prices will not be too high. Therefore, if the violinist plays the concerto, crowd will come.

Propositional logic …

Validity of an Argument

A: Violinist plays the concerto .

B: Crowds will come.

C: Prices are not too high.

Argument 1

Premises

1. $A \rightarrow (C \rightarrow B)$
2. $A \rightarrow C$

Conclusion $A \rightarrow B$

Propositional logic … validity

A	B	C	A $\rightarrow$ B	A $\rightarrow$ C	C $\rightarrow$ B	A $\rightarrow$ (C $\rightarrow$ B)
F	F	F	T	T	T	T
F	F	T	T	T	F	T
F	T	F	T	T	T	T
F	T	T	T	T	T	T
T	F	F	F	F	T	T
T	F	T	F	T	F	F
T	T	F	T	F	T	T
T	T	T	T	T	T	T

Hence the argument is valid

Propositional logic …

Validity of Argument

Argument 2

Premises

1. $A \rightarrow (B \wedge \neg C)$
2. $A \wedge (B \rightarrow C)$

Conclusion A

Propositional logic … validity

A	B	C	$\neg C$	$B \wedge \neg C$	$A \rightarrow (B \wedge \neg C)$	$B \rightarrow C$	$A \neg (B \rightarrow C)$
F	F	F	T	F	T	T	F
F	F	T	F	F	T	T	F
F	T	F	T	T	T	F	F
F	T	T	F	F	T	T	F
T	F	F	T	F	F	T	T
T	F	T	F	F	F	T	T
T	T	F	T	T	T	F	F
T	T	T	F	F	F	T	T

The is no interpretation for which both the premises are true simultaneously. Hence the argument is valid.

Limitation of Propositional logic

Argument 1

If stuntmanship is a dangerous trade then it should be highly paid. Stuntmanship is a dangerous trade. Therefore stuntmanship should be highly paid.

$$A \rightarrow B$$

$$A$$

$$\therefore B$$

Valid

Propositional logic … Limitations

Argument 2

All dangerous trades should be highly paid and stuntmanship is a dangerous trade. Therefore stuntmanship should be highly paid.

$$\begin{array}{l} P \wedge Q \\ \hline \therefore\ R \end{array}$$

Invalid

Predicate Logic

Argument 2

All dangerous trades should be highly paid and stuntmanship is a dangerous trade. Therefore stuntmanship should be highly paid.

$(\forall X)\ \{DT(X) \rightarrow HP(X)] \wedge DT(S)$

$\therefore HP(S)$

Vaild

Predicate Logic…

Alternative notation

All dangerous trades should be highly paid and stuntmanship is a dangerous trade. Therefore stuntmanship should be highly paid.

$(\forall X)\ [\{T\ (X) \wedge D(X)\} \rightarrow HP(X)] \wedge \{T(s) \wedge D(S)\}$

$\therefore HP(S)$

Vaild

Well formed Formulae(wff)

Constituent elements

1	Individual constants	: A, B, C, …
2	Individual variables	: X, Y, Z, …
3	Predicate Constants	: P, Q, R, …
4	Function constants	: f, g, h, …
5	Quantifies	: $\forall$, $\exists$
6	Logical connectives	: $\neg$, $\wedge$, $\vee$, $\rightarrow$
7	Parenthesis	: (,)

Well formed Formulae(wff)…

Definition

1 If P is a predicate constant and x is an individual constant, or variable, then P(x) is a wff.

2 If W is a wff the ¬ (w) is also a wff.

3 If $w_1$and w_2 are two wffs, then so are (w_1)^(w_2), (w_1) ˅ (w_2), $(w_1) \rightarrow (w_2)$

4 If W is a wff containing a free variable x, then the following are also wffs:

(∀ X) W and (∃ X) W

Well formed Formulae(wff)…

Example 1

Statement

All employees earning Rs 1, 00, 000 or more per year pays tax.

Rephrase the statement as

For all X, X is an employee and X earns Rs 1,00,000 or more per year implies x pays tax.

Well Formed Formulae(wff)…

Example 1…

Statement

For all X, X is an employee and X earns Rs 1,00,000 or more per year implies x pays tax.

Predicates

E(X) : X is an employee

L(X) : X earns Rs 1, 00, 000 or more per year

PT(X) : X pays tax

$$(\forall X) \ (E(X) \wedge L(X)) \rightarrow PT(X)$$

Example 1………. Alternative formulation

Statement

For all X, X is an employee and X earns Rs 1,00,000 or more per year implies X pays tax.

Predicates

E(X) : X ix an employee

e(X) : yearly earning of X

GE(X,Y) : X is greater than or equal to y

PT(X) : X pays tax.

$$(\forall X)[(E(X) \wedge GE(e(X), 100000)] \rightarrow PT(X)$$

Well Formed Formulae(wff)…

Example 2

Statement

Some employees are sick today

Rephrase the statement as

There exists x such that x is an employee and x is sick today.

$(\exists x)\ E(x) \wedge S(x)$

Well Formed Formulae(wff)…

Example 3

Statement

No employee earns more than the president.

Rephrase the statement as

There does not exists x such that x is an employee and x’s earning is more than the President’s earning.

$$\neg(\exists x)\ (E(x) \wedge GE(e(x), e(President)))$$

or

$$\neg(\exists x)\ (\exists y)\ (E(x) \wedge P(y) \wedge GE(e(x), e(y)))$$

Rules of Inference

Rules of inference are rules using which new statements can be obtained from existing statements.

1 Modus ponens

2 Universal specialization

3 Chain Rule

4 Resolution

5 Etc.

Rules of Inference

Modus Ponens

1 $P \rightarrow Q$

2 P

Q

Universal Specialization

1 $(\forall x)W(x)$

$\therefore W(A)$

Rules of Inference

Chain Rule

1 $A \rightarrow B$

2 $B \rightarrow C$

$\therefore A \rightarrow C$

Resolution

1 $A \vee B \vee C \ldots$

2 $\neg A \vee P \vee Q \ldots$

$\therefore B \vee C \ldots \vee P \vee Q \ldots$

Rules of Inference

Given :

1 $P \rightarrow Q$

2 $R \rightarrow S$

3 $P \vee R$

Prove: $Q \vee S$

Proof:

1 $\neg P \vee Q$ [Premise 1]

2 $\neg R \vee S$ [Premise 2]

3 $P \vee R$ [Premise 3]

4 $Q \vee R$ [1,3; Resolution]

5 $Q \vee S$ [2,4; Resolution]

Rules of Inference … example

All dangerous trades should be highly paid and stuntmanship is a dangerous trade. Therefore stuntmanship should be highly paid.

Given

1 $(\forall x)\{T(x) \wedge D(x)\} \rightarrow H(x)$

2 T(Stuntmanship)

3 D(Stuntmanship)

Prove H(Stuntmanship)

Rules of Inference … example

1 $(\forall x)\{T(x) \wedge D(x)\} \rightarrow H(x)$	[Premise 1]
2 $T(S)$	[Premise 2]
3 $D(S)$	[Premise 3]
4 $\{T(S) \wedge D(S)\} \rightarrow H(S)$	[1; Universal specialization]
5 $\neg\{T(S) \wedge D(S)\} \vee H(S)$	[4; rephrasing]
6 $\neg T(S) \vee \neg D(S) \vee H(S)$	[5; De Morgan's law]
7 $\neg D(S) \vee H(S)$	[2,6; Resolution]
8 $H(S)$	[3,7; Resolution]