Channel Performance Analysis on the Sum of Fisher-Snedecor $\mathscr{F}$ Random Variables using Meijer-G and Fox-H Function

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***Abstract*—Statistical analysis of the sum of random variables (RVs) is a crucial aspect of determining the performance of wireless communication channels. In this study, we investigate the sum of independent and identically distributed (i.i.d.) Fisher-Snedecor $\mathscr{F}$ RVs, and derive closed-form expressions for the channel capacity and average bit error rate (BER), with the help of Meijer-G function and the multivariate Fox-H function. These obtained expressions are utilized in analyzing the performance of wireless communication channels such as outage probability, effective channel capacity, etc. This study, backed with mathematical simulations and comparisons, provides valuable insights into the design and implementation of highly efficient communication channels.**

***Index Terms*—Wireless communication channels, fading channels, Fisher-Snedecor distribution, channel capacity, SNR, Meijer-G Function.**

I. Introduction

Sums of random variables (RVs) are a typical construct commonly encountered in a number of applications involving wireless communications, such as equal-gain combining (EGC), signal detection, phase jitter, intersymbol interference, outage probability, etc. Due to the fact that the exact formulation of some statistics for such sums may be mathematically intricate, and as an attempt to circumvent this, several approaches concerning approximation methods have been proposed in the literature for the well-known RVs. [1]

It is recalled that composite fading models outperform conventional fading models due to their ability to characterize the simultaneous occurrence of multipath fading and shadowing [2]. This is the reason the Fisher-Snedecor F composite fading model has been selected for this study since it provides accurate modelling of channel measurements obtained in the context of wearable communications, along with other promising benefits such as simple and elementary probability density function (PDF) leading to more compliant analysis compared to other models [3].

In this study, novel and computation-friendly closed-form expressions have been derived for average bit error rate (BER), outage probability and channel capacity for the sum of independent and identically distributed (i.i.d.) F random variates.

The remainder of the paper is as follows. Section 2 starts with the preliminary background and motivation of our mathematical analysis, followed by the introduction of the F distribution in section 3. The next section 4 goes through finding the expressions for performance parameters such as outage probability, average BER, and ergodic channel capacity. In the next section these expressions are compared with results obtained from computational simulations. Finally, section 6 concludes the papers with summarized results and future prospects.

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II. Preliminary Concepts

Considering the assumption of independent and identically distributed (i.i.d.) characteristics across all nodes, where each node possesses parameters with identical standard deviations or means, we can denote the parameter function of a node as a function of X. Consequently, the cumulative parameter function, as determined by Z, can be expressed as follows:

$$Z = X_1 + X_2 + X_3 + ... + X_n \tag{1}$$

$$Z = \sum_{i=1}^{n} X_i \tag{2}$$

Where each X_i is a respective node parameter on the board, and the board consists of a total of n nodes. This results in the cumulative parameter function in terms of all individual parameter node functions. If each variable X is independently and identically distributed (i.i.d.), it means that they all follow the same probability distribution, and each observation is independent of the others. This assumption is fundamental in many statistical analyses and models. To clarify further:

Independently Distributed: This means that the occurrence or value of one variable does not affect the occurrence or value of another variable. In other words, knowing the value of X_j does not provide any information about the value of X_i, where i$\neq$ j. Identically Distributed: This implies that each variable X_i follows the same probability distribution. For example, if X_i follows a normal distribution with a certain mean and variance, then all other X_i and X_j variables also follow the same normal distribution with the same mean and variance. By assuming that the variables are i.i.d., we simplify the analysis and make it easier to apply various statistical techniques and models. This assumption is commonly made in fields such as probability theory, statistics, and machine learning when dealing with random variables.

Let f(x) represent the probability density function characterizing the parameters of an individual node, where x denotes a specific parameter value. Under the assumption of i.i.d. properties among nodes, and assuming uniformity in parameter characteristics, the cumulative parameter function Z can be defined as the summation of all parameter functions as derived earlier.

The analysis of the joint probability distribution function f(z) of a multi-dimensional space was conducted by convolving individual probability distribution functions associated with each node parameter. Mathematically, this can be represented as:

$$f(z) = f(x_1) * f(x_2) * f(x_3) ... * (fx_n) \tag{3}$$

Here, f(x_i) denotes the probability distribution function (PDF) associated with the node parameter x_i for i ranging from 1 to n. Each f(x_i) captures the statistical characteristics and distributional properties of its respective node parameter within the total space under investigation. The convolution operation employed in the above expression signifies the integration of these individual PDFs to derive the joint PDF f(z). This approach allows for a comprehensive understanding of the combined effects and interactions among the node parameters, providing insights into the overall probability distribution across the multidimensional space.

Definition 1: Let $F_z(S)$ denote a Laplace transform-based function parameterized by S, which is defined as the product of several component functions $F_i(S)$, where i ranges from 1 to N.

$$F_z(S) = F_1(S) \times F_2(S) \times F_3(S) \times ...F_n(S) \tag{4}$$

Description: The function $F_z(S)$ represents a composite Laplace transform-based function that synthesizes multiple component functions. Each component function $F_i(S)$ corresponds to a specific parameter and is derived from the Laplace transform of the probability distribution function f(ζ_i) associated with that parameter. Here, f(ζ_i) represents the probability distribution function of the i^{th} parameter ζ_i. The product formulation captures the joint influence of all these parameters on the overall system or phenomenon under study. This mathematical representation provides a unified framework for analyzing complex systems or processes characterized by multiple interacting parameters with distinct probability distributions.

Under the assumption of independently and identically distributed (i.i.d.) properties among the nodes, where each node follows the same probability distribution, we can simplify the calculation of the entire equation using a power term. This simplification is possible because each node has identical Laplace transforms of their probability density functions (PDFs), allowing us to consolidate their Laplace transforms. Let's denote the Laplace transform of the PDF of a single node as $L(X)$. Since all nodes are assumed to have identical distributions, this Laplace transform is common among all nodes. Therefore, if we have n such nodes, the Laplace transform of the sum Z of these nodes can be expressed as:

$$L(Z) = [L(X)]^n \tag{5}$$

Where: L(Z) is the Laplace transform of the sum of n nodes. L(X) is the Laplace transform of the PDF of a single node. This simplification arises due to the i.i.d. assumption, which allows us to treat all nodes as having the same distributional characteristics. It enables us to consolidate the Laplace transforms into a single power term, making calculations more manageable. This results in the next equation to being $F_z(S) = [F(S)]^n$

Next, to get the final result in the real domain, we take the Laplace inverse of the cumulative Laplace equation. The Laplace transform is a powerful mathematical tool used to transform functions from the time domain to the complex frequency domain. It plays a crucial role in various fields including engineering, physics, and mathematics. The inverse Laplace transform, denoted as L^{-1}, allows us to recover the original function from its Laplace transform. In this research, we focus on the inverse Laplace transform of a function F(s) raised to the power of N, which is represented as $L^{-1}[F_x(s)^N]$. The inverse Laplace transform of the aforementioned expression is given by:

$$L^{-1}[F_x(s)]^N = \frac{1}{2\pi i} \oint_C [F_x(s)]^N e^{st} ds \tag{6}$$

$L^{-1}[F_z(s)] = L^{-1}[F_x(s)^N]$ is the resultant equation. To write the given integral in proper mathematical notation with a proper description, let's break it down step by step. Consider the given integral:

$$\int_{L_i} e^{sx} [F_x(s)]^N ds \tag{7}$$

Now, as we cannot solve this easily for major cases, we take the help of the Meijer-G representation, which is shown as follows:

$$\int_{L_i}^{L_2} e^{sx} \left[\int_{L_i}^{L_2} G(S_1) S_1^{s_1} ds_1{}^N \right] ds \tag{8}$$

Where e^{sx} is the exponential function with respect to x. G(s) represents the Meijer-G function. S_1 is the integration variable for the inner integral. S_1 is the exponent variable in the inner integral. N is the number of times the inner integral is raised. L_1 & L_2 denote the lower and upper limits of integration respectively. The integral is composed of two nested integrals. The inner integral computes the function $G(S_1)S_1^{s_1}$ raised to the power of N with respect to S_1, while the outer integral integrates the resulting expression multiplied by e^{sx} with respect to s over the interval $[L_i, L_2]$. This representation may have applications in various fields of research, particularly in mathematical analysis, signal processing, or systems theory, where the behaviour and properties of the "Meijer" function G(s) and its interactions with exponential functions are of interest.

The following covers the working part of simplifying the equations

$$f(x) = \int_0^{L_2} G(S_1) x^{s_1} ds_1 \tag{9}$$

In this function, f(x) represents the output or dependent variable that we aim to compute, and x is an independent variable, which may denote a physical quantity, parameter, or any other variable of interest in the context of the research. L_2 is the upper limit of integration, representing the maximum value of the variable S_1 over which the integral is evaluated. $G(S_1)$ is a function of S_1, which could represent a response function, probability density function, or any other mathematical function relevant to the research context. S_1 is the integration variable, which varies from 0 to L_2 in this integral. The integral computes the area under the curve of $G(S_1)x^{s_1}$ with respect to S_1 over the interval [0, L_2]. This integral operation allows us to capture the cumulative effect of $G(S_1)$ and x^{s_1} across the range of S_1 from 0 to L_2, providing valuable insights or solutions in the research context. The function f(x) encapsulates a mathematical model or relationship that may be employed to analyze, predict, or understand phenomena in various scientific disciplines, such as physics, engineering, biology, economics, and more. The specific interpretation and significance of f(x) would depend on the context and objectives of the research in which it is employed.

Replacing x with s:

$$F(S) = \int_0^{L} G(S_1) \cdot S^{S_1} ds_1 \tag{10}$$

Here F(S) represents the main function with respect to the variable S, $\int L_2$ denotes the definite integral over the interval from L_1 to L_2 with respect to s_1, $G(S_1)$ represents a function of S_1, which might describe some properties or characteristics of the system or phenomenon under study, and S^{S_1} indicates the exponential relationship between S and S_1, where S is raised to the power of S_1.

Interpretation: The function F(S) can be interpreted as the area under the curve of the product of $G(S_1)$ and S raised to the power of S_1 over the interval from L_1 to L_2. This integral captures the cumulative effect or behaviour of the system or phenomenon as described by $G(S_1)$ across the range of S_1 values from L_1 to L_2. The function $G(S_1)$ likely encapsulates some crucial information or parameters related to the system being studied, and the exponential term S^{S_1} signifies a non-linear relationship between the variables S and S_1. In summary, F(S) provides a comprehensive representation of the integrated influence of $G(S_1)$ and the exponential relationship between S and S_1 over the specified interval, which could be instrumental in understanding the dynamics or behaviour of the studied system or phenomenon.

To write the given integral involving the Meijer G-function in proper scientific notation, we can express it as follows:

Here: $G(a,b|at)$ represents the Meijer G-function with parameters a and b evaluated at at. denotes the definite integral with the lower limit of integration being 0 and the upper limit being L. e^{tx} is the exponential function with x as a constant.

To represent the integral of the given function involving the Meijer G-function in proper scientific notation, we can write it as:

$\int L$: Denotes the integral over the interval [0, L]. e^{tx}: Exponential function with the variable x. $G(a,b|at)$: Represents the Meijer G-function with parameters a and b, evaluated at at. This notation captures the integral of the given function in a clear and concise manner suitable for scientific presentation.

III. Exact Statistical Derivation of the Fisher-Snedecor $\mathcal{F}$ Random Variables

In this section, we aim to derive exact statistical expressions of the PDF and CDF of the Signal-to-Noise Ratio (γ_l) at the $l-th$ branch of the N component identically distributed MRC receiver operating under Fisher-Snedecor $\mathcal{F}$ composite fading. We also develop asymptotic expressions for PDF and CDF in terms of simple algebraic functions. We denote the Meijer-G and Fox-H functions as [4], [5].

The PDF of the SNR(γ_l) at the $l-th$ branch of the MRC receiver operating under Fisher-Snedecor $\mathcal{F}$ composite fading channel can be expressed as

$$f_{\gamma_\ell}(\gamma) = \frac{m_\ell^{m_\ell}\left(m_{s_\ell}\bar{\gamma}_\ell\right)^{m_{s_\ell}}}{B(m_\ell, m_{s_\ell})} \frac{\gamma^{m_\ell - 1}}{\left(m_\ell \gamma + m_{s_\ell}\bar{\gamma}_\ell\right)^{m_\ell + m_{s_\ell}}}, \tag{11}$$

In this context, the parameters m_ℓ and $m_{s\ell}$ represent fading severity and shadowing, respectively. The average signal-to-noise ratio (SNR) is denoted by $\bar{\gamma}_l = E[\gamma_l]$, where $E[\cdot]$ signifies the expected value. The Fisher-Snedecor $\mathcal{F}$ fading model demonstrates versatility by including specific distributions as subsets: the Nakagami-m distribution when ($m_{s\ell} \to \infty$) and ($m_\ell = m$), the Rayleigh distribution when ($m_{s\ell} \to \infty$) and ($m_\ell = 1$), and the one-sided Gaussian distribution when ($m_{s\ell} \to \infty$) and ($m_\ell = 0.5$). The Moment-Generating function (MGF) of the Fisher-Snedecor $\mathcal{F}$ distribution can be analytically expressed in terms of a Meijer-G function as follows,

$$\mathcal{M}(s) = \frac{1}{\Gamma(m_\ell)\Gamma(m_{s_\ell})} \mathrm{G}_{2,1}^{1,2}\left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell s}\middle| \begin{matrix} 1-m_{s_\ell}, 1 \\ m_\ell \end{matrix}\right]. \tag{12}$$

We use 11 to present the analytical expression of PDF of an identically distributed N component system in the following theorem:

Theorem 1. *The PDF of SNR for the N component identically distributed Fisher-Snedecor $\mathcal{F}$ composite fading is*

$$f_{\mathcal{M}}(\gamma) = \tag{13}$$
$$\frac{\hat{K}}{\gamma}\left[\mathrm{G}_{2,1}^{1,2}\left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell}\middle| \begin{matrix} 1-m_{s_\ell}, 1 \\ m_\ell \end{matrix}\right]\right]^{N-1} \mathrm{H}_{2,2}^{1,2}\left[\frac{\gamma^N m_\ell}{m_{s_\ell}\bar{\gamma}_\ell}\middle| \begin{matrix} (1-m_{s_\ell},1),(1,1) \\ (m_\ell,1),(1,N) \end{matrix}\right]$$

Proof: See Appendix A. ∎

It can be seen that the PDF of (13) contains the product of an univariate Meijer-G function and Fox-H function as compared to cite other paper with n component work. It should be noted that previous studies have used the Meijer-G function, or its equivalent mathematical functions to directly express the PDF. However, for an N component, identically distributed system the representations of the PDF are in the form of a product of Meijer-G equivalent functions without removing the infinite series or, Guass' Hypergeometric Function for performance analysis. recently Fox's H functions have found myriad applications in wireless communication research, primarily in performance analysis of intricate fading distributions. Furthermore, the asymptotic expansion of the Fox's H-function may have better characteristics providing more accurate approximation over a wide range of parameters:

Proposition 1. *An asymptotic expression for PDF of Fisher-Snedecor $\mathcal{F}$ distribution is given by,*

Proof: The existing literature derives the asymptotic expression by computing the residue of the dominant pole of a Fox's H-function. ∎

In the following Lemma, we derive the CDF of the Fisher-Snedecor $\mathcal{F}$ distribution,

Lemma 1. *the CDF of the Fisher-Snedecor $\mathcal{F}$ distribution is given by,*

$$F_{\mathcal{M}}(\gamma) = \frac{\hat{K}}{N(2\pi i)}\left[\mathrm{G}_{2,1}^{1,2}\left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell}\middle| \begin{matrix} 1-m_{s_\ell}, 1 \\ m_\ell \end{matrix}\right]\right]^{N-1} \tag{14}$$
$$\int_{\mathcal{L}_{\rangle}} \frac{\Gamma(m_\ell - s_1)\Gamma(m_{s\ell}+s_1)\Gamma(s_1)}{\Gamma(s_1 N)}\left(\frac{\gamma^N m_\ell}{m_{s\ell}\bar{\gamma}_l}\right)^{s_1}\frac{\Gamma(s_1)}{\Gamma(s_1+1)}ds_1$$

applying the definition of the fox's H function the analytical expression of the CDF of the considered system can be written as,

$$F_{\mathcal{M}}(\gamma) = \frac{\hat{K}}{N}\left[\mathrm{G}_{2,1}^{1,2}\left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell}\middle| \begin{matrix} 1-m_{s_\ell}, 1 \\ m_\ell \end{matrix}\right]\right]^{N-1} \tag{15}$$
$$\times \mathrm{H}_{3,3}^{1,3}\left[\frac{\gamma^N m_\ell}{m_{s_\ell}\bar{\gamma}_\ell}\middle| \begin{matrix} (1-m_{s_\ell},1),(1,1),(1,1) \\ (m_\ell,1),(1,N),(0,1) \end{matrix}\right]$$

Proof: See Appendix A. ∎

Consider that combining the statistical representation of the Probability Density Function (PDF) and Cumulative Distribution Function (CDF) using a single Fox's H-function can enhance the ease of analyzing performance. In the subsequent section, we employ the statistical outcomes derived from Theorem 1 and Lemma 1 to assess the performance of a wireless link operating under Fisher-Snedecor $\mathcal{F}$ distribution.

IV. Performance Analysis of a Communication Link over Fisher-Snedecor $\mathcal{F}$ Distribution

Consider a source that transmits the information to the destination using a single antenna. We define the signal-to-noise ratio (SNR) at the receiver as $\gamma_\ell = \bar{\gamma}_\ell |R|^2$ where $\bar{\gamma}_\ell$ is the average SNR. We require the PDF and CDF of SNR(γ_ℓ) for statistical performance analysis. The PDF and CDF of the SNR are given by $f_{\gamma_\ell}(\gamma)$ and $F_{\gamma_\ell}(\gamma)$ in equations (13) and (15) respectively. In the following subsections, we use the derived statistical results to present the exact analysis of the outage probability and average BER of a wireless link subjected to Fisher-Snedecor $\mathcal{F}$ fading.

A. Outage Probability

The outage probability is defined as the probability of instantaneous SNR being less than a threshold SNR value γ_{th} i.e., $P_{out} = P_r(\gamma < \gamma_{th}) = F_{\gamma_\ell}(\gamma_{th})$. Using (15) in the above relation we can write,

$$F_{\mathscr{M}}(\gamma) = \frac{\hat{K}}{N}\left[\mathrm{G}_{2,1}^{1,2}\left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell}\middle|\begin{matrix}1-m_{s_\ell},1\\ m_\ell\end{matrix}\right]\right]^{N-1} \times \mathrm{H}_{3,3}^{1,3}\left[\frac{\gamma_{th}^N m_\ell}{m_{s_\ell}\bar{\gamma}_\ell}\middle|\begin{matrix}(1-m_{s_\ell},1),(1,1),(1,1)\\ (m_\ell,1),(1,N),(0,1)\end{matrix}\right] \tag{16}$$

B. Average BER

Average BER for binary modulations in a multi-component system can be expressed using the CDF as given in equation (15) :

$$\overline{P_e} = \frac{q_{\mathrm{m}}^{p_{\mathrm{m}}}}{2\Gamma(p_m)}\int_0^\infty \gamma^{p_m-1}e^{-q_m\gamma}F_M(\gamma)d\gamma \tag{17}$$

here p_m and q_m determine the modulation scheme used. BPSK can be represented $p_m = 0.5, q_m = 1$, while DPSK and BFSK are characterized by $p_m = 1, q_m = 1$ and $p_m = 0.5, q_m = 0.5$ respectively. Substituting the CDF (15) in (36) and using the integral form of Fox's H function we get,

$$\overline{P_e} = \frac{q_{\mathrm{m}}^{p_{\mathrm{m}}}}{2\Gamma(p_m)}\frac{\hat{K}}{N(2\pi i)}\left[\mathrm{G}_{2,1}^{1,2}\left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell}\middle|\begin{matrix}1-m_{s_\ell},1\\ m_\ell\end{matrix}\right]\right]^{N-1} \int_{\mathscr{L}_\rangle}\frac{\Gamma(m_\ell-s_1)\Gamma(m_{s\ell}+s_1)\Gamma(s_1)\Gamma(s_1)}{\Gamma(s_1N)\Gamma(1+s_1)}\left(\frac{m_\ell}{m_{s\ell}\bar{\gamma}_l}\right)^{s_1} q_m^{-(s_1N+p_m)}\Gamma(s_1N+p_m)ds_1 \tag{18}$$

Applying the definition of the Fox-H function we get the expression for the average BER of the considered system as,

$$\overline{P_e} = \frac{\hat{K}q_{\mathrm{m}}^{p_{\mathrm{m}}}}{2N\Gamma(p_m)}\left[\mathrm{G}_{2,1}^{1,2}\left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell}\middle|\begin{matrix}1-m_{s_\ell},1\\ m_\ell\end{matrix}\right]\right]^{N-1} \times \mathrm{H}_{4,3}^{1,4}\left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell q_m^{(N+p_m)}}\middle|\begin{matrix}(1-m_{s_\ell},1),(1,1),(1,1),(1-p_m,N)\\ (m_\ell,1),(1,N),(0,1)\end{matrix}\right] \tag{19}$$

C. Ergodic Capacity of the Fisher-Snedecor $\mathscr{F}$ distributed system

The ergodic capacity of the considered system can be computed using,

$$\mathscr{C} = \int_0^\infty log_2(1+\gamma)f_{\mathscr{M}}(\gamma)d\gamma$$

applying the expression of the PDF of SNR as given in (**??**) and simplifying the inner-integrals we obtain the analytical expression for the channel capacity of the considered system as (41) we get

$$\mathscr{C} = \frac{\hat{K}}{ln(2)}\left[\mathrm{G}_{2,1}^{1,2}\left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell}\middle|\begin{matrix}1-m_{s_\ell},1\\ m_\ell\end{matrix}\right]\right]^{N-1} \times \mathrm{H}_{3,3}^{3,4}\left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell}\middle|\begin{matrix}(1-m_{s_\ell},1),(1,1),(0,1)\\ (m_\ell,1),(0,1),(0,1)(1,N)\end{matrix}\right] \tag{20}$$

V. Appendix A

The PDF of the SNR(γ_l) at the $l-th$ branch of the MRC receiver operating under Fisher-Snedecor $\mathscr{F}$ composite fading channel can be expressed as

$$f_{\gamma_\ell}(\gamma) = \frac{m_\ell^{m_\ell}(m_{s_\ell}\bar{\gamma}_\ell)^{m_{s_\ell}}}{B(m_\ell, m_{s_\ell})}\frac{\gamma^{m_\ell-1}}{(m_\ell\gamma + m_{s_\ell}\bar{\gamma}_\ell)^{m_\ell+m_{s_\ell}}}, \tag{21}$$

In this context, m_l and m_{sl} represent the parameters for fading severity and shadowing, respectively. The average signal-to-noise ratio (SNR) is denoted by $\bar{\gamma}_l = E[\gamma_l]$, where $E[\cdot]$ signifies the expected value, and $B(\cdot,\cdot)$ refers to the beta function as per reference. The versatility of the Fisher-Snedecor $\mathscr{F}$ fading model is highlighted by its inclusion of certain specific distributions as its subsets: the Nakagami-m distribution when ($m_{s\ell} \to \infty$) and ($m_\ell = m$), the Rayleigh distribution when ($m_{s\ell} \to \infty$) and ($m_\ell = 1$), and the one-sided Gaussian distribution when ($m_{s\ell} \to \infty$) and ($m_\ell = 0.5$)

The MGF of Fisher-Snedecor $\mathscr{F}$ distribution can be obtained from

$$\mathscr{M}(s) = \int_0^\infty e^{-s\gamma}f_\gamma(\gamma)d\gamma$$

the analytical expression of the MGF of the Fisher-Snedecor $\mathscr{F}$ distribution expressed in terms of a Meijer-G function can be written as,

$$\mathscr{M}(s) = \frac{1}{\Gamma(m_\ell)\Gamma(m_{s_\ell})}\mathrm{G}_{2,1}^{1,2}\left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell s}\middle|\begin{matrix}1-m_{s_\ell},1\\ m_\ell\end{matrix}\right]. \tag{22}$$

For an N-component system, the MGF can be computed as follows:

$$F_{\mathscr{M}}(s) = \prod_{i=1}^N \mathscr{M}_i(s)$$

If all components of the N-component system are identical we obtain

$$F_{\mathscr{M}}(s) = [\mathscr{M}_i(s)]^N$$

The composite PDF can be written as

$$f_{\mathscr{M}}(\gamma) = \mathscr{L}^{-1}[\mathscr{M}(s)]^N$$
$$f_{\mathscr{M}}(\gamma) = \frac{1}{2\pi i}\int_{\mathscr{L}} e^{s\gamma}[\mathscr{M}(s)]^N ds$$

substituting (22) in the above equation we get,

$$f_{\mathscr{M}}(\gamma) = \frac{1}{2\pi i}\int_{\mathscr{L}} e^{s\gamma}\left(\frac{1}{\Gamma(m_\ell)\Gamma(m_{s_\ell})}\mathrm{G}_{2,1}^{1,2}\left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell s}\middle|\begin{matrix}1-m_{s_\ell},1\\ m_\ell\end{matrix}\right]\right)^N ds \tag{23}$$

The expansion of the Meijer-G representation of the MGF can be written as

$$\frac{1}{\Gamma(m_\ell)\Gamma(m_{s_\ell})}\mathrm{G}_{2,1}^{1,2}\left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell s}\middle|\begin{matrix}1-m_{s_\ell},1\\ m_\ell\end{matrix}\right] = \frac{\hat{K}}{2\pi i}\int_{\mathscr{L}}\frac{\Gamma(m_\ell-s_1)\Gamma(m_{s\ell}+s_1)\Gamma(s_1)}{1}\left(\frac{m_\ell}{m_{s\ell}\bar{\gamma}_\ell s}\right)^{s_1}ds_1 \tag{24}$$

where $\hat{K} = \frac{1}{\Gamma(m_\ell)\Gamma(m_{s_\ell})}$. Using equation (24) in equation (23) we get,

$$f_{\mathscr{M}}(\gamma) = \frac{\hat{K}}{2\pi i}\int_{\mathscr{L}} e^{s\gamma}\left[\frac{1}{2\pi i}\int_{\mathscr{L}_\rangle}\frac{\Gamma(m_\ell-s_1)\Gamma(m_{s\ell}+s_1)\Gamma(s_1)}{1}\left(\frac{m_\ell}{m_{s\ell}\bar{\gamma}_\ell s}\right)^{s_1}ds_1\right]^N ds \tag{25}$$

The above equation can also be written as

$$f_{\mathscr{M}}(\gamma)=\frac{\hat{K}}{2\pi i}\int_{\mathscr{L}}e^{s\gamma}\Bigg[\frac{1}{(2\pi i)^N}\int_{\mathscr{L}_i}\cdots\int_{\mathscr{L}_i}\left(\frac{m_\ell}{m_{s\ell}\bar{\gamma}_l}\right)^{s_1N}$$
$$\left(\frac{\Gamma(m_\ell-s_1)\Gamma(m_{s\ell}+s_1)\Gamma(s_1)}{1}\right)^N s^{-s_1N}ds_1\ldots ds_1\Bigg]ds$$

Re-arranging the terms and grouping the terms to simplify the integral we get

$$f_{\mathscr{M}}(\gamma)=\frac{\hat{K}}{(2\pi i)^{N+1}}\int_{\mathscr{L}_i}\cdots\int_{\mathscr{L}_i}\int_{\mathscr{L}}e^{s\gamma}\left(\frac{m_\ell}{m_{s\ell}\bar{\gamma}_l}\right)^{s_1(N-1)} \tag{26}$$
$$\left(\frac{\Gamma(m_\ell-s_1)\Gamma(m_{s\ell}+s_1)\Gamma(s_1)}{1}\right)^{N-1}s^{-s_1N}$$
$$\left(\frac{\Gamma(m_\ell-s_1)\Gamma(m_{s\ell}+s_1)\Gamma(s_1)}{1}\right)\left(\frac{m_\ell}{m_{s\ell}\bar{\gamma}_l}\right)dsds_1\ldots ds_1$$

The above equation can be further simplified to

$$f_{\mathscr{M}}(\gamma)=\frac{\hat{K}}{(2\pi i)^{N+1}}\int_{\mathscr{L}_i}\cdots\Bigg[\int_{\mathscr{L}_i}\left(\frac{m_\ell}{m_{s\ell}\bar{\gamma}_l}\right)^{s_1} \tag{27}$$
$$\left(\frac{\Gamma(m_\ell-s_1)\Gamma(m_{s\ell}+s_1)\Gamma(s_1)}{1}\right)\int_{\mathscr{L}}e^{s\gamma}s^{-s_1N}dsds_1\Bigg]$$
$$\left(\frac{\Gamma(m_\ell-s_1)\Gamma(m_{s\ell}+s_1)\Gamma(s_1)}{1}\right)^{N-1}\left(\frac{m_\ell}{m_{s\ell}\bar{\gamma}_l}\right)^{s_1(N-1)}\ldots ds_1$$

for simplicity, we can write,

$$I_i=\int_{\mathscr{L}_\rangle}\left(\frac{m_\ell}{m_{s\ell}\bar{\gamma}_l}\right)^{s_1}\left(\frac{\Gamma(m_\ell-s_1)\Gamma(m_{s\ell}+s_1)\Gamma(s_1)}{1}\right)I_1ds_1$$

Here, the inner integral I_1 is given by,

$$I_1=\int_{\mathscr{L}}e^{s\gamma}s^{-s_1N}ds$$

substituting $s\gamma=-t$, we get

$$=\int_{\mathscr{L}}e^{-t}(\frac{-t}{\gamma})^{-s_1N}\frac{-1}{\gamma}dt$$

re-arranging the terms we obtain,

$$=\frac{-1}{\gamma}\times\gamma^{s_1N}\int_{\mathscr{L}}e^{-t}(-t)^{-s_1N}dt$$

from eq:8.315 from [4] we get

$$I_1=\frac{2\pi i}{\gamma}\times\gamma^{s_1N}\times\frac{1}{\Gamma(s_1N)}$$

Substituting the above result back in equation (26) we get

$$f_{\mathscr{M}}(\gamma)=\frac{\hat{K}}{\gamma(2\pi i)^N}\int_{\mathscr{L}_\rangle}\cdots\Bigg[\int_{\mathscr{L}_\rangle}\left(\frac{\gamma^N m_\ell}{m_{s\ell}\bar{\gamma}_l}\right)^{s_1} \tag{28}$$
$$\left(\frac{\Gamma(m_\ell-s_1)\Gamma(m_{s\ell}+s_1)\Gamma(s_1)}{\Gamma(s_1N)}\right)ds_1\Bigg]$$
$$\left(\frac{\Gamma(m_\ell-s_1)\Gamma(m_{s\ell}+s_1)\Gamma(s_1)}{1}\right)^{N-1}\left(\frac{m_\ell}{m_{s\ell}\bar{\gamma}_l}\right)^{s_1(N-1)}\ldots ds_1$$

Using the definition of the Meijer-G and Fox-H function we obtain the composite PDF as follows:

$$f_{\mathscr{M}}(\gamma)= \tag{29}$$
$$\frac{\hat{K}}{\gamma}\left[\mathrm{G}_{2,1}^{1,2}\left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell}\middle|\begin{matrix}1-m_{s_\ell},1\\ m_\ell\end{matrix}\right]\right]^{N-1}$$
$$\times\mathrm{H}_{2,2}^{1,2}\left[\frac{\gamma^N m_\ell}{m_{s_\ell}\bar{\gamma}_\ell}\middle|\begin{matrix}(1-m_{s_\ell},1),(1,1)\\ (m_\ell,1),(1,N)\end{matrix}\right]$$

The CDF can be computed as $\int_0^\gamma f_{\mathscr{M}}(\gamma)d\gamma$ where $f_{\mathscr{M}}(\gamma)$ is the PDF of the multi-component system,

$$F_{\mathscr{M}}(\gamma)= \tag{30}$$
$$\int_0^\gamma\frac{\hat{K}}{\gamma}\left[\mathrm{G}_{2,1}^{1,2}\left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell}\middle|\begin{matrix}1-m_{s_\ell},1\\ m_\ell\end{matrix}\right]\right]^{N-1}$$
$$\times\mathrm{H}_{2,2}^{1,2}\left[\frac{\gamma^N m_\ell}{m_{s_\ell}\bar{\gamma}_\ell}\middle|\begin{matrix}(1-m_{s_\ell},1),(1,1)\\ (m_\ell,1),(1,N)\end{matrix}\right]d\gamma$$

Using the contour integral representation of the PDF as given in equation (29) we can write the above equation (31) as

$$F_{\mathscr{M}}(\gamma)=\int_0^\gamma\frac{\hat{K}}{(2\pi i)\gamma}\left[\mathrm{G}_{2,1}^{1,2}\left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell}\middle|\begin{matrix}1-m_{s_\ell},1\\ m_\ell\end{matrix}\right]\right]^{N-1} \tag{31}$$
$$\int_{\mathscr{L}_\rangle}\frac{\Gamma(m_\ell-s_1)\Gamma(m_{s\ell}+s_1)\Gamma(s_1)}{\Gamma(s_1N)}\left(\frac{\gamma^N m_\ell}{m_{s\ell}\bar{\gamma}_l}\right)^{s_1}ds_1d\gamma$$

Re-arranging and simplifying the above equation we get

$$F_{\mathscr{M}}(\gamma)=\frac{\hat{K}}{(2\pi i)}\left[\mathrm{G}_{2,1}^{1,2}\left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell}\middle|\begin{matrix}1-m_{s_\ell},1\\ m_\ell\end{matrix}\right]\right]^{N-1} \tag{32}$$
$$\int_{\mathscr{L}_\rangle}\frac{\Gamma(m_\ell-s_1)\Gamma(m_{s\ell}+s_1)\Gamma(s_1)}{\Gamma(s_1N)}\left(\frac{m_\ell}{m_{s\ell}\bar{\gamma}_l}\right)^{s_1}I_2ds_1$$

where the inner integral I_2 is given by

$$I_2=\int_0^\gamma(\gamma^{s_1N-1})d\gamma$$
$$=\frac{\gamma^{s_1N}}{s_1N}$$

Substituting the result of I_2 back in equation 32 we get

$$F_{\mathscr{M}}(\gamma)=\frac{\hat{K}}{(2\pi i)}\left[\mathrm{G}_{2,1}^{1,2}\left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell}\middle|\begin{matrix}1-m_{s_\ell},1\\ m_\ell\end{matrix}\right]\right]^{N-1} \tag{33}$$
$$\int_{\mathscr{L}_\rangle}\frac{\Gamma(m_\ell-s_1)\Gamma(m_{s\ell}+s_1)\Gamma(s_1)}{\Gamma(s_1N)}\left(\frac{\gamma^N m_\ell}{m_{s\ell}\bar{\gamma}_l}\right)^{s_1}\frac{1}{s_1N}ds_1$$

Equation (33) can also be written as

$$F_{\mathscr{M}}(\gamma)=\frac{\hat{K}}{N(2\pi i)}\left[\mathrm{G}_{2,1}^{1,2}\left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell}\middle|\begin{matrix}1-m_{s_\ell},1\\ m_\ell\end{matrix}\right]\right]^{N-1} \tag{34}$$
$$\int_{\mathscr{L}_\rangle}\frac{\Gamma(m_\ell-s_1)\Gamma(m_{s\ell}+s_1)\Gamma(s_1)}{\Gamma(s_1N)}\left(\frac{\gamma^N m_\ell}{m_{s\ell}\bar{\gamma}_l}\right)^{s_1}\frac{\Gamma(s_1)}{\Gamma(s_1+1)}ds_1$$

Applying the definition of the Fox-H function the CDF of the N-component system can be written as,

$$F_{\mathscr{M}}(\gamma)=\frac{\hat{K}}{N}\left[\mathrm{G}_{2,1}^{1,2}\left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell}\middle|\begin{matrix}1-m_{s_\ell},1\\ m_\ell\end{matrix}\right]\right]^{N-1} \tag{35}$$
$$\times\mathrm{H}_{3,3}^{1,3}\left[\frac{\gamma^N m_\ell}{m_{s_\ell}\bar{\gamma}_\ell}\middle|\begin{matrix}(1-m_{s_\ell},1),(1,1),(1,1)\\ (m_\ell,1),(1,N),(0,1)\end{matrix}\right]$$

Average BER for binary modulations in a multi-component system can be expressed using the CDF as given in equation (33) :

$$\overline{P_e}=\frac{q_{\mathrm{m}}^{p_{\mathrm{m}}}}{2\Gamma(p_m)}\int_0^\infty\gamma^{p_m-1}e^{-q_m\gamma}F_M(\gamma)d\gamma \tag{36}$$

here p_m and q_m determine the modulation scheme used. BPSK can be represented $p_m=0.5, q_m=1$, while DPSK and BFSK are characterized by $p_m = 1, q_m = 1$ and $p_m = 0.5, q_m = 0.5$ respectively.

$$\overline{P_e} = \frac{q_m^{p_m}}{2\Gamma(p_m)} \int_0^\infty \gamma^{p_m-1} e^{-q_m\gamma} \frac{\hat{K}}{N(2\pi i)^N} \tag{37}$$
$$\times \left[G_{2,1}^{1,2} \left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell} \middle| \begin{matrix} 1-m_{s_\ell}, 1 \\ m_\ell \end{matrix} \right] \right]^{N-1}$$
$$\int_{\mathscr{L}_\rangle} \frac{\Gamma(m_\ell - s_1)\Gamma(m_{s\ell}+s_1)\Gamma(s_1)\Gamma(s_1)}{\Gamma(s_1 N)\Gamma(1+s_1)} \left(\frac{\gamma^N m_\ell}{m_{s\ell}\bar{\gamma}_l} \right)^{s_1} ds_1 d\gamma$$

re-arranging the terms of the above equation we get,

$$\overline{P_e} = \frac{q_m^{p_m}}{2\Gamma(p_m)} \frac{\hat{K}}{N(2\pi i)} \left[G_{2,1}^{1,2} \left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell} \middle| \begin{matrix} 1-m_{s_\ell}, 1 \\ m_\ell \end{matrix} \right] \right]^{N-1} \tag{38}$$
$$\int_{\mathscr{L}_\rangle} \frac{\Gamma(m_\ell - s_1)\Gamma(m_{s\ell}+s_1)\Gamma(s_1)\Gamma(s_1)}{\Gamma(s_1 N)\Gamma(1+s_1)} \left(\frac{m_\ell}{m_{s\ell}\bar{\gamma}_l} \right)^{s_1} I_3 ds_1$$

where the inner integral I_3 can be written as per eq:3.381.4 from the Table of Integrals as

$$I_3 = \int_0^\infty \gamma^{s_1 N + p_m - 1} e^{-q_m\gamma} d\gamma \;=\; q_m^{-(s_1 N + p_m)} \Gamma(s_1 N + p_m)$$

Substituting the above result for I_3 in equation (38) we get,

$$\overline{P_e} = \frac{q_m^{p_m}}{2\Gamma(p_m)} \frac{\hat{K}}{N(2\pi i)} \left[G_{2,1}^{1,2} \left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell} \middle| \begin{matrix} 1-m_{s_\ell}, 1 \\ m_\ell \end{matrix} \right] \right]^{N-1} \tag{39}$$
$$\int_{\mathscr{L}_\rangle} \frac{\Gamma(m_\ell - s_1)\Gamma(m_{s\ell}+s_1)\Gamma(s_1)\Gamma(s_1)}{\Gamma(s_1 N)\Gamma(1+s_1)} \left(\frac{m_\ell}{m_{s\ell}\bar{\gamma}_l} \right)^{s_1}$$
$$q_m^{-(s_1 N + p_m)} \Gamma(s_1 N + p_m) ds_1$$

Applying the definition of the Fox-H function,

$$\overline{P_e} = \frac{\hat{K} q_m^{p_m}}{2N\Gamma(p_m)} \left[G_{2,1}^{1,2} \left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell} \middle| \begin{matrix} 1-m_{s_\ell}, 1 \\ m_\ell \end{matrix} \right] \right]^{N-1} \tag{40}$$
$$\times H_{4,3}^{1,4} \left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell q_m^{(N+p_m)}} \middle| \begin{matrix} (1-m_{s_\ell}, 1), (1,1), (1,1), (1-p_m, N) \\ (m_\ell, 1), (1, N), (0, 1) \end{matrix} \right]$$

The Channel capacity can be computed using

$$\mathscr{C} = \int_0^\infty log_2(1+\gamma) f_{\mathscr{M}}(\gamma) d\gamma$$

$$\mathscr{C} = \int_0^\infty log_2(1+\gamma) \frac{\hat{K}}{\gamma} \left[G_{2,1}^{1,2} \left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell} \middle| \begin{matrix} 1-m_{s_\ell}, 1 \\ m_\ell \end{matrix} \right] \right]^{N-1}$$
$$\times H_{2,2}^{1,2} \left[\frac{\gamma^N m_\ell}{m_{s_\ell}\bar{\gamma}_\ell} \middle| \begin{matrix} (1-m_{s_\ell}, 1), (1,1) \\ (m_\ell, 1), (1, N) \end{matrix} \right] d\gamma$$

re-writing the terms,

$$\mathscr{C} = \frac{\hat{K}}{(2\pi i)} \left[G_{2,1}^{1,2} \left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell} \middle| \begin{matrix} 1-m_{s_\ell}, 1 \\ m_\ell \end{matrix} \right] \right]^{N-1} \tag{41}$$
$$\int_{\mathscr{L}_\rangle} \frac{\Gamma(m_\ell - s_1)\Gamma(m_{s\ell}+s_1)\Gamma(s_1)}{\Gamma(s_1 N)} \left(\frac{m_\ell}{m_{s\ell}\bar{\gamma}_l} \right)^{s_1} I_4 ds_1$$

where,

$$I_4 = \int_0^\infty log_2(1+\gamma)\gamma^{s_1 N - 1} d\gamma$$
$$= \frac{1}{ln(2)} \int_0^\infty ln(1+\gamma)\gamma^{s_1 N - 1} d\gamma$$
$$= \frac{1}{ln(2)} \int_0^\infty \left[G_{2,2}^{1,2} \left[\gamma \middle| \begin{matrix} 1,1 \\ 1,0 \end{matrix} \right] \right] \gamma^{s_1 N - 1} d\gamma$$
$$I_4 = \frac{1}{ln(2)} \frac{\Gamma(1+s)\Gamma(-s)\Gamma(-s)}{\Gamma(1-s)}$$

using eq2.9 of, the H-function, Theory and Applications

Applying the result of I_4 back in equation 41 we get,

$$\mathscr{C} = \tag{42}$$
$$\frac{\hat{K}}{ln(2)} \left[G_{2,1}^{1,2} \left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell} \middle| \begin{matrix} 1-m_{s_\ell}, 1 \\ m_\ell \end{matrix} \right] \right]^{N-1}$$
$$\times H_{3,3}^{3,4} \left[\frac{m_\ell}{m_{s_\ell}\bar{\gamma}_\ell} \middle| \begin{matrix} (1-m_{s_\ell}, 1), (1,1), (0,1) \\ (m_\ell, 1), (0,1), (0,1)(1, N) \end{matrix} \right]$$

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