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Samuelson-Berkowitz and Modular Computation Algorithm

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Katowice 2024

Abstract

Gaussian elimination method is a fast and widely used algorithm for efficient computation of the determinant of square matrices. However, it involves division operation, which can result in coefficients that are not integers. In fact, it forces us to work in the rational domain $\mathbb{Q}$ instead of the integer domain $\mathbb{Z}$, we so much desired.

This project considers two algorithms, first, a division-free algorithm for obtaining the characteristics polynomial and the determinant of square matrices whose coefficients are in any unital commutative ring $\mathcal{R}$ which is not an integral domain.

The second method, a modular computation algorithm, which utilizes both division-free and division-based techniques to calculate the determinant of square matrices with integer coefficients. This algorithm is fast but can become very slow if too many primes are used, leading to an excessive number of unnecessary modular computations. We use fewer odd primes p_i to speed up the algorithm by defining a function that asks for the number of primes we would like to use in our computation.

Contents

1 Introduction

The study of determinant has a rich and long history because of its importance in linear algebra, geometry, computational mathematics etc.. Its background can be traced back to the work of Cardano, Leibnitz, Crammer, Vandermode, Binet, Cauchy, Jacobi, Gauss and many others. Given its importance in physical sciences and engineering, it is not surprising that a galaxy of great mathematicians investigated determinant from different point of views. Similarly, the concept of characteristic polynomials has a rich history spanning several centuries and various mathematical developments, namely, Viète, Newton, Cayley, Frobenius to mention but a few. The characteristics polynomial have applications in diverse areas such as quantum mechanics for understanding the possible outcomes of measurements and the evolution of quantum states over time and in data analysis to extract meaningful features or reduce the dimension of the data while preserving important structural information (see, [1], [2]).

The importance of determinants and characteristic polynomials reflects its utility in mathematics from theoretical developments to practical applications in computations that involves real world problems. The need for less costly computations, restricted domain of interest and fast computations in higher dimensions give rise to several methods in literature for computing the determinants and the characteristic polynomials of square matrices.

The most common algorithm for computing the determinant of square matrices is the Gaussian elimination method (GEM for short) and it requires $O(n^3)$ additions, multiplications and divisions (see, [3]). GEM is fast but requires division, so when applied to matrices with integers coefficient, it forces us to work in the field of rationals $\mathbb{Q}$, it is also affected by the growth sizes of coefficients because it uses division. Division-free methods avoid these problems, but their time complexity is higher than that of GEM. The Leibniz formula among methods for calculating determinants requires $O(n!)$ operations [4, page 127]. In fact, Leibniz formula approach shows that determinant can be obtained without divisions.

Moreover, avoiding divisions seems attractive when working over a commutative ring which is not a field. The notion of division free method birth the Samuelson–Berkowitz algorithm (SBA for short) which is used for computing the characteristics polynomial and the determinant of square matrices (constant term of the characteristics polynomial) whose entries belong to a unital commutative ring.

Modular reduction is an important tool for speeding up computations in computer arithmetic, symbolic computation, and elsewhere. The technique allows us to reduce a problem that involves large integer or polynomial coefficients to one or more similar problems that only involve small modular coefficients (see, [5]).

Modular computation algorithm (MCA for short) is a method that employs both division-free and non division-free algorithm in the the computation of the determinant of a square matrix whose coefficients are in $\mathbb{Z}$.

The project is organized as follows. Section 2 contains essential tools that are explored later. In particular, in Section 2.1 we recall some basic definitions from linear algebra, while in Section 2.2 we state some theorems from linear algebra that are needed for this project. The description, formulation with an example,

implementation and complexity of SBA is discussed in Section 3.1, Section 3.2, Section 3.3 and Section 3.4 of Section 3 respectively. We proceed to Section 4 where we describe the MCA in Section 4.1, while the formulation of MCA with an example, implementation of MCA and the description of the complexity of MCA is done in Section 4.2, Section 4.3 Section 4.4 respectively.

2 Background

This section collects the basic definitions and results needed to write this project.

2.1 Definition of Terms

We shall recall some basic definitions and facts from linear algebra.

An *integral domain* $\mathcal{D}$ is a commutative ring with unity but without a zero divisor.

A *Toeplitz matrix* M of size $n \times n$ is a matrix whose each entries depend on the difference $i - j$ and hence they are constant down all the diagonals.

$$M_{i,j} = \begin{pmatrix} a_0 & a_{-1} & \cdots & a_{-(n-1)} \\ a_1 & a_0 & \cdots & a_{-(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(n-1)} & a_{(n-2)} & \cdots & a_0 \end{pmatrix}$$

We say that a matrix is *lower triangular* if all the values above the main diagonal are zero

The *adjoint matrix* of a square matrix W of size $n \times n$ with coefficients in some ring $\mathcal{R}$ is the matrix

$$adj(W) = ((-1)^{i+j} \cdot detW_{ij})^T \text{ for } i, j \leq n$$

Let M be a matrix of size $n \times n$ with coefficients in some ring $\mathcal{R}$. The *determinant* of a square matrix W is defined by the formula

$$detW = \sum_{\sigma} sgn(\sigma) \cdot w_{1\sigma(1)} \cdots w_{n\sigma(n)}$$

where, $sgn(\sigma) = (-1)^{n-m}$ is the permutations over the set $\{1, 2, \ldots n\}$.

Let W be an $n \times n$ matrix with coefficients in some ring $\mathcal{R}$. A *principal sub-matrix* of a square matrix W_k is the matrix obtained by deleting any k rows and the corresponding k columns of the matrix W.

The $k \times k$ sub-matrix $W_{k,k}$ obtained from matrix W by deleting any $(n - k)$ columns of W and the corresponding $(n - k)$ rows of W is called *kth-order principal sub-matrix of* W. That is,

$$W_k = \left[\begin{array}{c|c} W_{k,k} & S_{k,n-k} \\ \hline T_{n-k,k} & W_{n-k,n-k} \end{array}\right]$$

where $T_{n-k,k}, S_{k,n-k}$ and $W_{n-k,n-k}$ are $(n-k) \times k$, $k \times (n-k)$ and $(n-k) \times (n-k)$ $(k = 0, 1, 2, \ldots, n)$ sub-matrices, respectively.

Let W be a square matrix of size $n \times n$ with coefficients in some ring $\mathcal{R}$ and I_n denotes the unit matrix of size n. The polynomial $P_W(x) \in \mathcal{R}[x]$ is called the *characteristic polynomial* of W.

$$P_W(x) = det(xI_n - W) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n$$

A polynomial $p(x) = \sum_{i=0}^{n} b_i x^i$ is called a *monic polynomial* of degree n if $b_n = 1$.

Let n be a positive integer, we say the integers a and b are congruent modulo n, and write $a \equiv b (mod\ n)$, if they have the same remainder on division by n.

2.2 Basic Theorems

In this part, several theorems about determinants and congruence relation are stated and proved. The first theorem is a useful tool for calculating the determinant of a matrix.

Theorem 2.1. *(Laplace expansion). Let $W = (w_{ij})$ be a square matrix of size $n \times n$ and W_{ij} a matrix obtained from W after deleting its i-th row and j-th column. Then*

$$detW = \begin{cases} \sum_{j=1}^{n}(-1)^{i+j}w_{ij}detW_{ij}, & \text{(j-th row expansion)} \\ \sum_{i=1}^{n}(-1)^{i+j}w_{ij}detW_{ij} & \text{(i-th column expansion)} \end{cases}$$

Proof. See [6] for a proof to this theorem. □

Colorally 2.2. *If W is a triangular matrix (either upper or lower-triangular), then det W is the product of elements on the main diagonal of W*

The following establish the fact that there is a relationship between the characteristics polynomial of a square matrix W and its determinant.

Remark 2.3. The constant term of the characteristic polynomial of a matrix W of size $n \times n$ equals $(-1)^n \cdot detW$.

Since, $P_W(x) = det(xI_n - W)$, then substituting $x = 0$ gives

$$P_W(0) = det(-W) = (-1)^n \cdot detW$$

The next result is a useful tool in the formulation of the Samuelson-Berkowitz algorithm, as it provides an alternative expression for the inverse of a matrix in terms of its adjugate and determinant.

Theorem 2.4. *Let W be a square matrix of size $n \times n$, then*

$$W \cdot adj(W) = adj(M) \cdot W = det(W) \cdot I_n \tag{2.1}$$

Proof. If W is a non-singular square matrix then there is nothing to prove. This is because, it is trivially true by definition that

$$W^{-1} = \frac{adj(W)}{det(W)}$$

However, if W is any matrix, then by Theorem 2.1 we have

$$detW = \sum_{i}^{n}(-1)^{i+j}w_{ij}detW_{ij},$$

$$(adjW)_{ij} = (-1)^{i+j}detW_{ji}$$

Multiplication of the coefficient of this adjoint by W gives

$$(W \cdot adj(M))_{kl} = \sum_{i}^{n}(-1)^{i+l}w_{ki}detW_{li}$$

if $k = l$ then

$$(W \cdot adj(W))_{kl} = (adj(W) \cdot W)_{kl} = \sum_{i}^{n} (-1)^{i+l} w_{li} det W_{li} = det(W) \cdot I$$

If $k \neq l$, then the result will be zero because it's the determinant of a matrix with two equal rows.

The proof is complete. □

Theorem 2.5. *(Cayley–Hamilton).Every square matrix satisfy its own characteristics equation. That is, if W is a square matrix and $P_W(x)$ is it's characteristic polynomial, then $P_W(W) = 0$*

Proof. Let W be a square matrix of size $n \times n$, then by (2.1) we have

$$(W - xI) \cdot adj(W - xI) = adj(W - xI) \cdot (W - xI) = det(W - xI) \cdot I,$$

and the adjoint has the form $adj(W - xI) = P_{ij}(x)$, where $p_{ij}(x)$ are polynomial of degree at most $n - 1$ for $1 \leq i, j \leq n$.
Therefore, the adjoint matrix can be written as

$$adj(W - xI) = B_0 + B_1 x + \ldots + B_{n-1} x^{n-1} \quad = (-1)^n (x^n + a_1 x^{n-1} + \ldots + a_n) \cdot I, \tag{2.2}$$

for some $n \times n$ matrices $B_0, B_1, \ldots, B_{n-1}$. Then it follows that

$$(W - xI)(B_0 + B_1 x + \ldots + B_{n-1} x^{n-1}) = (-1)^n (x^n + a_1 x^{n-1} + \ldots + a_n) \cdot I$$

Equating the coefficient of like powers of x we have

$$\begin{aligned} WB_0 &= (-1)^n a_n I \\ -B_0 + WB_1 a &= (-1)^n a_{n-1} I \\ &\vdots \\ -B_{n-2} + WB_{n-1} &= (-1)^n a_1 I \\ -B_{n-1} &= (-1)^n I \end{aligned} \tag{2.3}$$

Multiplying the left hand side of (2.3) by $I, W, W^2, \ldots, W^n$ respectively we obtain

$$\begin{aligned} WB_0 &= (-1)^n a_n I \\ -WB_0 + W^2 B_1 a &= (-1)^n a_{n-1} W \\ &\vdots \\ -W^{n-1} B_{n-2} + W^n B_{n-1} &= (-1)^n a_1 W^{n-1} \\ -W^n B_{n-1} &= (-1)^n W^n \end{aligned} \tag{2.4}$$

Clearly, the left hand side of (2.4) telescope, so adding all the equations gives

$$P_W(W) = (-1)^n (W^n + a_1 W^{n-1} + \ldots + W_n) = 0$$

□

An important consequence of the Cayley-Hamilton theorem is that any polynomial in a square matrix W of size $n \times n$ can be re-written as a polynomial whose degree is at most $n-1$.

The next theorem is the famous Chinese remainder theorem which is important in the computation of determinant through modular computation algorithm.

Theorem 2.6. *(Chinese remainder theorem) Let $t_1, t_2, \ldots, t_k$ be pairwise relatively prime positive integers. Denote $T = t_1 \cdot t_2 \cdots t_k$. Then, for every integers $a_1, a_2, \cdots, a_k$ the system of congruence's*

$$\begin{cases} a \equiv a_1 (mod\ t_1) \\ a \equiv a_2 (mod\ t_2) \\ \vdots \\ a \equiv a_k (mod\ t_k). \end{cases} \tag{2.5}$$

has a unique solution a in the set $\{0, 1, \ldots, T-1\}$.

Proof. See [7, Page 5] for a concise proof of Chinese remainder theorem. □

Lemma 2.7. *Let $m_1, m_2 \in \mathcal{R}$ be two relative prime moduli and $b_1, b_2 \in \mathcal{R}$ are two arbitrary elements. The following are equivalent*

1. *$b_1 \equiv b_2 (mod\ m_1)$ and $b_1 \equiv b_2 (mod\ m_2)$*
2. *$b_1 \equiv b_2 (mod\ m_1 \cdot m_2)$*

Proof. For the proof of this lemma (see, [4, page 82]). □

The next Lemma which express the adjoint of a matrix as a product of the coefficients and powers of a matrix, is a useful tool in the proof of Samuelson's formula

Lemma 2.8. *Let $P_W(x) = a_0 + a_1 x + \ldots + a_n x^n$ be the characteristic polynomial (monic) of a square matrix W of size $n \times n$. Then,*

$$adj(xI_n - W) = \sum_{j=1}^{n} \left(\sum_{i=1}^{j} a_{n-j+i} \cdot W^i \right) \cdot x^{n-j}$$

Proof. For a concise proof of this lemma (see, [4, page 137]) □

Theorem 2.9. *(Cauchy theorem). Let M and N be two matrices of the same size $n \times n$. Then*

$$det(M \cdot N) = det(M). \cdot det(N)$$

Proof. If at least one of the two matrices is singular, then the product MN is singular because

$$rank(MN) \leq min(rank(M), rank(N))$$

Therefore, $det(MN) = 0$ and at least one of $det(M) = 0$ or $det(N) = 0$ is zero, so that $det(M) \cdot det(N) = 0$

Thus, the theorem holds if at least one of the two matrices is singular.

Without any loss of generality, suppose neither M nor N is singular, then we can write them as products of elementary matrices as follows.
$M = U_1, U_2 \cdots U_n$ and $N = V_1, V_2 \cdots V_n$ where, $U_1, U_2 \cdots U_n$ and $V_1, V_2 \cdots V_n$ are elementary matrices. We recall that the determinant of a product of elementary matrices is equal to the products of their determinants. Therefore, we obtain

$$\begin{aligned} det(MN) &= det(U_1 \cdot U_2 \cdots U_n \cdot V_1 \cdot V_2 \cdots V_n) \\ &= det(U_1) \cdot det(U_2) \cdots det(U_n) \cdot det(V_1) \cdot det(V_2) \cdots det(V_n) \\ &= det(U_1 \cdot U_2 \cdots U_n) \cdot det(V_1 \cdot V_2 \cdots V_n)) \\ &= det(M) \cdot det(N) \end{aligned}$$

This completes the proof.

□

3 Samuelson-Berkowitz Algorithm (SBA)

In this section, we describe and formulate the Samuelson-Berkowitz algorithm for finding the characteristics polynomial and the determinant of any square matrix whose coefficients are in a unital commutative ring $\mathcal{R}$ which is not a field.

3.1 Description of Samuelson-Berkowitz Algorithm

The Samuelson-Berkowitz algorithm is an efficient method for computing the characteristic polynomial and the determinant of a square matrix. It is particularly useful in our case because the Samuelson-Berkowitz algorithm is well behaved when the entries of the matrix belong to a unital commutative ring $\mathcal{R}$ without zero divisors. Given any $n \times n$ matrix with coefficients in $\mathcal{R}$, it recursively partitions the matrix into principal sub-matrices until it reaches the 1×1 sub-matrix. It then assembles the coefficients of the characteristic polynomial by taking successively larger vector-matrix products.

The main idea behind Berkowitz's algorithm is Samuelson's formula, which relates the characteristics polynomial of a square matrix with the characteristics polynomial of its principal sub-matrix. Therefore, the coefficients of the characteristics polynomial of a square matrix W of size $n \times n$ are computed in terms of the coefficients of the characteristics polynomial of its sub-matrix.

3.2 Formulation of Samuelson-Berkowitz Algorithm

In this part of our project, we shall provide a comprehensive explanation for the formulation of Samuelson-Berkowitz algorithm (adapted from, [4, pages 135-140], [8, pages 46-48]).

Theorem 3.1. *(Samuelson Identity). Let W be a square matrix of size $n \times n$ and assume that the characteristics polynomial of sub-matrix k of W is $P_k = a_0 + a_1 x + \ldots + a_k x^k$, with $a_k = 1$, then*

$$P_{k+1} = (x - w_{k+1,k+1})P_k - \sum_{j=1}^{k}(\sum_{i=1}^{j} a_{k-j+1} T_k . W_k^{i-1}) \cdot x^{k-j} \tag{3.6}$$

Proof. By definition, $P_{k+1} = det(xI_{k+1} - W_{k+1})$, then using the proposed subdivision of W_{k+1} we write

$$P_{k+1} = det\left(\begin{pmatrix} xI_k & 0 \\ 0 & x \end{pmatrix} - \begin{pmatrix} W_k & S_k \\ T_k & w_{k+1,k+1} \end{pmatrix} \right)$$

Expanding the determinants along the last row using Laplace formula in Theorem 2.1 yield

$$= \sum_{j}^{k} (-1)^{j+k+1} \cdot (-w_{k+1,j} \cdot detD_j + (x - w_{k+1,k+1} \cdot det(xI_k - W_k)$$

Where, D_j is a $k \times k$ matrix obtained from $(xI_k - W_k|S_k)$ by dropping the j-th column. Expanding each determinant along the last column, again using Laplace expansion in Theorem 2.1 yield

$$= \sum_{j=1}^{k} \left((-1)^{j+k+1} \cdot (-w_{k+1,j}) \cdot \sum_{i=1}^{k} (-1)^{i+k} \cdot detW_{ij} \right) + (x - w_{k+1,k+1}) \cdot P_k$$

Next, the matrix W_{ij} is nothing else but $(xI_k - W_k)$ with the i-th row and j-th column deleted. Rearranging the terms we obtain

$$= (-1) \sum_{j=1}^{k} \left(w_{k+1,j} \cdot \sum_{i=1}^{k} (-1)^{i+j} \cdot detW_{ij} \cdot w_{i,k+1} \right) + (x - w_{k+1,k+1}) \cdot P_k$$

Observe that, $(-1)^{i+j} W_{ij}$ is the entry of $adj(xI_k - W_k)$ located in the j-th row and i-th column. The double sum can now be interpreted as a matrix multiplication so that we have

$$= (x - w_{k+1,k+1}) \cdot P_k - T_k \cdot adj(xI_k - W_k) \cdot S_k$$

It then follows by Lemma 2.8 that,

$$= (x - w_{k+1,k+1}) \cdot P_k - T_k \cdot \sum_{j=1}^{k} \left(\sum_{i=1}^{j} a_{k-j+1} W_k^{i-1} \right) \cdot x^{k-j} \cdot S_k$$

This completes the proof. □

Example 3.2. Compute the determinant and characteristics polynomial of the matrix

$$W = \begin{pmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{pmatrix}$$

using Samuelson's identity

The first principal minor W_1 consists of a single entry of W, that is $W_1 = 2$. The characteristic polynomial in this case is $p_1 = x - 2$.

We subdivide the second principal minor to obtain

$$\left[\begin{array}{c|c} 2 & 1 \\ \hline 5 & -7 \end{array} \right]$$

From which, $T_1 = (5)$ and $S_1 = -7$, then compute p_2 using the result in Theorem 3.1 and let $p_{i,j}$ denotes the coefficient at x^j in p_i.

$$p_2 = (x - w_{22}) \cdot p_1 - p_{1,1} \cdot T_1 W_1^0 S_1 \cdot x^0 = (x+7)(x-2) - 5 = x^2 + 5x - 19$$

Finally, we compute p_3

$$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 5 & -7 & 1 \\ \hline 3 & 0 & -6 \end{array}\right]$$

where, $T_2 = (3, 0)$, $S_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, $w_{33} = -6$, $p_{2,1} = 5$ and $p_{2,2} = 1$

$$\begin{aligned} p_3 &= (x - w_{33}) \cdot p_2 - (p_{2,2} \cdot T_2 W_2^0 S_2 x + (p_{2,1} \cdot T_2 W_2^0 S_2 + p_{2,2} \cdot T_2 W_2^1 S_2) \cdot x^0) \\ &= (x+6)(x^2 + 5x - 19) - (9x + (45 + 21)) = x^3 + 11x^2 - 2x - 180 \end{aligned}$$

The characteristics polynomial of W is $P_W(x) = x^3 + 11x^2 - 2x - 180$.
It follows by Remark 2.3 that

$$detW = (-1)^3 P_W(0) = 180$$

Berkowitz Algorithm

Given a square matrix W of size $n \times n$ with coefficients in $\mathcal{R}$, Berkowitz algorithm computes $(n+1) \times 1$ column vector P_W (characteristics polynomial). That is P_W is $(p_n, p_{n-1}, \ldots, p_0)$. The p_i are the coefficients of the n-th degree polynomial given by $det(xI - W)$.

The main idea in the standard proof of Berkowitz's algorithm (see, [9]) is Samuelson's identity, which relates the characteristics polynomial of a matrix to the characteristics polynomial of its principal sub-matrix.

Let $W = (w_{ij})$ be a fixed $n \times n$ matrix and W_k be the principal minor of W of size k, for any $k \in \{1, 2, \ldots, n\}$. That is, W_k is the sub-matrix of W consisting of elements located in the first k rows and columns. In particular, $W_1 = (w_{11})$ and $W_n = W$. Subdivide the minor W_{k+1} into four cells as shown below

$$\left[\begin{array}{c|c} W_k & S_k \\ \hline T_k & w_{k+1,k+1} \end{array}\right]$$

where, $S_k = (w_{k+1,1}, \ldots, w_{k+1,k})$ is a row vector and $T_k = (w_{1,k+1}, \ldots, w_{k,k+1})$ is a column vector. Let $P_k = P_{W_k}$ be the characteristic polynomial of the sub-matrix W_k. The idea behind Berkowitz algorithm is to express P_{k+1} in terms of P_k. This way we can recursively build all the successive characteristic polynomials.
Let τ_k be $(k+2) \times (k+1)$ lower-triangular Toeplitz matrix, where the entries in the i-th row and j-th column obey

i. $T_k \cdot W_k^{j-i-1} \cdot S_k$ if $i < j - 1$,

ii. $-w_{k+1,k+1}$ if $i = j - 1$,

iii. 1 if $i = j$,

iv. 0 if $i > j$.

The Toeplitz matrix τ_k whose the description of its entries explained above is written as

$$\tau_k = \begin{pmatrix} -w_{k+1,k+1} & -T_k S_k & \cdots & T_k W_k^{k-1} S_k \\ 1 & -w_{k+1,k+1} & \cdots & T_k W_k^{k-2} S_k \\ 0 & 1 & \ddots & T_k W_k^{k-3} S_k \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{3.7}$$

The coefficients of the characteristics polynomial, of the given square matrix W, is the output of Berkowitz's algorithm, i.e., to be the entries of the column vector $P_W = \tau_1 \cdot \tau_2 \cdots \tau_n$.

In other words, if $(l_0 \dots l_{k+1} = 1)$ are the coefficients of the characteristic polynomial P_{k+1} of sub-matrix W_{k+1} and $(f_0, \dots, f_{k+1} = 1)$ the coefficients of the characteristic polynomial P_k of sub-matrix W_{k+1}. The Samuelson's formula can now be interpreted in terms of matrix multiplication as

$$\begin{bmatrix} l_0 \\ l_2 \\ \vdots \\ l_{k+1} \end{bmatrix} = \tau_k \begin{bmatrix} f_0 \\ f_2 \\ \vdots \\ f_{k+1} \end{bmatrix}$$

3.3 Implementation of Samuelson-Berkowitz Algorithm

Given a square matrix W of size $n \times n$ whose coefficients are in a unital commutative ring $\mathcal{R}$, this algorithm computes the characteristic polynomial P_W and determinant of W.

1. Initialize $U = \begin{bmatrix} -w_{11} \\ 1 \end{bmatrix}$
2. for every $k \in \{1, \dots, n-1\}$ proceed as follows:
   i. let W_k be the k-th principal minor of W
   ii. set $T_k = (w_{k+1,1}, \dots, w_{k+1,k})$ and $S_k = Transpose((w_{1,k+1}, \dots, w_{k,k+1}))$
   iii. compute the products
   $$-T_k S_k, -T_k W_k^1 S_k \cdots - T_k W_k^{k-1} S_k$$
   iv. construct a Toeplitz matrix τ_k in (3.7)
   v. update U by setting $U = \tau_k \cdot U$
3. return $P_W(x) = (1, x, \dots, x^{n-1}, x^n) \cdot U$

3.4 Complexity of Samuelson-Berkowitz Algorithm

It is of utmost importance to expatiate on the computations of step (2) in Section 3.3 being the most important step that speed up the algorithm. It can be performed much more efficiently when executed serially by using matrix vector multiplication rather than matrix-matrix multiplications.

More precisely, instead of computing W_k^{k-1} for $1 \leq k \leq n$ we first compute matrix-vector product $W_k^{k-1}S_k$ and then the dot product $T_kW_k^{k-1}S_k$ for $1 \leq k \leq n$. In this case the algorithm is shown to involves less than $\frac{1}{2}n^4 - n^3 + \frac{5}{2}n^2$ arithmetic operations (additions, subtraction and multiplication) of coefficients in the commutative ring $\mathcal{R}$ (see, [10]).

4 Modular Computation Algorithm (MCA)

This section provides a comprehensive explanation of the formulation of a modular computation algorithm for calculating the determinant of any $n \times n$ matrix G with integral coefficients.

4.1 Description of Modular Computation Algorithm

Modular computation algorithm is a method centred around the methods of division-free algorithm and fast division algorithm, for example, Gaussian elimination method. The main idea behind modular computation algorithm is to compute the determinant of square matrix G with coefficients in $\mathbb{Z}$ modulo small number of distinct odd primes p_i, and then use the Gaussian elimination method or any method to find the determinant d_i of the systems obtained and finally use Chinese remainder theorem (see, Theorem 2.6) to obtain the determinant d of the given matrix G.

In other words, the determinant of a matrix G over $\mathbb{Z}$ is equal to the determinant of a system of matrices G_i $(G \ mod \ p_i)$ over a finite field $GF(p_i)$, where the scalar value associated with each of this systems G_i is obtained using GEM. In fact, we are less worried about the growth sizes and inverses of the coefficients of the systems G_i which was initially in $\mathbb{Z}$, because the algorithm makes us to work over a finite field $\mathbb{F}_p = \{0, \ldots, p-1\}$.

The immediate question begging for answer is that: **how do we choose $\mathbf{p_i}$?** The choice of p_i rest on the bound of the determinant we seek and so, we need an a-priori bound for the determinant of G which is obtained through the Hadamard's inequality.

4.2 Formulation of Modular Computation Algorithm

Let $G = (g_{ij})$ be a matrix of size $n \times n$ and fix a distinct odd primes p_i. For every primes p_i we denote the matrix G modulo p_i by $\overline{g_{ij}} = (g_{ij} \ mod \ p_i)$. We compute the determinant of $\overline{G} = (\overline{g_{ij}})$ using the fact that congruence's preserves sums and products. By definition of determinants we have

$$det(\overline{G}) = \sum_{\sigma} sgn(\sigma) \cdot \prod_{i=1}^{n} \overline{g_{i\sigma(i)}} \equiv \sum_{\sigma} sgn(\sigma) \cdot \prod_{i=1}^{n} g_{i\sigma(i)} = detG(mod \ p_i)$$

The following theorem is a step close to knowing how to determine the primes p_i which is the centre of focus of the modular computation algorithm.

Theorem 4.1. *(Hadamard inequality). Let $G = (g_{ij})$ be a matrix of size $n \times n$ with real coefficients, then*

$$|detG| \leq \prod_{j=1}^{n} \sqrt{\sum_{i=1}^{n} g_{ij}^2} \tag{4.8}$$

Proof. A detailed proof of this theorem which relies on the QR decomposition of a non-singular matrix G combined with the Cauchy theorem (see, Theorem 2.9) is provided in [4, page 144]). □

The following corollary is a less costly approach for finding the bound of the determinant (Hadamard bound) of a given square matrix G.

Colorally 4.2. *Let there be a constant $C > 0$ such that $|g_{ij}| \leq C$ for every $i, j \in \{1, \ldots, n\}$, then*

$$|det(G)| \leq C^n \cdot \sqrt{n^n}$$

Proof. Since g_{ij} are real coefficients then Theorem 4.1 holds and also we have $g_{ij} \leq |g_{ij}| \leq C$, and $g_{ij}^2 \leq C^2$, for every $i, j \leq n$.
Then

$$|detG| \leq \prod_{j=1}^{n} \sqrt{\sum_{i=1}^{n} m_{ij}^2} \leq \prod_{j=1}^{n} \sqrt{\sum_{i=1}^{n} C^2} \leq \prod_{j=1}^{n} \sqrt{n \cdot C^2} = C^n \cdot \sqrt{n^n}$$

This completes the proof. □

Next, we present an approach whose goal is to control the growth of the intermediate computations when calculating the determinant of G so that less unnecessary calculations are performed. Let $detG = d$ and suppose $p = p_1 . \cdot p_2 \cdots p_i$ be such that $p > 2 \cdot C^n \cdot \sqrt{n^n}$

We have already observed that a congruence $det(G \ mod \ p_i) \equiv detG(mod \ p_i)$ holds for every prime p_i . Lemma 2.7 asserts that $detG \equiv d \ (mod \ p)$. Hadamard's inequality says that if $d < \frac{p}{2}$, then the congruence $d - detG \equiv 0(mod \ p)$ implies that $d = detG > 0$. If on the other hand, we have $d > \frac{p}{2}$ (it cannot be equal $\frac{p}{2}$, since the primes p_i are odd and products of odd is odd), then $detG = d - p < 0$.

It is observed that, this method for computing determinants can be much faster than a direct computation, but will be slower when the number of primes p_i used for Chinese remainder theorem is large. That is, if applied to a big matrix (or matrix with big coefficients) and we used too many primes, then modular algorithm would do too many unnecessary modular computations and lost its beauty.

Remark 4.3. The modular computation algorithm can be speed up if fewer primes p_i are used.

4.3 Implementation of Modular Computation Algorithm

Given a matrix $G = (g_{ij})$ with integer entries, this algorithm computes the determinant $detG$.

1. find a bound C such that $C \geq |g_{ij}|$ for all $i, j \leq n$;

2. find odd primes $p_1 \cdots p_s$, whose product is greater than twice the Hadamard's bound, i.e.
$$p = p_1 \cdots p_s > 2 \cdot C^n \cdot \sqrt{n^n}$$

3. for every $k \in \{1, 2, \ldots, s\}$ compute the determinant over $\mathbb{F}_{p_k}$ using e.g. Gaussian elimination or any other method;

4. use Chinese remainder theorem in Theorem 2.6 to find a solution d to the system of congruence's $d \equiv d_k \ (mod\ p_k)$ for every $k \in \{1, 2, \ldots, s\}$;

5. if $d > \frac{p}{2}$, then replace it by $d - p$;

6. return the determinant $detG = d$.

Example 4.4. Compute the determinant of a matrix
$$G = \begin{pmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & -2 \end{pmatrix}$$
using algorithm in Section 4.3.

Clearly, from the matrix G, we have, $n = 3$, $C = 9$ and the Hadamard's inequality gives the bound
$$|detG| \leq 9^3 \cdot \sqrt{3^3} = 3787.995$$

To this end, we consider odd primes: $3, 5, 11,$ and 53, for which the product $p = 3 \cdot 5 \cdot 11 \cdot 53 = 8,745$ is greater than twice the Hadamard's bound. We will successively compute the determinants modulo each prime using GEM, and then use Chinese remainder theorem in Theorem 2.6 to incrementally solve the resulting system of congruence's.

Now, we compute $det(G\ mod\ p_i)$ and obtain
$$d_1 = det(G\ mod\ 3) = det \begin{pmatrix} 0 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} = -6 \equiv 0\ mod(3),$$
$$d_2 = det(G\ mod\ 5) = det \begin{pmatrix} 4 & 2 & 1 \\ 0 & 4 & 1 \\ 4 & 0 & 3 \end{pmatrix} = 40 \equiv 0\ (mod\ 5),$$
$$d_3 = det(G\ mod\ 11) = det \begin{pmatrix} 9 & 2 & 1 \\ 5 & 10 & 6 \\ 4 & 0 & 9 \end{pmatrix} = 728 \equiv 2\ (mod\ 11),$$
$$d_4 = det(G\ mod\ 53) = det \begin{pmatrix} 9 & 2 & 1 \\ 5 & 52 & 6 \\ 4 & 0 & 51 \end{pmatrix} = 23,198 \equiv 37\ (mod\ 53).$$

Therefore, it follows that

$$\begin{cases} d \equiv\ 0\ (mod\ 3) \\ d \equiv\ 0\ (mod\ 5) \\ d \equiv\ 2\ (mod\ 11) \\ d \equiv\ 37\ (mod\ 53), \end{cases} \tag{4.9}$$

Next, we apply the Chinese remainder theorem to (4.9) and obtain

$$d \equiv 90\ (mod\ 8745)$$

Therefore, $d = 90$ is a unique solution of the system and it is less than half times p. In consequence, $detG = 90$

4.4 Complexity of Modular Computation Algorithm

Prime numbers are frequent enough to find one with a word length in the same order of magnitude as $O(n^2)$ while computing the determinant of an $n \times n$ matrix over a finite field for each prime p_i using Gaussian elimination has a complexity of $O(n^3)$ and using the Chinese Remainder Theorem has a complexity of $O(n)$ involving simple arithmetic operations.

The complexities of each step demonstrate that the overall complexity of the algorithm is dominated by the step with the highest complexity, which is typically computing determinants $mod\ p_i$ (Step 3) with $O(n^3)$ complexity. For more explanations on the complexity of this algorithm (see, [11]). Therefore, the cost of computations in modular computation algorithm is $0(n^3)$.

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