MATA22 Booklet 1 Notes

Definitions:

1. If n is a positive integer, then the Euclidean n – space, denoted by R^n, is the collection of all n – tuples of real numbers.

   There are 2 kinds of n – tuples in R^n:
   1. $(a_1, a_2, \dots a_n)$ denotes a point in R^n.
   2. $[a_1, a_2, \dots a_n]$ denotes a vector in R^n.

2. The zero vector in R^n is denoted by **0** and is a vector with all of its entries consisting of 0s only.

3. A scalar only has magnitude. It does not have direction.

4. A vector has both magnitude and direction.

5. 2 vectors are parallel if they are both non – zero and they can be written as a non – zero multiple of each other.
   E.g. [1, 2, 3] and [2, 4, 6] are parallel because $[1, 2, 3] = \left(\frac{1}{2}\right)[2, 4, 6]$.
6. Given the vectors $\mathbf{v_1}, \mathbf{v_2}, \dots \mathbf{v_n}$ in R^n and the scalars $s_1, s_2, \dots s_n$ in R, the linear combination of those vectors with those scalars is
   $s_1\mathbf{v_1} + s_2\mathbf{v_2} + \dots s_n\mathbf{v_n}$.

7. The set of all linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \dots \mathbf{v_n}$ is called the span of $\mathbf{v_1}, \mathbf{v_2}, \dots \mathbf{v_n}$ and is denoted as $sp(\mathbf{v_1}, \mathbf{v_2}, \dots \mathbf{v_n})$.

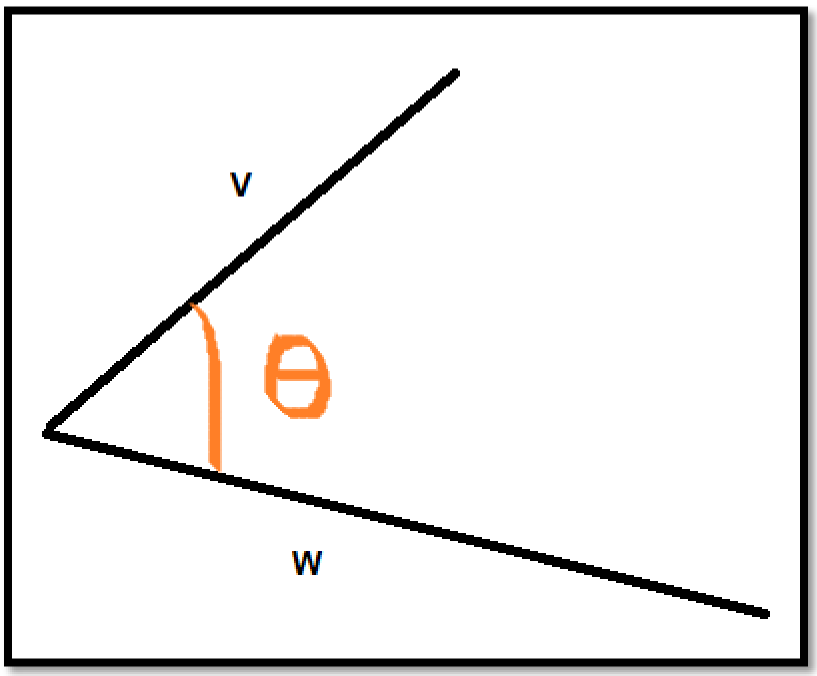
8. A vector is in standard position if it starts at the origin. If a vector does not start at the origin, then that vector has been translated (moved).

9. Two vectors are equal iff all of their entries are equal.
   I.e. $\mathbf{v} = \mathbf{w}$ iff $v_i = w_i$ for all i = 1, 2, … n
   E.g. [1, 2, 3] = [1, 2, 3] because all of their entries are equal.

10. The magnitude or norm of a vector, **v**, is denoted by $||\mathbf{v}||$.
    $||\mathbf{v}|| = \sqrt{(V_1)^2 + \dots + (V_n)^2}$
    E.g. Let $\mathbf{v} = [1, 2, 3]$ $||\mathbf{v}|| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

11. A unit vector is a vector with a magnitude of 1. It is equal to $\frac{\mathbf{v}}{||\mathbf{v}||}$.

12. The dot product of 2 vectors, **v** and **w** is defined by $\mathbf{v} * \mathbf{w} = (||\mathbf{v}||)(||\mathbf{w}||)(\cos(\Theta))$ where Θ is the angle between the vectors.



Let $\mathbf{v} = [v_1, v_2, \ldots, v_n]$.
Let $\mathbf{w} = [w_1, w_2, \ldots, w_n]$.
$\mathbf{v} * \mathbf{w} = (v_1)(w_1) + (v_2)(w_2) + \ldots + (v_n)(w_n)$

E.g.
Let $\mathbf{v} = [1, 2, 3]$.
Let $\mathbf{w} = [2, 3, 4]$.

$\mathbf{v} * \mathbf{w} = (1)(2) + (2)(3) + (3)(4)$
$= 20$

13. $\Theta = \cos^{-1}\left(\frac{\mathbf{v} * \mathbf{w}}{(||\mathbf{v}||)(||\mathbf{w}||)}\right)$

14. Two vectors are perpendicular if their dot product equals to 0.

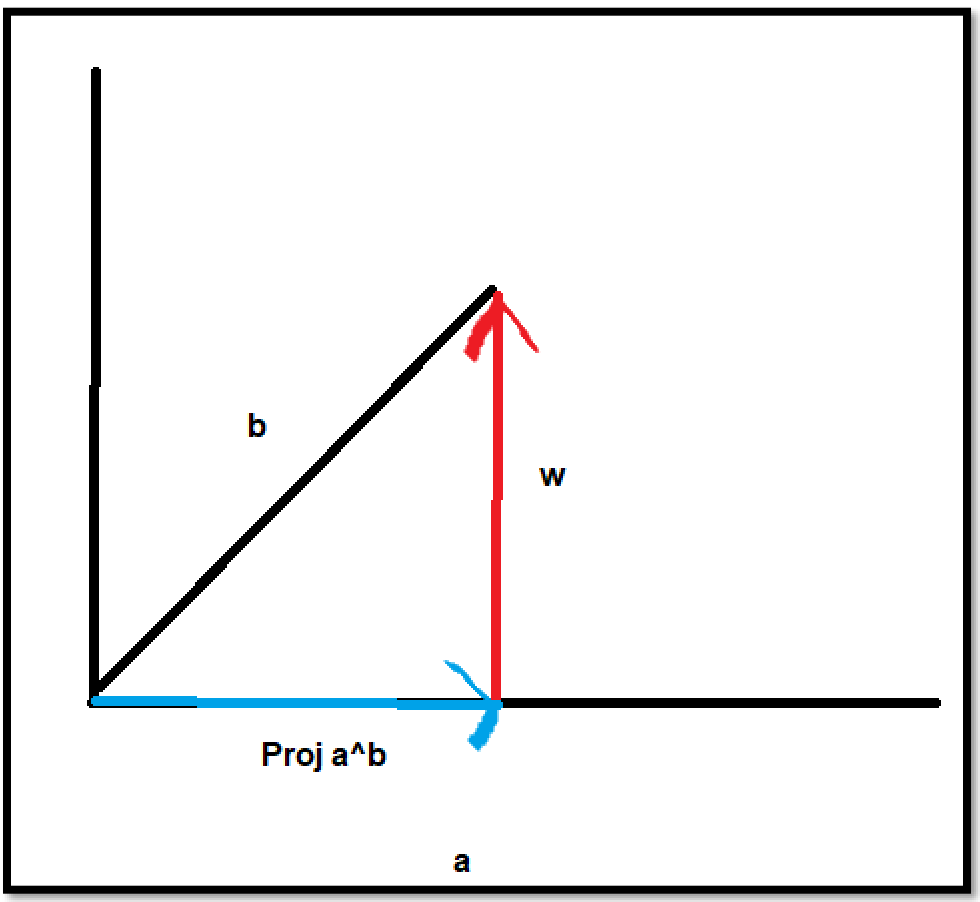
15. $\mathbf{v} * \mathbf{v} = (||\mathbf{v}||)^2$

16. The orthogonal projection of **b** onto **a** is denoted by $\text{Proj}_{\mathbf{a}}^{\mathbf{b}}$.

$$\text{Proj}_{\mathbf{a}}^{\mathbf{b}} = \left(\frac{\mathbf{a} * \mathbf{b}}{||\mathbf{a}||^2}\right)\mathbf{a}$$

E.g.
Find the orthogonal projection of [1, 2, 4] onto [3, 5, 2].

$$\text{Proj}_{\mathbf{v}}^{\mathbf{w}} = \left(\frac{[1,2,4] * [3,5,2]}{||[\mathbf{3,5,2}]||^2}\right)[3,5,2]$$
$$= \left(\frac{21}{38}\right)[3,5,2]$$



17. **w** is the vector component of **b** onto **a.**

$$\mathbf{w} = \mathbf{b} - \text{Proj}_{\mathbf{a}}^{\mathbf{b}}$$

Vector Math:

Let $\mathbf{v} = [v_1, v_2, \ldots, v_n]$.
Let $\mathbf{w} = [w_1, w_2, \ldots, w_n]$.
Let r be a scalar.

1. Vector Addition:

$$\mathbf{v} + \mathbf{w} = [v_1, v_2, \ldots, v_n] + [w_1, w_2, \ldots, w_n]$$
$$= [v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n]$$

2. Vector Subtraction:

$$\mathbf{v} - \mathbf{w} = [v_1, v_2, \ldots, v_n] - [w_1, w_2, \ldots, w_n]$$
$$= [v_1 - w_1, v_2 - w_2, \ldots, v_n - w_n]$$

3. Scalar Multiplication:

$$r(\mathbf{v}) = r([v_1, v_2, \ldots, v_n])$$
$$= [rv_1, rv_2, \ldots, rv_n])$$

Theorem:

Let $\mathbf{v} = [v_1, v_2, \ldots, v_n]$.
Let $\mathbf{w} = [w_1, w_2, \ldots, w_n]$.
Let $\mathbf{u} = [u_1, u_2, \ldots, u_n]$.
Let r and s be scalars.

1. $\mathbf{v} + (\mathbf{u} + \mathbf{w}) = (\mathbf{v} + \mathbf{u}) + \mathbf{w}$
2. $\mathbf{v} + \mathbf{u} = \mathbf{u} + \mathbf{v}$
3. $\mathbf{0} + \mathbf{v} = \mathbf{v}$
4. $r(\mathbf{v} + \mathbf{u}) = r\mathbf{v} + r\mathbf{u}$
5. $(r+s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$
6. $r(s\mathbf{v}) = (rs)\mathbf{v}$
7. $\mathbf{1v} = \mathbf{v}$
8. $||\mathbf{v}|| \geq 0$ and $||\mathbf{v}|| = 0$ iff $\mathbf{v} = \mathbf{0}$
9. $||r\mathbf{v}|| = (|r|)(||\mathbf{v}||)$
10. $||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$ (Triangle Inequality)
11. $\mathbf{v*w} = \mathbf{w*v}$
12. $\mathbf{u}*(\mathbf{v} + \mathbf{w}) = \mathbf{u*w} + \mathbf{u*v}$
13. $r(\mathbf{u*w}) = r\mathbf{u} + r\mathbf{w}$
14. $|\mathbf{u*w}| \leq (||\mathbf{u}||)(||\mathbf{w}||)$ (Cauchy – Schwartz Inequality)

MATA22 Booklet 2 Notes

Definitions:

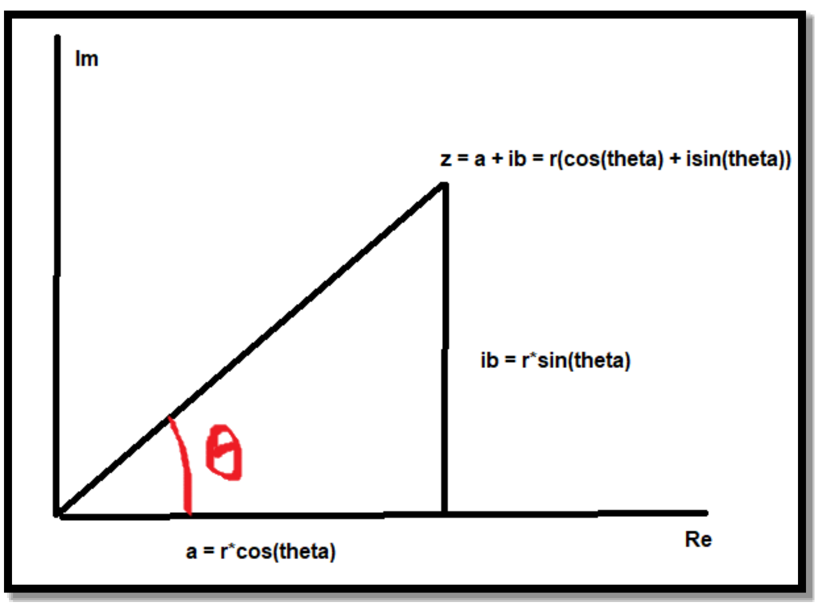
1. The set of all complex numbers, denoted by C, is the set of all numbers in the form of a + ib, where a and b are real numbers.
   a is the real part, denoted by Re.
   b is the imaginary part, denoted by Im.

   Let z = a + ib

2. The modulus or magnitude of z is denoted by |z| or r.
   $|z| = \sqrt{a^2 + b^2}$

3. The polar form of z is r(cos(Θ) + isin(Θ)).

4. The angle Θ is called the argument of z, denoted by Arg(z).
   If $-\pi \leq \Theta \leq \pi$, then Θ is the principal argument of z.
   $\Theta = \tan^{-1}(\frac{b}{a})$



5. The conjugate of z, denoted by $\bar{z}$ is a – ib.

   If x = a – ib, then $\bar{x}$ = a + ib.

<u>Complex Number Arithmetic:</u>

Let x = a + ib.
Let y = c + id.
Let r be a scalar.

1. **<u>Addition:</u>**
   $x + y = (a + ib) + (c + id)$
   $\quad\quad = (a + c) + i(b + d)$

2. **<u>Multiplication:</u>**
   $xy = (a + ib)(c + id)$
   $\quad = ac + iad + ibc + i^2bd$
   $\quad = (ac - bd) + i(ad + bc)$

3. **<u>Scalar Multiplication:</u>**
   $rx = r(a + ib)$
   $\quad = ra + irb$

<u>Theorem:</u>

Let $z_1 = a + ib = r_1(\cos(\Theta_1) + i\sin(\Theta_1))$
Let $z_2 = c + id = r_2(\cos(\Theta_2) + i\sin(\Theta_2))$

1. $(z_1)(z_2) = (r_1)(r_2)[\cos(\Theta_1 + \Theta_2) + i\sin(\Theta_1 + \Theta_2)]$
2. $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\Theta_1 - \Theta_2) + i\sin(\Theta_1 - \Theta_2)]$
3. $\bar{\bar{z}}_1 = Z_1$

4. $|z_1|^2 = (z_1)\boxed{\overline{z_1}}$

5. $$\boxed{\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}}$$

6. $$\boxed{z_1^{-1} = \frac{\overline{z_1}}{\|z_1\|^2}}$$

7. $$\boxed{\overline{z_1 / z_2} = \overline{z_1} / \overline{z_2}}$$

8. $|z_1 + z_2| \le |z_1| + |z_2|$ (Triangle Inequality)

Examples:

1. Express $(1 + i)^8$ in the form of a + ib.

   Step 1: Find r
   r = $\sqrt{a^2 + b^2}$
   = $\sqrt{1^2 + 1^2}$
   = $\sqrt{2}$

   Step 2: Find Θ
   Θ = $\tan^{-1}(\frac{b}{a})$
   = $\tan^{-1}(\frac{1}{1})$
   = $\frac{\pi}{4}$

   Step 3: Write it in polar form
   z = r(cos(Θ) + isin(Θ))
   = $\sqrt{2}(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}))$

   Step 4: Plug it into the equation
   $z^n = r^n(\cos(n\,\Theta) + i\sin(n\,\Theta))$
   $z^8 = (\sqrt{2})^8(\cos(\frac{8\pi}{4}) + i\sin(\frac{8\pi}{4}))$
   = 16

2. Find the four fourth roots of -16.

3. Step 1: Find r
   r = $\sqrt{(-16)^2}$
   = 16

   Step 2: Find Θ
   Θ = $\tan^{-1}(\frac{0}{-16})$
   = 0

   Step 3: Plug it into the equation
   $z^{1/n} = r^{1/n}(\cos(\frac{\Theta}{n} + \frac{2k\pi}{n}) + i\sin(\frac{\Theta}{n} + \frac{2k\pi}{n}))$
   $z^{1/4} = 2(\cos(\frac{k\pi}{2}) + i\sin(\frac{k\pi}{2}))$

$k = 0, 1, 2, \ldots n - 1$
Plug in the different values for k and solve.

MATA22 Booklet 3 Notes

Definition:

1. An m*n matrix is an ordered rectangular array of real entries with m rows and n columns.
   E.g.
   $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ ← A 2*2 matrix.

2. The set of all m*n matrices with real entries is denoted by $M_{m,n}(R)$.

3. Another way to denote a matrix is $A = [a_{ij}]$ where a_{ij} is the entry in the ith row and the jth column.

4. A m*1 matrix is a column vector.
   A 1*n matrix is a row vector.
   E.g.
   $\begin{bmatrix} 1 & 2 \end{bmatrix}$ ← A row vector.
   $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ← A column vector.

5. A square matrix is a matrix with the same number of rows and columns.
   E.g.
   $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ ← A square matrix.

6. The main diagonal in a square matrix is the diagonal that contains a_{ii}.
   E.g.
   $\begin{bmatrix} \mathbf{1} & 2 \\ 0 & \mathbf{2} \end{bmatrix}$ ← The highlighted diagonal is the main diagonal.

7. A diagonal matrix is a square matrix where $a_{ij} = 0$ for all $i \neq j$.
   I.e. All entries not on the main diagonal is 0.
   E.g.
   $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ ← A diagonal matrix.

8. The identity matrix, denoted by I, is a special type of a diagonal matrix. All of its entries along the main diagonal is 1.
   E.g.
   $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ← A 3*3 identity matrix.

9. An upper triangular matrix is a square matrix such that $a_{ij} = 0$ for all $i > j$.
   I.e. All entries below the main diagonal must be 0. Entries on and above the main diagonal could be 0.
   E.g.
   $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ ← An upper triangular matrix.

10. A lower triangular matrix is a square matrix such that $a_{ij} = 0$ for all $i < j$.
    I.e. All entries above the main diagonal must be 0. Entries on and below the main diagonal could be 0.
    E.g.
    $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$ ← A lower triangular matrix.

11. A zero matrix is a matrix with all of its entries 0.
    E.g.
    $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ← A zero matrix

12. The transpose of a matrix A_{ij} is denoted by $(A_{ij})^T$.
    $(A_{ij})^T = A_{ji}$
    I.e. The rows of matrix A become the columns of A^T.
    E.g.
    $A = \begin{bmatrix} 1 & 6 & 5 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 1 & 4 \\ 5 & 0 & 1 \end{bmatrix}$

    If $A = A^T$ then A is a symmetric matrix.
    If $A = -A^T$ then A is a skew-symmetric matrix.

13. The trace of a square matrix A, denoted by Tr(A), is the sum of the entries along A's main diagonal.
    E.g.
    $A = \begin{bmatrix} 1 & 6 & 5 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$

    $\text{Tr}(A) = 1 + 1 + 1$
    $= 3$

14. A square matrix, A, is invertible if there exists another square matrix, C, such that (A)(C) = (C)(A) = I. C is the inverse matrix of A and is denoted by A^{-1}.

    If a square matrix is not invertible, then it is singular.
    Not all matrices are invertible.

15. A m*n linear system of equations is a system of m linear equations in n variables.
    E.g.
    $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$
    $a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$
    |
    |
    |
    $a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$

    The above linear system of equations is equivalent to A**x** = **b** where

    $$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

    A**x** = **b** has solutions iff **b** is in the span of A.

    A is called the coefficient matrix.

    $(\mathrm{A}|\mathbf{b})$ is called the augmented/partitioned matrix.

    $$(\mathrm{A}|\mathbf{b}) = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right)$$

16. Elementary Row Operations:
    1. Row Interchange: Switches the i^{th} and j^{th} rows.
    2. Row Scaling: Multiplies a row by a non-zero scalar.
    3. Row Addition: Adds the i^{th} row to the j^{th} row.

17. If matrix B can be obtained by performing a sequence of elementary row operations on matrix A, then A and B are row equivalent. We denote this by A ~ B.

18. If $(\mathrm{A}|\mathbf{b}) \sim (\mathrm{H}|\mathbf{c})$, then A**x** = **b** and H**x** = **c** has the same solution set.

19. A matrix is in REF if the following conditions are satisfied:
    1. All the rows with only zeros are at the bottom.
    2. The first non-zero entry in any row (pivot) is to the right of all the pivots in the rows above it.

    E.g.
    $\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$ ← A matrix in REF (Reduced Echelon Form).

20. A matrix is in RREF if the following conditions are satisfied:
    1. The matrix is in REF.
    2. All the pivots are 1.
    3. Each pivot is the only non-zero entry in its column.

    E.g.
    $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ← A matrix in RREF (Reduced – Row Echelon Form).

    Note: REFs are unique, but RREfs are not unique.

21. The Gauss Reduction with Back Substitution Method solves A**x** = **b** by reducing $(\mathrm{A}|\mathbf{b})$ to REF and then using substitution.

    The Gauss – Jordan Method solves A**x** = **b** by reducing $(\mathrm{A}|\mathbf{b})$ to RREF.

22. A consistent linear system is a linear system with 1 or more solutions.
    An inconsistent linear system is a linear system with 0 solutions.

23. Let A**x** = **b** be a linear system and suppose $(\mathrm{A}|\mathbf{b}) \sim (\mathrm{H}|\mathbf{c})$ and suppose that $(\mathrm{H}|\mathbf{c})$ is in REF, then:

    1. H**x** = **c** is inconsistent. This means that $(\mathrm{H}|\mathbf{c})$ has zeroes to the left of the partition and non-zero entries to the right of the partition.
    I.e. This means that 0**x** = a non-zero number.
    0**x** = 4 doesn't have any solutions because there is no value of **x** that will make it so that 0**x** equals 4.

    2. H**x** = **c** is consistent with pivots in every column. Here, the solution is unique.

    3. H**x** = **c** is consistent but not every column has a pivot. This means that there is infinite set of solutions.
       I.e. 0**x** = 0. **x** can be any real number.

24. An elementary matrix is a matrix such that only 1 elementary row operation has been performed to an identity matrix.

Matrix Arithmetic:

1. **Matrix Addition and Subtraction:**
   2 matrices can only add/subtract each other if they have the same dimensions. I.e. Both matrices have the same number of rows and columns.
   E.g.
   $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 8 & 10 \end{bmatrix}$

   $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2. **Matrix Multiplication:**
   2 matrices can multiply if the number of columns of the first matrix equals to the number of rows of the second matrix. The resulting matrix will have the same number of rows as the first matrix and same number of columns as the second matrix.
   E.g.
   $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$

   (1)(1) + (2)(3) = 1 + 6 = 7
   (1)(2) + (2)(4) = 2 + 8 = 10
   (3)(1) + (4)(3) = 3 + 12 = 15
   (3)(2) + (4)(4) = 6 +16 = 22

   With matrix multiplication, you dot product the first row of the first matrix with each column of the second matrix, one at a time. Then, you move on to the second row and you dot product that with each column of the second matrix, one at a time. You continue this pattern until you finish the last row of the first matrix.

   Note: Matrix multiplication is usually not commutative. This means that AB most likely won't equal to BA. An exception to this is inverse matrices. As mentioned before, $A(A^{-1}) = (A^{-1})A = I$.

3. **Matrix Scalar Multiplication:**
   When you multiply a scalar to a matrix, you multiply that scalar to all of the matrix's entries.
   E.g.
   $3 * \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$

Theorems:

Let A be an m*n matrix.
Let B, C and D be n*j matrices.
Let E be a j*l matrix.
Let r and s be scalars.
Let $\underline{0}$ be the n*j zero matrix.
Let -B be an n*j matrix with its entries opposite of B's.

1. $A + A^T$ is a symmetric matrix.
2. $A – A^T$ is a skew-symmetric matrix.
3. $B + C = C + B$
4. $B + (C + D) = (B + C) + D$
5. $B + \underline{0} = B$
6. $B + (-B) = \underline{0}$
7. $r(sB) = (rs)B$
8. $(r + s)B = rB + sB$
9. $r(B + C) = rB + rC$
10. $A(rB) = r(AB) = (rA)B$
11. $A(B + C) = AB + AC$
12. $AI = IA = A$
13. $(A^T)^T = A$
14. $(B + C)^T = B^T + C^T$
15. $(rA)^T = r(A)^T$
16. $Tr(A^T) = Tr(A)$
17. $Tr(sA) = s(Tr(A))$
18. $A(BE) = (AB)E$

MATA22 Booklet 4

Definitions:

1. A homogeneous linear system of equations is of the form:
   $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0$
   $a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0$
   |
   |
   $a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0$
   or $A\mathbf{x} = \mathbf{0}$, where A is the coefficient matrix.

   Every homogeneous system is consistent because the zero vector is a solution. This is known as the trivial solution.

   If A ~ H and H has a pivot in every column, then $A\mathbf{x} = \mathbf{0}$ only has the trivial solution.

   If A has fewer rows than columns and A ~ H, then it is impossible for H to have a pivot in every column. Therefore, $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions.

2. The nullspace of A, denoted by N, is the set of solutions to $A\mathbf{x} = \mathbf{0}$.
   I.e. $N = \{\mathbf{x} \in R^n \mid A\mathbf{x} = \mathbf{0}\}$.

3. The row space of A is the span of the row vectors of A. If A is a m*n matrix, then the row space of A is a subset of R^n.

4. The column space of A is the span of the column vectors of A. If A is a m*n matrix, then the column space of A is a subset of R^m.

5. Let $\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n} \in R^n$.
   Let $s_1, s_2, \ldots s_n \in R$.

   If $s_1\mathbf{v_1} + s_2\mathbf{v_2} + \ldots s_n\mathbf{v_n} = \mathbf{0}$ has exactly 1 solution, then we say that the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n}$ are linearly independent.
   I.e. $s_1\mathbf{v_1} + s_2\mathbf{v_2} + \ldots s_n\mathbf{v_n} = \mathbf{0}$ is linearly independent iff $s_1 = s_2 = \ldots s_n = 0$.

   If $s_1\mathbf{v_1} + s_2\mathbf{v_2} + \ldots s_n\mathbf{v_n} = \mathbf{0}$ has more than 1 solution, then we say that the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n}$ are linearly dependent.

   2 vectors are linearly independent if they are non – zero and non – parallel.

6. W is a subset of R^n if W satisfies the following conditions:

   1. If **u**, **v** $\in$ W, then **u** + **v** $\in$ W. (Closure by addition.)
   2. If **v** $\in$ W and r $\in$ R, then r**v** $\in$ W. (Closure by scalar multiplication.)

7. Let W be a subset of R^n.
   W is a subspace of R^n if W satisfies the following conditions:

   1. W is non – empty. (Non – empty.)
   2. If **u**, **v** $\in$ W, then **u** + **v** $\in$ W. (Closure by addition.)
   3. If **v** $\in$ W and r $\in$ R, then r**v** $\in$ W. (Closure by scalar multiplication.)

   Note: The zero vector is in all subspaces. If you have a W that does not include the zero vector, then it is not a subspace of R^n.

8. Let W is a subspace of R^n. If B = {$\mathbf{b_1}$, $\mathbf{b_2}$, …, $\mathbf{b_n}$} is a subset of W, then B is a basis for W if every vector in W can be written uniquely as a linear combination of the vectors in B.
   $\mathbf{b_1}$, $\mathbf{b_2}$, …, $\mathbf{b_n}$ must be linearly independent in order for B to be a basis.
   I.e. B is a basis for W if B is the smallest set of vectors that spans W.

   How to find the basis for a few column vectors $\mathbf{b_1}$, $\mathbf{b_2}$, …, $\mathbf{b_n}$:
   1. Write the column vectors as a matrix like such: A = [$\mathbf{b_1}$, $\mathbf{b_2}$, …, $\mathbf{b_n}$]
   2. Row reduce A to either REF or RREF. Let A ~ H.
   3. The basis is the set of all columns in H that have a pivot.

9. Let W be a subspace of R^n. The number of elements in a basis for W is called the dimension of W, denoted as dim(W).

10. Let A be an m*n matrix. The dimension of the column space of A equals to the dimension of the row space of A. Both are denoted as rank(A).
    I.e. rank(A) is the number of columns with pivots.

11. Let A be an m*n matrix. The dimension of the nullspace is called the nullity of A. It is denoted as nullity(A).
    I.e. nullity(A) is the number of columns without pivots.

12. The rank-nullity equation states that rank(A) + nullity(A) = the number of columns A has.

Theorems:

1. Let **a** and **b** be solutions to A**x** = **0**. Then, S_1**a** + S_2**b** are also solutions to A**x** = **0**.

2. Let A**x** = **b** be a linear system with a solution **p**. Then:
   1. If **h** is in the nullspace of A, then **p** + **h** is also a solution to A**x** = **b**.
   2. If **q** is any solution to A**x** = **b**, then **q** = **p** + $\mathbf{h_1}$ for some $\mathbf{h_1} \in \mathbf{h}$.

3. Let A be a m*n matrix. Then, the following are equivalent:
   1. The linear system A**x** = **b** has a unique solution for each $\mathbf{b} \in R^n$.
   2. A is row equivalent to the n*n identity matrix I.
   3. A is invertible.
   4. The column vectors of A form a basis for R^n.

4. Let A be a m*n matrix such that m > n. Then, the following are equivalent:
   1. Each consistent system A**x** = **b** has a unique solution.
   2. The RREF of A consists of the n*n identity matrix on top followed by m – n rows of zeroes.
   3. The column vectors of A form a basis for the column space of A.

5. Let A be a m*n matrix such that m < n. Then, A**x** = **b** is a linear system with fewer equations than unknowns. If A**x** = **b** is consistent, then it has an infinite number of solutions.

6. B = {$\mathbf{b_1}$, $\mathbf{b_2}$, …, $\mathbf{b_n}$} is a basis for the subspace W of R^n iff B = {$\mathbf{b_1}$, $\mathbf{b_2}$, …, $\mathbf{b_n}$} is a linearly independent set of vectors and W = sp($\mathbf{b_1}$, $\mathbf{b_2}$, …, $\mathbf{b_n}$).

7. Let W be a subspace of R^n. Let $\mathbf{w_1}$, $\mathbf{w_2}$, …, $\mathbf{w_k}$ be vectors in W that span W. Let $\mathbf{w_1}$, $\mathbf{w_2}$, …, $\mathbf{w_m}$ be linearly independent vectors in W. Then, $k \geq m$.

8. Any 2 bases for a subspace W of R^n contain the same number of vectors.

9. Every subspace W of R^n has a basis and $\dim(W) \leq n$.

10. Every set of linearly independent vectors in R^n can be enlarged, if necessary, to become a basis for R^n.

11. If W is a subspace of R^n and dim(W) = k, then:
    1. Every linearly independent set of k vectors in W is a basis for W.
    2. Every set of k vectors that spans W is a basis for W.

12. Let A be a m*n matrix and let A ~ H.
Let C be the column space of A.
Let R be the row space of A.
Let N be the nullspace of A.
Then:

1. The non – zero rows of H form a basis of R^n.

2. A basis for c consists of all the columns of A corresponding to the columns of H that contains pivots.

3. To find a basis for N, we solve for the linear system H**x** = **0** and find a basis for the solution set.

4. dim(c) = dim(r) = number of columns in H with pivots.

5. dim(N) = number of columns in H without pivots.

13. A n*n matrix A is invertible iff rank(A) = n.

<u>Examples:</u>

1. Let A = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix}$ Find the nullspace and basis for A.

Solution:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 2 & 3 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right)$$

R2 – R1

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right)$$

R2/3

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right)$$

R1 – R2

$$\left(\begin{array}{ccc|c} 0 & 2 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right)$$

R3 – 2*R2

$$\left(\begin{array}{ccc|c} 0 & 2 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

R2 – 2R3

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

R3 – R1

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

Row Interchange

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

The basis for A is $\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}$.

The nullspace for A is {}.

2. Let A = $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{bmatrix}$. Find the basis and nullspace of A.

Solution:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 3 & 3 & 0 \end{array}\right)$$

R2 – R1

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

R1 – 2R2

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

The basis for A is $\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$.

Let x_3 = s.

x_1 = -3s

The nullspace of A is sp($\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$).

3. Determine if the vectors [1, 1, 3], [3, 0, 4] and [1, 4, -1] are linearly independent.

Solution:

We want to show whether or not $r_1\left(\begin{bmatrix}1\\1\\3\end{bmatrix}\right) + r_2\left(\begin{bmatrix}3\\0\\4\end{bmatrix}\right) + r_3\left(\begin{bmatrix}1\\4\\-1\end{bmatrix}\right) = \mathbf{0}$ only has 1 solution and that solution is $r_1 = r_2 = r_3 = 0$.

$$\left(\begin{array}{ccc|c}1 & 3 & 1 & 0\\1 & 0 & 4 & 0\\3 & 4 & -1 & 0\end{array}\right)$$

R1 – R2

$$\left(\begin{array}{ccc|c}0 & 3 & -3 & 0\\1 & 0 & 4 & 0\\3 & 4 & -1 & 0\end{array}\right)$$

R3 – 3R2

$$\left(\begin{array}{ccc|c}0 & 3 & -3 & 0\\1 & 0 & 4 & 0\\0 & 4 & -13 & 0\end{array}\right)$$

R1/3

$$\left(\begin{array}{ccc|c}0 & 1 & -1 & 0\\1 & 0 & 4 & 0\\0 & 4 & -13 & 0\end{array}\right)$$

R3 – 4R1

$$\left(\begin{array}{ccc|c}0 & 1 & -1 & 0\\1 & 0 & 4 & 0\\0 & 0 & -9 & 0\end{array}\right)$$

R3/-9

$$\left(\begin{array}{ccc|c}0 & 1 & -1 & 0\\1 & 0 & 4 & 0\\0 & 0 & 1 & 0\end{array}\right)$$

$$\left(\begin{array}{ccc|c}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{array}\right)$$

Here, you can see that the only solution is $r_1 = r_2 = r_3 = 0$. Therefore, the vectors are linearly independent.

MATA22 Booklet 5 Notes

Definitions:

1. A function T: $R^n \rightarrow R^m$ is a linear transformation if for all $\mathbf{v}$, $\mathbf{u} \in R^n$ and for all $r \in R$, the following conditions are satisfied.

   1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (Preservation of vector addition)
   2. $T(r\mathbf{v}) = r(T(\mathbf{v}))$ (Preservation of scalar multiplication)

2. If T: $R^n \rightarrow R^m$ is a linear transformation, then:
   1. R^n is the domain of T.
   2. R^m is the co-domain of T.

3. Let T: $R^n \rightarrow R^m$ be a linear transformation.
   If $W \subset R^n$, then the image of W under T, denoted by T[W], is $\{T(\mathbf{w}) \mid \mathbf{w} \in W\}$.
   The image is the span of the vectors in the linear transformation.
   Image = Range = Column Space

4. Let T: $R^n \rightarrow R^m$ be a linear transformation.
   The range of T, denoted by $T[R^n]$, is $\{T(\mathbf{v}) \mid \mathbf{v} \in R^n\}$.

5. Let T: $R^n \rightarrow R^m$ be a linear transformation.
   If $W' \subset R^m$, then the inverse image of W' under T, denoted by $T^{-1}[W']$, is $\{\mathbf{w} \in R^n \mid T(\mathbf{w}) \in W'\}$.

6. Let T: $R^n \rightarrow R^m$ be a linear transformation.
   The kernel of T, denoted by $T^{-1}[\mathbf{0'}]$, is $\{\mathbf{w} \in R^n \mid T(\mathbf{w}) \in 0\}$ and $\mathbf{0'} \in R^m$.
   The kernel is the nullspace of the linear transformation.

7. The rank of the linear transformation = dim(Image)
   = dim(Range)
   = dim(Column Space)

8. The nullity of the linear transformation = dim(kernel).

9. Rank(Linear Transformation) + Nullity(Linear Transformation) equals to the number of columns in the linear transformation.

10. Let T: $R^n \rightarrow R^m$ be a linear transformation.
    Let A be a m*n matrix such that A =

$$\begin{bmatrix} | & | & & | \\ T(\mathbf{e1}) & T(\mathbf{e2}) & \cdots & T(\mathbf{en}) \\ | & | & & | \end{bmatrix}$$

    Then, T(**x**) = A**x** for all **x** $\in R^n$. A is called the standard matrix representation of T.

    Note: T is invertible if m = n and if A is invertible.

    Note: To find the inverse image of T, find the inverse matrix of A.

11. Let T: $R^n \rightarrow R^m$ be a linear transformation.
    1. T is one – to – one if T(**v**) = T(**u**) implies that **v** = **u**.
       I.e. If **v** ≠ **u**, then T(**v**) ≠ T(**u**).
       I.e. T is one – to – one if the kernel is empty.

    2. T is onto if $T[R^n] = R^m$.
       I.e. $\forall$ **v'** $\in R^m$ $\exists$ **v** $\in R^n$ such that T(**v**) = **v'**.
       I.e. T is onto if the rank of the domain equals the rank of the co-domain.

    3. T is isomorphic if T is both one – to – one and onto.

Theorems:

1. Let T: $R^n \rightarrow R^m$ be a linear transformation.
   1. If $\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n}, \in R^n$ and $r_1, r_2, \ldots r_n, \in R$, then $T(r_1 \mathbf{v_1} + r_2 \mathbf{v_2} + \ldots r_n \mathbf{v_n}) = T(r_1 \mathbf{v_1}) + T(r_2 \mathbf{v_2}) + \ldots T(r_n \mathbf{v_n})$.

   2. $T(\mathbf{0}) = \mathbf{0'}$ where $\mathbf{0} \in R^n$ and $\mathbf{0'} \in R^m$.

2. If T: $R^n \rightarrow R^m$ be a linear transformation and $\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n}, \in R^n$ such that $\{T(\mathbf{v_1}), T(\mathbf{v_2}), \ldots T(\mathbf{v_n})\}$ is a linearly independent set in R^m, then $\{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n}\}$ is also linearly independent.

3. If T: $R^n \rightarrow R^m$ be a linear transformation and $B = \{\mathbf{b_1}, \mathbf{b_2}, \ldots \mathbf{b_n}\}$ is a basis for R^n, then if $\mathbf{v} \in R^n$, $T(\mathbf{v})$ is determined by $T(\mathbf{b_1}), T(\mathbf{b_2}), \ldots T(\mathbf{b_n})$.

4. Let T: $R^n \rightarrow R^m$ be a linear transformation. Then:
   1. If W is a subspace of $R^{n,}$ then T[W'] is a subspace of R^m.
      I.e. If W is a subspace of R^n, then the image of W is a subspace of R^m.

   2. If W' is a subspace of $R^{m,}$ then T^{-1}[W] is a subspace of R^n.
      I.e. If W' is a subspace of R^m, then the inverse image of W' is a subspace of R^n.

Booklet 6 Notes

1. The determinant of a 1x1 matrix is its sole entry.

The determinant of a 2x2 matrix is:

$$\det(A) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1, \quad A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

The determinant of a 3x3 matrix is:

$$\det(A) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Cross Product:

Let $\vec{a} = [a_1, a_2, a_3]$

Let $\vec{B} = [b_1, b_2, b_3]$

The cross product of $\vec{a}$ and $\vec{b}$, which is $\vec{a} \times \vec{b}$ is

$$= \left[\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right],$$ which is the same as

$$[a_2b_3 - a_3b_2, \; a_3b_1 - a_1b_3, \; a_1b_2 - a_2b_1]$$

Note, $\vec{a} \times \vec{b}$ is $\perp$ to both $\vec{a}$ and $\vec{b}$.

Proof:

Let $\vec{a} = [a_1, a_2, a_3]$

Let $\vec{b} = [b_1, b_2, b_3]$

$\vec{a} \times \vec{b} = [a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1]$

$\vec{a} \cdot (\vec{a} \times \vec{b})$

$= [a_1, a_2, a_3] \cdot [a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1]$

$= [a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1)]$

$= [\cancel{a_1a_2b_3} - \cancel{a_1a_3b_2} + \cancel{a_2a_3b_1} - \cancel{a_1a_2b_3} + \cancel{a_1a_3b_2} - \cancel{a_2a_3b_1}]$

$= 0$

$\therefore\ \vec{a} \perp (\vec{a} \times \vec{b})$

QED

2. Thm:

1. If a parallelogram is determined by 2 non-zero vectors, $\vec{a} = [a_1, a_2]$ and $\vec{b} = [b_1, b_2]$, then its area is given by $|a_1b_2 - a_2b_1|$

2. If a parallelogram is determined by 2 non-zero vectors, $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$ in R^3, then its area is $\|\vec{a} \times \vec{b}\|$.

3. If a parallelpiped is determined by 3 non-zero vectors, $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$ and $\vec{c} = [c_1, c_2, c_3]$ in R^3, then its volume is $|\vec{a} \cdot (\vec{b} \times \vec{c})|$.

Proof of 1.

$$\text{Area} = ||\vec{a}|| \cdot ||\vec{h}||$$
$$= ||\vec{a}|| \cdot ||\vec{b}|| \sin\theta$$
$$(\text{Area})^2 = ||\vec{a}||^2 \; ||\vec{b}||^2 \sin^2\theta$$
$$= ||\vec{a}||^2 \; ||\vec{b}||^2 (1-\cos^2\theta)$$
$$= ||\vec{a}||^2 \; ||\vec{b}||^2 - ||\vec{a}||^2 \cdot ||\vec{b}||^2 \cos^2\theta$$
$$= ||\vec{a}|| \; ||\vec{b}|| - (\vec{a}\cdot\vec{b})^2$$
$$= (a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1b_1 + a_2b_2)^2$$
$$= \cancel{a_1^2 b_1^2} + a_1^2 b_2^2 + a_2^2 b_1^2 + \cancel{a_2^2 b_2^2} - \cancel{a_1^2 b_1^2} - 2a_1b_1a_2b_2 - \cancel{a_2^2 b_2^2}$$
$$= a_1^2 b_2^2 - 2a_1b_1a_2b_2 + a_2^2 b_1^2$$
$$= (a_1b_2 - a_2b_1)^2$$
$$\text{Area} = |a_1b_2 - a_2b_1|, \text{ as wanted}$$

QED

Proof of 3.

$$\text{Vol} = \text{Base} \times \text{Height}$$
$$= |\vec{a}| \cos\theta \, |\vec{b} \times \vec{c}|$$
$$= |\vec{a} \cdot (\vec{b} \times \vec{c})|, \text{ as wanted}$$

QED

If $\vec{a} \cdot \vec{b} \times \vec{c} = 0$, then the vol $= 0$.

3. We've already discussed the determinant of 1×1, 2×2 and 3×3 matrices. Let $n>1$ and suppose that the det of $(n-1)\times(n-1)$ matrices is defined. Let $A = [a_{ij}]$ be a $n\times n$ matrix.

The minor matrix, A_{ij}, is the $(n-1)\times(n-1)$ matrix obtained by removing the i^{th} row and j^{th} col of A.

The cofactor of a_{ij} of A is: $a'_{ij} = (-1)^{i+j} \det(A_{ij})$
$= (-1)^{i+j} |A_{ij}|$

The det of A can be computed by a cofactor expansion across any row or any column

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \dots & a_{nn} \end{vmatrix}$$

$$= a_{11}(a'_{11}) + a_{12}(a'_{12}) + \dots + a_{1n}(a'_{1n})$$

4. Thm: Let A be a $n \times n$ matrix. Then:
   1. $\det(A) = \det(A^T)$
   2. If A is a triangular matrix, then $\det(A)$ is the product of all of its entries along the main diagonal.

Proof of 1.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\det(A) = ad - bc$

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$\det(A) = ad - bc$

$\therefore$ The claim is true for 2×2 matrices. Suppose that the claim is true for all $A_{n-1 \times n-1}$ matrices, $n \geq 3$.

WTS: It's true for $A_{n\times n}$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{n1} & \dots & & a_{nn} \end{bmatrix}$$

$$A^T = \begin{bmatrix} a_{11} & \dots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & & a_{nn} \end{bmatrix}$$

$$\begin{aligned} \det(A) &= a_{11}(a_{11})' + a_{12}(a_{12})' + \dots a_{1n}(a_{1n})' \\ &= a_{11}\det(A^T)_{11} + a_{12}(\det(A^T))_{21} + \dots a_{1n}(\det(A^T))_{n1} \\ &= \det(A^T) \end{aligned}$$

QED

Proof of 2

Suppose A is an upper triangular matrix.

$$\text{I.e. } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & a_{22} & & \vdots \\ 0 & & \ddots & a_{nn} \end{bmatrix}$$

$$\begin{aligned} \det(A) &= a_{11}(a_{11})' \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} & \dots & a_{2n} \\ & \ddots & & \vdots \\ & & & a_{nn} \end{vmatrix} \\ &= a_{11} \cdot a_{22} \cdot \dots a_{nn} \end{aligned}$$

QED

5. Thm: Let A be a $n \times n$ matrix. Then:

1. $A \xrightarrow{R_i \leftrightarrow R_j} B$, then $\det(B) = -\det(A)$
2. $A \xrightarrow{R_i \to rR_i} B$, then $\det(B) = r\det(A)$
3. $A \xrightarrow{R_i \to R_i + rR_j} B$, then $\det(B) = \det(A)$

6. Let A, B be $n \times n$ matrices. Then:

1. If A contains proportional rows or cols, $\det(A) = 0$
2. A is invertible $\iff \det(A) \neq 0$

7. Let A, B be $n \times n$ matrices. Then:

1. $\det(AB) = \det(A) \cdot \det(B)$
2. If A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

8. Let A be a $n \times n$ matrix.

1. Let A_{ij} denoted the $(n-1) \times (n-1)$ matrix obtained by removing the i^{th} row and j^{th} col of A.

2. The cofactor of A is: $a'_{ij} = (-1)^{i+j} |A_{ij}|$

3. The transpose of the cofactor matrix of A is called the adjoint matrix of A. Let $A' = [a_{ij}]$ be the matrix with the ij^{th} entry of the ij^{th} cofactor of A. Then, the adjoint matrix of A, denoted as $\operatorname{adj}(A)$, is the $n \times n$ matrix equal to $(A')^T$.
I.e. $\operatorname{adj}(A) = (A')^T$

9. Thm: If A is a $n \times n$ matrix, then:

1. $a_{i1}a'_{j1} + a_{i2}a'_{j2} + \dots a_{in}a'_{jn} = \begin{cases} \det(A) & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

2. $a_{1i}a'_{1j} + a_{2i}a'_{2j} + \dots + a_{ni}a'_{nj} = \begin{cases} \det(A) & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

10. If A is a $n \times n$ matrix, then:
$(\text{adj}(A))A = A(\text{adj}(A)) = \det(A)(I)$

11. If A is a $n \times n$ matrix and $\det(A) \neq 0$, then:
$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

12. Cramer's Rule.
If $A\vec{x} = \vec{b}$ is a system of n linear eqns with n unknowns and $\det(A) \neq 0$, then the unique soln $\vec{x} = [x_1, x_2, \dots x_n]$ is of the form

$$x_k = \frac{\det(B_k)}{\det(A)} \quad \text{for } k = 1, 2, \dots n$$

B_k is the matrix A with the k^{th} col replaced by the col vector b.

E.g. Use Cramer's Rule to solve this system:

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 2x_1 - x_2 &= 11 \\ x_2 + 4x_3 &= 3 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 11 \\ 3 \end{bmatrix}$$

$$\det(A) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= (-1)(4) - (1)(0) - \left[(2)(4) - (0)(0)\right] + (2)(1) - (-1)(0)$$

$$= -4 - 8 + 2$$

$$= -10$$

Replaced A's col 1 with B because k=1

↓

$$B_1 = \begin{bmatrix} 0 & 1 & 1 \\ 11 & -1 & 0 \\ 3 & 1 & 4 \end{bmatrix}$$

$$\det(B_1) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= (-1)\left[(11)(4) - (0)(3)\right] + \left[(11)(1) - (-1)(3)\right]$$

$$= -44 + 14$$

$$= -30$$

$$x_1 = \frac{\det(B_1)}{\det(A)} = \frac{-30}{-10} = 3$$

$$B_2 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 11 & 0 \\ 0 & 3 & 4 \end{bmatrix}$$

Now, I replaced A's col 2 because k=2.

$$\det(B_2) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \left[(11)(4) - (0)(3)\right] + \left[(2)(3) - (11)(0)\right]$$

$$= 44 + 6$$

$$= 50$$

$$x_2 = \frac{\det(B_2)}{\det(A)} = \frac{50}{-10} = -5$$

$$B_3 = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 11 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\det(B_3) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \left[(-1)(3) - (11)(1)\right] - \left[(2)(3) - (11)(0)\right]$$

$$= -20$$

This time, I replaced A's 3rd col with b because k=3.

$$x_3 = \frac{\det(B_3)}{\det(A)} = \frac{-20}{-10} = 2$$

$\therefore$ The answer is $\begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}$.

Booklet 7 Notes

Note: Since linear recurrence isn't on the exam, I will exclude them from here.

1. Let A be a $n \times n$ matrix. Let λ be a scalar. λ is the eigenvalue of A if there's a non-zero vector, $\vec{v} \in \mathbb{R}^n$ s.t. $A\vec{v} = \lambda\vec{v}$. $\vec{v}$ is the eigenvector of A.
I.e. Suppose we have a linear transformation that changes a vector by multiplying it to a scalar. The scalar is the eigenvalue and the vector is the eigenvector.

$A\vec{v} = \lambda\vec{v}$
$A\vec{v} - \lambda\vec{v} = 0$
$\vec{v}(A - \lambda I) = 0$

2. Let A be a $n \times n$ matrix. The characteristic polynomial of A is given by $P(\lambda) = \det|A - \lambda I|$. If λ is the eigenvalue of A, then $E_\lambda = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\}$ is the eigenspace of A.
E_λ = nullspace of $(A - \lambda I)$.

3. Let A be a $n \times n$ matrix. Let λ be the eigenvalue of A and let $\vec{v}$ be the eigenvector of A.

1. λ^k is an eigenvalue of A^k and $\vec{v}$ is an eigenvector of A^k corresponding to A^k, $k \geq 0$, $k \in \mathbb{Z}$.

2. A is invertible iff $\lambda \neq 0$.

3. If A is invertible, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} and $\vec{v}$ is an eigenvector of A^{-1} corresponding to $\frac{1}{\lambda}$.

4. If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, then the eigenvalues of T are the eigenvalues of its standard matrix rep.

4. Diagonizable Matrices

Let A and B be nxn matrices.

1. A is a diagonal matrix if all its entries are on its main diagonal and every other entry is 0.

E.g.
$$\begin{bmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{bmatrix}$$

2. A is diagonizable if $\exists$ an invertible nxn matrix, P, s.t. PAP^{-1} is a diagonal matrix.

3. Let $\vec{v_1}, \vec{v_2}, \dots \vec{v_k}$ be eigenvectors corresponding to the distinct eigenvalues $\lambda_1, \lambda_2, \dots \lambda_k$ of the square matrix, A. Then $\vec{v_1}, \vec{v_2}, \dots \vec{v_k}$ are lin indep.

5. 3 ways to tell if a matrix is diagonizable.

1. If A is symmetric, $A^T = A$, then A is diagonziable. This does not mean that if $A^T \neq A$, then A is not diagonizable.

2. Let A be a nxn matrix. Let $A \sim H$. If there is a pivot in every col of H, then A is diagonizable. Otherwise, A is not.

3. The algebraic multiplicity of $A \geq$ geometric multiplicity of A. Algebraic multiplicity is the number of times λ equals a value. Geometric multiplicity is the $\dim(E_\lambda)$, I.e. the nullspace of the eigenspace.

If geo = alge, then the matrix is diagonalizable. Otherwise, it's not.

6. If A, B are $n \times n$ matrices s.t. $B = PAP^{-1}$ for some invertible $n \times n$ matrix, P, then A and B are similar matrices.

Properties of Similar Matrices:

1. $\det(A) = \det(B)$
2. A is invertible iff B is invertible
3. $\text{rank}(A) = \text{rank}(B)$
4. $\text{Nullity}(A) = \text{Nullity}(B)$
5. $\det(A - \lambda I) = \det(B - \lambda I)$

7. If A is a $n \times n$ matrix similar to a diagonal matrix, D, s.t. $A = PDP^{-1}$, for some invertible matrix P, then $A^k = PD^kP^{-1}$.

8. Cayley - Hamiliton Theorem:

Let A be an $n \times n$ matrix with characteristic polynominal

$P(\lambda) = |A - \lambda I| = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_0$, then

$P(A) = a_n A^n + a_{n-1} A^{n-1} + \dots a_0 I = \vec{0}$

zero vector

9. If A is a $n \times n$ matrix with $\det(A - \lambda I) = \lambda^n + c_1 \lambda^{n-1} + \dots c_{n-1}\lambda + c_n$, where $c_n \neq 0$, then A is invertible and $A^{-1} = \frac{-1}{c_n}[A^{n-1} + c_1 A^{n-2} + \dots c_{n-1} I]$

Examples

1. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$

a) Find the characteristic polynominal of A

$P(\lambda) = |A - \lambda I|$

$= \left| \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right|$

$= \left| \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & -1 \\ 1 & 3 & -2-\lambda \end{bmatrix} \right|$

$= (2-\lambda) \begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix}$

$= (2-\lambda)\left[(2-\lambda)(-2-\lambda) - (3)(-1)\right]$

$= (2-\lambda)\left[(2-\lambda)(-2-\lambda) + 3\right]$

$= (2-\lambda)(\lambda^2-1)$

$= (2-\lambda)(\lambda-1)(\lambda+1)$

b) Find the eigenvalues of A

$0 = (2-\lambda)(\lambda-1)(\lambda+1)$

$\lambda = -1, 1, 2$

Because each value of λ occurs once, they all have an algs multiplicity of 1.

c) For each eigenvalue λ of A, find its eigenspace.

When $\lambda = -1$

$$\begin{bmatrix} 2-(-1) & 0 & 0 \\ 1 & 2-(-1) & -1 \\ 1 & 3 & -2-(-1) \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & -1 \\ 1 & 3 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The geometric multiplicity of the eigenspace of $\lambda = 1$ is 1, because there's 1 row of 0's.

geo = alge

Let $x_3 = s$

$3x_2 - x_3 = 0$

$3x_2 = x_3$

$x_2 = \frac{x_3}{3}$

/ Sp $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

$x_1 = 0$

When $\lambda = 1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 3 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Geo multi of eigenspace of $\lambda = 1$ is 1.

Geo = alge

Let $x_3 = s$

$x_1 = 0$

$x_2 - x_3 = 0$

$x_2 = x_3$

$$sp\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$$

When $\lambda = 2$

The resultant matrix is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Geo multi = alge multi = 1

Let $x_3 = s$ $\quad x_1 = x_3$

$x_2 = x_3$

$$sp\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

d) Find D and P such that $D = PAP^{-1}$

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

To find D, take the values of λ and form a diagonal matrix with them. The order doesn't matter.

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

To find P, take the eigenspaces of each λ and put them in the same col as you put the eigenvalues in D.

I.e. Say we got $\lambda = A, B, C$ and the eigenspace of A is $\vec{x}$, the eigenspace of B is $\vec{y}$ and the eigenspace of C is $\vec{z}$.

If $D = \begin{bmatrix} A & & 0 \\ & B & \\ 0 & & C \end{bmatrix}$, $P = \begin{bmatrix} | & | & | \\ \vec{x} & \vec{y} & \vec{z} \\ | & | & | \end{bmatrix}$

If $D = \begin{bmatrix} B & & 0 \\ & C & \\ 0 & & A \end{bmatrix}$, $P = \begin{bmatrix} | & | & | \\ \vec{y} & \vec{z} & \vec{x} \\ | & | & | \end{bmatrix}$

The order of the eigenspaces in P must correspond with the orders of its eigenvalue in D.

Other Facts.

1. Let A be a $n \times n$ matrix. Then, $AA^{-1} = I_{n \times n}$.

2. If you divide a matrix, you are multiplying by its inverse.
   E.g. $AB = C$
   $B = A^{-1}C$

3. Multiplying a matrix by the identity matrix gets you the same matrix.
   E.g. Let A be a $n \times n$ matrix.
   $AI_{n \times n} = A$

4. Let A be a $n \times n$ matrix.
   Let c be a constant.
   $\det(cA) = c^n \cdot \det(A)$
   E.g. Let A be a 3×3 matrix.
   Let $\det(A) = 2$
   $\det(3A) = 3^3 \cdot \det(A)$
   $= 27 \cdot 2$
   $= 54$

5. $\det(A^{-1}) = \dfrac{\det(I)}{\det(A)}$
   $= \dfrac{1}{\det(A)}$

E.g. If $\det(A) = 2$, find $\det(A^{-1})$
$\det(A \cdot A^{-1}) = \det(I) = 1$
$1 = \det(A) \cdot \det(A^{-1})$
$\dfrac{1}{\det(A)} = \det(A^{-1})$
$\dfrac{1}{2} = \det(A^{-1})$

6. A singular matrix is a matrix that is not invertible.
If A is a singular matrix, then $\det(A) = 0$.

7. To find the inverse image of a linear transformation, find the inverse of the standard matrix rep of that linear transformation.