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MATB42 Week 4-5 Notes

1. Fourier Series:

- For a differentiable function $f(x)$ defined on $(-\ell, \ell)$, the Full Fourier Series is given by:

$$f(x) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{n\pi x}{\ell}\right) + \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{\ell}\right)$$

$$= C_0 + \sum_{n=1}^{\infty} \left(C_n \cos\left(\frac{n\pi x}{\ell}\right) + D_n \sin\left(\frac{n\pi x}{\ell}\right)\right)$$

Note: I separated the "$n=0$"-th term from the first summation so that both summations start at 1 and go to infinity. That way, I can combine the 2 summations.

Note: The Fourier Sine and Cosine Series on $(0, \ell)$ are both special cases of the Full Fourier Series on $(-\ell, \ell)$.

Recall the Orthogonal Relations for Fourier Series:

1. $$\int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \cdot \sin\left(\frac{m\pi x}{\ell}\right) = \begin{cases} 0, & \text{if } m \neq n \\ \frac{\ell}{2}, & \text{if } m = n \end{cases}$$

2. $$\int_0^{\ell} \cos\left(\frac{n\pi x}{\ell}\right) \cdot \cos\left(\frac{m\pi x}{\ell}\right) = \begin{cases} 0, & \text{if } m \neq n \\ \frac{\ell}{2}, & \text{if } (m=n) \neq 0 \\ \ell, & \text{if } m = n = 0 \end{cases}$$

Now, we're on $(-\ell, \ell)$. Thus, the eqns change.

$$\int_{-\ell}^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \cdot \sin\left(\frac{m\pi x}{\ell}\right) = \begin{cases} 0, & \text{if } m \neq n \\ L, & \text{if } m = n \end{cases}$$

$$\int_{-\ell}^{\ell} \cos\left(\frac{n\pi x}{\ell}\right) \cdot \cos\left(\frac{m\pi x}{\ell}\right) = \begin{cases} 0, & \text{if } m \neq n \\ L, & \text{if } (m=n) \neq 0 \\ 2L, & \text{if } m=n=0 \end{cases}$$

$$\int_{-\ell}^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \cdot \cos\left(\frac{m\pi x}{\ell}\right) = 0 \text{ for all } m, n.$$

Note: The last one is the same as before. You can go to page 8 of my week 1 notes to see it listed under the Orthogonal Relations for Fourier Series.

Using the new formulas, we get:

1. $C_0 = \frac{1}{2L} \int_{-\ell}^{\ell} f(x)$

2. $C_n = \frac{1}{L} \int_{-\ell}^{\ell} f(x) \cos\left(\frac{n\pi x}{\ell}\right)$

3. $D_n = \frac{1}{L} \int_{-\ell}^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right)$

2. Parseval's Equality

- For notational purposes, let's write a Fourier Series using the following function:

$$f(x) = \sum_{n=0}^{\infty} a_k X_k(x), \quad a<x<b$$

- For the Fourier Sine Series, $X_k(x) = \sin\left(\frac{k\pi x}{l}\right)$.

- For the Fourier Cosine Series, $X_k(x) = \cos\left(\frac{k\pi x}{l}\right)$.

- For a Full Fourier Series, we have:

$a_0 = C_0, \quad X_0 = 1$

$a_1 = C_1, \quad X_1 = \cos\left(\frac{\pi x}{l}\right)$

$a_2 = D_1, \quad X_2 = \sin\left(\frac{\pi x}{l}\right)$

$a_3 = C_2, \quad X_3 = \cos\left(\frac{2\pi x}{l}\right)$

$a_4 = D_2, \quad X_4 = \sin\left(\frac{2\pi x}{l}\right)$

$\vdots$

- Parseval's Equality states that if $\int_a^b |f(x)|^2$ is finite, then

$$\int_a^b |f(x)|^2 = \sum_{n=0}^{\infty} |a_k|^2 \int_a^b |X_k(x)|^2$$

3. Integration of Fourier Series:

- We can integrate the Full Fourier Series term by term.

- More precisely, for a differentiable function $f(x)$ defined on $(-\ell, \ell)$, we can consider its anti-derivative, $F(x)$ where $F'(x) = f(x)$ and $F(0) = 0$. The integral of the Fourier Series would be exactly equal to $F(x)$.

$$f(x) = C_0 + \sum_{n=0}^{\infty} \left(C_n \cos\left(\frac{n\pi x}{\ell}\right) + D_n \sin\left(\frac{n\pi x}{\ell}\right) \right)$$

$$F(x) = C + C_0 x + \sum_{n=0}^{\infty} \left(\frac{C_n \ell}{n\pi} \sin\left(\frac{n\pi x}{\ell}\right) - \frac{D_n \ell}{n\pi} \cos\left(\frac{n\pi x}{\ell}\right) \right)$$

Note: C is the constant of integration and is equal to $\frac{1}{2\ell} \int_{-\ell}^{\ell} F(x)$.

- Similarly, we can integrate the Fourier Sine/Cosine series term by term.

- Note: The new series that results from term by term integration may not be the Fourier Series of the integral of the function.

Differentiation of Fourier Series

- In general, we cannot differentiate a Fourier Series.

- Consider the example of the Fourier Series of $f(x) = 1$ on $(0, \pi)$:

$$1 = \sum_{n \text{ odd}}^{\infty} \frac{4}{n\pi} \sin(nx)$$

If we attempt to take the derivative of the right, we get

$$\sum_{n \text{ odd}}^{\infty} \frac{4}{\pi} \cos(nx)$$

This new summation no longer convergences.

If we take the derivative again, we get

$$\sum_{n \text{ odd}}^{\infty} \frac{-4n}{\pi} \sin(nx)$$

This new summation increases as n increases.

- We can take the derivative of a Fourier Series if we assume a new condition.

– The condition: For a differentiable function $f(x)$ defined on $(-\ell, \ell)$ with $f(-\ell) = f(\ell)$, we can consider its derivative, $f'(x)$. The derivative of the Fourier Series would be exactly equal to the derivative, $f'(x)$.

$$f(x) = C_0 + \sum_{n=1}^{\infty} \left(C_n \cos\left(\frac{n\pi x}{\ell}\right) + D_n \sin\left(\frac{n\pi x}{\ell}\right)\right)$$

$$f'(x) = \sum_{n=1}^{\infty} \left(\frac{-C_n n\pi}{\ell} \sin\left(\frac{n\pi x}{\ell}\right) + \frac{D_n \pi x}{\ell} \cos\left(\frac{n\pi x}{\ell}\right)\right)$$

Note: For a function $f(x)$ defined on $(0, \ell)$ having a Fourier Sine or Cosine series, we must extend $f(x)$ onto $(-\ell, \ell)$ before using the result above.

Examples:

1. Find the Fourier Sine Series of $f(x)=1$ on $(0, \pi)$.

Soln:

Note: $\ell = \pi$ in this example

$f(x) = 1$ on $(0, \pi)$

$$1 = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\int_0^{\ell} \sin\left(\frac{m\pi x}{\ell}\right) = \sum_{n=1}^{\infty} A_n \int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \cdot \sin\left(\frac{m\pi x}{\ell}\right)$$

$$\int_0^{\ell} \sin(mx) = A_n \cdot \frac{\ell}{2}$$

$$\frac{2}{\pi} \int_0^{\ell} \sin(mx) = A_n$$

$$\frac{-2}{\pi m} \left[\cos(mx) \Big|_0^{\ell} \right] = A_n$$

$$\frac{-2}{\pi m} \left[\cos(m\pi) - 1 \right] = A_n$$

Note: $\cos(n\pi) = (-1)^n$

$$\frac{2}{\pi m} \left[1 - (-1)^m \right] = A_n$$

$$\therefore A_n = \begin{cases} 0, & \text{if } m \text{ is even} \\ \frac{4}{\pi m}, & \text{if } m \text{ is odd} \end{cases}$$

Note: $m=n$ here

$$1 = \frac{4}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \cdots\right)$$

when $x = \frac{\pi}{2}$:

$$1 = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \leftarrow$$ One of the most important series for π.

2. Find the Fourier Sine series for $f(x) = x$ on $-L \leq x \leq L$.

Soln:

$$x = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)$$

$$A_n = \frac{1}{L}\int_{-\ell}^{\ell} x \sin\left(\frac{m\pi x}{\ell}\right)$$

Let $u = \frac{m\pi x}{\ell} \longrightarrow$ $x = -\ell \rightarrow u = -m\pi$, $x = \ell \rightarrow u = m\pi$

$$x = \frac{u\ell}{m\pi}$$

$$du = \frac{m\pi}{\ell}\,dx$$

$$du \cdot \frac{\ell}{m\pi} = dx$$

$$A_n = \frac{\ell}{(m\pi)^2}\int_{-m\pi}^{m\pi} u \cdot \sin(u)$$

$$= \frac{\ell}{m^2\pi^2}\left[\sin(u) - u\cos(u) \Big|_{-m\pi}^{m\pi} \right]$$

$$= \frac{-\ell m\pi}{m^2\pi^2}\left[\cos(m\pi) - \cos(-m\pi)\right]$$

$$= \frac{-2\ell}{m\pi}\cos(m\pi)$$

$$= \frac{2\ell}{m\pi}(-1)^{m+1}$$

3. Find the Fourier Sine Series of $f(x) = x$ on $(0, \ell)$.

Soln:

$$x = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)$$

$$A_n = \frac{2}{L}\int_0^{\ell} x \sin\left(\frac{n\pi x}{\ell}\right)$$

let $u = \frac{n\pi x}{\ell} \longrightarrow$ $x = 0 \rightarrow u = 0$, $x = \ell \rightarrow u = n\pi$

$$x = \frac{u\ell}{n\pi}$$

$$du = \frac{n\pi}{\ell} \, dx$$

$$du \cdot \frac{\ell}{n\pi} = dx$$

$$An = \frac{2\ell^2}{\ell n^2 \pi^2} \int_0^{n\pi} u \sin(u)$$

$$= \frac{2\ell}{n^2\pi^2} \left[\sin(u) - u\cos(u) \Big|_0^{n\pi} \right]$$

$$= \frac{-2\ell}{n^2\pi^2} \left[(n\pi) \cos(n\pi) \right]$$

$$= \frac{-2\ell}{n\pi} \left[\cos(n\pi) \right]$$

$$= \frac{2\ell}{n\pi} \qquad (-1)^{n+1}$$

Let's plug $\frac{\ell}{2}$ for x.

$$\frac{\ell}{2} = \sum_{n=1}^{\infty} An \sin\left(\frac{n\pi}{2}\right)$$

When n is even, $\sin\left(\frac{n\pi}{2}\right) = 0$

Let $n = 2k+1$. (I.e. Make n odd)
when $n = 2k+1$, $\sin\left(\frac{n\pi}{2}\right) = \sin\left(\frac{(2k+1)\pi}{2}\right) = (-1)^k$

$$\frac{\ell}{2} = \sum_{k=0}^{\infty} A_{2k+1} \cdot \sin\left(\frac{(2k+1)\pi}{2}\right)$$

$$= \sum_{k=0}^{\infty} \underbrace{(-1)^{2k+1+1} \cdot \frac{2\ell}{(2k+1)\pi}}_{A_{2k+1}} \cdot \underbrace{(-1)^k}_{\sin\left(\frac{(2k+1)\pi}{2}\right)}$$

$$= \sum_{k=0}^{\infty} (-1)^{2k+2} \cdot \frac{2\ell}{(2k+1)\pi} \cdot (-1)^k$$

$$= \sum_{k=0}^{\infty} (-1)^k \cdot \frac{2\ell}{(2k+1)\pi}$$

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{(2k+1)}$$

$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$ $\leftarrow$ Saw this earlier

4. Integrate and find the Fourier Sine Series of $f(x) = x$ on $(0, \ell)$.

Soln:

$$x = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)$$

$$A_n = \frac{2}{\ell}\int_0^{\ell} x \sin\left(\frac{n\pi x}{\ell}\right)$$

Let $u = \frac{n\pi x}{\ell} \longrightarrow x = \frac{u\ell}{n\pi}$, $\quad x=0 \rightarrow u=0$
$\quad x=\ell \rightarrow u=n\pi$

$$du = \frac{n\pi}{\ell}\, dx$$

$$du \cdot \frac{\ell}{n\pi} = dx$$

$$A_n = \frac{2\ell^2}{n^2\pi^2\ell}\int_0^{n\pi} u\sin(u)$$

Let $x=u$
Let $v=\sin(u)$
$\int u\sin(u) = x\int v - \int x'\int v \leftarrow$ Integration by parts
$\quad = -u\cos(u) + \sin(u)$

$$A_n = \frac{2\ell}{(n\pi)^2}\left[\sin(u) - u\cos(u)\Big|_0^{n\pi}\right]$$

$$= \frac{2\ell}{(n\pi)^2}\Big(\sin(n\pi) - \sin(0) - \big((n\pi)\cos(n\pi) - 0\cos(0)\big)\Big)$$

$$= \frac{-2\ell}{(n\pi)^2}(n\pi)\cos(n\pi)$$

$$= \frac{2\ell}{n\pi}(-1)^{n+1}$$

$$\int x = c + \sum_{n=1}^{\infty} A_n \int \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\frac{x^2}{2} = c + \sum_{n=1}^{\infty} A_n \left(\frac{-\ell}{n\pi}\right) \cos\left(\frac{n\pi x}{\ell}\right)$$

$$x^2 = c - \sum_{n=1}^{\infty} A_n \left(\frac{2\ell}{n\pi}\right) \cos\left(\frac{n\pi x}{\ell}\right)$$

5. Integrate the Fourier Cosine Series of $f(x) = x^2$ on $(0, \ell)$ as an indefinite integral.

Soln:

$$x^2 = \sum_{n=0}^{\infty} B_n \cos\left(\frac{n\pi x}{\ell}\right)$$

$$= B_0 + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi x}{\ell}\right)$$

$$B_0 = \frac{1}{\ell} \int_0^{\ell} x^2$$

$$= \frac{1}{\ell} \left[\frac{x^3}{3} \Big|_0^{\ell} \right]$$

$$= \frac{1}{\ell} \left[\frac{\ell^3}{3} \right]$$

$$= \frac{\ell^2}{3}$$

$$B_n = \frac{2}{\ell}\int_0^{\ell} x^2 \cos\left(\frac{n\pi x}{\ell}\right)$$

Let $u = \frac{n\pi x}{\ell} \rightarrow x = \frac{u \cdot \ell}{n\pi}$, $\begin{array}{l} x=0 \rightarrow u=0 \\ x=\ell \rightarrow u=n\pi \end{array}$

$$du = \frac{n\pi}{\ell}\ dx$$

$$du \cdot \frac{\ell}{n\pi} = dx$$

$$B_n = \frac{2}{\ell}\int_0^{n\pi} \frac{\ell^3}{(n\pi)^3} \cdot u^2 \cos(u)$$

$$= \frac{2\ell^2}{(n\pi)^3}\int_0^{n\pi} u^2 \cos(u)$$

$$= \frac{2\ell^2}{(n\pi)^3}\left[(u^2-2)\sin(u) + 2u\cos(u) \Big|_0^{n\pi}\right]$$

$$= \frac{2\ell^2}{(n\pi)^3}\left[2(n\pi)\cos(n\pi)\right]$$

$$= \frac{4\ell^2}{(n\pi)^2}(-1)^n$$

$$x^2 = \frac{\ell^2}{3} + \sum_{n=1}^{\infty} \frac{4\ell^2}{(n\pi)^2}(-1)^n \cos\left(\frac{n\pi x}{\ell}\right)$$

Let's plug $x = \ell$.

$$\ell^2 = \frac{\ell^2}{3} + \sum_{n=1}^{\infty} \frac{4\ell^2}{(n\pi)^2} (-1)^n \cos(n\pi)$$

$$\ell^2 - \frac{\ell^2}{3} = \sum_{n=1}^{\infty} \frac{4\ell^2}{(n\pi)^2} (-1)^n (-1)^n$$

$$\frac{2\ell^2}{3} = \sum_{n=1}^{\infty} \frac{4\ell^2}{(n\pi)^2}$$

$$\frac{1}{6} = \sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} \leftarrow$$ This is the p-series when $p = 2$.

Note: A p-series follows the following form:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise.

Don't confuse p-series with geometric series.

$\sum \frac{1}{n^p} \leftarrow$ P-series

$\sum \left(\frac{1}{a}\right)^n \leftarrow$ Geometric Series

Let's plug in $x=0$.

$$0 = \frac{\ell^2}{3} + \sum_{n=1}^{\infty} \frac{4\ell^2}{(n\pi)^2} (-1)^n \cos(0)$$

$$-\frac{\ell^2}{3} = \sum_{n=1}^{\infty} \frac{4\ell^2}{(n\pi)^2} (-1)^n$$

$$-\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$ ← Alternating p-series

Now let's integrate x^2.

$$x^2 = \frac{\ell^2}{3} + \sum_{n=1}^{\infty} \frac{4\ell^2}{(n\pi)^2} (-1)^n \cos\left(\frac{n\pi x}{\ell}\right)$$

$$\int x^2 = C + \int \frac{\ell^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4\ell^2}{(n\pi)^2} \int \cos\left(\frac{n\pi x}{\ell}\right)$$

$$\frac{x^3}{3} = C + \frac{\ell^2 x}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4\ell^3}{(n\pi)^3} \sin\left(\frac{n\pi x}{\ell}\right)$$

$$x^3 = C + \ell^2 x + \sum_{n=1}^{\infty} (-1)^n \frac{12\ell^3}{(n\pi)^3} \sin\left(\frac{n\pi x}{\ell}\right)$$

If we take $n=0$, we get $C=0$. ← $C = \int_0^\ell x^3 \cdot \sin\left(\frac{0\pi x}{\ell}\right) = 0$

$$x^3 = \ell^2 x + \sum_{n=1}^{\infty} (-1)^n \frac{12\ell^3}{(n\pi)^3} \sin\left(\frac{n\pi x}{\ell}\right)$$

Note: The RHS of the last line on page 16 is not a Fourier Series. To change it to a Fourier Series, we need to convert x to its own corresponding Fourier Series.

6. Integrate the Fourier Sine Series of $f(x) = x^3$ on $(0, \ell)$ as an indefinite integral.

Soln:

$$x^3 = \ell^2 x + \sum_{n=1}^{\infty} (-1)^n \frac{12\ell^3}{(n\pi)^3} \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\int x^3 = c + \int \ell^2 x + \sum_{n=1}^{\infty} (-1)^n \frac{12\ell^3}{(n\pi)^3} \int \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\frac{x^4}{4} = c + \frac{\ell^2 x^2}{2} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{12\ell^4}{n^4 \pi^4} \cos\left(\frac{n\pi x}{\ell}\right)$$

$$x^4 = c + 2\ell^2 x^2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{48\ell^4}{n^4 \pi^4} \cos\left(\frac{n\pi x}{\ell}\right)$$

To find c, we multiply both sides by $\cos\left(\frac{0\pi x}{\ell}\right)$ and integrate from 0 to ℓ.

$$\int_0^\ell x^4 \cos(0) = \int_0^\ell c\cos(0) + \int_0^\ell 2\ell^2 x^2 \cos(0) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{48\ell^4}{(n\pi)^4} \int_0^\ell \cos\left(\frac{n\pi x}{\ell}\right)\cos(0)$$

$$\int_0^\ell x^4 = \int_0^\ell c + \int_0^\ell 2\ell^2 x^2$$

$$\frac{x^5}{5}\Big|_0^\ell = Cx\Big|_0^\ell + \frac{2\ell^2 x^3}{3}\Big|_0^\ell$$

$$\frac{\ell^5}{5} = C\ell + \frac{2\ell^5}{3}$$

$$\frac{\ell^5}{5} - \frac{2\ell^5}{3} = C\ell$$

$$C = \frac{3\ell^4 - 10\ell^4}{15}$$

$$= \frac{-7\ell^4}{15}$$

Now, we'll plug in $x = \ell$ to get the value of the p-series when $p=4$.

$$\ell^4 = \frac{-7\ell^4}{15} + 2\ell^4 + \sum_{n=1}^{\infty} \frac{48\ell^4}{(n\pi)^4} (-1)^{n+1} \cos(n\pi)$$

$$1 = \frac{-7}{15} + 2 + \sum_{n=1}^{\infty} \frac{48}{(n\pi)^4} (-1)^{n+1} (-1)^n$$

$$1 + \frac{7}{15} - 2 = \sum_{n=1}^{\infty} \frac{48}{(n\pi)^4} (-1)^{2n+1}$$

$$\frac{-8}{15} = \sum_{n=1}^{\infty} -\frac{48}{(n\pi)^4} \longrightarrow \frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

7. Find the Full Fourier Series of $f(x) = x^2$ on $(-\ell, \ell)$ and differentiate the series to get the Full Fourier Series of $f(x) = x$.

Soln:

The first thing to note is that x^2 is an even function. Even functions are always represented by cosine.

Proof:

$$f(x) = C_0 + \sum_{n=1}^{\infty} \left(C_n \cos\left(\frac{n\pi x}{\ell}\right) + D_n \sin\left(\frac{n\pi x}{\ell}\right) \right)$$

$$f(-x) = C_0 + \sum_{n=1}^{\infty} \left(C_n \cos\left(\frac{-n\pi x}{\ell}\right) + D_n \sin\left(\frac{-n\pi x}{\ell}\right) \right)$$

$$= C_0 + \sum_{n=1}^{\infty} \left(C_n \cos\left(\frac{n\pi x}{\ell}\right) - D_n \sin\left(\frac{n\pi x}{\ell}\right) \right)$$

$$f(x) = f(-x)$$

$$f(x) \rightarrow \cancel{C_0} + \cancel{\sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{\ell}\right)} + \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{\ell}\right) =$$

$$f(-x) \rightarrow \cancel{C_0} + \cancel{\sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{\ell}\right)} - \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{\ell}\right) = -\sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{\ell}\right)$$

$$2\sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{\ell}\right) = 0 \longrightarrow \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{\ell}\right) = 0$$

As you can see, when we have an even function, the sine terms add up to 0 and we're left with only the cosine terms.

Similarly, odd functions are always represented by sine.

Proof:

$$-f(x) = -C_0 - \sum_{n=1}^{\infty}\left(C_n \cos\left(\frac{n\pi x}{\ell}\right) + D_n \sin\left(\frac{n\pi x}{\ell}\right)\right)$$

$$f(-x) = C_0 + \sum_{n=1}^{\infty}\left(C_n \cos\left(\frac{-n\pi x}{\ell}\right) + D_n \sin\left(\frac{-n\pi x}{\ell}\right)\right)$$

$$= C_0 + \sum_{n=1}^{\infty}\left(C_n \cos\left(\frac{n\pi x}{\ell}\right) - D_n \sin\left(\frac{n\pi x}{\ell}\right)\right)$$

$$f(-x) = -f(x)$$

$$f(-x) \rightarrow C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{\ell}\right) - \cancel{\sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{\ell}\right)} =$$

$$-f(x) \rightarrow -C_0 - \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{\ell}\right) - \cancel{\sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{\ell}\right)}$$

$$-2\left(C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{\ell}\right)\right) = 0$$

$$C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{\ell}\right) = 0$$

As you can see, for odd functions, all the cosine terms add up to 0 and we're left with the sine terms only.

Recap:
1. An even function represented using the Fourier Series will have cosine terms only.

2. An odd function represented using the Fourier Series will have sine terms only.

∴ The Full Fourier Series of x^2 is:

$$\frac{\ell^2}{3} + \sum_{n=1}^{\infty} \frac{4\ell^2}{(n\pi)^2} (-1)^n \cos\left(\frac{n\pi x}{\ell}\right)$$

We found this on page 14.

Now, we'll differentiate it.

$$2x = \sum_{n=1}^{\infty} \frac{4\ell^2}{(n\pi)^2} (-1)^n (-1) \left(\frac{n\pi}{\ell}\right) \sin\left(\frac{n\pi x}{\ell}\right)$$

$$x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2\ell}{n\pi} \sin\left(\frac{n\pi x}{\ell}\right) \leftarrow$$ Found this on pg 8 and 9.

Now, using Parseval's Equality, we will find the value of the p-series with p=2 and p=4.

Recall: Parseval's Equality states that

$$\int_a^b |f(x)|^2 = \sum_{n=1}^{\infty} |a_k|^2 \int_a^b |X_k(x)|^2$$

In our case:

- $f(x) = x$
- $a_k = (-1)^{n+1} \left(\frac{2\ell}{n\pi}\right)$
- $X_k = \sin\left(\frac{n\pi x}{\ell}\right)$
- $a = -\ell$
- $b = \ell$

$$\int_{-\ell}^{\ell} x^2 = \sum_{n=1}^{\infty} \left((-1)^{n+1}\right)^2 \left(\frac{2\ell}{n\pi}\right)^2 \int_{-\ell}^{\ell} \sin^2\left(\frac{n\pi x}{\ell}\right)$$

$$\frac{x^3}{3}\Big|_{-\ell}^{\ell} = \sum_{n=1}^{\infty} \frac{4\ell^2}{n^2\pi^2} \cdot \ell$$

$$\frac{\ell^3 - (-\ell)^3}{3} = \sum_{n=1}^{\infty} \frac{4\ell^3}{n^2\pi^2}$$

$$\frac{2\ell^3}{3} = \sum_{n=1}^{\infty} \frac{4\ell^3}{n^2\pi^2}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$ ← Check pg 15