MATB61 Week 2 Notes

1. Linear Programming Theorem:

- Let f be a linear function. Let U be a non-empty region in R^2 s.t. U is defined by linear inequalities and it includes its boundaries.

  a) If U is bounded, then f has a max and a min on U and these values occur at corner points of U.

  b) If U is unbounded and if f has a max or min, then this occurs at a corner point of U.

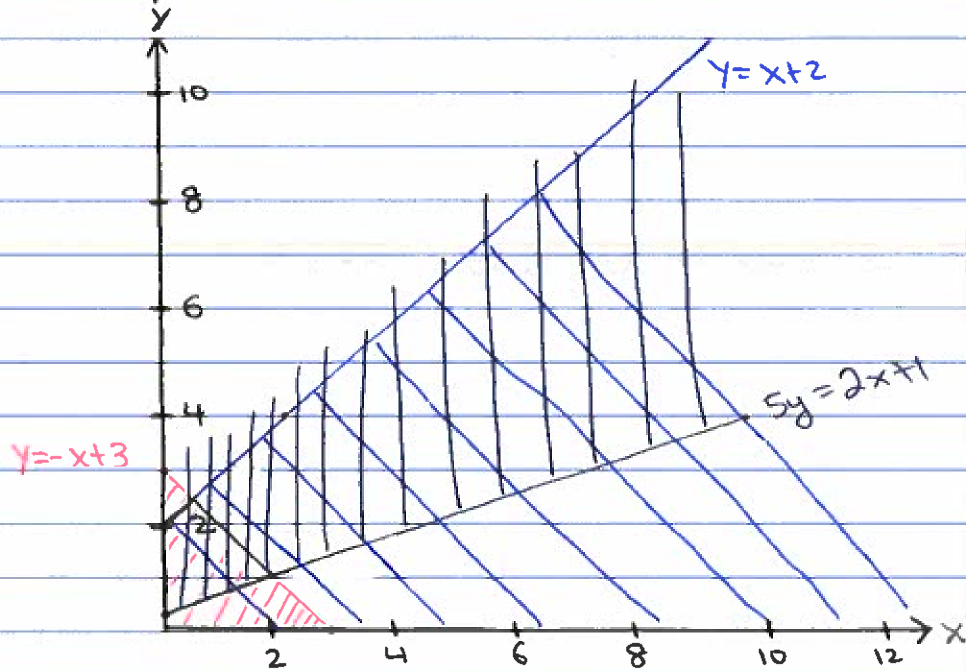
2. Graphic Soln of LP Problems:

a) Find the max and min of $f(x,y) = 3x+2y$ subject to the constraints:

1. $y \leq x+2$
2. $y \leq -x+3$
3. $5y \geq 2x+1$
4. $x \geq 0,\ y \geq 0$

Soln:

1. Graph the lines. Since $x \geq 0$ and $y \geq 0$, we just need the first quadrant.



2. Find the region that satisfies the first 3 constraints. I've shaded the region that satisfies each constraint and the region that has all 3 colours is the region that satisfies all 3 constraints. I've also highlighted the border in green.

3. Once you have the region, determine if its bounded or unbounded. In this case, it's bounded, so the max and min must be one of the 4 corner points, each.

4. Find all the corner points and plug them into the eqn to see which is the max and which is the min.

We can find 2 corner points by looking at the graph: $(0,2)$ and $(0,\frac{1}{3})$. We need to solve for the remaining ones.

1. $-x+3 = x+2$

$2x = 1$

$x = \frac{1}{2}$

$y = x+2$

$= \frac{1}{2}+2$

$= 5/2$

Corner Point: $(\frac{1}{2}, \frac{5}{2})$

2. $-x+3 = \frac{1}{3}(2x+1)$

$-5x+15 = 2x+1$

$-7x = -14$

$x = 2$

$Y = -x+3$

$= -2+3$

$= 1$

Corner Point: $(2,1)$

Now that we have all the corner points, we will plug them into $f(x,y)=3x+2y$ to find the max and min point.

$$f(0,2) = 3(0)+2(2)$$
$$=4$$

$$f(0,1/5) = 3(0)+2(1/5)$$
$$=2/5$$

$$f(\tfrac{1}{2},\tfrac{5}{2}) = 3(\tfrac{1}{2})+2(\tfrac{5}{2})$$
$$=\frac{3}{2}+5$$
$$=\frac{13}{2}$$

$$f(2,1) = 3(2)+2(1)$$
$$=6+2$$
$$=8$$

$\therefore$ $f(x,y)=3x+2y$ subject to the constraints

$y \le x+2$

$y \le -x+3$

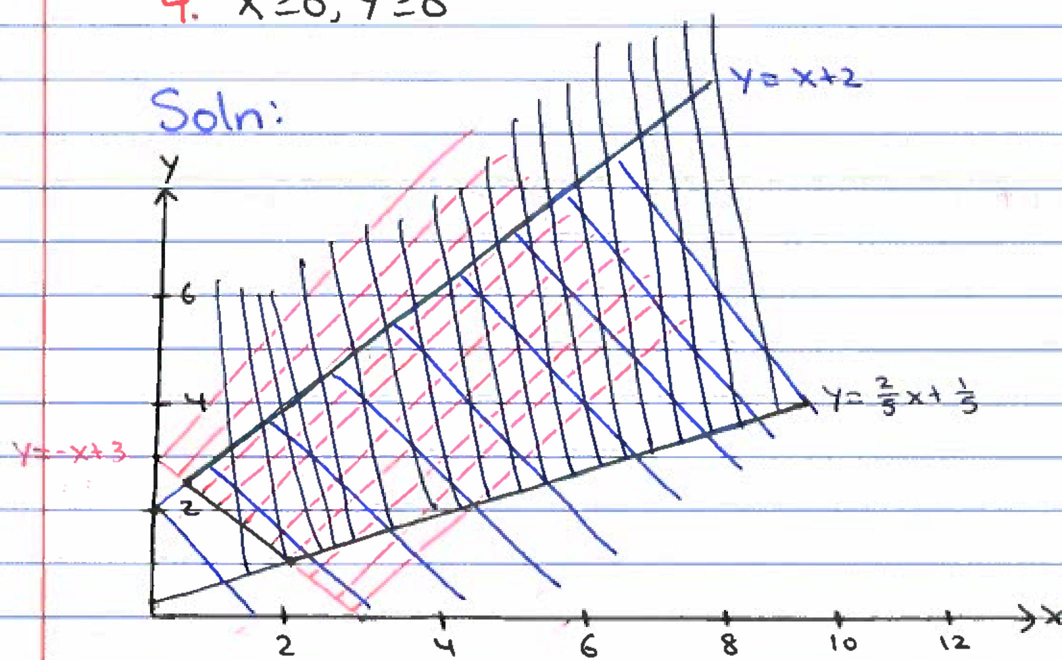
$y \ge \frac{2}{3}x + 1/5$

$x \ge 0,\ y \ge 0$

has a min of $2/5$ at $(0,1/5)$ and a max of 8 at $(2,1)$.

b) Find the min and max of
$f(x,y) = 3x+2y$ subject to
1. $y \leq x+2$
2. $y \geq -x+3$
3. $y \geq \frac{2}{5}x + \frac{1}{5}$
4. $x \geq 0, y \geq 0$

Soln:



Note that this region is unbounded. If the coefficients of the obj func are all positive, then an unbounded feasible region will have a min but no max. This is because the isoprofit lines increases in value and since the region is unbounded, there's no limit on how big it can get.

Isoprofit lines are parallel straight lines that represent constant values of the obj func.

To get the isoprofit lines, let p= 3x+2y, and then isolate y.

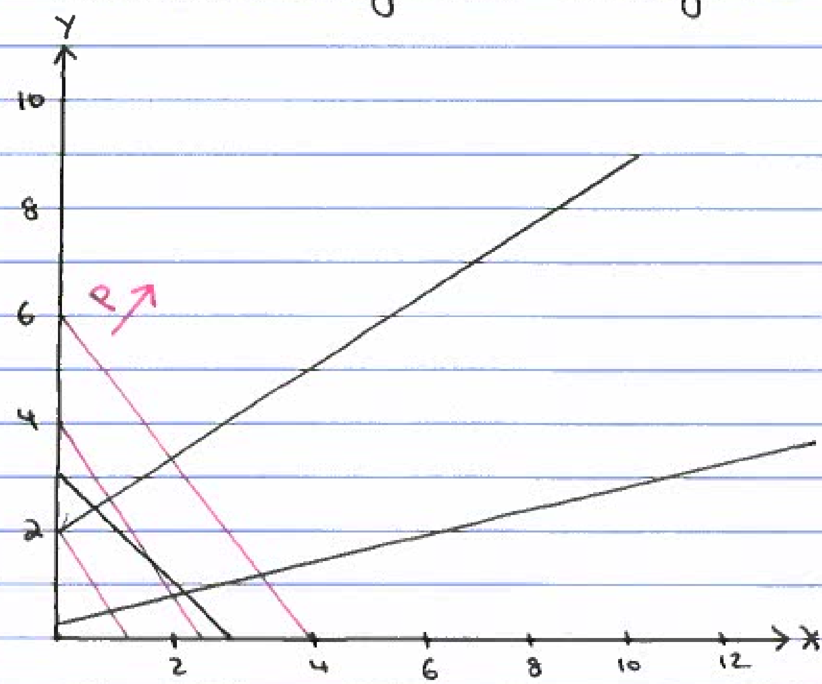
$$P = 3x + 2y$$
$$2y = -3x + P$$
$$y = \frac{-3x}{2} + \frac{P}{2}$$

If $P=2$, $Y = -\frac{3x}{2}$

If $P=10$, $Y = \frac{-3x}{2} + 5$

If $P=20$, $Y = \frac{-3x}{2} + 10$

Note: You must show a few isoprofit lines as well as the direction they are moving to.



To find the min value, we find all the corner points and plug them into the obj func. The 2 corner points are $(\frac{1}{2}, \frac{5}{2})$ and $(2,1)$.

$$f(\tfrac{1}{2}, \tfrac{5}{2}) = 3(\tfrac{1}{2}) + 2(\tfrac{5}{2}) = \tfrac{13}{2}$$

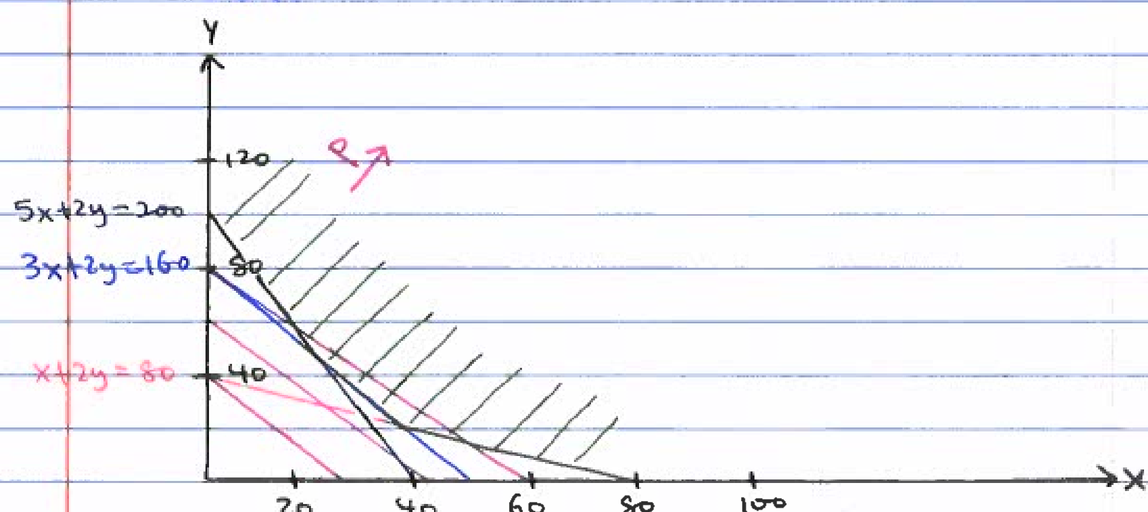
$$f(2,1) = 3(2) + 2(1) = 8$$

While there is no max value, the min value is $\frac{13}{2}$ at $(\frac{1}{2}, \frac{5}{2})$.

c) Find the min and max of $f(x,y) = 8x + 6y$ subject to

1. $3x + 2y \geq 160$
2. $5x + 2y \geq 200$
3. $x + 2y \geq 80$
4. $x \geq 0,\ y \geq 0$

Soln:



Once again, the feasible region is unbounded. Since the coefficients of the obj func are all positive, there is no max and only a min.

Isoprofit lines:

$P = 8x + 6y$

$6y = -8x + P$

$y = -\frac{4x}{3} + \frac{P}{6}$

$P = 0 \longrightarrow y = \frac{-4x}{3}$

$P = 6 \longrightarrow y = \frac{-4x}{3} + 1$

$P = 60 \longrightarrow y = \frac{-4x}{3} + 10$

To find the min value, find all corner points and plug them into the obj func.

Corner points: $(0,100)$, $(20,50)$, $(40,20)$, $(80,0)$.

$f(0,100) = 8(0) + 6(100)$
$= 600$

$f(20,50) = 8(20) + 6(50)$
$= 460$

$f(40,20) = 8(40)+6(20)$
$= 440$

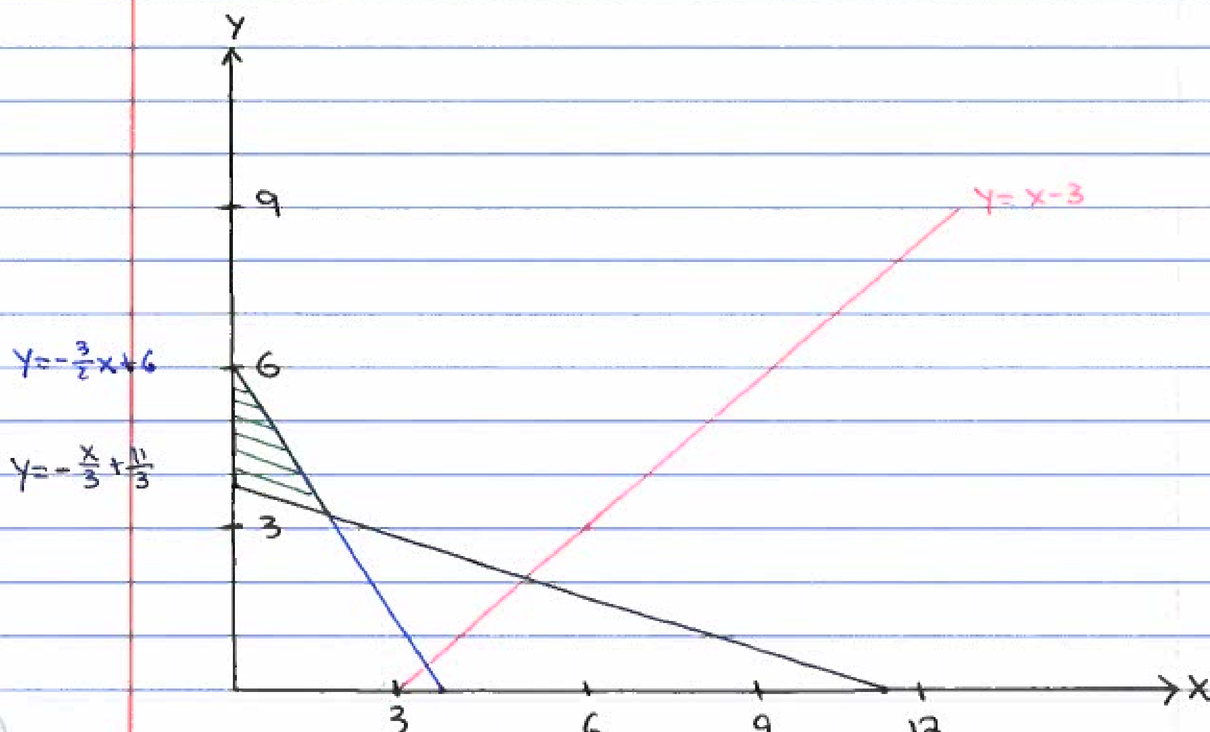
$f(80,0) = 8(80)+6(0)$
$= 640$

$\therefore$ $f(x,y) = 8x+6y$ has no max and has a min of 440 at $(40,20)$.

d) Min $f(x,y) = 3x+9y$ subject to

1. $y \le -\frac{3}{2}x+6$
2. $y \ge -\frac{1}{3}x+\frac{11}{3}$
3. $y \ge x-3$
4. $x \ge 0,\ y \ge 0$

Soln:



The corner pts are: $(0,6)$, $(0, \frac{11}{3})$ and $(2,3)$.

$f(0,6) = 54$
$f(0, \frac{11}{3}) = 33$
$f(2,3) = 33$

Any pt on the line $y = -\frac{1}{3}x + \frac{11}{3}$ btwn the pts $(0, \frac{11}{3})$ and $(2,3)$ gets the optimal soln.

Thm:
If (x_1, y_1) and (x_2, y_2) are 2 corner pts at which the obj func is optimum, then the obj func will also be optimum at all pts (x,y) where
$x = (1-t)x_1 + t x_2$
$y = (1-t)y_1 + t y_2$
$0 \le t \le 1$

$\therefore$ The optimal soln is $f(x,y) = 33$ at all points on the line $y = -\frac{1}{3}x + \frac{11}{3}$ btwn the points $(0, \frac{11}{3})$ and $(2,3)$.

3. Geometry of LP Problems:

a) Geometry of a constraint of a LP Problem:

- $a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq b_i$
  or $a^T x \leq b_i$ where
  $a^T = [a_{i1}, a_{i2}, \dots, a_{in}]$.

- The set of pts $x = (x_1, x_2, \dots, x_n)$ in R^n that satisfy this constraint is called a closed half-space.

- The set of points $x = (x_1, x_2, \dots, x_n)$ in R^n that satisfy $a^T x = b_i$ is called a hyperplane. A hyperplane is a boundary of a closed half-space.

- The set of feasible solns to a LP problem is the intersection of all the closed half-spaces determined by the constraints.

b) Geometry of the obj func:

- An obj func is $z = c^T x$.

- Let k be a constant. $c^T x = k$ is a hyperplane.

- Geometrically, the optimal Soln is the hyperplane that intersects the set of feasible solns and for which k is a max or min.

c) Geometry of the set of feasible Solns:

- Let x_1 and x_2 be feasible solns. The line segment connecting x_1 and x_2 is $\{x \in R^n \mid x = \lambda x_1 + (1-\lambda) x_2,\ 0 \le \lambda\ 1\}$.

- If $a^T x \le b_i$ is a constant of the problem and $a^T x_1 \le b_i$ and $a^T x_2 \le b_i$, for any interior pt of the line segment, then $x = \lambda x_1 + (1-\lambda) x_2$ should satisfy $a^T x \le b_i$.

Proof:

$$a^T x = \lambda a^T x_1 + (1-\lambda) a^T x_2, \quad \lambda \ge 0,\ (1-\lambda) \ge 0$$
$$\le \lambda b_i + (1-\lambda) b_i$$
$$= b_i$$

- If $c^T x_1 \le c^T x_2$, then $c^T x_1 \le c^T x \le c^T x_2$.

Proof:

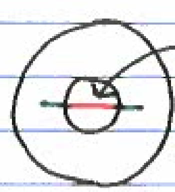
$$c^T x = \lambda c^T x_1 + (1-\lambda) c^T x_2$$
$$\ge \lambda c^T x_1 + (1-\lambda) c^T x_1$$
$$= c^T x_1$$

$$c^T x = \lambda c^T x_1 + (1-\lambda) c^T x_2$$
$$\le \lambda c^T x_2 + (1-\lambda) c^T x_2$$
$$= c^T x_2$$

- An interior point is a feasible soln but not an optimal soln.

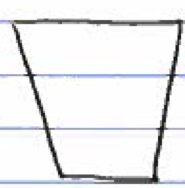
- A subset k of $\mathbb{R}^n$ is convex if for any $x_1, x_2 \in k$, $x = \lambda x_1 + (1-\lambda) x_2 \in k$.
I.e. k is convex if for any $x_1, x_2 \in k$, the line segment btwn x_1 and x_2 is included in k.

E.g.



Not on k.

is not convex.



is convex.

- Determine whether or not the below sets are convex:

a) A hyperplane $a_1x_1 + \dots + a_nx_n = b$

Yes, it is convex.

Proof:

$a^T x = b$

$\forall y_1, y_2 \in X,\ a^T y_1 = b$ and $a^T y_2 = b$

$y = \lambda y_1 + (1-\lambda) y_2,\ 0 \le \lambda \le 1$

$$a^T y = \lambda a^T y_1 + (1-\lambda) a^T y_2$$
$$= \lambda b + (1-\lambda) b$$
$$= b$$

$\therefore$ y is on the hyperplane.

b) A closed half-space $a_1x_1 + \dots + a_nx_n \le b$.

Yes, it is convex.

Proof:

$a^T x \le b$

$\forall y_1, y_2 \in X,\ a^T y_1 \le b,\ a^T y_2 \le b$

$y = \lambda y_1 + (1-\lambda) y_2,\ 0 \le \lambda \le 1$

$$a^T y = \lambda a^T y_1 + (1-\lambda) a^T y_2$$
$$\le \lambda b + (1-\lambda) b$$
$$= b$$

$\therefore$ y is on the hyperplane.

c) $\{\|x\| \ge 1 \mid x \in R^n\}$ and $\{\|x\| = 1 \mid x \in R^n\}$

No, it is not convex.

In R^2, $\|x\| \ge 1$ has a hole.

In R^2, $\|x\| = 1$ is just a boundary.

- Thm: The intersection of a finite collection of convex sets is convex.

  Proof:
  Let $k_1, k_2, \dots, k_n$ be convex sets.
  Let $k = \cap k_i$ and k is non-empty.
  Assume that $x, y \in k$.
  Since $k = \cap k_i$, $x, y \in k_i$.
  Since k_i is a convex set, this means that $\lambda x + (1-\lambda) y \in k_i$, $0 \le \lambda \le 1$.
  Then, this means that $\lambda x + (1-\lambda) y \in k$.
  $\therefore$ k is convex.

4. Extreme Point Thm:

- A point $x \in \mathbb{R}^n$ is a convex combination of the points $x_1, x_2, \dots, x_r$ in $\mathbb{R}^n$ if for some real numbers $c_1, c_2, \dots, c_r$ which satisfy $\sum_{i=1}^{r} c_i = 1$ and $c_i \ge 0$ for i in $1, 2, \dots, r$, we have $x = \sum_{i=1}^{r} c_i x_i$.

  I.e. A convex combination is a linear combination of points where all the coefficients are non-negative and sum to 1.

- A convex set is bounded if it can be enclosed in a rectangle $\{x \in \mathbb{R}^n \mid a_i \le x_i \le b_i\}$ in $\mathbb{R}^n$. Otherwise, it's unbounded.

- Thm: The set of all convex combinations of a finite set of points in R^n is a convex set.

Proof:

Let S be a set of finite points.

$S = \{x_i \mid i=1,2,\ldots,r\} \in R^n$

Let $k = \{x \mid x = c_1x_1 + \ldots + c_rx_r,\ c_i \geq 0,\ \sum_{i=1}^{r} c_i = 1,\ i=1,2,\ldots,r\}$

$\forall y_1, y_2 \in k,\ y_1 = a_1x_1 + \ldots + a_rx_r,\ a_i \geq 0,\ \sum_{i=1}^{r} a_i = 1$

$y_2 = b_1x_1 + \ldots + b_rx_r,\ b_i \geq 0,\ \sum_{i=1}^{r} b_i = 1$

Let $y = \lambda y_1 + (1-\lambda)y_2,\ 0 \leq \lambda \leq 1$

$= \lambda[a_1x_1 + \ldots + a_rx_r] + (1-\lambda)[b_1x_1 + \ldots + b_rx_r]$

$= (\lambda a_1 + (1-\lambda)b_1)x_1 + \ldots + (\lambda a_r + (1-\lambda)b_r)x_r$

As $\lambda, 1-\lambda, a_i, b_i \geq 0,\ \lambda a_i + (1-\lambda)b_i \geq 0$

$$\sum_{i=1}^{r} \lambda a_i + (1-\lambda)b_i$$

$$= \sum_{i=1}^{r} \lambda a_i + \sum_{i=1}^{r} (1-\lambda)b_i$$

$$= \lambda \underbrace{\sum_{i=1}^{r} a_i}_{1} + (1-\lambda) \underbrace{\sum_{i=1}^{r} b_i}_{1}$$

$= \lambda + (1-\lambda)$

$= 1$

$\therefore$ y is a convex comb of $x_1, \ldots, x_r$.

$\therefore$ $y \in k$

$\therefore$ k is a convex set.