MATB42 Week 3 Notes

1. Review of Complex Numbers:

- The standard form of a complex num is $a+ib$ where a and b are both real numbers. a is called the real part and b is called the imaginary part.

- $i^2 = -1 \Longleftrightarrow i = \sqrt{-1}$

- Let $z = a+ib$. The conjugate of z, denoted as $\bar{z}$, is $a-ib$.
$\bar{z} = a - ib$

- Let $z_1 = a+ib$ and $z_2 = c+id$. Then:

a) $z_1 \pm z_2$
$= (a+ib) \pm (c+id)$
$= (a \pm c) + i(b \pm d)$

E.g. Calculate $(1+3i) + (3+5i)$
Soln:
$(1+3) + (3+5)i$
$= 4+8i$

E.g. Calculate $(-4+7i) + (5-10i)$
Soln:
$(-4+5) + (7-10)i$
$= 1-3i$

E.g. Calculate $(4+12i) - (3-15i)$
Soln:
$(4-3) + (12-(-15))i$
$= 1+27i$

b) $z_1 \cdot z_2$

$= (a+ib)\cdot(c+id)$

$= ac + iad + ibc + i^2 bd$

$= (ac-bd) + i(ad+bc)$

E.g. Calculate $(1+3i)\cdot(2+2i)$

Soln:

$(1+3i)\cdot(2+2i)$

$= 2 + 2i + 6i - 6$

$= -4 + 8i$

E.g. Calculate $(1-5i)\cdot(-9+2i)$

Soln:

$(1-5i)(-9+2i)$

$= -9 + 2i + 45i + 10$

$= 1 + 47i$

c) $z \cdot \bar{z} = |z|^2$

Proof:

$z \cdot \bar{z}$

$= (a+ib)(a-ib)$

$= a^2 - iab + iab + b^2$

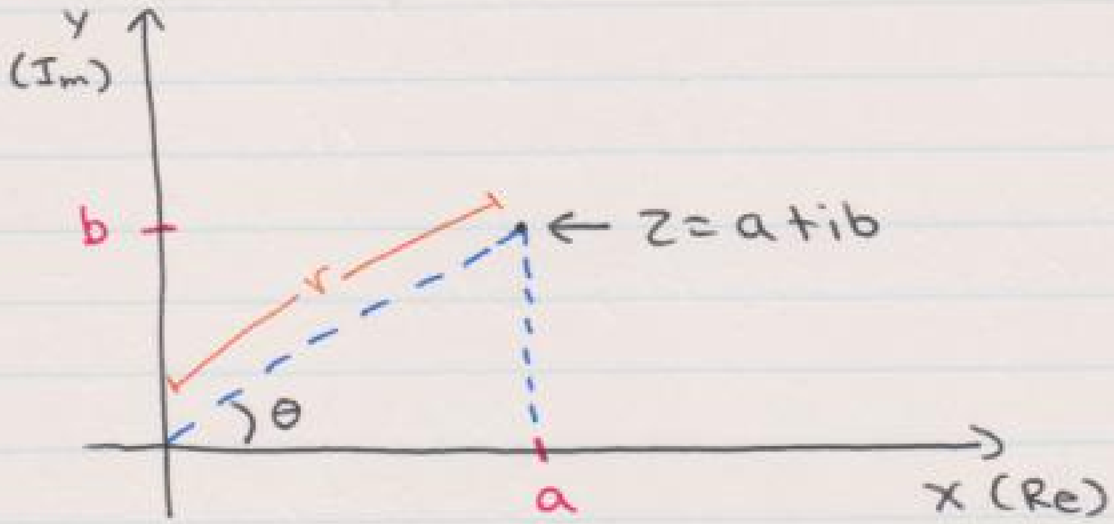
$= a^2 + b^2$

$|z| = \sqrt{a^2+b^2}$

$|z|^2 = a^2+b^2$

$\therefore\ z \cdot \bar{z} = |z|^2$

– Let $z = a + ib$. We will now graph z.

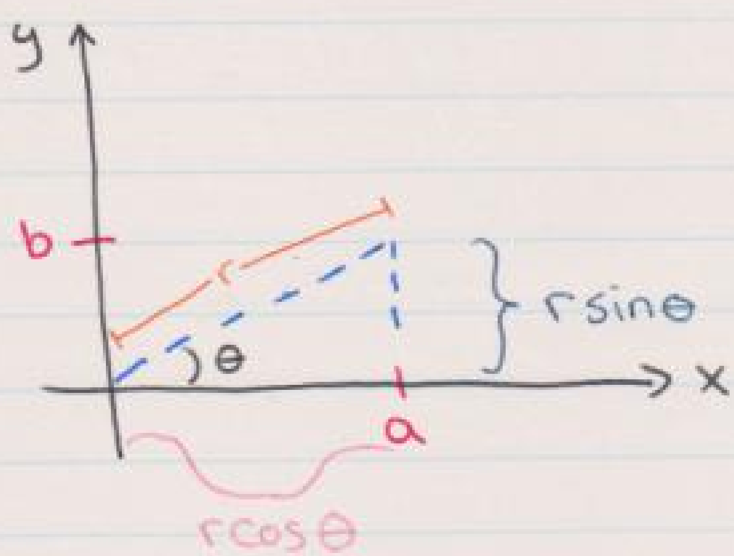


The x-axis denotes the real part of the complex number while the y-axis denotes the imaginary part.

r is the magnitude of the complex number.
I.e. $r = |z|$
$= \sqrt{a^2 + b^2}$

$r^2 = a^2 + b^2$ ← From Pythagorean Thm

Note: Another way to think about this is through polar coordinates.



$$\begin{aligned} z &= a+ib \\ &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

Polar Form of z

$$\left.\begin{aligned} a &= r\cos\theta \\ b &= r\sin\theta \end{aligned}\right\} \quad \begin{aligned} a^2+b^2 &= r^2\cos^2\theta + r^2\sin^2\theta \\ &= r^2(\cos^2\theta + \sin^2\theta) \\ &= r^2 \end{aligned}$$

2. Schrödinger Equation

- The PDE $i\hbar \psi_t = -\frac{\hbar^2}{2m}\psi_{xx} + V\psi$

is called the time dependent Schrödinger Equation.

$\psi(x,t)$ is an unknown complex function, which somewhat describes the position of a particle in Quantum Mechanics.

i is the complex number, with $i^2 = -1$.

$\hbar$, called h bar, is a physical constant, approximately 10^{-34}.

m is a constant, being the mass of the particle.

$V = V(x)$ is the potential function which only depends on x.

Given a complicated V, this eqn is almost impossible to solve. Thus, we will take $V(x)$ to be of the form:

$$V(x) = \begin{cases} 0, & \text{if } 0<x<\ell \\ \infty, & \text{otherwise} \end{cases}$$

This means that the particle will be allowed anywhere with 0 and ℓ but will not be allowed outside the domain.

With this choice of V, we can simplify the PDE to $i\bar{h}\,\psi_t = -\frac{\bar{h}^2}{2m}\psi_{xx}$.

- We aim to solve $\psi(x,t)$ in the finite domain $0<x<\ell$.

- $\psi(x,t)$ is called the wave function. In general, it is a function that involves complex numbers. $\psi(x,t)$ describes the position of the particle as it is the probability amplitude for the position of the particle. Furthermore, $|\psi(x,t)|^2$ is the probability density for the position of the particle.

$\int_a^b |\psi(x,t)|^2\,dx =$ The prob of finding the particle between a and b at time t.

where $|\psi|^2 = \underbrace{\bar{\psi}}_{\text{Complex conjugate}} \cdot \psi$

- Since we said that the particle must be in the domain $0<x<\ell$, $\psi(x,t)=0$ outside the domain.

$\int_{-\infty}^{0} |0|^2 = 0, \quad \int_{\ell}^{\infty} |0|^2 = 0$

The probability of finding the particle outside the domain is 0.

The boundary conditions are:

1. $\psi(0,t)=0$
2. $\psi(\ell,t)=0$

The initial condition is $\psi(x,0)=\phi(x)$, where $\phi(x)$ describes the initial state of the particle at time $t=0$.

- Since $|\psi(x,t)^2|$ is a probability density, it must be normalized.

I.e.

$$\int_0^\ell |\psi(x,t)|^2 = 1 \xrightarrow[t=0]{} \int_0^\ell |\phi(x)|^2 = 1$$

- We will use Separation of Variable now.

Assume that $\psi(x,t) = X(x)\cdot T(t)$.

Then, the PDE becomes $i\hbar X\cdot T' = \frac{-\hbar^2}{2m} X''\cdot T$.

After dividing XT on both sides, the PDE becomes

$$i\hbar \frac{T'}{T} = -\frac{\hbar^2}{2m}\frac{X''}{X} = E$$

By convention, we set the separation constant to be E, where $E>0$. E is the energy of the soln.

We can split the above eqn into 2 eqns:

1. $i\hbar \frac{T'}{T} = E$

$$T' = \frac{ET}{i\hbar}$$

$$= \frac{ET}{i\hbar} \cdot \frac{i}{i}$$

$$= -\frac{iET}{\hbar}$$ Recall: $i^2 = -1$

$$T' + \frac{iET}{\hbar} = 0$$

$$T(t) = Ae^{-\frac{iEt}{\hbar}}$$

2. $-\frac{\hbar^2}{2m} \frac{X''}{X} = E$

$$X'' = -\frac{2mEX}{\hbar^2}$$

$$X'' + \frac{2mEX}{\hbar^2} = 0$$

$$X(x) = C \cdot \cos\left(\sqrt{\frac{2mE}{\hbar^2}}\, x\right) + D \cdot \sin\left(\sqrt{\frac{2mE}{\hbar^2}}\, x\right)$$

Note: The soln for T is sometimes called the time evolution term. It is a complex exponential.
Recall that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ (Euler Formula)
This means that T(t) is sinusoidal, so $\psi(x,t)$ oscillates as time goes on, hence the name "wave function."

Now, we plug in the boundary condition.

$$\psi(0,t)=0 \rightarrow X(0)\cdot T(t)=0 \rightarrow X(0)=0$$

$$X(0) = C\cdot\cos(0) + D\cdot\sin(0)$$
$$= C$$

$$X(0) = 0$$
$$\therefore C = 0$$

$$\psi(\ell,t)=0 \rightarrow X(\ell)\cdot T(t)=0 \rightarrow X(\ell)=0$$
$$X(\ell) = D\sin\left(\sqrt{\frac{2mE}{\hbar^2}}\,\ell\right)=0$$

If $D=0$, we get the trivial soln, which we don't want. Furthermore, it can't satisfy $\int_0^\ell |\psi(x,t)|^2=1$.

Hence, $\sin\left(\sqrt{\frac{2mE}{\hbar^2}}\,\ell\right)=0$

$$\rightarrow \sqrt{\frac{2mE}{\hbar^2}}\,\ell = n\pi,\ n>0$$

$$\rightarrow \sqrt{\frac{2mE}{\hbar^2}} = \frac{n\pi}{\ell}$$

$$\rightarrow E = \frac{n^2\pi^2\hbar^2}{2m\ell^2}$$

Side note: If $D=0$, then $\psi(x,t)=0$ for all t. Therefore, $\int_0^\ell |\psi(x,t)|^2 \neq 1$.

For each value of n, we get a soln. To get the general soln, we need a linear combination of the solns.

$$\psi_n(x,t) = A_n \sin\left(\frac{n\pi x}{\ell}\right)e^{-\frac{iE_nt}{\hbar}} \leftarrow \text{Individual Soln}$$

$$\psi(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)e^{-\frac{iE_nt}{\hbar}} \leftarrow \text{General Soln}$$

Now, we will apply the initial condition to solve for An.

Recall: $\phi(x) = \psi(x,0)$

$$\psi(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right) e^0$$

$$= \underbrace{\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)}_{\text{Fourier (Sine) Series of } \phi(x)} \quad \text{Note: } e^0 = 1$$

Hence, $A_n = \frac{2}{\ell}\int_0^{\ell} \phi(x) \sin\left(\frac{n\pi x}{\ell}\right)$

Note: Since $\phi(x)$ may be complex, An may also be complex.

Examples

Define the expected value, avg position of the particle, at time t to be

$$\langle x \rangle = \int_0^\ell \overline{\psi(x,t)}\, x\, \psi(x,t)\, dx$$

$$= \int_0^\ell x\, |\psi(x,t)|^2\, dx$$

Given $\phi(x)$, find:

- The real normalization constant N
- $\psi(x,t)$ and $|\psi(x,t)|^2$
- $\langle x \rangle$ for all time
- Does $\langle x \rangle$ oscillate in time? If so, at what frequency?

a) $\phi(x) = N \sin\left(\frac{n\pi x}{\ell}\right)$, the n^{th} stationary state.

Soln:

We know that $\int_0^\ell |\phi(x)|^2 = 1$.

$$1 = \int_0^\ell \left|N \sin\left(\frac{n\pi x}{\ell}\right)\right|^2$$

$$= \int_0^\ell |N|^2 \left|\sin\left(\frac{n\pi x}{\ell}\right)\right|^2$$

$$= \int_0^\ell N^2 \sin^2\left(\frac{n\pi x}{\ell}\right)$$

Note: We assume that $N \in \mathbb{R}$ and we know $\sin(\theta) \in \mathbb{R}$, so $|N|^2 = N^2$ and $\left|\sin\left(\frac{n\pi x}{\ell}\right)\right|^2 = \sin^2\left(\frac{n\pi x}{\ell}\right)$.

$$= N^2 \int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \cdot \sin\left(\frac{n\pi x}{\ell}\right)$$

$$= \frac{N^2 \cdot \ell}{2}$$

$$N^2 = \frac{2}{\ell}$$

$$N = \sqrt{\frac{2}{\ell}}$$

$$\phi(x) = \sqrt{\frac{2}{\ell}} \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\psi(x,t) = \sqrt{\frac{2}{\ell}} \sin\left(\frac{n\pi x}{\ell}\right) e^{-\frac{iE_n t}{\hbar}}$$

$$|\psi(x,t)|^2 = \left| \sqrt{\frac{2}{\ell}} \sin\left(\frac{n\pi x}{\ell}\right) e^{\frac{-iE_n t}{\hbar}} \right|^2$$

$$= \frac{2\sin^2\left(\frac{n\pi x}{\ell}\right)}{\ell} \underbrace{\left| e^{\frac{-iE_n t}{\hbar}} \right|^2}_{1}$$

Recall: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$,
$|\cos(\theta) + i\sin(\theta)|^2 = \cos^2(\theta) + \sin^2(\theta)$
$= 1$

$$|\psi(x,t)|^2 = \frac{2}{\ell} \sin^2\left(\frac{n\pi x}{\ell}\right)$$

Note that the value of $|\psi(x,t)|^2$ is independent of t. This sort of means that the particle does not move. This is why $\Phi(x)$ is called the n^{th} stationary state.

$$\langle X \rangle = \int_0^{\ell} x\,|\psi(x,t)|^2$$

$$= \int_0^{\ell} x\left(\frac{2}{\ell}\right)\sin^2\left(\frac{n\pi x}{\ell}\right)$$

Let $u = \frac{n\pi x}{\ell}$ ← u-sub

$$du = \frac{n\pi}{\ell}\,dx$$

$$dx = \frac{du}{\frac{n\pi}{\ell}}$$

$$= du \cdot \frac{\ell}{n\pi}$$

Furthermore, $x = \frac{u\ell}{n\pi}$

Lastly, $x = 0 \rightarrow u = 0$
$x = \ell \rightarrow u = n\pi$

$$= \int_0^{n\pi}\left(\frac{u\ell}{n\pi}\right)\left(\frac{2}{\ell}\right)\left(\sin^2(u)\right)\left(\frac{\ell}{n\pi}\right)du$$

$$= \int_0^{n\pi} \left(\frac{\cancel{2}\ell}{n^2\pi^2}\right)(u)\underbrace{\left(\frac{1-\cos(2u)}{\cancel{2}}\right)}_{\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}} du$$

$$= \frac{\ell}{n^2\pi^2}\int_0^{n\pi} u - u\cos(2u)\, du$$

$$= \frac{\ell}{n^2\pi^2}\left[\frac{u^2}{2}\Big|_0^{n\pi} - \frac{2u\sin(2u)+\cos(2u)}{4}\Big|_0^{n\pi}\right]$$

$$= \frac{\ell}{n^2\pi^2}\left[\frac{n^2\pi^2}{2} - \left(\frac{\overbrace{2n\pi\sin(2n\pi)}^{0} + \overbrace{\cos(2n\pi)}^{1}}{4} - \frac{\cos(0)}{4}\right)\right]$$

$$= \frac{\ell}{n^2\pi^2}\left[\frac{n^2\pi^2}{2} - \left(\frac{1}{4} - \frac{1}{4}\right)\right]$$

$$= \frac{\ell}{\cancel{n^2\pi^2}}\left(\frac{\cancel{n^2\pi^2}}{2}\right)$$

$$= \frac{\ell}{2}$$

Note: For the stationary states, the expected value of x is the center of the box for all time.

Note: $\int u\cos(2u)$ can be solved using integration by parts.

Let $x = u$

Let $v = \cos(2u)$

$x\int v - \int x' \int v$

$= u\int\cos(2u) - \int u' \int \cos(2u)$

$= \frac{u\sin(2u)}{2} - \int \frac{\sin(2u)}{2}$

$= \frac{u\sin(2u)}{2} + \frac{\cos(2u)}{4}$

$= \frac{2u\sin(2u)+\cos(2u)}{4}$

b) $\phi(x) = N\left(\sin\left(\frac{\pi x}{\ell}\right) + \sin\left(\frac{2\pi x}{\ell}\right)\right)$

Soln:

$$1 = \int_0^{\ell} |\phi(x)|^2$$

$$= \int_0^{\ell} N^2 \left(\sin\left(\frac{\pi x}{\ell}\right) + \sin\left(\frac{2\pi x}{\ell}\right)\right)^2$$

$$= N^2 \int_0^{\ell} \underbrace{\sin^2\left(\frac{\pi x}{\ell}\right)}_{\frac{\ell}{2}} + 2\underbrace{\sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{2\pi x}{\ell}\right)}_{0} + \underbrace{\sin^2\left(\frac{2\pi x}{\ell}\right)}_{\frac{\ell}{2}}$$

$$= N^2 \ell$$

$$N^2 = \frac{1}{\ell}$$

$$N = \sqrt{\frac{1}{\ell}}$$

$$\phi(x) = \sqrt{\frac{1}{\ell}}\left(\sin\left(\frac{\pi x}{\ell}\right) + \sin\left(\frac{2\pi x}{\ell}\right)\right)$$

$$\psi(x,t) = \sqrt{\frac{1}{\ell}}\left(\sin\left(\frac{\pi x}{\ell}\right)e^{\frac{-iE_1 t}{\hbar}} + \sin\left(\frac{2\pi x}{\ell}\right)e^{\frac{-iE_2 t}{\hbar}}\right)$$

$$|\psi(x,t)|^2 = \frac{1}{\ell}\left(\sin\left(\frac{\pi x}{\ell}\right)e^{\frac{iE_1 t}{\hbar}} + \sin\left(\frac{2\pi x}{\ell}\right)e^{\frac{iE_2 t}{\hbar}}\right)$$
$$\left(\sin\left(\frac{\pi x}{\ell}\right)e^{\frac{-iE_1 t}{\hbar}} + \sin\left(\frac{2\pi x}{\ell}\right)e^{\frac{-iE_2 t}{\hbar}}\right)$$

$$\langle x \rangle = \int_0^{\ell} x \, |\psi(x,t)|^2$$

$$= \int_0^{\ell} x \left| \frac{1}{\sqrt{\ell}} \left(\sin\left(\frac{\pi x}{\ell}\right) e^{\frac{-iE_1 t}{\hbar}} + \sin\left(\frac{2\pi x}{\ell}\right) e^{\frac{-iE_2 t}{\hbar}} \right) \right|^2$$

$$= \frac{1}{\ell} \int_0^{\ell} x \left(\sin\left(\frac{\pi x}{\ell}\right) e^{\frac{iE_1 t}{\hbar}} + \sin\left(\frac{2\pi x}{\ell}\right) e^{\frac{iE_2 t}{\hbar}} \right) \left(\sin\left(\frac{\pi x}{\ell}\right) e^{\frac{-iE_1 t}{\hbar}} + \sin\left(\frac{2\pi x}{\ell}\right) e^{\frac{-iE_2 t}{\hbar}} \right)$$

Note: $|z|^2 = \bar{z} \cdot z$, where z is a complex number.

$$= \frac{1}{\ell} \int_0^{\ell} x \Bigg(\sin^2\left(\frac{\pi x}{\ell}\right) + \sin^2\left(\frac{2\pi x}{\ell}\right) + \sin\left(\frac{\pi x}{\ell}\right) e^{\frac{iE_1 t}{\hbar}} \sin\left(\frac{2\pi x}{\ell}\right) e^{\frac{-iE_2 t}{\hbar}} + \sin\left(\frac{2\pi x}{\ell}\right) e^{\frac{iE_2 t}{\hbar}} \sin\left(\frac{\pi x}{\ell}\right) e^{\frac{-iE_1 t}{\hbar}} \Bigg)$$

Note: $e^{x} \cdot e^{-x} = e^{x+(-x)} = e^0 = 1$

$$= \frac{1}{\ell} \int_0^{\ell} x \Bigg(\sin^2\left(\frac{\pi x}{\ell}\right) + \sin^2\left(\frac{2\pi x}{\ell}\right) + \sin\left(\frac{\pi x}{\ell}\right) \sin\left(\frac{2\pi x}{\ell}\right) e^{\frac{i(E_1 - E_2)t}{\hbar}} + \sin\left(\frac{\pi x}{\ell}\right) \sin\left(\frac{2\pi x}{\ell}\right) e^{\frac{-i(E_1 - E_2)t}{\hbar}} \Bigg)$$

Note: $\cos(\theta) = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

We can factor $\sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{2\pi x}{\ell}\right)$ in the last 2 terms in the bracket. Then, if we substitute $\frac{(E_1-E_2)t}{\hbar}$ as θ, we get

$$\sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{2\pi x}{\ell}\right)\underbrace{\left(e^{i\theta}+e^{-i\theta}\right)}_{2\cos(\theta)}$$

$$=\frac{1}{\ell}\int_0^\ell x\left(\sin^2\left(\frac{\pi x}{\ell}\right)+\sin^2\left(\frac{2\pi x}{\ell}\right)+\sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{2\pi x}{\ell}\right)\underbrace{\left(e^{\frac{i(E_1-E_2)t}{\hbar}}+e^{\frac{-i(E_1-E_2)t}{\hbar}}\right)}_{2\cos\left(\frac{(E_1-E_2)t}{\hbar}\right)}\right)$$

$$=\frac{1}{\ell}\int_0^\ell x\left(\sin^2\left(\frac{\pi x}{\ell}\right)+\sin^2\left(\frac{2\pi x}{\ell}\right)+\sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{2\pi x}{\ell}\right)\left(2\cos\left(\frac{(E_1-E_2)t}{\hbar}\right)\right)\right)$$

$$=\frac{1}{\ell}\left[\underline{\int_0^\ell x\sin^2\left(\frac{\pi x}{\ell}\right)}+\underline{\int_0^\ell x\sin^2\left(\frac{2\pi x}{\ell}\right)}+\underline{\int_0^\ell x\sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{2\pi x}{\ell}\right)2\cos\left(\frac{(E_1-E_2)t}{\hbar}\right)}\right]$$

$$\int_0^\ell x\sin^2\left(\frac{\pi x}{\ell}\right)$$

Let $u=\frac{\pi x}{\ell}\longrightarrow x=\frac{u\cdot\ell}{\pi}$

$du=\frac{\pi}{\ell}\,dx$

$du\cdot\frac{\ell}{\pi}=dx$

$$\int_0^\ell \frac{u\ell}{\pi}\sin^2(u)\frac{\ell}{\pi}$$

$$=\int_0^\ell \frac{\ell^2}{\pi^2}u\sin^2(u)$$

$$=\frac{\ell^2}{\pi^2}\int_0^\ell u\sin^2(u)$$

$$=\frac{\ell^2}{\pi^2}\int_0^\ell u\left(\frac{1-\cos(2u)}{2}\right)$$

$$=\frac{\ell^2}{\pi^2}\int_0^\ell \frac{u}{2}-\frac{u\cos(2u)}{2}$$

$$=\frac{\ell^2}{\pi^2}\left[\int_0^\ell \frac{u}{2}-\int_0^\ell \frac{u\cos(2u)}{2}\right]$$

$$=\frac{\ell^2}{\pi^2}\left[\frac{u^2}{4}\Big|_0^\ell-\frac{2u\sin(2u)+\cos(2u)}{8}\Big|_0^\ell\right]$$

Before plugging in 0 and ℓ, we need to change u back to $\frac{\pi x}{\ell}$.

$$=\frac{\ell^2}{\pi^2}\left[\frac{\left(\frac{\pi x}{\ell}\right)^2}{4}\Big|_0^\ell-\frac{2\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{2\pi x}{\ell}\right)+\cos\left(\frac{2\pi x}{\ell}\right)}{8}\Big|_0^\ell\right]$$

$$=\frac{\ell^2}{\pi^2}\left[\frac{\pi^2}{4}-\left(\frac{1}{8}-\frac{1}{8}\right)\right]$$

$$=\frac{\ell^2}{4}$$

Recall: $\sin(n\pi)=0$ and $\cos(n\pi)=1$

$$\int_0^{\ell} x\sin^2\left(\frac{2\pi x}{\ell}\right)$$

Let $u = \frac{2\pi x}{\ell} \rightarrow x = \frac{u\ell}{2\pi}$

$du = \frac{2\pi}{\ell}\, dx$

$dx = du \cdot \frac{\ell}{2\pi}$

when $x=0$, $u=0$.
When $x=\ell$, $u=2\pi$.

The integral becomes:

$$\int_0^{2\pi} \frac{u\ell}{2\pi} \sin^2(u) \frac{\ell}{2\pi}$$

$$= \frac{\ell^2}{4\pi^2}\int_0^{2\pi} u\sin^2(u)$$

$$= \frac{\ell^2}{4\pi^2}\int_0^{2\pi} u\left(\frac{1-\cos(2u)}{2}\right)$$

$$= \frac{\ell^2}{4\pi^2}\int_0^{2\pi} \frac{u}{2} - \frac{u\cos(2u)}{2}$$

$$= \frac{\ell^2}{4\pi^2}\left[\frac{u^2}{4}\Big|_0^{2\pi} - \frac{2u\sin(2u)+\cos(2u)}{8}\Big|_0^{2\pi}\right]$$

Note: Because we modified the boundaries of the integral earlier, we don't change u back to $\frac{2\pi x}{\ell}$. You can either modify the boundaries and use "u" or keep the old boundaries but revert "u" to what it's substituting for.

$$= \frac{\ell^2}{4\pi^2}\left[\frac{4\pi^2}{4} - \left(\frac{1}{8} - \frac{1}{8}\right)\right]$$
$$= \frac{\ell^2}{4\pi^2}\left(\pi^2\right)$$
$$= \frac{\ell^2}{4}$$

$$\int_0^{\ell} x\sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{2\pi x}{\ell}\right) 2\cos\left(\frac{(E_1 - E_2)t}{\hbar}\right)$$

To solve this, we will use the fact that

$$\sin A \cdot \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$

$$= \cancel{2}\cos\left(\frac{(E_1 - E_2)t}{\hbar}\right)\int_0^{\ell} x\left(\cancel{\tfrac{1}{2}}\left(\cos\left(\frac{-\pi x}{\ell}\right) - \cos\left(\frac{3\pi x}{\ell}\right)\right)\right)$$

$$= \cos\left(\frac{(E_1 - E_2)t}{\hbar}\right)\int_0^{\ell} x\cos\left(\frac{-\pi x}{\ell}\right) - x\cos\left(\frac{3\pi x}{\ell}\right)$$

$$\int x\cos\left(\frac{-\pi x}{\ell}\right)$$

let $u = x$

let $v = \cos\left(\frac{-\pi x}{\ell}\right)$ Note: cos is an even function, so $\cos(-\theta) = \cos(\theta)$.

$$u\int v - \int u' \int v$$
$$= x\int\cos\left(\frac{\pi x}{\ell}\right) - \int\int\cos\left(\frac{\pi x}{\ell}\right)$$
$$= \frac{x\sin\left(\frac{\pi x}{\ell}\right)\ell}{\pi} + \frac{\cos\left(\frac{\pi x}{\ell}\right)\ell^2}{\pi^2}$$

$$\int x\cos\left(\frac{3\pi x}{\ell}\right)$$

let $u = x$

let $v = \cos\left(\frac{3\pi x}{\ell}\right)$

$$u\int v - \int u' \int v$$

$$= x\int \cos\left(\frac{3\pi x}{\ell}\right) - \int\int \cos\left(\frac{3\pi x}{\ell}\right)$$

$$= \frac{x\sin\left(\frac{3\pi x}{\ell}\right)\ell}{3\pi} + \frac{\cos\left(\frac{3\pi x}{\ell}\right)\ell^2}{9\pi^2}$$

Substituting everything into the integral from where we left off, we get

$$= \cos\left(\frac{(E_1 - E_2)t}{\hbar}\right)\left[\frac{x\sin\left(\frac{\pi x}{\ell}\right)\ell}{\pi} + \frac{\cos\left(\frac{\pi x}{\ell}\right)\ell^2}{\pi^2} - \frac{x\sin\left(\frac{3\pi x}{\ell}\right)\ell}{3\pi} - \frac{\cos\left(\frac{3\pi x}{\ell}\right)\ell^2}{9\pi^2}\right]\Bigg|_0^{\ell}$$

$$= \cos\left(\frac{(E_1 - E_2)t}{\hbar}\right)\left[\frac{\cos(\pi)\ell^2}{\pi^2} - \frac{\cos(0)\ell^2}{\pi^2} - \left(\frac{\cos(3\pi)\ell^2}{9\pi^2} - \frac{\cos(0)\ell^2}{9\pi^2}\right)\right]$$

$$= \cos\left(\frac{(E_1 - E_2)t}{\hbar}\right)\left[\frac{-\ell^2}{\pi^2} - \frac{\ell^2}{\pi^2} - \left(\frac{-\ell^2}{9\pi^2} - \frac{\ell^2}{9\pi^2}\right)\right]$$

$$= \cos\left(\frac{(E_1 - E_2)t}{\hbar}\right)\left[-\frac{2\ell^2}{\pi^2} - \frac{-2\ell^2}{9\pi^2}\right]$$

$$= \cos\left(\frac{(E_1 - E_2)t}{\hbar}\right)\left[\frac{-2\ell^2}{\pi^2}\left(1 - \frac{1}{9}\right)\right]$$

$$= \cos\left(\frac{(E_1 - E_2)t}{\hbar}\right)\left(\frac{-2\ell^2}{\pi^2}\right)\left(\frac{8}{9}\right)$$

Now that we found each individual integral, we can put them together.

$$\frac{1}{\ell}\left[\underline{\int_0^\ell x\sin^2\left(\frac{\pi x}{\ell}\right)} + \underline{\int_0^\ell x\sin^2\left(\frac{2\pi x}{\ell}\right)} + \underline{\int_0^\ell x\sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{2\pi x}{\ell}\right)2\cos\left(\frac{(E_1-E_2)t}{\hbar}\right)}\right]$$

Where we left off

$$= \frac{1}{\cancel{\ell}}\left[\frac{\ell^{\cancel{2}}}{4} + \frac{\ell^{\cancel{2}}}{4} + \cos\left(\frac{(E_1-E_2)t}{\hbar}\right)\left(\frac{\ell^{\cancel{2}}}{\pi^2}\right)\left(\frac{-16}{9}\right)\right]$$

$$= \frac{\ell}{2} - \frac{16\ell}{9\pi^2}\cos\left(\frac{(E_1-E_2)t}{\hbar}\right)$$

Notice that the expected value of x oscillates with time due to the cosine term.

The center of the oscillation is when the cosine term is 0, which gives $\frac{\ell}{2}$.

The amplitude of the oscillation is $\frac{16}{9\pi^2}\ell$.

The frequency of the oscillation is $\frac{E_1 - E_2}{\hbar}$.

c) $\phi(x) = N\left(\sin\left(\frac{\pi x}{\ell}\right) + e^{i\alpha}\sin\left(\frac{2\pi x}{\ell}\right)\right)$

Soln:

$$1 = \int_0^\ell |\phi(x)|^2$$

$$= \int_0^\ell N^2 \left|\sin\left(\frac{\pi x}{\ell}\right) + e^{i\alpha}\sin\left(\frac{2\pi x}{\ell}\right)\right|^2$$

$$= N^2 \int_0^\ell \left(\sin\left(\frac{\pi x}{\ell}\right) + e^{-i\alpha}\sin\left(\frac{2\pi x}{\ell}\right)\right)\left(\sin\left(\frac{\pi x}{\ell}\right) + e^{i\alpha}\sin\left(\frac{2\pi x}{\ell}\right)\right)$$

Recall: $|z|^2 = \bar{z}\cdot z$

$$= N^2 \int_0^\ell \sin^2\left(\frac{\pi x}{\ell}\right) + e^{i\alpha}\sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{2\pi x}{\ell}\right) + e^{-i\alpha}\sin\left(\frac{2\pi x}{\ell}\right)\sin\left(\frac{\pi x}{\ell}\right) + e^{-i\alpha}e^{i\alpha}\sin^2\left(\frac{2\pi x}{\ell}\right)$$

$= \quad N^2\ell$ Note: $e^{-i\alpha}\cdot e^{i\alpha} = e^{-i\alpha+i\alpha} = e^0 = 1$

$$N = \sqrt{\frac{1}{\ell}}$$

$$\phi(x) = \sqrt{\frac{1}{\ell}}\left(\sin\left(\frac{\pi x}{\ell}\right) + e^{i\alpha}\sin\left(\frac{2\pi x}{\ell}\right)\right)$$

$$\psi(x,t) = \sqrt{\frac{1}{\ell}}\left(\sin\left(\frac{\pi x}{\ell}\right)e^{\frac{-iE_1 t}{\hbar}} + e^{i\alpha}\sin\left(\frac{2\pi x}{\ell}\right)e^{\frac{-iE_2 t}{\hbar}}\right)$$

$$|\psi(x,t)|^2 = \frac{1}{\ell}\left|\sin\left(\frac{\pi x}{\ell}\right)e^{\frac{-iE_1t}{\hbar}} + e^{i\alpha}\sin\left(\frac{2\pi x}{\ell}\right)e^{\frac{-iE_2t}{\hbar}}\right|^2$$

$$= \frac{1}{\ell}\left(\overline{\sin\left(\frac{\pi x}{\ell}\right)e^{\frac{-iE_1t}{\hbar}} + e^{i\alpha}\sin\left(\frac{2\pi x}{\ell}\right)e^{\frac{-iE_2t}{\hbar}}}\right)$$
$$\left(\sin\left(\frac{\pi x}{\ell}\right)e^{\frac{-iE_1t}{\hbar}} + e^{i\alpha}\sin\left(\frac{2\pi x}{\ell}\right)e^{\frac{-iE_2t}{\hbar}}\right)$$

$$= \frac{1}{\ell}\left(\sin\left(\frac{\pi x}{\ell}\right)e^{\frac{iE_1t}{\hbar}} + e^{-i\alpha}\sin\left(\frac{2\pi x}{\ell}\right)e^{\frac{iE_2t}{\hbar}}\right)$$
$$\left(\sin\left(\frac{\pi x}{\ell}\right)e^{\frac{-iE_1t}{\hbar}} + e^{i\alpha}\sin\left(\frac{2\pi x}{\ell}\right)e^{\frac{-iE_2t}{\hbar}}\right)$$

$$= \frac{1}{\ell}\Bigg(\sin^2\left(\frac{\pi x}{\ell}\right) + \sin^2\left(\frac{2\pi x}{\ell}\right) +$$
$$e^{i\alpha}\sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{2\pi x}{\ell}\right)e^{\frac{i(E_1-E_2)t}{\hbar}} +$$
$$e^{-i\alpha}\sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{2\pi x}{\ell}\right)e^{\frac{-i(E_1-E_2)t}{\hbar}}\Bigg)$$

$$= \frac{1}{\ell}\Bigg(\sin^2\left(\frac{\pi x}{\ell}\right) + \sin^2\left(\frac{2\pi x}{\ell}\right) +$$
$$\sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{2\pi x}{\ell}\right)\left[e^{\frac{i(E_1-E_2)t}{\hbar}+\alpha} +\right.$$
$$\left.e^{\frac{-i(E_1-E_2)t}{\hbar}+\alpha}\right]\Bigg)$$

Recall: $\cos(\theta) = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

Therefore: $e^{\frac{i(E_1-E_2)t}{\hbar}+\alpha} + e^{\frac{-i(E_1-E_2)}{\hbar}+\alpha}$
$= 2\cos\left(\frac{(E_1-E_2)t}{\hbar}+\alpha\right)$

$$= \frac{1}{\ell}\left(\sin^2\left(\frac{\pi x}{\ell}\right) + \sin^2\left(\frac{2\pi x}{\ell}\right) + \sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{2\pi x}{\ell}\right) 2\cos\left(\frac{(E_1 - E_2)t}{\hbar} + \alpha\right)\right)$$

$$\langle x \rangle = \int_0^\ell x\,|\psi(x,t)|^2$$

$$= \int_0^\ell x\left(\frac{1}{\ell}\right)\left(\sin^2\left(\frac{\pi x}{\ell}\right) + \sin^2\left(\frac{2\pi x}{\ell}\right) + \sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{2\pi x}{\ell}\right) 2\cos\left(\frac{(E_1 - E_2)t}{\hbar} + \alpha\right)\right)$$

$$= \frac{1}{\ell}\left[\underbrace{\int_0^\ell x\sin^2\left(\frac{\pi x}{\ell}\right)}_{\frac{\ell^2}{4}} + \underbrace{\int_0^\ell x\sin^2\left(\frac{2\pi x}{\ell}\right)}_{\frac{\ell^2}{4}} + \int_0^\ell x\sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{2\pi x}{\ell}\right) 2\cos\left(\frac{(E_1 - E_2)t}{\hbar} + \alpha\right)\right]$$

$$= \frac{1}{\ell}\left[\frac{\ell^2}{2} + 2\cos\left(\frac{(E_1 - E_2)t}{\hbar} + \alpha\right)\underline{\int_0^\ell x\sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{2\pi x}{\ell}\right)}\right]$$

$$\int_0^\ell x\sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{2\pi x}{\ell}\right)$$

Recall: $\sin(A)\cdot\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B))$

$$= \int_0^\ell x\left(\frac{1}{2}\right)\left(\cos\left(-\frac{\pi x}{\ell}\right) - \cos\left(\frac{3\pi x}{\ell}\right)\right)$$

Because cos is an even function.

$$= \frac{1}{2}\left(\int_0^\ell x\cos\left(\frac{\pi x}{\ell}\right) - \int_0^\ell x\cos\left(\frac{3\pi x}{\ell}\right)\right)$$

Note: $\int_0^\ell x\cos\left(\frac{n\pi x}{\ell}\right) = \frac{\ell^2(\cos(n\pi) - 1)}{n^2\pi^2}$

$$= \frac{1}{2}\left(\frac{\ell^2(\cos(\pi)-1)}{\pi^2} - \frac{\ell^2(\cos(3\pi)-1)}{9\pi^2}\right)$$

Now back to the main equation:

$$\langle x \rangle = \frac{1}{\ell}\left[\frac{\ell^2}{2} + \cancel{2}\cos\left(\frac{(E_1 - E_2)t}{\hbar} + \alpha\right)\frac{\cancel{1}}{\cancel{2}}\left(\frac{-2\ell^2}{\pi^2} - \frac{-2\ell^2}{9\pi^2}\right)\right]$$

$$= \frac{1}{\cancel{\ell}}\left[\frac{\ell^{\cancel{2}}}{2} + \cos\left(\frac{(E_1 - E_2)t}{\hbar} + \alpha\right)\frac{-16\ell^{\cancel{2}}}{9\pi^2}\right]$$

$$= \frac{\ell}{2} - \frac{16\ell}{9\pi^2}\cos\left(\frac{(E_1 - E_2)t}{\hbar} + \alpha\right)$$

The center of the oscillation occurs when the cos term is 0, which gives us $\frac{\ell}{2}$.

The amplitude of the oscillation is $\frac{16\ell}{9\pi^2}$.

The frequency of the oscillation is $\frac{E_1 - E_2}{\hbar}$.