MATB61 Week 1 Notes

1. Overview:

- Optimization is a mathematical problem in which some function is either maximized or minimized relative to a given set of alternatives.
- The function to be maximized or minimized is called the objective function.
- The optimum solution is a soln to the system of constraints that give the max/min value of the obj function.
- A linear program (LP) is an optimization problem in which there are finitely many variables having a linear objective function and a constraint region determined by a finite number of linear equality and/or inequality constraints.

2. LP Forms:

a) General LP Problem:

- For the value of $x_1, x_2, \ldots, x_n$, max or min $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ subject to the restrictions/constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq (\geq)(=)\ b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq (\geq)(=)\ b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq (\geq)(=)\ b_m$$

b) LP in Standard form:

- For the values of $X_1, X_2, \ldots, X_n$, max $Z = C_1X_1 + C_2X_2 + \ldots + C_nX_n$ subject to the restrictions

$$a_{11}X_1 + a_{12}X_2 + \ldots + a_{1n}X_n \leq b_1$$
$$a_{21}X_1 + a_{22}X_2 + \ldots + a_{2n}X_n \leq b_2$$
$$\vdots$$
$$a_{m1}X_1 + a_{m2}X_2 + \ldots + a_{mn}X_n \leq b_m$$

and $X_i \geq 0$ for i in $1, 2, \ldots n$

- Note that we are only maximizing $Z = C_1X_1 + \ldots + C_nX_n$, and the restrictions are always $\leq$, and X_i must be greater than or equal to 0.

c) LP in Canonical form:

- For the values of $X_1, X_2, \ldots, X_n$, max $Z = C_1X_1 + C_2X_2 + \ldots + C_nX_n$ Subject to the restrictions

$$a_{11}X_1 + a_{12}X_2 + \ldots + a_{1n}X_n = b_1$$
$$a_{21}X_1 + a_{22}X_2 + \ldots + a_{2n}X_n = b_2$$
$$\vdots$$
$$a_{m1}X_1 + a_{m2}X_2 + \ldots + a_{mn}X_n = b_m$$

and $X_i \geq 0$ for i in $1, 2, \ldots, n$

- Note that we are only maximizing $Z = C_1X_1 + C_2X_2 + \ldots + C_nX_n$, the restrictions are always $=$, and X_i must be greater than or equal to 0.

3. Converting Different Forms of LP:

a) Converting GLP to SLP:

1. To change from a minimization problem to a maximization problem, multiply the objective function by -1.

2. To reverse an inequality, multiply both sides by -1.

   E.g. $a_1x_1 + a_2x_2 + \ldots + a_nx_n \geq b$
   is equivalent to $-a_1x_n - \ldots - a_nx_n \leq b$

3. To change an equality into an inequality, rewrite the equality as two linear inequalities.

   E.g. $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$ can be written as:
   1. $a_1x_1 + a_2x_2 + \ldots + a_nx_n \leq b$ and
   2. $a_1x_1 + a_2x_2 + \ldots + a_nx_n \geq b$

   or equivalently:
   1. $a_1x_1 + a_2x_2 + \ldots + a_nx_n \leq b$ and
   2. $-a_1x_1 - a_2x_2 - \ldots - a_nx_n \leq -b$

4. To replace an unconstrained variable, you can create 2 new variables that are greater than or equal to 0 and make the original variable equal to the difference of the 2 newly created vars.

E.g. Suppose x is unrestricted.
Let $x_1, x_2 \geq 0$
Let $x = x_2 - x_1$

However, you don't always need to create 2 variables. If there are multiple unrestricted vars, you can create 2 new vars for the first unrestricted var and then 1 new var for each of the remaining unrestricted vars.

E.g. Suppose x_1 and x_2 are unrestricted.
Let $x_0', x_1', x_2' \geq 0$
Let $x_1 = x_1' - x_0'$
Let $x_2 = x_2' - x_0'$

Remember to substitute your unrestricted vars with the new ones in the equations.

5. To change an inequality to an equality, you can add a slack var.

E.g. $a_1 x_1 + a_2 x_2 + \dots + a_n x_n \leq b$
Let $s \geq 0$
$a_1 x_1 + a_2 x_2 + \dots + a_n x_n + s = b$

b) Converting SLP into CLP:

To convert a SLP into a CLP, just add a slack variable to each restraint equation.

E.g. Convert the below SLP to CLP.

SLP ⇒ For the values $x_1, x_2, \ldots, x_n$, max $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ subject to the restrictions

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m$$
$$x_i \geq 0,\ i = 1, 2, \ldots, n$$

Soln:

For the values $x_1, x_2, \ldots, x_n$, max $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ subject to the restrictions

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + s_1 = b_1$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n + s_m = b_m$$
$$x_i \geq 0,\ i = 1, 2, \ldots, n$$
$$s_j \geq 0,\ j = 1, 2, \ldots, m$$

E.g. 1 Let LP be min $z = 3x + 2y$ subject to the constraints:

1. $y \geq x + 2$
2. $y \leq -x + 3$
3. $y \geq \frac{2}{5}x + \frac{1}{5}$

Convert it to a SLP and then a CLP.

Soln:

1. Convert GLP to SLP.

Recall that for SLP, we are always maximizing $z = c_1 x_1 + \dots + c_n x_n$, the restrictions are always less than or equal to and that x_i must be greater than or equal to 0.

Max $Z = c_1 x_1 + \dots + c_n x_n$

In GLP form, we are to min $Z = 3x + 2y$. Recall that to change from a min problem to a max problem, we multiply the objective function by -1.

Let $z' = -z$

$= -3x - 2y$

$\therefore$ We want to max z'.

Changing Inequality

1. $Y \geq x + 2 \Rightarrow x - y \leq 2$
2. $Y \leq -x + 3 \Rightarrow x + y \leq 3$
3. $Y \geq \frac{2}{5}x + \frac{1}{5} \Rightarrow \frac{2}{5}x - y \leq -\frac{1}{5}$

Dealing with unrestricted vars

In the original GLP form, there is no restriction for x_i to be greater than or equal to 0. However, we need this restriction for SLP.

Set $x_0, x', y' \geq 0$
Let $x = x' - x_0$
Let $y = y' - y_0$

Now, we need to replace x with $x' - x_0$ and y with $y' - y_0$.

Max $z' = -3x - 2y$ becomes:

$$\text{Max } z' = -3(x' - x_0) - 2(y' - y_0)$$
$$= -3x' + 3x_0 - 2y' + 2y_0$$
$$= -3x' - 2y' + 5x_0$$

$x - y \leq 2$ becomes:

$$(x' - x_0) - (y' - y_0) \leq 2$$
$$x' - y' \leq 2$$

$x + y \leq 3$ becomes:

$$(x' - x_0) + (y' - x_0) \leq 3$$
$$x' - y' - 2x_0 \leq 3$$

$\frac{2}{5}x - y \leq \frac{1}{5}$ becomes:

$$\frac{2}{5}(x' - x_0) - (y' - x_0) \leq \frac{1}{5}$$

$$\frac{2}{5}x' - y' + \frac{3}{5}x_0 \leq \frac{1}{5}$$

Putting it all together

SLP:

Max $z' = -3x' - 2y' + 5x_0$ subject to the constraints

1. $x' - y' \leq 2$
2. $x' - y' - 2x_0 \leq 3$
3. $\frac{2}{5}x' - y' + \frac{3}{5}x_0 \leq \frac{1}{5}$

$x, y \geq 0$

2. Converting from SLP to CLP:

Recall that for GLP, we are always maximizing $z = c_1x_1 + \dots + c_nx_n$, the restrictions are always $=$ and $x_i \geq 0$. The only difference between SLP and GLP is that the restrictions are always $\leq$ for SLP and $=$ for GLP.

To change/convert from SLP to GLP, we need to add a slack var for each restriction.

$x' - y' \leq 2 \Rightarrow x' - y' + S_1 = 2$

$x' - y' - 2x_0 \leq 3 \Rightarrow x' - y' - 2x_0 + S_2 = 3$

$\frac{2}{5}x' - y' + \frac{3}{5}x_0 \leq \frac{1}{5} \Rightarrow \frac{2}{5}x' - y' + \frac{3}{5}x_0 + S_3 = \frac{1}{5}$

$S_1, S_2, S_3 \geq 0$

Everything else is the same.

E.g. 2 Convert the following GLP problem into SLP:

min $z = -x_1 + 2x_2 - 3x_3$ subject to the restrictions:

1. $x_1 - x_2 + x_3 = -2$
2. $|x_1 - x_3| \leq 1$
3. $x_2 \geq 0$

Soln:

To convert a min problem to a max problem, we need to multiply the objective function by -1.

Let $z' = -z \rightarrow z' = x_1 - 2x_2 + 3x_3$

We need to max z'.

We can convert $x_1 - x_2 + x_3 = -2$ into 2 inequalities:

1. $x_1 - x_2 + x_3 \leq -2$
2. $x_1 - x_2 + x_3 \geq -2$

We can then turn $x_1 - x_2 + x_3 \geq -2$ into $-x_1 + x_2 - x_3 \leq 2$.

We can turn $|x_1 - x_3| \leq 1$ into $-1 \leq x_1 - x_3 \leq 1$ and then split it into:

1. $-1 \leq x_1 - x_3$
2. $x_1 - x_3 \leq 1$

By multiplying (1) with -1, we get $x_3 - x_1 \leq 1$

Right now, we have:
Max $z' = x_1 - 2x_2 + 3x_3$ subject to the constraints:
1. $x_1 - x_2 + x_3 \leq -2$
2. $-x_1 + x_2 - x_3 \leq 2$
3. $x_3 - x_1 \leq 1$
4. $x_1 - x_3 \leq 1$
5. $x_2 \geq 0$

Note: x_1 and x_3 are still unrestricted.
Let $x_0', x_1', x_3' \geq 0$
$x_3 = x_3' - x_0'$
$x_1 = x_1' - x_0'$
Now, we replace x_1 with $x_1' - x_0'$ and x_3 with $x_3' - x_0'$.

Max $z' = x_1' - 2x_2 + 3x_3' - 4x_0'$
subject to the constraints:
1. $x_1' - x_2 + x_3' - 2x_0' \leq -2$
2. $-x_1' + x_2 - x_3' + 2x_0' \leq 2$
3. $x_3' - x_1' \leq 1$
4. $x_1' - x_3' \leq 1$
5. $x_i \geq 0, \; i = 1, 2, 3$

4. Creating LP Equality/Inequality:

- often, creating a table helps with creating the equality/inequality.

E.g. 1 A produce grower is purchasing fertilizer containing 3 nutrients, A, B and C. The min needs are 160 units of A, 200 units of B, and 80 units of C. There are 2 popular brands of fertilizer on the market.
Fast Grow: Contains 3 units of A, 5 units of B, 1 unit of C and costs \$8/bag.
Easy Grow: Contains 2 units of each nutrient and costs \$6/bag.

If the producer wants to min costs, while still maintaining the nutrients required, how many bags of each brand should be bought?

Soln:
Let x be the number of fast grow bags bought.
Let y be the number of easy grow bags bought.

	A	B	C	Cost
x	3	5	1	\$8
y	2	2	2	\$6

$$3x + 2y \geq 160$$
$$5x + 2y \geq 200$$
$$x + 2y \geq 80$$
$$8x + 6y = \text{cost}$$

E.g. 2 A furniture maker has a line of 4 types of desks. They vary in the manufacturing process and their profitability. The furniture maker has 6000 hours of time in the carpentry shop and 4000 hours of time in the finishing shop.

Each desk of type 1 requires 4 hours of carpentry and 1 hour of finishing.

Each desk of type 2 requires 9 hrs of carpentry and 1 hr of finishing.

Each desk of type 3 requires 7 hrs of carpentry and 3 hrs of finishing.

Each desk of type 4 requires 10 hrs of carpentry and 40 hrs of finishing.

The profit is $12 for each desk of type 1, $20 for each desk of type 2, $28 for each desk of type 3 and $40 for each desk of type 4. How should the production be scheduled to max profit?

Soln:

Assume the producer is creating x_i amounts of type i desk.

	Carpentry	Finish	Profit
X_1	4	1	\$12
X_2	9	1	\$20
X_3	7	3	\$28
X_4	10	40	\$40

$$4X_1 + 9X_2 + 7X_3 + 10X_4 \le 6000$$
$$X_1 + X_2 + 3X_3 + 40X_4 \le 4000$$
$$\text{Profit} = 12X_1 + 20X_2 + 28X_3 + 40X_4$$

E.g. 3 A PO requires different numbers of employees on diff days of the week. Union rules states each employee must work 5 consecutive days and then receive 2 days off. Find the min number of employees needed.

Day	Mon	Tues	Wed	Thur	Fri	Sat	Sun
Staff needed	13	15	19	14	16	11	17

Soln:
Assume there are X_i employees on the ith day, where 1 = Mon, 2 = Tues, ..., 7 = Sun.

$$X_1 + X_2 + X_3 + X_4 + X_5 \ge 16$$
$$X_2 + X_3 + X_4 + X_5 + X_6 \ge 11$$
$$X_3 + X_4 + X_5 + X_6 + X_7 \ge 17$$
$$X_4 + X_5 + X_6 + X_7 + X_1 \ge 13$$
$$X_5 + X_6 + X_7 + X_1 + X_2 \ge 15$$
$$X_6 + X_7 + X_1 + X_2 + X_3 \ge 19$$

$X_7 + X_1 + X_2 + X_3 + X_4 \geq 14$

Min number of employees $= X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7$

E.g. 4 A nutritionist is planning a menu consisting of 2 main foods, A and B. Each ounce of A contains 2 units of fat, 1 unit of carbs and 4 units of protein. Each ounce of B contains 3 units of fat, 3 units of carbs, and 3 units of protein. The nutritionist wants the meal to provide at least 18 units of fat and 12 units of carbs, and 24 units of proteins. If an ounce of A costs \$2, and an ounce of B costs \$2.5, how many ounces of each food should be served to min the cost of the meal yet satisfy the nutritionist's requirements?

Soln:

Let x be the amount of ounces of A.
Let Y be the amount of ounces of B

	Fat	Carb	Protein	Cost
X	2	1	4	2
Y	3	3	3	2.5

$2x + 3y \geq 18$ $\quad$ $2x + 2.5y = \text{cost}$

$x + 3y \geq 12$

$4x + 3y \geq 24$

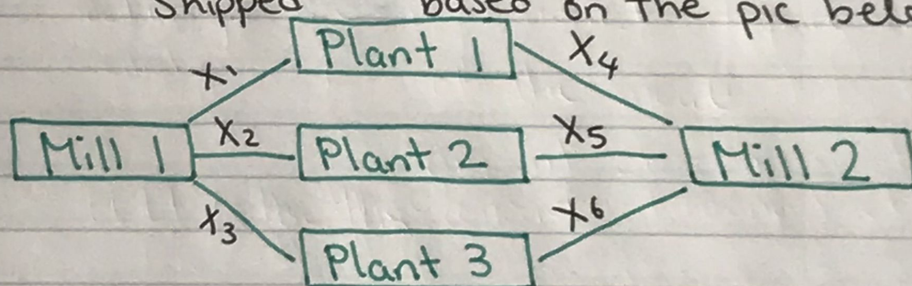
E.g. 5 A paper manufacturer with 2 mills must supply weekly 3 printing plants with newsprint. Mill 1 produces 350 tons of newsprint a week and Mill 2 produces 550 tons. Plant 1 requires 275 tons/week, plant 2 325 tons and plant 3 300 tons. The shipping costs in dollars per tons are:

	Plant 1	Plant 2	Plant 3
Mill 1	17	22	15
Mill 2	18	16	12

Determine how many tons each mill should ship to each plant so that the total transportion cost is minimal.

Soln:

Let x_i be the tonnes of newsprint shipped based on the pic below.



$x_1 + x_4 \geq 275$

$x_2 + x_5 \geq 325$

$x_3 + x_6 \geq 300$

$x_1 + x_2 + x_3 \leq 350$

$x_4 + x_5 + x_6 \leq 550$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

$\text{Cost} = 17x_1 + 22x_2 + 15x_3 + 18x_4 + 16x_5 + 12x_6$

5. Matrix Notation:

- The SLP form: For the values of $x_1, x_2, \ldots,$ x_n max $z = c_1x_1 + \ldots + c_nx_n$ subject to the restrictions

$$a_{11}x_1 + \ldots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + \ldots + a_{2n}x_n \leq b_2$$
$$\vdots$$
$$a_{m1}x_1 + \ldots + a_{mn}x_n \leq b_m$$
$$x_j \geq 0 \text{ for } j \text{ in } 1, 2, \ldots n$$

Let

1. $$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

2. $$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

3. $$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

4. $$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Then

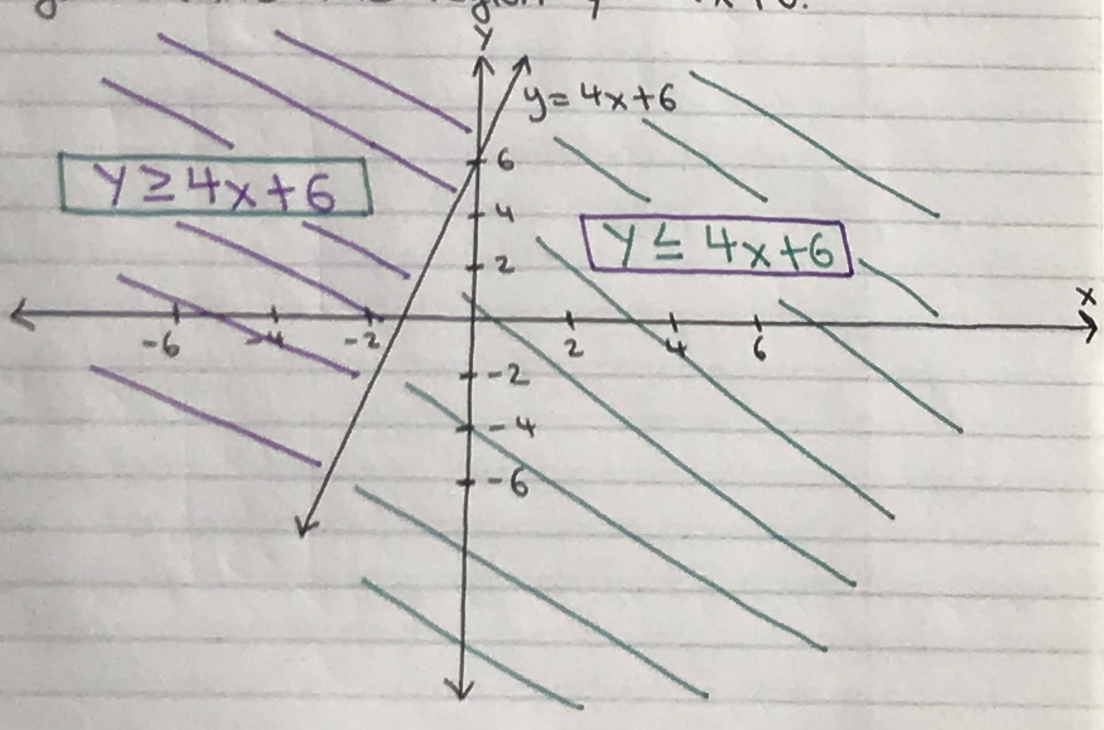
1. The SLP in matrix form maximizes
   $z = c^T x$ subject to $Ax \leq b$ and $x \geq 0$.

2. The CLP in matrix form maximizes
   $z = c^T x$ subject to $Ax = b$ and $x \geq 0$

6. Theorem of LP Problems:

- To tell if a region is less than or greater than a line, pick any point that is not on the line and plug it into the eqn of the line and compare the Y value with the result of the eqn. If the Y value of the point is bigger, then the region the point lies on is greater than the line and if the Y value of the point is smaller, then the region the point lies on is smaller than the line.

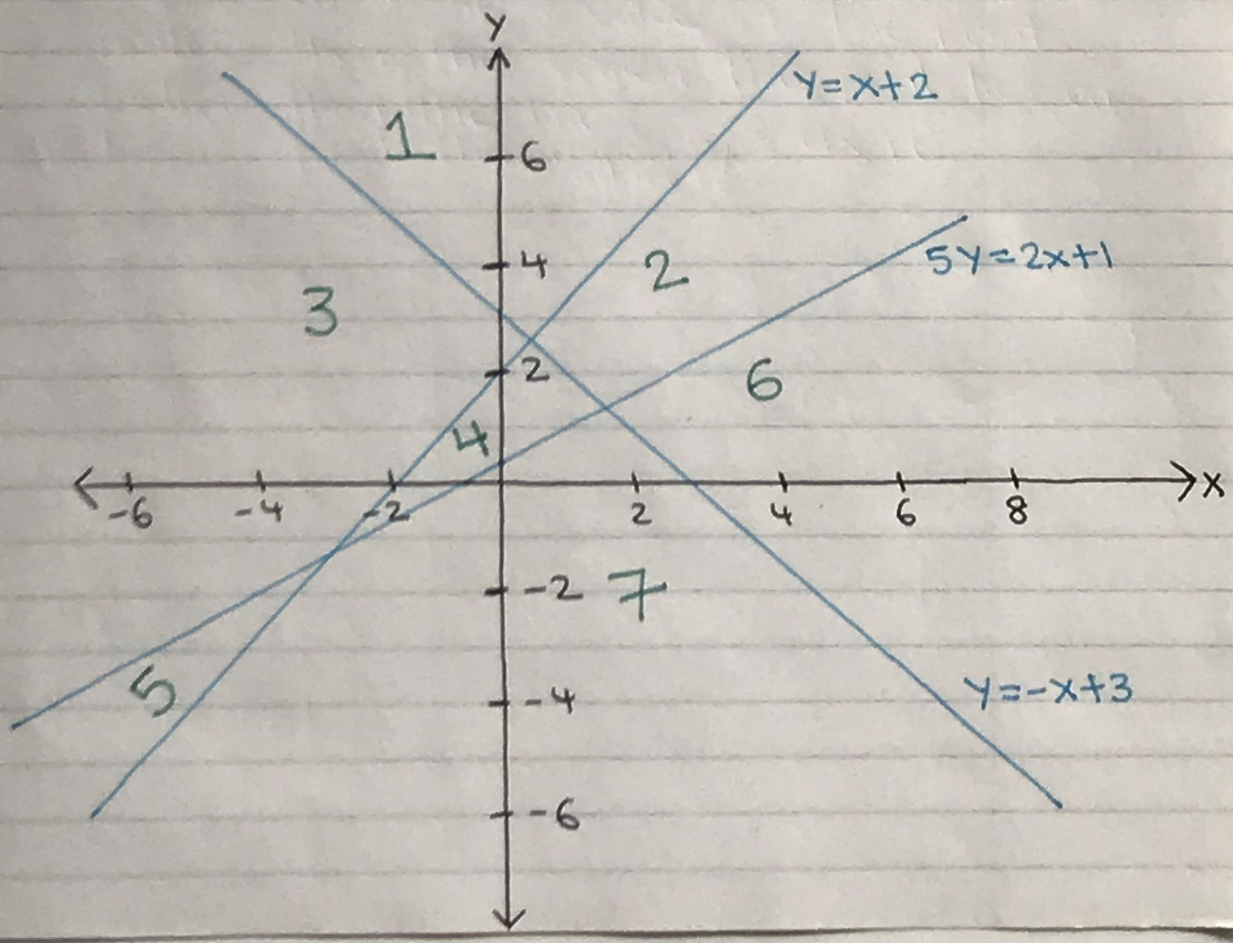
E.g. 1. Find the region $y \geq 4x+6$.



Soln: Lets pick (0,0) as our arbitrary point. If we plug (0,0) into the equation $Y = 4x + 6$, we get 6. Since $6 > 0$, the region (0,0) on is smaller than the line.
$\therefore$ The other region must be
$y \geq 4x + 6$.

E.g. 2. Given the 3 equations below, label the graph with where each set of inequalities belong to.

Eqns:
1. $Y = x + 2$
2. $Y = -x + 3$
3. $5Y = 2x + 1 \rightarrow Y = \frac{2}{5}x + \frac{1}{5}$



Inequalities:

1. $y \geq x+2$
   $y \geq -x+3$
   $y \geq \frac{2}{5}x+\frac{1}{5}$

2. $y \leq x+2$
   $y \geq -x+3$
   $y \geq \frac{2}{5}x+\frac{1}{5}$

3. $y \geq x+2$
   $y \leq -x+3$
   $y \geq \frac{2}{5}x+\frac{1}{5}$

4. $y \leq x+2$
   $y \leq -x+3$
   $y \geq \frac{2}{5}x+\frac{1}{5}$

5. $y \geq x+2$
   $y \leq -x+3$
   $y \leq \frac{2}{5}x+\frac{1}{5}$

6. $y \leq x+2$
   $y \geq -x+3$
   $y \leq \frac{2}{5}x+\frac{1}{5}$

7. $y \leq x+2$
   $y \leq -x+3$
   $y \leq \frac{2}{5}x+\frac{1}{5}$

Soln:
Lets find which region is greater than the line for each eqn.

1. $y = x+2$
   Lets pick $(0,0)$ as the arbitrary point. Plugging $(0,0)$ into the eqn, we get 2. $2 > 0$.
   $\therefore$ The region that contains $(0,0)$ is $y \leq x+2$ and the other region is $y \geq x+2$.

2. $y = -x+3$
   Using $(0,0)$ as the arbitrary point, we see that the region that contains $(0,0)$ is $y \leq -x+3$ and the other region is $y \geq -x+3$.

3. $5y = 2x+1$

Using $(0,0)$, we see that the region with $(0,0)$ is $5y \leq 2x+1$ and the other region is $5y \geq 2x+1$.

Now that we know which region has which inequality for each line, we just need to find the common region for each set of inequalities.

- A feasible solution to a LP problem is $\{x_1, x_2, \ldots, x_n\}$ satisfying the constraints of the problem.

- $\{x_1, x_2, \ldots, x_n\}$ is a feasible solution of SLP iff $\{x_1, x_2, \ldots, x_{n+m}\}$ is a feasible solution of CLP.

- The feasible region is the set of all the feasible solutions.

- If a feasible region can be contained within a circle, then it is bounded. Otherwise, it's unbounded. In the previous example, region 4 is bounded and all other regions are unbounded.

- If a feasible region contains at least one point, it is non-empty. Otherwise, it is empty.

E.g. $\left.\begin{array}{l} y \geq -x+3 \\ y \geq x+2 \\ y \leq \frac{2}{5}x + \frac{1}{5} \end{array}\right\}$ is empty because there are no points.