

Erdos-Szekeres Theorem

1. Theorem:

Given $n^2 + 1$ real numbers, there is always an inc or dec sub-sequence consisting of $n+1$ numbers.

Sequence: An enumerated collection of objects in which order matters.

I.e. $1, 3, 5, 6 \neq 1, 5, 3, 6$

Sub-Sequence: What remains after we remove some terms from a sequence.

I.e. $1, 3$ is a sub-seq of $1, 3, 5, 6$

Increasing Sequence: A seq is inc if each term \geq than the prev term.

I.e. $x_1 \leq x_2 \leq \dots \leq x_n$

Decreasing Sequence: A seq is dec if each term is \leq than the prev term.

I.e. $x_1 \geq x_2 \geq \dots \geq x_n$

2. General Proof:

Consider the $n^2 + 1$ nums $a_1, a_2, \dots, a_{n^2+1}$.

Suppose that we want to find $n+1$ numbers s.t. they form a decreasing sub-sequence.

There are 2 cases:

Case 1: Consider the longest decreasing sub-seq starting from a_1 . The max possible length the sub-seq can have is n^2+1 .

I.e. $a_1 \geq a_2 \geq \dots \geq a_{n^2+1}$

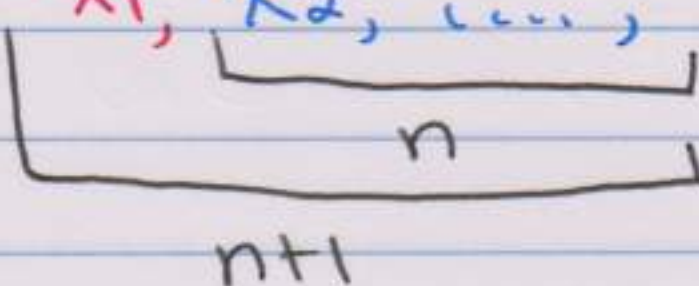
If there exists a sub-sequence with length $\geq n+1$, then we are done.

Case 2: Suppose that there is no sub-seq starting from a_i , $i \in \{1, 2, \dots, n^2+1\}$ that has length $\geq n+1$.

I.e. Suppose all the sub-seqs have a length between 1 and n , inclusive.

Since there are n^2+1 sub-sequences and n different possible lengths, there will be $n+1$ sub-seq with the same length (By P.P.). Let these $n+1$ sub-seqs be X_1, X_2, \dots, X_{n+1} and suppose that they all have a length of n . Then, $X_1 \leq X_2 \leq \dots \leq X_{n+1}$. Here is why: Consider X_1 and X_2 and suppose $X_1 > X_2$. We know that X_2 has a sub-seq of length n . If we were to add X_1 to that sub-seq, then the length would be $n+1$, violating our assumption that the lengths of the sub-seq is n .

I.e. : $X_1, X_2, \dots,$



Hence $x_1 \leq x_2$. The same logic applies to all other x_i 's.

3. Examples:

- a) Prove that given 5 numbers, there must be 3 that form an inc or dec sub-seq.

Soln:

Let a_1, a_2, a_3, a_4, a_5 be the 5 numbers. Suppose we want to find a dec sub-seq.

Case 1:

Consider the longest dec sub-seq starting with a_1 . The length is 5.

Consider the longest dec sub-seq starting with a_2 . The length is 4.

Consider the longest dec sub-seq starting with a_3 . The length is 3.

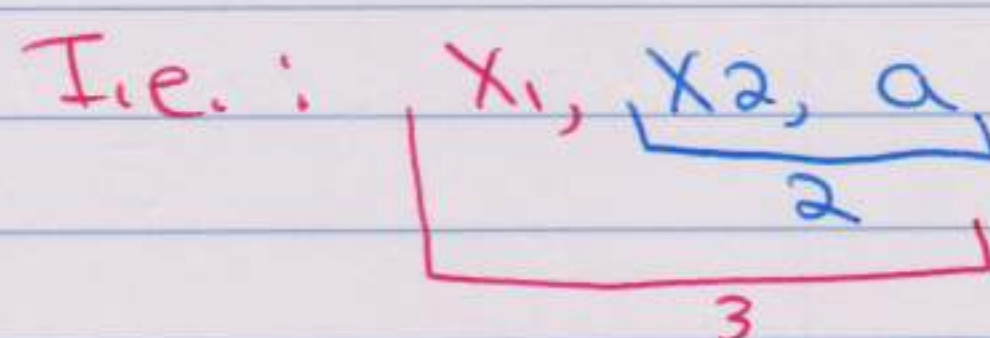
If there exists a dec sub-seq with length ≥ 3 , we're done.

Case 2:

Assume that the lengths of all dec sub-seq are either 1 or 2. By P.P., there are 3 dec sub-seq with the same length, say 2. Let x_1, x_2, x_3 be the head of these 3 dec sub-seq. Then: $x_1 \leq x_2 \leq x_3$

Why:

Consider x_1 and x_2 and suppose $x_1 \geq x_2$. Since x_2 is the head of a dec sub-seq with length 2, appending it to x_1 would make the length 3, contradicting our assumption.



The same logic applies for x_2 and x_3 .

- b) In any sub-seq of 17 nums, there is always an inc or dec sub-seq of 5 nums.

Soln:

Let the 17 nums be a_1, a_2, \dots, a_{17} . Suppose we want to find a dec sub-seq.

Case 1:

Consider the max dec sub-seq starting from a_1 . The length is 17.

We want to show that there exists a dec sub-seq with length ≥ 5 .

Case 2:

Suppose that no dec sub-seq has length ≥ 5 . Then, the possible lengths are 1, 2, 3, 4. By P.P., there must be 5 nums with the same length, say 4.

Let these 5 nums be x_1, x_2, \dots, x_5 .

Then $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$.

Why:

Consider x_1 and x_2 . Suppose $x_1 > x_2$.

Then, because x_2 is at the start of a dec sub-seq with length 4, adding x_1 to it will make

it length 5, contradicting our assumption. The same logic applies to x_2, x_3, x_4, x_5 .

- c) Is the following statement correct:
 "In any seq of 4 nums, there is always an inc or dec sub-seq of 3 nums."?

Soln:

$$\text{No } n^2 + 1 = 4 \rightarrow n^2 = 3 \rightarrow n = \pm\sqrt{3}$$

$$n + 1 = \pm\sqrt{3} + 1 \neq 3$$

Counterpoint: 3, 4, 1, 2