MATB42 Week 2 Notes

1. HEAT Equation:

- The PDE $u_t = k u_{xx}$ is called the heat eqn / diffusion eqn where k is a constant called the rate of diffusion.

- The heat eqn describes the diffusion of heat through a 1-D medium, such as a metal rod. The unknown function $u(x,t)$ is the temperature at point x at time t.

- We aim to solve $u(x,t)$ in the finite domain $0<x<\ell$ where ℓ is the length of the domain.

- Initial Condition:
$u(x,0) = \phi(x)$ where $\phi(x)$ describes the initial heat distribution at time $t=0$.

- Boundary Condition:
$u_x(0,t)=0$
$u_x(\ell,t)=0$

The boundary conditions indicate that no heat is leaving or entering the domain.

To solve the heat eqn, we'll use Separation of Variables.

Assume $u(x,t) = X(x) \cdot T(t)$
Then, the PDE $u_t = k u_{xx}$ becomes
$X \cdot T' = k \cdot X'' \cdot T$.

Recall: By convention, we move all the constants and terms with t to the LHS and all the terms with x to the RHS.

Thus, we get $\frac{T'}{k \cdot T} = \frac{X''}{X} = F$

The separation constant. Is always negative.

$$\frac{T'}{k \cdot T} = \frac{X''}{X} = -\lambda, \text{ where } \lambda > 0$$

$$\frac{T'}{k \cdot T} = -\lambda$$

$$T' = -\lambda k T$$

Recall: $(\ln(f(x)))' = \frac{f'(x)}{f(x)}$

$\frac{T'}{T} = -\lambda k$

Hence, $\frac{T'}{T} = (\ln(T))'$

$(\ln(T))' = -\lambda k$
$\ln(T) = \int -\lambda k$
$= -\lambda k t + C$
$T = e^{-\lambda k t + C}$
$T = Ae^{-\lambda k t}$

To get rid of the derivative, we integrate both sides with respect to t.

Now, we have $\ln(T) = -\lambda k t$

To get rid of ln, raise both sides by e.

$e^{\ln f(x)} = f(x)$

$$\frac{X''}{X} = -\lambda$$

$X'' = -\lambda X$
$X'' + \lambda X = 0$
$r^2 + \lambda = 0$
$r^2 = -\lambda$
$r = \pm\sqrt{\lambda}\, i$
Take $r = \sqrt{\lambda}\, i$
$e^{rx} = e^{(\sqrt{\lambda}x)i}$
$= \cos(\sqrt{\lambda}x) + i\sin(\sqrt{\lambda}x)$

$X = C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)$

$$u(x,t) = \underbrace{X(x)\cdot T(t)}_{\text{We found both terms}}$$

Now, we plug in the boundary equations. Recall that the boundary conditions are
1. $u_x(0,t) = 0$
2. $u_x(\ell,t) = 0$

This means that we need to differentiate u with respect to (w.r.t) x first.

$$u_x = \frac{\partial u}{\partial x}$$
$$= \frac{\partial}{\partial x}\left(X(x)\cdot T(t)\right)$$
$$= T(t)\cdot\frac{\partial}{\partial x}(X(x)) \leftarrow T \text{ is treated as a constant.}$$
$$= T(t)\underbrace{\left(-C\sin(\sqrt{\lambda}x) + D\cos(\sqrt{\lambda}x)\right)(\sqrt{\lambda})}_{X'}$$

$u_x(0,t)=0 \rightarrow X'(0)\cdot T(t)=0 \rightarrow X'(0)=0$

Recall: We ignore the case when $T(t)=0$ bc that gives us the trivial soln.

$X'(0) = (-C\sin(0) + D\cos(0))(\sqrt{\lambda})$
$X'(0) = 0$
$D\sqrt{\lambda} = 0$

Since we assumed that $\lambda>0$, this means $D=0$.

Now, we'll plug in the other boundary condition.
$u_x(\ell,t)=0 \rightarrow X'(\ell)\cdot T(t)=0 \rightarrow X'(\ell)=0$

$X'(\ell) = -C\sin(\sqrt{\lambda}\,\ell)\sqrt{\lambda}$
$X'(\ell) = 0$
$-C\sin(\sqrt{\lambda}\,\ell)\sqrt{\lambda} = 0$
$C\sin(\sqrt{\lambda}\,\ell) = 0$

Either $C=0$ or $\sin(\sqrt{\lambda}\,\ell)=0$.
We ignore the case when $C=0$. (Trivial soln)
$\sin(\sqrt{\lambda}\,\ell)=0 \rightarrow \sqrt{\lambda}\,\ell = n\pi,\ n \geq 0$
$\rightarrow \sqrt{\lambda} = \frac{n\pi}{\ell}$

For each value of n, we have a soln, denoted as U_n.

$$U_n(x,t) = A_n \cos\left(\frac{n\pi x}{\ell}\right) e^{-\left(\frac{n\pi}{\ell}\right)^2 kt}$$

Recall:

$$X = C\cos(\sqrt{\lambda}x) + \underbrace{D\sin(\sqrt{\lambda}x)}_{D=0}$$
$$= C\cos\left(\frac{n\pi x}{\ell}\right) \leftarrow \sqrt{\lambda} = \frac{n\pi}{\ell}$$

$$T = Ae^{-\lambda kt} \leftarrow \sqrt{\lambda} = \frac{n\pi}{\ell} \rightarrow \lambda = \left(\frac{n\pi}{\ell}\right)^2$$
$$= Ae^{-\left(\frac{n\pi}{\ell}\right)^2 kt}$$

Note: Because $\cos(0) = 1 \neq 0$, it does not give the trivial soln and we can't ignore the case when $n=0$.

Since each value of n gives a soln to the PDE, the general soln is a linear comb of all the U_n's.

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{\ell}\right) e^{-\left(\frac{n\pi}{\ell}\right)^2 kt}$$

Notice we start at $n=0$.

We now apply the initial condition $u(x,0) = \phi(x)$.

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{\ell}\right) e^{-\left(\frac{n\pi}{\ell}\right)^2 kt}$$

$$u(x,0) = \phi(x) = \underbrace{\sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{\ell}\right)}$$

This is called the Fourier (cosine) Series of $\phi(x)$. The reason we got cosine instead of sine is bc our boundary conditions involve the derivative of u.

Recall the Orthogonal Relations for Fourier Series:

$$\int_0^{\ell} \cos\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{m\pi x}{\ell}\right) = \begin{cases} 0, & \text{if } m \neq n \\ \frac{\ell}{2}, & \text{if } m = n \neq 0 \end{cases}$$

Since we have $m = n \neq 0$, we have to distinguish when $m=0$ and when $m \neq 0$.

1. When $m=0$:

$$\phi(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{\ell}\right) \leftarrow \text{From before}$$

Multiply both sides by $\cos\left(\frac{m\pi x}{\ell}\right)$ and integrate from 0 to ℓ.

$$\int_0^{\ell} \phi(x) \cos\left(\frac{m\pi x}{\ell}\right) = \sum_{n=0}^{\infty} A_n \underbrace{\int_0^{\ell} \cos\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{m\pi x}{\ell}\right)}_{\text{This integral equals 0 for when } n \neq 0\text{, and is equal to } \ell \text{ when } n=0}$$

$$= A_n \int_0^{\ell} 1 + 0 + 0 + \dots 0$$
$$= A_n \ell$$

$$A_n = \frac{1}{\ell}\int_0^{\ell} \phi(x) \underbrace{\cos\left(\frac{m\pi x}{\ell}\right)}_{\text{Equals 1 since } \cos(0)=1}, \text{ when } m=0$$
$$= \frac{1}{\ell}\int_0^{\ell} \phi(x)$$

2. when $m \neq 0$:

$$\phi(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{\ell}\right) \leftarrow \text{From Before}$$

Multiply both sides by $\cos\left(\frac{m\pi x}{\ell}\right)$ and integrate from 0 to ℓ.

$$\int_0^{\ell} \phi(x) \cos\left(\frac{m\pi x}{\ell}\right) = \sum_{n=0}^{\infty} A_n \underbrace{\int_0^{\ell} \cos\left(\frac{n\pi x}{\ell}\right) \cos\left(\frac{m\pi x}{\ell}\right)}_{\text{The integral equals 0 when } n \neq m \text{ and } \frac{\ell}{2} \text{ when } n=m}$$

$$= 0 + \ldots + 0 + A_n \frac{\ell}{2} + 0 \ldots + 0$$

$$A_n = \frac{2}{\ell} \int_0^{\ell} \phi(x) \cos\left(\frac{m\pi x}{\ell}\right)$$

– Earlier, we said that the separation constant is always negative, and we denoted it as $-\lambda$, where $\lambda \geq 0$. We'll see why here.

If $\lambda = 0$:

$T' = 0$ and $X'' = 0$

$T(t) = A$ $\quad$ $X(x) = Cx + D$

$u_x(0,t) = 0 \rightarrow X'(0) \cdot T(t) = 0 \rightarrow X'(0) = 0$

$X'(0) = C$

$\therefore C = 0$

$u_x(\ell,t) = 0 \rightarrow X'(\ell) \cdot T(t) = 0 \rightarrow X'(\ell) = 0$

$X'(\ell) = C$

$\therefore C = 0$

Both boundary conditions show that $C = 0$.

Furthermore, we see no restrictions on A or D. This means that, when $\lambda=0$, we get a constant A.D. This is the exact constant we get when $n=0$.

I.e.
$$u_0(x,t) = A_0 e^0 \cdot D_0 \cos(0)$$
$$= A_0 \cdot D_0$$

If $\lambda<0$:
Let $\lambda = -\mu$, where $\mu>0$.
Now, we have: $T' - \mu k T = 0$ and $X'' - \mu X = 0$.

From $T' - \mu k T = 0$, we get $T(t) = Ae^{\mu k t}$.
From $X'' - \mu X = 0$, we get $X(x) = Ce^{\sqrt{\mu}x} + De^{-\sqrt{\mu}x}$.

We now apply the boundary eqns.
First, we apply $u_x(0,t)=0$.
$$u_x = X' \cdot T$$
$$= \underbrace{(Ce^{\sqrt{\mu}x} - De^{-\sqrt{\mu}x})(\sqrt{\mu})}_{X'} \cdot T$$
$u_x(0,t)=0 \rightarrow X'(0) \cdot T(t) = 0 \rightarrow X'(0)=0$
$X'(0) = (C-D)\sqrt{\mu}$
$(C-D)\sqrt{\mu} = 0$
$C-D=0$ (we know $\mu \neq 0$)
$C=D$

Now, we use the other boundary condition.
$u_x(\ell,t)=0 \rightarrow X'(\ell) \cdot T(t) = 0 \rightarrow X'(\ell)=0$
$X'(\ell) = (Ce^{\sqrt{\mu}\ell} - De^{-\sqrt{\mu}\ell})\sqrt{\mu} = 0$
Using $C=D$, we get $C(e^{\sqrt{\mu}\ell} - e^{-\sqrt{\mu}\ell}) = 0$

This means that either $C=0$ or $e^{\sqrt{\mu}\ell} - e^{-\sqrt{\mu}\ell} = 0$.
$C=0$ gives us the trivial soln, so we ignore it.

$e^{\sqrt{\mu}\ell} - e^{-\sqrt{\mu}\ell} = 0$

$e^{\sqrt{\mu}\ell} = e^{-\sqrt{\mu}\ell}$

$\sqrt{\mu}\ell = -\sqrt{\mu}\ell$ (we took the ln of both sides.)

This means that $\mu = 0$, which is a contradiction.
Hence, there are no non-trivial solns for $\lambda < 0$.

$\therefore$ λ is never negative but may be equal to 0.

2. Important Formulas:

- $u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{\ell}\right) e^{-\left(\frac{n\pi}{\ell}\right)^2 kt}$

- If $m=0$, $A_n = \frac{1}{\ell}\int_0^{\ell} \phi(x)$

- If $m \neq 0$, and $n=m$, $A_n = \frac{2}{\ell}\int_0^{\ell} \phi(x)\cos\left(\frac{m\pi x}{\ell}\right)$

- $\int_0^{\ell} \cos\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{m\pi x}{\ell}\right) = \begin{cases} 0, & \text{if } m \neq n \\ \frac{\ell}{2}, & \text{if } m = n \neq 0 \end{cases}$

3. Examples

E.g. 1 Find the soln to the heat eqn on $0<x<\ell$ with $u_x(0,t)=0$, $u_x(\ell,t)=0$ and $\phi(x) = \cos\left(\frac{2\pi x}{\ell}\right)$.

Soln:

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{\ell}\right) e^{-\left(\frac{n\pi}{\ell}\right)^2 kt}$$

Because n could equal m, which equals 0, we need to split this into 2 cases.

Case 1 ($n=m=0$):

$$\begin{aligned} A_n &= \frac{1}{\ell}\int_0^\ell \phi(x)\, dx \\ &= \frac{1}{\ell}\int_0^\ell \cos\left(\frac{2\pi x}{\ell}\right) dx \\ &= \frac{1}{\ell}\left(\frac{\ell}{2\pi}\right)\left[\sin\left(\frac{2\pi x}{\ell}\right)\Big|_0^\ell\right] \\ &= \frac{1}{2\pi}\left(\underbrace{\sin(2\pi)}_{0} - \underbrace{\sin(0)}_{0}\right) \\ &= 0 \end{aligned}$$

Case 2 ($n=m\neq 0$):

$$\begin{aligned} A_n &= \frac{2}{\ell}\int_0^\ell \phi(x)\cos\left(\frac{m\pi x}{\ell}\right) \\ &= \frac{2}{\ell}\int \cos\left(\frac{2\pi x}{\ell}\right)\cos\left(\frac{m\pi x}{\ell}\right) \\ &= 0 \text{ if } n\neq 2, \text{ and } 1, \text{ if } n=2. \end{aligned}$$

$$\therefore \; A_n = \begin{cases} 0, & \text{if } n=m=0 \\ 0, & \text{if } n\neq 2 \\ 1, & \text{if } n=2 \end{cases}$$

E.g. 2 Find the soln to the heat eqn on $0<x<l$ with $u_x(0,t)=0$, $u_x(l,t)=0$ and $\phi(x)=1$

Soln:

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi}{l}\right)^2 kt}$$

If $n=0$: $(n=m=0)$

$$A_n = \frac{1}{l}\int_0^l \phi(x)$$

$$= \frac{1}{l}\int_0^l 1$$

$$= \frac{1}{l}(l-0)$$

$$=1$$

If $n\neq 0$: $(n=m\neq 0)$

$$A_n = \frac{2}{l}\int_0^l \phi(x)\cos\left(\frac{m\pi x}{l}\right)$$

$$= \frac{2}{l}\int_0^l \cos\left(\frac{m\pi x}{l}\right)$$

$$= \frac{2}{m\pi}\left(\sin(m\pi) - \sin(0)\right)$$

$$=0$$

$$\therefore \; A_n = \begin{cases} 1, & \text{if } n=0 \\ 0, & \text{if } n\neq 0 \end{cases}$$

E.g. 3 Find the soln to the heat eqn on $0<x<\ell$ with $u(0,t)=0$, $u(\ell,t)=0$ and $u(x,0)=\phi(x)$.

Soln:
Notice that the boundary conditions do not differentiate u w.r.t x.
I.e. we have $u(0,t)=0$ instead of $u_x(0,t)=0$.

Assume $u(x,t)=X(x)\cdot T(t)$ ← Separation of var
The PDE $u_t=ku_{xx}$ now becomes
$X\cdot T' = k\cdot X''\cdot T$

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda$$

$T' = -\lambda kT$

$T = Ae^{-\lambda kt}$

$X'' = \lambda X$
$X'' - \lambda X = 0$
$X = C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)$

Now, lets plug in the boundary conditions.
$u(0,t)=0 \rightarrow X(0)\cdot T(t)=0 \rightarrow X(0)=0$
$X(0)=C$
$X(0)=0$
$\therefore C=0$

$u(\ell,t)=0 \rightarrow X(\ell)\cdot T(t)=0 \rightarrow X(\ell)=0$
$X(\ell)=D\sin(\sqrt{\lambda}\ell)$
$X(\ell)=0$
$D\sin(\sqrt{\lambda}\ell)=0$
$\sqrt{\lambda}\ell=n\pi,\ n>0$
$\sqrt{\lambda}=\frac{n\pi}{\ell}$

For each n, we have $u_n(x,t) = A_n \sin\left(\frac{n\pi x}{\ell}\right) e^{-\left(\frac{n\pi}{\ell}\right)^2 kt}$
To get the general soln, we need to sum up all the u_n's.

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right) e^{-\left(\frac{n\pi}{\ell}\right)^2 kt}$$

To solve for A_n, we'll plug in the initial condition.

$$u(x,0) = \phi(x)$$

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)$$

Now, we multiply both sides by $\sin\left(\frac{m\pi x}{\ell}\right)$ and integrate from 0 to ℓ.

$$\int_0^{\ell} \phi(x) \sin\left(\frac{m\pi x}{\ell}\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{m\pi x}{\ell}\right) \sin\left(\frac{n\pi x}{\ell}\right)$$
$$= A_n \cdot \frac{\ell}{2}$$

$$A_n = \frac{2}{\ell} \int_0^{\ell} \phi(x) \sin\left(\frac{m\pi x}{\ell}\right)$$

E.g. 4 Find the soln to the heat eqn on $0<x<\ell$ with $u(0,t)=0$, $u_x(\ell,t)=0$ and $u(x,0)=\phi(x)$.

Soln:
Note: This is sometimes called a mixed boundary eqn.

Assume $u(x,t)=X(x)\cdot T(t)$
The PDE $u_t=ku_{xx}$ is now: $T'X=k\cdot T\cdot X''$.
$$\frac{T'}{k\cdot T}=\frac{X''}{X}=-\lambda$$

$$T=Ae^{-\lambda kt}$$

$$X=C\cos(\sqrt{\lambda}x)+D\sin(\sqrt{\lambda}x)$$

Now, let's plug in the boundary conditions.
First, we'll use $u(0,t)=0$.
$u(0,t)=0 \rightarrow X(0)\cdot T(t)=0 \rightarrow X(0)=0$
$X(0)=C$
$X(0)=0$
$\therefore C=0$

Next, we'll use $u_x(\ell,t)=0$.
$$u_x=X'(x)\cdot T(t)$$
$$=\underbrace{D\cos(\sqrt{\lambda}x)(\sqrt{\lambda})}_{X'}\cdot T(t)$$
$u_x(\ell,t)=0 \rightarrow X'(\ell)\cdot T(t)=0 \rightarrow X'(\ell)=0$
$X'(\ell)=\sqrt{\lambda}D\cos(\sqrt{\lambda}\ell)$
$X'(\ell)=0$
$\sqrt{\lambda}D\cos(\sqrt{\lambda}\ell)=0$
Since we assume that $\lambda>0$, and $D\neq 0$, we get
$\cos(\sqrt{\lambda}\ell)=0$

$$\sqrt{\lambda}\, \ell = \frac{(2n+1)\pi}{2}, \quad n \geq 0$$

$$\sqrt{\lambda} = \frac{(2n+1)\pi}{2\ell}$$

Each value of n gets us a soln, denoted as $U_n(x,t)$.

$$U_n(x,t) = A_n \sin\left(\frac{(2n+1)\pi x}{2\ell}\right) e^{-\left(\frac{(2n+1)\pi}{2\ell}\right)^2 kt}$$

To get the general soln of the PDE, we need a linear combination of each of the U_n's.

$$u(x,t) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{(2n+1)\pi x}{2\ell}\right) e^{-\left(\frac{(2n+1)\pi}{2\ell}\right)^2 kt}$$

To solve for A_n, we'll use the initial condition.

$u(x,0) = \phi(x)$

$$u(x,0) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{(2n+1)\pi x}{2\ell}\right)$$

$$\phi(x) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{(2n+1)\pi x}{2\ell}\right)$$

Now, we'll multiply both sides by $\sin\left(\frac{(2m+1)\pi x}{2\ell}\right)$ and integrate from 0 to ℓ.

$$\int_0^{\ell} \phi(x) \sin\left(\frac{(2m+1)\pi x}{2\ell}\right) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{(2n+1)\pi x}{2\ell}\right) \sin\left(\frac{(2m+1)\pi x}{2\ell}\right)$$

$$= A_n \cdot \frac{\ell}{2}$$

$$A_n = \frac{2}{\ell} \int_0^{\ell} \phi(x) \sin\left(\frac{(2m+1)\pi x}{2\ell}\right)$$

Note: Since we have a sine function, the above also includes the case when $n=m=0$.