Orthogonal Relations for Fourier Series Proofs

1. Review:

   1. $\sin(n\pi) = 0, \; n \in \mathbb{N}$

   2. $\int \cos(x) = \sin(x)$

   3. $$\int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \cdot \sin\left(\frac{m\pi x}{\ell}\right) = \begin{cases} 0, & \text{if } m \neq n \\ \frac{\ell}{2}, & \text{if } m = n \end{cases}$$

   4. $$\int_0^{\ell} \cos\left(\frac{n\pi x}{\ell}\right) \cdot \cos\left(\frac{m\pi x}{\ell}\right) = \begin{cases} 0, & \text{if } m \neq n \\ \frac{\ell}{2}, & \text{if } (m=n) \neq 0 \\ \ell, & \text{if } m=n=0 \end{cases}$$

2. Proof of (3):

$$\int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \cdot \sin\left(\frac{m\pi x}{\ell}\right) = \begin{cases} 0, & \text{if } m \neq n \\ \frac{\ell}{2}, & \text{if } m = n \end{cases}$$

Case 1 $(m=n)$:

$$\int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \cdot \sin\left(\frac{m\pi x}{\ell}\right)$$

$$= \int_0^{\ell} \sin^2\left(\frac{n\pi x}{\ell}\right)$$

Now, we'll use u-sub.

Let $u = \frac{n\pi x}{\ell} \longrightarrow \begin{array}{l} x=0 \rightarrow u=0 \\ x=\ell \rightarrow u=n\pi \end{array}$

$$du = \frac{n\pi}{\ell}\, dx$$

$$du \cdot \frac{\ell}{n\pi} = dx$$

$$= \frac{\ell}{n\pi} \int_0^{n\pi} \sin^2(u)\, du$$

$$= \frac{\ell}{n\pi} \int_0^{n\pi} \left(\frac{1}{2} - \frac{\cos(2u)}{2}\right) du$$

$$= \frac{\ell}{n\pi} \left[\int_0^{n\pi} \frac{1}{2} - \frac{1}{2}\int_0^{n\pi} \cos(2u)\right]$$

$$= \frac{\ell}{n\pi} \left[\frac{x}{2}\Big|_0^{n\pi} - \frac{1}{4}\sin(2u)\Big|_0^{n\pi}\right]$$

$$= \frac{\ell}{n\pi} \left[\frac{n\pi}{2} - \frac{1}{4}\underbrace{\left(\underbrace{\sin(2n\pi)}_{0} - \underbrace{\sin(0)}_{0}\right)}_{0}\right]$$

$$= \frac{\ell}{\cancel{n\pi}} \cdot \frac{\cancel{n\pi}}{2}$$

$$= \frac{\ell}{2}$$

Case 2 $(m \neq n)$:

$$\int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \cdot \sin\left(\frac{m\pi x}{\ell}\right)$$

$$\sin(A) \cdot \sin(B) = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$

$$= \frac{1}{2}\int_0^{\ell} \cos\left(\frac{(n-m)\pi x}{\ell}\right) - \cos\left(\frac{(n+m)\pi x}{\ell}\right)$$

let $\alpha = n-m$

let $\beta = n+m$

$$= \frac{1}{2}\left[\int_0^{\ell} \cos\left(\frac{\alpha\pi x}{\ell}\right) - \int_0^{\ell} \cos\left(\frac{\beta\pi x}{\ell}\right)\right]$$

$$\int_0^{\ell} \cos\left(\frac{\alpha\pi x}{\ell}\right)$$

let $u = \frac{\alpha\pi x}{\ell} \longrightarrow \begin{array}{l} x=0 \rightarrow u=0 \\ x=\ell \rightarrow u=\alpha\pi \end{array}$

$$du = \frac{\alpha\pi}{\ell}\,dx$$

$$du \cdot \frac{\ell}{\alpha\pi} = dx$$

$$\int_0^{\alpha\pi} \frac{\ell}{\alpha\pi} \cos(u)$$

$$= \frac{\ell}{\alpha\pi} \int_0^{\alpha\pi} \cos(u)$$

$$= \frac{\ell}{\alpha\pi}\left[\sin(u)\Big|_0^{\alpha\pi}\right]$$

$$= \frac{\ell}{\alpha\pi}\left[\sin(\alpha\pi) - \sin(0)\right]$$

$$= 0$$

$$\int_0^{\ell} \cos\left(\frac{\beta \pi x}{\ell}\right)$$

let $u = \dfrac{\beta \pi x}{\ell} \longrightarrow$ $x=0 \rightarrow u=0$
$x=\ell \rightarrow u=\beta\pi$

$$du = \frac{\beta \pi}{\ell} \quad dx$$

$$du = \frac{\ell}{\beta \pi} \; dx$$

$$\int_0^{\beta\pi} \frac{\ell}{\beta\pi} \cos(u)$$

$$= \frac{\ell}{\beta\pi} \int_0^{\beta\pi} \cos(u)$$

$$= \frac{\ell}{\beta\pi}\left[\sin(u) \Big|_0^{\beta\pi}\right]$$

$$= 0$$

$$\therefore \quad \frac{1}{2}\left[\int_0^{\ell} \cos\left(\frac{\alpha \pi x}{\ell}\right) - \int_0^{\ell} \cos\left(\frac{\beta \pi x}{\ell}\right)\right] = 0$$

3. Proof of (4):

$$\int_0^{\ell} \cos\left(\frac{m\pi x}{\ell}\right) \cdot \cos\left(\frac{n\pi x}{\ell}\right) = \begin{cases} 0, & \text{if } m \neq n \\ \ell, & \text{if } m = n = 0 \\ \frac{\ell}{2}, & \text{if } (m=n) \neq 0 \end{cases}$$

Case 1 $(m=n=0)$:

$\cos(0) = 1$

$$\int_0^{\ell} 1 = x \Big|_0^{\ell}$$

$$= \ell - 0$$

$$= \ell$$

Case 2 $((m=n) \neq 0)$:

$$\int_0^{\ell} \cos^2\left(\frac{m\pi x}{\ell}\right)$$

Let $u = \frac{m\pi x}{\ell} \rightarrow$ $x = 0 \rightarrow u = 0$
$x = \ell \rightarrow u = m\pi$

$$du = \frac{m\pi}{\ell} \; dx$$

$$du \cdot \frac{\ell}{m\pi} = dx$$

$$\int_0^{m\pi} \frac{\ell}{m\pi} \cos^2(u)$$

$$= \frac{\ell}{m\pi} \int_0^{m\pi} \frac{1 + \cos(2u)}{2}$$

Note:
$$\cos^2(x) = 1 - \sin^2(x)$$
$$= 1 - \left(\frac{1 - \cos(2x)}{2}\right)$$
$$= 1 - \frac{1}{2} + \frac{\cos(2x)}{2}$$
$$= \frac{1}{2} + \frac{\cos(2x)}{2}$$

$$= \frac{\ell}{m\pi}\left[\int_0^{m\pi} \frac{1}{2} + \int_0^{m\pi} \frac{\cos(2u)}{2}\right]$$

$$= \frac{\ell}{m\pi}\left[\frac{x}{2}\Big|_0^{m\pi} + \underbrace{\frac{1}{4}\sin(2u)\Big|_0^{m\pi}}_{\text{Equals } 0}\right]$$

$$= \frac{\ell}{\cancel{m\pi}}\left(\frac{\cancel{m\pi}}{2}\right)$$

$$= \frac{\ell}{2}$$

Case 3 ($m \neq n$):

$$\int_0^{\ell} \cos\left(\frac{m\pi x}{\ell}\right) \cdot \cos\left(\frac{n\pi x}{\ell}\right)$$

$$\cos(A) \cdot \cos(B) = \frac{\cos(A-B) + \cos(A+B)}{2}$$

$$= \frac{1}{2}\int_0^{\ell} \cos\left(\frac{(m-n)\pi x}{\ell}\right) + \cos\left(\frac{(m+n)\pi x}{\ell}\right)$$

let $\alpha = m-n$

let $\beta = m+n$

$$= \frac{1}{2}\left[\underbrace{\int_0^{\ell} \cos\left(\frac{\alpha\pi x}{\ell}\right)}_{0} + \underbrace{\int_0^{\ell} \cos\left(\frac{\beta\pi x}{\ell}\right)}_{0}\right]$$

$$= 0$$

Check pages 3 and 4 for the calculations.