MATA22 Booklet 4

Definitions:

1. A homogeneous linear system of equations is of the form:
   $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0$
   $a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0$
   |
   |
   $a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0$
   or $A\mathbf{x} = \mathbf{0}$, where A is the coefficient matrix.

   Every homogeneous system is consistent because the zero vector is a solution. This is known as the trivial solution.

   If A ~ H and H has a pivot in every column, then $A\mathbf{x} = \mathbf{0}$ only has the trivial solution.

   If A has fewer rows than columns and A ~ H, then it is impossible for H to have a pivot in every column. Therefore, $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions.

2. The nullspace of A, denoted by N, is the set of solutions to $A\mathbf{x} = \mathbf{0}$.
   I.e. $N = \{\mathbf{x} \in R^n \mid A\mathbf{x} = \mathbf{0}\}$.

3. The row space of A is the span of the row vectors of A. If A is a m*n matrix, then the row space of A is a subset of R^n.

4. The column space of A is the span of the column vectors of A. If A is a m*n matrix, then the column space of A is a subset of R^m.

5. Let $\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n} \in R^n$.
   Let $s_1, s_2, \ldots s_n \in R$.

   If $s_1\mathbf{v_1} + s_2\mathbf{v_2} + \ldots s_n\mathbf{v_n} = \mathbf{0}$ has exactly 1 solution, then we say that the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n}$ are linearly independent.
   I.e. $s_1\mathbf{v_1} + s_2\mathbf{v_2} + \ldots s_n\mathbf{v_n} = \mathbf{0}$ is linearly independent iff $s_1 = s_2 = \ldots s_n = 0$.

   If $s_1\mathbf{v_1} + s_2\mathbf{v_2} + \ldots s_n\mathbf{v_n} = \mathbf{0}$ has more than 1 solution, then we say that the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n}$ are linearly dependent.

   2 vectors are linearly independent if they are non – zero and non – parallel.

6. W is a subset of R^n if W satisfies the following conditions:

   1. If **u**, **v** $\in$ W, then **u** + **v** $\in$ W. (Closure by addition.)
   2. If **v** $\in$ W and r $\in$ R, then r**v** $\in$ W. (Closure by scalar multiplication.)

7. Let W be a subset of R^n.
   W is a subspace of R^n if W satisfies the following conditions:

   1. W is non – empty. (Non – empty.)
   2. If **u**, **v** $\in$ W, then **u** + **v** $\in$ W. (Closure by addition.)
   3. If **v** $\in$ W and r $\in$ R, then r**v** $\in$ W. (Closure by scalar multiplication.)

   Note: The zero vector is in all subspaces. If you have a W that does not include the zero vector, then it is not a subspace of R^n.

8. Let W is a subspace of R^n. If B = {$\mathbf{b_1}$, $\mathbf{b_2}$, …, $\mathbf{b_n}$} is a subset of W, then B is a basis for W if every vector in W can be written uniquely as a linear combination of the vectors in B.
   $\mathbf{b_1}$, $\mathbf{b_2}$, …, $\mathbf{b_n}$ must be linearly independent in order for B to be a basis.
   I.e. B is a basis for W if B is the smallest set of vectors that spans W.

   How to find the basis for a few column vectors $\mathbf{b_1}$, $\mathbf{b_2}$, …, $\mathbf{b_n}$:
   1. Write the column vectors as a matrix like such: A = [$\mathbf{b_1}$, $\mathbf{b_2}$, …, $\mathbf{b_n}$]
   2. Row reduce A to either REF or RREF. Let A ~ H.
   3. The basis is the set of all columns in H that have a pivot.

9. Let W be a subspace of R^n. The number of elements in a basis for W is called the dimension of W, denoted as dim(W).

10. Let A be an m*n matrix. The dimension of the column space of A equals to the dimension of the row space of A. Both are denoted as rank(A).
    I.e. rank(A) is the number of columns with pivots.

11. Let A be an m*n matrix. The dimension of the nullspace is called the nullity of A. It is denoted as nullity(A).
    I.e. nullity(A) is the number of columns without pivots.

12. The rank-nullity equation states that rank(A) + nullity(A) = the number of columns A has.

Theorems:

1. Let $\mathbf{a}$ and $\mathbf{b}$ be solutions to $A\mathbf{x} = \mathbf{0}$. Then, $S_1\mathbf{a} + S_2\mathbf{b}$ are also solutions to $A\mathbf{x} = \mathbf{0}$.

2. Let $A\mathbf{x} = \mathbf{b}$ be a linear system with a solution $\mathbf{p}$. Then:
   1. If $\mathbf{h}$ is in the nullspace of A, then $\mathbf{p} + \mathbf{h}$ is also a solution to $A\mathbf{x} = \mathbf{b}$.
   2. If $\mathbf{q}$ is any solution to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{q} = \mathbf{p} + \mathbf{h_1}$ for some $\mathbf{h_1} \in \mathbf{h}$.

3. Let A be a m*n matrix. Then, the following are equivalent:
   1. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b} \in R^n$.
   2. A is row equivalent to the n*n identity matrix I.
   3. A is invertible.
   4. The column vectors of A form a basis for R^n.

4. Let A be a m*n matrix such that m > n. Then, the following are equivalent:
   1. Each consistent system $A\mathbf{x} = \mathbf{b}$ has a unique solution.
   2. The RREF of A consists of the n*n identity matrix on top followed by m – n rows of zeroes.
   3. The column vectors of A form a basis for the column space of A.

5. Let A be a m*n matrix such that m < n. Then, $A\mathbf{x} = \mathbf{b}$ is a linear system with fewer equations than unknowns. If $A\mathbf{x} = \mathbf{b}$ is consistent, then it has an infinite number of solutions.

6. $B = \{\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_n}\}$ is a basis for the subspace W of R^n iff $B = \{\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_n}\}$ is a linearly independent set of vectors and $W = sp(\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_n})$.

7. Let W be a subspace of R^n. Let $\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_k}$ be vectors in W that span W. Let $\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_m}$ be linearly independent vectors in W. Then, $k \geq m$.

8. Any 2 bases for a subspace W of R^n contain the same number of vectors.

9. Every subspace W of R^n has a basis and $\dim(W) \leq n$.

10. Every set of linearly independent vectors in R^n can be enlarged, if necessary, to become a basis for R^n.

11. If W is a subspace of R^n and $\dim(W) = k$, then:
    1. Every linearly independent set of k vectors in W is a basis for W.
    2. Every set of k vectors that spans W is a basis for W.

12. Let A be a m*n matrix and let A ~ H.
    Let C be the column space of A.
    Let R be the row space of A.
    Let N be the nullspace of A.
    Then:

    1. The non – zero rows of H form a basis of R^n.

    2. A basis for c consists of all the columns of A corresponding to the columns of H that contains pivots.

    3. To find a basis for N, we solve for the linear system H**x** = **0** and find a basis for the solution set.

    4. dim(c) = dim(r) = number of columns in H with pivots.

    5. dim(N) = number of columns in H without pivots.

13. A n*n matrix A is invertible iff rank(A) = n.

<u>Examples:</u>

1. Let A = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix}$ Find the nullspace and basis for A.

   Solution:

   $$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 2 & 3 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right)$$

   R2 – R1

   $$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right)$$

   R2/3

   $$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right)$$

   R1 – R2

   $$\left(\begin{array}{ccc|c} 0 & 2 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right)$$

R3 – 2*R2
$$\left(\begin{array}{ccc|c} 0 & 2 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

R2 – 2R3
$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

R3 – R1
$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

Row Interchange
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

The basis for A is {$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$}.

The nullspace for A is {}.

2. Let A = $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{bmatrix}$. Find the basis and nullspace of A.

Solution:
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 3 & 3 & 0 \end{array}\right)$$

R2 – R1
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

R1 – 2R2
$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

The basis for A is {$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$}.

Let x_3 = s.

x_1 = -3s

The nullspace of A is sp($\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$).

3. Determine if the vectors [1, 1, 3], [3, 0, 4] and [1, 4, -1] are linearly independent.

Solution:

We want to show whether or not $r_1\left(\begin{bmatrix}1\\1\\3\end{bmatrix}\right) + r_2\left(\begin{bmatrix}3\\0\\4\end{bmatrix}\right) + r_3\left(\begin{bmatrix}1\\4\\-1\end{bmatrix}\right) = \mathbf{0}$ only has 1 solution and that solution is $r_1 = r_2 = r_3 = 0$.

$$\left(\begin{array}{ccc|c}1 & 3 & 1 & 0\\1 & 0 & 4 & 0\\3 & 4 & -1 & 0\end{array}\right)$$

R1 – R2

$$\left(\begin{array}{ccc|c}0 & 3 & -3 & 0\\1 & 0 & 4 & 0\\3 & 4 & -1 & 0\end{array}\right)$$

R3 – 3R2

$$\left(\begin{array}{ccc|c}0 & 3 & -3 & 0\\1 & 0 & 4 & 0\\0 & 4 & -13 & 0\end{array}\right)$$

R1/3

$$\left(\begin{array}{ccc|c}0 & 1 & -1 & 0\\1 & 0 & 4 & 0\\0 & 4 & -13 & 0\end{array}\right)$$

R3 – 4R1

$$\left(\begin{array}{ccc|c}0 & 1 & -1 & 0\\1 & 0 & 4 & 0\\0 & 0 & -9 & 0\end{array}\right)$$

R3/-9

$$\left(\begin{array}{ccc|c}0 & 1 & -1 & 0\\1 & 0 & 4 & 0\\0 & 0 & 1 & 0\end{array}\right)$$

$$\left(\begin{array}{ccc|c}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{array}\right)$$

Here, you can see that the only solution is $r_1 = r_2 = r_3 = 0$. Therefore, the vectors are linearly independent.