Techniques of Integration

1. Substitution:

E.g. Evaluate $\int \sin(x+3)\,dx$

Soln:

Let $u = x+3$

$du = dx$

$\int \sin(u)\,du$

$= -\cos(u) + C$

$= -\cos(x+3) + C$

2. By Parts:

E.g. Evaluate $\int xe^x\,dx$

Soln:

Let $u = x \rightarrow du = dx$

Let $dv = e^x \rightarrow v = e^x$

$uv - \int v\,du$

$= xe^x - \int e^x\,dx$

$= xe^x - e^x$

Note: If there are lower and upper bounds, then the eqn becomes

$$uv\Big|_a^b - \int_a^b v\,du$$

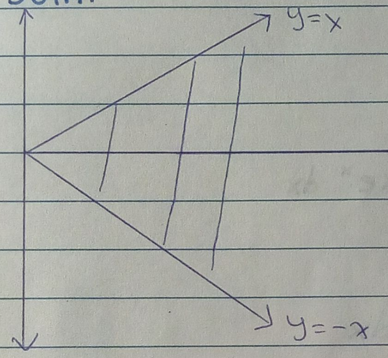
3. Improper Integrals:

Type 1:

- when either the lower bound or the upper bound or both is/are ∞ or $-\infty$.

- E.g. Evaluate $\int_D e^{-x^2} dA$ where D is the region where $x \geq 0$ and $-x \leq y \leq x$.

Soln:



$$\int_0^\infty \int_{-x}^{x} e^{-x^2} \, dy \, dx$$

$$= \int_0^\infty (2x)(e^{-x^2}) \, dx$$

Let $u = -x^2$

$du = -2x \, dx$

$\frac{du}{-2x} = dx$

$$\lim_{A\to\infty}\int_0^A -e^u\,du$$

$$= -\lim_{A\to\infty}\int_0^A e^u\,du$$

$$= -\lim_{A\to\infty}\left[e^u\Big|_0^A\right]$$

$$= -\lim_{A\to\infty}\left[e^{-x^2}\Big|_0^A\right]$$

$$= -\lim_{A\to\infty}\left[e^{-A^2}-1\right]$$

$$= -(-1)$$

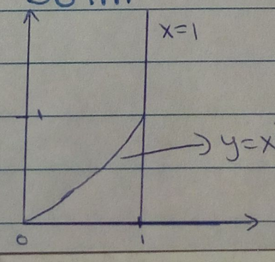
$$= 1$$

Type 2:

- when either the lower bound, upper bound or a number between the lower and upper bound would make the eqn undefined.

- E.g. Evaluate $\int_D \frac{1}{(x+y)^2}\,dA$ where D is the set $\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 1,\ 0\le y\le x^2\}$.

Soln:



Here, $\frac{1}{(x+y)^2}$ is undefined at the point

$(0,0)$.

$$\lim_{A\to 0^+} \int_A^1 \int_0^{x^2} \frac{1}{(x+y)^2}\, dy\, dx$$

$$= -\lim_{A\to 0^+} \int_A^1 \left[\frac{1}{x+y} \Big|_0^{x^2} \right] dx$$

$$= -\lim_{A\to 0^+} \int_A^1 \frac{1}{x^2+x} - \frac{1}{x}\, dx$$

$$= \lim_{A\to 0^+} \int_A^1 \frac{(x+1)-1}{x(x+1)}\, dx$$

$$= \lim_{A\to 0^+} \int_A^1 \frac{1}{x+1}\, dx$$

$$= \lim_{A\to 0^+} \left(\ln|x+1| \Big|_A^1 \right)$$

$$= \lim_{A\to 0^+} \left(\ln(2) - \ln(A+1) \right)$$

$$= \ln(2)$$