

MATB44 Week 4 Notes

I. Review of Week 3 Material

a) Homogeneous Eqns With Constant Coefficients

- Rule 1: $y = e^{rt}$
- Rule 2: We can combine 2 solns to get a new soln.

b) Homogeneous Eqns With Constant Coefficients and 2 Real Distinct Roots

- Suppose $ay'' + by' + cy = 0$.

From rule 1, we know that $y = e^{rt}$.

$$y' = re^{rt}$$

$$y'' = r^2 e^{rt}$$

$$ar^2(e^{rt}) + br(e^{rt}) + c(e^{rt}) = 0$$

$$(e^{rt})(ar^2 + br + c) = 0$$

Since $e^{rt} \neq 0$, we can divide both sides by it.

$ar^2 + br + c = 0 \leftarrow$ Called characteristic equation

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$y_1 = e^{r_1 t}, \quad y_2 = e^{r_2 t}$$

From rule 2, we know that we can combine 2 solns to get a new one.

$$y = c_1 y_1 + c_2 y_2$$

c) Wronksian

- Useful for finding the second soln in homogeneous eqns with constant coefficients and repeated roots as well as determining if a pair of solns is a fundamental set of soln.
- Consider $p(t)y'' + q(t)y' + r(t)y = 0$ and $y(t_0) = y_0$ and $y'(t_0) = y'_0$. Suppose y_1 and y_2 are solns. Then, $y = c_1 y_1 + c_2 y_2$ is also a general soln. We want to know if $y = c_1 y_1 + c_2 y_2$ is also a general soln. In order to be a general soln, it must satisfy the initial conditions.

$$y_0 = c_1 y_1(t_0) + c_2 y_2(t_0)$$

$$y'_0 = c_1 y'_1(t_0) + c_2 y'_2(t_0)$$

We can use Cramer's rule to find c_1 and c_2 .

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y'_0 & y'_2(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}} \quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y'_1(t_0) & y'_0 \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}}$$

The denominator is the **Wronksian**.

I.e. If y_1 and y_2 are 2 solns to a linear homogeneous eqn with constant coefficients, then

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} \\ &= y_1 y'_2 - y'_1 y_2 \end{aligned}$$

Notice that the only thing preventing us from finding C_1 and C_2 is if $\omega=0$.

If $\omega \neq 0$ and y_1 and y_2 are solns, then

y_1 and y_2 are a **fundamental set/pair**

of solns and $y = C_1y_1 + C_2y_2$ is a general soln.

I.e. Take $y = C_1y_1 + C_2y_2$ and the initial conditions $y(t_0) = y_0$ and $y'(t_0) = y'_0$.

$$\begin{aligned} C_1y_1(t_0) + C_2y_2(t_0) &= y_0 \\ C_1y'_1(t_0) + C_2y'_2(t_0) &= y'_0 \end{aligned} \quad \left. \begin{array}{l} \text{This system has a} \\ \text{soln for any RHS iff} \\ \omega \neq 0. \end{array} \right\}$$

Note: Recall from linear algebra that a matrix is linearly independent iff the determinant of the matrix $\neq 0$. Hence, if $\omega \neq 0$, y_1 and y_2 are linearly independent. otherwise, they are linearly dependent.

Note: If $\omega = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} \neq 0$, then y_1 and y_2 are a fundamental pair of solns.

E.g. 1 Let $f(t) = e^{2t}$, $\omega = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = 3e^{4t}$.

Solve for $g(t)$.

Soln:

$$\omega = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

$$fg' - f'g = 3e^{4t}$$

$$e^{2t}g' - 2e^{2t}g = 3e^{4t}$$

$$e^{2t}(y' - 2y) = 3e^{4t}$$

$$y' - 2y = 3e^{2t} \leftarrow \text{Linear first order}$$

$$y = 3te^{2t} + ce^{2t}$$

We keep the c.

- We can solve w without solving the eqn.

$$w = y_1 y_2' - y_1' y_2$$

$$\begin{aligned}\frac{dw}{dt} &= y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2 \\ &= y_1 y_2'' - y_1'' y_2\end{aligned}$$

Recall that

$$y_1'' + p(t)y_1' + q(t)y_1 = 0 \rightarrow y_1'' = -p(t)y_1' - q(t)y_1$$

$$y_2'' + p(t)y_2' + q(t)y_2 = 0 \rightarrow y_2'' = -p(t)y_2' - q(t)y_2$$

$$\begin{aligned}\frac{dw}{dt} &= y_1(-p(t)y_2' - q(t)y_2) - y_2(-p(t)y_1' - q(t)y_1) \\ &= -p(t)y_1 y_2' - q(t)y_1 y_2 + p(t)y_1' y_2 + q(t)y_1 y_2\end{aligned}$$

$$= -p(t)(y_1 y_2' - y_1' y_2)$$

$$= -p(t)w$$

$$\frac{1}{w} dw = -p(t) dt$$

$$\int \frac{1}{w} dw = \int -p(t) dt$$

$$\ln(w) + C = \int -p(t) dt$$

$$\ln(w) = \int -p(t) dt + C$$

$$w = e^{-\int -p(t) dt + C}$$

$$= e^{-\int -p(t) dt} \cdot e^C$$

$$= C'e^{-\int -p(t) dt} \leftarrow \text{Abel's Formula}$$

- The Wronksian Dichotomy for 2 Solns states that for 2 solns, $w=0$ for all t or $w \neq 0$ for all t .

Abel's formula proves the dichotomy.
Either $c'=0$ and $w=0$ everywhere or $c' \neq 0$ and $w \neq 0$ everywhere.

E.g. 2 Let $x^2 y'' - x(x+2)y' + (x+2)y = 0$

a) Verify $y_1 = x$ and $y_2 = xe^x$

Soln:

Simply plug y_1 and y_2 into the eqn above and see if LHS=RHS.

y_1 :

$$\begin{aligned} \text{LHS} &= -x(x+2) + (x+2)x \\ &= 0 \end{aligned}$$

RHS = 0

LHS = RHS

y_2 :

$$\begin{aligned} \text{LHS} &= x^2(xe^x)'' - x(x+2)(xe^x)' + (x+2)(xe^x) \\ &= x^2(x+2)e^x - x(x+2)(x+1)e^x + (x+2)(x)(e^x) \\ &= e^x(x^3 + 2x^2 - x^3 - 3x^2 - 2x + x^2 + 2x) \\ &= e^x(0) \\ &= 0 \end{aligned}$$

RHS = 0

LHS = RHS

b) Do y_1 and y_2 make a fundamental pair of solns?

Soln:

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} \\ &= \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} \\ &= x(e^x + xe^x) - xe^x \\ &= xe^x + x^2e^x - xe^x \\ &= x^2e^x \end{aligned}$$

$W \neq 0$ IFF $x \neq 0$

Note: We can't use Abel's formula here. All it says is that $W=0$ iff $c'=0$, which doesn't give us enough information.

E.g. 3 Suppose we have $y_1 = e^{-t}$ and $y_2 = 2e^{-t}$. Prove that they are not a fundamental pair of solns.

Solns:

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} \\ &= \begin{vmatrix} e^{-t} & 2e^{-t} \\ -e^{-t} & -2e^{-t} \end{vmatrix} \\ &= (e^{-t})(-2e^{-t}) - (2e^{-t})(-e^{-t}) \\ &= 0 \end{aligned}$$

E.g. 4 Let $y_1 = t$ and $y_2 = \sin t$. Are y_1 and y_2 solns? a fundamental pair of

Soln:

$$\begin{aligned} w &= \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} \\ &= \begin{vmatrix} t & \sin t \\ 1 & \cos t \end{vmatrix} \\ &= t \cos t - \sin t \end{aligned}$$

If $t=0$, $w=0$.

Furthermore, $w\left(\frac{\pi}{2}\right) = -1 < 0$ } Contradiction
 $w(2\pi) = 2\pi > 0$ }

Abel's formula either gives all positives or all negatives.

Hence, y_1 and y_2 cannot be a fundamental pair of solns.

d) Homogeneous Eqns with Constant Coefficients and Repeated Roots

- Here $r_1 = r_2$. However, this poses a problem. We need y_1 and y_2 , and right now, we just have y_1 .
- E.g. 5 Solve $y'' + 2y' + y = 0$

Soln:

$$r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0$$

$$r = -1$$

$$y_1 = e^{-t}$$

To find y_2 , we'll use the Wronksian.

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} \\ &= y_1 y'_2 - y'_1 y_2 \\ &= (e^{-t}) y'_2 - (-e^{-t}) y_2 \\ &= (e^{-t}) y'_2 + (e^{-t}) y_2 \\ &= (e^{-t})(y'_2 + y_2) \end{aligned}$$

$$\begin{aligned} \text{Also, } W &= C' \cdot e^{-\int p dt} \\ &= C' \cdot e^{-\int 2 dt} \\ &= C' \cdot e^{-2t + C_1} \\ &= e^{-2t} \end{aligned}$$

$$(e^{-t})(y'_2 + y_2) = e^{-2t}$$

$$y'_2 + y_2 = e^{-t} \leftarrow \text{Linear Differential eqn}$$

$$y_2 = t e^{-t}$$

Note: If we have $y'' + b y' + c y = 0$ and $r_1 = r_2$ then $y_1 = e^{r_1 t}$ and $y_2 = t e^{r_1 t}$.

This is called the **Repeated Roots Rule**.

e) Homogeneous Eqns With Constant Coefficients and Complex Roots

- Z is a complex number if it can be written in the form: $z = a + i b$.

a and b are real numbers.

$$i = \sqrt{-1} \Leftrightarrow i^2 = -1$$

a is the **real part**.

b is the **imaginary part**. (Does not include i).

i is the **imaginary unit**.

- $(a+ib) + (c+id) = (a+c) + i(b+d)$

E.g. 6 $(1+2i) + (3+4i) = 4+6i$

- $(a+ib) \times (c+id) = ac + iad + ibc + i^2 bd$

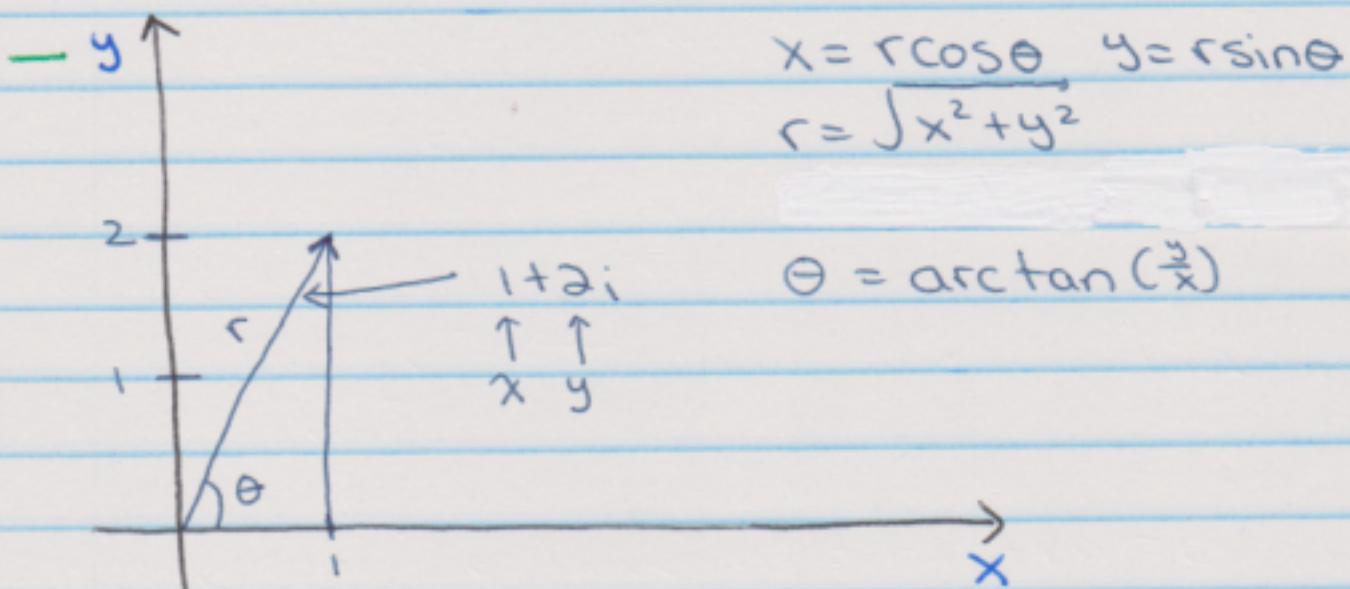
$$= ac + i(ad+bc) - bd$$

$$= (ac - bd) + i(ad + bc)$$

E.g. 7 $(1+2i) \times (3+4i)$

$$= 3+4i+6i+8i^2$$

$$= -5+10i$$



y is the imaginary part.

x is the real part.

$$z = x + iy$$

$$= r \cos \theta + i r \sin \theta$$

$= r(\cos \theta + i \sin \theta)$ ← Polar form of
Complex Numbers

- Euler's Formula: $e^{i\theta} = \cos \theta + \boxed{} i \sin \theta$

Consider

$$e^{i\alpha} = \cos \alpha + i \sin \alpha$$

$$e^{i\beta} = \cos \beta + i \sin \beta$$

$$\begin{aligned}
 e^{ia} \times e^{ib} &= (\cos \alpha + i \sin \alpha) \times (\cos \beta + i \sin \beta) \\
 &= \cos \alpha \cos \beta + i \cos \alpha \sin \beta + i \sin \alpha \cos \beta + \\
 &\quad i^2 \sin \alpha \sin \beta \\
 &= (\cos \alpha \cos \beta - \sin \alpha \sin \beta) + i(\cos \alpha \sin \beta + \sin \alpha \cos \beta) \\
 &= \cos(\alpha + \beta) + i \sin(\alpha + \beta) \\
 &= e^{i(\alpha + \beta)}
 \end{aligned}$$

Hence, $e^{at+ib} = e^a \cdot e^{ib}$

If $R_1 \neq R_2$ and $R_1, R_2 \in \mathbb{C}$, $R_1, R_2 = \lambda \pm i\mu$,
 $y_1 = e^{\lambda t} \cos(\mu t)$, $y_2 = e^{\lambda t} \sin(\mu t)$.

E.g. 8 Solve $y'' + y' + 9.25y = 0$

Soln:

$$r^2 + r + 9.25 = 0$$

$$r = \frac{-1}{2} \pm 3i$$

Recall that $y = e^{rt}$

$$\begin{aligned}
 y &= e^{rt} \\
 &= e^{(-\frac{1}{2} + 3i)t} \\
 &= e^{-\frac{t}{2}} \cdot e^{(3t)i} \\
 &= e^{-t/2} \cdot e^{(3t)i} \\
 &= e^{-t/2} (\cos(3t) + i \sin(3t)) \text{ By Euler's Formula} \\
 &= e^{-t/2} (\cos(3t)) + i e^{-t/2} (\sin(3t))
 \end{aligned}$$

Solns

Note: We're only taking the real parts.

$$e^{-\frac{t}{2}} \cos(3t) = e^{\lambda} \cos(ut)$$

$$e^{-t/2} \sin(3t) = e^{\lambda} \sin(ut)$$

Note: We don't need to consider $r = \frac{-1}{2} - 3i$ because it gives us redundant solns.

$$r_2 = \frac{-1}{2} - 3i$$

$$y = e^{-\frac{t}{2}} ((\cos(-3t)) + i(\sin(-3t)))$$

$$= e^{-\frac{t}{2}} (\cos(-3t)) + ie^{-\frac{t}{2}} (\sin(-3t))$$

However, because cos is an even function,
 $\cos(-3t) = \cos(3t)$.

Furthermore, because sin is an odd function,
 $\sin(-3t) = -\sin(3t)$.

Hence, we get no new solns.

2 Reduction of Order

- Now we will focus on homogeneous eqns with non-constant coefficients.

- Rule 1: $y_1(t) = \frac{1}{t}$

- Rule 2: $y_2(t) = v(t) y_1(t)$ where $v(t)$ is an unknown function.

- E.g. 9 Solve $2t^2y'' + 3ty' - y = 0$

Soln:

$$y_1(t) = \frac{1}{t}$$

$$y_2(t) = v(t) y_1(t)$$

$$\begin{aligned} y_2' &= (v y_1)' \\ &= v'y_1 + v y_1' \end{aligned}$$

$$\begin{aligned} y_2'' &= (v'y_1 + v y_1')' \\ &= v''y_1 + v'y_1' + v'y_1' + v y_1'' \\ &= v''y_1 + 2v'y_1' + v y_1'' \end{aligned}$$

Now, put y_2 , y_2' and y_2'' back into the original eqn.

$$\begin{aligned} 2t^2(v''y_1 + 2v'y_1' + v y_1'') + 3t(v'y_1 + v y_1') \\ - v y_1 = 0 \end{aligned}$$

Now, expand the above eqn.

$$\begin{aligned} 2t^2v''y_1 + 4t^2v'y_1' + 2t^2v y_1'' + 3tv'y_1 + 3tv y_1' \\ - v y_1 = 0 \end{aligned}$$

Now, collect all the terms that has v in it.
We do not collect the terms with v' or v'' .

$$(2t^2v'y_1'' + 3tv'y_1 - vy_1) + (2t^2v''y_1 + 4t^2v'y_1' + 3tv'y_1) = 0$$

All the terms with a v .

$$v(2t^2y_1'' + 3tv'y_1 - y_1) + (2t^2v''y_1 + 4t^2v'y_1' + 3tv'y_1) = 0$$

Notice that this is our original eqn but with y_1 instead of y . Since we know that y_1 is a soln, then this equals 0.

Note: If you did your calculations correctly, at some point all the terms with v will go away. V must go.

Now, we're left with

$$2t^2v''y_1 + 4t^2v'y_1' + 3tv'y_1 = 0 \quad \} \text{ No term with } v.$$

Now, let $w = v'$ and $w' = v''$.

So, we have: $2t^2w'y_1 + 4t^2w'y_1' + 3tw'y_1 = 0$

$$\text{Plug in } y_1 = \frac{1}{t} \rightarrow \frac{2t^2w'}{t} + \frac{4t^2w}{-t^2} + \frac{3tw}{t} = 0$$

$$2tw' - 4w + 3w = 0$$

$$2tw' - w = 0$$

$$2t \frac{dw}{dt} = w$$

$$2t dw = w dt$$

$$\frac{1}{w} dw = \frac{1}{2t} dt \quad \leftarrow \begin{matrix} \text{Separable} \\ \text{Eqns} \end{matrix}$$

$$\int \frac{1}{\omega} dw = \int \frac{1}{2t} dt$$

$$\ln |w| + C_1 = \frac{\ln |t|}{2} + C_2$$

$$\begin{aligned}\ln |w| &= \frac{\ln |t|}{2} + \frac{C_2 - C_1}{2} \\ &= \frac{\ln |t|}{2} + C\end{aligned}$$

$$e^{\ln |w|} = e^{\frac{\ln |t|}{2} + C}$$

$$\begin{aligned}w &= e^c \cdot e^{\frac{\ln |t|}{2}} \\ &= c^2 \cdot (e^{\ln |t|})^{1/2} \\ &= c^2 \cdot t^{1/2}\end{aligned}$$

Note: If we just want y_2 , we can let $c' = 1$.

$$w = t^{1/2}$$

Now, we will solve for v .

$$v' = w$$

$$v = \int w dt$$

$$\begin{aligned}&= \int t^{1/2} dt \rightarrow \int x^n dx = \frac{x^{n+1}}{n+1} \\ &= \frac{2}{3} t^{3/2}\end{aligned}$$

$$y_2 = v y_1$$

$$= \left(\frac{2}{3} t^{3/2} \right) \left(\frac{1}{t} \right)$$

$$= \frac{2}{3} t^{1/2}$$

Note: Letting $y_2 = \nu y_1$ is called D'Alembert's step. When you do it, y_1 and y_2 always make a fundamental pair of solns.

Note: If the question also asks to verify that y_1 and y_2 are a fundamental pair of solns, show that $w \neq 0$.

I.e. In our example $y_1 = \frac{1}{t}$, $y_2 = \frac{2}{3} t^{1/2}$

$$\begin{aligned} w &= \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} \\ &= \begin{vmatrix} \frac{1}{t} & \frac{2}{3} t^{1/2} \\ -\frac{1}{t^2} & \frac{1}{3} t^{-1/2} \end{vmatrix} \\ &= \left(\frac{1}{t}\right)\left(\frac{1}{3} t^{-1/2}\right) - \left(-\frac{1}{t^2}\right)\left(\frac{2}{3} t^{1/2}\right) \\ &= \frac{1}{3t^{3/2}} + \frac{2}{3t^{3/2}} \\ &= \frac{1}{t^{3/2}} \\ &\neq 0 \end{aligned}$$

Note: The steps I did to find y_2 were used that ν has to go. On assignments/quizzes/tests, you don't have to show all those steps. Here are the steps you should do on tests/quizzes/assignments.

$$2t^2 \left(\nu''\left(\frac{1}{t}\right) + 2\nu'\left(\frac{1}{t}\right)' + \nu\left(\frac{1}{t}\right)'' \right) + 3t \left(\nu'\left(\frac{1}{t}\right) + \nu\left(\frac{1}{t}\right)' \right) - \nu\left(\frac{1}{t}\right) = 0$$

Ignore all terms with a v.

Note: If you expand and simplify, you'll find that the terms with v cancels out.

$$\frac{2t^2v''}{t} + \frac{4t^2v'}{-t^2} + \frac{3tv'}{t} = 0$$

$$2tv'' - 4v' + 3v' = 0$$

$$2tv'' - v' = 0$$

Let $w = v'$ and $\dot{w} = v''$.

$$2t \frac{dw}{dt} = w$$

$$\frac{1}{w} dw = \frac{1}{2t} dt$$

$$\int \frac{1}{w} dw = \int \frac{1}{2t} dt$$

$$\ln|w| + C_1 = \frac{\ln|t|}{2} + C_2$$

$$\ln|w| = \frac{\ln|t|}{2} + c$$

$$w = c \cdot t^{1/2} \\ = t^{1/2}$$

$$v' = w$$

$$v = \int w dt$$

$$= \int t^{1/2} dt$$

$$= \frac{2}{3} t^{3/2}$$

$$y_2 = \frac{2t^{3/2}}{3}, \quad y_1$$

$$= \frac{2}{3} t^{3/2} \cdot \frac{1}{t}$$

$$= \frac{2}{3} t^{1/2}$$

Note: Another way to solve for y_2 is to use Abel's formula. Here, it is crucial that the coefficient of y'' is 1.

$$2t^2y'' + 3ty' - y = 0$$

$$y'' + \frac{3y'}{2t} - \frac{y}{2t^2} = 0$$

Now, apply Abel's formula.

$$\begin{aligned} w_1 &= C'e^{-\int p(t)dt} \\ &= C'e^{-\int \frac{3}{2t} dt} \\ &= C'e^{-\frac{3}{2} \int \frac{1}{t} dt} \\ &= C'e^{-\frac{3}{2} \ln|t| + C_1} \\ &= C' \cdot C_1 \cdot t^{-3/2} \\ &= \frac{C}{t^{3/2}} \quad \leftarrow \text{Take } C=1 \\ &= \frac{1}{t^{3/2}} \end{aligned}$$

Another way to compute w_1 .

$$\begin{aligned} w_1 &= \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} \\ &= y_1 y'_2 - y'_1 y_2 \\ &= \frac{y'_2}{t} + \frac{y_2}{t^2} \end{aligned}$$

$$\omega_1 = \omega_2$$

$$\frac{1}{t^{3/2}} = \frac{y'_2}{t} + \frac{y_2}{t^2}$$

$$y'_2 + \frac{y_2}{t} = \frac{1}{t^{1/2}} \quad \leftarrow \text{Linear Diff Eqn First order}$$

$$\mu(y'_2) + \frac{\mu y_2}{t} = \frac{\mu}{t^{1/2}}$$

$$\text{LHS} = (\mu y_2)'$$

$$\cancel{\mu(y'_2)} + \frac{\mu y_2}{t} = (\mu y_2)' \\ = \mu' y_2 + \cancel{\mu y'_2}$$

$$\frac{\mu y_2}{t} = \mu' y_2$$

$$\frac{1}{t} = \frac{\mu'}{\mu}$$

$$= (\ln(\mu))'$$

$$\int \frac{1}{t} dt = \ln(\mu)$$

$$\ln|t| + C = \ln(\mu)$$

$$\mu = t$$

$$(\mu y_2)' = \frac{\mu}{t^{1/2}}$$

$$(t y_2)' = \frac{t}{t^{1/2}} \\ = t^{1/2}$$

$$t y_2 = \int t^{1/2} dt \\ = \frac{2}{3} t^{3/2} + C$$

$$y_2 = \frac{2}{3} t^{1/2} + \frac{C}{t} \longrightarrow y_2 = \frac{2}{3} t^{1/2}$$

Let this be 0.