MATB44 Week 5 Notes

1. Euler's Equation:

– Consider $t^2 y'' + \alpha t y' + \beta y = 0$ where α and β are known constants.

We let $y = t^r$, where r is an unknown constant.
$y' = r t^{r-1}$ and $y'' = r(r-1)t^{r-2}$.

We now have $t^2 (r)(r-1)(t^{r-2}) + \alpha t r t^{r-1} + \beta t^r = 0$

$r(r-1)(t^r) + \alpha r t^r + \beta t^r = 0$

$t^r (r^2 - r + \alpha r + \beta) = 0$

Since we know that $t^r \neq 0$, we can divide both sides of the eqn by it.

$r^2 + (\alpha - 1) r + \beta = 0$ ← Called characteristic eqn for the Euler eqn/ Indical Eqn

– We can use the quadratic eqn to solve for r. Since $b^2 - 4ac$ has 3 possibilities:

a) > 0

b) $= 0$

c) < 0

there are 3 different cases we need to look at.

Case 1: $b^2 - 4ac > 0$

- Here, $y_1 = t^{r_1}$ and $y_2 = t^{r_2}$ $(R_1 \neq R_2)$
- E.g. 1 Solve $t^2y'' + ty' - 2y = 0$

Soln:

1. $\alpha = 1$ (It's the numerical coefficient of ty')
   $\beta = -2$ (It's the numerical coefficient of y)
   Rewrite the eqn into $r^2 + (\alpha - 1)r + \beta = 0$.
   We have $r^2 + (1-1)r - 2 = 0$.
   $r^2 - 2 = 0$
   $r^2 = 2$
   $r = \pm\sqrt{2} \rightarrow r_1 = \sqrt{2},\ r_2 = -\sqrt{2}$

2. $y_1 = t^{r_1}$, $y_2 = t^{r_2}$
   $= t^{\sqrt{2}}$ $\quad = t^{-\sqrt{2}}$
   $y = C_1 y_1 + C_2 y_2$
   $= C_1 t^{\sqrt{2}} + C_2 t^{-\sqrt{2}}$

Note: If $R_1 \neq R_2$ and $R_1, R_2 \in \mathbb{R}$, then $y_1 = t^{r_1}$ and $y_2 = t^{r_2}$ is always a fundamental pair of solns.

Case 2: $b^2 - 4ac = 0$

- Here, $R_1 = R_2$ (Repeated Roots)
- Here, $y_1 = t^{r_1}$ and $y_2 = \ln(t)\, t^{r_1}$
- E.g. 2 Solve $t^2y'' - 5ty' + 9y = 0$

Soln:

1. $\alpha = -5$, $\beta = 9$
   $r^2 + (\alpha - 1)r + \beta = 0$
   $r^2 + (-5-1)r + 9 = 0$
   $r^2 + (-6)r + 9 = 0$

$r^2 - 6r + 9 = 0$
$(r-3)^2 = 0$
$r_1 = r_2 = 3$
$y_1 = t^{r_1}$
$= t^3$

2. To find y_2, do $y_2 = v y_1$ (D'Alembert)

$t^2 y_2'' - 5t y_2' + 9y_2 = 0$
$t^2(v'' y_1 + 2v' y_1' + v y_1'') - 5t(v' y_1 + v y_1') +$
$9v y_1 = 0$

$t^2 v'' y_1 + 2v' y_1' t^2 + t^2 \cancel{v} y_1'' - 5t v' y_1 - 5\cancel{t v} y_1' + 9\cancel{v} y_1 = 0$

Collect all the terms with v.
$v\underbrace{(t^2 y_1'' - 5t y_1' + 9y_1)}=0$
Equals to 0 because y_1 is a solution.
Remember: "V must go"

We are left with
$t^2 v'' y_1 + 2t^2 v' y_1' - 5t v' y_1 = 0$
Let $w = v'$, $w' = v''$
$w' t^2 y_1 + 2t^2 w y_1' - 5t w y_1 = 0$
Recall that we found $y_1 = t^3$.
$w' t^5 + 6w t^4 - 5w t^4 = 0$
$w' t + 6w - 5w = 0$
$w' t + w = 0$
$t \frac{dw}{dt} = -w$
$t\, dw = -w\, dt$
$\frac{1}{w} dw = -\frac{1}{t} dt$

$$\int \frac{1}{w}\,dw = \int -\frac{1}{t}\,dt$$

$\ln|w| + C_1 = -\ln|t| + C_2$

$\ln|w| = -\ln|t| + C_2 - C_1$

$= -\ln|t| + C$

$w = e^{-\ln|t| + C}$

$= e^{C}(e^{-\ln|t|})$

$= C'(e^{\ln|t|})^{-1}$

$= \dfrac{C'}{t}$

let $C' = 1$

$w = \dfrac{1}{t}$

$v' = w$

$v = \int w\,dt$

$= \int \frac{1}{t}\,dt$

$= \ln|t| + C$

let $C = 0$

$v = \ln|t|$

$y_2 = v y_1$

$= (\ln|t|)\,t^3$

Note: $y_2 = \ln(t) \cdot y_1$

$= \ln(t) \cdot t^{r_1}$

Do NOT do D'Alembert on assignments/quizzes/tests/etc unless instructed to. Just do $y_2 = \ln(t)\,t^{r_1}$. I only did D'Alembert to show why $y_2 = \ln(t) \cdot t^{r_1}$.

Case 3: $b^2 - 4ac < 0$

- Here, we have complex roots.
- E.g. 3 Solve $t^2y'' + 3ty' + 2y = 0$

Soln:

$$r^2 + (3-1)r + 2 = 0$$

$$r^2 + 2r + 2 = 0$$

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm 2i}{2}$$

$$= -1 \pm i \qquad r_1 = -1+i, \ r_2 = -1-i$$

$$y = t^r$$

$$= t^{-1+i}$$

$$= t^{-1} \cdot t^i$$

Recall: $t = e^{\ln t} \rightarrow t^i = e^{i \ln t}$

$$= \underbrace{\cos(\ln|t|) + i\sin(\ln|t|)}_{\text{Euler's Formula}}$$

$$y = \frac{\cos(\ln|t|) + i\sin(\ln|t|)}{x}$$

$$y_1 = \frac{\cos(\ln|t|)}{x}, \quad y_2 = \frac{\sin(\ln|t|)}{x}$$

2. Non-Homogeneous Linear Eqns

- A non-homogeneous linear eqn has the form $y'' + p(t)y' + q(t)y = g(t)$ where p, q, g are given functions.
- Rule: If y_1 and y_2 are solns to the non-homogeneous eqn, then $y = y_2 - y_1$ solves the homogeneous eqn.

Proof:

$(y_2 - y_1)'' + p(t)(y_2 - y_1)' + q(t)(y_2 - y_1)$
$= y_2'' + p(t)y_2' + q(t)y_2 - (y_1'' + p(t)y_1' + q(t)y_1)$
$= g(t) - g(t)$
$= 0$

- Rule: The general soln to a non-homogeneous eqn = General soln of the homogeneous eqn + particular soln of the homogeneous eqn.

- To find the particular soln of the homogeneous eqn, we will use the method of undetermined coefficients.

- Undetermined Coefficients
  - Note: This method only works for some functions.

  - E.g. 4 Find a particular soln to $y'' - 3y' - 4y = 3e^{2t}$

    Here, let $y = Ae^{2t}$, where A is an unknown constant.

$(Ae^{2t})'' - 3(Ae^{2t})' - 4(Ae^{2t}) = 3e^{2t}$

$\cancel{4Ae^{2t}} - 6Ae^{2t} - \cancel{4Ae^{2t}} = 3e^{2t}$

$-6Ae^{2t} = 3e^{2t}$

$-6A = 3$

$A = -\frac{1}{2}$

$y_p = -\frac{1}{2}e^{2t}$ is a particular soln of the non-homogeneous eqn.

To find the general soln of the non-homogeneous eqn, we also need to find the general soln of the homogeneous eqn.

$y'' - 3y' - 4y = 0$

$r^2 - 3r - 4 = 0$

$r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{3 \pm 5}{2}$

$= 4$ or -1 ← Important that $R_1, R_2 \neq 2$

$y_1 = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

$= C_1 e^{4t} + C_2 e^{-t}$

Hence, the general soln of the non-homogeneous soln is

$$y = \underbrace{\frac{-e^{2t}}{2}}_{\text{Particular Soln}} + \underbrace{C_1 e^{4t} + C_2 e^{-t}}_{\text{General soln of homogeneous eqn}}$$

- E.g. 5 Find a particular soln to
$y'' - 4y' - 12y = 3e^{5t}$.

Let $y = Ae^{5t}$

$(Ae^{5t})'' - 4(Ae^{5t})' - 12Ae^{5t} = 3e^{5t}$

$25A\cancel{e^{5t}} - 20A\cancel{e^{5t}} - 12A\cancel{e^{5t}} = 3\cancel{e^{5t}}$

$(25 - 20 - 12)A = 3$

$-7A = 3$

$A = -3/7$

Hence, the particular soln is $y_0 = -3/7\, e^{5t}$

The general soln to the homogeneous eqn is

$y'' - 4y' - 12y = 0$

$r^2 - 4r - 12 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm 8}{2}$$

$= -2$ or 6 ← Important that $R_1, R_2 \neq 5$

$y_1 = C_1e^{-2t} + C_2e^{6t}$

Hence, the general soln of the homogeneous eqn is $y = -\frac{3}{7}e^{5t} + C_1e^{-2t} + C_2e^{6t}$

Now, if we apply the initial conditions

$y(0) = \frac{18}{7} \quad y'(0) = \frac{-1}{7}$

$y(0) = \frac{18}{7}$

$\frac{18}{7} = -\frac{3}{7} + C_1 + C_2$

$3 = C_1 + C_2$

$y'(0) = -\frac{1}{7}$

$-\frac{1}{7} = -\frac{15}{7} + (-2C_1) + 6C_2$

$2 = -2C_1 + 6C_2$

$1 = -C_1 + 3C_2$

$C_2 = 1,\ C_1 = 2$

- E.g. 6 Find a particular soln to
$y'' - 3y' - 4y = 2\sin(t)$

Let $y = A\cos t + B\sin t$

$(A\cos t + B\sin t)'' - 3(A\cos t + B\sin t)'$
$-4(A\cos t + B\sin t) = 2\sin(t)$

$-A\cos t - B\sin t - 3(-A\sin t + B\cos t)$
$-4A\cos t - 4B\sin t = 2\sin t$

$-A\cos t - B\sin t + 3A\sin t - 3B\cos t - 4A\cos t$
$-4B\sin t = 2\sin t$

Collect all the terms with cost and all the terms with sint

$-A\cos t - 3B\cos t - 4A\cos t = 0\cos t$ ← There's no cost in RHS.
$-B\sin t + 3A\sin t - 4B\sin t = 2\sin t$

$$\left.\begin{aligned} -A - 3B - 4A &= 0 \\ -B + 3A - 4B &= 2 \end{aligned}\right\}$$ Took the coefficients from above.

$$\left.\begin{aligned} -5A - 3B &= 0 \\ 3A - 5B &= 2 \end{aligned}\right\} A = \frac{3}{17},\ B = \frac{-5}{17}$$

Hence, the particular soln is
$\frac{3}{17}\cos t - \frac{5}{17}\sin t.$

- E.g. 7 Find a particular soln to
$y'' - 4y' - 12y = \sin(2t).$

Let $y = A\cos(2t) + B\sin(2t)$

$(A\cos(2t) + B\sin(2t))'' - 4(A\cos(2t)$
$+ B\sin(2t))' - 12(A\cos(2t) + B\sin(2t)) = \sin(2t)$

$-4A\cos(2t) - 4B\sin(2t) + 8A\sin(2t)$
$- 8B\cos(2t) - 12A\cos(2t) - 12B\sin(2t) = \sin(2t)$

$-4A - 8B - 12A = 0$
$-4B + 8A - 12B = 1$

$-16A - 8B = 0$
$8A - 16B = 1$

$-2A - B = 0 \rightarrow B = -2A$
$8A - 16B = 1$

$8A - 16(-2A) = 1$
$8A + 32A = 1$
$40A = 1$
$A = \frac{1}{40}$
$B = \frac{-1}{20}$

Hence, $\frac{\cos(2t)}{40} - \frac{\sin(2t)}{20}$ is the particular soln.

– E.g. 8 Find a particular soln to $y'' - 3y' - 4y = -8e^{t}\cos(2t)$.

Let $y = Ae^{t}\cos(2t) + Be^{t}\sin(2t)$

$(Ae^{t}\cos(2t) + Be^{t}\sin(2t))'' -$
$3(Ae^{t}\cos(2t) + Be^{t}\sin(2t))' -$
$4(Ae^{t}\cos(2t) + Be^{t}\sin(2t)) = -8e^{t}\cos(2t)$

$Ae^t\cos(2t) + 2Ae^t(-2\sin(2t)) +$
$Ae^t(-4\cos(2t)) + Be^t\sin(2t) +$
$Be^t(\cos(2t)2) + Be^t(-4\sin(2t)) -$
$3(Ae^t\cos(2t) - 2Ae^t\sin(2t) +$
$\quad Be^t\sin(2t) + 2Be^t\cos(2t)) -$
$4(Ae^t\cos(2t) + Be^t\sin(2t)) = -8e^t\cos(2t)$

$(-10A - 2B)\cos(2t) + (2A - 10B)\sin(2t) =$
$-8e^t\cos(2t)$

$$\left.\begin{aligned} -10A - 2B &= -8 \\ 2A - 10B &= 0 \end{aligned}\right\} A = \frac{10}{13},\ B = \frac{2}{13}$$

– E.g. 9 Find a particular soln for
$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t\cos(2t)$.

This is called superposition of particular solns. We just add up the solns for each individual term on the RHS.
Note: In practice, we will never do this for our course.
I.e. We won't be tested on it.
It's just good to know.

Rule: If Y_1 is a particular soln for $y'' + p(t)y' + q(t)y = g_1(t)$ and Y_2 is a particular soln for $y'' + p(t)y' + q(t)y = g_2(t)$, then $Y_1 + Y_2$ is a particular soln to $y'' + p(t)y' + q(t)y = g_1(t) + g_2(t)$.

- E.g. 10 Find a particular soln to
$y'' - 3y' - 4y = 2e^{-t}$.

Let $y = Ae^{-t}$

$(Ae^{-t})'' - 3(Ae^{-t})' - 4Ae^{-t} = 2e^{-t}$

$Ae^{-t} + 3Ae^{-t} - 4Ae^{-t} = 2e^{-t}$

$0 = 2e^{-t}$

To figure out why we got $0 = 2e^{-t}$,
consider this: $y'' - 3y' - 4y = 0$

$r^2 - 3r - 4 = 0$

$(r-4)(r+1) = 0$

$r_1 = 4,\ r_2 = -1$

Hence, we get

$y = C_1 e^{4t} + \underbrace{C_2 e^{-t}}_{\text{Same as } Ae^{-t}}$

Since Ae^{-t} solves the homogeneous eqn, it can't also solve the non-homogeneous eqn. This is called a resonance because the homogeneous part resonates with the RHS. In this case, let $y = Ate^{-t}$. The proof follows from D'Alembert.

Let $y = Ate^{-t}$

$(Ate^{-t})'' - 3(Ate^{-t})' - 4Ate^{-t} = 2e^{-t}$

$-2Ae^{-t} + Ate^{-t} - 3Ae^{-t} + 3Ate^{-t} - 4Ate^{-t} = 2e^{-t}$

Collect all terms with t in it.

$t\underbrace{(Ae^{-t} + 3Ae^{-t} - 4Ae^{-t})}_{0}$

It should cancel out to 0.
"t must go"

$-2Ae^{-t} - 3Ae^{-t} = 2e^{-t}$
$-5Ae^{-t} = 2e^{-t}$
$A = \frac{-2}{5}$

Hence, the particular soln is
$-\frac{2}{5}te^{-t}$.

– E.g. 11 Find a particular soln to
$y'' + 4y' + 4y = e^{-2t}$.

Consider the homogeneous eqn
$y'' + 4y' + 4y = 0$
$r^2 + 4r + 4 = 0$
$(r+2)^2 = 0$
$r_1 = r_2 = -2$
$y_1 = e^{-2t}$, $y_2 = \underline{te^{-2t}}$

So, Ate^{-2t} solves the homogeneous eqn.
Hence, it can't solve the non-homogeneous eqn. This is called a double resonance.
In this case, let $y = At^2e^{-2t}$.

$(At^2e^{-2t})'' + 4(At^2e^{-2t})' + 4At^2e^{-2t} = e^{-2t}$
$2Ae^{-2t} - 8Ate^{-2t} + 4At^2e^{-2t} + 8Ate^{-2t}$
$- 8At^2e^{-2t} + 4At^2e^{-2t} = e^{-2t}$

Collect all the terms with t and t^2, individually.

$$t\underbrace{(-8Ae^{-2t} + 8Ae^{-2t})}_{0} \rightarrow 0$$

$$t^2\underbrace{(4Ae^{-2t} - 8Ae^{-2t} + 4Ae^{-2t})}_{0} \rightarrow 0$$

Note: Everything with t and t^2 should cancel out.

$-2Ae^{-2t} = e^{-2t}$
$A = \frac{-1}{2}$

Hence, the particular soln is $\frac{-1}{2}t^2e^{-2t}$

- E.g. 12 Find a particular soln to
$y'' + y' + 9.25y = -6e^{-t/2}\cos 3t$

Consider $r^2 + r + 9.25 = 0$

$r = \frac{-1 \pm \sqrt{1-37}}{2}$

$= -\frac{1}{2} \pm 3i$

$\lambda = \frac{-1}{2}, \ u = 3$

$$y_1 = e^{\lambda t}\cos(ut) \qquad y_2 = e^{\lambda t}\sin(ut)$$
$$= e^{-t/2}\cos(\underset{3t}{\;\;}) \qquad = e^{-t/2}\sin(\underset{3t}{\;\;})$$

This is called complex resonance.

Let $y = Aty_1 + Bty_2$

$$(Aty_1 + Bty_2)'' + (Aty_1 + Bty_2)' + 9.25(Aty_1 + Bty_2) = -6e^{-t/2}\cos(3t)$$

$$2Ay_1' + Aty_1'' + 2By_2' + Bty_2'' + Ay_1 + Aty_1' + By_2 + Bty_2' + 9.25Aty_1 + 9.25Bty_2 = -6e^{-t/2}\cos(3t)$$

Collect all terms with At and Bt.

$$At\underbrace{(y_1'' + y_1' + 9.25y_1)}_{0} \to 0$$

$$Bt\underbrace{(y_2'' + y_2' + 9.25y_2)}_{0} \to 0$$

Everything with t cancels out.

$$2Ay_1' + 2By_2' + Ay_1 + By_2 = -6e^{-t/2}\cos(3t)$$

From above, $y_1 = e^{-t/2}\cos(3t)$ and $y_2 = e^{-t/2}\sin(3t)$.

$$2A\left(e^{-t/2}\cos(3t)\right)' + 2B\left(e^{-t/2}\sin(3t)\right)' +$$
$$A\left(e^{-t/2}\cos(3t)\right) + Be^{-t/2}\sin(3t) = -6e^{-t/2}\cos(3t)$$

$$2A\left(-\tfrac{1}{2}e^{-t/2}\cos(3t) + e^{-t/2}(-3\sin(3t))\right) +$$
$$2B\left(-\tfrac{1}{2}e^{-t/2}\sin(3t) + e^{-t/2}\,3\cos(3t)\right) +$$
$$Ae^{-t/2}\cos(3t) + Be^{-t/2}\sin(3t) = -6e^{-t/2}\cos(3t)$$

$$6B\cos 3t - 6A\sin 3t = -6\cos 3t$$

$B = -1$, $A = 0$

Hence, the particular soln is $-e^{-\frac{t}{2}}\cos(3t)$.