MATB42 Week 1 Notes:

Wave Equations:

- The partial differential eqn (PDE) $u_{tt} = c^2 u_{xx}$ is called the 1-D wave eqn.

Note: c is a constant called the speed of the wave.

- The unknown function $u(x,t)$ can be used to describe the height of a wave relative to the equilibrium $u=0$, over a region with x being the position variable. The function $u(x,t)$ depends on t, which indicates that the height of the wave would change over time.

- $u(x,t) = f(x-ct) + g(x+ct)$ where f and g are 2 arbitrary functions.
  $f(x-ct)$ is $f(x)$ but moving to the right for t units at speed c.
  $g(x+ct)$ is $g(x)$ but moving to the left for t units at speed c.
  Recall: $f(x-a)$ is $f(x)$ that's moved a units to the right. $f(x+b)$ is $f(x)$ that's moved b units to the left.

- We want to solve $u(x,t)$ in the finite domain $0<x<\ell$ where ℓ is a constant that's the length of the domain.

- Suppose we are given the initial conditions $u(x,0) = \phi(x)$, where $\phi(x)$ describes the shape of the wave at time $t=0$ and $u_t(x,0) = \psi(x)$, where $\psi(x)$ describes the verticle velocity of the wave at time $t=0$.

- Furthermore, because have a finite domain, we must also set the boundary conditions. We will use the easiest type of boundary conditions:
$u(0,t)=0$ and $u(\ell,t)=0$
These boundary conditions indicate that the wave is fixed at both ends of the interval.

- Note: The boundary conditions represent the endpoints while the initial conditions represent the wave at time = 0.

- We will use separation of variables to solve this PDE.

Separation of Variables:
Assume that $u(x,t) = X(x) \cdot T(t)$ where $X(x)$ depends on x only and $T(t)$ depends on t only.

Now, we plug $u(x,t)$ into the PDE $u_{tt} = c^2 u_{xx}$.

$$u_t = \frac{\partial}{\partial t}(u(x,t))$$
$$= \frac{\partial}{\partial t}(X \cdot T) \leftarrow X \text{ is treated as a constant.}$$
$$= X \cdot T'$$

$$u_{tt} = \frac{\partial}{\partial t}(u_t)$$
$$= \frac{\partial}{\partial t}(X \cdot T') \leftarrow \text{Again, } X \text{ is treated as a constant.}$$
$$= X \cdot T''$$

$$u_x = \frac{\partial}{\partial x}(u(x,t))$$
$$= \frac{\partial}{\partial x}(X \cdot T) \leftarrow$$ Here, T is treated as a constant.
$$= T \cdot X'$$

$$u_{xx} = \frac{\partial}{\partial x}(u_x)$$
$$= \frac{\partial}{\partial x}(T \cdot X')$$
$$= T \cdot X''$$

Now, we have $\underbrace{X \cdot T''}_{u_{tt}} = \underbrace{c^2 \cdot T \cdot X''}_{u_{xx}}$.

We will now separate the vars onto 2 sides of the eqn. By convention, we put the T's on the LHS and the X's on the RHS. Furthermore, we put all constants on the LHS.

$$\frac{T''}{c^2 T} = \frac{X''}{X} = F$$

We will now show that F is a constant and doesn't depend on x or t.

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left(\frac{T''}{c^2 T}\right) \leftarrow$$ We used $\frac{T''}{c^2 T}$ for F here.
$$= 0$$

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial t}\left(\frac{X''}{X}\right) \leftarrow$$ We used $\frac{X''}{X}$ for F here.

$$= 0$$

Since both derivatives are 0, F must be a constant. We call F the separation constant. The separation constant is always negative, and by convention, we set it to be $-\lambda$, where λ is a positive number. (I.e. $\lambda > 0$).

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda \leftarrow$$ We'll split this into 2 eqns.

$\frac{T''}{c^2 T} = -\lambda$	$\frac{X''}{X} = -\lambda$
$T'' = -\lambda T c^2$	$X'' = -\lambda X$
$T'' + c^2 \lambda T = 0$	$X'' + \lambda X = 0$

Both eqns are 2nd order linear homogeneous diff eqns.

To solve $T'' + \lambda c^2 T = 0$:

$T = e^{rt}$

$T' = re^{rt}$, $T'' = r^2 e^{rt}$

$r^2 e^{rt} + \lambda c^2 e^{rt} = 0$

$r^2 + \lambda c^2 = 0 \leftarrow$ Because we know $e^{rt} \neq 0$, we can divide both sides by it.

$r^2 = -\lambda c^2$

$r = \pm\sqrt{-\lambda c^2}$

Take $r = +\sqrt{-\lambda c^2}$

$r = \sqrt{\lambda} c i \leftarrow$ Recall: $i^2 = -1 \rightarrow i = \sqrt{-1}$

Plug r into e^{rt}.
We get

$e^{(\sqrt{\lambda} ct)i}$

By Euler Formula, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.
Here, we have $\cos(\sqrt{\lambda}\, ct) + i\sin(\sqrt{\lambda}\, ct)$.

$T_1 = \cos(\sqrt{\lambda}\, ct)$, $T_2 = \sin(\sqrt{\lambda}\, ct)$
Note: We only take the real parts.

$T(t) = AT_1 + BT_2$
$\quad = A\cos(\sqrt{\lambda}\, ct) + B\sin(\sqrt{\lambda}\, ct)$
A and B are arbitrary constants.

To solve $X'' + \lambda X = 0$:
$X = e^{rx}$
$X' = re^{rx}$, $X'' = r^2 e^{rx}$
$r^2 e^{rx} + \lambda e^{rx} = 0$
$r^2 + \lambda = 0$
$r^2 = -\lambda$
$r = \pm\sqrt{-\lambda}$
$\quad = +\sqrt{\lambda} i$

$e^{rx} = e^{(\sqrt{\lambda} x)i}$
$\quad = \cos(\sqrt{\lambda}\, x) + i\sin(\sqrt{\lambda}\, x)$

$X_1 = \cos(\sqrt{\lambda}\, x)$, $X_2 = \sin(\sqrt{\lambda}\, x)$
$X(x) = CX_1 + DX_2$
$\quad = C\cos(\sqrt{\lambda}\, x) + D\sin(\sqrt{\lambda}\, x)$

Now, we'll plug in the boundary conditions.
Recall the following:

a) Our guess is $u(x,t) = X \cdot T$

b) Our boundary conditions are:
   i) $u(0,t)=0$
   ii) $u(\ell,t)=0$

Let's use the boundary condition $u(0,t)=0$ first.

$u(0,t)=0 \rightarrow X(0)\cdot T(t)=0 \rightarrow X(0)=0$

Note: We assume that $T(t)\neq 0$. If $T(t)=0$ for all t, then we get the trivial soln. We don't care about trivial solns.

$$X(0)=0 \rightarrow C\cos(\sqrt{\lambda}\,0)+D\sin(\sqrt{\lambda}\,0)=0$$
$$C\cdot\underbrace{\cos(0)}_{\cos(0)=1}+D\cdot\underbrace{\sin(0)}_{\sin(0)=0}=0$$
$$C=0$$

Now, we'll use the other boundary condition.
$u(\ell,t)=0 \rightarrow X(\ell)\cdot T(t)=0 \rightarrow X(\ell)=0$
Once again, we disregard the case that $T(t)=0$.

$X(\ell)=0 \rightarrow D\sin(\sqrt{\lambda}\,\ell)=0$ Remember that $C=0$.
Either $D=0$ or $\sin(\sqrt{\lambda}\,\ell)=0$.
We ignore the case $D=0$. (Trivial Soln)
$\sin(\sqrt{\lambda}\,\ell)=0 \rightarrow \sqrt{\lambda}\,\ell = n\pi \quad , n>0$
$$\sqrt{\lambda}=\frac{n\pi}{\ell}$$

For each n, we have a soln, denoted as U_n.
$$U_n(x,t) = X\cdot T$$
$$= D_n\sin\left(\frac{n\pi x}{\ell}\right)\left(A_n\cos\left(\frac{n\pi ct}{\ell}\right)+B_n\sin\left(\frac{n\pi ct}{\ell}\right)\right)$$

Note:

- We ignore the case when $n=0$ because $\sin(0)=0$. (Trivial soln)
- Cos is an even function while sin is an odd function. This means that $\cos(-\theta)=\cos(\theta)$ and $\sin(-\theta)=-\sin(\theta)$. Hence, if $n<0$, the cos part wouldn't change and the negative sign from the sign part would get absorbed by the coefficient.
- We can ignore D_n since D_nA_n and D_nB_n are just constant coefficients. I.e. D_n gets absorbed into A_n and B_n.

Since each value of n gives a soln to the PDE, the general soln must be a linear combination of each of these solns.

$$u(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{\ell}\right)\left(A_n\cos\left(\frac{n\pi ct}{\ell}\right)+B_n\sin\left(\frac{n\pi ct}{\ell}\right)\right)$$

Now, we'll apply the initial conditions to find A_n and B_n.

To find A_n, we'll use the initial condition $u(x,0)=\phi(x)$.

$$u(x,0)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{\ell}\right)$$

$$\therefore\ \phi(x)=\underbrace{\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{\ell}\right)}_{\text{Fourier (Sine) Series of }\phi(x)}$$

For Fourier Series, we approximate a function, $f(x)$, with an infinite sum of sine and cosine with different frequencies which increase with n.

To calculate An and Bn, we'll use these 3 eqns:

1. $\int_0^{\ell} \sin\frac{n\pi x}{\ell} \cdot \sin\frac{m\pi x}{\ell} = \begin{cases} 0, & \text{if } m \neq n \\ \frac{\ell}{2}, & \text{if } m = n \end{cases}$

2. $\int_0^{\ell} \cos\frac{n\pi x}{\ell} \cdot \cos\frac{m\pi x}{\ell} = \begin{cases} 0, & \text{if } m \neq n \\ \frac{\ell}{2}, & \text{if } m = n \neq 0 \end{cases}$

3. $\int_{-\ell}^{\ell} \sin\frac{n\pi x}{\ell} \cdot \cos\frac{m\pi x}{\ell} = 0$, for any m, n

The above 3 eqns are called the Orthogonal Relations for Fourier Series.

Take $f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)$

Choose an integer m. We will compute A_m by multiplying both sides of the eqn by $\sin\left(\frac{m\pi x}{\ell}\right)$ and then integrating over 0 to ℓ.

Now, we have

$$\int_0^{\ell} f(x) \sin\left(\frac{m\pi x}{\ell}\right) = \sum_{n=1}^{\infty} A_n \int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \sin\left(\frac{m\pi x}{\ell}\right)$$

The right side has an infinite sum of integrals but every single one of the integrals is 0 except for 1.

Recall that $\int_0^{\ell} \sin \frac{n\pi x}{\ell} \cdot \sin \frac{m\pi x}{\ell} = \begin{cases} 0, & \text{if } m \neq n \\ \frac{\ell}{2}, & \text{if } m = n \end{cases}$

Hence, when $n \neq m$, and we randomly chose m, we'd get 0. The one time when $n = m$, we'd get $\frac{\ell}{2}$.

Hence, $\int_0^{\ell} f(x) \cdot \sin\left(\frac{m\pi x}{\ell}\right) = 0 + 0 + \ldots + A_m \frac{\ell}{2} + 0 + \ldots 0$

$$= A_m \cdot \frac{\ell}{2}$$

$$A_m = \frac{2}{\ell} \int_0^{\ell} f(x) \cdot \sin\left(\frac{m\pi x}{2}\right)$$

Using this process and the fact that

$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)$, we get

$$A_n = \frac{2}{\ell} \int_0^{\ell} \phi(x) \cdot \sin\left(\frac{n\pi x}{2}\right)$$

To solve for B_n, we apply the other initial condition $u_t(x,0) = \psi(x)$.

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{\ell}\right)\left(A_n \cos\left(\frac{n\pi ct}{\ell}\right) + B_n \sin\left(\frac{n\pi ct}{\ell}\right)\right)$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{\ell}\right)\left(-A_n \sin\left(\frac{n\pi ct}{\ell}\right) + B_n \cos\left(\frac{n\pi ct}{\ell}\right)\right)\left(\frac{n\pi c}{\ell}\right)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{\ell}\right) \cdot B_n \cdot \frac{n\pi c}{\ell} = \psi(x)$$

To solve for B_n, we use the same process as we did for finding A_n.

$$\psi(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{\ell}\right) B_n \left(\frac{n\pi c}{\ell}\right)$$

$$\int_0^{\ell} \psi(x) \cdot \sin\left(\frac{m\pi x}{\ell}\right) = \sum_{n=0}^{\infty} B_n \left(\frac{n\pi c}{\ell}\right) \int_0^{\ell} \sin\left(\frac{m\pi x}{\ell}\right) \cdot \sin\left(\frac{n\pi x}{\ell}\right)$$

$$= B_n \left(\frac{n\pi c}{\ell}\right)\left(\frac{\ell}{2}\right)$$

$$= B_n \left(\frac{n\pi c}{2}\right)$$

$$B_n = \frac{2}{n\pi c} \int_0^{\ell} \psi(x) \cdot \sin\left(\frac{n\pi x}{\ell}\right)$$

– An Important Note:

Consider when $u_t(x,0) = \psi(x) = 0$. Recall that $\psi(x)$ describes the verticle velocity of the wave at time $t=0$. If $\psi(x) = 0$, this means that the string was held at a shape $\phi(x)$ and then released with no initial velocity given. In this scenario, $B_n = 0$ for all n. Furthermore, if we choose $\phi(x) = \sin\left(\frac{m\pi x}{\ell}\right)$ for some int m,

then $A_n = 0$ for all n except when $n = m$. Then, $A_n = 1$. In this case, we get only 1 term for $u(x,t)$. $u(x,t) = \sin\left(\frac{m\pi x}{\ell}\right) \cos\left(\frac{m\pi c t}{\ell}\right)$.

This is called the m-th harmonic where the string vibrates at a frequency of $\frac{m\pi c}{\ell}$.

Note: $\sin\left(\frac{m\pi x}{\ell}\right)$ gives the shape while $\cos\left(\frac{m\pi ct}{\ell}\right)$ gives the amplitude.

— Earlier, on pg 4, we said that the separation variable is always negative. Furthermore, we said to let $-\lambda$ represent this negative number where $\lambda > 0$. Here, we'll see why.

Consider the case when $\lambda = 0$:

$T'' + \lambda c^2 T = 0$

$T'' = 0$

$r^2 = 0$

$r_1 = r_2 = 0 \leftarrow$ Repeated Roots

$T_1 = e^{r_1 t} = 1 \qquad T_2 = t \cdot T_1 = t \cdot e^{r_1 t} = t$

$T = AT_2 + BT_1 = At + B$

Similarly, $X'' + \lambda X = 0$

$X'' = 0$

$r^2 = 0$

$r_1 = r_2 = 0$

$X_1 = e^{r_1 x} = 1 \qquad X_2 = x \cdot X_1 = x$

$X = CX_2 + DX_1 = Cx + D$

$u(x,t) = X \cdot T$

Let's plug the boundary conditions.

Let's start with $u(0,t)=0$.

$u(0,t)=0 \rightarrow X(0) \cdot T(t) = 0$

Once again, we ignore the case when $T(t)=0$.

$X(0)=0$

$C \cdot 0 + D = 0$

$D=0$

Now, let's use $u(\ell,t)=0$.

$u(\ell,t)=0 \rightarrow X(\ell) \cdot T(t)=0 \rightarrow X(\ell)=0$

$X(\ell)=0 \rightarrow C \cdot \ell = 0$

Either $C=0$ or $\ell=0$.

If $C=0$, we get the trivial soln.

If $\ell=0$, that means the length of the domain is 0. Furthermore, this is a contradiction. We stated before, on pg 1, that $0<x<\ell$.

$\therefore$ There is no non-trivial soln if $\lambda=0$.

Consider the case when $\lambda<0$:

Let $\lambda=-\mu$, $\mu>0$

Then, we have:

$T'' - \mu c^2 T = 0$

$r^2 - \mu c^2 = 0$

$r^2 = \mu c^2$

$r = \pm\sqrt{\mu c^2}$

$\;\;= \pm\sqrt{\mu}\, c$

$r_1 = \sqrt{\mu}\, c$, $r_2 = -\sqrt{\mu}\, c$

$T_1 = e^{r_1 t}$

$\;\;= e^{\sqrt{\mu} c t}$

$T_2 = e^{r_2 t}$

$\;\;= e^{-\sqrt{\mu} c t}$

$T = AT_1 + BT_2$

$\;\;= Ae^{\sqrt{\mu} c t} + Be^{-\sqrt{\mu} c t}$

$X'' - \mu X = 0$

$r^2 - \mu = 0$

$r^2 = \mu$

$r = \pm\sqrt{\mu}$ $\quad r_1 = \sqrt{\mu}$, $r_2 = -\sqrt{\mu}$

$X_1 = e^{r_1 x}$

$\;\;= e^{\sqrt{\mu} x}$

$X_2 = e^{r_2 x}$

$\;\;= e^{-\sqrt{\mu} x}$

$X = CX_1 + DX_2$

$\;\;= Ce^{\sqrt{\mu} x} + De^{-\sqrt{\mu} x}$

Now, let's apply the boundary eqns.

First, we use $u(0,t)=0$.
$u(0,t)=0 \rightarrow X(0)\cdot T(t)=0 \rightarrow X(0)=0$
$X(0)=0 \rightarrow C+D=0$
This means that $D=-C$.

Now, we use $u(\ell,t)=0$.
$u(\ell,t)=0 \rightarrow X(\ell)\cdot T(t)=0 \rightarrow X(\ell)=0$
$X(\ell)=0 \rightarrow Ce^{\sqrt{\mu}\ell}+De^{-\sqrt{\mu}\ell}=0$
$Ce^{\sqrt{\mu}\ell}-Ce^{-\sqrt{\mu}\ell}=0$ (Because we got $D=-C$)
$C(e^{\sqrt{\mu}\ell}-e^{-\sqrt{\mu}\ell})=0$

Either $C=0$ or $e^{\sqrt{\mu}\ell}-e^{-\sqrt{\mu}\ell}=0$.
If $C=0$, then $D=0$ and we get the trivial soln.
This means that $e^{\sqrt{\mu}\ell}-e^{-\sqrt{\mu}\ell}=0$.

$e^{\sqrt{\mu}\ell}-e^{-\sqrt{\mu}\ell}=0$
$e^{\sqrt{\mu}\ell}=e^{-\sqrt{\mu}\ell}$
$\sqrt{\mu}\ell=-\sqrt{\mu}\ell \leftarrow$ Take the ln of both sides.
$\ln e^{f(x)}=f(x)$.
Recall that $\log_a a^x=x$.
$\ln=\log_e$

$\sqrt{\mu}=-\sqrt{\mu}$
$2\sqrt{\mu}=0$
This means that $\sqrt{\mu}=0$, which is a contradiction since we stated that $\mu>0$.

There are no non-trivial solns for when $\lambda<0$.

$\therefore \lambda>0$ and the separation constant is negative.