MATA22 Booklet 5 Notes

Definitions:

1. A function $T: R^n \rightarrow R^m$ is a linear transformation if for all **v**, **u** $\in R^n$ and for all $r \in R$, the following conditions are satisfied.

   1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (Preservation of vector addition)
   2. $T(r\mathbf{v}) = r(T(\mathbf{v}))$ (Preservation of scalar multiplication)

2. If $T: R^n \rightarrow R^m$ is a linear transformation, then:
   1. R^n is the domain of T.
   2. R^m is the co-domain of T.

3. Let $T: R^n \rightarrow R^m$ be a linear transformation.
   If $W \subset R^n$, then the image of W under T, denoted by T[W], is $\{T(\mathbf{w}) \mid \mathbf{w} \in W\}$.
   The image is the span of the vectors in the linear transformation.
   Image = Range = Column Space

4. Let $T: R^n \rightarrow R^m$ be a linear transformation.
   The range of T, denoted by $T[R^n]$, is $\{T(\mathbf{v}) \mid \mathbf{v} \in R^n\}$.

5. Let $T: R^n \rightarrow R^m$ be a linear transformation.
   If $W' \subset R^m$, then the inverse image of W' under T, denoted by $T^{-1}[W']$, is $\{\mathbf{w} \in R^n \mid T(\mathbf{w}) \in W'\}$.

6. Let $T: R^n \rightarrow R^m$ be a linear transformation.
   The kernel of T, denoted by $T^{-1}[\mathbf{0'}]$, is $\{\mathbf{w} \in R^n \mid T(\mathbf{w}) \in 0\}$ and $\mathbf{0'} \in R^m$.
   The kernel is the nullspace of the linear transformation.

7. The rank of the linear transformation = dim(Image)
   = dim(Range)
   = dim(Column Space)

8. The nullity of the linear transformation = dim(kernel).

9. Rank(Linear Transformation) + Nullity(Linear Transformation) equals to the number of columns in the linear transformation.

10. Let T: $R^n \rightarrow R^m$ be a linear transformation.
   Let A be a m*n matrix such that A =

$$\begin{bmatrix} | & | & & | \\ T(\mathbf{e1}) & T(\mathbf{e2}) & \cdots & T(\mathbf{en}) \\ | & | & & | \end{bmatrix}$$

   Then, T($\mathbf{x}$) = A$\mathbf{x}$ for all $\mathbf{x} \in R^n$. A is called the standard matrix representation of T.

   Note: T is invertible if m = n and if A is invertible.

   Note: To find the inverse image of T, find the inverse matrix of A.

11. Let T: $R^n \rightarrow R^m$ be a linear transformation.
   1. T is one – to – one if T($\mathbf{v}$) = T($\mathbf{u}$) implies that $\mathbf{v}$ = $\mathbf{u}$.
      I.e. If $\mathbf{v} \neq \mathbf{u}$, then T($\mathbf{v}$) ≠ T($\mathbf{u}$).
      I.e. T is one – to – one if the kernel is empty.

   2. T is onto if T[R^n] = R^m.
      I.e. $\forall \mathbf{v'} \in R^m \ \exists \mathbf{v} \in R^n$ such that T($\mathbf{v}$) = $\mathbf{v'}$.
      I.e. T is onto if the rank of the domain equals the rank of the co-domain.

   3. T is isomorphic if T is both one – to – one and onto.

Theorems:

1. Let $T: R^n \rightarrow R^m$ be a linear transformation.
    1. If $\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n}, \in R^n$ and $r_1, r_2, \ldots r_n, \in R$, then
       $T(r_1 \mathbf{v_1} + r_2 \mathbf{v_2} + \ldots r_n \mathbf{v_n}) = T(r_1 \mathbf{v_1}) + T(r_2 \mathbf{v_2}) + \ldots T(r_n \mathbf{v_n})$.

    2. $T(\mathbf{0}) = \mathbf{0'}$ where $\mathbf{0} \in R^n$ and $\mathbf{0'} \in R^m$.

2. If $T: R^n \rightarrow R^m$ be a linear transformation and $\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n}, \in R^n$ such that $\{T(\mathbf{v_1}), T(\mathbf{v_2}), \ldots T(\mathbf{v_n})\}$ is a linearly independent set in R^m, then $\{ \mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n}\}$ is also linearly independent.

3. If $T: R^n \rightarrow R^m$ be a linear transformation and $B = \{\mathbf{b_1}, \mathbf{b_2}, \ldots \mathbf{b_n}\}$ is a basis for R^n, then if $\mathbf{v} \in R^n$, $T(\mathbf{v})$ is determined by $T(\mathbf{b_1}), T(\mathbf{b_2}), \ldots T(\mathbf{b_n})$.

4. Let $T: R^n \rightarrow R^m$ be a linear transformation. Then:
    1. If W is a subspace of $R^{n,}$ then T[W'] is a subspace of R^m.
       I.e. If W is a subspace of R^n, then the image of W is a subspace of R^m.

    2. If W' is a subspace of $R^{m,}$ then T^{-1}[W] is a subspace of R^n.
       I.e. If W' is a subspace of R^m, then the inverse image of W' is a subspace of R^n.