MATB44 Week 3 Notes

1. Existence and Uniqueness Theorem:

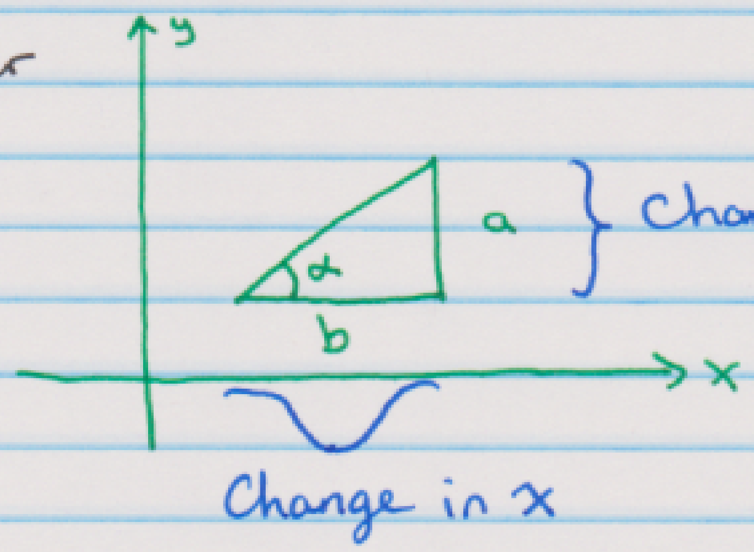
- Note: We won't be tested on this.
- Let $y' = f(t,y)$ where $f = t^2 + y^2$
  Lets take some arbitrary values for t and y and find f.
  E.g.
  $f(1,1) = 2$
  $= \tan \alpha_{(1,1)}$

  $f(1,2) = 5$
  $= \tan \alpha_{(1,2)}$

Recall: The derivative of a function is the slope of that function. Once we start talking about slopes, we start talking about tangent. This is because

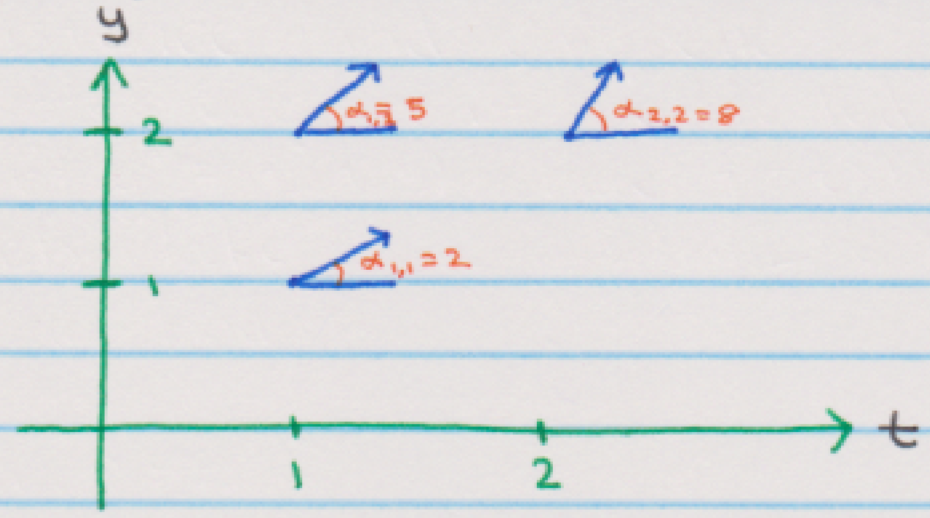
slope $= \dfrac{\text{Change in } y}{\text{Change in } x}$ and $\tan \alpha = \dfrac{\text{Change in } y}{\text{Change in } x}$.

Consider



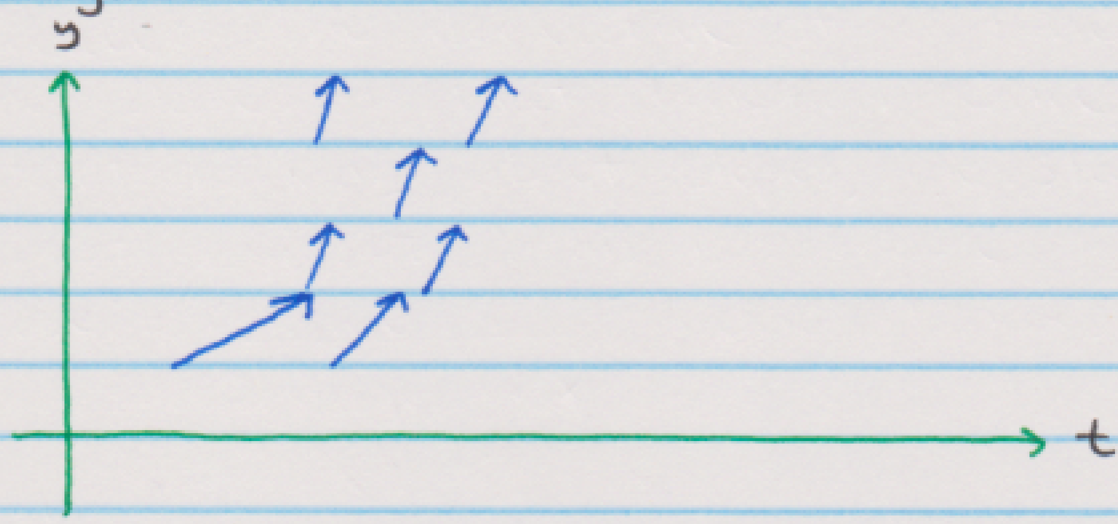
$$\tan \alpha = \frac{a}{b} = \frac{\text{Change in } y}{\text{Change in } x}$$

Now, consider this vector field



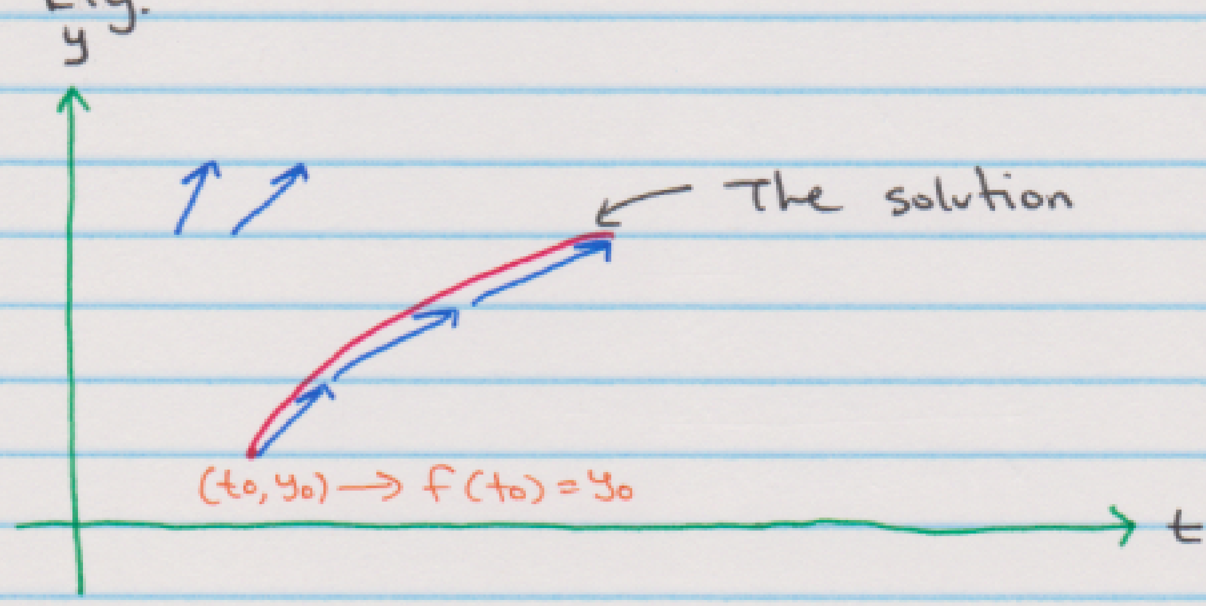
In a vector field, at each point, there is a unit vector attached to it.

E.g.



The solution is given based on the vector field.

E.g.

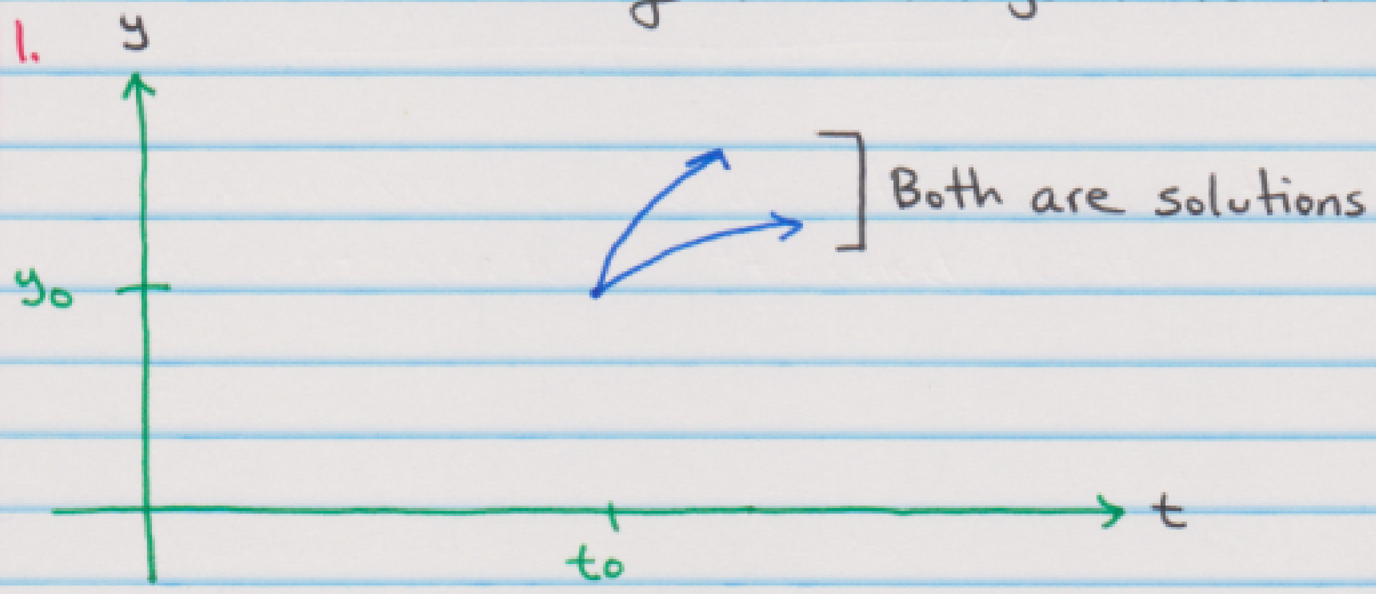


– Existence and Uniqueness Theorem:

Given an eqn $y' = f(t,y)$ with the initial condition $y(t_0) = y_0$, we can find an unique soln defined for some interval $t_0 < t < t_1$.

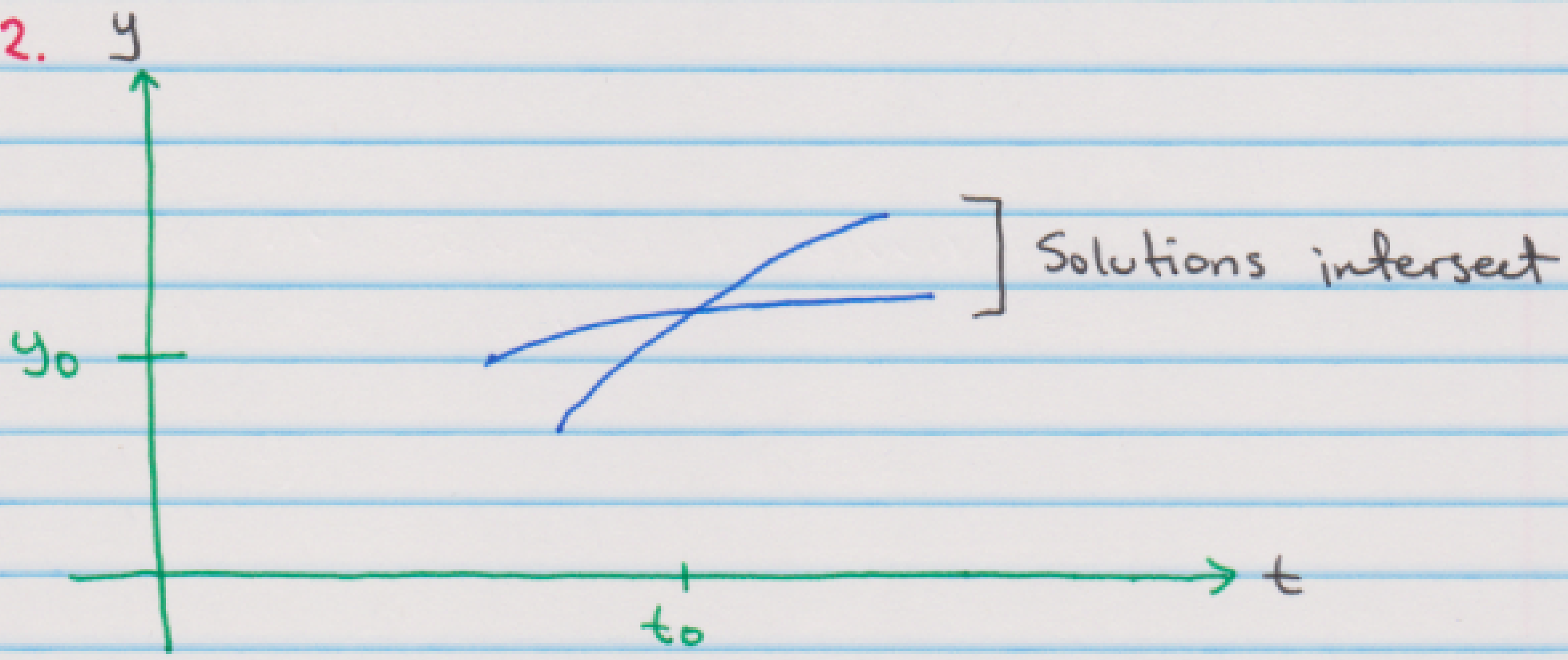
I.e. The following two things never happen.

1\.



This never happens because solutions are unique.

2\.



This can't happen because the solns are parallel. Furthermore, if they did intersect, that means that there would be 2 slopes at one point.

Consider the question $y' = y^2$, $y(0) = 1$

$$\frac{dy}{dt} = y^2$$

$$\frac{1}{y^2}\,dy = 1\,dt$$

$$\int \frac{1}{y^2}\,dy = \int 1\,dt$$

$$-\frac{1}{y} = t + c$$

$$y = \frac{-1}{t+c}$$

Plug in 0 for t and 1 for y.

$$1 = \frac{-1}{0+c}$$

$$c = -1$$

$$y = \frac{-1}{t-1}$$

$$= \frac{1}{1-t}$$

Now, we know $t_0 = 0$, but how far can it go? $t_0 = 0 \longrightarrow$ up to $t=1$. At $t=1$ there's a problem. The denominator is 0. The soln escapes to infinity. Hence, the soln is only good for $0 \le t < 1$. After $t=1$, the soln changes. Hence, solns are defined for some interval $t_0 < t < t_1$.

2. Homogeneous Equations With Constant Coefficients:

a) Introduction:

- Many second order differential equations have the form $\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$.

We say that the above eqn is linear if f has the form $f\left(t, y, \frac{dy}{dt}\right) = g(t) - p(t)\frac{dy}{dt} - q(t)y$.

In this case, we can rewrite the first eqn as
$y'' + p(t)y' + q(t)y = g(t)$.
We may also see the eqn in this form
$P(t)y'' + Q(t)y' + R(t)y = G(t)$.
If $P(t) \neq 0$, then we can get $y'' + p(t)y' + q(t)y = g(t)$ from $P(t)y'' + Q(t)y' + R(t)y = G(t)$ by dividing the latter eqn by $P(t)$.
I.e.
$p(t) = \frac{Q(t)}{P(t)}$, $q(t) = \frac{R(t)}{P(t)}$, $g(t) = \frac{G(t)}{P(t)}$
If the eqn is not of the form
$y'' + p(t)y' + q(t)y = g(t)$ or $P(t)y'' + Q(t)y' + R(t)y = G(t)$
then it is non-linear.

- A second order linear differential eqn is homogeneous if $g(t)$ or $G(t)$ or RHS is 0.
Otherwise, the eqn is nonhomogeneous.

b) Solving Homogeneous Eqns with Constant Coefficients

Rule 1: Homogeneous eqns always have a soln of this form $y=e^{rt}$ where r is an unknown constant and t is an unknown function.

Rule 2: We can combine the solns to get a new soln.

c) Solving Homogeneous Eqns with Constant Coefficients and 2 Real Distinct Roots

E.g. 1 Solve $y''-y=0$, $y(0)=2$ and $y'(0)=-1$

Soln:

$$\left.\begin{aligned} y&=e^{rt}\\ y'&=re^{rt}\\ y''&=r^2e^{rt}\end{aligned}\right\}\text{ Applying rule 1}$$

$$r^2e^{rt}-e^{rt}=0$$
$$e^{rt}(r^2-1)=0$$
$$r^2-1=0$$
$$r^2=1$$
$$r=\pm 1$$

$y_1=e^{t}$, $y_2=e^{-t}$

$y=C_1e^{t}+C_2e^{-t}$ ← Applying rule 2

Note: C_1 and C_2 are arbitrary constants. There are 2 constants because it's a 2nd order diff eqn.

Now, we'll solve for C_1 and C_2.

$y(0) = 2 \rightarrow 2 = C_1 e^{0} + C_2 e^{-0}$
$= C_1 + C_2$

$y'(0) = -1 \rightarrow y' = (C_1 e^{t} + C_2 e^{-t})'$
$= C_1 e^{t} - C_2 e^{-t}$
$-1 = C_1 e^{0} - C_2 e^{-0}$
$= C_1 - C_2$

$2 = C_1 + C_2$
$-1 = C_1 - C_2$
$C_1 = \frac{1}{2}, \ C_2 = \frac{3}{2}$

$\therefore \ y = \frac{e^{t}}{2} + \frac{3e^{-t}}{2}$

- From example 1, we can come to this conclusion:
  $ay'' + by' + cy = 0 \rightarrow (ar^2 + br + c)e^{rt} = 0$
  Since $e^{rt} \neq 0$,
  $ar^2 + br + c = 0$.

Note: $ar^2 + br + c = 0$ is called the characteristic equation.

Assuming that $r_1 \neq r_2$, $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$
and $y = C_1 \cdot e^{r_1 t} + C_2 \cdot e^{r_2 t}$.

E.g. 2 Solve $y'' + 5y' + 6y = 0$

Soln:

$r^2 + 5r + 6 = 0$

$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-5 \pm \sqrt{25-24}}{2}$

$= \frac{-5 \pm 1}{2}$

$= -2$ or -3

Aside: Consider $b^2 - 4ac$.
It has 3 possible values:
a) >0, b) $=0$, c) <0.
For this section, $b^2 - 4ac > 0$.
This means that there will be 2 distinct, real roots.

$y_1 = e^{-2t}$, $y_2 = e^{-3t}$

$y = C_1 e^{-2t} + C_2 e^{-3t}$ ← General Solution

E.g. 3 Solve $y'' + 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = 3$

Soln:

We know from above that $y = C_1 e^{-2t} + C_2 e^{-3t}$

$y(0) = 2 \rightarrow 2 = C_1 + C_2$

$y'(0) = 3 \rightarrow y' = -2C_1 e^{-2t} - 3C_2 e^{-3t}$

$3 = -2C_1 - 3C_2$

$C_1 + C_2 = 2$

$-2C_1 - 3C_2 = 3$

$C_1 = 9$ and $C_2 = -7$

$y = 9e^{-2t} - 7e^{-3t}$

E.g. 4 Solve $4y'' - 8y' + 3y = 0$, $y(0) = 2$, $y'(0) = \frac{1}{2}$

Soln:

$4r^2 - 8r + 3 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{8 \pm \sqrt{64 - 48}}{8}$$

$$= \frac{8 \pm \sqrt{16}}{8}$$

$$= \frac{8 \pm 4}{8}$$

$$= 3/2 \text{ or } 1/2$$

$y_1 = e^{\frac{1}{2}t}$, $y_2 = e^{\frac{3}{2}t}$

$y = C_1 \cdot e^{t/2} + C_2 \cdot e^{\frac{3t}{2}}$

$y(0) = 2 \longrightarrow 2 = C_1 + C_2$

$y'(0) = \frac{1}{2} \longrightarrow y' = \frac{C_1}{2} \cdot e^{t/2} + \frac{3C_2}{2} \cdot e^{\frac{3t}{2}}$

$$\frac{1}{2} = \frac{C_1}{2} + \frac{3C_2}{2}$$

$$1 = C_1 + 3C_2$$

$$\left.\begin{array}{l} C_1 + C_2 = 2 \\ C_1 + 3C_2 = 1 \end{array}\right\} C_1 = \frac{5}{2}, \quad C_2 = \frac{-1}{2}$$

$$y = \frac{5}{2} e^{t/2} - \frac{1}{2} e^{\frac{3t}{2}}$$

E.g. 5 Solve $y'' + 2y' - 3y = 0$

Soln:

$r^2 + 2r - 3 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4 - (-12)}}{2}$$

$$= \frac{-2 \pm 4}{2}$$

$= 1$ or -3

$y_1 = e^t$, $y_2 = e^{-3t}$

$y = C_1 \cdot e^t + C_2 \cdot e^{-3t}$

E.g. 6 Solve $y'' + 3y' + 2y = 0$

Soln:

$r^2 + 3r + 2 = 0$

$(r+2)(r+1) = 0$

$r = -2$ or -1

$y_1 = e^{-2t}$, $y_2 = e^{-t}$

$y = C_1 \cdot e^{-2t} + C_2 \cdot e^{-t}$

E.g. 7 Solve $y'' + 3y' - 10y = 0$, $y(0) = 4$, $y'(0) = -2$

Soln:

$r^2 + 3r - 10 = 0$ $\rightarrow$ $y_1 = e^{2t}$, $y_2 = e^{-5t}$

$(r-2)(r+5) = 0$ $\quad y = C_1 \cdot e^{2t} + C_2 \cdot e^{-5t}$

$r = 2$ or -5

$y(0)=4 \rightarrow 4=C_1+C_2$

$y'(0)=-2 \rightarrow y'=2C_1\cdot e^{2t}-5C_2\cdot e^{-5t}$

$-2=2C_1-5C_2$

$\left.\begin{array}{l} C_1+C_2=4 \\ 2C_1-5C_2=-2 \end{array}\right\} C_1=\frac{18}{7},\ C_2=\frac{10}{7}$

$y=\frac{18}{7}\cdot e^{2t}+\frac{10}{7}e^{-5t}$

d) Solving Homogeneous Eqns with Constant Coefficients and Complex Roots

- So far, we have been dealing with eqns s.t. $b^2-4ac>0$. Now, we will look at what happens if $b^2-4ac<0$.
- If $b^2-4ac<0$, then

$r_1=\lambda+iu$, $r_2=\lambda-iu$ where λ and u are real numbers.

Furthermore, $y_1=e^{r_1 t}=e^{(\lambda+iu)t}$ $\quad y_2=e^{r_2 t}=e^{(\lambda-iu)t}$

I.e. If $\lambda=-1$, $u=2$ and $t=3$, then

$y_1=e^{(-1+2i)(3)}=e^{-3+6i}$, $\quad Y_2=e^{(-1-2i)3}=e^{-3-6i}$

- Euler's Formula: $e^{it}=\cos(t)+i\sin(t)$

Note: If t is replaced with $-t$, we get $e^{-it}=\cos(t)-i\sin(t)$.

Note: If t is replaced with μt, we get $e^{i\mu t}=\cos(\mu t)-i\sin(\mu t)$.

- With Euler's Formula in mind,

$$
\begin{aligned}
e^{(\lambda + iu)t} &= e^{\lambda t + iut} \\
&= e^{\lambda t}(e^{iut}) \\
&= e^{\lambda t}(\cos(ut) + i\sin(ut)) \\
&= e^{\lambda t}\cos(ut) + ie^{\lambda t}\sin(ut)
\end{aligned}
$$

AND

$$
\begin{aligned}
e^{(\lambda - iu)t} &= e^{\lambda t - iut} \\
&= e^{\lambda t}(e^{-iut}) \\
&= e^{\lambda t}(\cos(ut) - i\sin(ut)) \\
&= e^{\lambda t}(\cos(ut)) - ie^{\lambda t}(\sin(ut))
\end{aligned}
$$

Recall that $y_1 = e^{(\lambda + iu)t}$ and $y_2 = e^{(\lambda - iu)t}$. Hence, $y_1 = e^{\lambda t}(\cos(ut)) + ie^{\lambda t}(\sin(ut))$ and $y_2 = e^{\lambda t}(\cos(ut)) - ie^{\lambda t}(\sin(ut))$.

Thm 1. Consider $y'' + p(t)y' + q(t)y = 0$. If $y = u(t) + iv(t)$ is a complex-valued soln, then both u and v are also solutions.

$$
\begin{aligned}
y_1 &= e^{\lambda t}(\cos(ut)) + ie^{\lambda t}(\sin(ut)) \\
y_2 &= e^{\lambda t}(\cos(ut)) - ie^{\lambda t}(\sin(ut))
\end{aligned}
$$

Let $a(t) = e^{\lambda t}(\cos(ut))$
Let $b(t) = e^{\lambda t}(\sin(ut))$ } By Thm 1.

$y = C_1(a(t)) + C_2(b(t))$
$= C_1(e^{\lambda t}(\cos(ut))) + C_2(e^{\lambda t}(\sin(ut)))$

E.g. 8 Solve $y'' + y' + 9.25y = 0$, $y(0) = 2$, $y'(0) = 8$

Soln:
$r^2 + r + 9.25 = 0$
$4r^2 + 4r + 37 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-4 \pm \sqrt{16 - 592}}{8}$$
$$= \frac{-4 \pm \sqrt{-576}}{8}$$
$$= \frac{-4 \pm 24i}{8}$$
$$= -\frac{1}{2} \pm 3i$$

$r_1 = \frac{-1}{2} + 3i$, $r_2 = \frac{-1}{2} - 3i$

$\lambda = -\frac{1}{2}$, $u = 3$

$y_1 = e^{\lambda t}(\cos(ut) + i\sin(ut))$
$= e^{-t/2}(\cos(3t) + i\sin(3t))$

$y_2 = e^{\lambda t}(\cos(ut) - i\sin(ut))$
$= e^{-t/2}(\cos(3t) - i\sin(3t))$

$a(t) = e^{-\frac{t}{2}}(\cos(3t))$, $b(t) = e^{-\frac{t}{2}}(\sin(3t))$

$y = C_1(a(t)) + C_2(b(t))$
$= C_1(e^{-t/2}(\cos(3t))) + C_2(e^{-t/2}(\sin(3t)))$

$y(0) = 2 \rightarrow 2 = C_1(e^0(\cos(0))) + C_2(e^0(\sin(0)))$
$= C_1$

$y'(0) = 8 \longrightarrow y' = (C_1(e^{-\frac{t}{2}}(\cos(3t)) + C_2(e^{-\frac{t}{2}}(\sin(3t))))'$

$$= -\frac{C_1(e^{-\frac{t}{2}})(6\sin(3t) + \cos(3t))}{2} - \frac{C_2(e^{-\frac{t}{2}})(\sin(3t) - 6\cos(3t))}{2}$$

$8 = (-C_1(1) - [C_2(-6)])(\frac{1}{2})$
$16 = -C_1 + 6C_2$
$= -2 + 6C_2$
$18 = 6C_2$
$C_2 = 3$ $\quad\quad C_1 = 2,\ C_2 = 3$

$y = 2e^{-t/2}\cos(3t) + 3e^{-t/2}\sin(3t)$
$= e^{-t/2}(2\cos(3t) + 3\sin(3t))$

E.g. 9 Solve $16y'' - 8y' + 145y = 0$, $y(0) = -2$, $y'(0) = 1$

Soln :
$16r^2 - 8r + 145 = 0$
$r = \frac{1}{4} \pm 3i$
$\lambda = \frac{1}{4},\ \mu = 3$

$y_1 = e^{\lambda t}(\cos(ut) + i\sin(ut))$

$= e^{\frac{t}{4}}(\cos(3i) + i\sin(3i))$

$y_2 = e^{\lambda t}(\cos(ut) - i\sin(ut))$

$= e^{\frac{t}{4}}(\cos(3t) - i\sin(3t))$

$a(t) = e^{\lambda t}(\cos(ut))$ $\quad$ $b(t) = e^{\lambda t}(\sin(ut))$

$= e^{\frac{t}{4}}(\cos(3t))$ $\quad$ $= e^{\frac{t}{4}}(\sin(3t))$

$y = C_1\, a(t) + C_2\, b(t)$

$= C_1(e^{t/4})(\cos(3t)) + C_2(e^{t/4})(\sin(3t))$

$y(0) = -2 \longrightarrow -2 = C_1$

$y'(0) = 1 \longrightarrow 1 = \dfrac{-C_1(-1)}{4} + \dfrac{C_2(12)}{4}$

$4 = C_1 + 12C_2$

$6 = 12C_2$

$C_2 = \dfrac{1}{2}$

$y = -2(e^{t/4})(\cos(3t)) + \frac{1}{2}(e^{t/4})(\sin(3t))$

E.g. 10 Solve $y''+9y=0$

Soln:

$r^2+9r=0$

$r=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$=\dfrac{0\pm\sqrt{-36}}{2}$

$=\dfrac{\pm 6i}{2}$

$=\pm 3i \longrightarrow r_1=3i,\ r_2=-3i$

$\lambda=0$ and $u=3$

$a(t)=e^{\lambda t}(\cos(ut))$ $b(t)=e^{\lambda t}(\sin(ut))$

$=\cos(3t)$ $=\sin(3t)$

$y=c_1a(t)+c_2b(t)$

$=c_1\cdot\cos(3t)+c_2\cdot\sin(3t)$

E.g. 11 Solve $y''-2y'+2y=0$

Soln:

$r^2-2r+2=0$

$r=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$=\dfrac{2\pm\sqrt{4-8}}{2}$

$=\dfrac{2\pm 2i}{2}$

$=1+i$ or $1-i$

$\lambda = 1, \ u = 1$

$a(t) = e^{\lambda t}(\cos(ut))$ $\quad$ $b(t) = e^{\lambda t}(\sin(ut))$

$\quad = e^{t}(\cos(t))$ $\quad\quad$ $= e^{t}(\sin(t))$

$y = C_1 a(t) + C_2 b(t)$

$\quad = C_1 (e^t)(\cos(t)) + C_2 (e^t)(\cos(t))$

E.g. 12 Solve $y'' - 2y' + 6y = 0$

Soln:

$r^2 - 2r + 6 = 0$

$r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\quad = \dfrac{2 \pm \sqrt{4 - 24}}{2}$

$\quad = \dfrac{2 \pm 2\sqrt{5}i}{2}$

$\quad = 1 \pm \sqrt{5}i$

$\lambda = 1, \ u = \sqrt{5}$

$a(t) = e^{\lambda t}(\cos(ut))$ $\quad$ $b(t) = e^{\lambda t}(\sin(ut))$

$\quad = e^{t}(\cos(\sqrt{5}t))$ $\quad\quad$ $= e^{t}(\sin(\sqrt{5}t))$

$y = C_1 a(t) + C_2 b(t)$

$\quad = C_1 (e^t)(\cos(\sqrt{5}t)) + C_2 (e^t)(\sin(\sqrt{5}t))$

e) Solving Homogeneous Eqns With Constant Coefficients and 1 Real Distinct Root

- We will now look at the case when $b^2 - 4ac = 0$.

If $b^2 - 4ac = 0$, then $r = \frac{-b \pm 0}{2a}$

$$= \frac{-b}{2a}$$

Hence, if $b^2 - 4ac = 0$, there is 1 distinct real root.

However, this poses a problem. We need both y_1 and y_2, and right now, we just have y_1.

- To find y_2, we will use the **Wronksian**.
- The Wronksian is also important for seeing if a particular soln is also a general soln. We will look at this first.

Consider the following:

$p(t)y'' + q(t)y' + r(t)y = 0$ and $y(t_0) = y_0$ and $y'(t_0) = y'_0$

Suppose that y_1 and y_2 are solutions to this eqn.

We know that $y = c_1 y_1 + c_2 y_2$ is also a soln. What we want to know is if this will be a general soln. In order for this to be a general soln, it must satisfy the initial conditions.

$y_0 = c_1 y_1(t_0) + c_2 y_2(t_0)$
$y'_0 = c_1 y'_1(t_0) + c_2 y'_2(t_0)$

We can use Cramer's Rule to find c_1 and c_2.

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y'_0 & y'_2(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}} \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y'(t_0) & y'_0 \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}}$$

The denominator is the Wronksian.
I.e. If y_1 and y_2 are 2 solns of a linear homogeneous eqn, then

$$w(t) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1 y'_2 - y_2 y'_1$$

Notice that the only thing preventing us from finding c_1 and c_2 is if $w(t) = 0$. If $w(t) \neq 0$ and y_1 and y_2 are solns, then the 2 solns are called a fundamental set of solns and $y = c_1 y_1 + c_2 y_2$ is a general soln.

Note: The Wronksian Dichotomy for Two Solns states that for 2 solns, $w(t) = 0$ for all t or $w(t) \neq 0$ for all t.

- Now, we will see how the Wronksian is used to find y_2.

$$W = y_1 y_2' - y_1' y_2$$
$$w' = (y_1 y_2' - y_1' y_2)'$$
$$= \cancel{y_1' y_2'} + y_1 y_2'' - y_1'' y_2 - \cancel{y_1' y_2'}$$
$$= y_1 y_2'' - y_1'' y_2$$

Note that

a) $y_1'' + p(t) y_1' + q(t) y_1 = 0 \rightarrow y_1'' = -p(t) y_1' - q(t) y_1$

b) $y_2'' + p(t) y_2' + q(t) y_2 = 0 \rightarrow y_2'' = -p(t) y_2' - q(t) y_2$

$$w' = y_1(-p(t) y_2' - q(t) y_2) - y_2(-p(t) y_1' - q(t) y_1)$$
$$= -y_1 p(t) y_2' - y_1 q(t) y_2 + y_2 p(t) y_1' + y_2 q(t) y_1$$
$$= -y_1 p(t) y_2' + y_2 p(t) y_1'$$
$$= p(t)(-y_1 y_2' + y_2 y_1')$$
$$= -p(t)(y_1 y_2' - y_2 y_1')$$
$$= -p(t) \cdot w$$

$$\frac{dw}{dt} = -p(t) w$$

$$\frac{1}{w} dw = -p(t)\, dt \leftarrow \text{Separable Eqn}$$

$$\int \frac{1}{w} dw = \int -p(t)\, dt$$

$$\ln|w| = \int -p(t)\, dt + c$$

$$w = e^{\int -p + c}$$

$\omega = e^{c} \cdot e^{\int p}$

$= c' \cdot e^{\int p} \leftarrow$ Abel's Formula

Note: Abel's formula proves the dichotomy. Either $c=0$ and $\omega = 0$ everywhere or $c \neq 0$ and $\omega \neq 0$ everywhere.

Note: It is very important for this calculation that there is no coefficient in front of y''.

E.g. 13 Solve $y'' + 2y' + y = 0$

Soln:

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r = -1$

$y_1 = e^{-t}$

$\omega = c' e^{-\int p \, dt}$

$= c' e^{-\int 2 \, dt}$

$= c' e^{2t + c_1}$

Let $c' = 1$

$\omega = e^{-2t} \cdot e^{c_1}$

Let e^{c_1} be c_1'.

$\omega = c_1' \cdot e^{2t}$

Let $c_1' = 1$.

$\omega = e^{-2t}$

Also, $\omega = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= y_1 y_2' - y_2 y_1'$

$= e^{-t} y_2' - (e^{-t})' y_2$

$(e^{-t}) y_2' - (e^{-t})' y_2 = e^{-2t}$

$(e^{-t}) y_2' + (e^{-t}) y_2 = e^{-2t}$

$(e^{-t})(y_2' + y_2) = e^{-2t}$

$y_2' + y_2 = e^{-t}$ ← Linear Differential Eqn

$y_2 = te^{-t}$

Note: If we have $y'' + by' + cy = 0$ and $r_1 = r_2$, then $y_1 = e^{r_1 t}$ and $y_2 = te^{r_1 t}$. This is called the Repeated Root's Rule.

E.g. 14 Solve $y'' - 4y' + 4y = 0$

Soln

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r = 2$

$y_1 = e^{2t}$

$y_2 = te^{2t}$

$y = c_1 e^{2t} + c_2 te^{2t}$

E.g. 15 Solve $y'' + 14y' + 49y = 0$

Soln:

$r^2 + 14r + 49 = 0$

$(r+7)^2 = 0 \rightarrow r = -7$

$\left.\begin{array}{l} y_1 = e^{-7t} \\ y_2 = te^{-7t} \end{array}\right\} y = c_1 e^{-7t} + c_2 te^{-7t}$

E.g. 16 Let $f(t) = e^{2t}$, $g(t)$ be an unknown function and $w(f,g) = 3e^{4t}$. Find $g(t)$.

Soln:

$$w = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = 3e^{4t}$$

$$f \cdot g' - f' \cdot g = 3e^{4t}$$
$$e^{2t} \cdot g' - 2e^{2t} \cdot g = 3e^{4t}$$
$$e^{2t}(g' - 2g) = 3e^{4t}$$
$$g' - 2g = 3e^{2t} \leftarrow \text{Linear First order}$$
$$g = 3te^{2t} + \underline{ce^{2t}}$$

↑ Note: Here, we keep the c. We don't let it be 0 or 1.

E.g. 17 Solve $y'' - 8y' + 17y = 0$

Soln:

$$r^2 - 8r + 17 = 0$$
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{8 \pm \sqrt{64 - 68}}{2}$$
$$= \frac{8 \pm 2i}{2}$$
$$= 4 \pm i$$

$r_1 = 4 + i$, $r_2 = 4 - i$

$\lambda = 4$, $u = 1$

$a(t) = e^{\lambda t}(\cos(ut))$ $\quad$ $b(t) = e^{\lambda t}(\sin(ut))$

$= e^{4t}(\cos t)$ $\quad$ $= e^{4t}(\sin t)$

$y = c_1 a(t) + c_2 b(t) = c_1 e^{4t}(\cos t) + c_2 e^{4t}(\sin t)$