CSC 373 Week 2 Notes

Greedy Algorithm:

1. Introduction:

- With greedy algorithms, you want to get the piece with the most immediate benefit at each step.

- Note: You can't go back.
  I.e. After you make a choice, it's final.

- E.g. Suppose we have coins of 1, 7, and 10 denomination and we want to make \$18 with as little coins as possible.

  Using a greedy algorithm, we would first choose the \$10 coin, then \$7 and then \$1.

  However, if we want to make \$15, then we run into a problem. We first choose a \$10 coin, so we have \$4 left over. That means we have to use 4 \$1 coins.

  \$10, \$1, \$1, \$1, \$1 ⇒ \$15

  However, we can make \$15 from 2 \$7 coins and 1 \$1 coin, using 3 coins instead of 5.

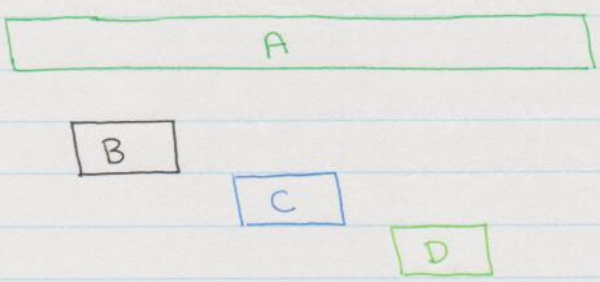
2. Interval Scheduling:

- Problem: We have a list of jobs and each job has a start time and a finish time.
  E.g. For job J, S_J denotes its start time while F_J denotes its end time.
  2 jobs, I and J, are compatible if $[S_i, F_i)$ and $[S_j$ and $F_j)$ don't overlap. (We allow a job to start right away when another finishes.) We want to find the maximum number of mutually compatible jobs.

- Here are a few ways we can order the jobs:
  1. Earliest start time: Ascending order of S_j.
  2. Earliest Finish time: Ascending order of F_j.
  3. Shortest Interval: Ascending order of $f_j - s_j$.
  4. Fewest conflicts: Ascending order of c_j, where c_j is the number of remaining jobs that conflict with j.

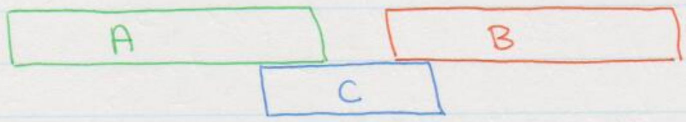
However, out of the 4 ways above, only "Earliest Finish Time" works. Here are some counterexamples.

1. For "Earliest Start Time", consider this:



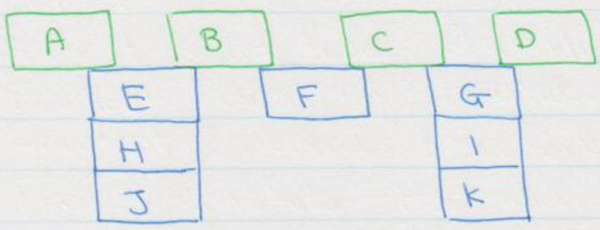
Notice how even though job A starts the earliest, it blocks 3 other jobs.

2. For "Shortest Interval"



Notice how even though C has the shortest interval, it's blocking 2 other jobs.

3. For "Fewest Conflicts"



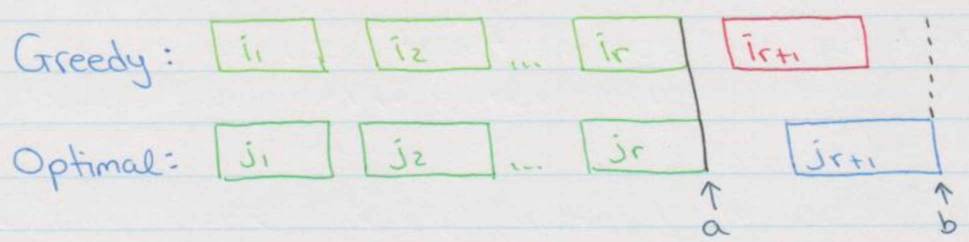
Here, if we use "Fewest conflicts", we get FAD. However, we could've gotten ABCD.

- The only viable ordering system is to use earliest finish time.

  Sorting will take $O(n\lg n)$.
  For each job j, we only need to check if it's compatible with the end time of the last added job. We can perform each check in $O(1)$.

  $\therefore$ The overall running time is $O(n\lg n)$.

- Proof of Optimality by Contradiction:
  - Suppose for contradiction that greedy soln is not optimal.
  - Say the greedy algo selects jobs $i_1, i_2, \ldots, i_k$ sorted by finish time.
  - Consider an optimal soln $J_1, J_2, \ldots, J_m$ also sorted by finish time and matches the greedy soln for as many indices as possible.
    I.e. We want $i_1 = j_1, \ldots, i_r = j_r$ for the greatest possible value of r.
  - We know both i_{r+1} and j_{r+1} must be compatible with the prev selection.



We know that both i_{r+1} and j_{r+1} must be between points a and b and we also know that i_{r+1} must end before J_{r+1}. This is bc we used the greedy algo to get i_{r+1}.

  - Suppose we switch jobs i_x and J_x for $1 \le x \le r+1$.
    I.e. We get this new soln: $i_1, i_2, \ldots, i_r, i_{r+1}, J_{r+2}, \ldots, J_m$
    This is still feasible because $f_{i_{r+1}} \le f_{j_{r+1}} \le S_{j_t}$ for $t \ge 2$.

    This is still optimal cause m jobs are selected, but it matches the greedy soln in $r+1$ indices.

- Proof of Optimality by Induction:
  - We will define S_j to be the subset of jobs picked by the greedy algo.

    Note: $S_0 = \emptyset$

  - We call this partial soln promising if there is a way to extend it to an optimal soln by picking some subset of jobs $j+1, \dots, n$.
    I.e. $\exists\, t \subseteq \{j+1, \dots, n\}$ s.t. $O_j = S_j \cup t$ is optimal.

  - WTP: $\forall t \in \{0, \dots, n\}$, S_t is promising.

  - Proof:

    Base Case ($t=0$):
    Let $t=0$.
    $S_t = \emptyset$ and is promising bc any optimal soln extends it.

    Induction Hypothesis:
    Suppose the claim holds for $t=j-1$ and optimal soln O_{j-1} extends S_{j-1}.

    Induction Step:
    At $t=j$, we have 2 possibilities:
    1. Greedy did not select job j so $S_j = S_{j-1}$.
       Job j must conflict with some job in S_{j-1}.
       Since $S_{j-1} \subseteq O_{j-1}$, it also cannot include job j.
       $O_j = O_{j-1}$ extends $S_j = S_{j-1}$.

    2. Greedy selects job j.
       $S_j = S_{j-1} \cup \{j\}$
       Consider the earliest job r between S_{j-1} and O_{j-1}.
       Consider O_j obtained by replacing r with j in O_{j-1}.
       O_j is still feasible and extends S_j as desired.

3. Interval Partitioning Problem:

- Problem: Job j starts at s_j and finishes at f_j. 2 jobs are compatible if they don't overlap. The goal is to group the jobs into the fewest partitions s.t. jobs in the same partition don't overlap (I.e. They're compatible)

- We'll be ordering the jobs based on their earliest start time.

- Pseudo-Code:

```
def partition (s1, s2, ..., sn , f1, ..., fn):
    Sort the jobs by start time s.t. s1 ≤ s2 ≤ ... ≤ sn.
    p=0 ← Number of partitions

    for j=1 to n:
        if job j is compatible with some partition:
            put job j in that partition
        else:
            create a new partition, p+1, and put job j
            in there
            p=p+1

    return p
```

- Running Time
  - Sorting will take $O(n\lg n)$.
  - We can use a priority queue to store the end times of each partition. We will do n compares and each compare is $\lg n$, so in total, we have $O(n\lg n)$.
  - $\therefore$ The total running time complexity is $O(n\lg n)$.

- Proof of Optimality:
  - Let d be the # of partitions used by the greedy algo.
  - Let depth be the max num of jobs running at any time.

1. Lower Bound:
   $d \geq$ depth (Have at least 1 partition per job)

2. Upper Bound:
   Partition d was opened bc there's a job j that's incompatible with some job in the other $d-1$ partitions. This means that these jobs end after S_j. However, bc we're sorting by start time, we know that they start before or at S_j.

   Hence, at time S_j, there are d overlapping jobs. This means that depth $\geq d$.

   Since we have $d \geq$ depth and depth $\geq d$, depth $= d$.

   $\therefore$ The greedy algo uses exactly as many partitions as the depth.

4. Minimizing Lateness:

- Problem: We have a single machine. Each job j requires t_j units of time to complete and is due by d_j. If it's scheduled to start at s_j, it will finish at $f_j = s_j + t_j$. The lateness of a job is $\ell_j = \max\{0, f_j - d_j\}$. The goal is to minimize the max lateness.

- We'll sort the jobs in ascending order of due time.

- Pseudo-Code:

```
def EarliestDueFirst(n, t1, ..., tn, d1, ..., dn):
    Sort the jobs in ascending order of due time.
    I.e. d1 ≤ d2 ≤ ... ≤ dn

    t = 0

    for j = 1 to n:
        Assign job j to interval [t, t + tj]
        sj = t
        fj = t + tj
        t = t + tj

    return [s1, f1], [s2, f2], ..., [sn, fn]
```

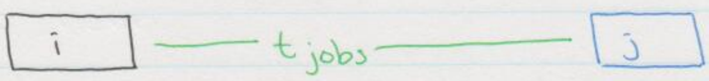
- Some observations:
  1. There's an optimal schedule with no idle time
  2. The job with the earliest deadline has no idle time.

Let an inversion be (i, j) s.t. $d_i < d_j$ but j is scheduled before i.
I.e. Job i is due before job j but job j is scheduled before job i.

3. The earliest deadline algo has no inversions.

4. If a schedule with no idle time has at least 1 inversion, then it has a pair of inverted jobs scheduled consecutively.

Proof:
Let jobs i and j be inverted and suppose they are the only 2 inverted jobs and that there are t jobs in between.



Job j is due before i but is scheduled after i.

Let's consider job i+1.
There are 2 possibilities:

1. Job i+1 is due before Job j. 2
   In this case, we now have inversions (i, i+1) and (i, j)
   which contradicts our assumption.

2. Job i+1 is due after Job j.
   In this case, we also get more than 1 inversion, which contradicts our assumption.

∴ i and j must be together.

5. Swapping adj scheduled inverted jobs doesn't increase lateness but it does reduce the num of inversions by 1.

Proof:
1. Reducing the num of inversions by 1 is easy to see.
   Suppose j and i are inverted, meaning that j is due before i but scheduled after. By switching them, j is now scheduled before i.

2. Let ℓ_k and ℓ'_k denote the lateness of job k before and after swap.

Let $L = \max_k \ell_k$ and $L' = \max_k \ell'_k$.

We know that:

1. $\ell_k = \ell'_k \quad \forall k \neq i, j$
2. $\ell'_i \leq \ell_i$ (Since i is moved early)
3. $\ell'_j = f'_j - d_j$
$= f_i - d_j$
$\leq f_i - d_i$
$= \ell_i$

$$\therefore \; L' = \max \{ \ell'_i, \ell'_j, \max_{k \neq i,j} \ell'_k \}$$

$$\leq \max \{ \ell_i, \ell_i, \max_{k \neq i,j} \ell'_k \}$$

$$\leq L$$