More Homogeneous Linear System Examples

1. Solve $\bar{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} 3-r & -2 \\ 2 & -2-r \end{vmatrix} = 0$$

$$(3-r)(-2-r)+4=0$$
$$-6-3r+2r+r^2+4=0$$
$$r^2-r-2=0$$

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$
$$= \frac{1 \pm \sqrt{1+8}}{2}$$
$$= \frac{1 \pm 3}{2}$$
$$= -1 \text{ or } 2$$

$$(A-rI)\bar{z} = \bar{0}$$
when $r = -1$

$$\begin{bmatrix} 3+1 & -2 \\ 2 & -2+1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4z_1 - 2z_2 = 0$
$2z_1 - z_2 = 0 \leftarrow$ Redundant

$2z_1 - z_2 = 0$
$2z_1 = z_2$
Let $z_1 = 1$. $z_2 = 2$

$$\overline{z^1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

When $r = 2$

$$\begin{bmatrix} 3-2 & -2 \\ 2 & -2-2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$z_1 - 2z_2 = 0$
$2z_1 - 4z_2 = 0 \leftarrow$ Redundant

$z_1 = 2z_2$

Let $z_1 = 1$. $z_2 = 1/2$

$$\overline{x} = c_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

2. Solve $\overline{x}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \overline{x}$

Soln:

$$\begin{vmatrix} 1-r & -2 \\ 3 & -4-r \end{vmatrix} = 0$$

$(1-r)(-4-r) + 6 = 0$
$-4 - r + 4r + r^2 + 6 = 0$
$r^2 + 3r + 2 = 0$
$(r+2)(r+1) = 0$
$r_1 = -2 \quad r_2 = -1$

$(A - rI)\bar{z} = \bar{0}$

When $r = -2$

$$\begin{bmatrix} 1+2 & -2 \\ 3 & -4+2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3z_1 - 2z_2 = 0$
$3z_1 - 2z_2 = 0 \leftarrow$ Redundant

$3z_1 = 2z_2$
$\frac{3}{2} z_1 = z_2$
Let $z_1 = 1$. $z_2 = 3/2$

$$\overline{z^1} = \begin{bmatrix} 1 \\ 3/2 \end{bmatrix}$$

When $r = -1$

$$\begin{bmatrix} 1+1 & -2 \\ 3 & -4+1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2z_1 - 2z_2 = 0$
$3z_1 - 3z_2 = 0 \leftarrow$ Redundant

$z_1 = z_2$
Let $z_1 = 1$. $z_2 = 1$

$$\overline{z^2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\bar{x} = c_1 e^{-2t} \begin{bmatrix} 1 \\ 3/2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

3. Solve $\bar{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} 2-r & -1 \\ 3 & -2-r \end{vmatrix} = 0$$

$(2-r)(-2-r)+3=0$
$-4-2r+2r+r^2+3=0$
$r^2-1=0$
$r^2=1$
$r_1=1,\ r_2=-1$

$(A-rI)\bar{z}=\bar{0}$
When $r=1$

$$\begin{bmatrix} 2-1 & -1 \\ 3 & -2-1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$z_1 - z_2 = 0$
$3z_1 - 3z_2 = 0$ ← Redundant

$z_1 = z_2$
Let $z_1 = 1.\ z_2 = 1.$

$$\overline{z^1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

When $r=-1$

$$\begin{bmatrix} 2+1 & -1 \\ 3 & -2+1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3z_1 - z_2 = 0$
$3z_1 - z_2 = 0$ ← Redundant

$3z_1 = z_2$
Let $z_1 = 1$. $z_2 = 3$.

$$\overline{z^2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\bar{x} = c_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

4. Solve $\bar{x}' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \bar{x}$, $\bar{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Soln:

$$\begin{vmatrix} 5-r & -1 \\ 3 & 1-r \end{vmatrix} = 0$$

$(5-r)(1-r)+3=0$
$5-5r-r+r^2+3=0$
$r^2-6r+8=0$
$(r-2)(r-4)=0$
$r_1=2,\ r_2=4$

$(A - rI)\bar{z} = \bar{0}$

When $r = 2$

$$\begin{bmatrix} 5-2 & -1 \\ 3 & 1-2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3z_1 - z_2 = 0$

$3z_1 = z_2$

Let $z_1 = 1$. $z_2 = 3$

$$\overline{z^1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

When $r = 4$

$$\begin{bmatrix} 5-4 & -1 \\ 3 & 1-4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$z_1 - z_2 = 0$

$z_1 = z_2$

Let $z_1 = 1$. $z_2 = 1$.

$$\overline{z^2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\bar{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\bar{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$c_1 + c_2 = 2$

$3c_1 + c_2 = -1$

$-2C_1 = 3 \rightarrow C_1 = -\frac{3}{2}$

$$\begin{aligned} C_2 &= 2 - C_1 \\ &= 2 + 3/2 \\ &= \frac{7}{2} \end{aligned}$$

$$\bar{x} = \frac{-3}{2} e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{7}{2} e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

5. Solve $\bar{x}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} -1-r & -4 \\ 1 & -1-r \end{vmatrix} = 0$$

$$(1+r)^2 + 4 = 0$$
$$r^2 + 2r + 1 + 4 = 0$$
$$r^2 + 2r + 5 = 0$$

$$\begin{aligned} r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{4 - 20}}{2} \\ &= \frac{-2 \pm 4i}{2} \\ &= -1 \pm 2i \end{aligned}$$

$(A - rI)\bar{z} = \bar{0}$

when $r = -1 + 2i$

$$\begin{bmatrix} -1-(-1+2i) & -4 \\ 1 & -1-(-1+2i) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(-1+1-2i)z_1 - 4z_2 = 0$

Note: In $A - rI$, even for complex numbers, the 2 rows are still linearly dependent.

$-2iz_1 = 4z_2$
$-iz_1 = 2z_2$

$\frac{-iz_1}{2} = z_2$

Let $z_1 = 2$. $z_2 = -i$

$$\overline{z^1} = \begin{bmatrix} 2 \\ -i \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Recall that $\bar{x} = e^{rt}\bar{z}$.
In this case, because we have complex eigenvalues, $r = \lambda + iu$.

$\bar{x} = e^{\lambda t + iut}\bar{z}$
$= e^{\lambda t} \cdot \underbrace{e^{iut}} \cdot \bar{z}$

Euler's Formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Hence, $\bar{x} = e^{\lambda t}(\cos(ut) + i\sin(ut))\bar{z}$.
In this example, $\lambda = -1$ and $u = 2$.

$$\bar{x} = e^{-t}(\cos(2t) + i\sin(2t))\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$

$$= e^{-t}\cos(2t)\begin{bmatrix} 2 \\ 0 \end{bmatrix} - e^{-t}\sin(2t)\begin{bmatrix} 0 \\ -1 \end{bmatrix} +$$
$$i\left(e^{-t}\cos(2t)\begin{bmatrix} 0 \\ -1 \end{bmatrix} + e^{-t}\sin(2t)\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)$$

$$\bar{x} = c_1\left(e^{-t}\cos(2t)\begin{bmatrix}2\\0\end{bmatrix} - e^{-t}\sin(2t)\begin{bmatrix}0\\-1\end{bmatrix}\right) +$$

$$c_2\left(e^{-t}\cos(2t)\begin{bmatrix}0\\-1\end{bmatrix} + e^{-t}\sin(2t)\begin{bmatrix}2\\0\end{bmatrix}\right)$$