MATA22 Booklet 3 Notes

Definition:

1. An m*n matrix is an ordered rectangular array of real entries with m rows and n columns.
   E.g.
   $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ ← A 2*2 matrix.

2. The set of all m*n matrices with real entries is denoted by $M_{m,n}(R)$.

3. Another way to denote a matrix is A = $[a_{ij}]$ where a_{ij} is the entry in the ith row and the jth column.

4. A m*1 matrix is a column vector.
   A 1*n matrix is a row vector.
   E.g.
   $\begin{bmatrix} 1 & 2 \end{bmatrix}$ ← A row vector.
   $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ← A column vector.

5. A square matrix is a matrix with the same number of rows and columns.
   E.g.
   $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ ← A square matrix.

6. The main diagonal in a square matrix is the diagonal that contains a_{ii}.
   E.g.
   $\begin{bmatrix} \mathbf{1} & 2 \\ 0 & \mathbf{2} \end{bmatrix}$ ← The highlighted diagonal is the main diagonal.

7. A diagonal matrix is a square matrix where a_{ij} = 0 for all i ≠ j.
   I.e. All entries not on the main diagonal is 0.
   E.g.
   $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ ← A diagonal matrix.

8. The identity matrix, denoted by I, is a special type of a diagonal matrix. All of its entries along the main diagonal is 1.
   E.g.
   $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ← A 3*3 identity matrix.

9. An upper triangular matrix is a square matrix such that $a_{ij} = 0$ for all $i > j$.
I.e. All entries below the main diagonal must be 0. Entries on and above the main diagonal could be 0.
E.g.
$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ ← An upper triangular matrix.

10. A lower triangular matrix is a square matrix such that $a_{ij} = 0$ for all $i < j$.
I.e. All entries above the main diagonal must be 0. Entries on and below the main diagonal could be 0.
E.g.
$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$ ← A lower triangular matrix.

11. A zero matrix is a matrix with all of its entries 0.
E.g.
$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ← A zero matrix

12. The transpose of a matrix A_{ij} is denoted by $(A_{ij})^T$.
$(A_{ij})^T = A_{ji}$
I.e. The rows of matrix A become the columns of A^T.
E.g.
$A = \begin{bmatrix} 1 & 6 & 5 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 1 & 4 \\ 5 & 0 & 1 \end{bmatrix}$

If $A = A^T$ then A is a symmetric matrix.
If $A = -A^T$ then A is a skew-symmetric matrix.

13. The trace of a square matrix A, denoted by Tr(A), is the sum of the entries along A's main diagonal.
E.g.
$A = \begin{bmatrix} 1 & 6 & 5 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$

$\text{Tr}(A) = 1 + 1 + 1$
$= 3$

14. A square matrix, A, is invertible if there exists another square matrix, C, such that (A)(C) = (C)(A) = I. C is the inverse matrix of A and is denoted by A^{-1}.

    If a square matrix is not invertible, then it is singular.
    Not all matrices are invertible.

15. A m*n linear system of equations is a system of m linear equations in n variables.
    E.g.
    $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$
    $a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$
    |
    |
    |
    $a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$

    The above linear system of equations is equivalent to A**x** = **b** where

    $$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

    A**x** = **b** has solutions iff **b** is in the span of A.

    A is called the coefficient matrix.

    $(\mathrm{A}|\mathbf{b})$ is called the augmented/partitioned matrix.

    $$(\mathrm{A}|\mathbf{b}) = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right)$$

16. Elementary Row Operations:
    1. Row Interchange: Switches the i^{th} and j^{th} rows.
    2. Row Scaling: Multiplies a row by a non-zero scalar.
    3. Row Addition: Adds the i^{th} row to the j^{th} row.

17. If matrix B can be obtained by performing a sequence of elementary row operations on matrix A, then A and B are row equivalent. We denote this by A ~ B.

18. If $(\mathrm{A}|\mathbf{b}) \sim (\mathrm{H}|\mathbf{c})$, then A**x** = **b** and H**x** = **c** has the same solution set.

19. A matrix is in REF if the following conditions are satisfied:
    1. All the rows with only zeros are at the bottom.
    2. The first non-zero entry in any row (pivot) is to the right of all the pivots in the rows above it.

    E.g.

    $\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$ ← A matrix in REF (Reduced Echelon Form).

20. A matrix is in RREF if the following conditions are satisfied:
    1. The matrix is in REF.
    2. All the pivots are 1.
    3. Each pivot is the only non-zero entry in its column.

    E.g.

    $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ← A matrix in RREF (Reduced – Row Echelon Form).

    Note: REFs are unique, but RREfs are not unique.

21. The Gauss Reduction with Back Substitution Method solves A**x** = **b** by reducing $(\mathrm{A}|\mathbf{b})$ to REF and then using substitution.

    The Gauss – Jordan Method solves A**x** = **b** by reducing $(\mathrm{A}|\mathbf{b})$ to RREF.

22. A consistent linear system is a linear system with 1 or more solutions.
    An inconsistent linear system is a linear system with 0 solutions.

23. Let A**x** = **b** be a linear system and suppose $(\mathrm{A}|\mathbf{b}) \sim (\mathrm{H}|\mathbf{c})$ and suppose that $(\mathrm{H}|\mathbf{c})$ is in REF, then:

    1. H**x** = **c** is inconsistent. This means that $(\mathrm{H}|\mathbf{c})$ has zeroes to the left of the partition and non-zero entries to the right of the partition.
    I.e. This means that 0**x** = a non-zero number.
    0**x** = 4 doesn't have any solutions because there is no value of **x** that will make it so that 0**x** equals 4.

    2. H**x** = **c** is consistent with pivots in every column. Here, the solution is unique.

    3. H**x** = **c** is consistent but not every column has a pivot. This means that there is infinite set of solutions.
    I.e. 0**x** = 0. **x** can be any real number.

24. An elementary matrix is a matrix such that only 1 elementary row operation has been performed to an identity matrix.

Matrix Arithmetic:

1. **Matrix Addition and Subtraction:**
   2 matrices can only add/subtract each other if they have the same dimensions. I.e. Both matrices have the same number of rows and columns.
   E.g.
   $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 8 & 10 \end{bmatrix}$

   $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2. **Matrix Multiplication:**
   2 matrices can multiply if the number of columns of the first matrix equals to the number of rows of the second matrix. The resulting matrix will have the same number of rows as the first matrix and same number of columns as the second matrix.
   E.g.
   $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$

   (1)(1) + (2)(3) = 1 + 6 = 7
   (1)(2) + (2)(4) = 2 + 8 = 10
   (3)(1) + (4)(3) = 3 + 12 = 15
   (3)(2) + (4)(4) = 6 +16 = 22

   With matrix multiplication, you dot product the first row of the first matrix with each column of the second matrix, one at a time. Then, you move on to the second row and you dot product that with each column of the second matrix, one at a time. You continue this pattern until you finish the last row of the first matrix.

   Note: Matrix multiplication is usually not commutative. This means that AB most likely won't equal to BA. An exception to this is inverse matrices. As mentioned before, $A(A^{-1}) = (A^{-1})A = I$.

3. **Matrix Scalar Multiplication:**
   When you multiply a scalar to a matrix, you multiply that scalar to all of the matrix's entries.
   E.g.
   $3 * \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$

Theorems:

Let A be an m*n matrix.
Let B, C and D be n*j matrices.
Let E be a j*l matrix.
Let r and s be scalars.
Let $\underline{0}$ be the n*j zero matrix.
Let -B be an n*j matrix with its entries opposite of B's.

1. $A + A^T$ is a symmetric matrix.
2. $A – A^T$ is a skew-symmetric matrix.
3. B + C = C + B
4. B + (C + D) = (B + C) + D
5. B + $\underline{0}$ = B
6. B + (-B) = $\underline{0}$
7. r(sB) = (rs)B
8. (r + s)B = rB + sB
9. r(B + C) = rB + rC
10. A(rB) = r(AB) = (rA)B
11. A(B + C) = AB + AC
12. AI = IA = A
13. $(A^T)^T = A$
14. $(B + C)^T = B^T + C^T$
15. $(rA)^T = r(A)^T$
16. $Tr(A^T) = Tr(A)$
17. Tr(sA) = s(Tr(A))
18. A(BE) = (AB)E