

MATB44 Week 2 Notes

1. Exact Equations:

- Suppose we have the following eqn
 $M(x,y) + N(x,y)y' = 0$.

If there's a function $F(x,y)$ s.t.

a) $F_x = M(x,y)$ and

b) $F_y = N(x,y)$

then, we call these equations **exact**.

Recall that $dF = F_x dx + F_y dy$.

$$F_x + F_y \frac{dy}{dx} = 0$$

$$\underbrace{F_x dx + F_y dy}_{dF} = 0$$

$$dF = 0$$

$dF = 0$ implies that $F(x,y) = C$ for some constant C .

- **Note:** Finding $F(x,y)$ is the central task in determining if a differential equation is exact and in finding its solution. Since finding $F(x,y)$ is a lengthy process, we can use the below test to see if the differential eqn is exact or not before doing any of the calculation steps.

The test is: If $F(x,y)$ is an exact differential eqn, then $M_y = N_x$.

Proof:

Assume that $F(x,y)$ is an exact differential eqn.

Then:

a) $M = F_x$

b) $N = F_y$

c) $F_{xy} = F_{yx}$ (Mixed derivative rule)

$$\begin{aligned} F_{xy} &= (F_x)_y \\ &= M_y \end{aligned}$$

$$\begin{aligned} F_{yx} &= (F_y)_x \\ &= N_x \end{aligned}$$

Therefore, if $F(x,y)$ is an exact differential eqn, then $M_y = N_x$.

- E.g.1 Solve $(y \cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0$.

Soln:

1. See if $M_y = N_x$.

$$\begin{aligned} M_y &= \frac{\partial}{\partial y} (y \cos x + 2xe^y) \\ &= \cos x + 2xe^y \end{aligned}$$

$$\begin{aligned} N_x &= \frac{\partial}{\partial x} (\sin x + x^2e^y - 1) \\ &= \cos x + 2xe^y \end{aligned}$$

$$M_y = N_x$$

This is an exact differential eqn.

2. Let $M = \frac{\partial}{\partial x} F$ and $N = \frac{\partial}{\partial y} F$.

Now,

$$\frac{\partial}{\partial x} F = y \cos x + 2xe^y$$

$$\frac{\partial}{\partial y} F = \sin x + x^2e^y - 1$$

Take the integral of the first eqn to find F.

$$\begin{aligned}
 F &= \int y \cos x + 2xe^y dx \\
 &= \int y \cos x dx + \int 2xe^y dx \\
 &= y \int \cos x dx + e^y \int 2x dx \\
 &= y \sin x + x^2 e^y + C(y)
 \end{aligned}$$

Note: When taking the integral w.r.t x , you treat y as a constant and vice versa.

Note: C might be a function of y . This is because, when we differentiate w.r.t x , y is treated as a constant.

Now, let's use the second eqn.

$$\partial_y F = \sin x + x^2 e^y - 1$$

Plug the eqn we just got for F into $\partial_y F$.

$$\begin{aligned}
 \partial_y (y \sin x + x^2 e^y + C(y)) \\
 = \sin x + x^2 e^y + C'(y)
 \end{aligned}$$

$$\sin x + x^2 e^y + C'(y) = \sin x + x^2 e^y - 1$$

$$C'(y) = -1$$

Note: At this point, if you have not made any mistakes, all of the terms with x 's should be gone.

$$C = \int -1 \, dy$$

$$= -y + C_1$$

$$F = y \sin x + x^2 e^y - y + C_1$$

Here, any C_1 works. Since it is an additive constant, we can take $C_1 = 0$.
If it was multiplicative, we take $C_1 = 1$.

The final answer is

$$F = y \sin x + x^2 e^y - y = C$$

- E.g. 2 Solve $\left(\frac{y}{x} + 6x\right) + (\log x - 2)y' = 0$

Soln:

$$\left(\frac{y}{x} + 6x\right) dx + (\log x - 2) dy = 0$$

$$M = \frac{y}{x} + 6x, \quad N = \log x - 2$$

1. Check to see if $M_y = N_x$

$$\begin{aligned} M_y &= \partial_y M \\ &= \partial_y \left(\frac{y}{x} + 6x\right) \\ &= \frac{1}{x} \end{aligned}$$

$$\begin{aligned} N_x &= \partial_x N \\ &= \partial_x (\log x - 2) \\ &= \frac{1}{x} \end{aligned}$$

$M_y = N_x$
F is an exact eqn.

2. Let $M = \partial_x F$ and $N = \partial_y F$
and solve for F .

$$\partial_x F = M$$

$$\begin{aligned} F &= \int M \, dx \\ &= \int y/x + 6x \, dx \\ &= \int y/x \, dx + \int 6x \, dx \\ &= y \int \frac{1}{x} \, dx + 6 \int x \, dx \\ &= y \ln|x| + 3x^2 + C(y) \end{aligned}$$

$$\begin{aligned} \partial_y F &= N \\ &= \log x - 2 \end{aligned}$$

$$\begin{aligned} \partial_y F &= \partial_y (y \ln|x| + 3x^2 + C(y)) \\ &= \ln|x| + C'(y) \end{aligned}$$

$$\ln|x| + C'(y) = \log x - 2$$

Note: In this course, we always use \log_e . Hence, in this course, $\ln = \log$.

$$C'(y) = -2$$

$$\begin{aligned} C &= \int -2 \, dy \\ &= -2y + C_1 \end{aligned}$$

$$F = y \ln|x| + 3x^2 - 2y + C_1$$

$$\text{Let } C_1 = 0$$

$$F = y \ln|x| + 3x^2 - 2y = C$$

- E.g. 3 Solve $(2xy - 9x^2) + (2y + x^2 + 1)y' = 0$

Soln:

$$(2xy - 9x^2)dx + (2y + x^2 + 1)dy = 0$$

$$M = 2xy - 9x^2, \quad N = 2y + x^2 + 1$$

1. Check if $M_y = N_x$.

$$\begin{aligned} M_y &= \partial_y M \\ &= \partial_y (2xy - 9x^2) \\ &= 2x \end{aligned}$$

$$\begin{aligned} N_x &= \partial_x N \\ &= \partial_x (2y + x^2 + 1) \\ &= 2x \end{aligned}$$

$$M_y = N_x$$

Hence, F is an exact differential eqn.

2. Let $M = \partial_x F$ and $N = \partial_y F$ and solve for F .

$$\begin{aligned} \partial_x F &= M \\ &= 2xy - 9x^2 \\ F &= \int 2xy - 9x^2 dx \\ &= x^2 y - 3x^3 + c(y) \end{aligned}$$

$$\partial_y F = N$$

$$\begin{aligned} \text{RHS} &= N \\ &= 2y + x^2 + 1 \end{aligned}$$

$$\begin{aligned}
 \text{LHS} &= \partial_y F \\
 &= \partial_y (x^2 y - 3x^3 + c(y)) \\
 &= x^2 + c'(y)
 \end{aligned}$$

$$\text{LHS} = \text{RHS}$$

$$x^2 + c'(y) = 2y + x^2 + 1$$

$$c'(y) = 2y + 1$$

$$\begin{aligned}
 C &= \int 2y + 1 \, dy \\
 &= y^2 + y + C_1
 \end{aligned}$$

$$F = x^2 y - 3x^3 + y^2 + y + C_1$$

$$\text{Let } C_1 = 0$$

$$F = y^2 + (x^2 + 1)y - 3x^3 = C$$

2. Exact Equation with Integrating Factor:

- So far, all the eqns we've seen have $M_y = N_x$. Not all eqns are like this. If $M_y \neq N_x$, we need to multiply both sides of the eqn by μ , the integrating factor.

- E.g. 4 Solve $(3xy + y^2) + (x^2 + xy)y' = 0$

Soln:

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

$$\rightarrow (3xy + y^2) dx + (x^2 + xy) dy = 0$$

$$\underbrace{\hspace{1.5cm}}_M$$

$$\underbrace{\hspace{1.5cm}}_N$$

M

N

1. Check if $M_y = N_x$.

$$\begin{aligned} M_y &= \partial_y M \\ &= \partial_y (3xy + y^2) \\ &= 3x + 2y \end{aligned}$$

$$\begin{aligned} N_x &= \partial_x N \\ &= \partial_x (x^2 + xy) \\ &= 2x + y \end{aligned}$$

Here, $M_y \neq N_x$

2. Multiply both sides of the eqn by $\mu(x)$.

$$\mu(x)(3xy + y^2) dx + \mu(x)(x^2 + xy) dy = 0$$

3. Solve for $\mu(x)$.

To solve for $\mu(x)$, we assume that $\partial_y [\mu(x)M] = \partial_x [\mu(x)N]$.

$$\partial_y [\mu(x)M] = \mu(x) [\partial_y M]$$

$$\partial_x [\mu(x)N] = \mu'(x)N + \mu(x) [\partial_x N]$$

$$\mu(x) (\partial_y M) = \mu'(x) N + \mu(x) (\partial_x N)$$

$$\begin{aligned} \mu'(x) N &= \mu(x) (\partial_y M) - \mu(x) (\partial_x N) \\ &= \mu(x) [\partial_y M - \partial_x N] \end{aligned}$$

$$\frac{\mu'(x)}{\mu(x)} = \frac{\partial_y M - \partial_x N}{N}$$

Note: For this to be solvable,
 $\partial_y M - \partial_x N$
 N CANNOT have any

"y" in it.

I.e. $\partial_y M - \partial_x N$
 N depends on "x" only.

$$\begin{aligned} \partial_y M &= 3x + 2y \\ \partial_x N &= 2x + y \\ N &= x^2 + xy \end{aligned}$$

$$\frac{\partial_y M - \partial_x N}{N}$$

$$= \frac{3x + 2y - (2x + y)}{x^2 + xy}$$

$$= \frac{3x + 2y - 2x - y}{x^2 + xy}$$

$$= \frac{x + y}{x(x + y)}$$

$$= \frac{1}{x} \quad \leftarrow \text{No "y"s left.}$$

$$\frac{\mu'(x)}{\mu(x)} = \frac{1}{x} \rightarrow \text{This is separable eqn.}$$

$$\frac{\mu'(x)}{\mu(x)} = \frac{1}{x}$$

$$\frac{1}{\mu(x)} d\mu(x) = \frac{1}{x} dx$$

$$\int (\ln(\mu(x)))' dx = \int \frac{1}{x} dx$$

$$\ln(\mu(x)) + C_1 = \ln|x| + C_2$$

$$\ln(\mu(x)) = \ln|x| + C_2 - C_1$$

$$= \ln|x| + C$$

$$\mu(x) = e^{\ln|x| + C}$$

$$= e^{\ln|x|} \cdot e^C$$

$$\text{Let } c' = e^C$$

$$\mu(x) = x \cdot c'$$

$$\text{Let } c' = 1$$

$$\mu(x) = x$$

So now we have:

$$x(3xy + y^2) dx + x(x^2 + xy) dy = 0$$

Note: After we find $\mu(x)$, we don't need to show that $\partial_y [\mu(x)M] = \partial_x [\mu(x)N]$.

4. Solve for F.

Now, we solve for F using the steps from when F is an exact eqn from the start.

$$M_1 = x(3xy + y^2)$$

$$N_1 = x(x^2 + xy)$$

Let $M_1 = \partial_x F$ and $N_1 = \partial_y F$.

Now,

$$\partial_x F = x(3xy + y^2)$$

$$\partial_y F = x(x^2 + xy)$$

Take the integral of $\partial_x F$ w.r.t. x to find F .

$$\begin{aligned} F &= \int x(3xy + y^2) dx \\ &= \int 3x^2y + xy^2 dx \\ &= \int 3x^2y dx + \int xy^2 dx \\ &= x^3y + \frac{x^2y^2}{2} + C(y) \end{aligned}$$

Now, we will use the second eqn.

$$\partial_y F = x^3 + x^2y$$

$$\text{Also } \partial_y F = \partial_y \left(x^3y + \frac{x^2y^2}{2} + C(y) \right)$$

$$= x^3 + x^2y + C'(y)$$

$$x^3 + x^2y + C'(y) = x^3 + x^2y$$

$$C'(y) = 0$$

$$C = \int 0 dy = 0$$

$$F = x^3y + \frac{x^2y^2}{2} = C$$

Note: If the question is "Verify that this eqn has an integrating factor that depends on x only.", you must end up with something without y .

Note: If the question is "Check whether this eqn has an integrating factor that depends on x only.", it's trickier than the previous question. In the previous question, you know right away that the integrating factor depends on x only, where here, you don't.

- E.g. 5. Find $\mu(y)$ and solve $y + (2xy - e^{-2y})y' = 0$.

Soln:

$$y dx + (2xy - e^{-2y}) dy = 0$$

1. Multiply both sides of the eqn by $\mu(y)$ and solve for $\mu(y)$.

$$\mu(y)y dx + \mu(y)(2xy - e^{-2y}) dy = 0$$

Now, assume that $\partial_y [\mu(y)M] = \partial_x [\mu(y)N]$.

$$\begin{aligned} \partial_y [\mu(y)M] \\ = \mu'(y)M + \mu(y)[\partial_y M] \end{aligned}$$

$$\begin{aligned} \partial_x [\mu(y)N] \\ = \mu(y)(\partial_x N) \end{aligned}$$

$$\mu'(y)M + \mu(y)[\partial_y M] = \mu(y)(\partial_x N)$$

$$\begin{aligned}\mu'(y)M &= \mu(y)(\partial_x N) - \mu(y)[\partial_y M] \\ &= \mu(y)[\partial_x N - \partial_y M]\end{aligned}$$

$$\frac{\mu'(y)}{\mu(y)} = \frac{\partial_x N - \partial_y M}{M}$$

Note: In the previous example, μ is a function of x . At this step, I said that there should be no "y"'s after you simplify $\frac{\partial_y M - \partial_x N}{N}$.

However, since μ is a function of y in this example, there should be no "x"'s after you simplify $\frac{\partial_x N - \partial_y M}{M}$.

$$\begin{aligned}\partial_y M &= \partial_y(y) \\ &= 1\end{aligned}$$

$$\begin{aligned}\partial_x N &= \partial_x(2xy - e^{-2y}) \\ &= 2y\end{aligned}$$

$$\begin{aligned}\frac{\partial_x N - \partial_y M}{M} &= \frac{2y - 1}{y} \\ &= 2 - \frac{1}{y}\end{aligned}$$

$$\frac{\mu'(y)}{\mu(y)} = 2 - \frac{1}{y}$$

$$(\ln(\mu(y)))' = 2 - \frac{1}{y}$$

$$\int (\ln(\mu(y)))' dy = \int 2 - \frac{1}{y} dy$$

$$\ln(\mu(y)) + C_1 = 2y - \ln|y| + C_2$$

$$\ln(\mu(y)) = 2y - \ln|y| + C_2 - C_1$$

$$= 2y - \ln|y| + C$$

$$\mu(y) = e^{2y - \ln|y| + C}$$

$$= \frac{e^{2y} \cdot e^C}{e^{\ln|y|}}$$

$$\text{Let } C' = e^C$$

$$\mu(y) = \frac{e^{2y} \cdot C'}{y}$$

$$\text{Let } C' = 1$$

$$\mu(y) = \frac{e^{2y}}{y}$$

2. Solve for F.

The eqn we now have is:

$$\underbrace{\frac{e^{2y}}{y} \cdot y}_{M_1} + \underbrace{\frac{e^{2y}}{y} (2xy - e^{-2y})}_{N_1}$$

$$\text{Let } \partial_x F = M_1$$

$$\partial_x F = e^{2y}$$

$$F = \int e^{2y} dx$$

$$= e^{2y} x + C(y)$$

$$\begin{aligned}\partial_y F &= N_1 \\ &= \frac{e^{2y}}{y} (2xy - e^{-2y})\end{aligned}$$

$$\begin{aligned}\text{Also } \partial_y F &= \partial_y (e^{2y} x + c(y)) \\ &= 2xe^{2y} + c'(y)\end{aligned}$$

$$\begin{aligned}2xe^{2y} + c'(y) &= \frac{e^{2y}}{y} (2xy - e^{-2y}) \\ &= 2xe^{2y} - \frac{1}{y}\end{aligned}$$

$$c'(y) = -\frac{1}{y}$$

$$c = \int -\frac{1}{y} dy$$

$$= -\log|y| + C_1$$

$$\text{Let } C_1 = 0$$

$$F = e^{2y} x - \log|y| = C$$

3. More Examples

a) Solve $(2x+3) + (2y-2)y' = 0$

Soln:

$$(2x+3)dx + (2y-2)dy = 0$$

1. Check if $M_y = N_x$

$$M = 2x+3$$

$$N = 2y-2$$

$$\begin{aligned}\partial_x M &= \partial_x (2x+3) \\ &= 2\end{aligned}$$

$$\begin{aligned}\partial_y N &= \partial_y (2y-2) \\ &= 2\end{aligned}$$

$$M_y = N_x$$

This is an exact equation.

2. Let $\partial_x F = M$ and $\partial_y F = N$.

$$\partial_x F = 2x+3$$

$$F = \int 2x+3 \, dx$$

$$= x^2 + c(y) + 3x$$

$$\partial_y F = N$$

$$= 2y-2$$

$$\text{Also } \partial_y F = \partial_y (x^2 + c(y) + 3x)$$

$$= c'(y)$$

$$c'(y) = 2y-2$$

$$c = \int 2y-2 \, dy$$

$$= y^2 - 2y + c_1$$

$$\text{Let } c_1 = 0$$

$$F = x^2 + 3x + y^2 - 2y = C$$

b) Solve $(3x^2 - 2xy + 2) + (6y^2 - x^2 + 3)y' = 0$

Soln:

$$(3x^2 - 2xy + 2) dx + (6y^2 - x^2 + 3) dy = 0$$

$$M = 3x^2 - 2xy + 2$$

$$N = 6y^2 - x^2 + 3$$

1. Check if $M_y = N_x$

$$M_y = \partial_y M$$

$$= \partial_y (3x^2 - 2xy + 2)$$

$$= -2x$$

$$N_x = \partial_x N$$

$$= \partial_x (6y^2 - x^2 + 3)$$

$$= -2x$$

$$M_y = N_x$$

Hence, this is an exact equation.

2. Let $\partial_x F = M$ and $\partial_y F = N$ and solve for F .

$$\begin{aligned}\partial_x F &= M \\ &= 3x^2 - 2xy + 2 \\ F &= \int 3x^2 - 2xy + 2 \, dx \\ &= x^3 - x^2y + 2x + c(y)\end{aligned}$$

$$\begin{aligned}\partial_y F &= N \\ &= 6y^2 - x^2 + 3\end{aligned}$$

$$\begin{aligned}\text{Also } \partial_y F &= \partial_y (x^3 - x^2y + 2x + c(y)) \\ &= -x^2 + c'(y)\end{aligned}$$

$$\begin{aligned}-x^2 + c'(y) &= 6y^2 - x^2 + 3 \\ c'(y) &= 6y^2 + 3 \\ c &= \int 6y^2 + 3 \, dy \\ &= 2y^3 + 3y + c_1 \\ \text{Let } c_1 &= 0\end{aligned}$$

$$F = x^3 - x^2y + 2x + 2y^3 + 3y = c$$

c) Solve $(3x^2y + 2xy + y^3) + (x^2 + y^2)y' = 0$

Soln:

$$(3x^2y + 2xy + y^3) \, dx + (x^2 + y^2) \, dy = 0$$

$$M = 3x^2y + 2xy + y^3$$

$$N = x^2 + y^2$$

1. Check if $M_y = N_x$

$$\begin{aligned} M_y &= \partial_y M \\ &= \partial_y (3x^2y + 2xy + y^3) \\ &= 3x^2 + 2x + 3y^2 \end{aligned}$$

$$\begin{aligned} N_x &= \partial_x N \\ &= \partial_x (x^2 + y^2) \\ &= 2x \end{aligned}$$

$M_y \neq N_x \leftarrow$ This is not an exact equation.

2. Multiply both sides of the eqn by μ .
 $\mu(3x^2y + 2xy + y^3) dx + \mu(x^2 + y^2) dy = 0$

3. Assume that $\partial_y(\mu M) = \partial_x(\mu N)$

$$\begin{aligned} \partial_y(\mu M) &= \mu(\partial_y M) \\ \partial_x(\mu N) &= \mu'N + \mu(\partial_x N) \end{aligned}$$

$$\mu(\partial_y M) = \mu'N + \mu(\partial_x N)$$

$$\mu(\partial_y M) - \mu(\partial_x N) = \mu'N$$

$$\mu(\partial_y M - \partial_x N) = \mu'N$$

$$\frac{\mu'}{\mu} = \frac{\partial_y M - \partial_x N}{N}$$

$$\partial_y M = 3x^2 + 2x + 3y^2$$

$$\partial_x N = 2x$$

$$\frac{\mu'}{\mu} = \frac{3x^2 + 2x + 3y^2 - 2x}{x^2 + y^2}$$

$$= \frac{3(x^2 + y^2)}{x^2 + y^2}$$

$$= 3$$

$$(\ln(\mu))' = 3$$

$$\int (\ln(\mu))' dx = \int 3 dx$$

$$\ln(\mu) = 3x + c$$

$$\mu = e^{3x+c}$$

$$= e^{3x} \cdot e^c$$

$$\text{Let } c' = e^c$$

$$\mu = e^{3x} \cdot c'$$

$$\text{Let } c' = 1$$

$$\mu = e^{3x}$$

4. Assume the question is like a normal exact eqn question and solve for F.

$$\underbrace{e^{3x}(3x^2y + 2xy + y^3)}_{M_1} dx + \underbrace{e^{3x}(x^2 + y^2)}_{N_1} dy = 0$$

$$\text{Let } \partial_x F = M_1$$

$$\text{Let } \partial_y F = N_1$$

$$\partial_x F = M_1$$

$$= e^{3x}(3x^2y + 2xy + y^3)$$

$$F = \int e^{3x}(3x^2y + 2xy + y^3) dx$$

$$= \frac{y(3x^2 + y^2)e^{3x}}{3} + c(y)$$

$$\begin{aligned}\partial_y F &= N_1 \\ &= e^{3x}(x^2+y^2)\end{aligned}$$

$$\text{Also, } \partial_y F = \partial_y \left(\frac{y(3x^2+y^2)e^{3x}}{3} + c(y) \right)$$

$$= \partial_y \left(\frac{y(3x^2+y^2)e^{3x}}{3} \right) + \partial_y (c(y))$$

$$= \frac{e^{3x}}{3} \left(\partial_y (3x^2y + y^3) \right) + c'(y)$$

$$= \frac{e^{3x}}{3} (3x^2 + 3y^2) + c'(y)$$

$$= e^{3x}(x^2+y^2) + c'(y)$$

$$e^{3x}(x^2+y^2) + c'(y) = e^{3x}(x^2+y^2)$$

$$c'(y) = 0$$

$$c = 0$$

$$F = \frac{y(3x^2+y^2)e^{3x}}{3} = c$$

d) Find an integrating factor depending on xy and solve the eqn

$$\left(3x + \frac{6}{y}\right) + \left(\frac{x^2}{y} + \frac{3y}{x}\right)y' = 0.$$

Soln:

$$\left(3x + \frac{6}{y}\right) dx + \left(\frac{x^2}{y} + \frac{3y}{x}\right) dy = 0$$

$$M = 3x + \frac{6}{y}$$

$$N = \frac{x^2}{y} + \frac{3y}{x}$$

1. Multiply both sides of the eqn by $\mu(xy)$.

$$\mu(xy) \left(3x + \frac{6}{y}\right) dx + \mu(xy) \left(\frac{x^2}{y} + \frac{3y}{x}\right) dy = 0$$

2. Assume that $\partial_y(\mu M) = \partial_x(\mu N)$.

$$\begin{aligned} \partial_y(\mu M) &= \partial_y(\mu(xy)M) \\ &= x\mu'(xy)M + \mu(xy)(\partial_y M) \end{aligned}$$

$$\begin{aligned} \partial_x(\mu N) &= \partial_x(\mu(xy)N) \\ &= y\mu'(xy)N + \mu(xy)(\partial_x N) \end{aligned}$$

$$x\mu'(xy)M + \mu(xy)(\partial_y M) = y\mu'(xy)N + \mu(xy)(\partial_x N)$$

$$x\mu'(xy)M - y\mu'(xy)N = \mu(xy)(\partial_x N) - \mu(xy)(\partial_y M)$$

$$\mu'(xy)[xM - yN] = \mu(xy)[\partial_x N - \partial_y M]$$

$$\frac{\mu'(xy)}{\mu(xy)} = \frac{\partial_x N - \partial_y M}{xM - yN}$$

We need $\frac{\partial_x N - \partial_y M}{xM - yN}$ to depend on xy only.

$$\begin{aligned}\partial_x N &= \partial_x \left(\frac{x^2}{y} + \frac{3y}{x} \right) \\ &= \frac{2x}{y} - \frac{3y}{x^2}\end{aligned}$$

$$\begin{aligned}\partial_y M &= \partial_y \left(3x + \frac{6}{y} \right) \\ &= -\frac{6}{y^2}\end{aligned}$$

$$\frac{\partial_x N - \partial_y M}{xM - yN} = \frac{1}{xy}$$

$$\frac{\mu'(xy)}{\mu(xy)} = \frac{1}{xy}$$

$$(\ln(\mu(xy)))' = \frac{1}{xy}$$

$$\text{Let } t = xy$$

$$(\ln(\mu(t)))' = \frac{1}{t}$$

$$\int (\ln(\mu(t)))' dt = \int \frac{1}{t} dt$$

$$\ln(\mu(t)) + C_1 = \ln|t| + C_2$$

$$\ln(\mu(t)) = \ln|t| + C_2 - C_1$$

$$= \ln|t| + C$$

$$\mu(t) = e^{\ln|t| + C}$$

$$= e^{\ln|t|} \cdot e^C$$

$$\text{Let } C' = e^C$$

$$\mu(t) = t \cdot C'$$

$$\text{Let } C' = 1$$

$$\mu(t) = t$$

$$= xy$$

3. Treat the eqn as an exact eqn and solve for F.

$$\mu(xy) M dx + \mu(xy) N dy = 0$$

$$(xy) \left(3x + \frac{6}{y} \right) dx + (xy) \left(\frac{x^2}{y} + \frac{3y}{x} \right) dy = 0$$

$$\underbrace{(xy) \left(3x + \frac{6}{y} \right)}_{M_1} dx + \underbrace{(xy) \left(\frac{x^2}{y} + \frac{3y}{x} \right)}_{N_1} dy = 0$$

$$\text{Let } \partial_x F = M_1$$

$$\text{Let } \partial_y F = N_1$$

$$\partial_x F = M_1$$

$$= (xy) \left(3x + \frac{6}{y} \right)$$

$$F = \int 3x^2y + 6x dx$$

$$= x^3y + 3x^2 + C(y)$$

$$\begin{aligned}\partial_y F &= N_1 \\ &= x^3 + 3y^2\end{aligned}$$

$$\begin{aligned}\text{Also } \partial_y F &= \partial_y (x^3 y + 3x^2 + c(y)) \\ &= x^3 + c'(y)\end{aligned}$$

$$\begin{aligned}x^3 + c'(y) &= x^3 + 3y^2 \\ c'(y) &= 3y^2 \\ C &= \int 3y^2 dy \\ &= y^3 + C_1 \\ \text{Let } C_1 &= 0\end{aligned}$$

$$F = x^3 y + 3x^2 + y^3 = C$$