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Wave Eqn Examples

E.g. 1. Find the soln to the wave eqn on $0 < x < \ell$ with $u(0,t) = 0$, $u(\ell,t) = 0$, $\phi(x) = \sin\left(\frac{2\pi x}{\ell}\right)$ and $\psi(x) = \sin\left(\frac{3\pi x}{\ell}\right)$

Soln:

Initial conditions:

1. $u(x,0) = \phi(x) = \sin\left(\frac{2\pi x}{\ell}\right)$
2. $u_t(x,0) = \psi(x) = \sin\left(\frac{3\pi x}{\ell}\right)$

Boundary Conditions:

1. $u(0,t) = 0$
2. $u(\ell,t) = 0$

We want to solve for $u(x,t)$.
Assume that $u(x,t) = X(x) \cdot T(t)$
Plug $u(x,t)$ into $u_{tt} = c^2 u_{xx}$.

$$u_{tt} = \frac{\partial^2}{\partial t^2}(X \cdot T)$$

$$= \frac{\partial}{\partial t}(X \cdot T') \leftarrow$$ X is treated as a constant.

$$= X \cdot T''$$

$$u_{xx} = \frac{\partial^2}{\partial x^2}(X \cdot T)$$

$$= \frac{\partial}{\partial x}(X' \cdot T)$$

$$= X'' \cdot T$$

Now, we have $X \cdot T'' = c^2 \cdot X'' \cdot T$.
By convention, we move all terms with t or is a constant to the LHS and all terms with x to the RHS.

$$\frac{T''}{c^2 \cdot T} = \frac{X''}{X} = F$$

We can prove that F is a constant and doesn't depend on x or t.

$$\left.\begin{aligned} \frac{\partial F}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{X''}{X}\right) = 0 \\ \frac{\partial F}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{T''}{c^2 \cdot T}\right) = 0 \end{aligned}\right\} \text{Both derivatives are 0}$$

This shows that F is a constant and not dependent on x or t.

F is called the separation constant and is always negative. Hence, we'll use $-\lambda$ to represent F where $\lambda > 0$.

$$\frac{T''}{c^2 \cdot T} = \frac{X''}{X} = -\lambda \leftarrow \text{Will split into 2 eqns.}$$

$\frac{T''}{c^2 \cdot T} = -\lambda$	$\frac{X''}{X} = -\lambda$
$T'' = -c^2 \lambda T$	$X'' = -\lambda X$
$T'' + c^2 \lambda T = 0$	$X'' + \lambda X = 0$

Both eqns are second order homogeneous linear diff eqns.

To solve $T'' + c^2 \lambda T = 0$:
Let $T = e^{rt}$. Then, $T' = re^{rt}$, $T'' = r^2 e^{rt}$

$r^2 e^{rt} + c^2 \lambda e^{rt} = 0$

$r^2 + c^2 \lambda = 0$

$r^2 = -c^2 \lambda$

$r = \pm \sqrt{-c^2 \lambda}$

$= \sqrt{\lambda} c i$

$e^{rt} = e^{(\sqrt{\lambda} ct) i}$

$= \cos(\sqrt{\lambda} ct) + i \sin(\sqrt{\lambda} ct)$ ← Euler Formula

$T(t) = A \cos(\sqrt{\lambda} ct) + B \sin(\sqrt{\lambda} ct)$

To solve $X'' + \lambda X = 0$:
Let $X = e^{rx}$. Then, $X' = re^{rx}$, $X'' = r^2 e^{rx}$

$r^2 e^{rx} + \lambda e^{rx} = 0$

$r^2 + \lambda = 0$

$r^2 = -\lambda$

$r = \pm \sqrt{-\lambda}$

$= \sqrt{\lambda} i$

$e^{rx} = e^{(\sqrt{\lambda} x) i}$

$= \cos(\sqrt{\lambda} x) + i \sin(\sqrt{\lambda} x)$ ← Euler Formula

$X(x) = C \cos(\sqrt{\lambda} x) + D \sin(\sqrt{\lambda} x)$

$$u(x,t) = X(x) \cdot T(t)$$
$$= (A\cos(\sqrt{\lambda}\,ct) + B\sin(\sqrt{\lambda}\,ct)) \leftarrow T(t)$$
$$(C\cos(\sqrt{\lambda}\,x) + D\sin(\sqrt{\lambda}\,x)) \leftarrow X(x)$$

Now, we'll plug in our boundary conditions.
$u(0,t)=0 \rightarrow X(0)\cdot T(t)=0 \rightarrow X(0)=0$

$$X(0)=0 \rightarrow C\underbrace{\cos(\sqrt{\lambda}0)}_{1} + D\underbrace{\sin(\sqrt{\lambda}0)}_{0} = 0$$

$$C=0$$

$u(l,t)=0 \rightarrow X(l)\cdot T(t)=0 \rightarrow X(l)=0$
$X(l)=0 \rightarrow D\sin(\sqrt{\lambda}l)=0$
Either $D=0$ or $\sin(\sqrt{\lambda}l)=0$.
We ignore the case when $D=0$. It's the trivial case.
$\sin(\sqrt{\lambda}l)=0 \rightarrow \sqrt{\lambda}l=n\pi,\ n>0$

$$\sqrt{\lambda} = \frac{n\pi}{l}$$

For each n, we have the soln
$$U_n = D_n\sin\left(\frac{n\pi x}{l}\right)\left(A_n\cos\left(\frac{n\pi ct}{l}\right)+B_n\sin\left(\frac{n\pi ct}{l}\right)\right)$$

We can ignore D_n as D_n will get absorbed into A_n and B_n.

Since each value of n gives a soln to the PDE, the general soln must be a linear combination of each of these solns.

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right)\left(A_n\cos\left(\frac{n\pi ct}{l}\right)+B_n\sin\left(\frac{n\pi ct}{l}\right)\right)$$

Now, we will use the initial conditions to solve for A_n and B_n.

To find A_n, we'll use the initial condition $u(x,0) = \phi(x)$.

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\phi(x) = \sin\left(\frac{2\pi x}{\ell}\right)$$

$$A_n = \frac{2}{\ell}\int_0^{\ell} \phi(x) \cdot \sin\left(\frac{n\pi x}{\ell}\right)$$

$$= \frac{2}{\ell}\int_0^{\ell} \sin\left(\frac{2\pi x}{\ell}\right) \cdot \sin\left(\frac{n\pi x}{\ell}\right)$$

$= 1$ if $n=2$ and 0 otherwise.

Recall: $$\int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \cdot \sin\left(\frac{m\pi x}{\ell}\right) = \begin{cases} 0 & \text{if } m \neq n \\ \frac{\ell}{2} & \text{if } m = n \end{cases}$$

To solve for B_n, we'll use $u_t(x,0) = \psi(x)$

$$u_t = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{\ell}\right)\left(-A_n \sin\left(\frac{n\pi ct}{\ell}\right) + B_n \cos\left(\frac{n\pi ct}{\ell}\right)\right) \frac{n\pi c}{\ell}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{\ell}\right) \cdot B_n \cdot \frac{n\pi c}{\ell}$$

$$\psi(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{\ell}\right) \cdot B_n \cdot \frac{n\pi c}{\ell}$$

$$\psi(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{\ell}\right) \cdot B_n \cdot \frac{n\pi c}{\ell}$$

$$B_n = \frac{2}{n\pi c} \int_0^\ell \psi(x) \cdot \sin\left(\frac{n\pi x}{\ell}\right)$$

$$= \frac{2}{n\pi c} \int_0^\ell \sin\left(\frac{3\pi x}{\ell}\right) \cdot \sin\left(\frac{n\pi x}{\ell}\right)$$

$= \frac{\ell}{n\pi c}$ if $n=3$ and 0 otherwise.

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{\ell}\right)\left(A_n \cos\left(\frac{n\pi c t}{\ell}\right) + B_n \sin\left(\frac{n\pi c t}{\ell}\right)\right)$$

We know that when $n=2$, $A_n = 1$, and when $n=3$, $B_n = \frac{\ell}{n\pi c}$.

$$u(x,t) = \sin\left(\frac{2\pi x}{\ell}\right) \cos\left(\frac{2\pi c t}{\ell}\right) +$$

$$\sin\left(\frac{3\pi x}{\ell}\right) \frac{\ell}{3\pi c} \sin\left(\frac{3\pi c t}{\ell}\right)$$

E.g. 2. Find the soln to the wave eqn on $0 < x < \ell$, with $u(0,t) = 0$, $u(\ell,t) = 0$ and $\phi(x) = 1$, $\psi(x) = 0$

Soln:

$U_{tt} = c^2 U_{xx}$

$$\left.\begin{array}{l} u(0,t) = 0 \\ u(\ell,t) = 0 \end{array}\right\} \text{Boundary Conditions}$$

$$\left.\begin{array}{l} u(x,0) = \phi(x) = 1 \\ u_t(x,0) = \psi(x) = 0 \end{array}\right\} \text{Initial Conditions}$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{\ell}\right)\left(A_n \cos\left(\frac{n\pi ct}{\ell}\right) + B_n \sin\left(\frac{n\pi ct}{\ell}\right)\right)$$

We will use the initial conditions to solve for A_n and B_n.

$$A_n = \frac{2}{\ell}\int_0^{\ell} \phi(x) \sin\left(\frac{n\pi x}{\ell}\right)$$

$$= \frac{2}{\ell}\int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right)$$

$$= \frac{2}{\ell}\left(\frac{\ell}{n\pi}\right)\left(-\cos\left(\frac{n\pi \ell}{\ell}\right) - (-\cos(0))\right)$$

$$= \frac{-2}{n\pi}\left(\cos(n\pi) - 1\right)$$

$$= \frac{-2}{n\pi}\left((-1)^n - 1\right)$$ Note: $\cos(n\pi) = (-1)^n$

$$\therefore A_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

$$B_n = \frac{2}{n\pi c} \int_0^{\ell} \psi(x) \cdot \sin\left(\frac{n\pi x}{\ell}\right)$$

$$= \frac{2}{n\pi c} \int_0^{\ell} 0$$

$$= 0$$

E.g. 3 Find the soln to the wave eqn on $0 < x < \ell$ with $u(0,t) = 0$, $u(\ell,t) = 0$, $\phi(x) = x$ and $\psi(x) = 0$.

Soln:

$$\left.\begin{aligned} u(x,0) &= \phi(x) = x \\ u_t(x,0) &= \psi(x) = 0 \end{aligned}\right\} \quad \text{Initial Conditions}$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{\ell}\right)\left(A_n \cos\left(\frac{n\pi c t}{\ell}\right) + B_n \sin\left(\frac{n\pi c t}{\ell}\right)\right)$$

$$A_n = \frac{2}{\ell} \int_0^{\ell} \phi(x) \cdot \sin\left(\frac{n\pi x}{\ell}\right)$$

$$= \frac{2}{\ell} \underbrace{\int_0^{\ell} x \cdot \sin\left(\frac{n\pi x}{\ell}\right)}_{\text{Integration By Parts}}$$

$$\int u \cdot v = u \int v - \int u' \int v$$

Let $u = x$, $v = \sin\left(\frac{n\pi x}{\ell}\right)$

$$\left[\left[x\int \sin\left(\frac{n\pi x}{\ell}\right)\right] - \left[\int x' \int \sin\left(\frac{n\pi x}{\ell}\right)\right]\right]\Big|_0^\ell$$

$$= \left[\frac{-x\ell}{n\pi}\cos\left(\frac{n\pi x}{\ell}\right) - \int\int \sin\left(\frac{n\pi x}{\ell}\right)\right]\Big|_0^\ell$$

$$= \left[\frac{-x\ell}{n\pi}\cos\left(\frac{n\pi x}{\ell}\right) - \frac{\ell^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{\ell}\right)\right]\Big|_0^\ell$$

$$= \frac{-\ell^2}{n\pi}\cos(n\pi) - \frac{\ell^2}{n^2\pi^2}\underbrace{\sin(n\pi)}_{0} - 0$$

$$= \frac{-\ell^2}{n\pi}(-1)^n$$

$$\frac{2}{\ell}\left(\frac{\ell^2}{n\pi}\right)(-1)^{n+1}$$

$$= \frac{2\ell}{n\pi}(-1)^{n+1} = A_n$$

$$B_n = \frac{2}{n\pi c}\int_0^\ell \psi(x) \cdot \sin\left(\frac{n\pi x}{\ell}\right)$$

$$= 0$$