Booklet 6 Notes

1. The determinant of a 1x1 matrix is its sole entry.

The determinant of a 2x2 matrix is:

$$\det(A) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1, \quad A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

The determinant of a 3x3 matrix is:

$$\det(A) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Cross Product:

Let $\vec{a} = [a_1, a_2, a_3]$

Let $\vec{B} = [b_1, b_2, b_3]$

The cross product of $\vec{a}$ and $\vec{b}$, which is $\vec{a} \times \vec{b}$ is

$$= \left[\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right],$$ which is the same as

$$[a_2b_3 - a_3b_2, \ a_3b_1 - a_1b_3, \ a_1b_2 - a_2b_1]$$

Note, $\vec{a} \times \vec{b}$ is $\perp$ to both $\vec{a}$ and $\vec{b}$.

Proof:

Let $\vec{a} = [a_1, a_2, a_3]$

Let $\vec{b} = [b_1, b_2, b_3]$

$\vec{a} \times \vec{b} = [a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1]$

$\vec{a} \cdot (\vec{a} \times \vec{b})$

$= [a_1, a_2, a_3] \cdot [a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1]$

$= [a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1)]$

$= [\cancel{a_1a_2b_3} - \cancel{a_1a_3b_2} + \cancel{a_2a_3b_1} - \cancel{a_1a_2b_3} + \cancel{a_1a_3b_2} - \cancel{a_2a_3b_1}]$

$= 0$

$\therefore\ \vec{a} \perp (\vec{a} \times \vec{b})$

QED

2. Thm:

1. If a parallelogram is determined by 2 non-zero vectors, $\vec{a} = [a_1, a_2]$ and $\vec{b} = [b_1, b_2]$, then its area is given by $|a_1b_2 - a_2b_1|$

2. If a parallelogram is determined by 2 non-zero vectors, $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$ in R^3, then its area is $\|\vec{a} \times \vec{b}\|$.

3. If a parallelpiped is determined by 3 non-zero vectors, $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$ and $\vec{c} = [c_1, c_2, c_3]$ in R^3, then its volume is $|\vec{a} \cdot (\vec{b} \times \vec{c})|$.

Proof of 1.

$$\text{Area} = \|\vec{a}\| \cdot \|\vec{h}\|$$
$$= \|\vec{a}\| \cdot \|\vec{b}\| \sin\theta$$
$$(\text{Area})^2 = \|\vec{a}\|^2 \, \|\vec{b}\|^2 \sin^2\theta$$
$$= \|\vec{a}\|^2 \, \|\vec{b}\|^2 (1-\cos^2\theta)$$
$$= \|\vec{a}\|^2 \, \|\vec{b}\|^2 - \|\vec{a}\|^2 \cdot \|\vec{b}\|^2 \cos^2\theta$$
$$= \|\vec{a}\| \, \|\vec{b}\| - (\vec{a}\cdot\vec{b})^2$$
$$= (a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1b_1 + a_2b_2)^2$$
$$= \cancel{a_1^2 b_1^2} + a_1^2 b_2^2 + a_2^2 b_1^2 + \cancel{a_2^2 b_2^2} - \cancel{a_1^2 b_1^2} - 2a_1b_1a_2b_2 - \cancel{a_2^2 b_2^2}$$
$$= a_1^2 b_2^2 - 2a_1b_1a_2b_2 + a_2^2 b_1^2$$
$$= (a_1b_2 - a_2b_1)^2$$
$$\text{Area} = |a_1b_2 - a_2b_1|, \text{ as wanted}$$

QED

Proof of 3.

$$\text{Vol} = \text{Base} \times \text{Height}$$
$$= |\vec{a}| \cos\theta \, |\vec{b} \times \vec{c}|$$
$$= |\vec{a} \cdot (\vec{b} \times \vec{c})|, \text{ as wanted}$$

QED

If $\vec{a} \cdot \vec{b} \times \vec{c} = 0$, then the vol $= 0$.

3. We've already discussed the determinant of 1×1, 2×2 and 3×3 matrices. Let $n>1$ and suppose that the det of $(n-1)\times(n-1)$ matrices is defined. Let $A = [a_{ij}]$ be a $n\times n$ matrix.

The minor matrix, A_{ij}, is the $(n-1)\times(n-1)$ matrix obtained by removing the i^{th} row and j^{th} col of A.

The cofactor of a_{ij} of A is: $a'_{ij} = (-1)^{i+j} \det(A_{ij})$
$= (-1)^{i+j} |A_{ij}|$

The det of A can be computed by a cofactor expansion across any row or any column

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \dots & a_{nn} \end{vmatrix}$$

$$= a_{11}(a'_{11}) + a_{12}(a'_{12}) + \dots + a_{1n}(a'_{1n})$$

4. Thm: Let A be a nxn matrix. Then:
   1. $\det(A) = \det(A^T)$
   2. If A is a triangular matrix, then $\det(A)$ is the product of all of its entries along the main diagonal.

Proof of 1.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\det(A) = ad - bc$

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$\det(A) = ad - bc$

$\therefore$ The claim is true for 2x2 matrices. Suppose that the claim is true for all $A_{n-1 \times n-1}$ matrices, $n \geq 3$.

WTS: It's true for $A_{n\times n}$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{n1} & \dots & & a_{nn} \end{bmatrix}$$

$$A^T = \begin{bmatrix} a_{11} & \dots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & & a_{nn} \end{bmatrix}$$

$$\begin{aligned} \det(A) &= a_{11}(a_{11})' + a_{12}(a_{12})' + \dots a_{1n}(a_{1n})' \\ &= a_{11}\det(A^T)_{11} + a_{12}(\det(A^T))_{21} + \dots a_{1n}(\det(A^T))_{n1} \\ &= \det(A^T) \end{aligned}$$

QED

Proof of 2

Suppose A is an upper triangular matrix.

I.e. $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & a_{21} & & \vdots \\ 0 & & \ddots & a_{nn} \end{bmatrix}$

$$\begin{aligned} \det(A) &= a_{11}(a_{11})' \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} & \dots & a_{2n} \\ & \ddots & & \vdots \\ & & & a_{nn} \end{vmatrix} \\ &= a_{11} \cdot a_{22} \cdot \dots a_{nn} \end{aligned}$$

QED

5. Thm: Let A be a $n \times n$ matrix. Then:

1. $A \xrightarrow{R_i \leftrightarrow R_j} B$, then $\det(B) = -\det(A)$
2. $A \xrightarrow{R_i \to rR_i} B$, then $\det(B) = r\det(A)$
3. $A \xrightarrow{R_i \to R_i + rR_j} B$, then $\det(B) = \det(A)$

6. Let A, B be $n \times n$ matrices. Then:

1. If A contains proportional rows or cols, $\det(A) = 0$
2. A is invertible $\Leftrightarrow$ $\det(A) \neq 0$

7. Let A, B be $n \times n$ matrices. Then:

1. $\det(AB) = \det(A) \cdot \det(B)$
2. If A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

8. Let A be a $n \times n$ matrix.

1. Let A_{ij} denoted the $(n-1) \times (n-1)$ matrix obtained by removing the i^{th} row and j^{th} col of A.

2. The cofactor of A is: $a'_{ij} = (-1)^{i+j} |A_{ij}|$

3. The transpose of the cofactor matrix of A is called the adjoint matrix of A. Let $A' = [a'_{ij}]$ be the matrix with the ij^{th} entry of the ij^{th} cofactor of A. Then, the adjoint matrix of A, denoted as $\operatorname{adj}(A)$, is the $n \times n$ matrix equal to $(A')^T$.
I.e. $\operatorname{adj}(A) = (A')^T$

9. Thm: If A is a nxn matrix, then:

1. $a_{i1}a'_{j1} + a_{i2}a'_{j2} + \dots a_{in}a'_{jn} = \begin{cases} \det(A) & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

2. $a_{1i}a'_{1j} + a_{2i}a'_{2j} + \dots + a_{ni}a'_{nj} = \begin{cases} \det(A) & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

10. If A is a nxn matrix, then:
$(\text{adj}(A))A = A(\text{adj}(A)) = \det(A)(I)$

11. If A is a nxn matrix and $\det(A) \neq 0$, then:
$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$$

12. Cramer's Rule.
If $A\vec{x} = \vec{b}$ is a system of n linear eqns with n unknowns and $\det(A) \neq 0$, then the unique soln $\vec{x} = [x_1, x_2, \dots x_n]$ is of the form

$$x_k = \frac{\det(B_k)}{\det(A)} \quad \text{for } k = 1, 2, \dots n$$

B_k is the matrix A with the k^{th} col replaced by the col vector b.

E.g. Use Cramer's Rule to solve this system:

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 2x_1 - x_2 \quad &= 11 \\ x_2 + 4x_3 &= 3 \end{aligned}$$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 11 \\ 3 \end{bmatrix}$

$$\det(A) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= (-1)(4) - (1)(0) - \left[(2)(4) - (0)(0)\right] + (2)(1) - (-1)(0)$$

$$= -4 - 8 + 2$$

$$= -10$$

Replaced A's col 1 with B because k=1

$$B_1 = \begin{bmatrix} 0 & 1 & 1 \\ 11 & -1 & 0 \\ 3 & 1 & 4 \end{bmatrix}$$

$$\det(B_1) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$
$$= (-1)\left[(11)(4) - (0)(3)\right] + \left[(11)(1) - (-1)(3)\right]$$
$$= -44 + 14$$
$$= -30$$

$$X_1 = \frac{\det(B_1)}{\det(A)} = \frac{-30}{-10} = 3$$

$$B_2 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 11 & 0 \\ 0 & 3 & 4 \end{bmatrix}$$

Now, I replaced A's col 2 because k=2.

$$\det(B_2) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$
$$= \left[(11)(4) - (0)(3)\right] + \left[(2)(3) - (11)(0)\right]$$
$$= 44 + 6$$
$$= 50$$

$$X_2 = \frac{\det(B_2)}{\det(A)} = \frac{50}{-10} = -5$$

$$B_3 = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 11 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\det(B_3) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$
$$= \left[(-1)(3) - (11)(1)\right] - \left[(2)(3) - (11)(0)\right]$$
$$= -20$$

This time, I replaced A's 3rd col with b because k=3.

$$X_3 = \frac{\det(B_3)}{\det(A)} = \frac{-20}{-10} = 2$$

$\therefore$ The answer is $\begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}$.