Booklet 7 Notes

Note: Since linear recurrence isn't on the exam, I will exclude them from here.

1. Let A be a $n \times n$ matrix. Let λ be a scalar. λ is the eigenvalue of A if there's a non-zero vector, $\vec{v} \in \mathbb{R}^n$ s.t. $A\vec{v} = \lambda\vec{v}$. $\vec{v}$ is the eigenvector of A.
I.e. Suppose we have a linear transformation that changes a vector by multiplying it to a scalar. The scalar is the eigenvalue and the vector is the eigenvector.

$A\vec{v} = \lambda\vec{v}$

$A\vec{v} - \lambda\vec{v} = 0$

$\vec{v}(A - \lambda I) = 0$

2. Let A be a $n \times n$ matrix. The characteristic polynomial of A is given by $P(\lambda) = \det|A - \lambda I|$. If λ is the eigenvalue of A, then $E_\lambda = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\}$ is the eigenspace of A.
$E_\lambda =$ nullspace of $(A - \lambda I)$.

3. Let A be a $n \times n$ matrix. Let λ be the eigenvalue of A and let $\vec{v}$ be the eigenvector of A.

   1. λ^k is an eigenvalue of A^k and $\vec{v}$ is an eigenvector of A^k corresponding to A^k, $k \geq 0$, $k \in \mathbb{Z}$.

   2. A is invertible iff $\lambda \neq 0$.

   3. If A is invertible, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} and $\vec{v}$ is an eigenvector of A^{-1} corresponding to $\frac{1}{\lambda}$.

4. If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, then the eigenvalues of T are the eigenvalues of its standard matrix rep.

4. Diagonizable Matrices

Let A and B be nxn matrices.

1. A is a diagonal matrix if all its entries are on its main diagonal and every other entry is 0.

E.g. $\begin{bmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{bmatrix}$

2. A is diagonizable if $\exists$ an invertible nxn matrix, P, s.t. PAP^{-1} is a diagonal matrix.

3. Let $\vec{v_1}, \vec{v_2}, \ldots \vec{v_k}$ be eigenvectors corresponding to the distinct eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_k$ of the square matrix, A. Then $\vec{v_1}, \vec{v_2}, \ldots \vec{v_k}$ are lin indep.

5. 3 ways to tell if a matrix is diagonizable.

1. If A is symmetric, $A^T = A$, then A is diagonziable. This does not mean that if $A^T \neq A$, then A is not diagonizable.

2. Let A be a nxn matrix. Let $A \sim H$. If there is a pivot in every col of H, then A is diagonizable. Otherwise, A is not.

3. The algebraic multiplicity of $A \geq$ geometric multiplicity of A. Algebraic multiplicity is the number of times λ equals a value. Geometric multiplicity is the $\dim(E_\lambda)$, I.e. the nullspace of the eigenspace.

If geo = alge, then the matrix is diagonalizable. Otherwise, it's not.

6. If A, B are $n \times n$ matrices s.t. $B = PAP^{-1}$ for some invertible $n \times n$ matrix, P, then A and B are similar matrices.

Properties of Similar Matrices:

1. $\det(A) = \det(B)$
2. A is invertible iff B is invertible
3. $\operatorname{rank}(A) = \operatorname{rank}(B)$
4. Nullity (A) = Nullity (B)
5. $\det(A - \lambda I) = \det(B - \lambda I)$

7. If A is a $n \times n$ matrix similar to a diagonal matrix, D, s.t. $A = PDP^{-1}$, for some invertible matrix P, then $A^k = PD^kP^{-1}$.

8. Cayley - Hamiliton Theorem:

Let A be an $n \times n$ matrix with characteristic polynominal

$P(\lambda) = |A - \lambda I| = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_0$, then

$$P(A) = a_n A^n + a_{n-1} A^{n-1} + \ldots \; a_0 I = \underbrace{\vec{0}}_{\text{zero vector}}$$

9. If A is a $n \times n$ matrix with $\det(A - \lambda I) = \lambda^n + c_1 \lambda^{n-1} + \ldots \; c_{n-1}\lambda + c_n$, where $c_n \neq 0$, then A is invertible and $A^{-1} = \dfrac{-1}{c_n}\left[A^{n-1} + c_1 A^{n-2} + \ldots \; c_{n-1} I\right]$

Examples

1. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$

a) Find the characteristic polynominal of A

$P(\lambda) = |A - \lambda I|$

$$= \left| \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & -1 \\ 1 & 3 & -2-\lambda \end{bmatrix} \right|$$

$$= (2-\lambda) \begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left[(2-\lambda)(-2-\lambda) - (3)(-1)\right]$$

$$= (2-\lambda)\left[(2-\lambda)(-2-\lambda) + 3\right]$$

$$= (2-\lambda)(\lambda^2-1)$$

$$= (2-\lambda)(\lambda-1)(\lambda+1)$$

b) Find the eigenvalues of A

$0 = (2-\lambda)(\lambda-1)(\lambda+1)$

$\lambda = -1, 1, 2$

Because each value of λ occurs once, they all have an algе multiplicity of 1.

c) For each eigenvalue λ of A, find its eigenspace.

When $\lambda = -1$

$$\begin{bmatrix} 2-(-1) & 0 & 0 \\ 1 & 2-(-1) & -1 \\ 1 & 3 & -2-(-1) \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & -1 \\ 1 & 3 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The geometric multiplicity of the eigenspace of $\lambda = 1$ is 1, because there's 1 row of 0's.

geo = alge

Let $x_3 = s$

$3x_2 - x_3 = 0$

$3x_2 = x_3$

$x_2 = \frac{x_3}{3}$

/ $Sp \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

$x_1 = 0$

When $\lambda = 1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 3 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Geo multi of eigenspace of $\lambda = 1$ is 1.

Geo = alge

Let $x_3 = s$

$x_1 = 0$

$x_2 - x_3 = 0$

$x_2 = x_3$

$$sp\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$$

When $\lambda = 2$

The resultant matrix is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Geo multi = alge multi = 1

Let $x_3 = s$ $\quad$ $x_1 = x_3$

$x_2 = x_3$

$$sp\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

d) Find D and P such that $D = PAP^{-1}$

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

To find D, take the values of λ and form a diagonal matrix with them. The order doesn't matter.

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

To find P, take the eigenspaces of each λ and put them in the same col as you put the eigenvalues in D.

I.e. Say we got $\lambda = A, B, C$ and the eigenspace of A is $\vec{x}$, the eigenspace of B is $\vec{y}$ and the eigenspace of C is $\vec{z}$.

If $D = \begin{bmatrix} A & & 0 \\ & B & \\ 0 & & C \end{bmatrix}$, $P = \begin{bmatrix} | & | & | \\ \vec{x} & \vec{y} & \vec{z} \\ | & | & | \end{bmatrix}$

If $D = \begin{bmatrix} B & & 0 \\ & C & \\ 0 & & A \end{bmatrix}$, $P = \begin{bmatrix} | & | & | \\ \vec{y} & \vec{z} & \vec{x} \\ | & | & | \end{bmatrix}$

The order of the eigenspaces in P must correspond with the orders of its eigenvalue in D.