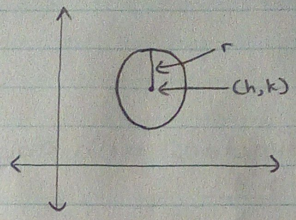
Level Curves and Domains

1. Useful Formulas:

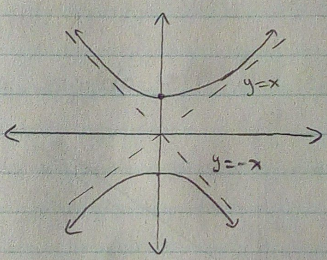
1. Circle : $(x-h)^2 + (y-k)^2 = r^2$



This is the eqn of a circle centered at (h,k) with a radius of r.

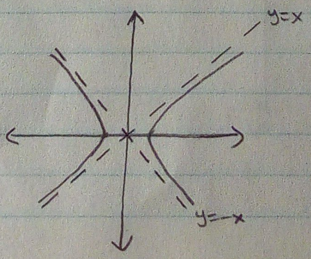
2. Hyperbola:

1. $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$



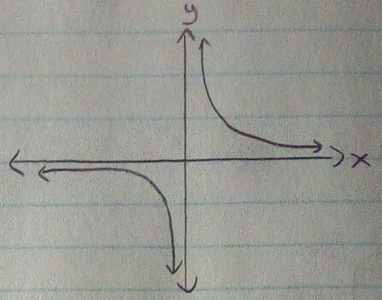
The graph of $y^2 - x^2 = 1$. All hyperbolas have 2 asymptotes, $y = \frac{a}{b}x$ and $y = -\frac{a}{b}x$.

2. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$



The graph of $x^2 - y^2 = 1$.

3. Rational Functions: $y = \frac{1}{x}$



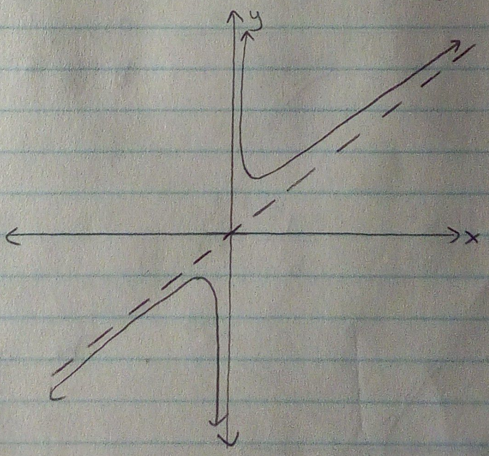
4. Rational functions with oblique asymptotes:

- E.g. $y = \frac{x^2+1}{x}$

To find the oblique asymptote, you need to do polynominal long division.

$$\begin{array}{r} x \text{ R}1 \\ x \overline{\smash{)}\, x^2+1} \\ -(x^2) \\ \hline 1 \end{array}$$

$y = x$ is the oblique asymptote.



The graph of $y = \frac{x^2+1}{x}$.

5. Ellipse: $\left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) = 1$

Note: If a^2 equals to b^2, then this is the equation of a circle.

2. Level Curves:

E.g. Sketch the level curves for $f(x,y) = \frac{2x}{x^2+y^2}$

Solution:

Let $c = \frac{2x}{x^2+y^2}$, $c \in \mathbb{R}$

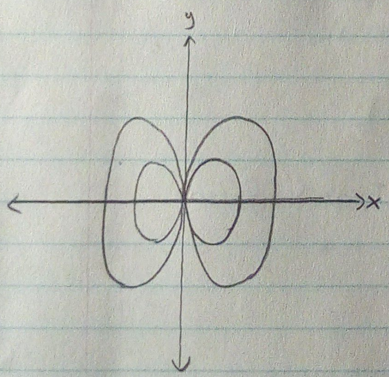
If $c = 0$, $\frac{2x}{x^2+y^2} = 0$

$$2x = 0$$
$$x = 0$$

This is the eqn of a straight line.

If $c \neq 0$, $\frac{2x}{x^2+y^2} = c$

$$2x = c(x^2+y^2)$$
$$2x = cx^2 + cy^2$$
$$0 = cx^2 - 2x + cy^2$$
$$= c(x^2 - \tfrac{2}{c}x + y^2)$$
$$= x^2 - \tfrac{2}{c}x + y^2$$
$$= x^2 - \tfrac{2}{c}x + \tfrac{1}{c^2} - \tfrac{1}{c^2} + y^2$$
$$\left(\tfrac{1}{c}\right)^2 = \left(x - \tfrac{1}{c}\right)^2 + y^2$$



Level Curves for $f(x,y) = \frac{2x}{x^2+y^2}$.

This is the eqn of a circle centered at $(\frac{1}{c}, 0)$ with a radius of $\frac{1}{c}$.

3. Completing the square:

E.g. Complete the square for x^2+6x.

This means that we want to change x^2+6x into $(x+b)^2$. We know that $(x+b)^2 = x^2+2bx+b^2$. We already have the x^2 and the $2bx$ terms. To get the b^2 term, divide $2b$ by 2, to get b, and then square it.

$$
\begin{aligned}
& x^2+6x+\left(\tfrac{6}{2}\right)^2-\left(\tfrac{6}{2}\right)^2 \\
&= x^2+6x+9-9 \\
&= (x+3)^2-9 \quad \longleftarrow \text{Final Form}
\end{aligned}
$$

E.g. Complete the square for $3x^2+6x$.

If the coefficient of x^2 is not 1, we first factor the coefficient.

$3(x^2+2x)$

Now, we divide the coefficient of x by 2 and square it.

$$
\begin{aligned}
& 3(x^2+2x+1-1) \\
&= 3(x^2+2x+1)-3 \\
&= 3(x+1)^2-3 \quad \longleftarrow \text{Final form}
\end{aligned}
$$

4. Finding and Sketching Domains:

E.g. Sketch the domain of

$$f(x,y) = \frac{1}{\sqrt{(x+y)(y-x^2+1)}}$$

Solution:

We know that $\sqrt{(x+y)(y-x^2+1)} > 0$.

This means that $(x+y)(y-x^2+1) > 0$.

We have 2 cases:

1. $(x+y) > 0$ and $(y-x^2+1) > 0$

   $y > -x$ $\quad$ $y > x^2-1$

2. $(x+y) < 0$ and $(y-x^2+1) < 0$

   $y < -x$ $\quad$ $y < x^2-1$