ON THE REMAINDER IN THE TAYLOR THEOREM

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ABSTRACT. We give a short straightforward proof for the bound of the reminder term in the Taylor theorem. The proof uses only induction and the fact that $f' \geq 0$ implies the monotonicity of f, so it might be an attractive proof to give to undergraduate students.

1. Introduction

Let f be an n-times differentiable function in a neighborhood of $a \in \mathbb{R}$. Recall that the Taylor polynomial of order n of f at a is the polynomial

$$P_n(x) = f(a) + f'(a)(x - a) + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

It will be convenient to define $P_{-1}(x) = 0$. Let $R_n = f - P_n$ be the remainder term. Then

Theorem 1 (Lagrange's formula for the remainder). If f has an (n+1)th derivative in [a,b] then there is some $a \le \xi \le b$ such that

$$R_n(b) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1}.$$

This formula is the main tool for bounding the remainder term of the Taylor expansion in calculus classes, especially when this subject is taught before integration. Therefore, one would like to have some "natural" proof for it. In [3] it is suggested that induction seems suitable, since P'_n is the Taylor polynomial of f' of order n-1, hence $R'_n(x)$ is given by induction. The reason that this approach fails is that one cannot integrate $R'_n(x)$, since $\xi = \xi(x)$ is implicit.

While we were teaching a first calculus course for chemistry and physics majors, we observed that this obstacle can be removed if we slightly change the problem to finding a **bound** of the remainder, which is all that is needed in order to show that the Taylor series does in fact converge to the function. From our personal experience, it seems that this approach enables students to grasp the material more easily. Finally we mark that Lagrange's formula can be deduced from the bound, as we show at the end of this note.

The only fact needed in the proof is that a function with a positive derivative is increasing. This can be easily proved with the mean value theorem or without it (see [1, 2]). As a direct corollary one gets:

Lemma 2. Let f, g be differentiable in a closed segment [a, b]. If f(a) = g(a) and $f'(x) \leq g'(x)$ for every $x \in (a, b)$, then $f(x) \leq g(x)$ for every $a \leq x \leq b$.

Proof. Take
$$h = g - f$$
. Thus $h' \ge 0$ and $h(a) = 0$, hence $0 \le h(x)$, i.e. $f(x) \le g(x)$.

2. The main result

Theorem 3. Suppose that f has an (n+1)th derivative in [a,b] and that $m \leq f^{(n+1)}(x) \leq M$ for every a < x < b. Then for any $a \leq x \leq b$

(1)
$$\frac{m}{(n+1)!}(x-a)^{n+1} \le R_n(x) \le \frac{M}{(n+1)!}(x-a)^{n+1}.$$

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Proof. By induction on n. For n = -1 the result is trivial.

For $n \ge 0$ write $f(x) = P_n(x) + R_n(x)$. Then $f'(x) = P'_n(x) + R'_n(x)$. Note that P'_n is the Taylor polynomial of f' of order n-1 and that $(f')^{(n)} = f^{(n+1)}$. Hence by induction we have

(2)
$$\frac{m}{n!}(x-a)^n \le R'_n(x) \le \frac{M}{n!}(x-a)^n,$$

for every $a \le x \le b$. Hence Lemma 2 gives the required inequality (since $\left(\frac{(x-a)^{n+1}}{(n+1)!}\right)' = \frac{(x-a)^n}{n!}$).

We conclude with a proof of Lagrange's classical formula. As mentioned before, this might be omitted in calculus classes.

Proof. Choose $m = \inf_{a \le x \le b} \{f^{(n+1)}(x)\}$ and $M = \sup_{a \le x \le b} \{f^{(n+1)}(x)\}$ (if $f^{(n+1)}$ is unbounded, we allow $m, M = \pm \infty$). Thus by Theorem 3 $R_n(b) = \frac{k}{(n+1)!} (b-a)^{n+1}$ for some $m \le k \le M$. If one of equalities holds, then the result is immediate from Theorem 3. Otherwise it follows directly from Darboux's intermediate value theorem.

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