68. Writing Let A be a nonsingular matrix of order 3. Prove that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set in $M_{3,1}$, then the set $\{A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3\}$ is also linearly independent. Explain, by means of an example, why this is not true if A is singular.

69. Prove the corollary to Theorem 4.8: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent if and only if one is a scalar multiple of the other.

4.5 Basis and Dimension

In this section you will continue your study of spanning sets. In particular, you will look at spanning sets (in a vector space) that both are linearly independent *and* span the entire space. Such a set forms a **basis** for the vector space. (The plural of *basis* is *bases.*)

Definition of Basis

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V is called a **basis** for V if the following conditions are true.

1. S spans V. 2. S is linearly independent.

REMARK: This definition tells you that a basis has two features. A basis S must have *enough vectors* to span V, but *not so many vectors* that one of them could be written as a linear combination of the other vectors in S.

This definition does not imply that every vector space has a basis consisting of a finite number of vectors. In this text, however, the discussion of bases is restricted to those consisting of a finite number of vectors. Moreover, if a vector space V has a basis consisting of a finite number of vectors, then V is **finite dimensional.** Otherwise, V is called **infinite dimensional.** [The vector space P of *all* polynomials is infinite dimensional, as is the vector space $C(-\infty, \infty)$ of all continuous functions defined on the real line.] The vector space $V = \{\mathbf{0}\}$, consisting of the zero vector alone, is finite dimensional.

EXAMPLE 1 The Standard Basis for R^3

Show that the following set is a basis for R^3.

$$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

SOLUTION

Example 4(a) in Section 4.4 showed that S spans R^3. Furthermore, S is linearly independent because the vector equation

$$c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) = (0, 0, 0)$$

has only the trivial solution $c_1 = c_2 = c_3 = 0$. (Try verifying this.) So, S is a basis for R^3. (See Figure 4.18.)

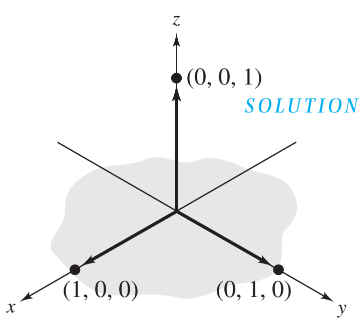


Figure 4.18

The basis $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is called the **standard basis** for R^3. This result can be generalized to n-space. That is, the vectors

$$\begin{aligned}
\mathbf{e}_1 &= (1, 0, \ldots, 0)\\
\mathbf{e}_2 &= (0, 1, \ldots, 0)\\
&\vdots\\
\mathbf{e}_n &= (0, 0, \ldots, 1)
\end{aligned}$$

form a basis for R^n called the **standard basis** for R^n.

The next two examples describe nonstandard bases for R^2 and R^3.

EXAMPLE 2 The Nonstandard Basis for R^2

Show that the set

$$S = \{\overset{\mathbf{v}_1}{(1, 1)}, \overset{\mathbf{v}_2}{(1, -1)}\}$$

is a basis for R^2.

SOLUTION According to the definition of a basis for a vector space, you must show that S spans R^2 and S is linearly independent.

To verify that S spans R^2, let

$$\mathbf{x} = (x_1, x_2)$$

represent an arbitrary vector in R^2. To show that $\mathbf{x}$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, consider the equation

$$\begin{aligned}
c_1\mathbf{v}_1 + c_2\mathbf{v}_2 &= \mathbf{x}\\
c_1(1, 1) + c_2(1, -1) &= (x_1, x_2)\\
(c_1 + c_2, c_1 - c_2) &= (x_1, x_2).
\end{aligned}$$

Equating corresponding components yields the system of linear equations shown below.

$$\begin{aligned}
c_1 + c_2 &= x_1\\
c_1 - c_2 &= x_2
\end{aligned}$$

Because the coefficient matrix of this system has a nonzero determinant, you know that the system has a unique solution. You can now conclude that S spans R^2.

To show that S is linearly independent, consider the linear combination

$$\begin{aligned}
c_1\mathbf{v}_1 + c_2\mathbf{v}_2 &= \mathbf{0}\\
c_1(1, 1) + c_2(1, -1) &= (0, 0)\\
(c_1 + c_2, c_1 - c_2) &= (0, 0).
\end{aligned}$$

Equating corresponding components yields the homogeneous system

$$\begin{aligned}
c_1 + c_2 &= 0\\
c_1 - c_2 &= 0.
\end{aligned}$$

Because the coefficient matrix of this system has a nonzero determinant, you know that the system has only the trivial solution

$$c_1 = c_2 = 0.$$

So, you can conclude that S is linearly independent.

You can conclude that S is a basis for R^2 because it is a linearly independent spanning set for R^2.

EXAMPLE 3 A Nonstandard Basis for R^3

From Examples 5 and 8 in the preceding section, you know that

$$S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$$

spans R^3 and is linearly independent. So, S is a basis for R^3.

EXAMPLE 4 A Basis for Polynomials

Show that the vector space P_3 has the basis

$$S = \{1, x, x^2, x^3\}.$$

SOLUTION It is clear that S spans P_3 because the span of S consists of all polynomials of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3, \quad a_0, a_1, a_2, \text{ and } a_3 \text{ are real,}$$

which is precisely the form of all polynomials in P_3.

To verify the linear independence of S, recall that the zero vector $\mathbf{0}$ in P_3 is the polynomial $\mathbf{0}(x) = 0$ for all x. The test for linear independence yields the equation

$$a_0 + a_1x + a_2x^2 + a_3x^3 = \mathbf{0}(x) = 0, \quad \text{for all } x.$$

This third-degree polynomial is said to be *identically equal to zero*. From algebra you know that for a polynomial to be identically equal to zero, all of its coefficients must be zero; that is,

$$a_0 = a_1 = a_2 = a_3 = 0.$$

So, S is linearly independent and is a basis for P_3.

REMARK: The basis $S = \{1, x, x^2, x^3\}$ is called the **standard basis** for P_3. Similarly, the **standard basis** for P_n is

$$S = \{1, x, x^2, . . . , x^n\}.$$

EXAMPLE 5 **A Basis for $M_{2,2}$**

The set

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis for $M_{2,2}$. This set is called the **standard basis** for $M_{2,2}$. In a similar manner, the standard basis for the vector space $M_{m,n}$ consists of the mn distinct $m \times n$ matrices having a single 1 and all the other entries equal to zero.

THEOREM 4.9
Uniqueness of Basis Representation

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of vectors in S.

PROOF The existence portion of the proof is straightforward. That is, because S spans V, you know that an arbitrary vector $\mathbf{u}$ in V can be expressed as $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$.

To prove uniqueness (that a vector can be represented in only one way), suppose $\mathbf{u}$ has another representation $\mathbf{u} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n$. Subtracting the second representation from the first produces

$$\mathbf{u} - \mathbf{u} = (c_1 - b_1)\mathbf{v}_1 + (c_2 - b_2)\mathbf{v}_2 + \cdots + (c_n - b_n)\mathbf{v}_n = \mathbf{0}.$$

Because S is linearly independent, however, the only solution to this equation is the trivial solution

$$c_1 - b_1 = 0, \quad c_2 - b_2 = 0, \quad \ldots, \quad c_n - b_n = 0,$$

which means that $c_i = b_i$ for all $i = 1, 2, \ldots, n$. So, $\mathbf{u}$ has only one representation for the basis S.

EXAMPLE 6 **Uniqueness of Basis Representation**

Let $\mathbf{u} = (u_1, u_2, u_3)$ be any vector in R^3. Show that the equation $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ has a unique solution for the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$.

SOLUTION From the equation

$$\begin{aligned}(u_1, u_2, u_3) &= c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-2, 0, 1) \\ &= (c_1 - 2c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3),\end{aligned}$$

the following system of linear equations is obtained.

$$\begin{aligned} c_1 - 2c_3 &= u_1 \\ 2c_1 + c_2 &= u_2 \\ 3c_1 + 2c_2 + c_3 &= u_3 \end{aligned} \qquad \underset{A}{\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}} \underset{c}{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}} = \underset{u}{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}$$

Because the matrix A is invertible, you know this system has a unique solution $\mathbf{c} = A^{-1}\mathbf{u}$. Solving for A^{-1} yields

$$A^{-1} = \begin{bmatrix} -1 & 4 & -2 \\ 2 & -7 & 4 \\ -1 & 2 & -1 \end{bmatrix},$$

which implies

$$\begin{aligned} c_1 &= -u_1 + 4u_2 - 2u_3 \\ c_2 &= 2u_1 - 7u_2 + 4u_3 \\ c_3 &= -u_1 + 2u_2 - u_3. \end{aligned}$$

For instance, the vector $\mathbf{u} = (1, 0, 0)$ can be represented uniquely as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ as follows.

$$(1, 0, 0) = -\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3$$

You will now study two important theorems concerning bases.

THEOREM 4.10 Bases and Linear Dependence

If $S = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_n\}$ is a basis for a vector space V, then every set containing more than n vectors in V is linearly dependent.

PROOF Let $S_1 = \{\mathbf{u}_1, \mathbf{u}_2, . . . , \mathbf{u}_m\}$ be any set of m vectors in V, where $m > n$. To show that S_1 is linearly *dependent,* you need to find scalars $k_1, k_2, . . . , k_m$ (not all zero) such that

$$k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdot \cdot \cdot + k_m\mathbf{u}_m = \mathbf{0}. \qquad \textbf{Equation 1}$$

Because S is a basis for V, it follows that each $\mathbf{u}_i$ is a linear combination of vectors in S, and you can write

$$\begin{aligned} \mathbf{u}_1 &= c_{11}\mathbf{v}_1 + c_{21}\mathbf{v}_2 + \cdot \cdot \cdot + c_{n1}\mathbf{v}_n \\ \mathbf{u}_2 &= c_{12}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + \cdot \cdot \cdot + c_{n2}\mathbf{v}_n \\ &\ \vdots \\ \mathbf{u}_m &= c_{1m}\mathbf{v}_1 + c_{2m}\mathbf{v}_2 + \cdot \cdot \cdot + c_{nm}\mathbf{v}_n. \end{aligned}$$

Substituting each of these representations of $\mathbf{u}_i$ into Equation 1 and regrouping terms produces

$$d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdot \cdot \cdot + d_n\mathbf{v}_n = \mathbf{0},$$

where $d_i = c_{i1}k_1 + c_{i2}k_2 + \cdots + c_{im}k_m$. Because the $\mathbf{v}_i$'s form a linearly independent set, you can conclude that each $d_i = 0$. So, the system of equations shown below is obtained.

$$\begin{aligned}
c_{11}k_1 + c_{12}k_2 + \cdots + c_{1m}k_m &= 0\\
c_{21}k_1 + c_{22}k_2 + \cdots + c_{2m}k_m &= 0\\
\vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots\\
c_{n1}k_1 + c_{n2}k_2 + \cdots + c_{nm}k_m &= 0
\end{aligned}$$

But this homogeneous system has fewer equations than variables $k_1, k_2, \ldots, k_m$, and from Theorem 1.1 you know it must have *nontrivial* solutions. Consequently, S_1 is linearly dependent.

EXAMPLE 7 Linearly Dependent Sets in R^3 and P_3

(a) Because R^3 has a basis consisting of three vectors, the set

$$S = \{(1, 2, -1), (1, 1, 0), (2, 3, 0), (5, 9, -1)\}$$

must be linearly dependent.

(b) Because P_3 has a basis consisting of four vectors, the set

$$S = \{1, 1 + x, 1 - x, 1 + x + x^2, 1 - x + x^2\}$$

must be linearly dependent.

Because R^n has the standard basis consisting of n vectors, it follows from Theorem 4.10 that every set of vectors in R^n containing more than n vectors must be linearly dependent. Another significant consequence of Theorem 4.10 is shown in the next theorem.

THEOREM 4.11 Number of Vectors in a Basis

If a vector space V has one basis with n vectors, then every basis for V has n vectors.

PROOF Let

$$S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

be the basis for V, and let

$$S_2 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$$

be any other basis for V. Because S_1 is a basis and S_2 is linearly independent, Theorem 4.10 implies that $m \le n$. Similarly, $n \le m$ because S_1 is linearly independent and S_2 is a basis. Consequently, $n = m$.

EXAMPLE 8 **Spanning Sets and Bases**

Use Theorem 4.11 to explain why each of the statements below is true.

(a) The set $S_1 = \{(3, 2, 1), (7, -1, 4)\}$ is not a basis for R^3.

(b) The set

$$S_2 = \{x + 2, x^2, x^3 - 1, 3x + 1, x^2 - 2x + 3\}$$

is not a basis for P_3.

SOLUTION (a) The standard basis for R^3 has three vectors, and S_1 has only two. By Theorem 4.11, S_1 cannot be a basis for R^3.

(b) The standard basis for P_3, $S = \{1, x, x^2, x^3\}$, has four elements. By Theorem 4.11, the set S_2 has too many elements to be a basis for P_3.

The Dimension of a Vector Space

The discussion of spanning sets, linear independence, and bases leads to an important notion in the study of vector spaces. By Theorem 4.11, you know that if a vector space V has a basis consisting of n vectors, then every other basis for the space also has n vectors. The number n is called the **dimension** of V.

Definition of Dimension of a Vector Space

If a vector space V has a basis consisting of n vectors, then the number n is called the **dimension** of V, denoted by $\dim(V) = n$. If V consists of the zero vector alone, the dimension of V is defined as zero.

This definition allows you to observe the characteristics of the dimensions of the familiar vector spaces listed below. In each case, the dimension is determined by simply counting the number of vectors in the standard basis.

1. The dimension of R^n with the standard operations is n.
2. The dimension of P_n with the standard operations is $n + 1$.
3. The dimension of $M_{m,n}$ with the standard operations is mn.

If W is a subspace of an n-dimensional vector space, then it can be shown that W is finite dimensional and the dimension of W is less than or equal to n. (See Exercise 81.) In the next three examples, you will look at a technique for determining the dimension of a subspace. Basically, you determine the dimension by finding a set of linearly independent vectors that spans the subspace. This set is a basis for the subspace, and the dimension of the subspace is the number of vectors in the basis.

EXAMPLE 9 **Finding the Dimension of a Subspace**

Determine the dimension of each subspace of R^3.

(a) $W = \{(d, c - d, c)\colon c$ and d are real numbers$\}$
(b) $W = \{(2b, b, 0)\colon b$ is a real number$\}$

SOLUTION The goal in each example is to find a set of linearly independent vectors that spans the subspace.

(a) By writing the representative vector $(d, c - d, c)$ as

$$\begin{aligned}(d, c - d, c) &= (0, c, c) + (d, -d, 0)\\ &= c(0, 1, 1) + d(1, -1, 0),\end{aligned}$$

you can see that W is spanned by the set

$$S = \{(0, 1, 1), (1, -1, 0)\}.$$

Using the techniques described in the preceding section, you can show that this set is linearly independent. So, it is a basis for W, and you can conclude that W is a two-dimensional subspace of R^3.

(b) By writing the representative vector $(2b, b, 0)$ as

$$(2b, b, 0) = b(2, 1, 0),$$

you can see that W is spanned by the set $S = \{(2, 1, 0)\}$. So, W is a one-dimensional subspace of R^3.

REMARK: In Example 9(a), the subspace W is a two-dimensional plane in R^3 determined by the vectors $(0, 1, 1)$ and $(1, -1, 0)$. In Example 9(b), the subspace is a one-dimensional line.

EXAMPLE 10 **Finding the Dimension of a Subspace**

Find the dimension of the subspace W of R^4 spanned by

$$S = \{\overset{\mathbf{v}_1}{(-1, 2, 5, 0)}, \overset{\mathbf{v}_2}{(3, 0, 1, -2)}, \overset{\mathbf{v}_3}{(-5, 4, 9, 2)}\}.$$

SOLUTION Although W is spanned by the set S, S is not a basis for W because S is a linearly dependent set. Specifically, $\mathbf{v}_3$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ as follows.

$$\mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2$$

This means that W is spanned by the set $S_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$. Moreover, S_1 is linearly independent because neither vector is a scalar multiple of the other, and you can conclude that the dimension of W is 2.

EXAMPLE 11 Finding the Dimension of a Subspace

Let W be the subspace of all symmetric matrices in $M_{2,2}$. What is the dimension of W?

SOLUTION Every 2×2 symmetric matrix has the form listed below.

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$$

$$= a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So, the set

$$S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

spans W. Moreover, S can be shown to be linearly independent, and you can conclude that the dimension of W is 3.

Usually, to conclude that a set

$$S = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_n\}$$

is a basis for a vector space V, you must show that S satisfies two conditions: S spans V and is linearly independent. If V is known to have a dimension of n, however, then the next theorem tells you that you do not need to check both conditions: either one will suffice. The proof is left as an exercise. (See Exercise 82.)

THEOREM 4.12
Basis Tests in an *n*-Dimensional Space

Let V be a vector space of dimension n.

1. If $S = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_n\}$ is a linearly independent set of vectors in V, then S is a basis for V.
2. If $S = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_n\}$ spans V, then S is a basis for V.

EXAMPLE 12 Testing for a Basis in an *n*-Dimensional Space

Show that the set of vectors is a basis for $M_{5,1}$.

$$S = \left\{ \overset{\mathbf{v}_1}{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}}, \overset{\mathbf{v}_2}{\begin{bmatrix} 0 \\ 1 \\ 3 \\ -2 \\ 3 \end{bmatrix}}, \overset{\mathbf{v}_3}{\begin{bmatrix} 0 \\ 0 \\ 2 \\ -1 \\ 5 \end{bmatrix}}, \overset{\mathbf{v}_4}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ -3 \end{bmatrix}}, \overset{\mathbf{v}_5}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}} \right\}$$

SOLUTION Because S has five vectors and the dimension of $M_{5,1}$ is five, you can apply Theorem 4.12 to verify that S is a basis by showing either that S is linearly independent or that S spans $M_{5,1}$. To show the first of these, form the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 = \mathbf{0},$$

which yields the homogeneous system of linear equations shown below.

$$\begin{aligned}
c_1 \qquad\qquad\qquad\qquad\qquad\qquad &= 0\\
2c_1 + c_2 \qquad\qquad\qquad\qquad\qquad &= 0\\
-c_1 + 3c_2 + 2c_3 \qquad\qquad\qquad &= 0\\
3c_1 - 2c_2 - c_3 + 2c_4 \qquad\quad &= 0\\
4c_1 + 3c_2 + 5c_3 - 3c_4 - 2c_5 &= 0
\end{aligned}$$

Because this system has only the trivial solution, S must be linearly independent. So, by Theorem 4.12, S is a basis for $M_{5,1}$.

SECTION 4.5 Exercises

In Exercises 1–6, write the standard basis for the vector space.

1. R^6 **2.** R^4 **3.** $M_{2,4}$

4. $M_{4,1}$ **5.** P_4 **6.** P_2

Writing In Exercises 7–14, explain why S is not a basis for R^2.

7. $S = \{(1, 2), (1, 0), (0, 1)\}$

8. $S = \{(-1, 2), (1, -2), (2, 4)\}$

9. $S = \{(-4, 5), (0, 0)\}$

10. $S = \{(2, 3), (6, 9)\}$

11. $S = \{(6, -5), (12, -10)\}$

12. $S = \{(4, -3), (8, -6)\}$

13. $S = \{(-3, 2)\}$

14. $S = \{(-1, 2)\}$

Writing In Exercises 15–20, explain why S is not a basis for R^3.

15. $S = \{(1, 3, 0), (4, 1, 2), (-2, 5, -2)\}$

16. $S = \{(2, 1, -2), (-2, -1, 2), (4, 2, -4)\}$

17. $S = \{(7, 0, 3), (8, -4, 1)\}$

18. $S = \{(1, 1, 2), (0, 2, 1)\}$

19. $S = \{(0, 0, 0), (1, 0, 0), (0, 1, 0)\}$

20. $S = \{(6, 4, 1), (3, -5, 1), (8, 13, 6), (0, 6, 9)\}$

Writing In Exercises 21–24, explain why S is not a basis for P_2.

21. $S = \{1, 2x, x^2 - 4, 5x\}$

22. $S = \{2, x, x + 3, 3x^2\}$

23. $S = \{1 - x, 1 - x^2, 3x^2 - 2x - 1\}$

24. $S = \{6x - 3, 3x^2, 1 - 2x - x^2\}$

Writing In Exercises 25–28, explain why S is not a basis for $M_{2,2}$.

25. $S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$

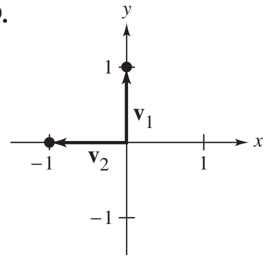
26. $S = \left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$

27. $S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 8 & -4 \\ -4 & 3 \end{bmatrix}\right\}$

28. $S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right\}$

In Exercises 29–34, determine whether the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for R^2.

29.



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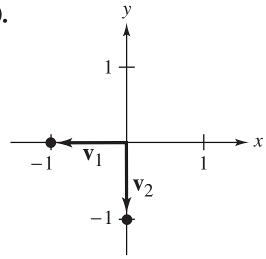

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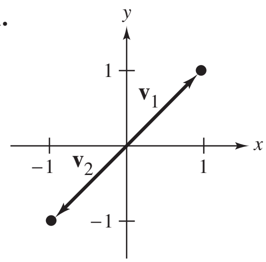

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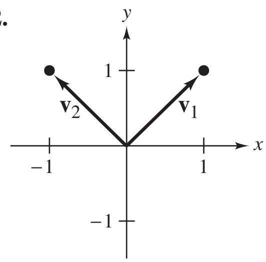

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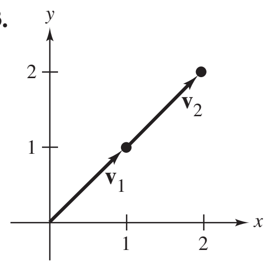

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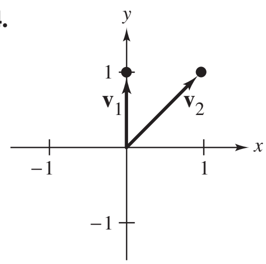

In Exercises 35–42, determine whether S is a basis for the indicated vector space.

35. $S = \{(3, -2), (4, 5)\}$ for R^2

36. $S = \{(1, 2), (1, -1)\}$ for R^2

37. $S = \{(1, 5, 3), (0, 1, 2), (0, 0, 6)\}$ for R^3

38. $S = \{(2, 1, 0), (0, -1, 1)\}$ for R^3

39. $S = \{(0, 3, -2), (4, 0, 3), (-8, 15, -16)\}$ for R^3

40. $S = \{(0, 0, 0), (1, 5, 6), (6, 2, 1)\}$ for R^3

41. $S = \{(-1, 2, 0, 0), (2, 0, -1, 0), (3, 0, 0, 4), (0, 0, 5, 0)\}$ for R^4

42. $S = \{(1, 0, 0, 1), (0, 2, 0, 2), (1, 0, 1, 0), (0, 2, 2, 0)\}$ for R^4

In Exercises 43 and 44, determine whether S is a basis for $M_{2,2}$.

43. $S = \left\{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\right\}$

44. $S = \left\{\begin{bmatrix} 1 & 2 \\ -5 & 4 \end{bmatrix}, \begin{bmatrix} 2 & -7 \\ 6 & 2 \end{bmatrix}, \begin{bmatrix} 4 & -9 \\ 11 & 12 \end{bmatrix}, \begin{bmatrix} 12 & -16 \\ 17 & 42 \end{bmatrix}\right\}$

In Exercises 45–48, determine whether S is a basis for P_3.

45. $S = \{t^3 - 2t^2 + 1, t^2 - 4, t^3 + 2t, 5t\}$

46. $S = \{4t - t^2, 5 + t^3, 3t + 5, 2t^3 - 3t^2\}$

47. $S = \{4 - t, t^3, 6t^2, t^3 + 3t, 4t - 1\}$

48. $S = \{t^3 - 1, 2t^2, t + 3, 5 + 2t + 2t^2 + t^3\}$

In Exercises 49–54, determine whether S is a basis for R^3. If it is, write $\mathbf{u} = (8, 3, 8)$ as a linear combination of the vectors in S.

49. $S = \{(4, 3, 2), (0, 3, 2), (0, 0, 2)\}$

50. $S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$

51. $S = \{(0, 0, 0), (1, 3, 4), (6, 1, -2)\}$

52. $S = \{(1, 0, 1), (0, 0, 0), (0, 1, 0)\}$

53. $S = \left\{\left(\frac{2}{3}, \frac{5}{2}, 1\right), \left(1, \frac{3}{2}, 0\right), (2, 12, 6)\right\}$

54. $S = \{(1, 4, 7), (3, 0, 1), (2, 1, 2)\}$

In Exercises 55–62, determine the dimension of the vector space.

55. R^6 **56.** R^4 **57.** R **58.** R^3

59. P_7 **60.** P_4 **61.** $M_{2,3}$ **62.** $M_{3,2}$

63. Find a basis for $D_{3,3}$ (the vector space of all 3×3 diagonal matrices). What is the dimension of this vector space?

64. Find a basis for the vector space of all 3×3 symmetric matrices. What is the dimension of this vector space?

65. Find all subsets of the set that forms a basis for R^2. $S = \{(1, 0), (0, 1), (1, 1)\}$

66. Find all subsets of the set that forms a basis for R^3. $S = \{(1, 3, -2), (-4, 1, 1), (-2, 7, -3), (2, 1, 1)\}$

67. Find a basis for R^2 that includes the vector $(1, 1)$.

68. Find a basis for R^3 that includes the set $S = \{(1, 0, 2), (0, 1, 1)\}$.

In Exercises 69 and 70, (a) give a geometric description of, (b) find a basis for, and (c) determine the dimension of the subspace W of R^2.

69. $W = \{(2t, t): t \text{ is a real number}\}$

70. $W = \{(0, t): t \text{ is a real number}\}$

In Exercises 71 and 72, (a) give a geometric description of, (b) find a basis for, and (c) determine the dimension of the subspace W of R^3.

71. $W = \{(2t, t, -t): t \text{ is a real number}\}$

72. $W = \{(2t - t, s, t): s \text{ and } t \text{ are real numbers}\}$

In Exercises 73–76, find (a) a basis for and (b) the dimension of the subspace W of R^4.

73. $W = \{(2s - t, s, t, s)$: s and t are real numbers$\}$

74. $W = \{(5t, -3t, t, t)$: t is a real number$\}$

75. $W = \{(0, 6t, t, -t)$: t is a real number$\}$

76. $W = \{(s + 4t, t, s, 2s - t)$: s and t are real numbers$\}$

True or False? In Exercises 77 and 78, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

77. (a) If $\dim(V) = n$, then there exists a set of $n - 1$ vectors in V that will span V.

(b) If $\dim(V) = n$, then there exists a set of $n + 1$ vectors in V that will span V.

78. (a) If $\dim(V) = n$, then any set of $n + 1$ vectors in V must be linearly dependent.

(b) If $\dim(V) = n$, then any set of $n - 1$ vectors in V must be linearly independent.

79. Prove that if $S = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_n\}$ is a basis for a vector space V and c is a nonzero scalar, then the set $S_1 = \{c\mathbf{v}_1, c\mathbf{v}_2, . . . , c\mathbf{v}_n\}$ is also a basis for V.

80. Prove that the vector space P of all polynomials is infinite dimensional.

81. Prove that if W is a subspace of a finite-dimensional vector space V, then (dimension of W) $\leq$ (dimension of V).

82. Prove Theorem 4.12.

83. Writing

(a) Let $S_1 = \text{span}((1, 0, 0), (1, 1, 0))$ and $S_2 = \text{span}((0, 0, 1), (0, 1, 0))$ be subspaces of R^3. Find a basis for and the dimension of each of the subspaces $S_1, S_2, S_1 \cap S_2$, and $S_1 + S_2$. (See Exercise 47 in Section 4.3)

(b) Let S_1 and S_2 be two-dimensional subspaces of R^3. Is it possible that $S_1 \cap S_2 = \{(0, 0, 0)\}$? Explain.

84. Guided Proof Let S be a spanning set for the finite dimensional vector space V. Prove that there exists a subset S' of S that forms a basis for V.

Getting Started: S is a spanning set, but it may not be a basis because it may be linearly dependent. You need to remove extra vectors so that a subset S' is a spanning set and is also linearly independent.

(i) If S is a linearly independent set, you are done. If not, remove some vector $\mathbf{v}$ from S that is a linear combination of the other vectors in S.

(ii) Call this set S_1. If S_1 is a linearly independent set, you are done. If not, continue to remove dependent vectors until you produce a linearly independent subset S'.

(iii) Conclude that this subset is the minimal spanning set S'.

85. Let S be a linearly independent set of vectors from the finite dimensional vector space V. Prove that there exists a basis for V containing S.

86. Let V be a vector space of dimension n. Prove that any set of less than n vectors cannot span V.

4.6 Rank of a Matrix and Systems of Linear Equations

In this section you will investigate the vector space spanned by the **row vectors** (or **column vectors**) of a matrix. Then you will see how such spaces relate to solutions of systems of linear equations.

To begin, you need to know some terminology. For an $m \times n$ matrix A, the n-tuples corresponding to the rows of A are called the **row vectors** of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Row Vectors of A

$(a_{11}, a_{12}, . . . , a_{1n})$
$(a_{21}, a_{22}, . . . , a_{2n})$
$\vdots$
$(a_{m1}, a_{m2}, . . . , a_{mn})$