3 Approximating a function using a polynomial

3.1 McLaurin series

Assume that $f(x)$ is a continuous function of x, then

$$f(x) = \sum_{i=0}^{\infty} a_i x^i = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i$$

is known as the McLaurin Series, where a_i's are the coefficients in the polynomial expansion given by

$$a_i = \frac{f^{(i)}(x)}{i!}|_{x=0}$$

and $f^{(i)}(x)$ is the i-th derivative of $f(x)$.

The McLaurin series is used to predict the value $f(x_1)$ (for any $x = x_1$) using the function's value $f(0)$ at the "reference point" ($x = 0$). The series is approximated by summing up to a suitably high value of i, which can lead to approximation or truncation errors.

Problem: When function $f(x)$ varies significantly over the interval from $x = 0$ to $x = x_1$, the approximation may not work well.

A better solution is to move the reference point closer to x_1, at which the function's polynomial expansion is needed. Then, $f(x_1)$ can be represented in terms of $f(x_r)$, $f^{(1)}(x_r)$, etc.

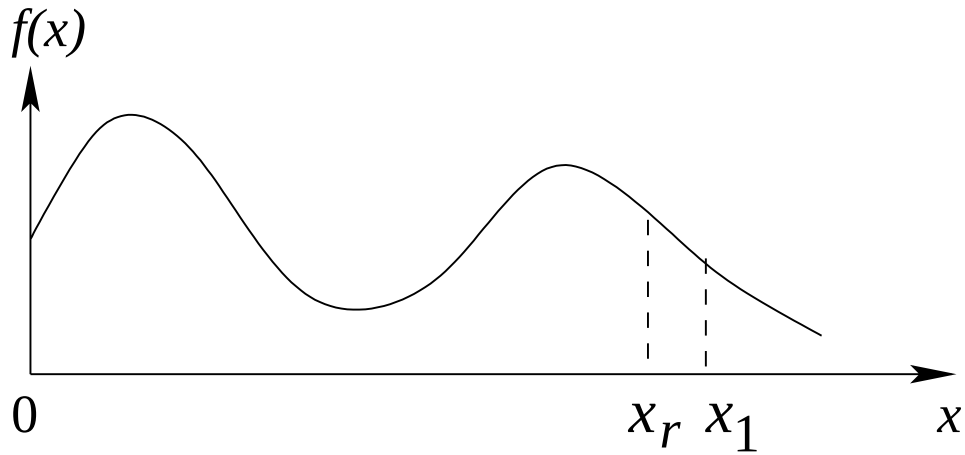

Figure 18: Taylor series

3.2 Taylor series

Assume that $f(x)$ is a continuous function of x, then

$$f(x) = \sum_{i=0}^{\infty} a_i (x - x_r)^i$$

where $a_i = \frac{f^{(i)}(x)}{i!}|_{x=x_r}$. Define $h = x - x_r$. Then,

$$f(x) = \sum_{i=0}^{\infty} a_i h^i = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_r)}{i!} h^i$$

which is known as the Taylor Series.

If x_r is sufficiently close to x, we can approximate $f(x)$ with a small number of coefficients since $(x - x_r)^i \to 0$ as i increases.

Question: What is the error when approximating function $f(x)$ at x by $f(x) = \sum_{i=0}^{n} a_i h^i$, where n is a finite number (the order of the Taylor series)?

Taylor theorem:

A function $f(x)$ can be represented exactly as

$$f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(x_r)}{i!} h^i + R_n$$

where R_n is the remainder (error) term, and can be calculated as

$$R_n = \frac{f^{(n+1)}(\alpha)}{(n+1)!} h^{n+1}$$

and α is an unknown value between x_r and x.

- Although α is unknown, Taylor's theorem tells us that the error R_n is proportional to h^{n+1}, which is denoted by

$$R_n = O(h^{n+1})$$

which reads "R_n is order h to the power of $n+1$".

- With n-th order Taylor series approximation, the error is proportional to step size h to the power $n+1$. Or equivalently, the truncation error goes to zero no slower than h^{n+1} does.

- With $h \ll 1$, an algorithm or numerical method with $O(h^2)$ is better than one with $O(h)$. If you half the step size h, the error is quartered in the former but is only halved in the latter.

Question: How to find R_n?

$$\begin{aligned} R_n &= \sum_{i=0}^{\infty} a_i h^i - \sum_{i=0}^{n} a_i h^i \\ &= \sum_{i=n+1}^{\infty} a_i h^i = \sum_{i=n+1}^{\infty} \frac{f^{(i)}(x_r)}{i!} h^i \end{aligned}$$

For small h ($h \ll 1$),

$$R_n \approx \frac{f^{(n+1)}(x_r)}{(n+1)!} h^{n+1}$$

The above expression can be used to evaluate the dominant error terms in the n-th order approximation.

For different values of n (the order of the Taylor series), a different trajectory is fitted (or approximated) to the function. For example:

- $n = 0$ (zero order approximation) $\rightarrow$ straight line with zero slope
- $n = 1$ (first order approximation) $\rightarrow$ straight line with some slope
- $n = 2$ (second order approximation) $\rightarrow$ quadratic function

Example 1: Expand $f(x) = e^x$ as a McLaurin series.

Solution:

$$a_0 = f(0) = e^0 = 1,$$
$$a_1 = \frac{f'(x)}{1!}\Big|_{x=0} = \frac{e^0}{1} = 1$$
$$a_i = \frac{f^{(i)}(x)}{i!}\Big|_{x=0} = \frac{e^0}{i!} = \frac{1}{i!}$$

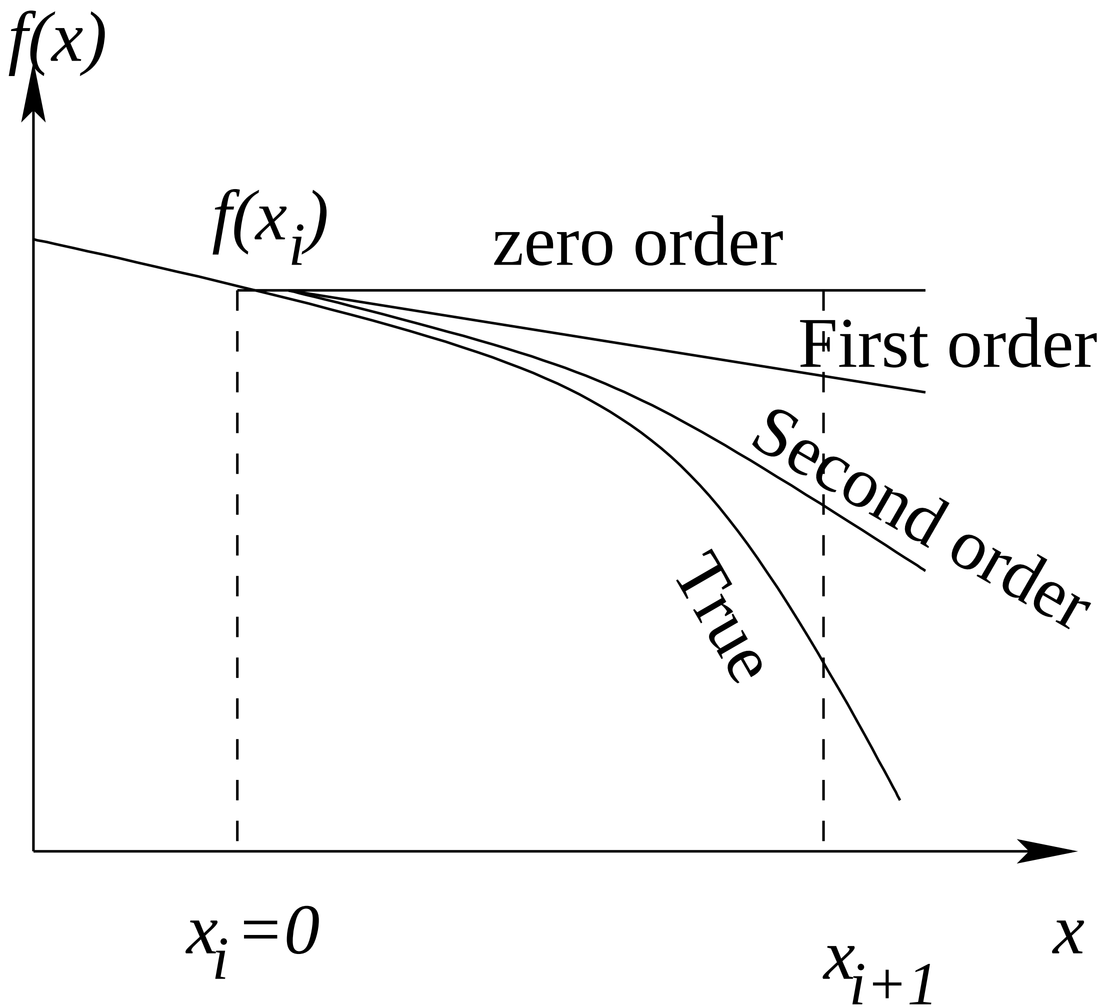

Figure 19: Taylor series

Then $f(x) = e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$.

Example 2: Find the McLaurin series up to order 4, Taylor series (around $x = 1$) up to order 4 of function $f(x) = x^3 - 2x^2 + 0.25x + 0.75$.

Solution:

$$
\begin{array}{lll}
f(x) = x^3 - 2x^2 + 0.25x + 0.75 & f(0) = 0.75 & f(1) = 0 \\
f^{'}(x) = 3x^2 - 4x + 0.25 & f^{'}(0) = 0.25 & f^{'}(1) = -0.75 \\
f^{''}(x) = 6x - 4 & f^{''}(0) = -4 & f^{''}(1) = 2 \\
f^{(3)}(x) = 6 & f^{(3)}(0) = 6 & f^{(3)}(1) = 6 \\
f^{(4)}(x) = 0 & f^{(4)}(0) = 0 & f^{(4)}(1) = 0
\end{array}
$$

The McLaurin series of $f(x) = x^3 - 2x^2 + 0.25x + 0.75$ can be written as

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i = \sum_{i=0}^{3} \frac{f^{(i)}(0)}{i!} x^i$$

Then the third order McLaurin series expansion is

$$
\begin{aligned}
f_{M3}(x) &= f(0) + f^{'}(0)x + \frac{1}{2} f^{''}(0)x^2 + \frac{1}{3!} f^{(3)}(0)x^3 \\
&= 0.75 + 0.25x - 2x^2 + x^3
\end{aligned}
$$

which is the same as the original polynomial function.

The lower order McLaurin series expansion may be written as

$$
\begin{aligned}
f_{M2}(x) &= f(0) + \frac{1}{2}f^{''}(0)x^2 \\
&= 0.75 + 0.25x - 2x^2 \\
f_{M1}(x) &= 0.75 + 0.25x \\
f_{M0}(x) &= 0.75
\end{aligned}
$$

The Taylor series can be written as

$$
f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_r)}{i!}(x - x_r)^i = \sum_{i=0}^{3} \frac{f^{(i)}(1)}{i!}(x - 1)^i
$$

Then the third order Taylor series of $f(x)$ is

$$
\begin{aligned}
f_{T3}(x) &= f(1) + f^{'}(1)(x-1) + \frac{1}{2}f^{''}(1)(x-1)^2 + \frac{1}{3!}f^{(3)}(1)(x-1)^3 \\
&= 0.75 + 0.25x - 2x^2 + x^3
\end{aligned}
$$

which is the same as the original function.

The lower order Taylor series expansion may be written as

$$
\begin{aligned}
f_{T2}(x) &= f(1) + \frac{1}{2}f^{''}(1)(x-1)^2 \\
&= -0.75(x-1) + (x-1)^2 \\
&= 1.75 - 2.75x + x^2 \\
f_{T1}(x) &= f(1) + f^{'}(x-1) \\
&= -0.75(x-1) = 0.75 - 0.75x \\
f_{T0}(x) &= f(1) = 0
\end{aligned}
$$

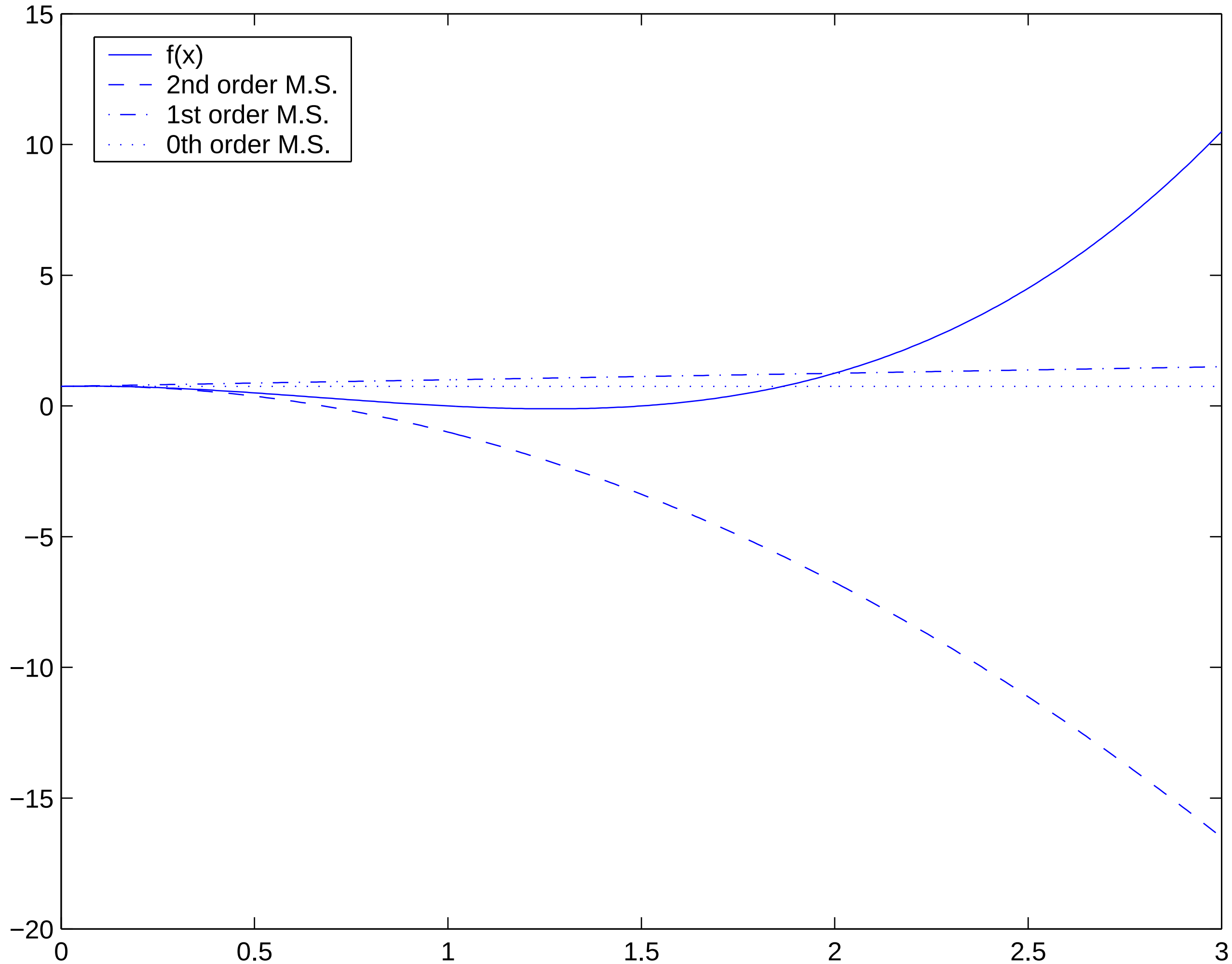

Figure 20: Example 1

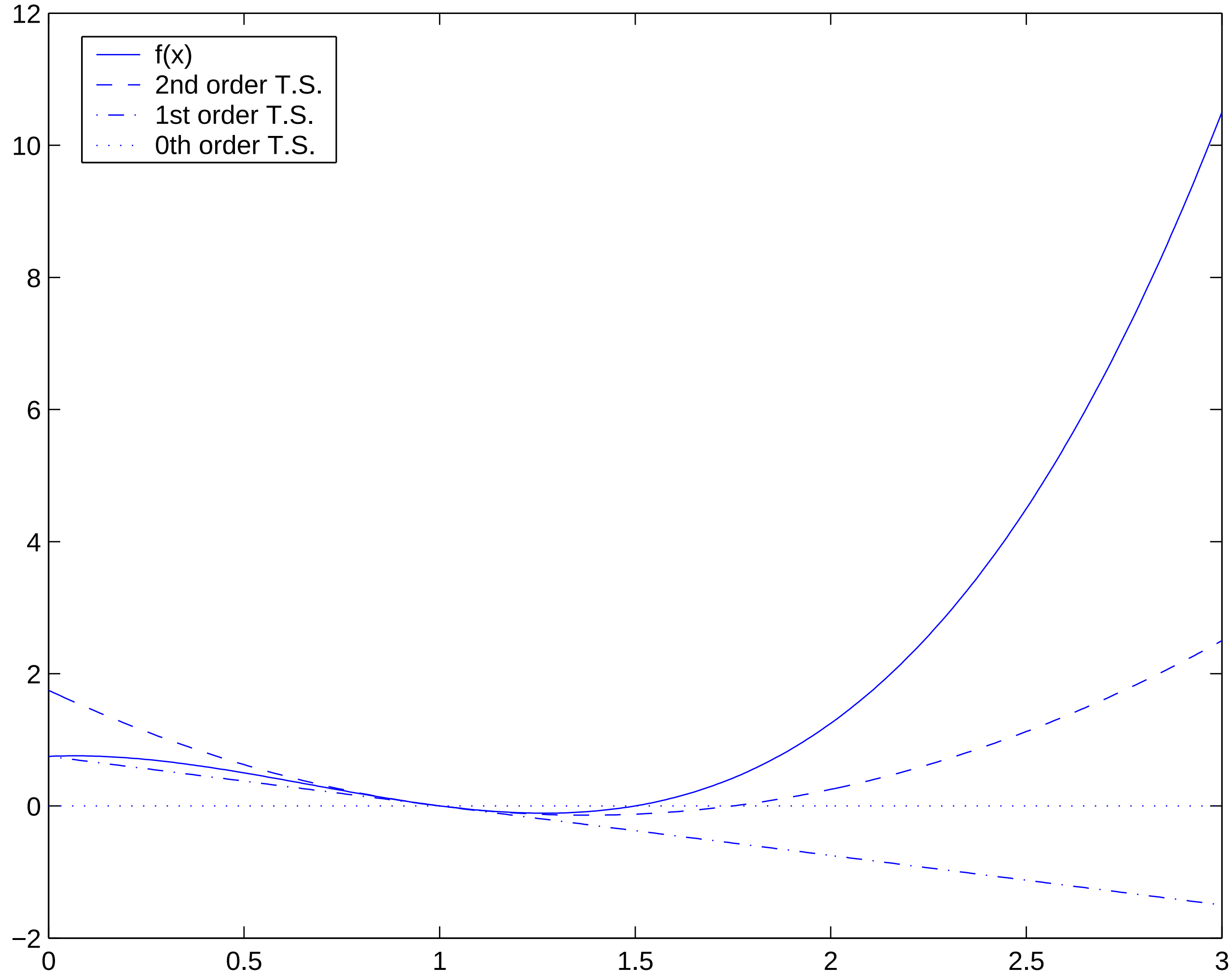

Figure 21: Example 2