Orthogonal Functions and Fourier Series

CSE 317

CHAPTER 12

Advanced Engineering Mathematics

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Contents

12.1 Orthogonal Functions

DEFINITION 12.1 **Inner Product of Function**

The ***inner product*** of two functions f_1 and f_2 on an interval $[a, b]$ is the number

$$< f_1, f_2 >= \int_a^b f_1(x) f_2(x) dx$$

DEFINITION 12.2 **Orthogonal Function**

Two functions f_1 and f_2 are said to be ***orthogonal*** on an interval $[a, b]$ if

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x)\, dx = 0$$

Example

- The function $f_1(x) = x^2$, $f_2(x) = x^3$ are orthogonal on the interval [−1, 1] since

$$(f_1, f_2) = \int_{-1}^{1} x^2 .\ x^3 dx = \frac{1}{6} x^6 \Big|_{-1}^{1} = 0$$

DEFINITION 12.3

Inner Product of Function

A set of real-valued functions $\{\phi_0(x),\ \phi_1(x),\ \phi_2(x),\ \ldots\}$ is said to be **orthogonal** on an interval $[a, b]$ if

$$(\phi_m, \phi_n) = \int_a^b \phi_m(x), \phi_n(x)\, dx = 0, \quad m \neq n \qquad (2)$$

Orthonormal Sets

- The expression (**u**, **u**) = $||\mathbf{u}||^2$ is called the ***square norm***. Thus we can define the square norm of a function as

$$\|\phi_n(x)\|^2 = \int_a^b \phi_n^2 dx, \quad \|\phi_n(x)\| = \sqrt{\int_a^b \phi_n^2(x) dx} \qquad (3)$$

If $\{\phi_n(x)\}$ is an orthogonal set on [*a*, *b*] with the property that $||\phi_n(x)|| = 1$ for all *n*, then it is called an ***orthonormal set*** on [*a, b*].

Example 1

Show that the set $\{1, \cos x, \cos 2x, \ldots\}$ is orthogonal on $[-\pi, \pi]$.

Solution

Let $\phi_0(x) = 1$, $\phi_n(x) = \cos nx$, we show that

$$(\phi_0, \phi_n) = \int_{-\pi}^{\pi} \phi_0(x)\phi_n(x)dx = \int_{-\pi}^{\pi} \cos nx dx$$

$$= \frac{1}{n}\sin nx\Big|_{-\pi}^{\pi} = 0, for\ n \neq 0$$

Example 1 (2)

and

$$(\phi_m,\phi_n)=\int_{-\pi}^{\pi}\phi_m(x)\phi_n(x)dx=\int_{-\pi}^{\pi}\cos mx\cdot\cos nx dx$$

$$=\frac{1}{2}\int_{-\pi}^{\pi}[\cos(m+n)x+\cos(m-n)x]dx$$

$$=\frac{1}{2}\left[\frac{\sin(m+n)x}{m+n}+\frac{\sin(m-n)x}{m-n}\right]\Bigg|_{-\pi}^{\pi}=0, m\neq n$$

Example 2

Find the norms of each functions in Example 1.

Solution

$$\phi_0 = 1, \|\phi_0\|^2 = \int_{-\pi}^{\pi} dx = 2\pi \Rightarrow \|\phi_0\| = \sqrt{2\pi}$$

$$\phi_n = \cos nx,$$

$$\| \phi_n \|^2 = \int_{-\pi}^{\pi} \cos^2 nx dx = \frac{1}{2} \int_{-\pi}^{\pi} (1 + \cos 2nx) dx = \pi$$

$$\| \phi_n \| = \sqrt{\pi}, n > 0$$

Vector Analogy

- Recalling from the vectors in 3-space that

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3, \tag{4}$$

we have

$$\mathbf{u} = \frac{(\mathbf{u}, \mathbf{v}_1)}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{(\mathbf{u}, \mathbf{v}_2)}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \frac{(\mathbf{u}, \mathbf{v}_3)}{\|\mathbf{v}_3\|^2}\mathbf{v}_3 = \sum_{n=1}^{3}\frac{(\mathbf{u}, \mathbf{v}_n)}{\|\mathbf{v}_n\|^2}\mathbf{v}_n \tag{5}$$

Thus we can make an analogy between vectors and functions.

Orthogonal Series Expansion

- Suppose $\{\phi_n(x)\}$ is an orthogonal set on $[a, b]$. If $f(x)$ is defined on $[a, b]$, we first write as

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) + \cdots ? \quad (6)$$

Then $\int_a^b f(x)\phi_m(x)\,dx$

$$= c_0\int_a^b \phi_0(x)\phi_m(x)\,dx + c_1\int_a^b \phi_1(x)\phi_m(x)\,dx + \cdots$$

$$+ c_n\int_a^b \phi_n(x)\phi_m(x)\,dx + \cdots$$

$$= c_0(\phi_0, \phi_m) + c_1(\phi_1, \phi_m) + \cdots + c_n(\phi_n, \phi_m) + \cdots$$

- Since $\{\phi_n(x)\}$ is an orthogonal set on $[a, b]$, each term on the right-hand side is zero except $m = n$. In this case we have

$$\int_a^b f(x)\phi_n(x)dx = c_n(\phi_n, \phi_n) = c_n \int_a^b \phi_n^{\,2}(x)dx$$

$$c_n = \frac{\int_a^b f(x)\phi_n(x)dx}{\int_a^b \phi_n^{\,2}(x)dx}, \quad n = 0,1,2,\ldots$$

In other words,

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x), \tag{7}$$

$$c_n = \frac{\int_a^b f(x)\phi_n(x)\,dx}{\|\phi_n(x)\|^2} \tag{8}$$

Then (7) becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{(f, \phi_n)}{\|\phi_n(x)\|^2} \phi_n(x) \tag{9}$$

DEFINITION 12.4 **Orthogonal Set/Weight Function**

A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is said to be orthogonal with respect to a weight function $w(x)$ on $[a, b]$, if

$$\int_a^b w(x)\phi_m(x)\phi_n(x)dx = 0, m \neq n$$

- Under the condition of the above definition, we have

$$c_n = \frac{\int_a^b f(x)w(x)\phi_n(x)\, dx}{\| \phi_n(x) \|^2} \tag{10}$$

$$\| \phi_n(x) \|^2 = \int_a^b w(x)\phi_n^2(x)\, dx \tag{11}$$

Complete Sets

- An orthogonal set is ***complete*** if the only continuous function orthogonal to each member of the set is the ***zero function***.
  - If f is orthogonal to every ϕ_n then $c_n = 0$ for all n

12.2 Fourier Series

- **Trigonometric Series**
  We can show that the set

$$\left\{1,\ \cos\frac{\pi}{p}x,\ \cos\frac{2\pi}{p}x,\ \cos\frac{3\pi}{p}x, \ldots,\ \sin\frac{\pi}{p}x,\ \sin\frac{2\pi}{p}x,\ \sin\frac{3\pi}{p}x, \ldots\right\} \quad (1)$$

is **orthogonal on [−*p*, *p*]**. Thus a function *f* defined on [−*p*, *p*] can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right) \qquad (2)$$

- Now we calculate the coefficients.

$$\int_{-p}^{p} f(x)dx = \frac{a_0}{2}\int_{-p}^{p} dx + \sum_{n=1}^{\infty}\left(a_n \int_{-p}^{p} \cos\frac{n\pi}{p} x dx + b_n \int_{-p}^{p} \sin\frac{n\pi}{p} x dx \right) \quad (3)$$

- Since $cos(n\pi x/p)$ and $sin(n\pi x/p)$ are orthogonal to 1 on this interval, then (3) becomes

$$\int_{-p}^{p} f(x)dx = \frac{a_0}{2}\int_{-p}^{p} dx = \frac{a_0}{2} x\Big|_{-p}^{p} = pa_0$$

- Thus we have

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx \quad (4)$$

- In addition,

$$\int_{-p}^{p} f(x)\cos\frac{m\pi}{p}x\,dx$$

$$= \frac{a_0}{2}\int_{-p}^{p}\cos\frac{m\pi}{p}x\,dx + \qquad (5)$$

$$\sum_{n=1}^{\infty}\left(a_n\int_{-p}^{p}\cos\frac{m\pi}{p}x\cos\frac{n\pi}{p}x\;dx + b_n\int_{-p}^{p}\cos\frac{m\pi}{p}x\sin\frac{n\pi}{p}x\;dx\right)$$

by orthogonality we have

$$\int_{-p}^{p}\cos\frac{m\pi}{p}x\,dx = 0, m > 0$$

$$\int_{-p}^{p}\cos\frac{m\pi}{p}x\sin\frac{n\pi}{p}x\,dx = 0$$

- and

$$\int_{-p}^{p} \cos\frac{m\pi}{p}x \cos\frac{n\pi}{p}x dx = \begin{cases} 0, & m \neq n \\ p, & m = n \end{cases}$$

Thus (5) reduces to

$$\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x dx = a_n p$$

and so

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\, dx \qquad (6)$$

- Finally, if we multiply (2) by sin($m\pi x/p$) and use

$$\int_{-p}^{p} \sin \frac{m\pi}{p} x dx = 0, \quad m > 0$$

and

$$\int_{-p}^{p} \sin \frac{m\pi}{p} x \sin \frac{n\pi}{p} x dx = 0$$

$$\int_{-p}^{p} \sin \frac{m\pi}{p} x \sin \frac{n\pi}{p} x dx = \begin{cases} 0, & m \neq n \\ p, & m = n \end{cases}$$

- we find that

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi}{p} x \, dx \qquad (7)$$

DEFINITION 12.5

Fourier Series

The **Fourier series** of a function *f* defined on the interval $(-p, p)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right) \tag{8}$$

where

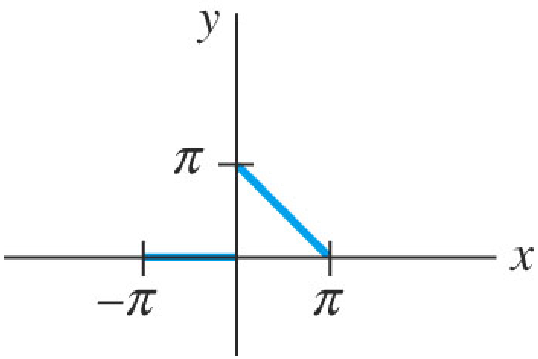
$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx \tag{9}$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx \tag{10}$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx \tag{11}$$

Example 1

Expand $$f(x)=\begin{cases}0, & -\pi<x<0\\ \pi-x, & 0\le x<\pi\end{cases}$$



in a Fourier series.

Solution

Here, $p = \pi$.

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}0dx+\int_{0}^{\pi}(\pi-x)\,dx\right]$$

$$=\frac{1}{\pi}\left[\pi x-\frac{x^2}{2}\right]_0^{\pi}=\frac{\pi}{2}$$

Example 1 (2)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi}(\pi - x)\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[(\pi - x)\frac{\sin nx}{n}\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\sin nx\, dx\right]$$

$$= -\frac{1}{n\pi}\frac{\cos nx}{n}\Big|_0^{\pi}$$ $\leftarrow \cos n\pi = (-1)^n$

$$= \frac{-\cos n\pi + 1}{n^2\pi} = \frac{1-(-1)^n}{n^2\pi}$$

Example 1 (3)

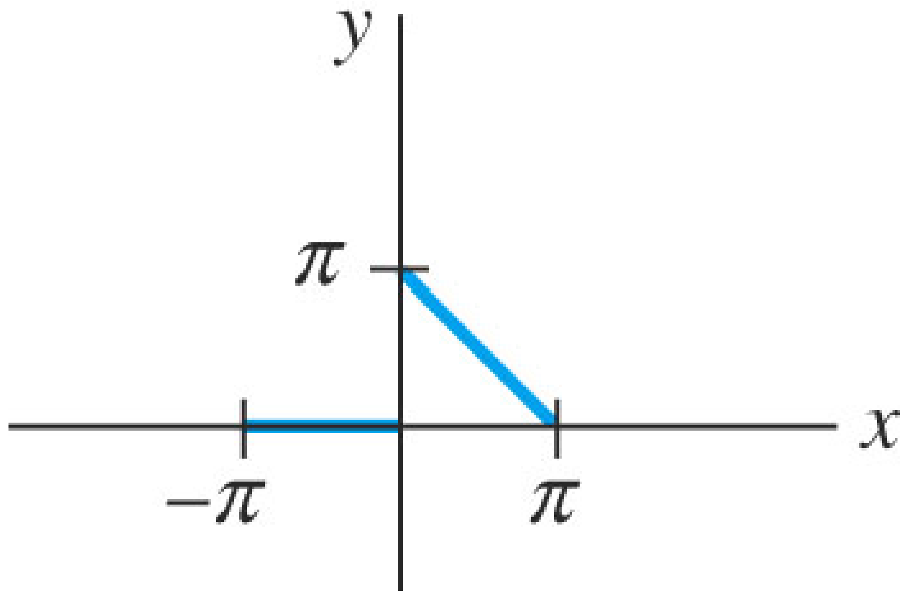
From (11) we have

$$b_n = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin nx dx = \frac{1}{n}$$

Therefore

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right\} \qquad (13)$$

Fig 12.1



Dirichlet Conditions

- Fourier series converges: If f is a real valued function over [-∞, ∞] and if it satisfies the Dirichlet conditions:
  - f is bounded over the closed subinterval [a,b]
  - f has a finite number of extreme values in [a,b]
  - f has only a finite number of (jump) discontinuities in [a,b]
  - f is periodic

THEOREM 12.1 Criterion for Convergence

Let f and f' be piecewise continuous on the interval $(-p, p)$; that is, let f and f' be continuous **except** at a finite number of points in the interval and have only finite discontinuities at these points. Then the Fourier series of f on the interval converge to $f(x)$ at a point of continuity. At a point of discontinuity, the Fourier series converges to the average

$$\frac{f(x^+) + f(x^-)}{2}$$

where $f(x^+)$ and $f(x^-)$ denote the limit of f at x from the right and from the left, respectively.

Example 2

- Referring to Example 1, function f is continuous on $(-\pi, \pi)$ except at $x = 0$. Thus the series (13) will converge to

$$\frac{f(0+) + f(0-)}{2} = \frac{\pi + 0}{2} = \frac{\pi}{2}$$

at $x = 0$.

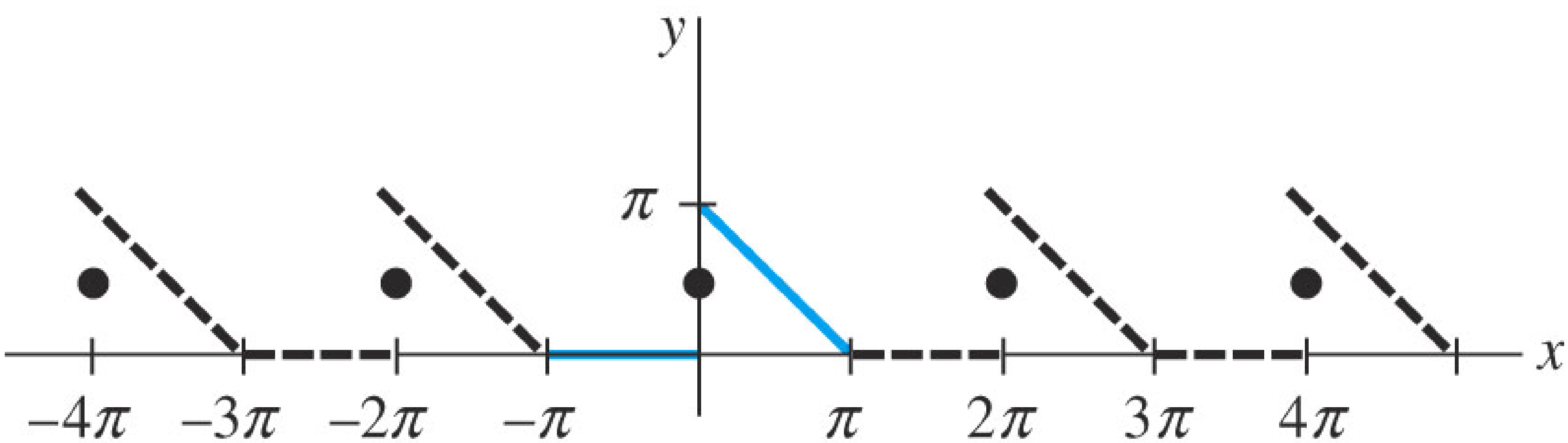
Periodic Extension

- A real-valued function *f* is said to be periodic with period T, if *f(x + T) = f(x)*
  - For example, *sin(x+4π)=sin(x)*, so *T = 4π* works
- The **smallest value of T** for which *f(x+ T)= f (x)* holds is called the **fundamental period** of *f*.
  - For example, **the fundamental period of *f(x) = sin x* is T = 2π**
- **Common period of all the functions in the following set is *2p [Fundamental period of each is 2p/n, n ≥ 1]***

$$\left\{1, \cos\frac{\pi}{p}x, \cos\frac{2\pi}{p}x, \cos\frac{3\pi}{p}x, \ldots, \sin\frac{\pi}{p}x, \sin\frac{2\pi}{p}x, \sin\frac{3\pi}{p}x, \ldots\right\}$$

Periodic Extension

- a Fourier series (fundamental period = 2p) not only represents the function on the interval (-p, p) but also gives the periodic extension off outside this interval
- the periodic extension of the function *f* in Example 1:



Thus the discontinuity at $x = 0, \pm 2\pi, \pm 4\pi$, ...will converge to

$$\frac{f(0+) + f(0-)}{2} = \frac{\pi}{2}$$

and at $x = \pm\pi, \pm 3\pi$, ... will converge to

$$\frac{f(\pi+) + f(\pi-)}{2} = 0$$

Sequence of Partial Sums

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left\{ \frac{1-(-1)^n}{n^2 \pi} \cos nx + \frac{1}{n} \sin nx \right\}$$

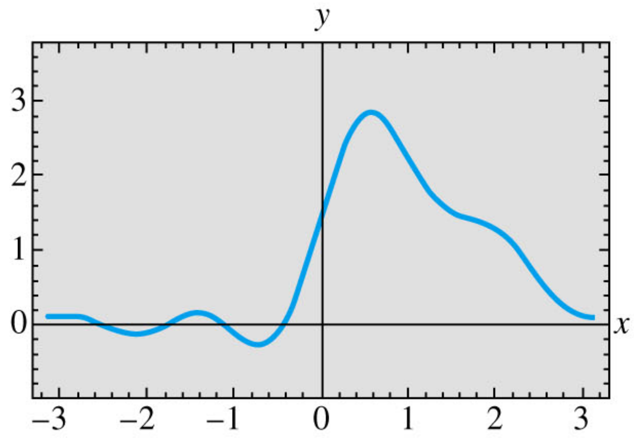
- we write the partial sums as

$$S_1 = \frac{\pi}{4}, \;\; S_2 = \frac{\pi}{4} + \frac{2}{\pi} \cos x + \sin x,$$

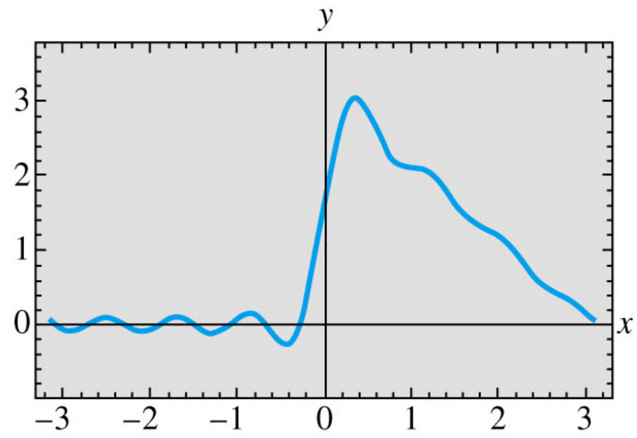
$$S_3 = \frac{\pi}{4} + \frac{2}{\pi} \cos x + \sin x + \frac{1}{2} \sin 2x$$

See Fig 12.3.

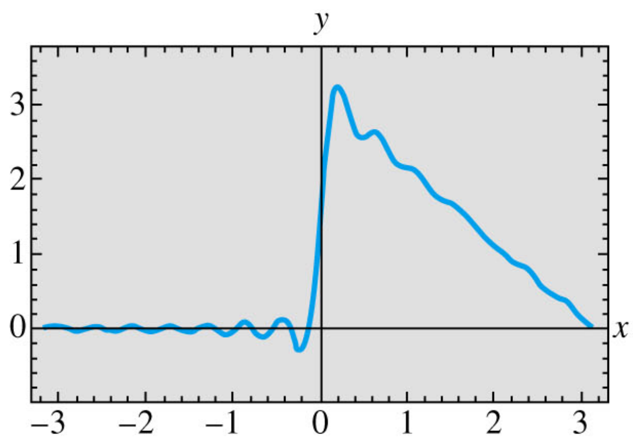
Fig 12.3



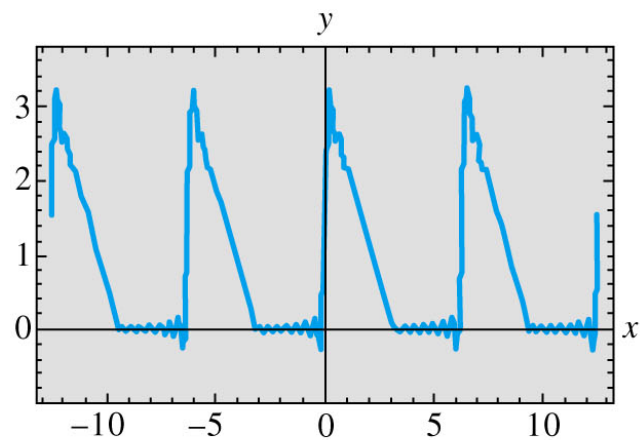
(a) $S_5(x)$ on $(-\pi, \pi)$



(b) $S_8(x)$ on $(-\pi, \pi)$



(c) $S_{15}(x)$ on $(-\pi, \pi)$



(d) $S_{15}(x)$ on $(-4\pi, 4\pi)$

12.3 Fourier Cosine and Sine Series

- **Even and Odd Functions**
  - even if $f(-x) = f(x)$
  - odd if $f(-x) = -f(x)$

Fig 12.4 Even function

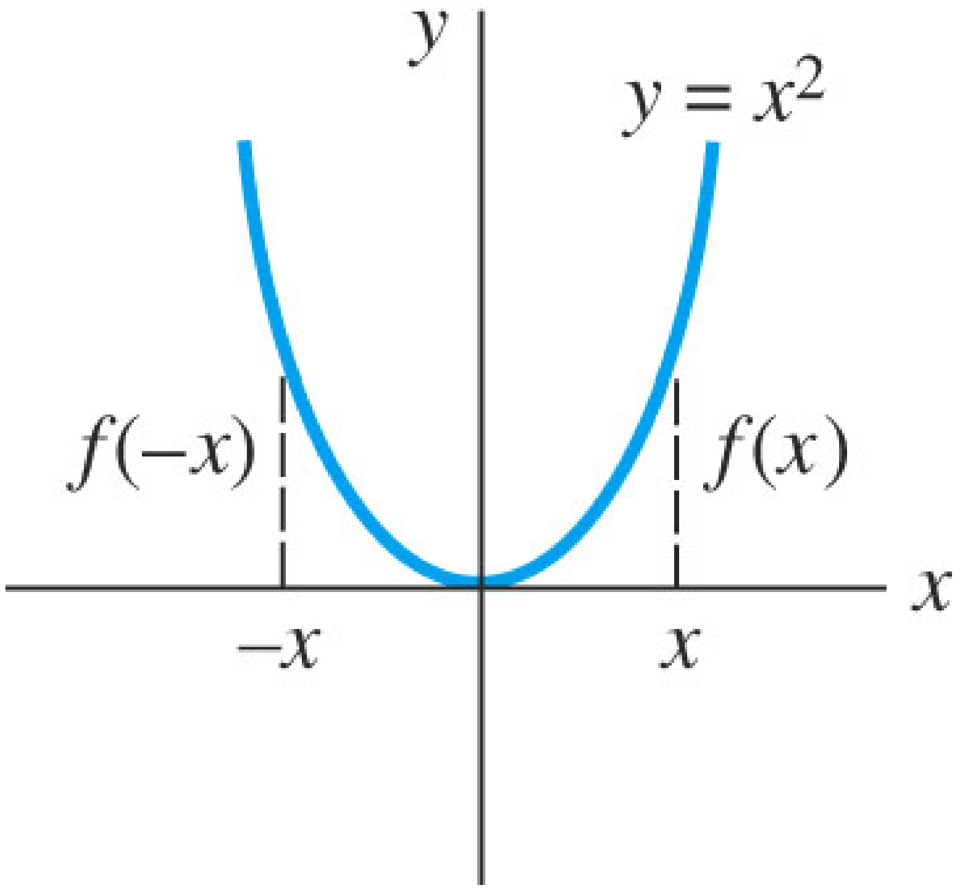
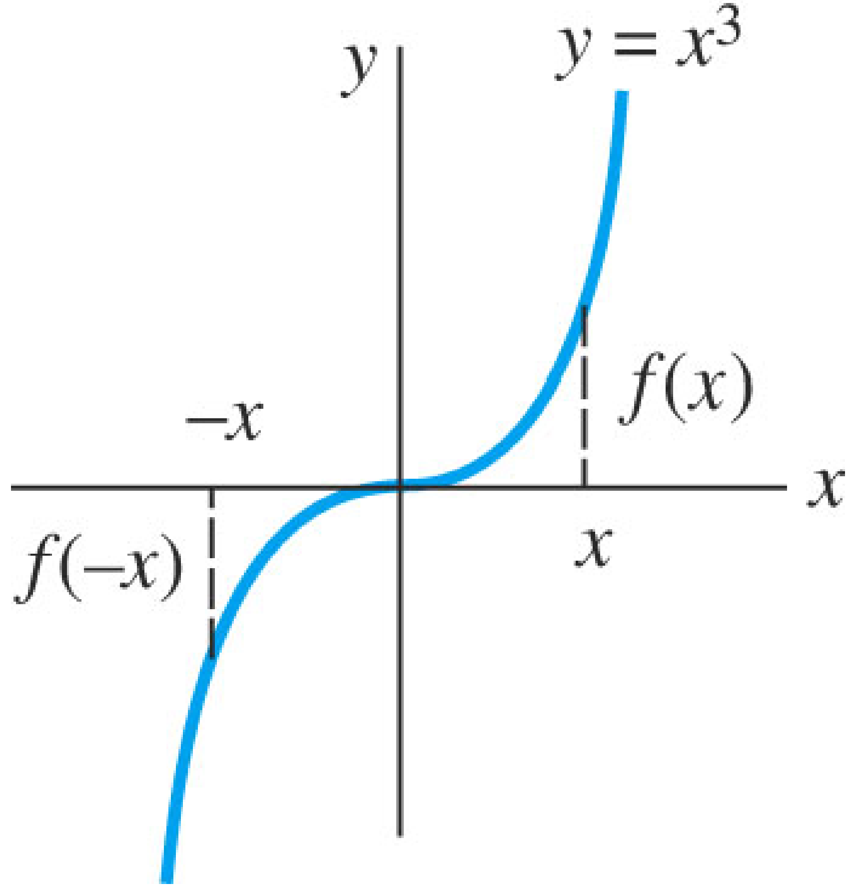


Fig 12.5 Odd function



THEOREM 12.2 **Properties of Even/Odd Functions**

(a) The product of two even functions is even.

(b) The product of two odd functions is even.

(c) The product of an even function and an odd function is odd.

(d) The sum (difference) of two even functions is even.

(e) The sum (difference) of two odd functions is odd.

(f) If f is even then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

(g) If f is odd then $\int_{-a}^{a} f(x)dx = 0$

Cosine and Sine Series

- If f is even on $(-p, p)$ then

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx = \frac{2}{p}\int_{0}^{p} f(x)dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}xdx = \frac{2}{p}\int_{0}^{p} f(x)\cos\frac{n\pi}{p}xdx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}xdx = 0$$

Similarly, if f is odd on $(-p, p)$ then

$$a_n = 0, n = 0,1,2,\ldots \qquad b_n = \frac{2}{p}\int_{0}^{p} f(x)\sin\frac{n\pi}{p}xdx$$

DEFINITION 12.6

Fourier Cosine and Sine Series

(i) The Fourier series of an **even** function f on the interval $(-p, p)$ is the **cosine series**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x \qquad (1)$$

where

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx \qquad (2)$$

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx \qquad (3)$$

(continued)

DEFINITION 12.6 **Fourier Cosine and Sine Series**

(ii) The Fourier series of an **odd** function f on the interval $(-p, p)$ is the **sine series**

$$f(x) = \sum_{i=1}^{\infty} b_n \sin \frac{n\pi}{p} x \tag{4}$$

where

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x \, dx \tag{5}$$

Example 1

Expand $f(x) = x$, $-2 < x < 2$ in a Fourier series.

Solution

Inspection of Fig 12.6, we find it is an odd function on $(-2, 2)$ and $p = 2$.

$$b_n = \int_0^2 x \sin\frac{n\pi}{2}x\,dx = \frac{4(-1)^{n+1}}{n\pi}$$

Thus

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{2}x \qquad (6)$$

Fig 12.7 is the periodic extension of the function in Example 1.

Fig 12.6

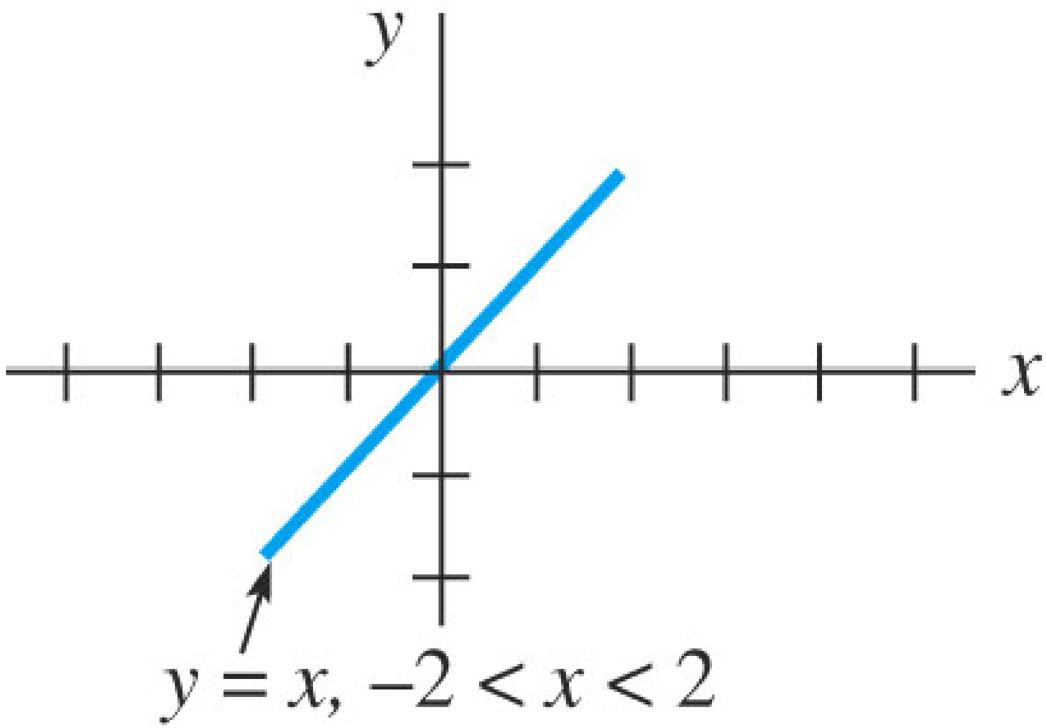
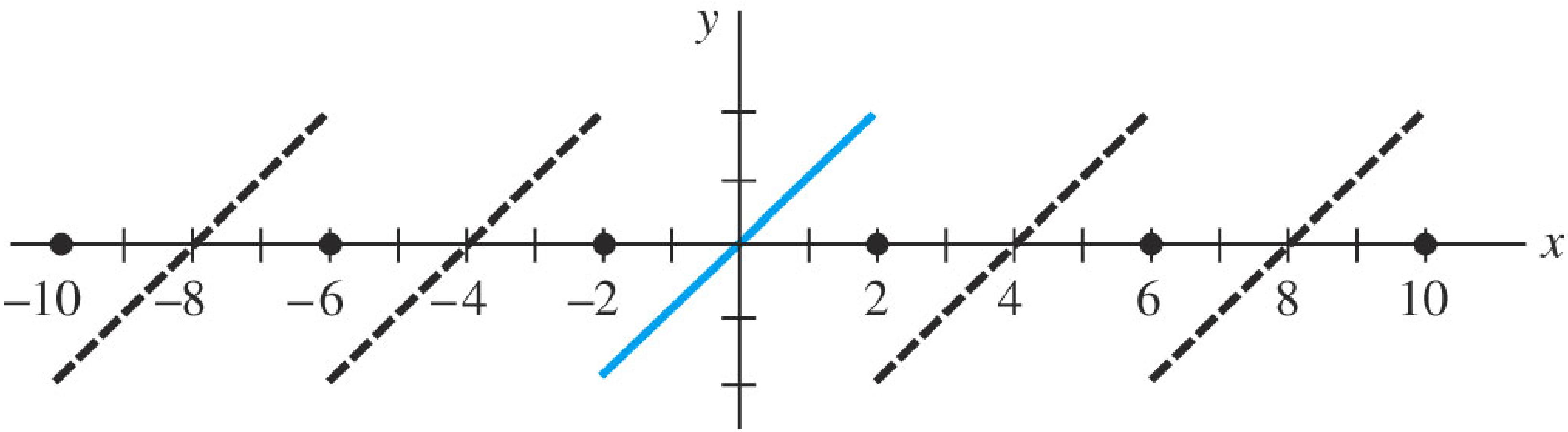


Fig 12.7



Example 2

- The function $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 \le x < \pi \end{cases}$

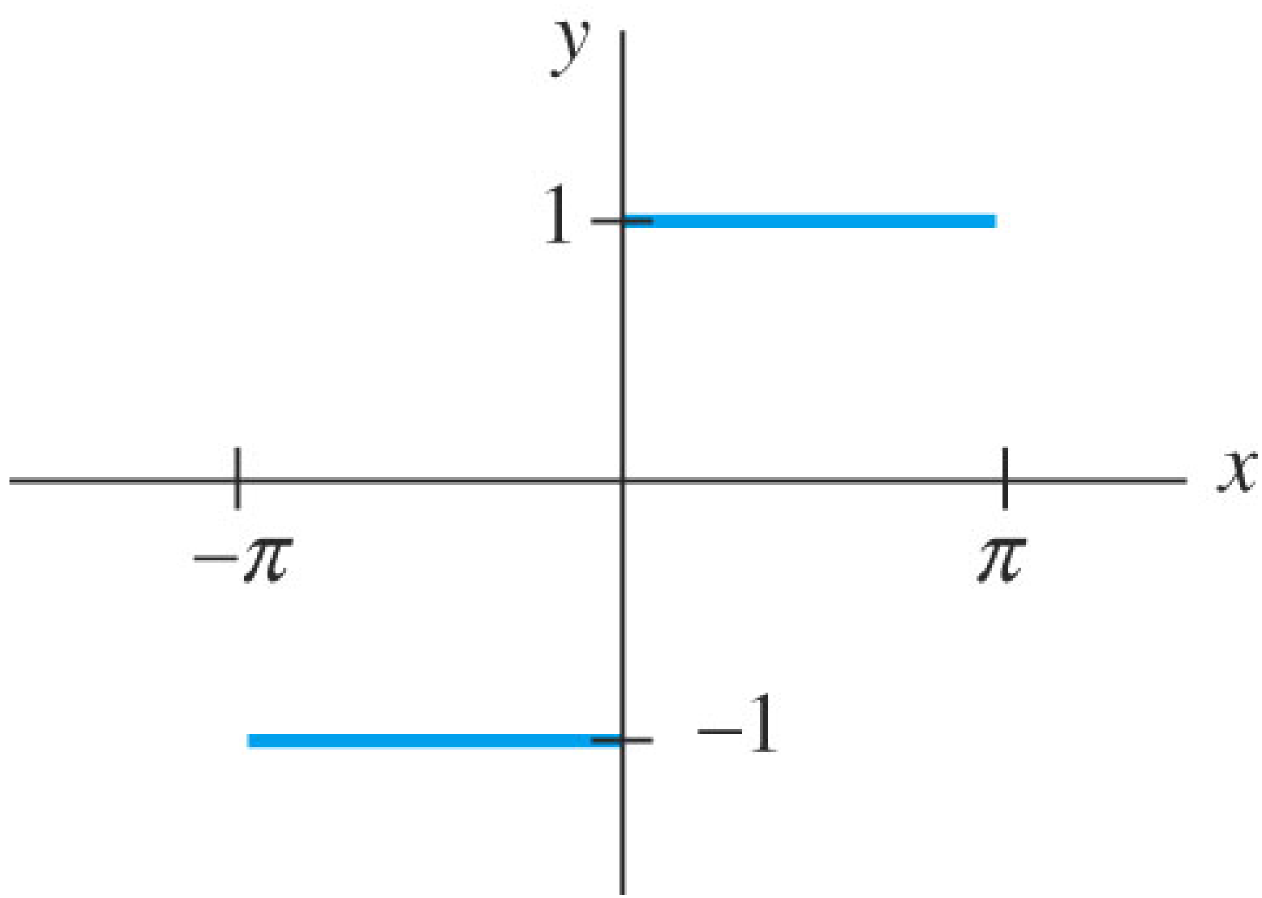
  shown in Fig 12.8 is odd on (−π, π) with $p = \pi$. From (5),

$$b_n = \frac{2}{\pi}\int_0^{\pi}(1)\sin nx\,dx = \frac{2}{\pi}\frac{1-(-1)^n}{n}$$

and so

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx \qquad (7)$$

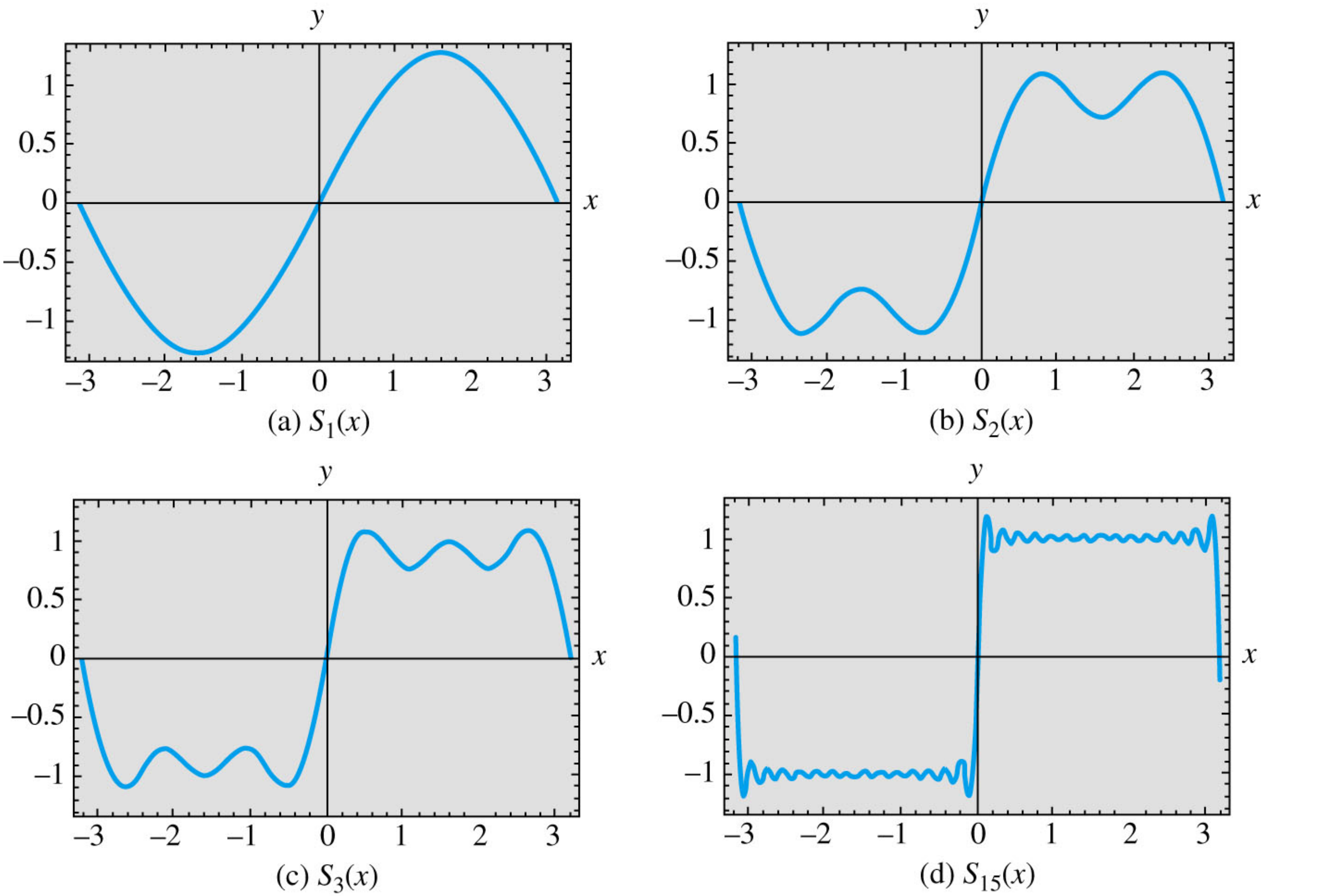
Fig 12.8



Gibbs Phenomenon

- Fig 12.9 shows the partial sums of (7).
  We can see there are pronounced spikes near the discontinuities. This overshooting of S_N does not smooth out but remains fairly constant even when N is large. This is so-called **Gibbs phenomenon**.

Fig 12.9



Half-Range Expansions

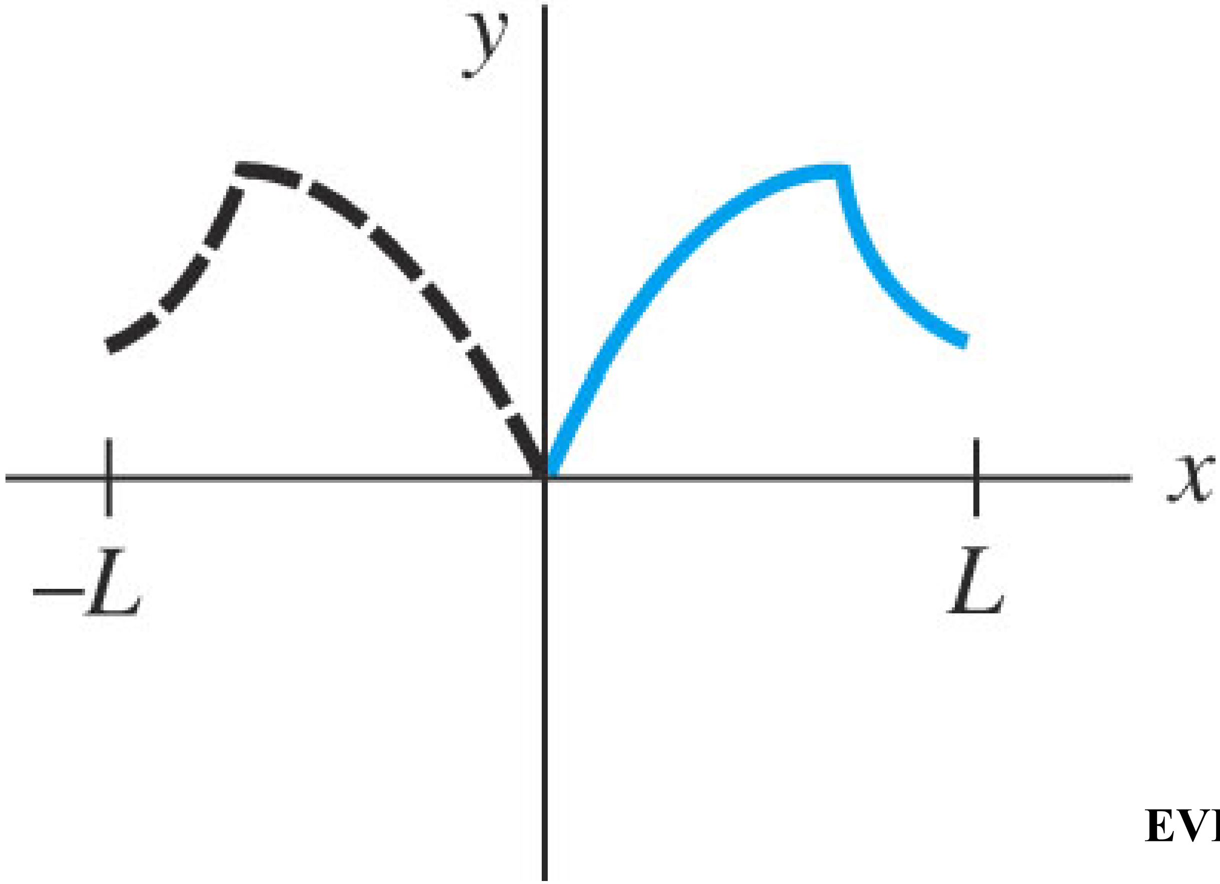
- If a function f is defined only on $0 < x < L$, we can make arbitrary definition of the function on $-L < x < 0$.
- If $y = f(x)$ is defined on $0 < x < L$,

(i) reflect the graph about the y-axis onto $-L < x < 0$; the function is now **even**. See Fig 12.10.

(ii) reflect the graph through the origin onto $-L < x < 0$; the function is now **odd**. See Fig 12.11.

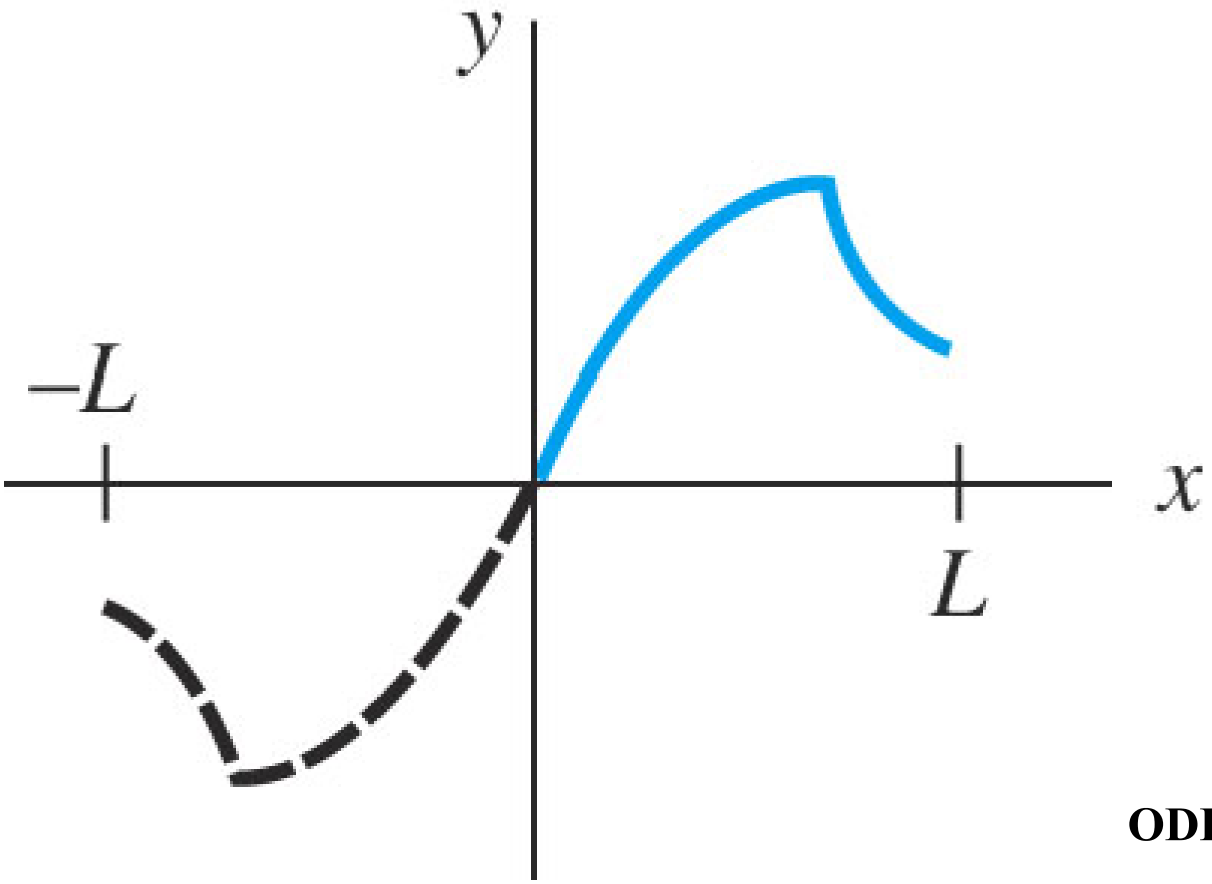
(iii) define f on $-L < x < 0$ by $\boldsymbol{f(x) = f(x + L)}$. See Fig 12.12.

Fig 12.10



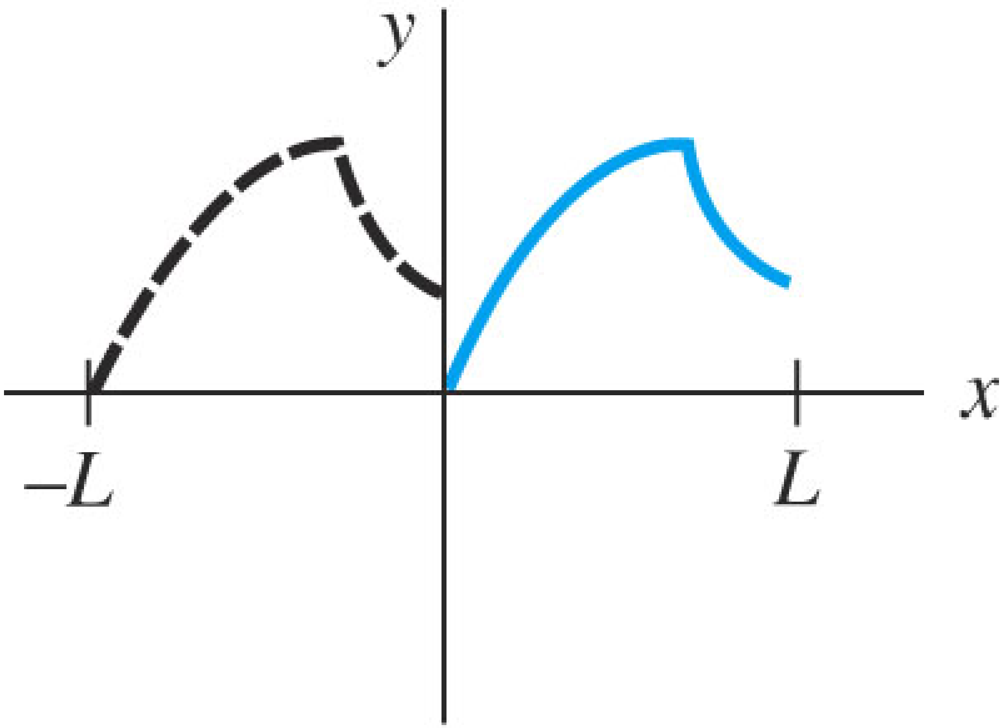
EVEN

Fig 12.11



ODD

Fig 12.12



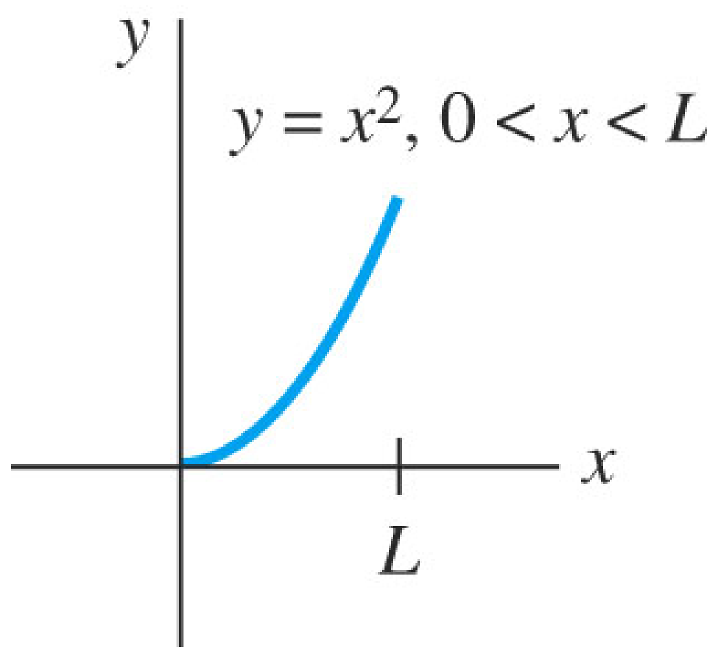
$$f(x) = f(x + L)$$

Example 3

Expand $f(x) = x^2$, $0 < x < L$, (a) in a cosine series, (b) in a sine series (c) in a Fourier series.

Solution

The graph is



Example 3 (2)

(a)

$$a_0 = \frac{2}{L}\int_0^L x^2 dx = \frac{2}{3}L^2,$$

$$a_n = \frac{2}{L}\int_0^L x^2 \cos\frac{n\pi}{L}x\, dx = \frac{4L^2(-1)^n}{n^2\pi^2}$$

Then

$$f(x) = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi}{L}x \qquad (8)$$

Example 3 (3)

(b)

$$b_n = \frac{2}{L}\int_0^L x^2 \sin\frac{n\pi}{L}x\,dx = \frac{2L^2(-1)^{n+1}}{n\pi} + \frac{4L^2}{n^3\pi^3}[(-1)^n - 1]$$

Hence

$$f(x) = \frac{2L^2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1}}{n} + \frac{2}{n^3\pi^2}[(-1)^n - 1]\right\}\sin\frac{n\pi}{L}x \qquad (9)$$

Example 3 (4)

(c) With *p* = *L*/2, *nπ*/*p* = 2*nπ*/*L*, we have

$$a_0 = \frac{2}{L}\int_0^L x^2 dx = \frac{2}{3}L^2, \quad a_n = \frac{2}{L}\int_0^L x^2 \cos\frac{2n\pi}{L}x dx = \frac{L^2}{n^2\pi^2}$$

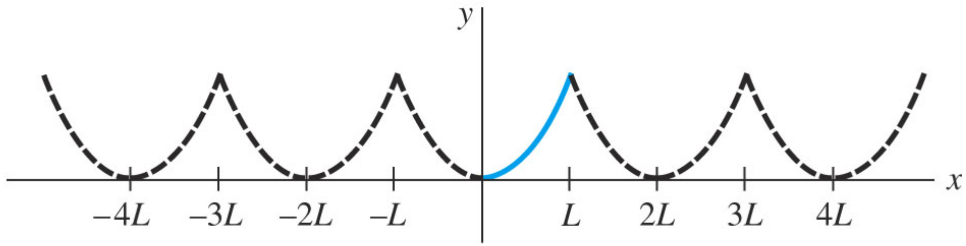
$$b_n = \frac{2}{L}\int_0^L x^2 \sin\frac{2n\pi}{L}x dx = \frac{-L^2}{n\pi}$$

Therefore

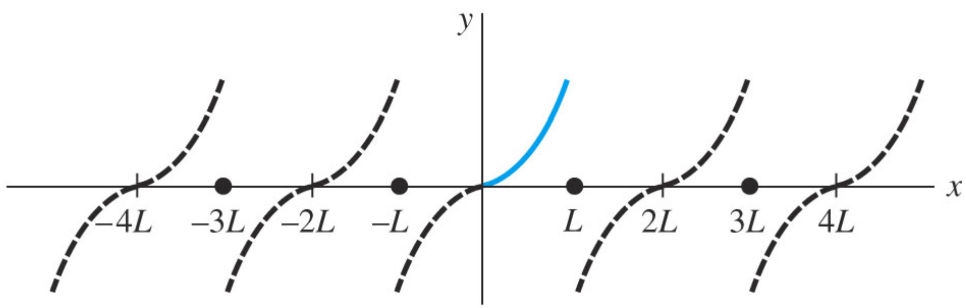
$$f(x) = \frac{L^2}{3} + \frac{L^2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{n^2\pi}\cos\frac{2n\pi}{L}x - \frac{1}{n}\sin\frac{2n\pi}{L}x\right\} \quad (10)$$

The graph of these periodic extension are shown in Fig 12.14.

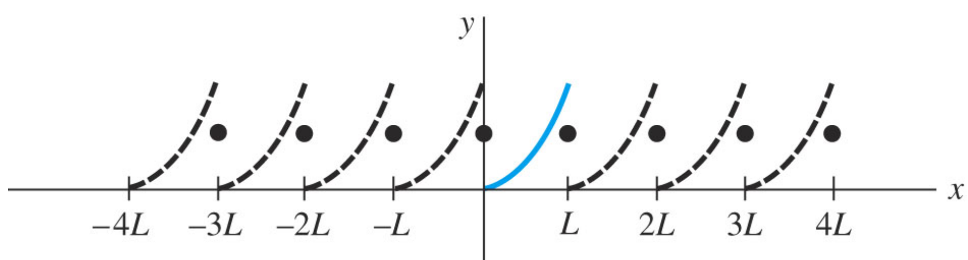
Fig 12.14



(a) Cosine series



(b) Sine series



(c) Fourier series

12.4 Complex Fourier Series

- **Euler's formula**

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos x - i \sin x \qquad (1)$$

Complex Fourier Series

- From (1), we have

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i} \tag{2}$$

Using (2) to replace cos($n\pi x/p$) and sin($n\pi x/p$), then

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \frac{e^{in\pi x/p} + e^{-in\pi x/p}}{2} + b_n \frac{e^{in\pi x/p} - e^{-in\pi x/p}}{2i}\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{2}(a_n - ib_n)e^{in\pi x/p} + \frac{1}{2}(a_n + ib_n)e^{-in\pi x/p}\right]$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/p} + \sum_{n=1}^{\infty} c_{-n} e^{-in\pi x/p} \tag{3}$$

where $c_0 = a_0/2$, $c_n = (a_n - ib_n)/2$, $c_{-n} = (a_n + ib_n)/2$.
When the function f is real, c_n and c_{-n} are complex conjugates.
We have

$$c_0 = \frac{1}{2} \cdot \frac{1}{p}\int_{-p}^{p} f(x)\,dx \tag{4}$$

$$
\begin{aligned}
c_n &= \frac{1}{2}(a_n - ib_n) \\
&= \frac{1}{2}\left(\frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx - i\frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx\right) \\
&= \frac{1}{2p}\int_{-p}^{p} f(x)\left[\cos\frac{n\pi}{p}x - i\sin\frac{n\pi}{p}x\right]dx \\
&= \frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}dx
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
c_{-n} &= \frac{1}{2}(a_n + ib_n) \\
&= \frac{1}{2}\left(\frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx + i\frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx\right) \\
&= \frac{1}{2p}\int_{-p}^{p} f(x)\left[\cos\frac{n\pi}{p}x + i\sin\frac{n\pi}{p}x\right]dx \\
&= \frac{1}{2p}\int_{-p}^{p} f(x)e^{in\pi x/p}\,dx \qquad (6)
\end{aligned}
$$

DEFINITION 12.7

Complex Fourier Series

The ***Complex Fourier Series*** of function f defined on an interval $(-p, p)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p} \tag{7}$$

where

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(x) e^{-in\pi x/p}\, dx, \; n = 0, \pm 1, \pm 2, \ldots \tag{8}$$

- If f satisfies the hypotheses of Theorem 12.1, a complex Fourier series converges to $f(x)$ at a point of continuity and to the average

$$\frac{f(x+)+f(x-)}{2}$$

at a point of discontinuity.

Example 1

Expand $f(x) = e^{-x}$, $-\pi < x < \pi$, in a complex Fourier series.

Solution

with $p = \pi$, (8) gives

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-x} e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-(in+1)x}\,dx$$

$$= \frac{1}{2\pi(in+1)}[e^{-(in+1)\pi} - e^{(in+1)\pi}]$$

Example 1 (2)

Using Euler's formula

$$e^{-(in+1)\pi} = e^{-\pi}(\cos n\pi - i\sin n\pi) = (-1)^n e^{-\pi}$$

$$e^{(in+1)\pi} = e^{\pi}(\cos n\pi + i\sin n\pi) = (-1)^n e^{\pi}$$

Hence

$$c_n = (-1)^n \frac{(e^{\pi} - e^{-\pi})}{2(in+1)\pi} = (-1)^n \frac{\sinh \pi}{\pi} \frac{1-in}{n^2+1} \qquad (9)$$

Example 1 (3)

The complex Fourier series is then

$$f(x) = \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{1 - in}{n^2 + 1} e^{inx} \qquad (10)$$

The series (10) converges to the 2π-periodic extension of *f*.

Fundamental Frequency

- The fundamental period is T = 2p and then p = T/2.
  The Fourier series becomes

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega x + b_n \sin n\omega x)$$

or, (11)

$$\sum_{n=-\infty}^{\infty} c_n e^{in\omega x}$$

where $\omega_0 = \omega = 2\pi/T$ is called the ***fundamental angular frequency***.

Frequency Spectrum

- If f is periodic and has fundamental period T, the plot of the points $(n\omega, |c_n|)$ is called the ***frequency spectrum*** of f.

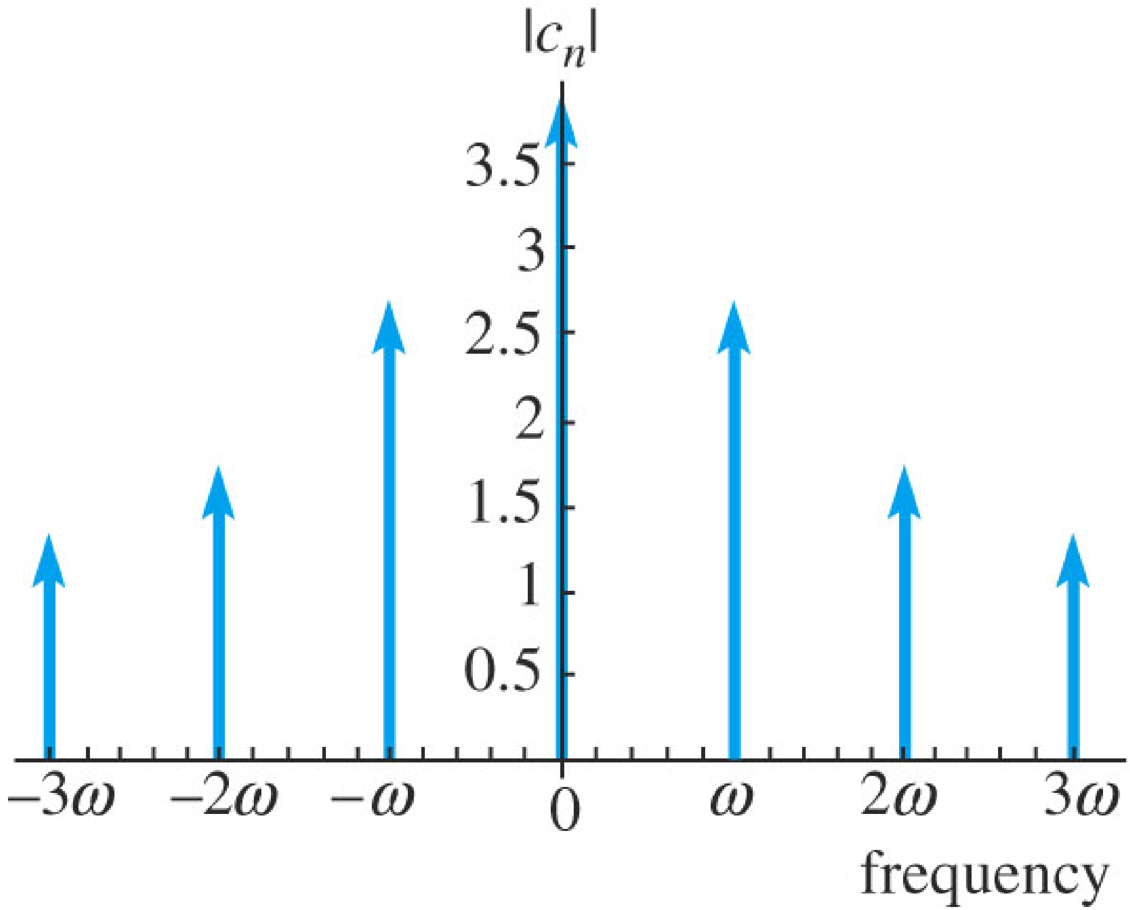
Example 2

- In Example 1, $\omega = 1$, so that $n\omega$ takes on the values 0, ± 1, ± 2, …
  Using $|\alpha + i\beta| = \sqrt{\alpha^2 + \beta^2}$, we see from (9) that

$$|c_n| = \frac{\sinh \pi}{\pi} \frac{1}{\sqrt{n^2 + 1}}$$

  See Fig 12.17.

Fig 12.17

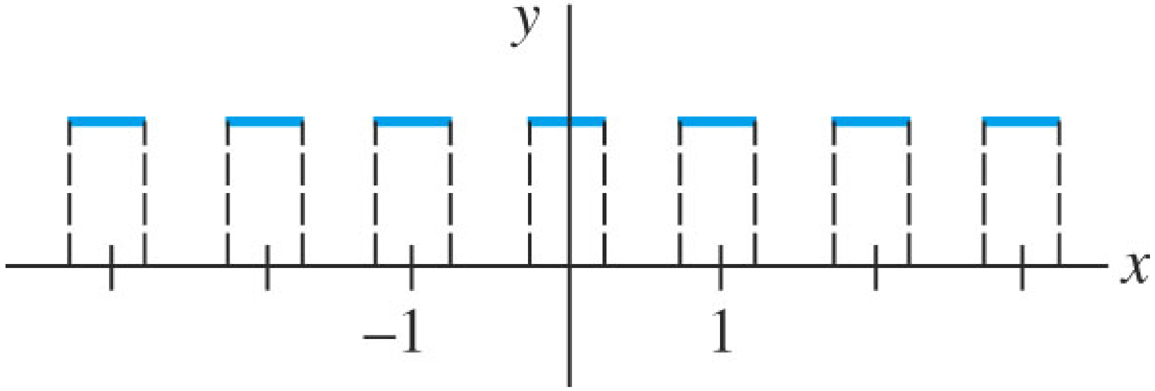


$|c_n|$
3.5
3
2.5
2
1.5
1
0.5
-3ω
-2ω
$-\omega$
0
ω
2ω
3ω
frequency

Example 3

- Find the spectrum of the wave shown in Fig12.18. The wave is the periodic extension of the function *f*:

$$f(x) = \begin{cases} 0, & -\frac{1}{2} < x < -\frac{1}{4} \\ 1, & -\frac{1}{4} < x < \frac{1}{4} \\ 0, & \frac{1}{4} < x < \frac{1}{2} \end{cases}$$



Example 3 (2)

Solution

Here $T = 1 = 2p$ so $p = ½$. Since f is 0 on $(–½, –¼)$ and $(¼, ½)$, (8) becomes

$$c_n = \int_{-1/2}^{1/2} f(x) e^{2in\pi x}\,dx = \int_{-1/4}^{1/4} (1) e^{2in\pi x}\,dx$$

$$= \left.\frac{e^{2in\pi x}}{2in\pi}\right|_{-1/4}^{1/4} = \frac{1}{n\pi}\frac{e^{in\pi/2} - e^{-in\pi/2}}{2i}$$

$$c_n = \frac{1}{n\pi}\sin\frac{n\pi}{2}$$

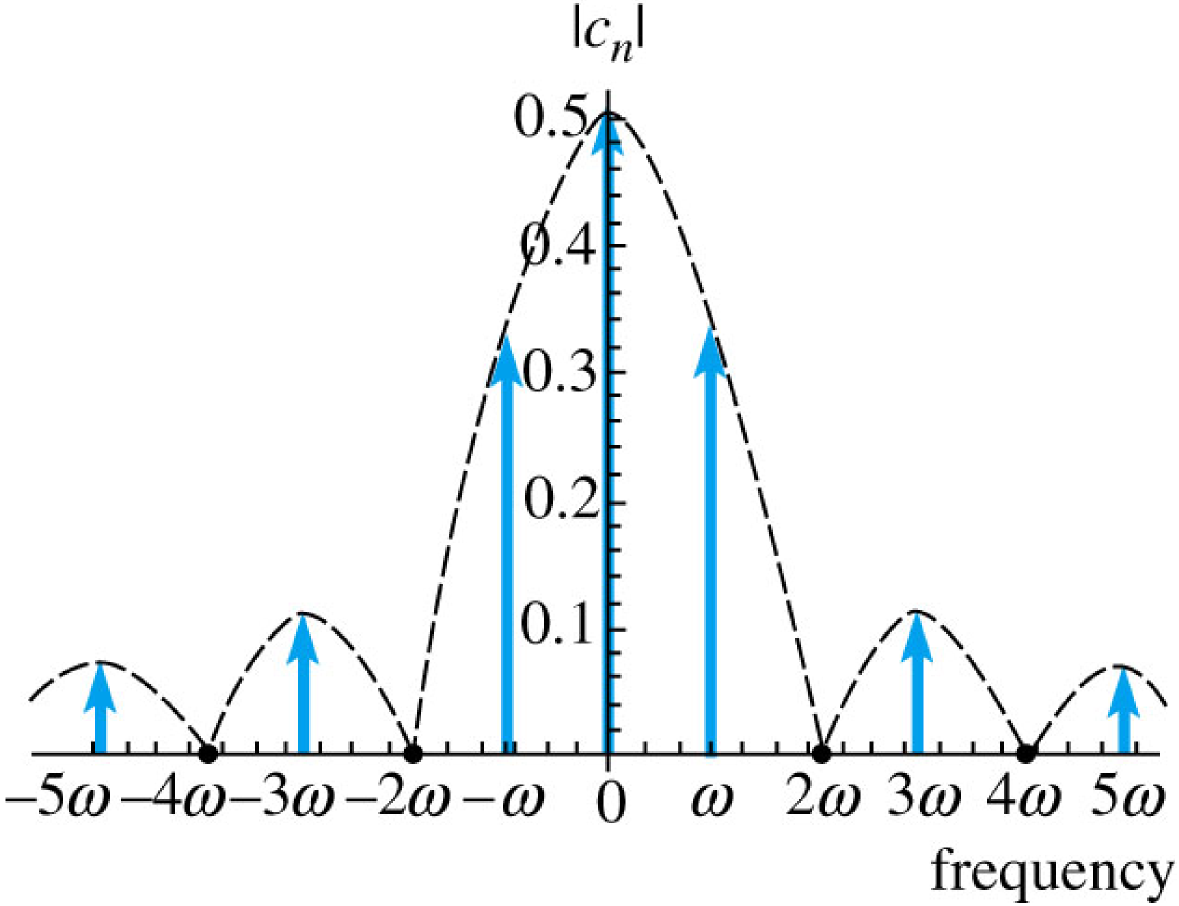
Example 3 (3)

It is easy to check that

$$c_0 = \int_{-1/4}^{1/4} dx = \frac{1}{2}$$

Fig 12.19 shows the frequency spectrum of *f*.

Fig 12.19



$|c_n|$
0.5
0.4
0.3
0.2
0.1
-5ω -4ω -3ω -2ω $-\omega$ 0 ω 2ω 3ω 4ω 5ω
frequency