Solving System of Linear Equations: Gauss Elimination

CSE 2105

Gauss Elimination

- **Solving small numbers of equations**
- **Determinants and Inverse**
- **Naive Gauss Elimination**
- **LU Decomposition**
- **Partial Pivoting and Permutation Matix**

System of Linear Equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots = b_3$$
$$\vdots$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix}$$

$$\mathbf{Ax} = \mathbf{b}$$

Solving System of Linear Equations

Solve Ax=b, where

A is an $n \times n$ square matrix and b is an $n \times 1$ column vector

Can also talk about non-square systems where

A is $m \times n$, b is $m \times 1$, and x is $n \times 1$

Overdetermined: if $m>n$ "more equations than unknowns" *A is tall*

Underdetermined: if $n>m$ "more unknowns than equations" *A is wide*

For every system of linear equations there can be

no solution
exactly one solution
or infinitely many solutions.

Infinitely many solutions?

Let, $A\mathbf{x} = \mathbf{b}$ has **at least two distinct** solutions $\mathbf{x}_1$ and $\mathbf{x}_2$

$A\,\mathbf{x}_1 = A\,\mathbf{x}_2 = \mathbf{b} \Rightarrow A(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0}$

So, $\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{x}_h$ is a solution for $A\mathbf{x} = \mathbf{0}$ (homogeneous system)

Then $\mathbf{x}_1 + c\mathbf{x}_h$ is also a solution of $A\mathbf{x} = \mathbf{b}$ for any scalar c, as

$$A(\mathbf{x}_1 + c\mathbf{x}_h) = A\mathbf{x}_1 + cA\mathbf{x}_h = \mathbf{b}$$

i.e., there are infinitely many solutions

But when will it have infinitely many solutions?

A Invertible => exactly one solution

If A is invertible i.e., A^{-1} exists, then the solution of

$A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x} = A^{-1}\mathbf{b}$

Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two distinct solutions, then

$A\mathbf{x}_1 = A\mathbf{x}_2$

Multiplying both sides with A^{-1} we get,

$\mathbf{x}_1 = \mathbf{x}_2$

Is the inverse of a matrix unique?

Does, $BA = AB = I = CA = AC \Rightarrow B = C$?

What about the converse of the statement?

If a system Ax = b has a unique solution, does that imply A is invertible?

– YES (we will see a proof later on)

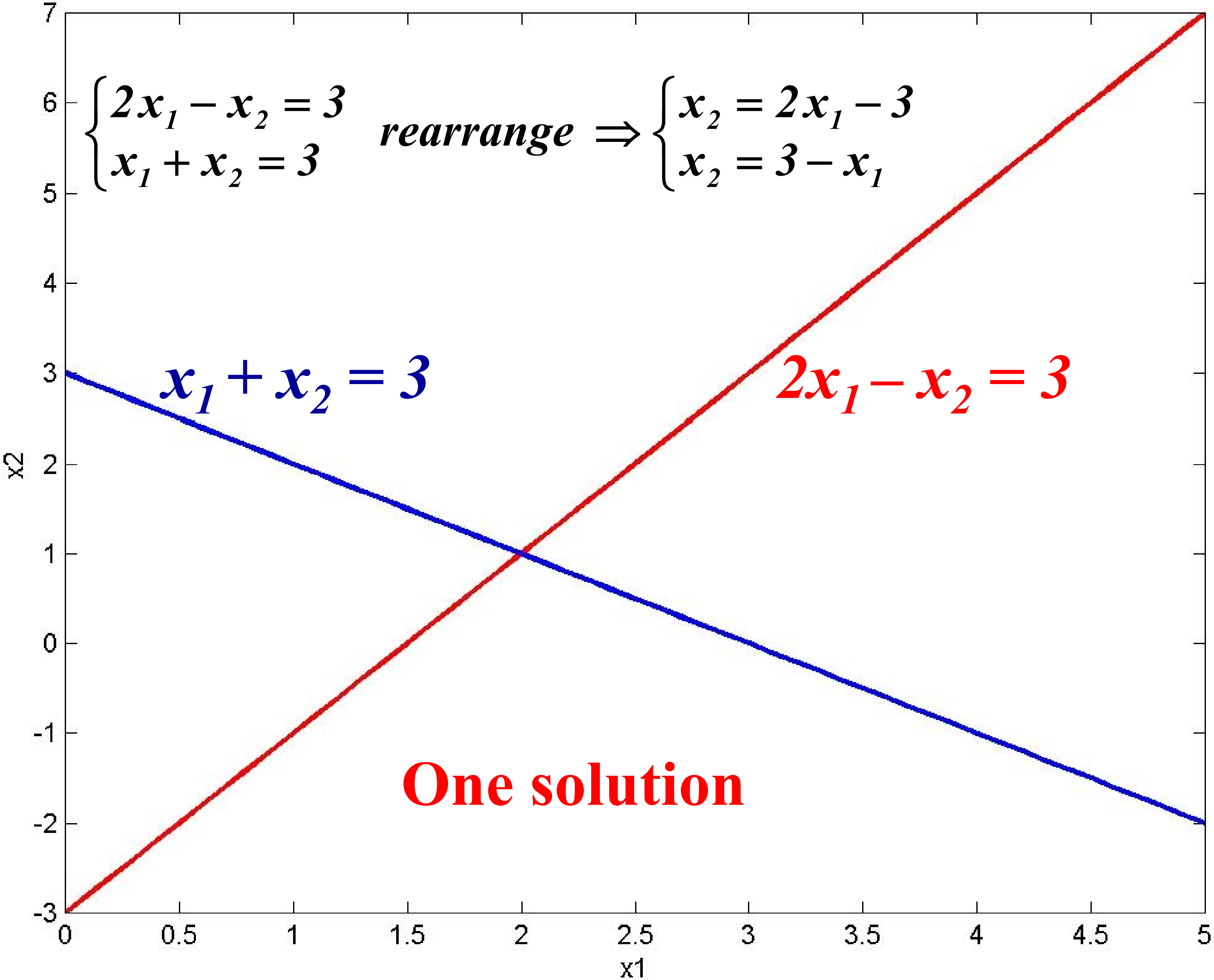
Solving for Small Matrices

$$\begin{cases} a_{11}x_1 & +a_{12}x_2 & +\cdots & +a_{1n}x_n & =b_1 \\ a_{21}x_1 & +a_{22}x_2 & +\cdots & +a_{2n}x_n & =b_2 \\ \vdots & & & & \vdots \\ a_{n1}x_1 & +a_{n2}x_2 & +\cdots & +a_{nn}x_n & =b_n \end{cases}$$

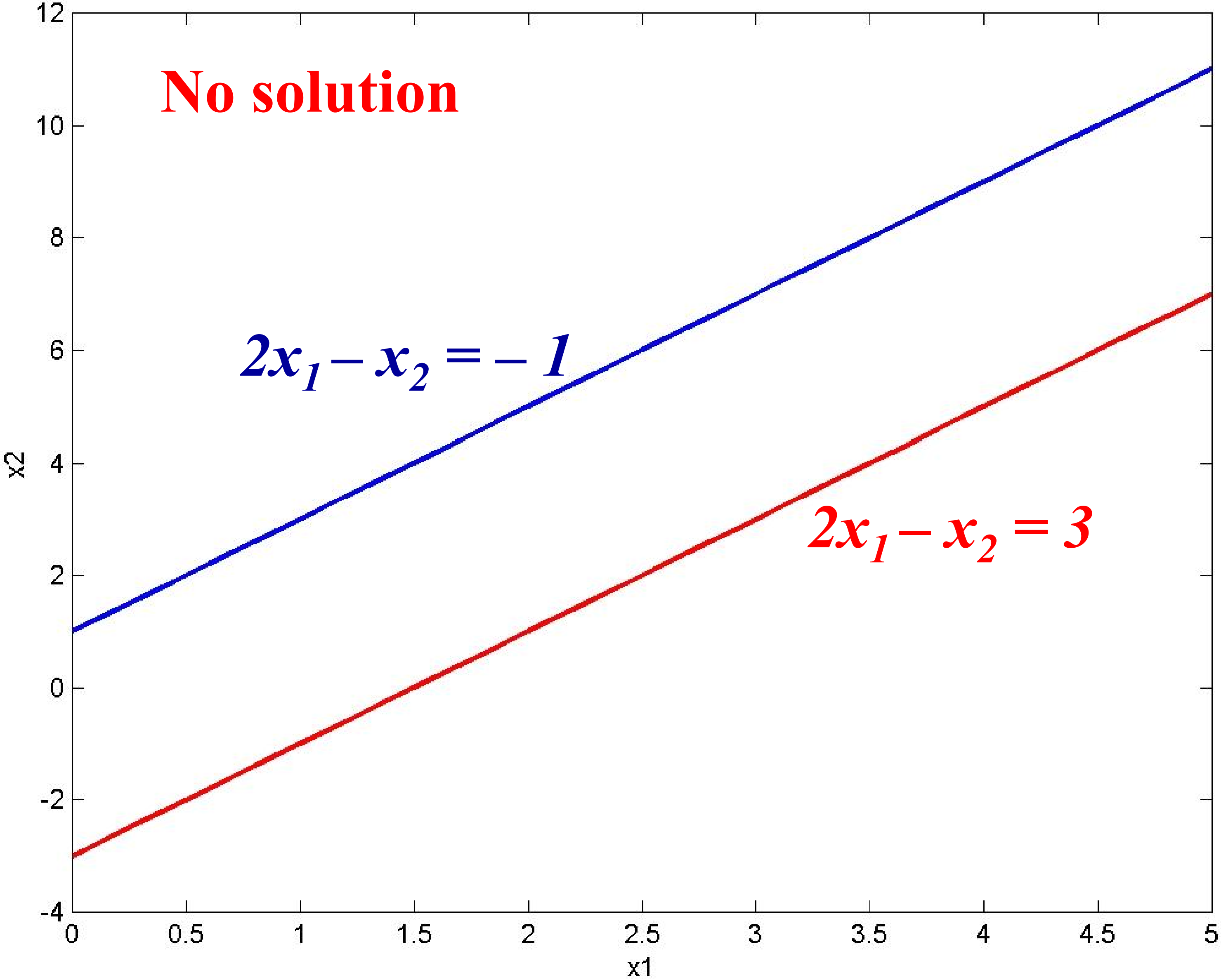
For small numbers of equations, can be solved by hand

- **Graphical**
- **Cramer's rule**
- **Elimination**

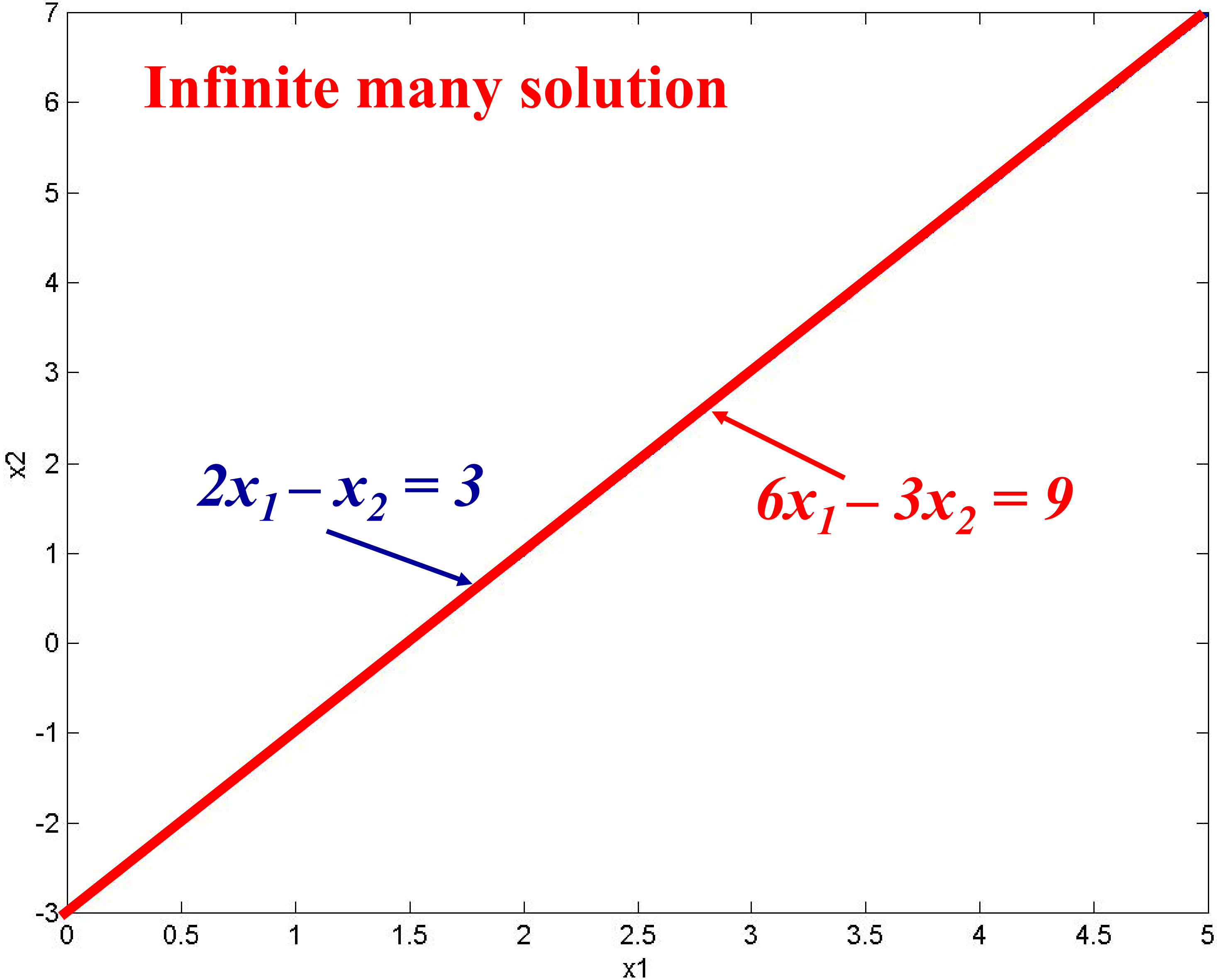
Graphical Method



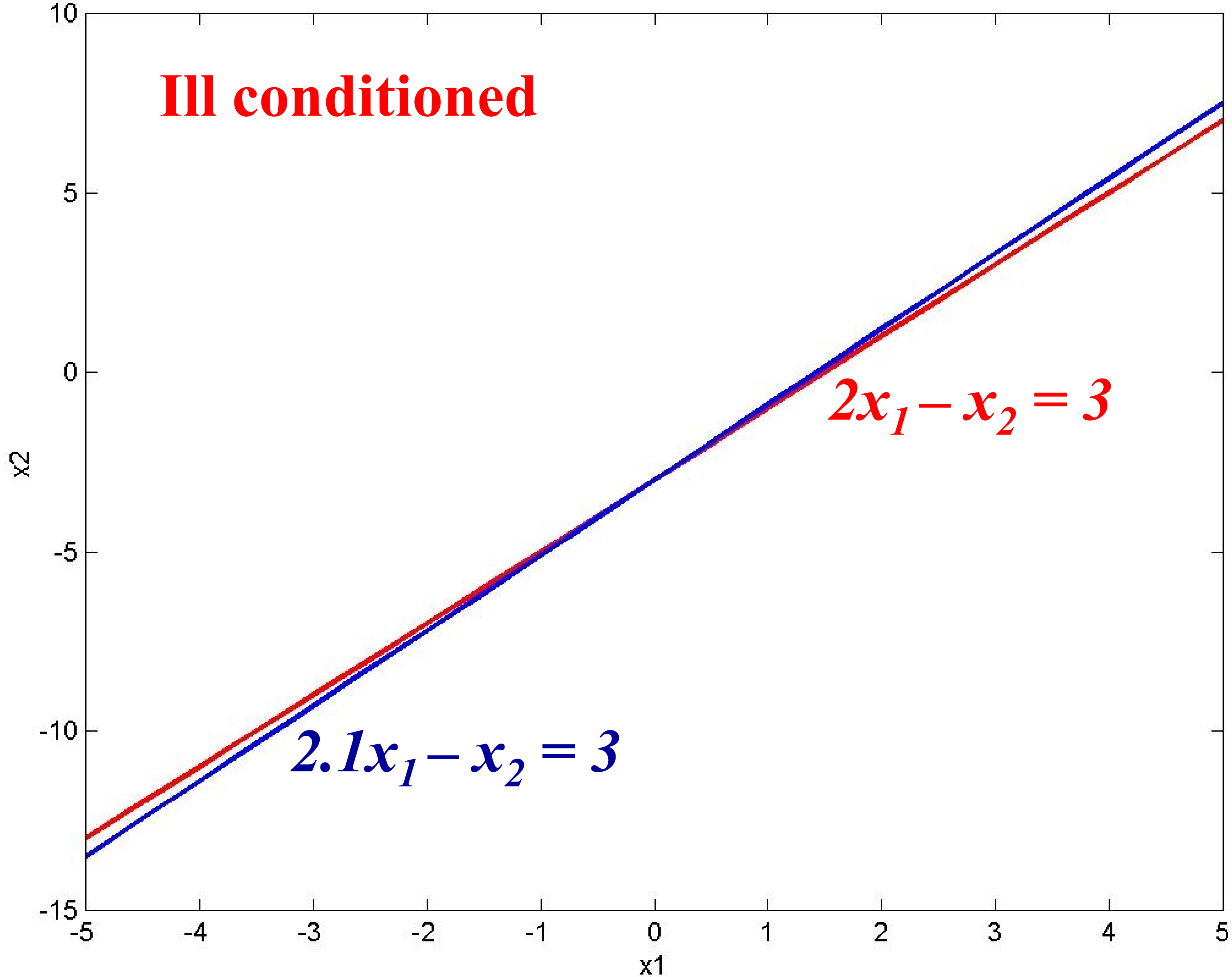
Graphical Method



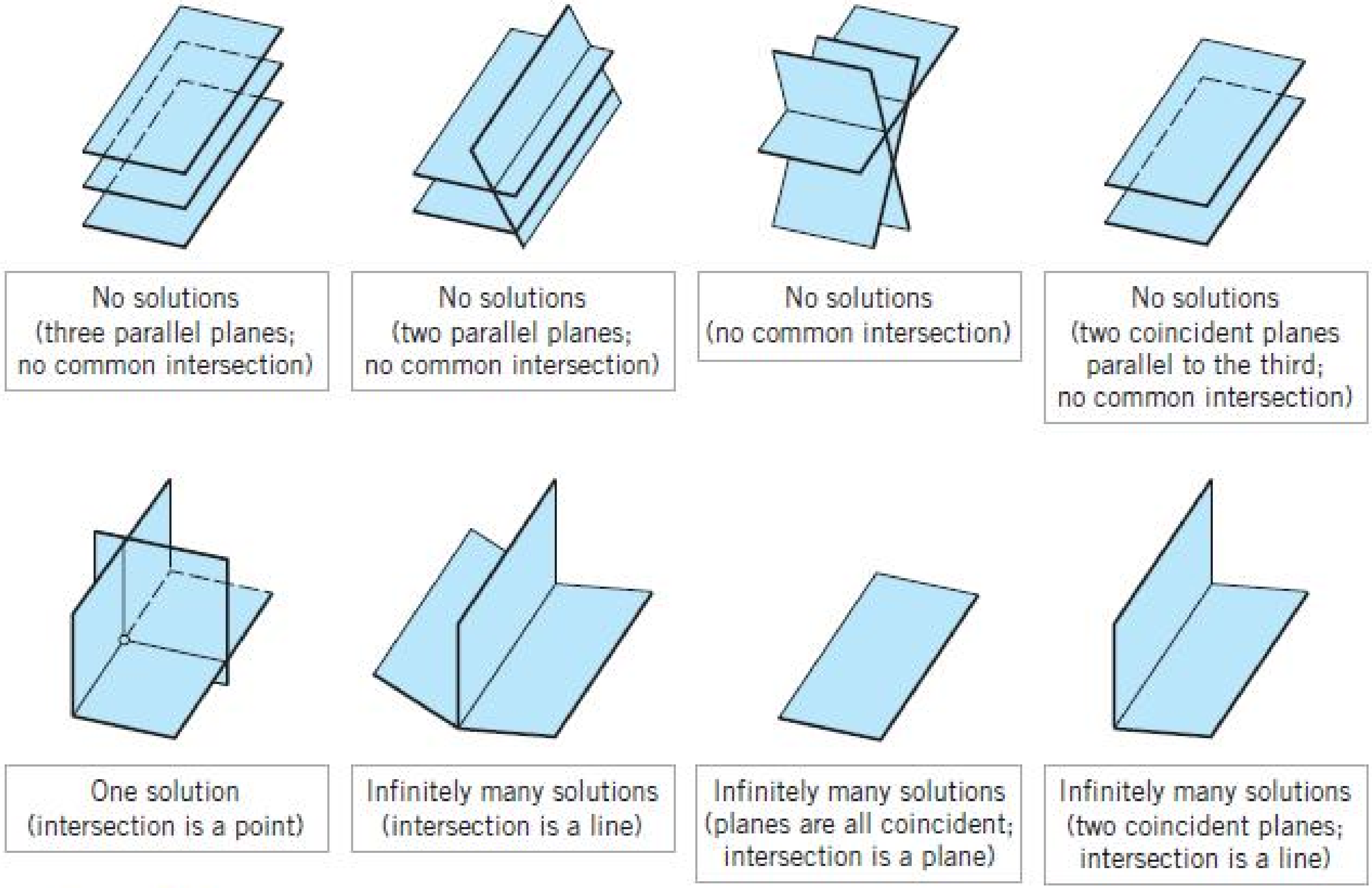
Graphical Method



Graphical Method



Cases in 3D



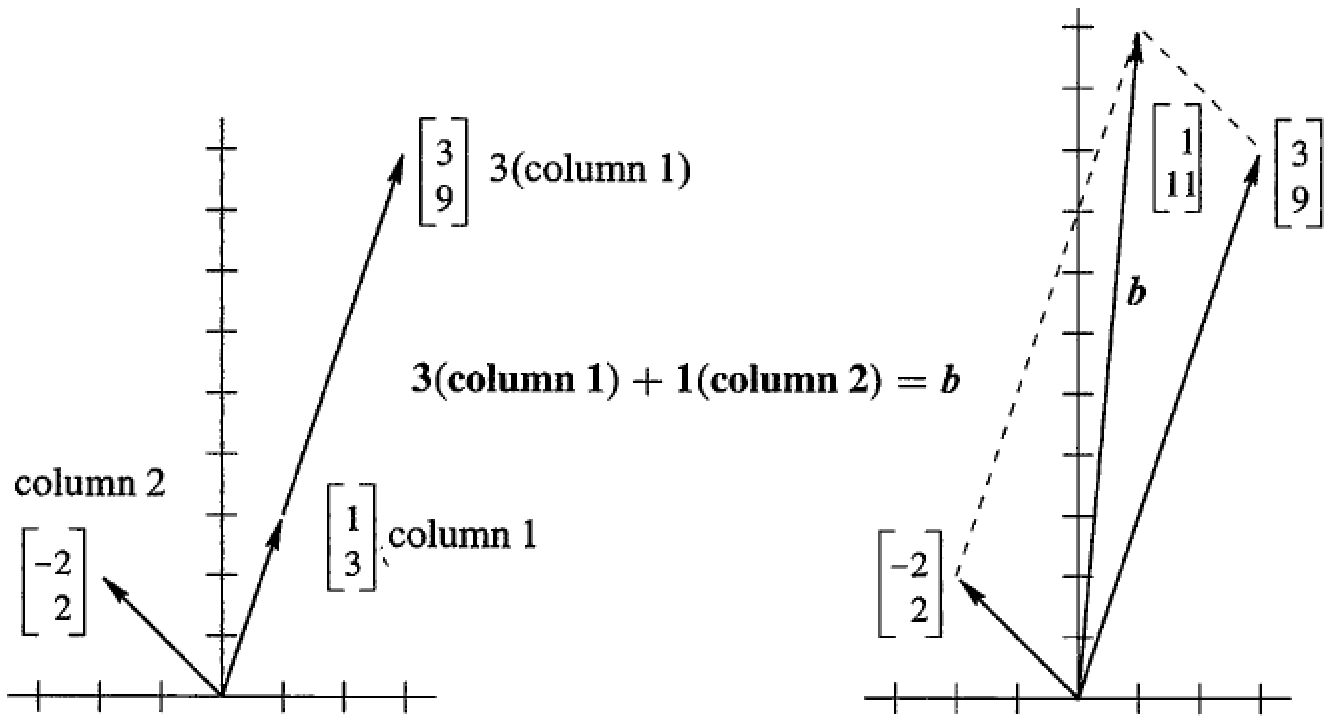
▲ Figure 1.1.2

Column Interpretation

x - 2y = 1

3x + 2y = 11

$$x\begin{bmatrix}1\\3\end{bmatrix}+y\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}\Rightarrow\begin{bmatrix}1&-2\\3&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}$$



$Ax = b$

- Row picture : intersection of planes
- Column picture : combination of columns
- Non-singular system (A is invertible/nonsingular)
  - One solution
- Singular systems (A is singular)
  - No solution
  - Infinitely many solutions
- Ill conditioned
  - close to being singular

Singular Systems

- **Singular systems can be underdetermined:**

$$2x_1 + 3x_2 = 5$$
$$4x_1 + 6x_2 = 10$$

- **or inconsistent:**

$$2x_1 + 3x_2 = 5$$
$$4x_1 + 6x_2 = 11$$

- **A is singular if some row of A is linear combination of other rows**

Cramer's Rule

- Compute the **determinant D**
- 2 x 2 matrix

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- 3 x 3 matrix

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

These determinants of the 2x2 matrices are called minors of A

Co-factors

$A = [a_{ij}]$ n x n matrix

(i,j)-Cofactor, $C_{ij} = (-1)^{i+j} M_{ij}$

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$ (expansion along row i)

$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$ (expansion along col j)

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0 \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 1$$

Cofactors

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$$

The cofactor for each element of matrix A:

$$A_{11} = \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 24 \qquad A_{12} = -\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = 5 \qquad A_{13} = \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = -4$$

$$A_{21} = \begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} = -12 \qquad A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \qquad A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

$$A_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2 \qquad A_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = -5 \qquad A_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$$

Cofactor Matrix

C = Cofactor matrix of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$ is then given by:

$$\begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}$$

Properties

- If A is **triangular**, det(A) = product of diagonal entries
- **det(AB) = det(A)det(B)**
- If A is a square matrix and any one of the following conditions is true, then **det(A) = 0**
  - An entire row (or an entire column) consists of zeros
  - Two rows (or columns) are equal
  - One row (or column) is a multiple of another row (or column)

Inverse using Determinant

- **$A^{-1} = C^T/\det(A) \Rightarrow A\,C^T = \det(A)I$**

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \dots & C_{n1} \\ \dots & \dots & \dots \\ C_{1n} & \dots & C_{nn} \end{bmatrix} = \begin{bmatrix} \det(A) & 0 & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \det(A) \end{bmatrix}$$

- **Proof outline:**
- Diagonal entries are det(A) bacause they represent the cofactor expansion of det(A)
- Why are off-diagonals all zero:
- Consider, row 1 of A, col 2 of C, $a_{11}C_{21} + a_{12}C_{22} + \ldots + a_{1n}C_{2n} = 0$?
  - Think this as a cofactor expansion of a new matrix B along row 2 where the 1st row of A is copied to row 2 of B
  - B has two rows that are same, det(B) = 0

Inverse of a 3×3 matrix

Inverse matrix of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$ is given by:

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}^T = \frac{1}{22}\begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 12/11 & -6/11 & -1/11 \\ 5/22 & 3/22 & -5/22 \\ -2/11 & 1/11 & 2/11 \end{bmatrix}$$

Cramer's Rule

- **To find x_k for the following system**

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n
\end{aligned}
$$

- **B_k = Replace k^{th} column of A with b(i.e., $a_{ik} \leftarrow b_i$)**

$$x = A^{-1}b = \frac{C^T b}{\det(A)}, \text{so } x_k = \frac{\det(B_k)}{\det(\mathrm{A})}$$

- x_k= k-th entry of $C^T b = b_1 C_{1k} + b_2 C_{2k} + \ldots + b_n C_{nk} = \det(B_k)$

2 x 2 case

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

Eliminate $x_2 \Rightarrow$ $\begin{cases} a_{22}a_{11}x_1 + a_{22}a_{12}x_2 = a_{22}b_1 \\ a_{12}a_{21}x_1 + a_{12}a_{22}x_2 = a_{12}b_2 \end{cases}$

Subtract to get

$$a_{22}a_{11}x_1 - a_{12}a_{21}x_1 = a_{22}b_1 - a_{12}b_2$$

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{22}a_{11} - a_{12}a_{21}} \Rightarrow x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

Not very practical for large number (> 4) of equations

3 x 3 matrix

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$x_1 = \frac{D_1}{D} = \frac{1}{D}\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$x_2 = \frac{D_2}{D} = \frac{1}{D}\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

$$x_3 = \frac{D_3}{D} = \frac{1}{D}\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Ill-Conditioned System

- **What happen if the determinant D is very small or zero?**

$$D = det\left[A\right] \approx 0$$

- **Divided by zero (linearly dependent system)**
  - **If A is invertible, det(A) ≠ 0, Is the converse true?**
- **Divided by a small number: Round-off error**
  - **Loss of significant digits**

Elimination Method

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

Eliminate $x_2 \Rightarrow \begin{cases} a_{22}a_{11}x_1 + a_{22}a_{12}x_2 = a_{22}b_1 \\ a_{12}a_{21}x_1 + a_{12}a_{22}x_2 = a_{12}b_2 \end{cases}$

Subtract to get

$$a_{22}a_{11}x_1 - a_{12}a_{21}x_1 = a_{22}b_1 - a_{12}b_2$$

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{22}a_{11} - a_{12}a_{21}} \Rightarrow x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

Not very practical for large number (> 4) of equations

MATLAB's Methods

- Forward slash (/)
- Back-slash (\ )
- Multiplication by the inverse of the quantity under the slash

$$Ax = b$$

$$x = A^{-1}b \Rightarrow x = A \backslash b$$

$$x = inv(A) * b$$

Gauss Elimination

- Manipulate equations to eliminate one of the unknowns
- Develop algorithm to do this repeatedly
- The goal is to set up upper triangular matrix

$$U = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & a_{22} & a_{23} & \cdots & a_{2n} \\ & & a_{33} & \cdots & a_{3n} \\ & & & \ddots & \vdots \\ & & & & a_{nn} \end{bmatrix}$$

- Back substitution to find solution (root)

Basic Gauss Elimination

- Direct Method (no iteration required)
- Forward elimination
- Column-by-column elimination of the below-diagonal elements
- Reduce to upper triangular matrix
- Back-substitution

Naive Gauss Elimination

- **Begin with**

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$$

- **Multiply the first equation by a_{21} / a_{11} and subtract from second equation**

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_n x_n = b_1$$
$$\cancel{\left(a_{21} - \frac{a_{21}}{a_{11}} a_{11}\right)} x_1 + \left(a_{22} - \frac{a_{21}}{a_{11}} a_{12}\right) x_2 + \ldots + \left(a_{2n} - \frac{a_{21}}{a_{11}} a_{1n}\right) x_n = b_2 - \frac{a_{21}}{a_{11}} b_1$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$$

Gauss Elimination

- Reduce to

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + \ldots + a'_{2n}x_n &= b'_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n
\end{aligned}
$$

- Repeat the forward elimination to get

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + \ldots + a'_{2n}x_n &= b'_2 \\
&\vdots \\
a'_{n2}x_2 + \ldots + a'_{nn}x_n &= b'_n
\end{aligned}
$$

Gauss Elimination

- First equation is pivot equation
- a_{11} is pivot element
- Now multiply second equation by a'_{32}/a'_{22} and subtract from third equation

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$\left(a'_{33} - \frac{a'_{32}}{a'_{22}}a'_{23}\right)x_3 + \ldots + \left(a'_{3n} - \frac{a'_{32}}{a'_{22}}a'_{2n}\right)x_n = \left(b'_3 - \frac{a'_{32}}{a'_{22}}b'_2\right)$$

$$\vdots$$

Gauss Elimination

- Repeat the elimination of a'_{i2} and get

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \ldots + a''_{3n}x_n &= b''_3 \\
&\vdots \\
a''_{n3}x_3 + \ldots + a''_{nn}x_n &= b''_n
\end{aligned}
$$

- Continue and get

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \ldots + a''_{3n}x_n &= b''_3 \\
&\vdots \\
a_{nn}^{(n-1)}x_n &= b_n^{(n-1)}
\end{aligned}
$$

Back Substitution

- Now we can perform back substitution to get $\{x\}$
- By simple division

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

- Substitute this into (n-1)th equation

$$a_{n-1,n-1}^{(n-2)} x_{n-1} + a_{n-1,n}^{(n-2)} x_n = b_{n-1}^{(n-2)}$$

- Solve for x_{n-1}
- Repeat the process to solve for $x_{n-2}, x_{n-3}, \ldots. x_2, x_1$

Back Substitution

- Back substitution: starting with x_n
- Solve for $x_{n-1}, x_{n-2}, \ldots, 3, 2, 1$

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \quad \text{for } i = n-1, n-2, \ldots, 1$$

$$\boxed{a_{ii}^{(i-1)} \neq 0}$$

Naive Gauss Elimination

Elimination of first column

$$\left[\begin{array}{cccc:c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & b_4 \end{array}\right] \qquad \begin{array}{l} \\ f_{21} = a_{21} / a_{11} \\ f_{31} = a_{31} / a_{11} \\ f_{41} = a_{41} / a_{11} \end{array}$$

$$\left[\begin{array}{cccc:c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ 0 & a'_{32} & a'_{33} & a'_{34} & b'_3 \\ 0 & a'_{42} & a'_{43} & a'_{44} & b'_4 \end{array}\right] \qquad \begin{array}{l} \\ (2) - f_{21} \times (1) \\ (3) - f_{31} \times (1) \\ (4) - f_{41} \times (1) \end{array}$$

Elimination of second column

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ 0 & a'_{32} & a'_{33} & a'_{34} & b'_3 \\ 0 & a'_{42} & a'_{43} & a'_{44} & b'_4 \end{array}\right]$$

$$f_{32} = a'_{32} / a'_{22}$$

$$f_{42} = a'_{42} / a'_{22}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ 0 & 0 & a''_{33} & a''_{34} & b''_3 \\ 0 & 0 & a''_{43} & a''_{44} & b''_4 \end{array}\right]$$

$$(3) - f_{32} \times (2)$$

$$(4) - f_{42} \times (2)$$

Elimination of third column

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ 0 & 0 & a''_{33} & a''_{34} & a''_3 \\ 0 & 0 & a''_{43} & a''_{44} & a''_4 \end{array}\right]$$

$$f_{43} = a''_{43} \,/\, a''_{33}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ 0 & 0 & a''_{33} & a''_{34} & b''_3 \\ 0 & 0 & 0 & a'''_{44} & b'''_4 \end{array}\right]$$

Upper triangular matrix

$$(4) - f_{43} \times (3)$$

Back-Substitution

$$\left[\begin{array}{cccc:c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ 0 & 0 & a''_{33} & a''_{34} & b''_3 \\ 0 & 0 & 0 & a'''_{44} & b'''_4 \end{array}\right]$$

Upper triangular matrix

$$x_4 = b'''_4 / a'''_{44}$$

$$x_3 = (b''_3 - a''_{34} x_4) / a''_{33}$$

$$x_2 = (b'_2 - a'_{23} x_3 - a'_{24} x_4) / a'_{22}$$

$$x_1 = (b_1 - a_{12} x_2 - a_{13} x_3 - a_{14} x_4) / a_{11}$$

$$\boxed{a_{11}, a'_{22}, a''_{33}, a'''_{44} \neq 0}$$

Example

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ -1 & 2 & 2 & -3 & -1 \\ 0 & 1 & 1 & 4 & 2 \\ 6 & 2 & 2 & 4 & 1 \end{array}\right] \qquad \begin{array}{l} \\ f_{21} = -1 \\ f_{31} = 0 \\ f_{41} = 6 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 0 \\ 0 & 1 & 1 & 4 & 2 \\ 0 & 2 & -10 & -14 & -5 \end{array}\right] \qquad \begin{array}{l} \\ (2) - (1) \times f_{21} \\ (3) - (1) \times f_{31} \\ (4) - (1) \times f_{41} \end{array}$$

Forward Elimination

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 0 \\ 0 & 1 & 1 & 4 & 2 \\ 0 & 2 & -10 & -14 & -5 \end{array}\right] \qquad \begin{array}{l} \\ \\ f_{32} = 1/2 \\ f_{42} = 1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & -1 & 4 & 2 \\ 0 & 0 & -14 & -14 & -5 \end{array}\right] \qquad \begin{array}{l} \\ \\ (3) - (2) \times f_{32} \\ (4) - (2) \times f_{42} \end{array}$$

Upper Triangular Matrix

$$\left[\begin{array}{cccc:c} 1 & 0 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 0 \\ 0 & 1 & -1 & 4 & 2 \\ 0 & 2 & -14 & -14 & -5 \end{array}\right] \qquad f_{43} = 14$$

$$\left[\begin{array}{cccc:c} 1 & 0 & 2 & 3 & 1 \\ \boxed{0} & 2 & 4 & 0 & 0 \\ \boxed{0} & \boxed{0} & -1 & 4 & 2 \\ \boxed{0} & \boxed{0} & \boxed{0} & -70 & -33 \end{array}\right] \qquad (4)-(3)\times f_{43}$$

Back-Substitution

$$\left[\begin{array}{cccc:c} 1 & 0 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & -1 & 4 & 2 \\ 0 & 0 & 0 & -70 & -33 \end{array}\right]$$

$$x_4 = -33/-70 = 33/70$$

$$x_3 = 4x_4 - 2 = -4/35$$

$$x_2 = -2x_3 = 8/35$$

$$x_1 = 1 - 2x_3 - 3x_4 = -13/70$$

$$\vec{x} = \begin{bmatrix} -13/70 \\ 8/35 \\ -4/35 \\ 33/70 \end{bmatrix}$$

MATLAB Script File: GaussNaive

```matlab
function x = GaussNaive(A,b)

% GaussNaive(A,b)  :
% Solve Ax =b using Gaussian elimination without pivoting
% Input:
%     A = coefficient matrix
%     b = right-hand-side matrix
%
% Output:
%     x = solution matrix

% compute the matrix sizes
[m, n] = size(A);
if m ~= n, error('Matrix A must be square'); end
nb = n + 1;
Aug = [A b];

% forward elimination
for k = 1 : n-1
  for i = k+1 : n
     factor = Aug(i,k) / Aug(k,k);
     Aug(i,k:nb) = Aug(i,k:nb) - factor*Aug(k,k:nb);
  end;
end

% back-substitution
x = zeros(n,1);
x(n) = Aug(n,nb) / Aug(n,n);
for i = n-1 : -1 : 1
     x(i) = (Aug(i,nb) - Aug(i,i+1:n)*x(i+1:n)) / Aug(i,i);
end
```

```
>> format short
>> x = GaussNaive(A,b)
m =
     4
n =
     4
Aug =
     1     0     2     3     1
    -1     2     2    -3    -1
     0     1     1     4     2
     6     2     2     4     1
factor =
    -1
Aug =
     1     0     2     3     1
     0     2     4     0     0
     0     1     1     4     2
     6     2     2     4     1
factor =
     0
Aug =
     1     0     2     3     1
     0     2     4     0     0
     0     1     1     4     2
     6     2     2     4     1
factor =
     6
Aug =
     1     0     2     3     1
     0     2     4     0     0
     0     1     1     4     2
     0     2   -10   -14    -5
```

Aug = [A, b]

Eliminate first column

```
factor =
    0.5000
Aug =
     1     0     2     3     1
     0     2     4     0     0
     0     0    -1     4     2
     0     2   -10   -14    -5
factor =
     1
Aug =
     1     0     2     3     1
     0     2     4     0     0
     0     0    -1     4     2
     0     0   -14   -14    -5
```

Eliminate second column

```
factor =
    14
Aug =
     1     0     2     3     1
     0     2     4     0     0
     0     0    -1     4     2
     0     0     0   -70   -33
```

Eliminate third column

```
x =
         0
         0
         0
    0.4714     x4
x =
         0
         0
   -0.1143     x3
    0.4714
x =
         0
    0.2286     x2
   -0.1143
    0.4714
x =
   -0.1857     x1
    0.2286
   -0.1143
    0.4714
```

Back-substitution

Print all `factor` and `Aug`

(do not suppress output)

Algorithm for Gauss elimination

Step 1. Forward elimination

- for each equation j, $j = 1$ to n-1

    for all equations k greater than j

    (a) multiply equation j by a_{kj}/a_{jj}

    (b) subtract the result from equation k

- This leads to an upper triangular matrix

Step 2. Back-Substitution

(a) determine x_n from $x_n = b_n^{(n-1)} / a_{nn}^{(n-1)}$

(b) put x_n into (n-1)th equation, solve for x_{n-1}

(c) repeat from (b), moving back to n-2, n-3, etc. until all equations are solved

Operation Count

- **Important as matrix gets large**
- **For Gauss elimination**
- **Elimination routine uses on the order of $O(n^3/3)$ operations**
- **Back-substitution uses $O(n^2/2)$**

Operation Count

Outer Loop k	Inner Loop i	Addition/Subtraction flops	Multiplication/Division flops
1	$2, n$	$(n-1)(n)$	$(n-1)(n+1)$
2	$3, n$	$(n-2)(n-1)$	$(n-2)(n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	$k+1, n$	$(n-k)(n-k+1)$	$(n-k)(n-k+2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	n, n	$(1)(2)$	$(1)(3)$

Total operation counts for elimination stage = $2n^3/3 + O(n^2)$

Total operation counts for back substitution stage = $n^2 + O(n)$

Operation Count

- **Number of flops (floating-point operations) for Naive Gauss elimination**

n	*Elimination*	*Back Sbustitution*	*Total Flops*	$\frac{2n^3}{3}$	*Percentage Due to Elimination*
10	*705*	*100*	*805*	*667*	*87.58%*
100	*671550*	*10000*	*681550*	*666667*	*98.53%*
1000	6.67×10^8	*1000000*	6.68×10^6	6.67×10^8	*99.85%*

- **Computation time increase rapidly with n**
- **Most of the effort is incurred in the elimination step**
- **Improve efficiency by reducing the elimination effort**

Matrix form of Elimination Steps

- Suppose $R_2' = R_2 - 2 * R_1$
  - Multiplying the first equation by 2 and subtracting form the 2nd equation
- This can be achieved by multiplying A with an elementary matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}, EA = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}$$

- Elementary matrix E_{ij} subtracts ℓ times row j from row i,
- i.e., $R_i' = R_i - \ell * R_j$ contains $-\ell$ in row i and col j
  - In the example above we have $E_{21,}$ with -2 in row 2, col 1 position

Gaussian Elimination

- A → U

$$[A:b]=\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

- 1st step: $R_2' = R_2 - 2 * R_1$, $R_3' = R_3 - (-1) * R_1$
- 2nd step: $R_3' = R_3 - (-1) * R_1$
- Elementary matrices

$$E_{21}=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, E_{32}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

LU Factorization

- **$E_{32} E_{31} E_{21} A = U \Rightarrow A = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} U = LU$**
- **$L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}$**
- **E_{ij}^{-1} can be obtained from E_{ij} by changing the sign of the ij-th element**

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

- And L becomes a lower triangular matrix

$$E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

Solving Ax=b by LU Decomposition

- Factor A into LU, where L is lower triangular and U is upper triangular

  $\mathrm{A}\mathbf{x} = \mathbf{b}$ => $\mathrm{LU}\mathbf{x} = \mathbf{b}$
  solve for y, $\mathrm{L}\mathbf{y} = \mathbf{b}$ **by forward substitution**
  then solve for x, $\mathrm{U}\mathbf{x} = \mathbf{y}$ **by backward substitution**

- Last 2 steps in $O(n^2)$ time, so total time dominated by decomposition

Example

$$\boxed{\begin{aligned}&2x_1 + x_2 + x_3 = 5\\&4x_1 - 6x_2 = -2\\&-2x_1 + 7x_2 + 2x_3 = 9\end{aligned}}$$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} = LU$$

$$Ly = b \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} \Rightarrow y = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix}$$

$$Ux = y \Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

A = LDU decomposition

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} u_{11} & 0 & 0 \\ 0 & u_{22} & 0 \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12}/u_{11} & u_{13}/u_{11} \\ 0 & 1 & u_{12}/u_{22} \\ 0 & 0 & 1 \end{bmatrix}$$

A = LDU where D is a diagonal matrix, and L and U are lower and upper triangular matrices with 1 in the diagonal positions.

LU factorization is unique, if A is invertible

if $A = L_1D_1U_1$, $A= L_2D_2U_2$ Show $L_1= L_2$, $D_1= D_2$, $U_1= U_2$

Outline: Show $D_1U_1U_2^{-1} = L_1^{-1}L_2D_2$

is left hand side is an upper and the right hand side is a lower triangular matrix?

Crout's Method

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

- More unknowns than equations!
- Let all l_{ii}=1 for all i

Crout's Method

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$u_{11} = a_{11}$$

$$l_{21}u_{11} = a_{21} \quad \Rightarrow \quad l_{21} = \frac{a_{21}}{u_{11}}$$

$$l_{31}u_{11} = a_{31} \quad \Rightarrow \quad l_{31} = \frac{a_{31}}{u_{11}}$$

$$u_{12} = a_{12}$$

$$l_{21}u_{12} + u_{22} = a_{22} \quad \Rightarrow \quad u_{22} = a_{22} - l_{21}u_{12}$$

$$l_{31}u_{12} + l_{32}u_{22} = a_{32} \quad \Rightarrow \quad l_{32} = \frac{a_{32} - l_{31}u_{12}}{u_{22}}$$

Crout's Method

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

- For i = 1..n
  - For j = 1..i

$$u_{ji} = a_{ji} - \sum_{k=1}^{j-1} l_{jk} u_{ki}$$

  - For j = i+1..n

$$l_{ji} = \frac{a_{ji} - \sum_{k=1}^{i-1} l_{jk} u_{ki}}{u_{ii}}$$

Partial Pivoting

Problems with Gauss elimination

- **division by zero**
- round off errors
- ill conditioned systems

Use “Pivoting” to avoid this

- **Find the row with largest absolute coefficient below the pivot element**
- **Switch rows (“partial pivoting”)**
- **complete pivoting switch columns also (rarely used)**

Round-off Errors

- A lot of chopping with more than $n^3/3$ operations
- More important - error is propagated
- For large systems (more than 100 equations), round-off error can be very important (machine dependent)
- Ill conditioned systems - small changes in coefficients lead to large changes in solution
- Round-off errors are especially important for ill-conditioned systems

Gauss Elimination with Partial Pivoting

- <u>Forward elimination</u>
- for each equation j, $j = 1$ to $n-1$
  - first scale each equation k greater than j
  - then pivot (switch rows)
  - Now perform the elimination

    (a) multiply equation j by a_{kj} / a_{jj}

    (b) subtract the result from equation

Partial (Row) Pivoting

$$\begin{aligned}
x_1 \quad\quad\quad &+ 2x_3 + 3x_4 = 1 \\
-x_1 + 2x_2 &+ 2x_3 - 3x_4 = -1 \\
x_2 &+ x_3 + 4x_4 = 2 \\
6x_1 + 2x_2 &+ 2x_3 + 4x_4 = 1
\end{aligned}$$

$$[A \,\vdots\, b] = \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ -1 & 2 & 2 & -3 & -1 \\ 0 & 1 & 1 & 4 & 2 \\ 6 & 2 & 2 & 4 & 1 \end{array}\right]$$

Forward Elimination

$$\left[\begin{array}{cccc|c} 6 & 2 & 2 & 4 & 1 \\ -1 & 2 & 2 & -3 & -1 \\ 0 & 1 & 1 & 4 & 2 \\ 1 & 0 & 2 & 3 & 1 \end{array}\right]$$

Interchange rows 1 & 4

$$f_{21} = -1/6$$
$$f_{31} = 0$$
$$f_{41} = 1/6$$

$$\left[\begin{array}{cccc|c} 6 & 2 & 2 & 4 & 1 \\ 0 & 7/3 & 7/3 & -7/3 & -5/6 \\ 0 & 1 & 1 & 4 & 2 \\ 0 & -1/3 & 5/3 & 7/3 & 5/6 \end{array}\right]$$

$$(2) - (1) \times f_{21}$$
$$(3) - (1) \times f_{31}$$
$$(4) - (1) \times f_{41}$$

Forward Elimination

$$\left[\begin{array}{cccc:c} 6 & 2 & 2 & 4 & 1 \\ 0 & 7/3 & 7/3 & -7/3 & -5/6 \\ 0 & 1 & 1 & 4 & 2 \\ 0 & -1/3 & 5/3 & 7/3 & 5/6 \end{array}\right]$$

No interchange required

$$f_{32} = 3/7$$

$$f_{42} = 1/7$$

$$\left[\begin{array}{cccc:c} 6 & 2 & 2 & 4 & 1 \\ 0 & 7/3 & 7/3 & -7/3 & -5/6 \\ 0 & 0 & 0 & 5 & 33/14 \\ 0 & 0 & 2 & 2 & 5/7 \end{array}\right]$$

$$(3) - (2) \times f_{32}$$

$$(4) - (2) \times f_{42}$$

Back-Substitution

$$\left[\begin{array}{cccc:c} 6 & 2 & 2 & 4 & 1 \\ 0 & 7/3 & 7/3 & -7/3 & -5/6 \\ 0 & 0 & 2 & 2 & 5/7 \\ 0 & 0 & 0 & 5 & 33/14 \end{array}\right]$$

Interchange rows 3 & 4

$$f_{43} = 0$$

$$x_4 = (33/14)/5 = 33/70$$

$$x_3 = (5/7 - 2\,x_4)/2 = -4/35$$

$$x_2 = (-5/6 + 7/3\,x_4 - 7/3\,x_3)/(7/3) = 8/35$$

$$x_1 = (1 - 4\,x_4 - 2\,x_3 - 2\,x_2)/6 = -13/70$$

$$\vec{x} = \begin{bmatrix} -13/70 \\ 8/35 \\ -4/35 \\ 33/70 \end{bmatrix}$$

Row Exchange and Permutation Matrix

$$\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$$

Now we can carry on with Gauss elimination

Solving, PAx = Pb

P is called a Permutation matrix: Let row *j be swapped into row k. Then the kth row of P must be a row of all zeroes except* for a 1 in the *jth position.*

Permutation Matrices

How many 3x3 permutation matrices are there?

$$I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} \quad P_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \quad P_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \quad P_{21}P_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.$$

PA = A with rows swapped

AP = ?

Single row swap

$P^{-1} = P^T$ for any permutation matrix because we should be able to reverse the swap

Let **P_{ij} be the matrix that swaps rows i and j**.

It has exactly two rows of I swapped, and in fact has those same two columns swapped.

In fact, ***$P = P^T = P^{-1}$ for a single row swap***.

But now any general P can be arrived at by swapping rows a pair at a time. So $P = P_1P_2 \ldots P_{r'}$. Then $P^{-1} = (P_1P_2 \ldots P_{r'})^T$

LU factorization and Permutations

Suppose we are going on with elimination and then suddenly encounter 0 in a pivot position

So, we need a permutation: PEEEA and continue elimination. But again encounter 0 in pivot and end up with something like EEPEEPEEA : *how to do factorization?*

It would be nice if all the swaps were done in the beginning then we would have EEEEEEPPA and have **P**A = LU

LU factorization and Permutations

First, let's look at a single **PE** pair in the string.

E* has its sole off-diagonal entry in *row i and column j, where $j < i$.

P swaps two *rows, k and l*.

Then ***both k and l are larger than j***.

This is because E eliminates in *column j*. The next swap happens in *some column past j, so must swap rows past j*.

Either could equal i, though.

LU factorization and Permutations

First, let's look at a single **PE** pair in the string.

***E* has its sole off-diagonal entry in *row i and column j, where $j < i$*.**

P swaps two *rows, k and l*.

Then ***both k and l are larger than j***.

This is because E eliminates in *column j*. The next swap happens in *some column past j, so must swap rows past j*.

Either could equal i, though.

LU factorization and Permutations

Suppose we have E**EP**EE = EE**PEP**$^{-1}$**P**E = EE**PEPP**E

PEP: first P swaps *row k and l* and the second P swaps *column k and l [both k and l are larger than j]*

the column swap doesn't move the column where the off-diagonal entry it located. **The first *P might change where it is* located within the column, but the *column that it is in doesn't change***

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\2&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}=\begin{bmatrix}1&0&0&0\\2&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\2&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\2&0&0&1\end{bmatrix}$$

Theorem: if *A is a non-singular square matrix, it can be factored as PA = LU*

So, If the E has its **off-diagonal entry in a row not affected** by P, then PE = PEPP = EP so the **P can just move to the right past E**.

If the off-diagonal entry is in a row affected by P then ***the entry moves to the other row affected by P***.

This happens to the *other E that has its off-diagonal entry in the other row affected by P. So in* the end analysis all we end up doing is swapping the two entries in *L!*

So we can move all the P's rightward next to A, and finish the factorization as usual.

Example

Example: find $PA = LU$ for $A = \begin{bmatrix} 2 & -3 & 4 & 3 \\ 6 & -9 & 12 & 8 \\ 4 & -5 & 6 & 7 \\ 2 & 1 & -1 & 8 \end{bmatrix}$

There is no need for a row swap to get things going, so simply eliminate in column 1:

$$P \text{ so far} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad L \text{ so far} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 2 & & 1 & 0 \\ 1 & & & 1 \end{bmatrix} \qquad U \text{ so far} = \begin{bmatrix} 2 & -3 & 4 & 3 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 4 & -5 & 5 \end{bmatrix}$$

We need to make a row swap. Let's swap rows 2 and 3. Remember to do the same in L and U!

$$P \text{ so far} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad L \text{ so far} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & & 1 & 0 \\ 1 & & & 1 \end{bmatrix} \qquad U \text{ so far} = \begin{bmatrix} 2 & -3 & 4 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 4 & -5 & 5 \end{bmatrix}$$

Example

Now eliminate in column 3. The multiplier goes into L as usual.

$$P \text{ so far} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad L \text{ so far} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 1 & 4 & & 1 \end{bmatrix} \qquad U \text{ so far} = \begin{bmatrix} 2 & -3 & 4 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

Now we must do another row swap, of rows 3 and 4. Remember to make the same swaps in the of-diagonal elements of L!

$$P \text{ so far} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad L \text{ so far} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 3 & 0 & & 1 \end{bmatrix} \qquad U \text{ so far} = \begin{bmatrix} 2 & -3 & 4 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

MATLAB M-File: GaussPivot

```
function x = GaussPivot(A,b)
% GaussPivot(A,b) :
% Solve Ax =b using Gaussian elimination with pivoting
% Input:
%     A = coefficient matrix
%     b = right-hand-side matrix
%
% Output:
%     x = solution matrix

% compute the matrix sizes
[m, n] = size(A);
if m ~= n, error('Matrix A must be square'); end
nb = n + 1;
Aug = [A b];

% forward elimination
for k = 1 : n-1
    % partial pivoting
    [big, i] = max(abs(Aug(k:n,k)));
    ipr = i+k-1;
    if ipr ~= k
        %pivot the rows
        Aug([k,ipr],:) = Aug([ipr,k],:);
    end
  for i = k+1 : n
     factor = Aug(i,k) / Aug(k,k);
     Aug(i,k:nb) = Aug(i,k:nb) - factor*Aug(k,k:nb);
  end;
end

% back-substitution
x = zeros(n,1);
x(n) = Aug(n,nb) / Aug(n,n);
for i = n-1 : -1 : 1
     x(i) = (Aug(i,nb) - Aug(i,i+1:n)*x(i+1:n)) / Aug(i,i);
end
```

Partial Pivoting

Partial Pivoting (switch rows)

largest element in {x}

`[big,i] = max(x)`

index of the largest element

```
>> format short
>> x=GaussPivot0(A,b)
Aug =
     1     0     2     3     1
    -1     2     2    -3    -1
     0     1     1     4     2
     6     2     2     4     1
big =
     6
i =
     4
ipr =
     4
Aug =
     6     2     2     4     1
    -1     2     2    -3    -1
     0     1     1     4     2
     1     0     2     3     1
factor =
   -0.1667
Aug =
    6.0000    2.0000    2.0000    4.0000    1.0000
         0    2.3333    2.3333   -2.3333   -0.8333
         0    1.0000    1.0000    4.0000    2.0000
    1.0000         0    2.0000    3.0000    1.0000
factor =
     0
Aug =
    6.0000    2.0000    2.0000    4.0000    1.0000
         0    2.3333    2.3333   -2.3333   -0.8333
         0    1.0000    1.0000    4.0000    2.0000
    1.0000         0    2.0000    3.0000    1.0000
factor =
    0.1667
Aug =
    6.0000    2.0000    2.0000    4.0000    1.0000
         0    2.3333    2.3333   -2.3333   -0.8333
         0    1.0000    1.0000    4.0000    2.0000
         0   -0.3333    1.6667    2.3333    0.8333
```

Aug = [A b]

Find the first pivot element and its index

Interchange rows 1 and 4

Eliminate first column

No need to interchange

```
big =
    2.3333
i =
     1
ipr =
     2
factor =
    0.4286
Aug =
    6.0000    2.0000    2.0000    4.0000    1.0000
         0    2.3333    2.3333   -2.3333   -0.8333
         0         0         0    5.0000    2.3571
         0   -0.3333    1.6667    2.3333    0.8333
factor =
   -0.1429
Aug =
    6.0000    2.0000    2.0000    4.0000    1.0000
         0    2.3333    2.3333   -2.3333   -0.8333
         0         0         0    5.0000    2.3571
         0         0    2.0000    2.0000    0.7143
big =
     2
i =
     2
ipr =
     4
Aug =
    6.0000    2.0000    2.0000    4.0000    1.0000
         0    2.3333    2.3333   -2.3333   -0.8333
         0         0    2.0000    2.0000    0.7143
         0         0         0    5.0000    2.3571
factor =
     0
Aug =
    6.0000    2.0000    2.0000    4.0000    1.0000
         0    2.3333    2.3333   -2.3333   -0.8333
         0         0    2.0000    2.0000    0.7143
         0         0         0    5.0000    2.3571
```

Second pivot element and index

No need to interchange

Eliminate second column

Third pivot element and index

Interchange rows 3 and 4

Eliminate third column

Back substitution

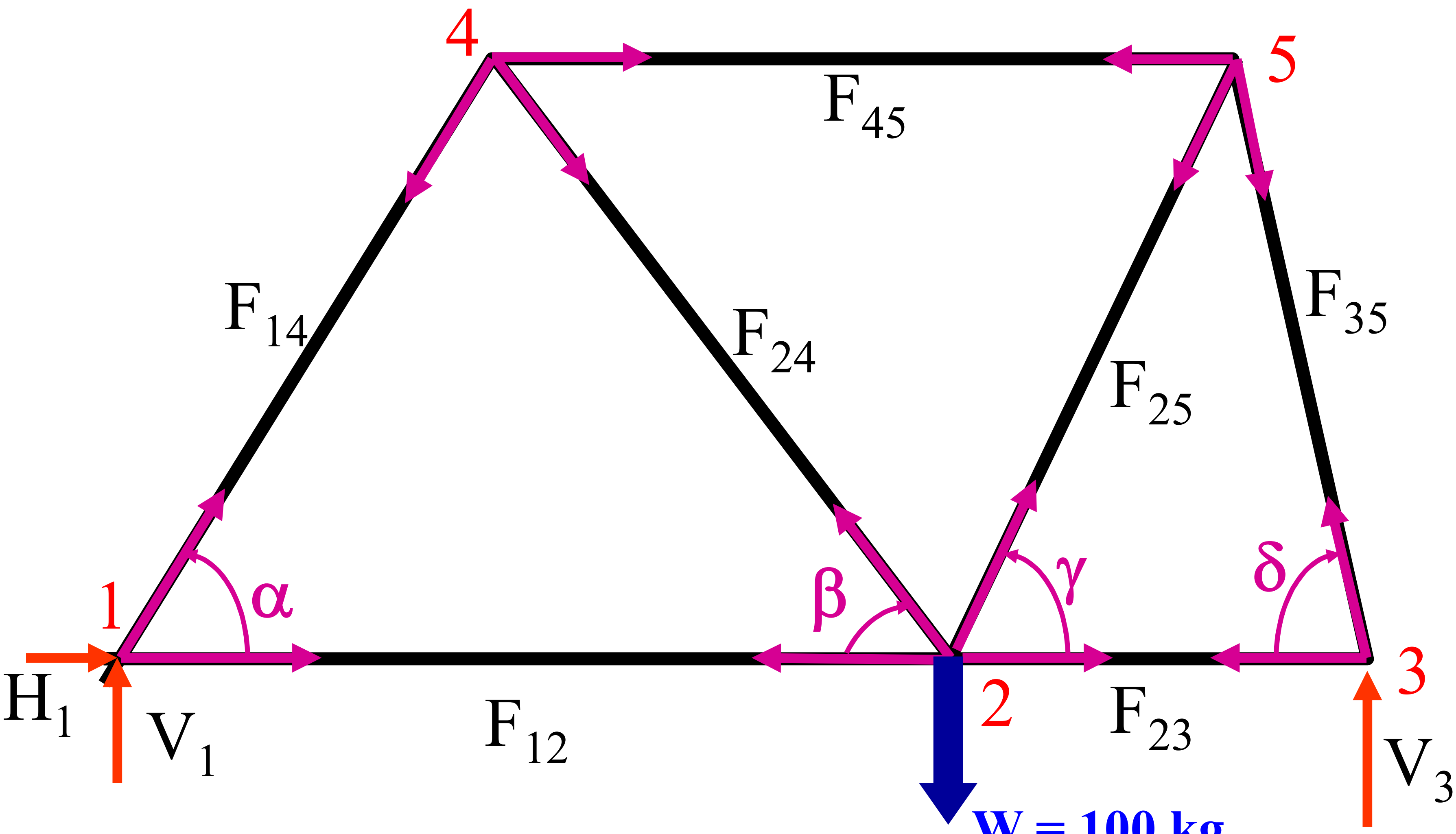
```
x =
         0
         0
         0
    0.4714
x =
         0
         0
   -0.1143
    0.4714
x =
         0
    0.2286
   -0.1143
    0.4714
x =

   -0.1857
    0.2286
   -0.1143
    0.4714
```

Save factors f_{ij} for

***LU* Decomposition**

TRUSS



4
5
F_{45}
F_{14}
F_{24}
F_{25}
F_{35}
α
β
γ
δ
1
2
3
H_1
V_1
F_{12}
F_{23}
V_3
W = 100 kg

Statics: Force Balance

Node 1
$$\begin{cases} \sum F_{y,1} = V_1 + F_{14}\sin\alpha = 0 \\ \sum F_{x,1} = H_1 + F_{12} + F_{14}\sin\alpha = 0 \end{cases}$$

Node 2
$$\begin{cases} \sum F_{y,2} = F_{24}\sin\beta + F_{25}\sin\gamma = 100 \\ \sum F_{x,2} = -F_{12} + F_{23} - F_{24}\cos\beta + F_{25}\cos\gamma = 0 \end{cases}$$

Node 3
$$\begin{cases} \sum F_{y,3} = V_3 + F_{35}\sin\delta = 0 \\ \sum F_{x,3} = -F_{23} - F_{35}\cos\delta = 0 \end{cases}$$

Node 4
$$\begin{cases} \sum F_{y,4} = -F_{14}\sin\alpha - F_{24}\sin\beta = 0 \\ \sum F_{x,4} = -F_{14}\cos\alpha + F_{24}\cos\beta + F_{45} = 0 \end{cases}$$

Node 5
$$\begin{cases} \sum F_{y,5} = -F_{25}\sin\gamma - F_{35}\sin\delta = 0 \\ \sum F_{x,5} = -F_{25}\cos\gamma + F_{35}\cos\delta - F_{45} = 0 \end{cases}$$

Example: Forces in a Simple Truss

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \sin\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \cos\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin\beta & \sin\gamma & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & \cos\beta & \cos\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \sin\delta & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -\cos\delta & 0 \\ 0 & 0 & 0 & 0 & -\sin\alpha & 0 & -\sin\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\cos\alpha & 0 & \cos\beta & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sin\gamma & \sin\delta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\gamma & \cos\delta & -1 \end{bmatrix} \begin{Bmatrix} V_1 \\ H_1 \\ V_3 \\ F_{12} \\ F_{14} \\ F_{23} \\ F_{24} \\ F_{25} \\ F_{35} \\ F_{45} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 100 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Define Matrices A and b in script file

```
function [A,b]=Truss(alpha,beta,gamma,delta)

A=zeros(10,10);
A(1,1)=1; A(1,5)=sin(alpha);
A(2,2)=1; A(2,4)=1; A(2,5)=cos(alpha);
A(3,7)=sin(beta); A(3,8)=sin(gamma);
A(4,4)=-1; A(4,6)=1; A(4,7)=-cos(beta);
A(4,8)=cos(gamma);
A(5,3)=1; A(5,9)=sin(gamma);
A(6,6)=-1; A(6,9)=-cos(delta);
A(7,5)=-sin(alpha); A(7,7)=-sin(beta);
A(8,5)=-cos(alpha); A(8,7)=cos(beta); A(8,10)=1;
A(9,8)=-sin(gamma); A(9,9)=-sin(delta);
A(10,8)=-cos(gamma); A(10,9)=cos(delta); A(10,10)=-1;

b=zeros(10,1); b(3,1)=100;
```

Gauss Elimination with Partial Pivoting

```
>> alpha=pi/6; beta=pi/3; gamma=pi/4; delta=pi/3;
>> [A,b] = Truss(alpha,beta,gamma,delta)
A =
    1.0000         0         0         0    0.5000         0         0         0         0         0
         0    1.0000         0    1.0000    0.8660         0         0         0         0         0
         0         0         0         0         0         0    0.8660    0.7071         0         0
         0         0         0   -1.0000         0    1.0000   -0.5000    0.7071         0         0
         0         0    1.0000         0         0         0         0         0    0.7071         0
         0         0         0         0         0   -1.0000         0         0   -0.5000         0
         0         0         0         0   -0.5000         0   -0.8660         0         0         0
         0         0         0         0   -0.8660         0    0.5000         0         0    1.0000
         0         0         0         0         0         0         0   -0.7071   -0.8660         0
         0         0         0         0         0         0         0   -0.7071    0.5000   -1.0000
b =
     0
     0
   100
     0
     0
     0
     0
     0
     0
     0
>> x = GaussPivot(A,b)
x =
   40.5827
         0
   48.5140
   70.2914
  -81.1655
   34.3046
   46.8609
   84.0287
  -68.6091
  -93.7218
```

Simple truss

Finding Determinant

- **Calculate determinant using Gauss elimination**

$$U = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ & a'_{22} & a'_{23} & \dots & a'_{2n} \\ & & a''_{33} & \dots & a''_{3n} \\ & & & & \vdots \\ & & & & a_{nn}^{(n-1)} \end{bmatrix}$$

$$\det(A) = \det(U) = a_{11} a'_{22} a''_{33} \dots a_{nn}^{(n-1)}$$