3. Linear equations

- linear equations
- example: polynomial interpolation
- applications
- geometrical interpretation

Linear equations

m equations in n variables $x_1, x_2, \ldots, x_n$:

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}$$

in matrix form: $Ax = b$, where

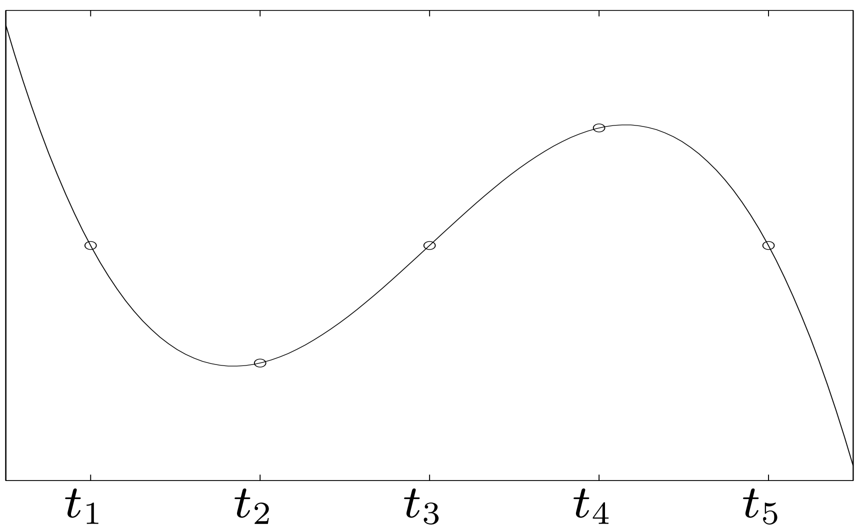
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Example: polynomial interpolation

fit a polynomial

$$p(t) = x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1}$$

through n points (t_1, y_1), . . . , (t_n, y_n)



problem data (parameters): t_1, . . . , t_n, y_1, . . . , y_n

problem variables: x_1, . . . , x_n

write out the conditions on x:

$$\begin{array}{rcl}
p(t_1) = x_1 + x_2 t_1 + x_3 t_1^2 + \cdots + x_n t_1^{n-1} & = & y_1 \\
p(t_2) = x_1 + x_2 t_2 + x_3 t_2^2 + \cdots + x_n t_2^{n-1} & = & y_2 \\
& \vdots & \\
p(t_n) = x_1 + x_2 t_n + x_3 t_n^2 + \cdots + x_n t_n^{n-1} & = & y_n
\end{array}$$

in matrix form: $Ax = b$ with

$$A = \begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-1} \end{bmatrix}, \qquad b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

n linear equations in n variables

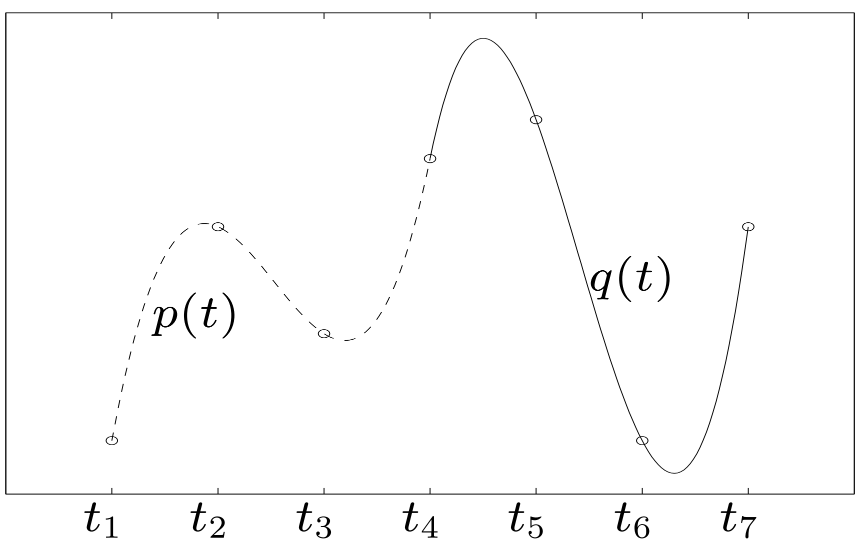
Exercise

express as a set of linear equations: find two cubic polynomials

$$p(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3, \qquad q(t) = d_0 + d_1 t + d_2 t^2 + d_3 t^3$$

that satisfy the following properties:

- $p(t_1) = y_1$, $p(t_2) = y_2$, $p(t_3) = y_3$ (t_i, y_i given for $i = 1, 2, 3$)
- $q(t_5) = y_5$, $q(t_6) = y_6$, $q(t_7) = y_7$ (t_i, y_i given for $i = 5, 6, 7$)
- $p(t_4) = q(t_4)$, $p'(t_4) = q'(t_4)$ (t_4 is given)



Applications

a set of linear equations $Ax = b$ (with A $m \times n$) is

- *square* if $m = n$
- *underdetermined* if $m < n$
- *overdetermined* if $m > n$

all three types arise in practice

broad categories of applications:

- analysis or simulation
- control or design
- estimation or inversion

Analysis or simulation

- $y = Ax$ is a linear (or linearized) model of a physical system
- x and y are quantities that describe the operation of the system ($e.g.$, currents & voltages, forces & displacements, ...)

we know y; we are interested in determining x

examples

- structure ($e.g.$, building, bridge, ...): y_i's are forces or loads, x_i's are the resulting displacements at specified points
- electrical circuit: y_i's are values of voltage and current sources, x_i's are the resulting voltages and currents in the circuit

Control or design

- x is vector of design parameters or inputs (which we can choose)
- y is vector of results, or outcomes
- $y = Ax$ describes how input choices affect results

problems

- find x so that $y = y_{\mathrm{des}}$
- find all x's that result in $y = y_{\mathrm{des}}$ (*i.e.*, all designs that meet the specs)
- among x's that satisfy $y = y_{\mathrm{des}}$, find a small one (*i.e.*, find a small or efficient x that meets specifications)

example: final position/velocity of mass from forces (see p. 1-10)

relation between forces and final velocity/position:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 19/2 & 17/2 & \cdots & 1/2 \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{10} \end{bmatrix}$$

- x: sequence of forces
- y_1: final position; y_2: final velocity

problems: given desired final position/velocity $y_{\rm des}$

- find force sequence x so that $y = y_{\rm des}$ (2 linear equations in 10 vars.)
- find all x's that result in $y = y_{\rm des}$
- among x's that satisfy $y = y_{\rm des}$, find one with small $\|x\|$

Estimation or inversion

- y_i is a measurement or sensor reading (which we know)
- x_j is parameter to be estimated or determined
- $y = Ax$ describes how parameters affect measurements
  (a_{ij} is the sensitivity of the ith sensor to the jth parameter)

problems

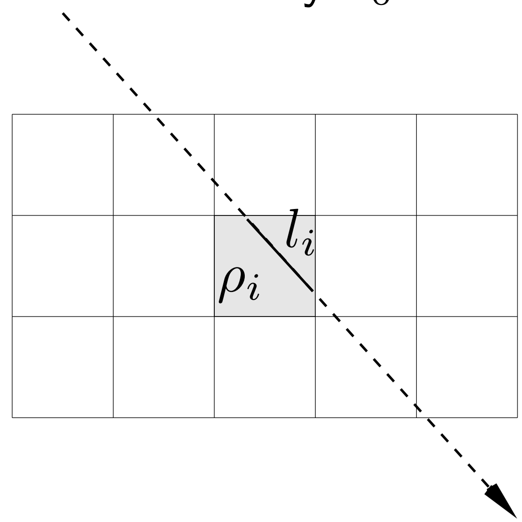
- find x, given y
- find all x's that result in y ($i.e.$, x's consistent with measurements)
- among all x's that result in y, find x closest to some prior guess x_{prior}
- if there is no x such that $y = Ax$, find x such that $y \approx Ax$ (if the sensor readings are inconsistent, find x which is almost consistent)

Image reconstruction from projections

problem: estimate density $\rho(x, y, z)$ of a 3-dimensional object

partition object in N small boxes ('pixels' or 'voxels') and assume density is constant on each pixel

X-ray beam with intensity I_0



measured intensity $I = I_0 \, e^{-\sum_{i=1}^{N} l_i \rho_i}$

ρ_i: (unknown) density in pixel i;
l_i: (known) length of path through pixel i (up to a physical constant)

after taking logs: one linear equation in the variables ρ_i:

$$\sum_{i=1}^{N} l_i \rho_i = \log(I_0/I)$$

repeat measurement with different source locations and beam directions: M measurements yield M linear equations

$$\sum_{i=1}^{N} l_{ki} \rho_i = \log(I_0/I_k), \quad k = 1, \dots, M$$

- usually a huge number of variables (millions)
- underdetermined set of linear equations ($M \ll N$): many solutions ρ_i
- a popular method: pick a solution ρ, consistent with the measurements, and close to a prior guess

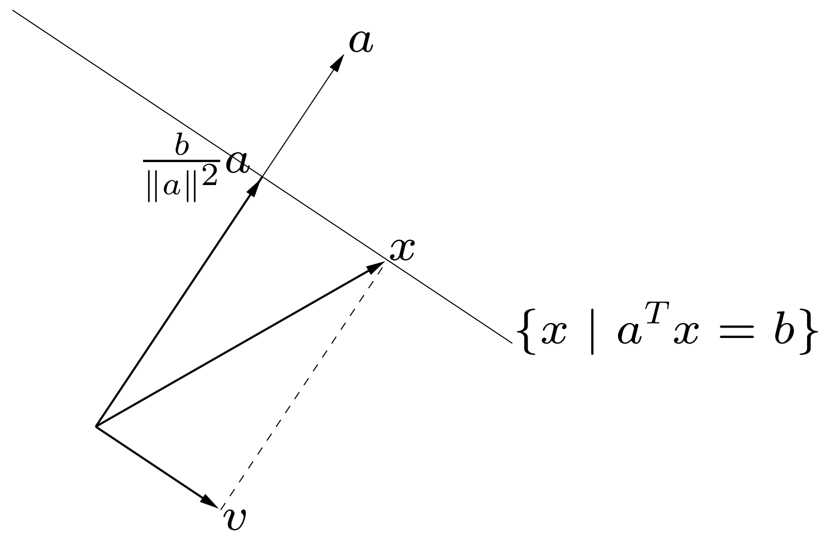
  for example, choose ρ_i that minimizes $\sum_{i=1}^{N}(\rho_i - \bar{\rho}_i)^2$ where $\bar{\rho}_i$ is a prior guess of the density ($e.g.$, uniform density)

Geometrical interpretation of $Ax = b$

special case: one equation in n variables ($m = 1$)

$$a^T x = b \qquad (a \neq 0)$$

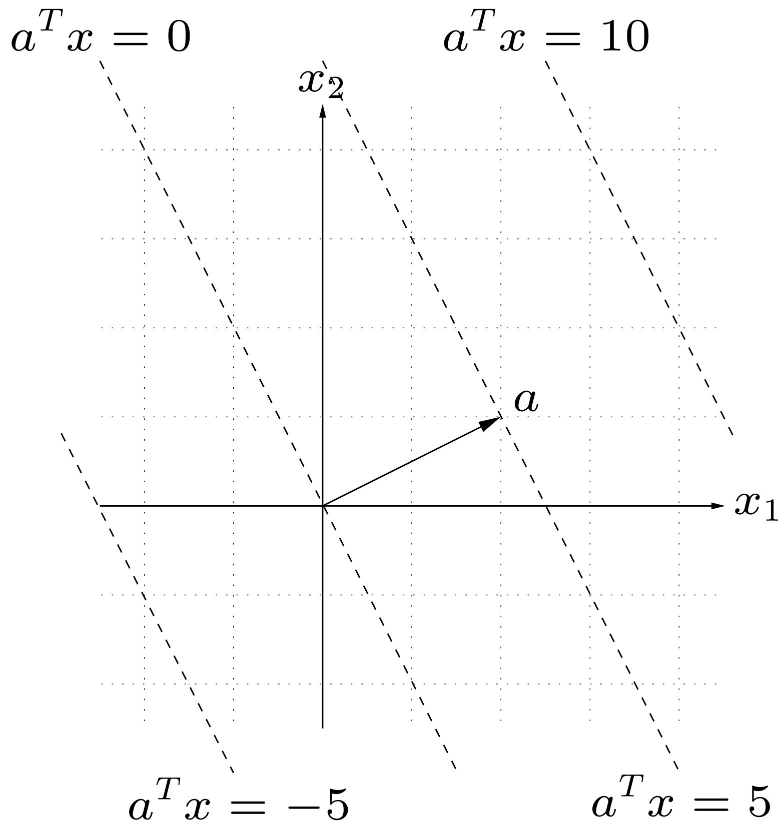
general solution: $x = \frac{b}{\|a\|^2} a + v$ where v is an arbitrary vector with $a^T v = 0$



the solution set is called a *(hyper-)plane*, a is the *normal vector*

for $n = 2$: a line perpendicular to a; $n = 3$: a plane perpendicular to a.

example $(n = 2,\ a = (2,1))$



general case: m equations in n variables

$$Ax = b$$

write A in terms of its rows:

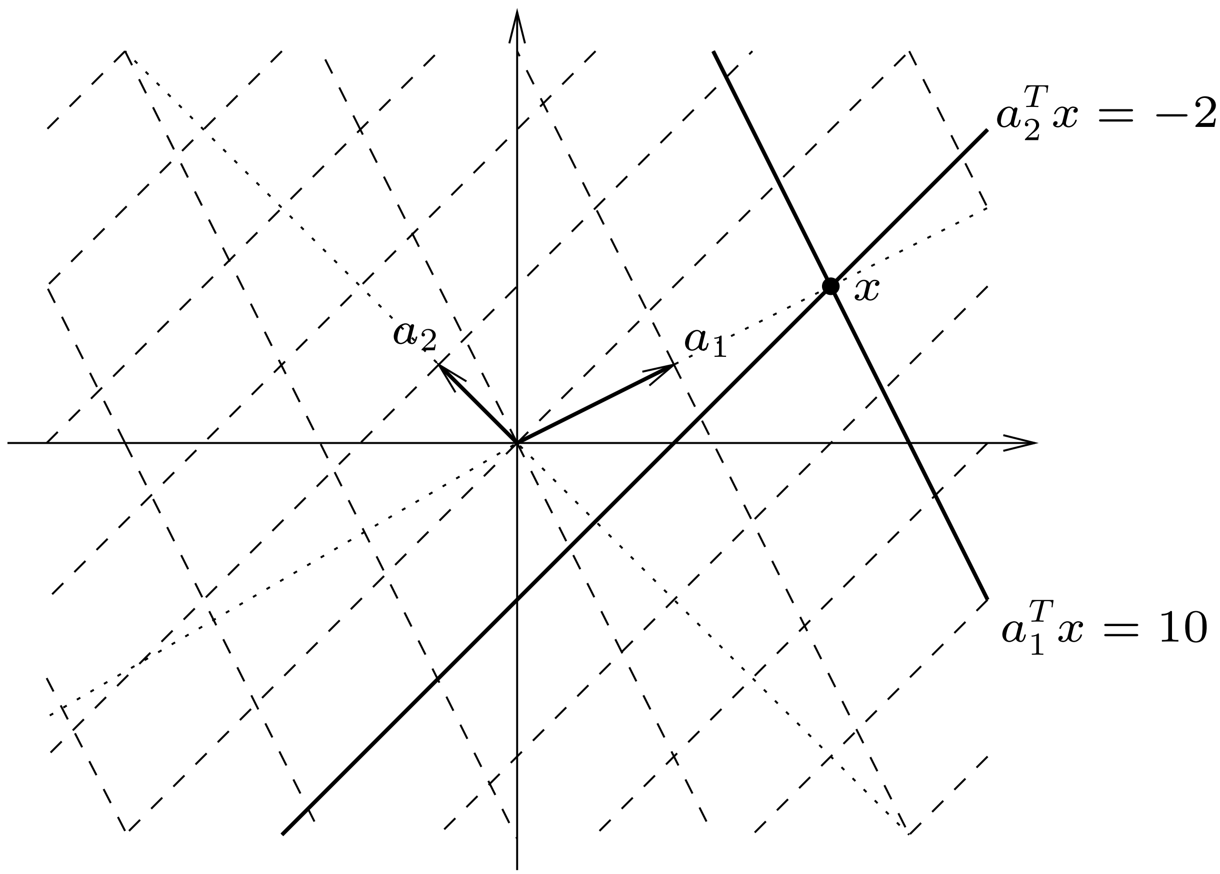
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} \quad \text{where } a_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix}$$

then $Ax = b$ can be written as

$$a_1^T x = b_1, \qquad a_2^T x = b_2, \qquad \ldots, \qquad a_m^T x = b_m$$

the solution set is the *intersection* of m hyperplanes $a_i^T x = b_i$
(assuming the rows a_i are nonzero)

example $(n = 2)$: $A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 10 \\ -2 \end{bmatrix}$



solution is $x = (4, 2)$