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Optimal Trajectories for Kinematic Planar Rigid Bodies with Switching Costs

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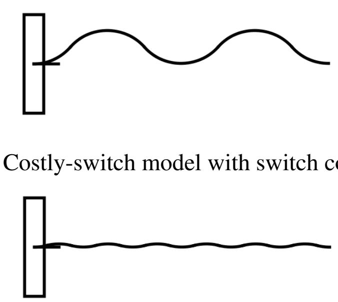
Abstract

The optimal trajectory with respect to some metric for a system with a discrete set of controls may require very many switches between controls, or even infinitely many, a phenomenon called *chattering*; this can be problematic for existing motion planning algorithms that plan using a finite set of motion primitives. One remedy is to add some penalty for switching between controls. This paper explores the implications of this *switching cost* for optimal trajectories, using kinematic rigid bodies in the plane (which have been studied extensively in the cost-free-switch model) as an example system. Blatt's Indifference Principle (BIP) is used to derive necessary conditions on optimal trajectories; Lipschitzian optimization techniques together with an A* search yield an algorithm for finding trajectories that can arbitrarily approximate the optimal trajectories.

1. Introduction

Consider the following problem. A mover would like to move a heavy park bench (modeled as a line segment) from one location and orientation to another, as efficiently as possible. Since the bench is heavy and there is only one mover, the bench can only be moved by lifting one end and rotating the bench around the end that is still on the ground, with rotational velocity of ± 1. We wish to find a sequence of durations and directions of rotations that bring the bench to the final configuration, while minimizing the total time of the trajectory (computed as the sum of the absolute values of the angles rotated through). This problem is very related to the Reeds-Shepp problem of finding the shortest path for a steered car ([25]), but with only four discrete controls.



(a) Costly-switch model with switch cost 1

(b) Chattering

Fig. 1. Trajectories for a bench starting at (-4, 0, 0). When the switch cost is one, the optimal trajectory takes 5 actions, shown on the top. However, if there is no switch cost, by increasing number of actions, it is always possible to create a faster trajectory, shown on the bottom.

For some configurations (moving the bench in a straight line), there exists no optimal trajectory with a finite number of actions: for any trajectory with finitely many switches, there is a faster trajectory with more switches, a phenomenon called *chattering*, see Figure 1. When chattering occurs, the bench mover is required to run back and forth between ends of the bench infinitely many times, rotating the bench by an infinitely small angle, see Figure 1.

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The chattering phenomenon is a fundamental problem in robot motion planning. Sussmann showed that an extension of the well-known Dubins car ([12]) to include bounds on angular acceleration leads to chattering ([30]). Desaulniers showed that chattering may occur if there are obstacles in the environment, even for Reeds-Shepp car that are well-behaved without obstacles ([11]).

One might argue that a motion planner does not need to find an optimal trajectory, and that it suffices to find a trajectory that is “good enough”. This is perhaps true, but systems that chatter also tend to expose weaknesses in the model that may not have been immediately apparent. A trajectory with very many turns is in fact quite bad for the bench mover, even if it is “short" in configuration space.

Choosing a set of primitive controls is a required first step for many general-purpose approaches to non-holonomic motion planning, including, for example, RRT-type motion planners ([15]). A natural, although certainly imperfect, approach to avoiding a very large number of discontinuous switches between discrete controls is to charge a fixed cost for switches; this approach has been used in practice at least as far back as ([4, 29]). This fixed cost both avoids chattering, and penalizes otherwise un-modeled costs (such as the cost of wearing out a switching mechanism, or the time cost of running between ends in the bench mover’s problem).

However, the implications of switching costs for optimal trajectories have perhaps not been thoroughly explored. In this paper, we limit the choice of controls to certain motion primitives. With this set of primitives, we associate with each pair of switch of controls a predetermined fixed cost. This predetermined cost may be suggested naturally by the design of the robot (for example, time cost of running between ends of a bench), or may be selected more arbitrarily to indicate a user preference for trajectories with fewer switches.

In order to make the consideration of switching costs more concrete, this paper focuses on time-optimal trajectories of kinematic rigid bodies in the plane. Rigid bodies are building blocks for many models of robotic locomotion or manipulation systems, and the time-optimal trajectories for the case when switching costs are zero are already well-understood [13].

A trajectory is defined by an initial configuration, and a sequence of motion primitives, each executed for some particular duration. We can view the optimal control problem as having two parts: selecting a sequence of primitives (the *discrete structure* of the trajectory), and choosing each duration (from a continuous interval).

For the problem of finding time-optimal trajectories of the cost-free-switch model studied in [13], Pontryagin’s Maximum Principle (PMP, [24]) was shown to give strong necessary conditions on both the discrete structure and continuous durations of optimal trajectories. A related principle, *Blatt’s Indifference Principle* (BIP), demonstrates existence of optimal trajectories for the costly-switch model, and also gives necessary conditions. However, the necessary conditions derived using Blatt’s Indifference Principle are weaker than those given by Pontryagin’s Maximum Principle, and although they tell us much about the continuous durations along trajectories, they do not constrain the discrete structures as strongly.

After deriving certain necessary conditions using BIP, we then work through an example of applying these necessary conditions directly to analyze time-optimal trajectories for the relatively simple bench-mover’s problem described above. However, due to the lack of constraints on trajectory structures, it appears very difficult to find similarly strong analytical results for more complicated systems, including other kinematic planar rigid bodies.

To attack the problem of finding optimal trajectories for kinematic planar rigid bodies with a specified set of primitives, we use BIP to classify trajectories into several types, which will be described later. For all but one of these classes, durations can be computed exactly, and we use an A* to search over trajectory structures. For the last remaining class, we show that Lipschitzian optimization techniques can be used to find good numerical approximations for the durations, while applying a different A* search over trajectory structures.

Although we believe that this work represents an interesting new exploration the connection between motion planning and optimal control, we admit that the algorithmic techniques presented in this paper suffer from some limitations. Many of these limitations do suggest rich problems for future study.

The focus on kinematic planar rigid-bodies with a time metric is limiting. Extending work on cost-free-switch models beyond simple systems or simple metrics using PMP has proved challenging, because the optimal trajectories for more complex systems may not be easy to describe analytically. However, choosing discrete (perhaps piecewise-constant) controls with a cost of switching ensures that optimal trajectories are describable by recognizable functions, and we believe the current techniques (particularly including application of Karush-Kuhn-Tucker conditions, which do not require integration of an adjoint vector) could be extended to more interesting systems.

Similarly, the current paper does not consider obstacles – but we do not believe this is a fundamental limitation. [13] has shown that optimal trajectories certainly exist for strictly positive switching costs even in the presence of obstacles (this not necessarily true for the cost-free-switch model), and although BIP has not been extended to allow for state constraints, we believe that it could be.

Perhaps the main limitation of the algorithms in this paper is computational cost. Due to the relative weakness of BIP w.r.t. PMP, the number of sequences of primitives generated in the algorithm may be exponential in the number of primitives, while they are only polynomial for the cost-free-switch model. Furthermore, the number of of sequences of primitives generated in the algorithm will increase when the cost of switch decreases. Hence, finding approximate optimal trajectories with many primitives is computationally infeasible by this method and we only use simple systems to demonstrate our technique in this paper. We believe that good heuristics for the A* search over discrete trajectory structures may ameliorate this issue.

1.1. Model and Notation

We use q to denote a configuration of the system and use u to denote a control in the control space U, which contains finite number of primitives.

At a configuration q, if we apply a control u, the instantaneous configuration space velocity can be expressed as a function f, such that $\dot{q} = f(q, u)$. A trajectory can be represented as a pair of sequences $(\mathbf{u}, \mathbf{t})$ with the initial configuration q_s, where $\mathbf{u} \in U^n$ is a sequence of controls and $\mathbf{t} \in \mathcal{R}_+^n$ is a sequence of durations. When the initial configuration is clear from the text, we use $(\mathbf{u}, \mathbf{t})$ to denote a trajectory.

We model the cost of control switches as a function $C : U \times U \to \mathcal{R}_+$ that depends on the control applied before and the control applied after. Furthermore, we assume that for any three controls u_a, u_b, and u_c, the cost of switching satisfies the *triangle inequality*, $C(u_a, u_b) + C(u_b, +u_c) \geq C(u_a, u_c)$, to ensure that switching from u_a to u_c directly is always faster than switching to u_c through other intermediate controls. The cost of a trajectory is the summation of all durations and all switch costs of a trajectory.

Problem statement: given a start configuration q_s, a final configuration q_f, a finite control set U, and a cost function C, find a trajectory $(\mathbf{u}, \mathbf{t})$ with minimum cost, connecting q_s to q_f.

1.2. Related Work

For some models of mobile robots in the plane, optimal trajectories can be found analytically, including Dubins car([12, 10] and Reeds-Shepp car([25, 31, 28]). We and many other researchers have tried to generalize techniques, typically based on Pontryagin's Maximum Principle ([24]), aiming to gain a greater understanding of optimal motion for mobile robots, including differential-drive ([26, 8, 7, 3, 27]), omni-directional vehicle ([2, 33]),and others([9, 1]). However, we are aware of little work in the robotics community providing strong results on optimal trajectories with a cost of switches; a notable exception is work by Stewart using a dynamic-programming approach to find optimal trajectories with a costly-switch model([29]).

The problem of costly switches has been studied in the optimal control community with results dating back as far as the 1970s. One of the most powerful tools for solving optimal control problems, Pontryagin's Maximum Principle (PMP), does not appear to be the right tool to characterize optimal trajectories in the costly-switch model due to the discontinuity with respect to time in the control and cost functions ([24]). In [6], Blatt proposed a model in which the control set contains certain primitives (a discrete set of actions), and there is some fixed cost associated with switching between controls. Blatt characterized a set of necessary conditions for optimal trajectories under this model; these necessary conditions are known as Blatt's Indifference Principle (BIP). Blatt showed that optimal trajectories always exist and the number of actions must be finite. Blatt's necessary conditions are similar to, but weaker than, those provided by PMP; using

BIP to solve an optimal control problem is more challenging than using PMP in the cost-free-switch model. In Blatt's model, the control set is a discrete set, but other models have been proposed ([21, 14, 20]).

Although the costly-switch model was proposed in the '70s, no algorithms for finding optimal trajectories in costly-switch model were proposed until the '90s ([32, 29]); several algorithms have been developed recently ([17, 34]). These recent approaches are based on approximating the control function as a piecewise-constant functions, and applying global optimization techniques to find optimal solutions. These algorithms converge to optimal solutions as the number of iterations approaches infinity, but cannot guarantee a bound of error within finite time. In this paper, we provide a stronger result for a particular system; the algorithm presented in this paper guarantees a bound of error within finite time, for the restricted problem of finding optimal trajectories of rigid bodies in the plane.

2. Mathematical Background

In this section, we review two mathematical tools for optimal control and non-linear programming: Blatt's Indifference Principle ([6]) and Karuhn-Kush-Tucker conditions.

For the rigid-body system studied in this paper, the Indifference Principle is sufficient. However, it is interesting that once a particular sequence of discrete, constant controls have been selected for a trajectory, the problem of selecting durations for each control is simply a finite constrained non-linear optimization problem for which KKT may be applied. Although we have used both approaches to derive similar results, the KKT approach is simpler in that it does not require the analytical integration of an adjoint vector – it is for this reason that we present both approaches.

2.1. *Blatt's Indifference Principle*

For the costly-switch model, BIP provides a set of necessary conditions for optimal trajectories for any finite dimension configuration space with finite control set ([6]). The configuration and control over time of an optimal trajectory $(\mathbf{u}^*, \mathbf{t}^*)$ from a configuration q_s can be represented as two functions $q^*(t)$ and $u^*(t)$, where $q^*(t)$ and $u^*(t)$ are the configuration of the robot and the control at time t respectively. BIP states that:

1. There exists a continuous adjoint function $\lambda(t)$, which is non-trivial.
2. The adjoint function satisfies

$$\frac{d\lambda}{dt} = \frac{\partial}{\partial q} H(\lambda(t), q^*(t), u^*(t))$$

where H is the *Hamiltonian*, which is the product of the velocity in the world frame and the adjoint function.

3. At the time $\hat{t}$ of switching control u to u', the Hamiltonian is indifferent to both u and u':

$$H(\lambda(\hat{t}), q^*(\hat{t}), u) = H(\lambda(\hat{t}), q^*(\hat{t}), u), \qquad (1)$$

and the Hamiltonian function is a positive constant along the trajectory.

For the cost-free-switch model, PMP also provides a set of necessary conditions on optimal trajectories, which has the same first two conditions, and different third condition: the control $u^*(t)$ maximizes the Hamiltonian along the trajectory.

2.2. *Karush-Kuhn-Tucker Conditions*

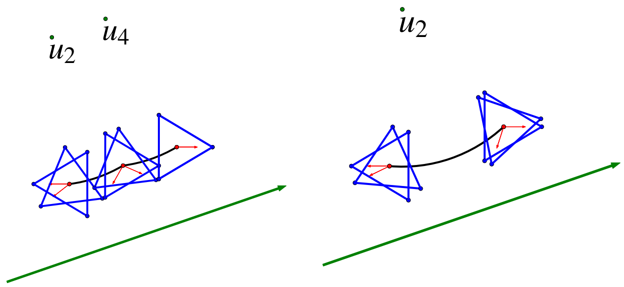
Karush-Kuhn-Tucker Conditions provides a set of necessary conditions for optimal solutions ([5]) of constrained non-linear optimization problems. Consider a non-linear optimization problem as follows

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & q(x) = 0 \\ & g(x) \leq 0 \\ & x \in \mathcal{R}^n, \text{with } f : \mathcal{R}^n \to \mathcal{R}, q : \mathcal{R}^n \to \mathcal{R}^m, \\ & \quad \text{and } g : \mathcal{R}^n \to \mathcal{R}^p \text{ differentiable.} \end{aligned}$$

The Karush-Kuhn-Tucker conditions are: If $\hat{x}$ is a local minimum and satisfies LICQ, then there exists $\lambda \in \mathcal{R}^m$ and $\mu \in \mathcal{R}^p$, such that

1. $\nabla f(\hat{x}) + \lambda \cdot \nabla q(\hat{x}) + \mu \cdot \nabla g(\hat{x}) = 0.$
2. $\mu \geq 0.$
3. $\mu \cdot g(\hat{x}) = 0.$

In order for these conditions to hold, certain *constraint qualification conditions* must be satisfied; since the primary focus of this paper is on BIP, we omit discussion of constraint qualification.



(a) Cost-free-switch model. (b) Costly-switch model with switch cost 1.

Fig. 2. Trajectories for an omni-directional vehicle starting at (-3, -1, π). For the cost-free-switch model, the optimal trajectory takes 5 actions. For the costly-switch model, the (approximately) optimal trajectory takes 3 actions. Thick lines are control lines.

3. Necessary Conditions for Optimal Trajectories

In this section, we will derive necessary conditions for optimal trajectories for rigid bodies in the plane in the costly-switch model. Based on these necessary conditions, we classify optimal trajectories into several classes and we also show that in order to find optimal trajectories, it suffices to find optimal trajectories in some trajectories classes. Since we focus on planar rigid-body robot, the configuration space is $SE(2)$ and we use $u = (v_x, v_y, \omega) \in \mathcal{R}^3$ to denote a control: x and y velocities in a frame attached to the body (robot frame), and angular velocity. Let U be the control space containing a finite number of primitives: constant-control actions. For example, one action might be $(v_x, v_y, \omega) = (1, 0, 0)$, corresponding to driving in a straight line.

Due to the similarity between BIP and PMP, several results in [13] in the cost-free-switch model can be extended to the costly-switch model by similar mechanisms.

Theorem 1. *For any rigid body in the plane in the costly-switch model, any optimal trajectory* $(\mathbf{u}^*, \mathbf{t}^*)$ *with* n *actions satisfies the following property: there exist four constants* $H > 0$*,* k_x*,* k_y*, and* k_θ*, such that for any control* u_i^**,* $1 \leq i \leq n$*, with the instantaneous velocity* (v_x, v_y, ω) *in the world frame when* u_i *is applied at a configuration* (x, y, θ)*, we have*

$$k_x v_x + k_y v_y + \omega(k_x y - k_y x + k_\theta) = H, \text{ where } k_x^2 + k_y^2 \in \{0, 1\}. \tag{2}$$

We also can derive the same result by applying KKT conditions (assuming constraint qualification holds), by fixing the sequence of controls in a trajectory, and showing that because the sequence is arbitrary, the result holds across all trajectory structures. The variables k_x, k_y, and k_θ are Lagrange multipliers from the KKT conditions; in Blatt's indifference principle, they arise as constants of integration.

A trajectory $(\mathbf{u}, \mathbf{t})$ is called *extremal* if there exist four constants $H > 0$, k_x, k_y, and k_θ, such that Equation 2 is satisfied.

Equation 2 is virtually identical to the necessary condition derived using PMP for the cost-free-switch problem, except that there is no requirement that controls maximize the Hamiltonian H. Instead, the Hamiltonian needs only be constant throughout the trajectory. Because of this similarity, a similar geometric structure to that which arises for the cost-free model exists. If $k_x^2 + k_y^2 = 1$, then the expression $k_x y - k_y x + k_\theta$ in Equation 2 can be interpreted as computing the distance of a point x, y from some line described by constants k_x, k_y, and k_θ. We therefore call such a trajectory a *control line trajectory*. An extremal trajectory with $k_x^2 + k_y^2 = 0$ is called a *whirl* trajectory because the angular velocity of body is constant and non-zero over the trajectory.

3.1. *Control Line Trajectories*

There is a nice geometric interpretation for Theorem 1 when $k_x^2 + k_y^2 = 1$, related to the control line interpretation in [13]. For a control line trajectory $(\mathbf{u}, \mathbf{t})$, we define its corresponding *control line*, represented as (k_x, k_y, k_θ) as a line in the plane with heading (k_x, k_y) and distance k_θ from the origin. Now, consider Equation 2. The term $k_x v_x + k_y v_y$ becomes the translational velocity along the vector (k_x, k_y) and the term $k_x y - k_y x + k_\theta$ becomes the *signed distance* from the reference point of the robot to the control line. By Corollary 1 in [13], when a rotation is applied, the signed distance from the rotation center to the control line is always H/ω. Similarly, when a translation is applied, the dot product between (k_x, k_y) and (v_x, v_y) must be the Hamiltonian value H. See Figure 2 for an (approximately) optimal trajectory for an omni-directional vehicle with control lines in the cost-free-switch model and in the costly-switch model. When the switch cost is introduced, optimal trajectories tend to use fewer number of switches.

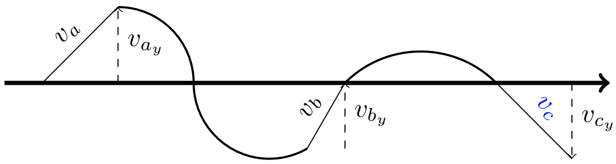


Fig. 3. Illustration of proof of theorem 2: a trajectory containing three actions of translations, v_a, v_b, and v_c. The sign of v_{ay} and v_{by} are the same.

Necessary conditions for control line trajectories. We can prove a further necessary condition for a control line trajectory to be optimal.

Theorem 2. *For any rigid body in the plane in the costly-switch model, any optimal control line trajectory has either zero translation actions, one translation action, or two non-parallel translation actions.*

Proof. Let $g = (\mathbf{u}, \mathbf{t})$ be a control line trajectory. Suppose that g is optimal but has two parallel translation actions. Let v_a and v_b be the velocity vectors in the world frame of two non-parallel translation actions of g. We can remove the action of v_b from g and increase the duration of v_a to $t_a + t_b$. The resulting trajectory still reaches the goal but has one fewer control and hence has smaller cost. This contradicts the optimality of g.

Suppose that g is optimal but has more than two non-parallel translation actions Let v_a, v_b, and v_c be the velocity vectors in the world frame of three translation actions of g. By Equation 2, we know that the projection of v_a, v_b, and v_c onto the control line must be the Hamiltonian value H. Let v_{a_y}, v_{b_y}, and v_{c_y} be the projection of v_a, v_b, and v_c onto the norm of the control line. By the Pigeonhole Principle, we know that at least two of v_{a_y}, v_{b_y}, and v_{c_y} have the same sign.

Without loss of generality, assume that v_{a_y} and v_{b_y} have the same sign; let their durations be t_a and t_b respectively. See Figure 3. If $v_{a_y} = v_{b_y}$, then the velocity vectors v_a and v_b are identical. This contradicts the assumption that v_a and v_b are non-parallel. If $v_{a_y} \neq v_{b_y}$, then without loss of generality, we assume $|v_{a_y}| > |v_{b_y}|$.

Since the projections of v_a and v_b onto the control line are the same, we can remove the actions of v_b from g and increase the duration of v_a to $t_a + \frac{t_b|v_{b_y}|}{|v_{a_y}|}$. Let u be the control corresponding to the translation vector v_b. Let u_p and u_q be the control before and after u in the trajectory. The new trajectory will decrease cost by $\frac{t_b|v_{b_y}|}{|v_{a_y}|} - t_b - C(u_p, u) - C(u, u_q) + C(u_p, u_q)$, which is strictly larger than zero. Hence, the resulting trajectory has smaller cost but still reaches the goal. This also contradicts the optimality of g. □

Singular, TGT, and regular trajectories. [13] classified control line trajectories into four classes: singular, TGT, generic, and regular.

A control line trajectory is called *singular* if there exists a non-zero measure interval along the trajectory that multiple controls have the same Hamiltonian value within the interval.

As an extension of a result in [13], any singular trajectory in the costly-switch model contains exactly one translation with velocity vector parallel to the control line, or contains exactly one switch from one translation to another translation. Hence, by Equation 2, the Hamiltonian value is either equal to the velocity of the only translation, or can be computed from the pair of consecutive translations. Since the control set U is a given finite set, the set of all possible Hamiltonian values for singular trajectories is finite.

A control line trajectory is called *generic* if the trajectory is not singular. For generic trajectories, switching between two translations can not occur, since switching between two translations only happens for singular trajectories. A generic trajectory is further called *TGT* if both the first control and the last control are translations, and *regular* otherwise. For a TGT trajectory, when the initial configuration and goal configuration are given, we can obtain the Hamiltonian value analytically, using methods from [13].

3.2. Whirl Trajectories

For whirl trajectories, Equation 2 only implies that all angular velocities are equal. We also can extend the result in [13] to the costly-switch model.

First we show that in order to compute an optimal whirl trajectory, it suffices to consider a smaller subclass, called *two-stage whirl trajectories*:

1. Move the last rotation center to the correct position in the goal configuration using the minimum cost.
2. Rotate around the last rotation center until the goal

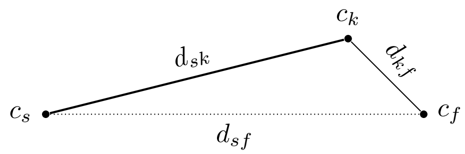


Fig. 4. Illustration of proof of Theorem 5. The thick line is the control direction where all rotations except the last one are on this line.

configuration is achieved.

Theorem 3. *For any rigid body in the plane in the costly-switch model, among all whirl trajectories, there exists one two-stage trajectory with the minimum cost.*

Proof. Let T_1 and T_2 be the durations corresponding to the first and the second stage respectively. Let T_f be the duration of an optimal trajectory. Since T_f is the duration of an optimal trajectory, $T_f \leq T_1 + T_2$. Moreover, since an optimal trajectory needs to place the last rotation center in the correct position, $T_f \geq T_1$. Since T_2 is strictly less than 2π, we have $T_f \leq T_1 + T_2 < T_f + 2\pi$. For any two admissible whirl trajectories, the difference of total durations must be a multiple of 2π. Therefore, T_f must equal $T_1 + T_2$. □

Furthermore, we extend the result in [13] to the costly-switch model.

Theorem 4. *Any two-stage trajectory must satisfy the following property: there exist three constants $H_\omega > 0$, k_α, and k_β, such that for any control u_i with the instantaneous velocity (v_x, v_y, ω) in the world frame when u_i,$1 \leq i < n$, is applied at configuration (x, y, θ), we have*

$$k_\alpha v_x + k_\beta v_y = H_\omega, \text{where } k_\alpha^2 + k_\beta^2 = 1. \quad (3)$$

Control direction interpretation for two-stage trajectories. For a two-stage trajectory, we define its *control direction* as a line heading (k_α, k_β) through the rotation center of the first control. By Equation 3, all rotation centers except the last one should have the same signed distance to this line. Since the first rotation is on the control direction, all rotation centers except the last one are parallel to the control direction.

Here is an extension of result [13] in the cost-free-switch model.

Theorem 5. *For the costly switch model, consider a two-stage trajectory $(\mathbf{u}, \mathbf{t})$ with $u_s = u_1, ..., u_{n-1} = u_k, u_n = u_f$. Let c_s, c_k, and c_n be the rotation centers of u_1, u_{n-1}, and u_n respectively. Let d_{sf} be the distance between c_s and c_f. Let d_{kf} be the distance between the rotation centers of u_k and u_f in the robot frame. Let l_i be the distance between the rotation centers of u_i and u_{i+1} in the robot frame. Let $d_{sk} = \sum_{i=1}^{n-2} l_i$. We have*

$$d_{sk} \in [|d_{sf} - d_{kf}|, d_{sf} + d_{kf}]. \quad (4)$$

Proof. By the geometric interpretation of Theorem 4, since all rotation centers from u_1 to u_{n-1} are on the same line, we know that d_{sf}, d_{kf}, and d_{sk} form a triangle. See Figure 4. Moreover, any sequence of controls $\mathbf{u}$ that all controls have the same angular velocity and satisfy Equation 4 can form a two-stage trajectory. □

3.3. Taxonomy of Optimal Trajectories

We summarize the taxonomy of optimal trajectories as Figure 5.

Since the Hamiltonian values for whirl, TGT, and singular trajectories can be determined, the problems of finding optimal trajectories in these three classes is equivalent to finding an optimal sequence of controls, a discrete search problem. For these three classes, we have designed three different A* search algorithms to find candidate optimal trajectories by searching over discrete trajectory structures.

The problem of finding optimal regular trajectories has two ingredients: one is finding the Hamiltonian value H, which is a continuous variable, and another one is finding the sequence of controls, chosen from a finite set.

4. The Bench Mover's Problem

In this section, as a working example, we demonstrate how to use necessary conditions of optimal trajectories to solve the simple bench mover's problem exactly.

4.1. Model and Trajectory Types

Consider a park bench with length 2. Let q_s be the initial configuration and $(0, 0, 0)$ be the goal configuration. Figure 6 gives an example with initial configuration $(-3, -3, \pi/4)$. Let the reference point be the center of the bench, $(0, 0)$ in the

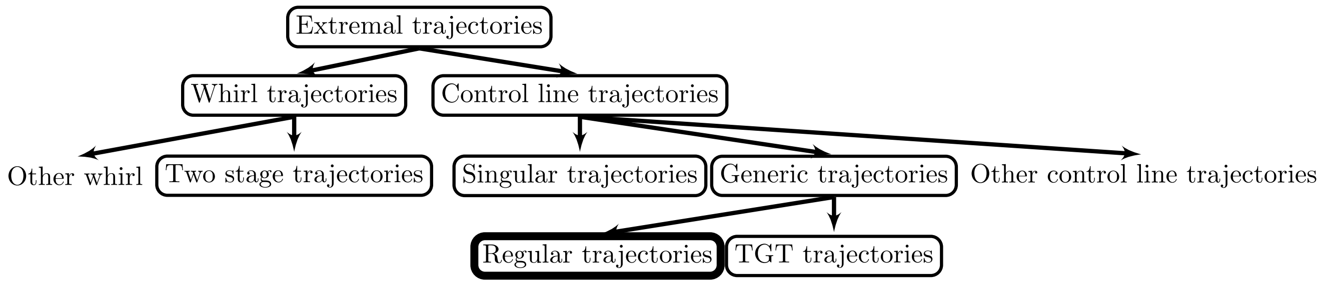


Fig. 5. Taxonomy of optimal trajectories. Each node corresponds to a type of optimal trajectories; each leaf node without border is not necessary for optimality. All leaf nodes with single border can be solved exactly. For the leaf node with double border, regular trajectories, we provide a search algorithm that can find a trajectory arbitrarily close to optimal trajectories.

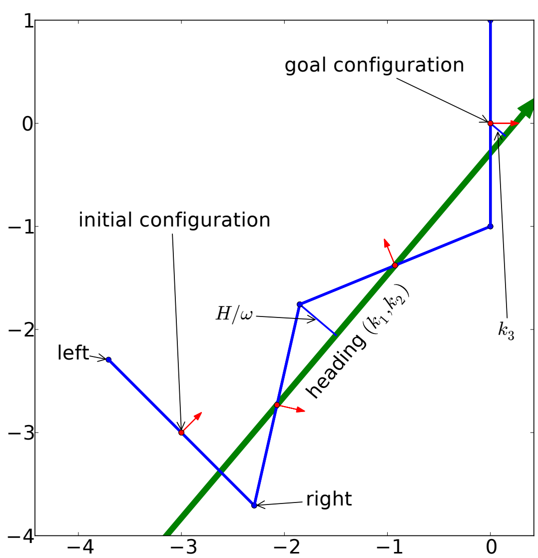


Fig. 6. Optimal trajectory for initial configuration $(-3, -3, \pi/4)$ with switching cost 1, where arrow represents the orientation of the bench. Thick line denotes the control line for this trajectory.

robot frame. There are two rotation centers: the left rotation center, $(0, 1)$ in the robot frame, and the right rotation center, $(0, -1)$ in the robot frame.

Let L be the set of controls containing $l^+ = (1, 0, 1)$ and $l^- = (-1, 0, -1)$ corresponding the left rotation center. Let R be the set of controls containing $r^+ = (-1, 0, 1)$ and $r^- = (1, 0, -1)$ corresponding to the right rotation center. The control set $U = L \cup R$. For two controls $u, u' \in U$, the cost of switching from u to u' is a constant c.

since we can determine optimal durations for a control sequence with length smaller than three easily, we focus below on the case where the length of the control sequence is at least three.

There are three broad types of trajectories:

1. Whirl: trajectories for which $k_x = k_y = 0$. All controls in the trajectory must have the same angular velocity.
2. Alternating sign: the control sequence contains controls alternating between l^+ and r^- or alternating between l^- and r^+.
3. Mixed: the control sequence contains controls alternating between L and R but not strictly alternating signs.

Our basic approach, given an initial configuration, is to compute an optimal trajectory of each of the three types, and then to compare to find the minimum. The following sections will demonstrate how to find an optimal trajectory for each type. For computing optimal trajectories of types 2 and 3, an upper bound on the number of control actions in the trajectory is required; this bound may be found by considering the cost of the optimal whirl.

4.2. *Whirl Trajectories*

For whirl trajectories, all rotation centers except possibly the last one are on the same line; see Figure 7. This section will show how this fact can be used to identify the minimum-cost whirl.

When the first control and the last control are fixed, the distance between their rotation centers is determined. Since the length of the bench is two and controls alternate between L and R, for any two consecutive controls, the distance between their rotation centers is two. Thus, in order to reach the goal, there is only one choice of the length of the control sequence, see Figure 8. Hence, a whirl trajectory can be described by its first and last control. Since there are only four choices for the first control and each has two choices

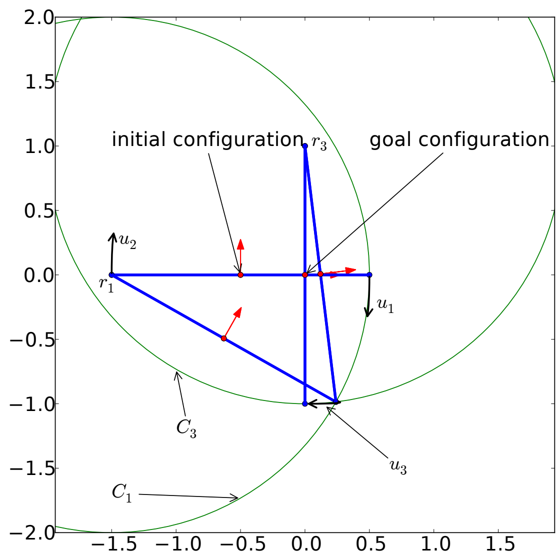


Fig. 7. Whirl trajectory with initial configuration $(-0.5, 0, \pi/2)$. All rotation centers except the last one are on the same line. This is the optimal trajectory for this initial configuration with switching cost 1.

for the last control, we can enumerate all possible pairs of first and last controls for whirl trajectories, see Figure 8.

Fix the first control u_1 and the last control u_n, with rotation centers r_1 and r_n respectively. Since the last control is fixed, the second to the last control u_{n-1} is also fixed and its rotation center r_{n-1} should be on a circle C_n centered at r_n with radius 2.

Since rotation centers r_i, $1 \leq i < n$, are on the same line, the distance L from r_1 to r_{n-1} is determined in the following way. Let D be the distance between the first and the last rotation centers. If $u_1 = u_n$ ($u_1 \neq u_{n-1}$), then L is multiple of four plus 2. Otherwise, L is multiple of four. Since the diameter of C is four and the difference between any choices is multiple of four, $L = 4\lceil (D-4)/4 \rceil + 2$ when $u_1 = u_n$, $L = 4\lceil (D-2)/4 \rceil$ otherwise.

After we determine L, we can find a circle C_1 centered at r_1 with radius L. The circle C_1 intersects with C_n at most two points and these points are possible locations of r_{n-1}. When the location of r_{n-1} is fixed, the durations for all controls can be determined easily.

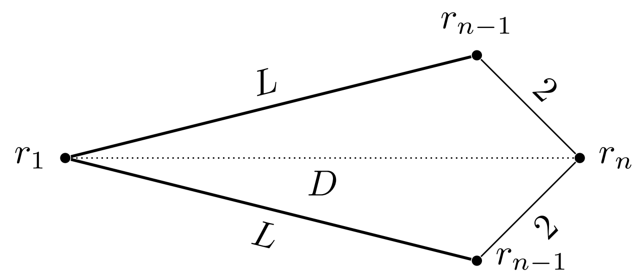


Fig. 8. When the first control and the last control are fixed, the rotation centers r_1 and r_n are fixed as well. The distance between r_1 and r_n is D. Since all rotation centers except for the last one is on the same line and distance between two consecutive rotation centers is two, the distance between r_1 and r_{n-1}, L, is a multiple of two. Since the distance between r_{n-1} and r_n is two, there is only one choice of L and two symmetric choices of locations of r_{n-1}

4.3. Alternating Sign Trajectories

Since all angular velocities have the same absolute value, all rotation centers must have equal distance to the control line; see Figure 6. An alternating sign trajectory can be described by its first control and the length of the sequence. There are four choices of first control u_1 in U, and the parity of n determines whether u_n is the same as u_1 or not. We will now show how to determine possible Hamiltonian values H, control lines, and durations based on u_1 and u_n.

Determining the Hamiltonian value H. Let r_1 and r_n be the first rotation center and the last rotation center, separated by distance D. If n is odd, $D = (2n-2)\sqrt{1-H^2}$ and $H = \sqrt{1 - \frac{D^2}{4(n-1)^2}}$. In this case, when $D^2 \geq 4(n-1)^2$, the control line exists and we can obtain a positive value of $H \leq 1$. When n is even, let X be $(2n-4)\sqrt{1-H^2}$, D^2 will be $X^2 + \sqrt{1-H^2} + 4$. Consequently, $D^2 = 4n(n-2)(1-H^2) + 4$ and $H = \sqrt{1 - \frac{D^2-4}{4n(n-2)}}$. In this case, when $D \geq 4$ and $D^2 \leq 4n(n-2)+4$ we can obtain a non-negative value of Hamiltonian values $H \leq 1$.

Determining control lines. After we determine the Hamiltonian value H, we determine the control line, which is represented by a tuple (k_x, k_y, k_θ) as follows. Since $k_x^2 + k_y^2 = 1$, we can use $(\cos\varphi, \sin\varphi)$ to represent (k_x, k_y). For one Hamiltonian value, H, there are two possible control lines. We determine (φ, k_θ) in a similar way as in [13]. Let $r'_{1_x} = r_{1_x} u_{1_\omega}, r'_{1_y} = r_{1_y} u_{1_\omega}, r'_{n_x} = r_{n_x} u_{n_\omega}$, and

$r'_{n_y} = r_{n_y} u_{n_\omega}$. Let $d_x = r'_{1_x} - r'_{n_x}$ and $d_y = r'_{1_y} - r'_{n_y}$. Let (α, β) be $(\mathrm{atan}(dx, dy), \pi/2)$ if the first control and the last control have the same angular velocity; otherwise $(\mathrm{atan}(d'_x, d'_y), \mathrm{acos}(\frac{H}{\sqrt{{d'_x}^2 + {d'_y}^2}}))$, where $d'_x = r'_{n_x} + d_x/2$ and $d'_y = r'_{n_y} + d_y/2$. Then, $\varphi = -\alpha \pm \beta$ and $k_\theta = \frac{H + r'_{n_x} \sin\varphi - r'_{n_y} \cos\varphi}{u_{n_\omega}}$.

Determining durations. For a given control line $L = (k_x, k_y, k_\theta)$ with a Hamiltonian value H, we can determine the durations as follows. For a given initial configuration q_0 with reference point at p, we can determine the angle α between the vector $p - r_1$ and L. Then, we determine the location of r_2 for u_2 as follows. By Theorem 1, all rotation centers have the same distance H to L. For two consecutive rotation centers, their distance must be 2, the length of the bench. Hence, the angle β between the vector $r_2 - r_1$ and L can only have two possible values: $\mathrm{asin}(H)$ and $\pi - \mathrm{asin}(H)$ if $u_{1_\omega} > 0$, otherwise $\pi + \mathrm{asin}(H)$ and $2\pi - \mathrm{asin}(H)$. For a fixed β, we can determine t_1.

For u_2, since we switch to u_2 at angle β with respect to L and $u_3 = u_1$, we also can switch to u_3 immediately with $t_2 = 0$. Since this null control will be examined by another control sequence, we ignore this choice. Consequently, we have only one choice of t_2. Similarly, all controls u_2 to u_{n-1} have the same duration. This duration will be either $2\mathrm{acos}H$ (if $\beta \in [\pi/2, \pi/2]$) or $2\pi - 2\mathrm{acos}H$ (otherwise). The duration t_n can be determined based on t_1 to t_{n-1} based on the constraint of reaching the goal configuration.

4.4. Mixed Trajectories

Consider a mixed trajectory that contains two consecutive controls, u and u', with the same angular velocity. By Theorem 1, these two controls' rotation centers, r and r', have equal distance to the control line, L, and are on the same side of the control line. Hence, the line through r and r' is parallel to L; see Figure 9. Consequently, if there are three consecutive controls with the same angular velocity, then the duration of the second one must be π.

For a mixed trajectory, subsequences of controls with the same angular velocity may appear anywhere in the control sequence. However, it is always possible to rearrange the controls in the control sequence without changing the cost, such that the prefix of the control sequence has controls

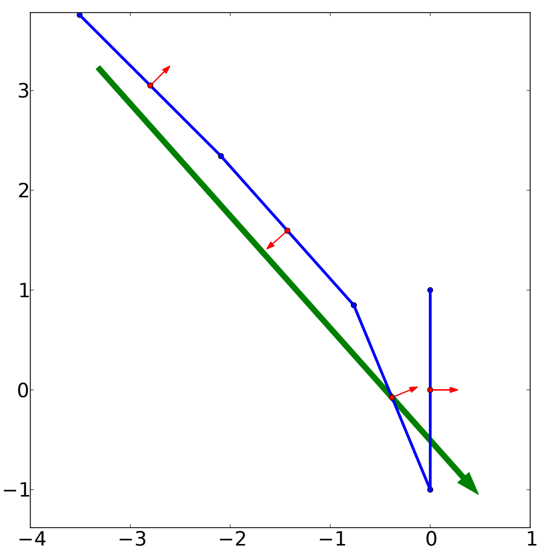


Fig. 9. Mixed trajectory with initial configuration $(-2.8, 3.05, \pi/4)$. First two controls have the same angular velocity and hence they are collinear and parallel to the control line. This is the optimal trajectory for this initial configuration with switching cost 1.

with the same angular velocity and the suffix has controls with alternating angular velocity. Hence, we only consider control sequences that can be decomposed into two parts in this way.

Let n be the length of the control sequence, $n_w < n$ be the number of controls with the same sign, and m be $n - n_w$. Let D be the distance between the first rotation center and the last rotation center. When m is even, H satisfies $|D - 2(n_w - 1)| = 2m\sqrt{1 - H^2}$ or $D + 2(n_w - 1) = 2m\sqrt{1 - H^2}$. Hence, $H = \sqrt{1 - \frac{(D - 2(n_w - 1))^2}{4m^2}}$ or $H = \sqrt{1 - \frac{(D + 2(n_w - 1))^2}{4m^2}}$. We can obtain at most two possible positive $H \leq 1$ values.

When m is odd, H satisfies $D^2/4 = (m^2 - 1)(1 - H^2) + 2m(n_w - 1)\sqrt{1 - H^2} + n_w^2 - 2n_w + 2$ or $D^2/4 = (m^2 - 1)(1 - H^2) - 2m(n_w - 1)\sqrt{1 - H^2} + n_w^2 - 2n_w + 2$. Hence, $1 - H^2 = |\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}|$, where $a = m^2 - 1$, $b = 2m(n_w - 1)$, and $c = n_w^2 - 2n_w + 2$. We can obtain at most two possible positive Hamiltonian values $H \leq 1$ values.

After we obtain Hamiltonian values H, we can compute the control line and durations by the method mentioned in the previous section.

Algorithm
`CostlySwitchOptimalMotionPlanner()`

$C \leftarrow$ feasible trajectories found by any planner.
$B \leftarrow$ upper bound of number switches.
```
// Find best two-stage
   trajectories, see Section 6
```
foreach *3-tuple of controls* (u_s, u_k, u_f) *with the same angular velocity.* **do**
  $C = C\cup$ best two-stage trajectory $(\mathbf{u}, \mathbf{t})$ with $u_1 = u_s$, $u_{n-1} = u_k$, $u_n = u_f$.
end
```
// Find best TGT trajectories, see
   Section 8
```
foreach *2-tuple of translation controls* (u_s, u_f) **do**
  $C = C\cup$ best TGT trajectory $(\mathbf{u}, \mathbf{t})$ with $u_1 = u_s$ and $u_n = u_f$.
end
```
// Find best singular trajectories,
   see Section 9
```
foreach *2-tuple of controls* (u_s, u_f) **do**
  foreach *singular value* H **do**
    $C = C\cup$ best singular trajectory $(\mathbf{u}, \mathbf{t})$ with $u_1 = u_s$, $u_n = u_f$ and Hamiltonian value H.
  end
end
```
// Find best regular trajectories,
   see Section 10
```
foreach *2-tuple of controls* (u_s, u_f) **do**
  foreach *interval* I *of Hamiltonian values not including singular values* **do**
    $C = C\cup$ approximately best regular trajectory $(\mathbf{u}, \mathbf{t})$ with $u_1 = u_s$ and $u_n = u_f$ in the interval I.
  end
end
return the optimal trajectory in C

Algorithm 1: Outline of the algorithm

5. Outline of an Algorithm for Finding Optimal Trajectories for Rigid-Body System in the Plane

In the previous section, we showed how to find optimal trajectories for a specific system, the bench mover's problem, exactly. However, for more complex systems, deriving an analytical result is challenging, since we not only need to determine the optimal Hamiltonian value but also the optimal control sequence. In this section, we design an algorithm to find (approximately) optimal trajectories for all rigid-body systems.

First, we want to find an upper bound of number of switches for optimal trajectories. The upper bound limits the length of control sequence that the algorithm needs to enumerate. We use the general planner described in [13] to obtain a feasible trajectory from q_s to q_f with cost M. Then, we can get an upper bound of number switches as $B = \lceil M/c_{\min} \rceil$, where $c_{\min}$ is the minimum switch cost.

By the taxonomy of optimal trajectories, we know that it is sufficient to find best two-stage, TGT, singular, and regular trajectories. For each of these four classes, we design algorithms separately in Section 6 to Section 10.

Since two-stage trajectories are whirl trajectories and all other three classes are control line trajectories, we explain how to find best two-stage trajectories first in Section 6. Then, we introduce several properties for control line trajectories in Appendix A. Finally, we show how to find best TGT trajectories in Section 8, singular trajectories in Section 9, and regular trajectories in Section 10.

The idea of finding best trajectories within each class is quite similar: decompose the problem into several sub-problems by enumerating the first control, the last control, and possibly the second to the last control (for two-stage trajectories only). For two-stage, TGT, and singular trajectories, the sub-problems are discrete optimization problems so that we use an A* search algorithm to find best trajectories exactly. For regular trajectories, the sub-problem is a mixed non-linear optimization problem; we combine Lipschitzian optimization techniques with an A* search algorithm over discrete trajectory structures to determine approximately best trajectories. We give the outline of the algorithm in Algorithm 1.

6. Finding Best Two-stage Whirl Trajectories

Recall that a two-stage trajectory is a trajectory such that all controls have the same angular velocity and all rotation centers except the last one are on the same line. Our approach is to enumerate all 3-tuple of controls (u_s, u_k, u_f) with the same angular velocity. For each $(u_s, u_k, u_f) \in U^3$, we determine a best two-stage trajectory $(\mathbf{u}, \mathbf{t})$, subject to $u_1 = u_s$, $u_{n-1} = u_k$, and $u_n = u_f$. Then, we pick the best trajectory among all best two-stage trajectories with respect to all 3-tuples of controls.

Fix a 3-tuple of controls (u_s, u_k, u_f) with the same angular velocities. Let U' be the reduced control set that all controls in U' have the same angular velocity as u_s. We want to find a best two-stage trajectory $(\mathbf{u}, \mathbf{t})$, subject to $u_s = u_1, ..., u_{n-1} = u_k, u_n = u_f$ and $u_i \in U'$ for all i.

Our method is to incrementally build sequences of controls that could possibly satisfy Equation 4 by using A* search. Each state is a sequence of controls $g = (u_1 = u_s, \ldots, u_{h-1} = u_k, u_h = u_f)$, where all controls have the same angular velocity. For a state $g = (u_1, \ldots, u_h)$, the duration of each control u_i, $1 < i < h-1$ is fully determined, since all rotation centers except the last one are on the same line. Hence, we use the summation of the switching costs and the durations for each control u_i, $1 < i < h-1$ as path cost. The neighbors of a state $g = (u_1, \ldots, u_h)$ are the states $g' = (u_1 = u_s, \ldots, u_{h-1}, u', u_h = u_k, u_{h+1} = u_f)$, where $u' \in U'$.

A state $g = (u_1, \ldots, u_h)$ reaches the goal if $(u_1, \ldots, u_h, u_k, u_f)$ satisfies Equation 4. When a state reaches the goal, we can solve for the duration of u_1 and u_h exactly with at most two solutions.

In order to speed up the A* search, we need an admissible heuristic. Let d_{sf} be the distance between the rotation center of u_s at q_s and u_f at q_f. For a state $g = (u_1, \ldots, u_h)$, let $l_g = \sum_{i=1}^{h-2} D(u_i, u_{i+1})$, where $D(u, u')$ is the distance between the rotation centers of u and u' in the robot frame. That is, l_g is the possible value of d_{sk} for the state in Equation 4. In order to satisfy Equation 4, we know that I_g must be at least $|d_{sf} - D(u_k, u_f)|$. Hence, we use $|d_{sf} - D(u_k, u_f)| - l_g$ as an admissible heuristic.

7. Properties of Control Line Trajectories

In this section, we briefly describe several properties of control line trajectories; additional details are in Appendix A and B.

Recall that a control line trajectory is an extremal trajectory with constants H, k_x, k_y, and k_θ such that $k_x^2 + k_y^2 = 1$. When the initial configuration q_s, the first control u_s, the final configuration q_f, the last control u_f, and the Hamiltonian value H are given, the control line can be constructed explicitly:

Theorem 6. *For a given initial configuration q_s, given goal configuration q_f, first control u_s, and the last control u_f, we can determine an interval of Hamiltonian values $I = (0, H_u)$, such that there exists mappings, $L_1(H)$ and $L_2(H)$, from Hamiltonian values in I to control lines.*

A *singular interval* is an interval of time within which at every time more than one control gives the same value for the Hamiltonian. Except within singular intervals, knowing the configuration of the rigid body, the location of the control line, and the current and next controls, essentially tells when the next switch will be:

Theorem 7. *Given a control line L and a non-singular interval, the duration of applying a control u from a configuration q until switching to another control u' has at most two possibilities such that the resulting motion can be a subtrajectory of a control line trajectory corresponding to the control line L. Moreover, these two possibilities can be determined analytically.*

Singular trajectories can only occur at particular critical values of the Hamiltonian. Except at these critical values, knowing the Hamiltonian value H and three consecutive controls is sufficient to compute the duration of the middle control. The following theorem will be proved in Appendix B:

Theorem 8. *Let u, u', and u'' be three consecutive controls in a control line trajectory. Given k_x, k_y, and a non-critical Hamiltonian value H, the duration of u' has at most two possibilities and can be determined analytically without knowing the configuration.*

Finally, for a given sequence of controls and a control line L, there is a way to determine the duration of each control when we are given the initial configuration q_s and the final configuration q_f:

Theorem 9. *Let $\mathbf{u} \in U^n$ be a sequence of controls. Let $\mathbf{b} \in \{1, 2\}^{n-1}$ be a sequence of* selector. *For a given control line L with a non-critical Hamiltonian value H, initial configuration q_s, and final configuration q_f, the duration of each control is fully determined by $\mathbf{u}$ and $\mathbf{b}$.*

By Theorem 6, the control line can be parametrized by the Hamiltonian value H. Suppose that the mapping from the Hamiltonian values to control lines is fixed. Given a sequence of control $\mathbf{u}$ and selectors, since the duration is fully determined by the control line L, the durations can also be parametrized by the Hamiltonian value H.

In order to find best trajectories corresponding to a control line L of known location, with given initial and final configurations of the rigid body, it suffices to search over only a finite set of trajectories. Since for a control line L, the number of corresponding trajectories is finite, if there is only a finite number of control lines as well, then we can possibly find best trajectories exactly by enumerating all possible control lines and corresponding trajectories. As it turns out, the number of control lines for TGT and singular trajectories are finite and we also can enumerate them. Hence, for TGT and singular trajectories, we can find best trajectories exactly. However, for regular trajectories, we do not have a method to reduce the number of control lines to a finite set and that's why we only find approximately best regular trajectories.

8. Finding Best TGT Trajectories

Remember that a TGT trajectory is a control line trajectory for which both the first control and the last control are non-parallel translations. Our approach is to enumerate all pairs of controls $(u_s, u_f) \in U^2$, where both u_s and u_f are translations. For each (u_s, u_f), we determine a best TGT trajectory $(\mathbf{u}, \mathbf{t})$, subject to $u_1 = u_s$, and $u_n = u_f$. Finally, we pick the best TGT trajectory among all best TGT trajectories with respect to all pairs of controls.

Fix a pair of translation controls (u_s, u_f). Since the initial configuration q_s and the final configuration q_f are given, we can compute the velocity vectors v_s and v_f for applying u_s at q_s and u_f at q_f respectively. Then, we substitute v_s and v_f in Equation 2 and we get a system of linear equations. Since we know $k_x^2 + k_y^2 = 1$ and $H > 0$, we can solve k_x, k_y, and the Hamiltonian value H exactly.

Our method is to incrementally build sequences of controls that could possibly satisfy Equation 2 by using A* search. Each state is sequences of controls and durations $g = (\mathbf{u}, \mathbf{t})$, where $|\mathbf{u}| = |\mathbf{t}| = h$, $u_1 = u_s$, and $u_h = u_f$. Each state also satisfies that each t_i, $1 < i < h$ is computed according to Theorem 7, but t_1 and t_h are undefined. We use the summation of the switch costs and the durations t_i, $1 < i < h$ as path cost. The neighbors of a state $g = (\mathbf{u}, \mathbf{t})$ are all states $g' = ((u_1, \ldots, u_{h-1}, u', u_{h+1} = u_f), (t_1, \ldots, t_{h-1}, t', t_{h+1})$, where $u' \in U$ and t' is computed according to Theorem 8.

We can test whether a state $g = (\mathbf{u}, \mathbf{t})$ reaches the goal as follows: For a state g, since for all t_i, $1 < i < h$ are determined, the displacement, $(\delta_x, \delta_y, \delta_\theta)$ in the configuration space of the sub-trajectory $u_2, \ldots, u_{h-1}$ can be computed. If $q_{s,\theta} + \delta_\theta \neq q_{f,\theta}$, then since u_s and u_f are translations, g cannot reach the goal. Otherwise, since we know the velocity v_s and v_f at the initial configuration and the final configuration respectively, we solve the following system of linear equations to get durations $t_1 = t_s$ and $t_h = t_f$ for u_s and u_f.

$$q_{s,x} + v_{s,x} t_s + \delta_x + v_{f,x} t_f = q_{f,x} \tag{5}$$

$$q_{s,y} + v_{s,y} t_s + \delta_y + v_{f,y} t_f = q_{f,y} \tag{6}$$

In order to speed up A* search, we need to design an admissible heuristic. A state $g = (\mathbf{u}, \mathbf{t})$ can reach the goal if, and only if, the change of orientation, δ_θ, equals $q_{f,\theta} - q_{s,\theta}$. Hence, we can use the difference between $|q_{f,\theta} - q_{s,\theta} - \delta_\theta|$ as an admissible heuristic.

9. Finding Best Singular Trajectories

Remember that a control line trajectory is called *singular* if there exists a non-zero measure interval along the trajectory that multiple controls have the same Hamiltonian value within this interval. Furthermore, the number of singular Hamiltonian value is a finite set when U is given. Our approach is to enumerate all pairs of controls $(u_s, u_f) \in U^2$ and singular Hamiltonian value H, where one of u_s and u_f is a rotation. For each (u_s, u_f) and H, we determine a best singular trajectory $(\mathbf{u}, \mathbf{t})$, subject to $u_1 = u_s$, $u_n = u_f$ and Hamiltonian value is H. Finally, we pick the best singular trajectory among all best singular trajectories with respect to all pairs of controls and singular Hamiltonian values.

Fix a pair of controls (u_s, u_f) and a singular Hamiltonian value H, where one of u_s and u_f is a rotation. Based on u_s, u_f, and H, we can construct two control lines according to Theorem 6. Fix a control line L, Our method is to incrementally build sequences of controls that could possibly satisfy Equation 2 by using bidirectional A* search.

There are two different states, S and F, denoting the state grow from q_s and q_f(in reverse) respectively. Each state is sequences of controls and durations $g = (\mathbf{u}, \mathbf{t})$, where $|\mathbf{u}| = |\mathbf{t}| = h$. If $g \in S$, then $u_1 = u_s$ and each t_i, $1 \leq i < h$ is computed according to Theorem 7 assuming the trajectory starts at q_s. If $g \in F$, then $u_h = u_f$ and each t_i, $1 < i \leq h$ is computed according to Theorem 7 assuming the trajectory is built from q_f backwards. In this way, each state has exactly one undefined duration and we use the summation of the switch cost and the defined durations as path cost.

The neighbors of a state $g = (\mathbf{u}, \mathbf{t}) \in S$ are all states $g' = ((u_1, \ldots, u_h, u'), (t_1, \ldots, t_h, t'))$, where $u' \in U$ and t_h is computed according to Theorem 7. We define the neighbor of a state $g \in F$ symmetrically.

For a pair of states $g = (\mathbf{u}, \mathbf{t}) \in S$ and $g' = (\mathbf{u}', \mathbf{t}') \in F$, where $|\mathbf{u}| = h$ and $|\mathbf{u}'| = h'$, if both u_h and u'_1 are translations, then we can combine the two states to construct a feasible singular trajectory by solving a system of linear equations similar to Equations 5 and 6. In order to speed up the search by using bi-directional A*, we design a heuristic as follows. For a state $g = (\mathbf{u}, \mathbf{t}) \in S$, let q_g be the configuration that start from q_s and apply all u_i with duration t_i in order for all $1 \leq i < h$. The distance between q_g and q_f is the lower bound of the length of the trajectory from the state to the goal. Hence, we use the Euclidean distance between q_g and q_f divide by the maximum velocity as an admissible heuristic. For a state $g \in F$, we design a similar heuristic.

10. Finding Best Regular Trajectories

Remember that a regular trajectory is a generic trajectory either starting or ending with a rotation. Our approach is to enumerate all pairs of controls $(u_s, u_f) \in U^2$, where one u_s of u_f is a rotation. Unlike TGT and singular trajectories, the number of potential control lines for regular trajectories is uncountably infinite. Hence, we use Lipschitzian optimization techniques to determine the best Hamiltonian value and its corresponding control line.

However, even when we fix a pair of controls (u_s, u_f), the trajectories may behave differently with respect to the Hamiltonian value H. For some Hamiltonian values H, control u can switch to another control u', but the switch cannot happen for some other Hamiltonian values H'. Therefore, for each (u_s, u_f), we partition the Hamiltonian values into several disjoint open intervals so that within each interval the change of trajectories with respect to the Hamiltonian value is Lipschitz continuous.

For each interval I, we use Lipschtzian optimization techniques to determine an best regular trajectory $(\mathbf{u}, \mathbf{t})$, subject to $u_1 = u_s$, $u_n = u_f$, and the Hamiltonian value $H \in I$. During this step, we need a method to determine a regular trajectory corresponding to a fixed control line L that approximately minimizes error and time. Similar to the idea we used for finding best TGT and singular trajectories, we use A* search to find a regular trajectory that approximately minimizes error and time. Finally, we pick the best regular trajectory among all best regular trajectories with respect to all pairs of controls and all interval of the Hamiltonian values.

In the following sections, we will explain how to partition the Hamiltonian values, reduce the problem to a Lipschtzian optimization problem, and determine best regular trajectories corresponding to a fixed control line.

10.1. Partition the Hamiltonian Values

Fix a pair of controls (u_s, u_f), where one of u_s and u_f is a rotation. We show how to partition the Hamiltonian values into several disjoint open intervals so that within each interval I, if u, u', and u'' are three consecutive controls in a control line trajectory for a Hamiltonian value $H \in I$, then u, u', and u'' will also be well-defined for another Hamiltonian value $H' \in I$.

According to Theorem 6, we have two continuous functions mapping from the Hamiltonian value to a control line. Consider one fixed mapping $L(H)$ of these two mappings. According to Theorem 8, for any three controls u, u', and u', if these three controls are consecutive in a regular trajectory, then the duration of u' can be determined by the Hamiltonian value H. Based on the calculation, we can figure out the range of the Hamiltonian values that the duration of u' is well defined and must be an interval. That is, for each triple $e = (u, u', u'') \in U^3$, we can determine the range of the Hamiltonian values I_e that the duration of u' is well-defined.

We collect the intervals I_e for all triple $e = (u, u', u'') \in U^3$ that the duration of u' is well-defined for all Hamiltonian values in I_e. Let S be the set of numbers containing all endpoints for all intervals I_e. . The set S partitions the domain of $L(H)$, determined by the parametrization and the control set, into several disjoint open intervals. We have the following theorem.

Theorem 10. *There exists a finite set of critical values of $\mathcal{R}$ that partition the Hamiltonian values into a finite set of open intervals, such that for each interval I, if u, u', and u'' are three consecutive controls in a control line trajectory for* *a Hamiltonian value $H \in I$, then u, u', and u'' will also be well defined for another Hamiltonian value $H' \in I$.*

10.2. Reduction to a Lipschitzian Optimization Problem

Fix a pair of controls (u_s, u_f), an interval of Hamiltonian values I constructed in Theorem 10, and a mapping $L(H)$ from the Hamiltonian values to control lines, we pose a Lipschitzian optimization problem to solve for the Hamiltonian value H with time and position error at most ϵ, for any desired $\epsilon > 0$. Here, we briefly introduce Lipschitzian optimization.

The goal of global optimization is to find optimal solutions of constrained optimization problem even for non-linear, non-continuous problems. A function $f : \mathcal{R} \to \mathcal{R}$ is called *Lipschitz continuous* if there exists a constant $L \geq 0$, such that for all pairs x, y in the domain we have $|f(x) - f(y)| \leq L|x - y|$, where L is called the *Lipschitz constant*. Given a Lipschitz continuous function $f(x)$, the problem of finding the global minimum $\min_x f(x)$ is called a *Lipschitzian optimization* problem. For Lipschitzian optimization problems, there exist efficient algorithms to find globally (approximately) optimal solutions with arbitrarily small error in finite time([22]).

One efficient algorithm for solving Lipschitzian optimization problem is Piyavskii's algorithm([23]). The idea of Piyavskii's algorithm is to iteratively subdivide a domain I into several intervals. For each interval, Piyavskii's algorithm determines the lower bound of the objective function based on Lipschitz constant, and decides whether to further subdivide this interval or disregard this interval based on the lower bound information. For any error bound $\epsilon > 0$, Piyavskii's algorithm is guaranteed to find a solution with additive an error at most ϵ within a finite number of iterations.

The Lipschitizian optimization for finding best regular trajectories is formulated as follows:

$$
\begin{aligned}
\min\ & c(L, \mathbf{u}, \mathbf{t}) \\
& d(L, \mathbf{u}, \mathbf{t}) = 0 \\
& L = L(H) \text{ for some } H \in I, && (7) \\
& \text{and } (\mathbf{u}, \mathbf{t}) \text{ is a regular trajectory.} && (8)
\end{aligned}
$$

The function c is the cost function that we want to minimize, which is the cost of the trajectory $(\mathbf{u}, \mathbf{t})$. The function d is the constraint that we want to satisfy, which should be the minimum distance from the trajectory $(\mathbf{u}, \mathbf{t})$ to the goal. We let $c(L(H), \mathbf{u}, \mathbf{t}) = d(L(H), \mathbf{u}, \mathbf{t}) = \infty$ if the trajectory $(\mathbf{u}, \mathbf{t})$ does not correspond to the control line $L(H)$.

In order to apply Lipschitzian optimization techniques, we need to show that functions c and d are Lipschitz continuous with respect to the change of the Hamiltonian value H. That is, we want to show that when the Hamiltonian values changes, the resulting distance and cost functions are Lipschitz continuous with respect to the Hamiltonian values. This differs from TGT and singular trajectories that we ignore all trajectories not reaching the goal exactly. We will prove the following theorem in Appendix B.

Theorem 11. *Let $I = (a, b)$ be an open interval of the partition of the Hamiltonian values. Let $\mathbf{u}$ be a fixed sequence of n controls. Let $t_i(H)$ be the duration for the u_i and $d_i(H)$ be the length of projection of the sub-trajectory corresponding to u_i onto the control line. For any $\delta > 0$, both functions $t_i(H)$ and $d_i(H)$ are Lipschitz continuous with respect to the Hamiltonian values $H \in (a, b - \delta)$ for all $1 \leq i \leq n$.*

Moreover, we need a method to find best regular trajectory corresponding to a control line L that approximately minimizes error and time, which will be explained in the next section.

10.3. Finding Optimal Trajectories for a Fixed Control Line L

Fix a pair of controls (u_s, u_f) and a control line L, we use A* search to find an best regular trajectory corresponding to the control line L approximately minimizing error. If it is possible to reach the goal with error at most ϵ, the result will be a regular trajectory approximately reaching the goal with approximately minimum cost. If it is impossible to reach the goal with error at most ϵ, the result will be a regular trajectory approximately minimizing the distance to the goal.

Our method is to incrementally build sequences of controls and durations that could possibly satisfy Equation 2 by using A* search. Each state is a sequence of controls, together with durations $g = (\mathbf{u}, \mathbf{t})$, where $|\mathbf{u}| = |\mathbf{t}| = h$ and $u_1 = u_s$ and $u_h = u_f$. Each state also satisfies that each t_i, $1 \leq i \leq h$ is computed according to Theorem 8 with an additional constraint that t_i is capped by M, the cost of a feasible trajectory. The neighbors of a state $g = (\mathbf{u}, \mathbf{t})$ are all states $g' = ((u_1, \ldots, u_{h-1}, u', u_{h+1} = u_f), (t_1, \ldots, t_{h-1}, t', t_{h+1}),$ where $u' \in U$ and t' is computed according to Theorem 8.

We use the summation of the switch cost and the duration $t_i, 1 \leq i < h$ as path cost. Note that we did not use t_h as part of the path cost; since the duration of $u_h = u_f$ depends on u_{h-1}, it may be possible the sum of the durations of a state g is larger than the sum of the durations of g's neighbor.

For a state $g = (\mathbf{u}, \mathbf{t})$, we define the distance as follows: Let q_g be the final configuration of applying controls $\mathbf{u}$ with durations $\mathbf{t}$. If u_f is a translation, the distance from the state g to the goal is the distance between q_g and q_f. If u_f is a rotation, then let r_f be the rotation center of applying u_f at q_f and r_g be the rotation center of applying u_f at q_g. The distance from the state g to the goal is the distance between r_f and r_g.

When the distance from the state g to the goal is zero, then the state g is at the goal. The distance divided by the maximum velocity can therefore also serve as an admissible heuristic for A* search.

There is one difficulty here: for a state $g = (\mathbf{u}, \mathbf{t})$, the switch from u_{h-1} to u_h may be impossible due to the constraint on the Hamiltonian values. In this case, we just pick the largest index i such that u_i can switch to u_{i+1} and let q_g be the configuration at which u_i switches to u_{i+1}. Then, use the Euclidean distance between q_g to q_f as the distance from the state g to the goal. In this case, we only use the distance to guide the search but will not use the trajectory in state g as a result, since the trajectory is infeasible.

11. Implemetation

We implemented the algorithm described in C++. Our testing environment is a desktop system with an Intel Xeon W3550 3.07 GHz CPU.

For the costly-switch model, we used three test cases. First, we used the bench mover's problem proposed in [18] as one test case. We compared our program's result with the results of the analytical solver. Except for some cases in which the Hamiltonian value is close to the upper bound

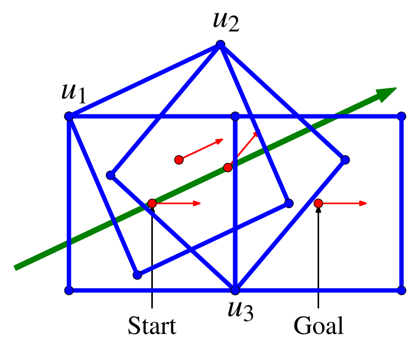


Fig. 10. An approximately optimal trajectory derived using the described approximation algorithm for a refrigerator robot starting at (-2, 0, 0), with unit cost for switching between any pair of controls. The green line is the control line, and the u_i labels show the sequence of rotation centers.

(for which numerical instability becomes a problem), the results coincide with the results from the exact solver.

We used the refrigerator-mover's problem as the second test case. The refrigerator-mover's problem is an extension of bench mover's problem, inspired by a problem from [19]: a mover wants to move a refrigerator from one location and orientation to another. The refrigerator is too heavy to move by lifting or pushing, but it can be lifted onto any of the four legs at the corners of the square base and rotated. One approximately optimal trajectory is shown in Figure 10. Third, we used omni-directional vehicle as a test case; one approximately optimal trajectory is show in Figure 2b.

The solver described can also be used as a general-purpose solver for time-optimal trajectories of rigid bodies in the plane in the special cost-free-switch case. In this case, we additionally constrain the structure of the trajectories using the maximization condition of PMP, and apply the Lipschitz optimizer to find best trajectories for each possible structure. We applied this approach to the problem of finding optimal trajectories for the omnidirectional robot described in [33], and found that the approach was only about one order of magnitude slower (on the order of 0.03 seconds per configuration) than the special-purpose analytical solver derived in [33]. One approximately optimal trajectory is show in Figure 2a.

12. Conclusion and Future Work

By adding a cost for switching between controls, we ensure existence of solutions for optimal control problems, and evade the problem of chattering. By applying Blatt's Indifference Principle and Lipschitzian optimization approach, we can find approximately optimal trajectories, and the error can be forced to be arbitrarily small.

The implemented approach does have some limitations, and these limitations do suggest rich problems for future study. One of the limitations is that when applying Lipschitzian optimization techniques, the algorithm reduces the search domain by a user controlled parameter in order to make the optimization problem behave smoothly. Although the controlled parameter can set to any arbitrarily small number, this algorithm may not find optimal trajectories for some scenarios if the controlled parameter is not small enough. We believe this that issue can be resolved by reformulating the Lipschitzian optimization problem with other parametrization.

Moreover, the potential number of optimal trajectory structures can be very very large in the costly-switch model. For the costly-switch model, an algorithm might potentially need to explore a number of structures that is exponential in the number of controls in order to find solutions. For example, in order to find approximately optimal trajectories for omni-directional vehicle, whose control set contains fourteen controls, it takes about an hour to find an high-precision approximately optimal trajectory for an initial configuration and goal configuration. We believe that better Lipschitz constants, use of the derivative of the objective function together with more sophisticated approaches to Lipschitzian optimization([16]), and a more directed A* search could dramatically reduce these costs.

Finally, in this paper, we assume that there are no obstacles. For the cost-free-switch model, Pontryagin's Maximum Principle can be extended to systems with state constraints. However, to the best of our knowledge, extending Blatt's Indifference Principle to systems with state constraints is still an unsolved task. Extending Blatt's Indifference Principle to systems with state constraints (or making use of the KKT approach) will be an interesting future direction of research.

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Appendix

Note to reviewers: We have included this appendix to simplify the review process, but because the appendix is quite long and detailed, we intend for the final version of the paper to submit a technical report to the arXiv, and reference that report in the journal paper.

A. Properties of Control Line Trajectories

In this section, we show several properties of control line trajectories. Remember that a trajectory $(\mathbf{u}, \mathbf{t})$ is called *extremal*, if there exist four constants $H > 0$, k_x, k_y, and k_θ, such that Equation 2 is satisfied. When the initial configuration q_s, the first control u_s, the final configuration q_f, the last control u_f, and the Hamiltonian value H are given, the control line can be constructed explicitly. Furthermore, when a control line L is fixed, we can show that the trajectory along the control line has only finite number of possibilities.

A.1. Notation

For a given configuration $q = (x, y, \theta)$, there is a corresponding transformation matrix representation:

$$T(q) = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}$$

For a given configuration q, if we apply a control (v_x, v_y, ω), we define the *homogeneous representation* of the rotation center of u at q is $T(q)u$.

For a given control line $L = (k_x, k_y, k_\theta)$, there is a corresponding transformation matrix T_L that transform a configuration in the world frame to the *control line frame*:

$$T_L = \begin{bmatrix} k_x & k_y & 0 \\ k_y & k_x & k_\theta \\ 0 & 0 & 1 \end{bmatrix}$$

For a configuration q in the world frame, we use $q^L = T_L q$ to denote its representation in the control line frame whenever the control line L is clear from the text.

A nice property of the control line frame is that, for a given control line L, a given configuration q in the world frame, and a control u, we can compute the corresponding Hamiltonian value of applying u at q along the control line L as $H = (T_L T(q) u)^T \cdot (0, 1, 0)$. Note that we will show how to construct the control line for a given Hamiltonian value in the next section. The Hamiltonian value of applying u at q along the control line L may be different from the Hamiltonian value used to construct the control line. If so, this shows that applying u at q will not satisfy necessary conditions for optimal trajectories.

In order to determine the duration of applying a control u before switch to another control u', it is convenient to define switch point. For two controls $u = (v_x, v_y, \omega)$ and $u' = (v'_x, v'_y, \omega')$, we define the *switch point* from u to u' be $p(u, u') = (v_y - v'_y, v'_x - v_x, \omega' - \omega)$.

A.2. Parametrization of the Control Lines

In this section, we show that when the initial configuration q_s, the first control u_s, the final configuration q_f, the last control u_f, where one of u_s and u_f is a rotation, we can construct at most two mappings from the Hamiltonian values to control

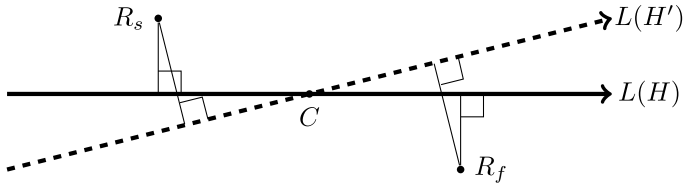

Fig. 11. Illustration of proof of Theorem 6 in the case that $g \neq 0$. The rotation centers at q_s and q_f are denoted by R_s and R_f respectively. The construction of the control line is only based on algebra. The geometrical meaning is that the control line will pass through a point $C = (a', b')$ and for any rotation with angular velocity ω, the distance from the rotation center to the control line is H/ω.

lines: $L_1(H)$ and $L_2(H)$. When both u_s and u_f are translations, the control line trajectories are TGT trajectories. Since we treat TGT trajectories separately, we do not deal with TGT trajectories in this section.

Theorem 6. *For a given initial configuration q_s, given goal configuration q_f, first control u_s, and the last control u_f, we can determine an interval of Hamiltonian values $I = (0, H_u)$, such that there exists mappings, $L_1(H)$ and $L_2(H)$, from Hamiltonian values in I to control lines.*

Proof. Since $k_x^2 + k_y^2 = 1$, we can represent the control line as $(k_x, k_y, k_\theta) = (\cos\varphi, \sin\varphi, k_\theta)$. Let $R_s = (a_s, b_s, g_s)$ be the homogeneous representation of the rotation center of u_s at q_s. Let $R_f = (a_f, b_f, g_f)$ be the homogeneous representation of the rotation center of u_f at q_f. Let $a = a_s - a_f$, $b = b_s - b_f$, and $g = g_s - g_f$. By Equation 2, we have:

$$-k_y a_s + k_x b_s + k_\theta g_s = H \tag{9}$$

$$-k_y a_f + k_x b_f + k_\theta g_f = H \tag{10}$$

$$-k_y a + k_x b + k_\theta g = 0 \tag{11}$$

There are two cases: $u_{s,\omega} \neq u_{f,\omega}$ and $u_{s,\omega} = u_{f,\omega}$.

The case of $u_{s,\omega} \neq u_{f,\omega}$ ($g \neq 0$). By Equation 11, we have $k_\theta = (k_y a - k_x b)/g$. By substitute back in Equation 10, we have

$$-k_y a_f + k_x b_f + (k_y a - k_x b) g_f / g = H \tag{12}$$

Combing with Equation 9, we have,

$$-k_y(a_f - (a g_f)/g) + k_x(b_f - (b g_f)/g) = H \tag{13}$$

By setting $a' = a_f - (a g_f)/g$ and $b' = b_f - (b g_f)/g$, we express Equation 13 differently as follows.

$$-a' \sin\varphi + b' \cos\varphi = H \tag{14}$$

The geometrical meaning is that the control line will pass through a point $C = (a', b')$ and for any rotation with angular velocity ω, the distance from the rotation center to the control line is H/ω.

Let $r^2 = a'^2 + b'^2$, $\alpha = \operatorname{atan}(a', b')$. We have $a' = r\sin\alpha$ and $b' = r\cos\alpha$. By trigonometric identities, we have

$$r\cos(\varphi + \alpha) = H$$

Since $\cos(\varphi+\alpha) = H/r$, $\sin(\varphi+\alpha) = \pm\sqrt{1-H^2/r^2} = \pm\sqrt{r^2-H^2}/r$.

$$k_x = \cos\varphi = \cos(\varphi+\alpha-\alpha) = \cos(\varphi+\alpha)\cos(\alpha) + \sin(\varphi+\alpha)\sin\alpha = \frac{b'H \pm a'\sqrt{r^2-H^2}}{r^2}$$

$$k_y = \sin\varphi = \sin(\varphi+\alpha-\alpha) = \sin(\varphi+\alpha)\cos(\alpha) - \cos(\varphi+\alpha)\sin\alpha = \frac{\pm b'\sqrt{r^2-H^2} - a'H}{r^2}$$

$$k_\theta = \frac{H + a_f k_y - b_f k_x}{g_f} = \frac{ak_y - bk_x}{g}$$

Thus, we have two control lines:

$$L_1(H) = (\frac{b'H + a'\sqrt{r^2-H^2}}{r^2}, \frac{b'\sqrt{r^2-H^2} - a'H}{r^2}, \frac{ak_y - bk_x}{g}) \text{ and}$$
$$L_2(H) = (\frac{b'H - a'\sqrt{r^2-H^2}}{r^2}, \frac{-b'\sqrt{r^2-H^2} - a'H}{r^2}, \frac{ak_y - bk_x}{g}).$$

The case of $u_{s,\omega} = u_{f,\omega}$ ($g = 0$). Let $r = a^2 + b^2$, $\alpha = \mathrm{atan}(a, b)$. We have

$$r\cos(\varphi+\alpha) = 0$$

Since $\cos(\varphi+\alpha) = 0$, then $\sin(\varphi+\alpha) = \pm 1$.

$$k_x = \cos\varphi = \cos(\varphi+\alpha-\alpha) = \cos(\varphi+\alpha)\cos(\alpha) + \sin(\varphi+\alpha)\sin\alpha = \pm a/r$$

$$k_y = \sin\varphi = \sin(\varphi+\alpha-\alpha) = \sin(\varphi+\alpha)\cos(\alpha) - \cos(\varphi+\alpha)\sin\alpha = \pm b/r$$

$$k_\theta = \frac{H + a_f k_y - b_f k_x}{g_f}$$

Thus, we have two control lines

$$L_1(H) = (a/r, b/r, \frac{H + a_f k_y - b_f k_x}{g_f}) \text{ and } L_2(H) = (-a/r, -b/r, \frac{H + a_f k_y - b_f k_x}{g_f}).$$

For a given control set, there exists an upper bound H_u of the Hamiltonian values that will correspond to non-trivial control line trajectories. Thus, we limit the range of $L_1(H)$ and $L_2(H)$ to be smaller than H_u. □

The parametrization we present here has different form from [13], which is based on trigonometric functions. Our parametrization is easier to analyze, but Furtuna's parametrization is more numerically stable, and hence we use Furtuna's parametrization in the implementation.

A.3. Durations of Control

In this section, we show that given a control line L, a configuration q, and two different controls u and u', the duration of applying u at a configuration q until switching to control u' can have at most two possibilities.

Theorem 7. *Given a control line L and a non-singular interval, the duration of applying a control u from a configuration q until switching to another control u' has at most two possibilities such that the resulting motion can be a subtrajectory of a control line trajectory corresponding to the control line L. Moreover, these two possibilities can be determined analytically.*

Proof. Suppose that a control line trajectory applies u at $q^L = T_L q$ and then switches to control u' at a configuration $\hat{q}^L$ in the control line frame. By BIP, at the moment of switching control at configuration $\hat{q}^L$, the Hamiltonian values for u and u' must be the same. Based on the analysis in [13], this implies $(T(\hat{q}^L)p(u,u'))^T \cdot [0,1,0] = 0$. That is, when we attach $p(u,u')$ to the robot, $p(u,u')$ will lie on the control line when the robot is at $\hat{q}^L$. Hence, we can solve for $\hat{q}^L$ as follows:

Let $p^L = T_L T(q) p(u,u') = (x_p, y_p, w_p)$ be the switching point in the control line frame when the robot is at q in the world frame. Let $r^L = T_L T(q) u = (x_r, y_r, \omega)$ be the homogeneous representation of the rotation center of u at q^L in the control line frame. If u is a translation and $w_p \neq 0$, then the duration $t = y_p/x_r$. If u is a rotation, then t must satisfy

$$b_1 \sin(\omega t) + b_2 \cos(\omega t) + b_3 = 0$$

where

$$\begin{aligned} b_1 &= x_p - w_p x_r/\omega \\ b_2 &= y_p - w_p y_r/\omega \\ b_3 &= w_p y_r/\omega \end{aligned}$$

The solution of Equation A.3 is

$$\omega t = \operatorname{atan}(b_1 \pm \sqrt{b_1^2 + b_2^2 - b_3^2}, b_2 - b_3) + 2\pi n, \forall n \in \mathcal{Z}$$

Since we are only interested in the solution with $t > 0$ and $|\omega t| < 2\pi$, there are at most two solutions. □

A.4. Durations for a Sequence of Controls

By the method described above, for a sequence of controls $\mathbf{u}$, $|\mathbf{u}| = n$, we can determine all possible durations $\mathbf{t}$, each t_i, $1 \leq i < n$, has at most two solutions, but the duration of the last control is still undetermined. When we are given the last configuration q_f, we can determine the duration of the last control as follows: If u_f is a rotation, we apply u_f until the configuration has the same orientation as q_f. If u_f is a translation, then we consider the control u_{n-1}. There are only two possible configuration of switch from u_{n-1} to u_f, such that applying u_f will reach q_f, and each configuration has different durations for u_f. Thus, there are two possible durations of u_f and we can choose the one which is closer to q_f.

Consequently, given a control line L and a sequence of control $\mathbf{u}$, $|\mathbf{u}| = n$, there are at most 2^{n-1} possible durations $\mathbf{t}$ so that the trajectory $(\mathbf{u}, \mathbf{t})$ corresponds to the control line L. Furthermore, if we fix the way to determine the duration according to *selectors* $b_1, \ldots, b_{n-1}$, $b_i \in \{1,2\}$ so that the duration t_i is the b_i-th solution, then the duration is fully determined by the control line L. Thus, we have the following theorem.

Theorem 9. *Let $\mathbf{u} \in U^n$ be a sequence of controls. Let $\mathbf{b} \in \{1,2\}^{n-1}$ be a sequence of* selectors*. For a given control line L with a non-critical Hamiltonian value H, initial configuration q_s, and final configuration q_f, the duration of each control is fully determined by $\mathbf{u}$ and $\mathbf{b}$.*

A.5. Finiteness of Trajectories Corresponding to a Control Line L

Since we have an upper bound for B for any optimal trajectories, we can limit ourselves to at most $|U|^B$ possible control sequences for optimal trajectories. When a control line L and a fixed control sequence $\mathbf{u}$ are given, since the duration for each control can be determined with at most two possibilities, there are at most 2^B possible corresponding durations for this control sequence with respect to the control line L. Thus, we can limit ourselves to these at most $(2|U|)^B$ possible trajectories corresponding to the control line L.

B. Lipschitz Continuity

Fix a pair of controls (u_s, u_f), an interval of Hamiltonian values I constructed in Theorem 10, and a mapping $L(H)$ from the Hamiltonian values to control lines, we show that the distance function computed in Section 10.3 is Lipschitz continuous with respect to the change of the Hamiltonian value H.

First consider the cost function c, which is the summation of the switch cost and the time cost of the trajectory. Consider a fixed sequence of controls $\mathbf{u}$ and we analyze the dependency of its duration $\mathbf{t}$ on the Hamiltonian value. By Theorem 9, when we fix a sequence of controls and selectors, the duration is fully determined by $L(H)$. Since Lipschitz continuity is closed under the minimum operation, it suffices to prove that for any sequence of controls $\mathbf{u}$ and selectors, the cost is Lipschitz continuous with respect to the change of the Hamiltonian value $H \in I$. Since the sequence is unchanged, the switch cost stays the same for any Hamiltonian value $H \in I$. Thus, it suffices to prove that for any sequence of controls $\mathbf{u}$ and selectors, the durations $\mathbf{t}$ is Lipschitz continuous with respect to the change of the Hamiltonian value $H \in I$. Similarly, we prove that for any sequence of controls $\mathbf{u}$ and selectors, the distance from the trajectory to the goal is Lipschitz continuous with respect to the change of the Hamiltonian value $H \in I$.

In order to simplify the analysis, in the remaining part of this section, we consider fixed selectors so that $\mathbf{t}$ is a function of the Hamiltonian value H directly without ambiguity.

B.1. Lipschitz Continuity of $d(L(H), \mathbf{u}, \mathbf{t})$ and $c(L(H), \mathbf{u}, \mathbf{t})$

Let $(\mathbf{u}, \mathbf{s})$, $|\mathbf{u}| = n$ be a regular trajectory corresponding to the control line $L(H)$. We first consider the cost function $c(L(H), \mathbf{u}, \mathbf{t})$, which depends on the durations of each control and switch cost. Since the sequence is unchanged, the switch cost will not change and hence we focus on durations. Let $t_i(H)$ be the duration for the i-th control u_i with respect to H. Since $c(L(H), \mathbf{u}, \mathbf{t})$ is the summation of all t_i, it suffices to prove that each $t_i(H)$ is Lipschitz continuous.

Second, we consider the distance function $d(L(H), \mathbf{u}, \mathbf{t})$. For control u_i and its corresponding sub-trajectory, we use d_i to denote the length of the sub-trajectory projection onto the control line. The distance function $d(L(H), \mathbf{u}, \mathbf{t})$ can be rewritten as $|q^L_{s,x} + \sum_{i=1}^n d_i - q^L_{f,x}|$. It suffices to show that each $d_i(H)$ and the mapping T_L is Lipschitz continuous.

Durations $t_i(H)$ and projections $d_i(H)$, $1 < i < n$ are easier to analyze, since they depend on H directly. However, durations $t_1(H)$ and $t_n(H)$ depend on H, initial configuration q^L_s, and final configuration q^L_f in the control line frame, which depends on H. Hence, $t_1(H)$ and $t_n(H)$ depend on H not only directly but also indirectly through q^L_s and q^L_f. Similarly, $d_1(H)$ and $d_n(H)$ also depend on H directly and indirectly. The analysis of $t_1(H)$, $t_n(H)$, $d_1(H)$, and $d_n(H)$ should be separated from the analysis of $t_i(H)$ and $d_i(H)$, $1 < i < n$. Thus, we analyze $t_i(H)$ and $d_i(H)$, $1 < i < n$ first and then show the mapping T_L is Lipschitz continuous. Finally, we give the analysis of $t_1(H)$, $t_n(H)$, $d_1(H)$, and $d_n(H)$.

B.2. Analysis of $t_i(H)$ and $d_i(H)$, $1 < i < n$

Theorem 12. *Let $I = (a, b)$ be an open interval of the partition of the Hamiltonian values. Let $\mathbf{u}$ be a fixed sequence of n controls. Let $t_i(H)$ be the duration for the u_i and $d_i(H)$ be the length of projection of the sub-trajectory corresponding to u_i onto the control line. For any $\delta > 0$, both functions $t_i(H)$ and $d_i(H)$ are Lipschitz continuous with respect to the Hamiltonian values $H \in (a, b - \delta)$ for all $1 < i < n$.*

Proof. The duration $t_i(H)$ and length $d_i(H)$ are fully determined by u_{i-1}, u_i, u_{i+1}, and H. Let q^L_i be the configuration in the control line frame at which the trajectory switches control from u_{i-1} to u_i. Let q^L_{i+1} be the configuration in the control line frame at which the trajectory switches control from u_i to u_{i+1}. Here, we use a result from [13] that there exists a point

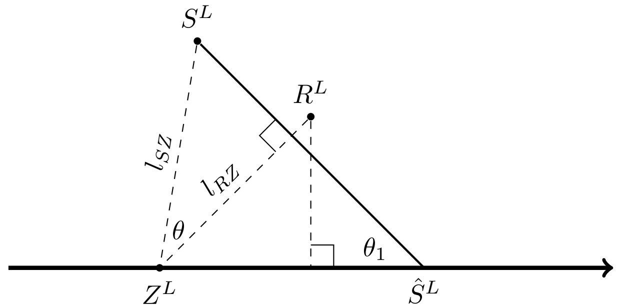

Fig. 12. Illustration of proof of Theorem 11 in the case that u_i is a translation. Initially, the switch point is located at S^L. A control line trajectory must apply u_i until the switch point collides with the control line at $\hat{S}^L$.

$p_i = p(u_{i-1}, u_i)$ rigidly attached to the robot, such that p_i will lie on the control line when the robot is at q_i^L. Similarly, when the robot is at q_{i+1}^L and switches from u_i to u_{i+1}, there exists a point $p_{i+1} = p(u_i, u_{i+1})$ attached to the robot such that p_{i+1} is on the control line.

We introduce some notation for the remainder of the proof. Let $Z^L = (Z_x^L, 0)$ be the location of p_i attached to the robot at q_i^L, which is on the control line. Let $S^L = (S_x^L, S_y^L)$ be the location of p_{i+1} attached to the robot at q_i^L. Let $\hat{S}^L = (\hat{S}_x^L, 0)$ be the location of p_{i+1} attached to the robot at q_{i+1}^L. By considering the position of S^L we can determine the $t_i(H)$ and $d_i(H)$.

Depending on whether u_i is a translation or not, there are two cases:

The case in which u_i is a translation. Let v_i be the velocity of u_i. By Theorem 1, the magnitude of the projection of the velocity onto the control line is H. Consequently, the magnitude of velocity in the y-coordinate in the control line frame is $v_y^L = \sqrt{v_i^2 - H^2}$. The duration of t_i can be computed as S_y^L / v_y^L. Consequently, the length of the projection of the trajectory onto the control line, $d_i(H)$, can be computed as $t_i H$. Hence, it suffices to prove t_i is Lipschitz continuous.

The control u_{i-1} must be a rotation, since if u_{i-1} is a translation, then u_i and u_{i-1} have the same Hamiltonian value along the sub-trajectory corresponding to u_i and the trajectory is a singular trajectory. Let $R^L = (R_x^L, R_y^L)$ be the location of the rotation center of control u_{i-1}. Let l_{SZ} be the distance between S^L and Z^L. Let l_{RZ} be the distance between R^L and Z^L. Let θ be the angle rotating from vector $Z^L S^L$ to vector $Z^L R^L$ counterclockwise. Since the the mutual distance among S^L, R^L and Z^L is independent from H, l_{RZ} and θ are independent from H.

Let θ_1 be the angle between segment $S^L \hat{S}^L$ and the control line; the value of θ_1 is $\operatorname{acos}(H/v_i)$. Furthermore, [13] shows that the line $Z^L R^L$ is perpendicular to the line $S^L \hat{S}^L$. By geometric reasoning, S_y^L can be computed as $l_{SZ} \cos(\theta - \operatorname{acos}(H/v_i)) = (l_{SZ}/v_i)(H \cos\theta + \sqrt{v_i^2 - H^2} \sin\theta)$. Remember all the durations is capped by M, the cost of a feasible trajectory. Hence, we have

$$t_i(H) = \min(M, \frac{S_y^L}{v_y^L} = \frac{l_{SZ}}{v_i}\left(\sin\theta + \frac{H\cos\theta}{\sqrt{v_i^2 - H^2}}\right)).$$

Since the second term is monotonically increasing in H, there exists a threshold γ that for all $H \geq \gamma$, $t_i = M$. Thus, within the interval $[\gamma, b)$, $t_i(H)$ is a constant. Hence, we only focus on the part (a, γ) that the minimum is taken from the second term.

A differentiable function is Lipschitz continuous if this function has a bounded first derivative.

$$\frac{\partial t_i(H)}{\partial H} = \left(\frac{v_i l_{SZ} \cos\theta}{(v_i^2 - H^2)^{1.5}}\right)$$

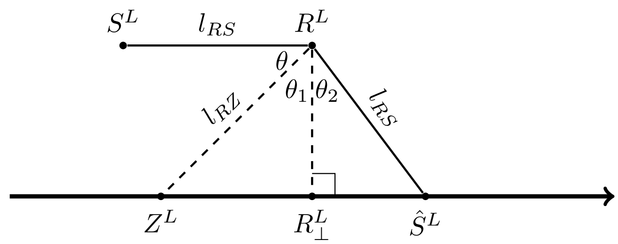

Fig. 13. Illustration of proof of Theorem 11 in the case that u_i is a rotation, $\omega_{i-1} > \omega_i$, and $\omega_{i+1} > \omega_i$. Initially, the switch point is located at S^L. A control line trajectory must apply u_i, a rotation around R^L counterclockwisely, until the switch point collides with the control line at $\hat{S}^L$.

Since $d_i = t_i H$, we have

$$\frac{\partial d_i(H)}{\partial H} = H\frac{\partial t_i}{\partial H} + t_i = \left(\frac{v_i l_{SZ}\cos\theta}{(v_i^2 - H^2)^{1.5}}\right) + \frac{l_{SZ}}{v_i}\left(\sin\theta + \frac{H\cos\theta}{\sqrt{v_i^2 - H^2}}\right).$$

For all $H \in (a, \gamma)$ and $H < v_i$, the derivatives of $t_i(H)$ and $d_i(H)$ are bounded.

The case in which u_i is a rotation. Let $R^L = (R^L_x, R^L_y)$ be the location of the rotation center of control u_i and let $R^L_\perp = (R^L_x, 0)$ be the projection of R^L on the control line. We want to compute the angle, φ_0, between the control line to the vector $R^L S^L$, and the angle φ_1, between the control line to the vector $R^L \hat{S}^L$; these angles are measured in counterclockwise direction. The duration $t_i(H)$ can be computed as $(\varphi_1 - \varphi_0)/\omega_i$, where the subtraction wrapping around 2π and the result has the same sign as ω_i. Let r be the distance between the reference point of the robot and R^L when robot is at q^L_i. The projection of the trajectory on the control line, $d_i(H)$, can be computed as $r(\cos\varphi_1 - \cos\varphi_0)$. Thus, it suffices to show that φ_0 and φ_1 are Lipschitz continuous with respect to H.

Let l_{RZ} be the distance between R^L and Z^L and let l_{RS} be the distance between R^L and S^L. Let θ be the angle rotating from vector $R^L Z^L$ to $R^L S^L$ counterclockwise. Note that θ, l_{RZ}, and l_{RS} are independent from H.

Let θ_1 be the angle between the segment $R^L Z^L$ and $R^L R^L_\perp$, which equals $\mathrm{acos}(H/(l_{RZ}\omega_i))$. Let θ_2 be the angle between the segment $R^L \hat{S}^L$ and $R^L R^L_\perp$, which equals $\mathrm{acos}(H/(l_{RS}\omega_i))$. Let ω_{i-1} and ω_{i+1} be the angular velocity of u_{i-1} and u_{i+1} respectively. Based on θ_1 and θ_2, we can compute φ_0 and φ_1 as follows:

		φ_0		φ_1
$\omega_i > 0$	$Z^L_x \le R^L_x$	$3\pi/2 - \theta_1 + \theta$	$\hat{S}^L_x \ge R^L_x$	$3\pi/2 + \theta_2$
$\omega_i > 0$	$Z^L_x > R^L_x$	$3\pi/2 + \theta_1 + \theta$	$\hat{S}^L_x < R^L_x$	$3\pi/2 - \theta_2$
$\omega_i < 0$	$Z^L_x > R^L_x$	$\pi/2 - \theta_1 + \theta$	$\hat{S}^L_x < R^L_x$	$\pi/2 + \theta_2$
$\omega_i < 0$	$Z^L_x \le R^L_x$	$\pi/2 + \theta_1 + \theta$	$\hat{S}^L_x \ge R^L_x$	$\pi/2 - \theta_2$

Thus, we have

$$\left|\frac{\partial\varphi_0(H)}{\partial H}\right| \le ((l_{RZ}\omega_i)^2 - H^2)^{-0.5} \text{ and } \left|\frac{\partial\varphi_1(H)}{\partial H}\right| \le ((l_{RS}\omega_i)^2 - H^2)^{-0.5}.$$

Consequently,

$$\left|\frac{\partial t_i(H)}{\partial H}\right| \le \frac{((l_{RZ}\omega_i)^2 - H^2)^{-0.5} + ((l_{RS}\omega_i)^2 - H^2)^{-0.5}}{|\omega_i|}.$$

$$\begin{aligned}\left|\frac{\partial d_i(H)}{\partial H}\right| \le \; & \frac{r}{l_{RZ}|\omega_i|}\left(|\sin\theta_1| + \left|\frac{H\cos\theta_1}{\sqrt{(l_{RZ}\omega_i)^2 - H^2}}\right|\right) \\ & + \frac{r}{l_{RS}|\omega_i|}\left(|\sin\theta_2| + \left|\frac{H\cos\theta_2}{\sqrt{(l_{RS}\omega_i)^2 - H^2}}\right|\right)\end{aligned}$$

By the construction of the partition of the Hamiltonian values, $b \leq |l_{RZ}\omega_i|$ and $b \leq |l_{RS}\omega_i|$ so that the switch of controls is feasible. For any $\delta > 0$, when $H \in (a, b-\delta)$, H is smaller than $|l_{RZ}\omega_i|$ and $|l_{RS}\omega_i|$, and the derivatives of $t_i(H)$ and $d_i(H)$ are bounded. □

During the analysis, we also fully analyze the duration for three consecutive controls u, u', and u'' in a control line trajectory with respect to a given H value.

Theorem 8. *Let u, u', and u'' be three consecutive controls in a control line trajectory. Given k_x, k_y, and a non-critical Hamiltonian value H, the duration of u' has at most two possibilities and can be determined analytically without knowing the configuration.*

Before we analyze d_1, t_1, d_n, and t_n, since they depend on the mapping $T_L(q) = q^L$, they indirectly depend on k_x, k_y, and k_θ. Hence, we show the analysis of k_x, k_y, k_θ first and then analyze q^L before we analyze d_1, t_1, d_n, and t_n.

B.3. Analysis of k_x, k_y, k_θ, and $\mathrm{atan}(k_y, k_x)$

Since the mapping T_L depends on k_x, k_y, and k_θ, we analyze the dependency of k_x, k_y, and k_θ on the Hamiltonian value H. Furthermore, we also analyze $\mathrm{atan}(k_y, k_x)$, since we will need it in the following sections.

Theorem 13. *For a mapping $L(H) = (k_x(H), k_y(H), k_\theta(H))$ from the Hamiltonian values to control lines with a domain $(0, H_u)$. The mapping $L(H)$ and $\mathrm{atan}(k_y(H), k_x(H))$ are Lipschitz continuous with respect to the Hamiltonian values in $(0, H_u - \delta)$ for any $\delta > 0$.*

Proof. Since $k_\theta(H) = \frac{H + a_f k_y - b_f k_x}{g_f}$, we have

$$\frac{\partial k_\theta(H)}{\partial H} = \frac{1 + a_f \frac{\partial k_y(H)}{\partial H} - b_f \frac{\partial k_x(H)}{\partial H}}{g_f}$$

By using the notation in Section A.2, there are two cases.

The case of $u_{s,\omega} = u_{f,\omega}$ ($g = 0$) In this case, k_x and k_y will not change with respect to H. Hence, $\frac{\partial k_x(H)}{\partial H} = \frac{\partial k_y(H)}{\partial H} = \frac{\partial \mathrm{atan}(k_y, k_x)}{\partial H} = 0$ and $\frac{\partial k_\theta(H)}{\partial H} = 1/g_f$.

The case of $u_{s,\omega} \neq u_{f,\omega}$ ($g \neq 0$)

$$\frac{\partial k_x(H)}{\partial H} \leq \frac{|b'| + |\frac{a'H}{\sqrt{r^2 - H^2}}|}{r^2}$$

$$\frac{\partial k_y(H)}{\partial H} \leq \frac{|a'| + |\frac{b'H}{\sqrt{r^2 - H^2}}|}{r^2}$$

$$
\begin{aligned}
\frac{\partial \operatorname{atan}(k_y(H), k_x(H))}{\partial H} &= \frac{1}{1 + k_y^2/k_x^2} \frac{\partial \frac{k_y(H)}{k_x(H)}}{\partial H} \\
&= k_x^2 \frac{\frac{\partial k_x(H)}{\partial H} k_y - k_x \frac{\partial k_y(H)}{\partial H}}{k_x^2} \\
&= \frac{\partial k_x(H)}{\partial H} k_y - k_x \frac{\partial k_y(H)}{\partial H} \\
&= \frac{b' + \frac{a'H}{\sqrt{r^2 - H^2}}}{r^2} \frac{b'\sqrt{r^2 - H^2} - a'H}{r^2} - \frac{b'H + a'\sqrt{r^2 - H^2}}{r^2} \frac{-a' + \frac{b'H}{\sqrt{r^2 - H^2}}}{r^2} \\
&= \frac{1}{r^4} \left((b' + \frac{a'H}{\sqrt{r^2 - H^2}})(b'\sqrt{r^2 - H^2} - a'H) - (b'H + a'\sqrt{r^2 - H^2})(-a' + \frac{b'H}{\sqrt{r^2 - H^2}})) \right) \\
&= \frac{1}{r^4} \left(b'^2\sqrt{r^2 - H^2} - a'b'H + a'b'H - \frac{a'^2H^2}{\sqrt{r^2 - H^2}} + a'b'H - \frac{b'^2H^2}{\sqrt{r^2 - H^2}} + a'^2\sqrt{r^2 - H^2} - a'b'H \right) \\
&= \frac{1}{r^4} \left((a'^2 + b'^2)(\sqrt{r^2 - H^2}) - \frac{(a'^2 + b'^2)H^2}{\sqrt{r^2 - H^2}} \right) \\
&= \frac{1}{r^4}(a'^2 + b'^2)(\sqrt{r^2 - H^2} - \frac{H^2}{\sqrt{r^2 - H^2}}) \\
&= \frac{r^2 - 2H^2}{r^2\sqrt{r^2 - H^2}} \\
&\leq \frac{1}{\sqrt{r^2 - H^2}}
\end{aligned}
$$

For any $\delta > 0$, since $r \geq H_u$, when $H \in (0, H_u - \delta)$, we have $H < r$. Thus, for any $\delta > 0$, $k_x(H)$, $k_y(H)$, $k_\theta(H)$, and $\operatorname{atan}(k_y(H), k_x(H))$ are Lipschitz continuous with respect to the Hamiltonian value $H \in (0, H_u - \delta)$. □

B.4. *Analysis of q^L*

For different Hamiltonian values, the mapping from q_s and q_f to the control line frame may be different. However, we can show that q_L is Lipschitz continuous with respect to the Hamiltonian value H.

Theorem 14. *For a mapping $L(H) = (k_x(H), k_y(H), k_\theta(H))$ from the Hamiltonian values to control lines with a domain $(0, H_u)$. The control line transformation $T_L(q)$ is Lipschitz continuous with respect to the Hamiltonian values in $(0, H_u - \delta)$ for any $\delta > 0$.*

Proof. For a configuration $q = (x, y, \cos\theta, \sin\theta)$ and a control line parametrized by H, $(k_x(H), k_y(H), k_\theta(H))$, the mapping from q to the control line frame is

$$
q_L(H) = \begin{pmatrix} k_x(H)x + k_y(H)y \\ -k_y(H)x + k_x(H)y + k_\theta \\ k_x(H)\cos\theta - k_y(H)\sin\theta \\ k_x(H)\sin\theta + k_y(H)\cos\theta \end{pmatrix}
$$

Hence,

$$
\frac{\partial q_L(H)}{\partial H} = \begin{pmatrix} \frac{\partial k_x(H)}{\partial H}x + \frac{\partial k_y(H)}{\partial H}y \\ -\frac{\partial k_y(H)}{\partial H}x + \frac{\partial k_x(H)}{\partial H}y + \frac{\partial k_\theta(H)}{\partial H} \\ \frac{\partial k_x(H)}{\partial H}\cos\theta - \frac{\partial k_y(H)}{\partial H}\sin\theta \\ \frac{\partial k_x(H)}{\partial H}\sin\theta + \frac{\partial k_y(H)}{\partial H}\cos\theta \end{pmatrix}
$$

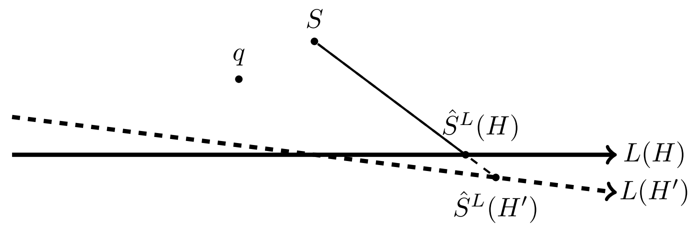

Fig. 14. Illustration of proof of Theorem 15 in the case that u_i is a translation. The reference point of the robot is at q and the switch point is at p in the world frame. When the Hamiltonian value H changes, the control line changes from $L(H)$ to $L(H')$. Hence, the endpoint of the translation changes from $\hat{S}^L(H)$ to $\hat{S}^L(H')$ accordingly.

For any $\delta > 0$, since $k_x(H)$, $k_y(H)$, and $k_\theta(H)$ are Lipschitz continuous with respect to the Hamiltonian value $H \in (0, H_u - \delta)$, q^L is Lipschitz continuous with respect to the Hamiltonian value $H \in (0, H_u - \delta)$ as well. □

B.5. Analysis of d_1, t_1, d_n, and t_n

In this section, we show that d_1, t_1, d_n, and t_n is Lipschitz continuous with respect to the change of the Hamiltonian value H.

Theorem 15. *Let $I = (a, b)$ be an open interval of the partition of the Hamiltonian values. Let $\mathbf{u}$ be a fixed sequence of n controls. Let $t_i(H)$ be the duration for the u_i and $d_i(H)$ be the length of projection of the sub-trajectory corresponding to u_i onto the control line. For any $\delta > 0$, d_1, t_1, d_n, and t_n are Lipschitz continuous with respect to the Hamiltonian values $H \in (a, b - \delta)$.*

Proof. Let $(0, H_u)$ be the domain of the mapping $L(H) = (k_x(H), k_y(H), k_\theta(H))$. For any $\delta > 0$, since $b \leq H_u$, k_x, k_y, k_θ, $\mathrm{atan}(k_y, k_x)$, and q^L are Lipschitz continuous with respect to the Hamiltonian values in $(a, b - \delta)$ by Theorems 13 and 14

Since the cases for t_1 and t_n are symmetric, we only analyze t_1 and similarly we only analyze d_1. Let the initial configuration be $q = (x, y, \cos\theta, \sin\theta)$ in the world frame. Fix a pair of controls (u, u'). Let $t(H)$ be the duration of applying u until switching to u'. Let $\hat{q}^L(H)$ be the configuration of the robot when the control switch from u to u'. Let $d(H) = \hat{q}^L_x - q^L_x$ be the projection of the motion on to the control line. There are two cases depending on whether u is a translation or a rotation.

The case in which u is a translation. Let $S = (p_x, p_y)$ be the switch point of between u and u' in the world frame. Let $v = (v_x, v_y, 0)$ be the velocity in the world frame.

The representation of S in the control line frame S^L is

$$S^L(H) = \begin{pmatrix} k_x(H)p_x + k_y(H)p_y \\ -k_y(H)p_x + k_x(H)p_y + k_\theta(H) \end{pmatrix}$$

We have

$$\frac{\partial S^L_y(H)}{\partial H} = -\frac{\partial k_y(H)}{\partial H}p_x + \frac{\partial k_x(H)}{\partial H}p_y + \frac{\partial k_\theta(H)}{\partial H}$$

The velocity v_L in the control line frame is

$$v^L(H) = \begin{bmatrix} k_x & k_y & 0 \\ -k_y & k_x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & x \\ -\sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} = \begin{bmatrix} k_x(\cos\theta v_x + \sin\theta v_y) + k_y(-\sin\theta v_x + \cos\theta v_y) \\ -k_y(\cos\theta v_x + \sin\theta v_y) + k_x(-\sin\theta v_x + \cos\theta v_y) \\ 0 \end{bmatrix}$$

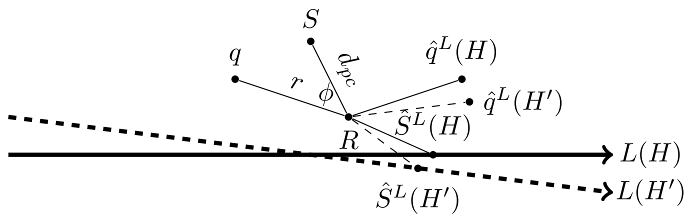

Fig. 15. Illustration of proof of Theorem 15 in the case that u_i is a rotation. The reference point of the robot is at q and the switch point is at p in the world frame. When the Hamiltonian value H changes, the control line changes from $L(H)$ to $L(H')$. Hence, the endpoint of the rotation changes from $\hat{p}(H)$ to $\hat{p}(H')$ accordingly.

Let $v' = (v'_x, v'_y) = (\cos\theta v_x + \sin\theta v_y, -\sin\theta v_x + \cos\theta v_y)$. We have

$$\frac{\partial v_x^L(H)}{\partial H} = \frac{\partial k_x(H)}{\partial H} v'_x + \frac{\partial k_y(H)}{\partial H} v'_y \text{ and } \frac{\partial v_y^L}{\partial H} = -\frac{\partial k_y(H)}{\partial H} v'_x + \frac{\partial k_x(H)}{\partial H} v'_y$$

The time is

$$t(H) = -S_y^L / v_y^L = \frac{-k_y(H)p_x + k_x(H)p_y + k_\theta(H)}{-k_y(H)(\cos\theta v_x + \sin\theta v_y) + k_x(H)(-\sin\theta v_x + \cos\theta v_y)}$$

Since $\hat{q}_x^L(H) = q_x^L(H) + v_x^L t(H)$, $d(H) = q_x^L(H) + v_x^L t(H) - q_x^L(H) = v_x^L t(H)$. Consequently,

$$\frac{\partial d(H)}{\partial H} = \frac{\partial v_x^L(H)}{\partial H} t(H) + v_x^L(H) \frac{\partial t(H)}{\partial H}$$

For any $\delta > 0$, since v_y^L and S_y^L are Lipschitz continuous with respect to the Hamiltonian values in $(a, b-\delta)$, $d(H)$ and $t(H)$ are Lipschitz continuous with respect to the Hamiltonian values in $(a, b-\delta)$ as well.

The case in which u is a rotation. Let $S = (p_x, p_y)$ be the switch point of between u and u' in the world frame. Let R be the rotation center of u in the world frame. Let r be the radius of rotation. Let d_{pc} be the distance between p and c. Let φ be the angle between the vector from c to p and the vector from c to q. Note that S, R, r, d_{pc}, q, and φ are independent from the change of the Hamiltonian value.

Let $\hat{S}^L(H)$ be the position of the switch point when the robot is at $\hat{q}^L(H)$, where $\hat{S}^L(H)$ should be on the control line. Let $\alpha(H)$ be the angle between the vector from R to $\hat{S}^L(H)$ and the control line.

$$\sin\alpha = \frac{H}{\omega d_{pc}}, \cos\alpha = \pm\frac{\sqrt{\omega^2 d_{pc}^2 - H^2}}{\omega d_{pc}}$$

The time $t(H) = (\varphi(H) - \alpha(H))/\omega$. Thus,

$$|\frac{\partial t(H)}{\partial H}| \leq \left(|\frac{\partial \mathrm{atan}(k_y, k_x)(H)}{\partial H}| + \frac{1}{\sqrt{1-(H^2)/(\omega d_{pc})^2}} \right) / |\omega| = \left(|\frac{\partial \mathrm{atan}(k_y, k_x)(H)}{\partial H}| + \frac{|\omega| d_{pc}}{\sqrt{(\omega d_{pc})^2 - H^2}} \right) / |\omega|$$

By the construction of the partition of Hamiltonian values, $b \leq \omega d_{pc}$ so that the switch of controls is feasible. For any $\delta > 0$, when $H \in (a, b-\delta)$, H is smaller than ωd_{pc}. Since $\mathrm{atan}(k_y, k_x)$ is Lipschitz continuous with respect to the Hamiltonian values in $(a, b-\delta)$, $d(H)$ is Lipschitz continuous with respect to the Hamiltonian values in $(a, b-\delta)$ as well.

Remember that $d(H) = q_x^L - \hat{q}_x^L$. We analyze $\hat{q}_x^L$ first.

$$\begin{aligned}
\hat{q}_x^L - R_x^L &= r\cos(\varphi + \alpha) \\
&= r(\cos\varphi\cos\alpha - \sin\varphi\sin\alpha) \\
&= r\left(\cos\varphi\frac{\sqrt{\omega^2 d_{pc}^2 - H^2}}{\omega d_{pc}} - \sin\varphi\frac{H}{\omega d_{pc}}\right) \\
&= \frac{r}{\omega d_{pc}}(\cos\varphi\sqrt{\omega^2 d_{pc}^2 - H^2} - \sin\varphi H)
\end{aligned}$$

Thus,

$$|\frac{\partial\hat{q}_x^L(H)}{\partial H}| \leq |\frac{\partial R_x^L(H)}{\partial H}| + \frac{r}{|\omega| d_{pc}}(|\sin\varphi| + |\cos\varphi|\frac{H}{\sqrt{(\omega d_{pc})^2 - H^2}})$$

Hence,

$$|\frac{\partial d(H)}{\partial H}| \leq |\frac{\partial q_x^L}{\partial H}| + |\frac{\partial R_x^L}{\partial H}| + \frac{r}{|\omega| d_{pc}}(|\sin\varphi| + |\cos\varphi|\frac{H}{\sqrt{\omega^2 d_{pc}^2 - H^2}})$$

For any $\delta > 0$, since $q_x^L(H)$ and $R_x^L(H)$ are Lipschitz continuous with respect to the Hamiltonian values in $(a, b-\delta)$, $d(H)$ is Lipschitz continuous with respect to the Hamiltonian values in $(a, b-\delta)$ as well. $\square$

By Theorem 12 and 15, we have the following theorem.

Theorem 11. *Let $I = (a, b)$ be an open interval of the partition of the Hamiltonian values. Let $\mathbf{u}$ be a fixed sequence of n controls. Let $t_i(H)$ be the duration for the u_i and $d_i(H)$ be the length of projection of the sub-trajectory corresponding to u_i onto the control line. For any $\delta > 0$, both functions $t_i(H)$ and $d_i(H)$ are Lipschitz continuous with respect to the Hamiltonian values $H \in (a, b-\delta)$ for all $1 \leq i \leq n$.*