IN7 INTEGRATION USING PARTIAL FRACTIONS

Adding fractions

To add fractions, rewrite the fractions with a common denominator then add the numerators.

For example:

$$\frac{3}{3x+4} + \frac{2}{x-5} = \frac{3}{3x+4} \times \frac{x-5}{x-5} + \frac{2}{x-5} \times \frac{3x+4}{3x+4} \qquad \text{[a common denominator is } (3x+4)(x-5)\text{]}$$

$$= \frac{3x-15}{(3x+4)(x-5)} + \frac{6x+8}{(3x+4)(x-5)}$$

$$= \frac{3x-15+6x+8}{(3x+4)(x-5)}$$

$$= \frac{9x-7}{(3x+4)(x-5)}$$

Finding Partial fractions

The reverse of this process is to split a fraction into partial fractions. In the above example

$$\frac{9x-7}{(3x+4)(x-5)} = \frac{3}{3x+4} + \frac{2}{x-5}$$

Algebraic fraction Partial fractions

The first step in finding partial fractions is to factorise the denominator. The factorisation will determine the form of the partial fractions:

$\frac{mx+k}{(x-a)(x-b)}$	(distinct linear factors)	$\frac{A}{x-a} + \frac{B}{x-b}$	A, B constant
$\frac{mx+k}{(x-a)^2}$	(repeated linear factor)	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$	A, B constant
$\frac{mx+k}{(ax^2+bx+c)(px+q)}$	(quadratic factor and linear factor)	$\frac{A}{px+q} + \frac{Bx+C}{ax^2+bx+c}$,	A, B, C constant

To express an algebraic fraction as partial fractions:

1. factorise the denominator.
2. write the algebraic fraction in partial fraction form with unknown constants as above
3. add the partial fractions
4. equate coefficients , or substitute values of x, to determine the value of the constants

Examples

1. (a) **Express $\frac{x-5}{x^2+2x-3}$ as the sum of partial fractions**

$\frac{x-5}{x^2+2x-3} = \frac{x-5}{(x-1)(x+3)} \Rightarrow \frac{x-5}{x^2+2x-3} = \frac{A}{x-a} + \frac{B}{x-b}$ [distinct linear factors]

$\Rightarrow \frac{x-5}{x^2+2x-3} = \frac{A}{x-1} + \frac{B}{x+3}$

$\Rightarrow \frac{x-5}{x^2+2x-3} = \frac{A(x+3)+B(x-1)}{x^2+2x-3}$ [common denominator is $(x-1)(x+3)$]

$\Rightarrow x-5 = A(x+3) + B(x-1)$ [Equating numerators]

$x = -3$: $-3 - 5 = A(0) + B(-3 - 1) \Rightarrow -8 = -4B$

$\Rightarrow B = -2$

$x = 1$: $1 - 5 = A(1 + 3)) + B(0) \Rightarrow -4 = 4A$

$\Rightarrow A = -1$

$\therefore \frac{x-5}{x^2+2x-3} = \frac{-1}{x-1} + \frac{-2}{x+3}$

Using the method of substituting convenient values for x

(b) **Find the integral of $\frac{x-5}{x^2+2x-3}$**

$\int \frac{x-5}{x^2+2x-3}\, dx$

$= \int \frac{-1}{x-1} + \frac{-2}{x+3}\, dx$ [using partial fractions]

$= -1 \ln|x-1| - 2\ln|x+3| + c$

$= \ln\left|\frac{1}{(x-1)(x+3)^2}\right| + c$ [The answer has been simplified using the laws of logarithms]

2. (a) **Express $\frac{x^2-3x+16}{x^3-5x^2+x-5}$ as the sum of partial fractions**

$$\frac{x^2-3x+16}{x^3-5x^2+x-5} = \frac{x^2-3x+16}{(x^2+1)(x-5)}$$

$$\frac{x^2-3x+16}{x^3-5x^2+x-5} = \frac{A}{x-5} + \frac{Bx+C}{x^2+1} \quad \text{[linear and quadratic factors]}$$

$$\frac{x^2-3x+16}{x^3-5x^2+x-5} = \frac{A(x^2+1)+(Bx+C)(x-5)}{x^3-5x^2+x-5} \qquad \text{[common denominator is } x^3-5x^2+x-5\text{]}$$

$$x^2-3x+16 = A(x^2+1)+(Bx+C)(x-5) \quad \text{[equating numerators]}$$

$$x^2-3x+16 = (A+B)x^2+(C-5B)x+A-5C \quad \text{[removing brackets and regrouping]}$$

$$\therefore \left.\begin{aligned} A+B &= 1 \\ C-5B &= -3 \\ A-5C &= 16 \end{aligned}\right\} \text{ Using the method of equating coefficients}$$

$$\text{and} \quad \left.\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -3 \end{aligned}\right\} \text{ by solving simultaneous equations}$$

$$\therefore \frac{x^2-3x+16}{x^3-5x^2+x-5} = \frac{1}{x-5} - \frac{3}{x^2+1}$$

(b) Find the integral of $\frac{x^2-3x+16}{x^3-5x^2+x-5}$

$$\int \frac{x^2-3x+16}{x^3-5x^2+x-5}\,dx$$

$$= \int \frac{1}{x-5} + \frac{-3}{x^2+1}\,dx \qquad \text{[using partial fractions]}$$

$$= \ln|x-5| - 3\,\mathrm{Tan}^{-1}x + c$$

The process of finding partial fractions can only be performed on fractions where the degree of the numerator of the algebraic fraction is greater than that of the denominator. If necessary, divide the denominator into the numerator then express the remaining fractional part as partial fractions.

For example $\frac{x^2+7x+7}{(x+1)(x+2)} = 1 + \frac{4x+5}{(x+1)(x+2)} = 1 + \frac{1}{x+1} + \frac{3}{x+2}$ [after expressing as partial fractions]

Exercises

1. Rewrite each of the following in the appropriate generalized (do not calculate the constants) partial fractions form.

   a) $\frac{x+6}{2x^2+5x-12}$ b) $\frac{2x}{(x^2+3)(x+1)}$ c) $\frac{2x}{x^2+8x+16}$ d) $\frac{x^2}{(x-1)(x+1)}$

2. Express the following as partial fractions

   a) $\frac{x+2}{x^2-5x+6}$ b) $\frac{3}{x^2-2x+1}$ c) $\frac{\boldsymbol{x^2-2x+2}}{\boldsymbol{x^3+x^2+x}}$ d) $\frac{x^2}{x^2-4}$

3. Perform the following integrations

   a) $\int \frac{4x+9}{x^2+x-12}$ dx

   b) $\int \frac{21-8x}{x^2-x-6}$ dx

   c) $\int \frac{5x^2-5x+2}{(x+1)(x-1)^2}$ dx

Answers

1. a) $\frac{A}{2x-3} + \frac{B}{x+4}$ b) $\frac{Ax+B}{x^2+3} + \frac{C}{x+1}$ c) $\frac{A}{x+4} + \frac{B}{(x+4)^2}$ d) $1 + \frac{A}{x+1} + \frac{B}{x-1}$

2. a) $\frac{5}{x-3} - \frac{4}{x-2}$ b) $\frac{3}{(x-1)^2}$ c) $\frac{2}{x} - \frac{x+4}{x^2+x+1}$ d) $1 - \frac{1}{x+2} + \frac{1}{x-2}$

3. a) $\ln|(x-3)^3(x+4)| + c$

b) $\ln|(x+3)^{-9}(x-2)| + c$

c) $\ln|(x+1)^3(x-1)^2| - \frac{1}{x-1} + c$