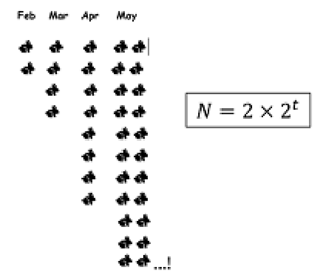
ILS2.2: Exponential Equations



Exponential equations have a term in which the variable appears as a power (index or exponent).

For example:

$$\begin{aligned} 2^{1-x} &= 4^{3x} \\ \text{or,} \quad 5^x &= 17. \end{aligned}$$

To solve exponential equations we may make use of the properties:

$$m^x = m^y \Rightarrow x = y \quad (1)$$

$$m^x = y \Rightarrow \log_a m^x = \log_a y \quad (2)$$

Examples

1. Solve $3^x = 27$

$$\begin{aligned} 3^x &= 27 \\ 3^x &= 3^3 \\ x &= 3, \text{ using property (1).} \end{aligned}$$

2. Solve $2^{1-x} = \frac{1}{8}$

$$\begin{aligned} 2^{1-x} &= \frac{1}{8} \\ 2^{1-x} &= 2^{-3} \\ 1 - x &= -3 \text{ using property (1)} \\ x &= 4. \end{aligned}$$

3. Solve $5^{\frac{1}{x}} = 25$

$$
\begin{aligned}
5^{\frac{1}{x}} &= 25 \\
5^{\frac{1}{x}} &= 5^2 \\
\frac{1}{x} &= 2 \text{ using property (1)} \\
x &= \frac{1}{2}.
\end{aligned}
$$

Non Integer Solutions

Not all equations have integer solutions.[1]

For example: $3^x = 10$ has a solution between 2 and 3 since $3^2 = 9$ and $3^3 = 27$.

Logarithms with base 10 or base e can be used to solve such equations with the calculator.

On the calculator the LOG button will calculate $\log_{10} x$ and the LN button will calculate $\log_e x$. Logarithms with base e are known as natural logarithms and sometimes the abbreviation $\ln x$ is used for $\log_e x$.

[1] An integer is a whole number. We denote the set of integers by the symbol $\mathbb{Z}$.

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

Examples

1. Solve $3^x = 10$ to 3 decimal places.
   Solution:

$$
\begin{aligned}
3^x &= 10 \\
\log_{10} 3^x &= \log_{10} 10 \\
x \log_{10} 3 &= 1 \\
x &= \frac{1}{\log_{10} 3} \\
x &= 2.095.
\end{aligned}
$$

2. Solve $2 \times 5^{x+1} = 15$ to two decimal places.
   Solution:

$$
\begin{aligned}
2 \times 5^{x+1} &= 15 \\
5^{x+1} &= 7.5 \\
\log_{10} 5^{x+1} &= \log_{10} 7.5 \\
(x+1) \log_{10} 5 &= \log_{10} 7.5 \\
x + 1 &= \frac{\log_{10} 7.5}{\log_{10} 5} \\
x + 1 &= 1.25 \\
x &= 0.25.
\end{aligned}
$$

3. Solve $2^{2x+1} = 5^{2-x}$ to three decimal places.
Solution:

$$\begin{aligned}
2^{2x+1} &= 5^{2-x} \\
\log_{10} 2^{2x+1} &= \log_{10} 5^{2-x} \\
(2x+1)\log_{10} 2 &= (2-x)\log_{10} 5 \\
(2x+1)0.301 &= (2-x)0.699 \\
0.602x + 0.301 &= 1.398 - 0.699x \\
1.301x &= 1.097 \\
x &= 0.843.
\end{aligned}$$

Growth and Decay

Exponential and logarithmic expressions occur in formulae used to model growth and decay in all branches of science.

Example

The number of bacteria present in a sample is given by $N = 800e^{0.2t}$, where t is the time in seconds.

Find a) the initial number of bacteria and b) the time taken for the bacteria count to reach 10000.

a) The initial number of bacteria is the count when $t = 0$:

$$\begin{aligned}
N &= 800e^{0.2t} \\
&= 800e^{0.2\times 0} \\
&= 800e^{0} \\
&= 800.
\end{aligned}$$

b) To find t when N= 10000. [2]

[2] In this solution we use the log law:

$$\ln a^b = b \ln a.$$

So that

$$\ln e^{0.2t} = 0.2t \ln e.$$

$$
\begin{aligned}
N &= 800e^{0.2t} \\
10000 &= 800e^{0.2t} \\
\frac{10000}{800} &= e^{0.2t} \\
12.5 &= e^{0.2t} \\
\ln 12.5 &= \ln e^{0.2t} (\text{use ln when the base is } e) \\
\ln 12.5 &= 0.2t \ln e \\
\ln 12.5 &= 0.2t \ (\text{remember} \ln e = \log_e e = 1) \\
t &= \frac{\ln 12.5}{0.2} \\
t &= 12.6.
\end{aligned}
$$

It takes 12.6 sec for the number of bacteria to reach 10000.

Exercises

1. Solve for x:
   a) $3^{1-x} = 27$
   b) $2^{2x-1} = 128$
   c) $9^{\frac{1}{x}} = 3^{-4}$
   d) $5^x = 12$
   e) $2^{x-3} = 9$
   f) $\frac{1}{2^{x+1}} = 4^{x+2}$

2. The decay rate for a radioactive element is given by $R = 400e^{-0.03t}$ where t is measured in seconds. Find
   i) the initial decay rate
   ii) the time for decay rate to reduce to half the initial decay rate.

3. The charge Q units on the plate of a condenser t seconds after it starts to discharge is given by $Q = Q_0 10^{-kt.}$. If the initial charge is 5076 units and $Q = 1840$ when $t = 0.5$sec, find
   i) the value of k
   ii) the time needed for the charge to fall to 1000 units
   iii) the charge after 2sec.

Answers

1.	*a*) $x = -2$	*b*) $x = 4$	*c*) $x = -1/2$
	d) $x = 1.54$	*e*) $x = 6.17$	*f*) $x = -1.67$
2.	i) $R = 400$	ii) $t = 23.1\,s$	
3.	i) $k = 0.881$	ii) $t = 0.8\,s$	iii) $Q = 87.8$ units