RMIT UNIVERSITY

IN1.4: INTEGRATION BY PARTS

The rule for integration by parts is the rule corresponding to the product rule for differentiation.

Formula for integration by parts.

$$\boxed{\int u\,dv = uv - \int v\,du}$$

The aim when using integration by parts is to form a new integral, (on the right), that is simpler than the original (on the left). The correct choice of terms for *u* and *dv/dx* is critical.

Choose *u* such that when it is differentiated it is removed, or simplifies the integral. (provided *dv/dx* can be easily integrated for *v*.)

If the integral involves a natural logarithm or inverse trig function this should be chosen as *u*

Examples

1. Find $\int x\cos x\,dx$ let u = x ∴ du = dx (x differentiates to 1)

and dv = cosxdx ∴ v = sinx (cosx is easily integrated)

$$\int x\cos x\,dx = \int u\,dv = uv - \int v\,du$$

$$= x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + c$$

2. Find $\int xe^{2x}dx$ let u = x ∴ du = dx and dv = $e^{2x}dx$ ∴ v = $\frac{1}{2}e^{2x}$

$$\int xe^{2x}dx = \int u\,dv = uv - \int v\,du$$

$$= x\,(\frac{1}{2}e^{2x}) - \int \frac{1}{2}e^{2x}dx$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$$

3.Find $\int x^3 \log_e x dx$ let u = $\log_e x$ $\therefore$ du = $\frac{1}{x}$ dx (need to let u = log function)

dv = $x^3 dx$ $\therefore$ v = $\frac{x^4}{4}$

$$\int x^3 \log_e x dx = \int udv = uv - \int vdu$$

$$= \log_e x (\frac{x^4}{4}) - \int \frac{x^4}{4} \times \frac{1}{x} dx$$

$$= \frac{x^4}{4} \log_e x - \int \frac{x^3}{4} dx$$

$$= \frac{x^4}{4} \log_e x - \frac{x^4}{16} +$$

4. Find $\int \arcsin x dx$

Consider $\int \arcsin x dx$ as $\int 1 \times \arcsin x dx$ let $u = \arcsin x$ $\therefore$ $du = \frac{1}{\sqrt{1-x^2}} dx$

$dv = 1dx$ $\therefore$ $v = x$

$$\int \arcsin x dx = \int udv = uv - \int vdu$$

$$= \arcsin x (x) - \int x \frac{1}{\sqrt{1-x^2}} dx \ldots\ldots$$

use a substitution $t = 1 - x^2$

$dt = -2xdx \therefore xdx = -\frac{1}{2} dt$

$$= x \arcsin x - -\frac{1}{2} \int \frac{1}{\sqrt{t}} dt$$

$$= x \arcsin x + \sqrt{1-x^2} + c$$

See Exercise 1

Repeated integration by parts

Sometimes after integrating by parts the new integral cannot be integrated except by another use of integration by parts.

Examples

1. Find $\int x^2 e^x dx$ let $u = x^2$ $\therefore$ $du = 2x\, dx$

and $dv = e^x dx$ $\therefore$ $v = e^x$

$$\int x^2 e^x dx = \int udv = uv - \int vdu$$

$$= x^2 e^x - \int e^x 2x dx$$

$$= x^2 e^x - 2\int xe^x dx$$ Integrate this new integral by parts.

$u = x$ $\therefore$ $du = dx$

$dv = e^x dx$ $\therefore$ $v = e^x$

$$= x^2 e^x - 2(xe^x - \int e^x dx)$$

$$= x^2 e^x - 2xe^x + 2e^x + c$$

2. Find $\int e^x \cos x dx$ $\quad u = e^x \quad \therefore\ du = e^x\, dx$

$dv = \cos x dx \ \therefore\ v = \sin x$

$$\int e^x \cos x dx = \int u dv = uv - \int v du$$

$$= e^x \sin x - \int \sin x e^x dx$$ Integrate this new integral by parts .

$u = e^x \quad \therefore\ du = e^x\, dx$

$dv = \sin x dx \ \therefore\ v = -\cos x$

$$= e^x \sin x - (e^x(-\cos x) - \int -\cos x e^x dx)$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x dx)$$

The new integral on the right side of the equation is the same as the original integral, on the left, multiplied by (–1). Shift this to the left side of the equation to give:

$$2\int e^x \cos x dx = e^x \sin x + e^x \cos x$$

$$\therefore \int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x)$$

When the integral is a product of trig functions, or exponential and trig functions it is often necessary to repeat integration by parts until the original integral appears on the right-hand side of the equation.

See Exercise 2

Exercises

Find

1. (a) $\int x \sin x dx$ (b) $\int x e^x dx$ (c) $\int x \cos 4x dx$

(d) $\int x \sin(\pi - 2x) dx$ (e) $\int \log_e x dx$ (f) $\int x^4 \log_e x dx$

(g) $\int \frac{1}{x^2} \log_e x dx$

2.

(a) $\int x^2 \cos x dx$ (b) $\int x^2 e^{3x} dx$ (c) $\int e^x \sin x dx$

(d) $\int \sqrt{x} \log_e x dx$ (e) $\int x^2 e^{-2x} dx$

(f) $\int x^5 e^{x^3} dx$ (*hint: write as* $\int x^3 x^2 e^{x^3} dx$)

Answers

1.(a) $-x\cos x+\sin x+c$ (b) $xe^{x}-e^{x}+c$ (c) $\frac{1}{4}x\sin 4x+\frac{1}{16}\cos 4x+c$

(d) $\frac{x}{2}\cos(\pi-2x)+\frac{1}{4}\sin(\pi-2x)+c$ (e) $xlog_{e}x-x+c$

(f) $\frac{x^{5}}{5}log_{e}x-\frac{x^{5}}{25}+c$ (g) $-\frac{1}{x}log_{e}x-\frac{1}{x}+c$

2. (a) $x^{2}\sin x+2x\cos x-2\sin x+c$ (b) $\frac{1}{3}x^{2}e^{3x}-\frac{2}{9}xe^{3x}+\frac{2}{27}e^{3x}+c$

(c) $\frac{1}{2}e^{x}\sin x-\frac{1}{2}e^{x}\cos x+c$ (d) $\frac{2}{3}x^{\frac{3}{2}}\log_{e}x-\frac{4}{9}x^{\frac{3}{2}}+c$

(e) $-\frac{1}{2}x^{2}e^{-2x}-\frac{1}{2}xe^{-2x}-\frac{1}{4}e^{-2x}+c$ (g) $\frac{1}{3}x^{3}e^{x^{3}}-\frac{1}{3}e^{x^{3}}+c$