CONDITIONAL PROBABILITY

Dependent events

Two events are dependent if the outcome or occurrence of the first affects the outcome or occurrence of the second so that the probability is changed.

Example

A card is chosen at random from a pack. If the first card chosen is the jack of diamonds and it is ***not replaced*** *what is the probability that the second card is*

(a) *a diamond?* $P(\diamondsuit) = \frac{12}{51} = \frac{4}{17}$

(b) *a jack?* $P(\text{jack}) = \frac{3}{51} = \frac{1}{17}$

(c) *the queen of clubs?* $P(Q\clubsuit) = \frac{1}{51}$

one less diamond in the pack

one less card in the pack

The events J_1 'jack of diamonds on the first draw' and D_2 'a diamond on the second draw' are dependent when there is no replacement. The probability of choosing a diamond on the second draw given that the jack of diamonds was chosen on the draw pick is called a **conditional probability**. We say $\Pr(D_2/J_1) = \frac{4}{17}$... "The probability of D_2 given J_1 is $\frac{4}{17}$ "

Multiplication Rule

When two events, A and B, are dependent, the probability of both occurring is:

$$\Pr(A \text{ and } B) = \Pr(A \cap B) = P(A) \cdot P(B|A)$$

Example

Find the probability of obtaining two jacks if two cards are drawn is succession from a pack

(a) with replacement

(b) without replacement

(a) If the cards are replaced then the events are independent:

$$\Pr(J_1 \cap J_2) = \Pr(J_1) \times \Pr(J_2) = \frac{4}{52} \times \frac{4}{52} = \frac{1}{169}$$

(b) If the cards are not replaced then the probability of the second draw depends on the first draw:

$$\Pr(J_1 \cap J_2) = \Pr(J_1) \times \Pr(J_2/J_1) = \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$$

Conditional probability

The multiplication rule for dependent events can be rearranged to find a conditional probability

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{P(A)} \quad \text{or} \quad \Pr(A|B) = \frac{\Pr(A \cap B)}{P(B)}$$

Examples

1. *Find the Pr(A|B) if Pr(A) = 0.7, Pr(B) = 0.5 and Pr(A ∪ B) = 0.8.*

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ [we must first find $\Pr(A \cap B)$]

$0.8 = 0.7 + 0.5 - \Pr(A \cap B)$

$\therefore \quad \Pr(A \cap B) = 0.4$

and $\Pr(A|B) = \frac{\Pr(A \cap B)}{P(B)} = \frac{0.4}{0.5} = 0.8$

2. *In a class of 15 boys and 12 girls two students are to be randomly chosen to collect homework. What is the probability that both students chosen are boys?*

$\Pr(B_1 \cap B_2) = \Pr(B_1) \times \Pr(B_2/B_1) = \frac{15}{27} \times \frac{14}{26} = \frac{210}{702} = \frac{35}{117}$

Another way to do conditional probability problems is to reduce the sample space:

3. *Given the information in the following table find the probability that someone was sunburnt given that they were not wearing a hat.*

		Sunburnt face		
		Yes	No	
Hat	Yes	3	77	80
	No	12	8	20
		15	85	100

Highlight the part of the table that satisfies the condition "not wearing a hat".
This becomes the sample space for the question.

$\Pr(S/\overline{H}) = \frac{12}{20} = \frac{3}{5}$

Exercise

1. The results of a survey of music preferences are displayed in the Venn diagram. Find the probability that a student likes rock music given that they like dance music.

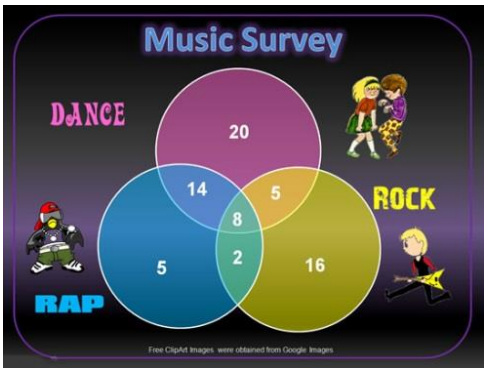

Image Source: Passy's World of Mathematics

2. Three cards are chosen at random from a pack without replacement. What is the probability of choosing 3 aces?
3. In a maths class of 20 students 5 failed the final exam. If two students are chosen at random without replacement, what is the probability that the first passed but the second failed?
4. If $\Pr(X) = 0.5$, $\Pr(Y) = 0.5$ and $\Pr(X \cap Y) = 0.2$ find the probability of
   (a) $\Pr(X/Y)$ (b) $\Pr(X \cup Y)$ (c) $\Pr(X) \times \Pr(Y/X)$
5. In a three child family what is the probability that all three children will be girls given that the first child is a girl. [Hint: Draw a tree diagram to find the sample space]

Answers

1. $\frac{5+8}{14+20+8+5} = \frac{13}{47}$
2. $\frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} = \frac{1}{5525}$
3. $\frac{15}{20} \times \frac{5}{19} = \frac{15}{76}$
4. (a) 0.4 (b) 0.8 (c) 0.2
5. 0.25