RMIT UNIVERSITY

IN1.3: INTEGRATION BY SUBSTITUTION

Direct Substitution

Many functions cannot be integrated using the methods previously discussed. Substitution is used to change the integral into a simpler one that *can* be integrated.

Substitution rule

If $u = g(x)$ then $du = g'(x)dx$ and

$$\boxed{\int f(g(x))g'(x)dx = \int f(u)du}$$

Examples

Find the following:

1. $\int (8x-5)^9 dx$

Let $u = 8x-5$ then $du = 8dx$

Substitute for *x* and *dx* the original integral

$$\int (8x-5)^9 dx = \int u^9 \frac{1}{8} du \text{ ....} \textit{because } dx = 1/8\, du$$

$$= \frac{1}{8}\frac{u^{10}}{10} + c \text{ ....} \textit{integrate}$$

$$\int (8x-5)^9 dx = \frac{(8x-5)^{10}}{80} + c \text{ ...} \textit{substitute for u to give the answer as a function of x.}$$

If possible choose u to be a function in the integrand whose derivative (or a multiple of) also occurs.

2. $\int x\left(x^2+2\right)^3 dx$

$$\int x\left(x^2+2\right)^3 dx = \int \left(x^2+2\right)^3 x dx \text{ ..} \textit{let } u = x^2+2 \quad du = 2xdx$$

$$= \frac{1}{2}\int u^3 du \text{ ....} \textit{substitute: } x^2+2 = u \textit{ and } xdx = \frac{1}{2}du$$

$$= \frac{1}{2}\left(\frac{u^4}{4}\right) + c \text{ ......} \textit{integrate}$$

$$= \frac{\left(x^2+2\right)^4}{8} + c \text{ ...} \textit{substitute for u}$$

3. $\int \frac{x}{\sqrt{1+2x^2}}dx$ ....let u = $1+2x^2$, du = $4x$ dx

$\int \frac{x}{\sqrt{1+2x^2}}dx = \frac{1}{4}\int \frac{1}{\sqrt{u}}du$ ....substitute: $1+2x^2$ =u, x dx = $\frac{1}{4}$ du

$= \frac{1}{4}\int u^{-\frac{1}{2}}du$

$= \frac{1}{2}u^{\frac{1}{2}} + c$ .... integrate

$= \frac{1}{2}\sqrt{1+2x^2} + c$ ..... substitute for u

4. $\int \sin x \cos x dx$ ....let u = sinx du = cosxdx

$\int \sin x \cos x dx = \int u du$ ....substitute: sinx = u , cosxdx = du

$= \frac{u^2}{2} + c$ .... integrate

$= \frac{\sin^2 x}{2} + c$ ....substitute for u

5. $\int x^2 e^{4-x^3} dx$ ....let u = $4-x^3$, du = $-3x^2$ dx

$\int x^2 e^{4-x^3} dx = \int e^{4-x^3} x^2 dx$

$= -\frac{1}{3}\int e^u du$ .... substitute: $4-x^3$ = u, x^2 dx = $-\frac{1}{3}$ du

$= -\frac{1}{3}e^u + c$ ....integrate

$= -\frac{1}{3}e^{4-x^3} + c$ .... substitute for u

6. $\int \frac{4x}{3-x^2}dx$ .... let u = $3-x^2$, du = $-2x$ dx

$\int \frac{4x}{3-x^2}dx = -2\int \frac{1}{u}du$ ....substitute: $3-x^2$ =u, $4x$ dx = -2 du

$= -2\log_e |u| + c$ ....integrate

$= -2\log_e |3-x^2| + c$ ..substitute for u

7. $\int \frac{2x-3}{x^2-3x+7}dx$ .... let u = x^2-3x+7 du = $(2x-3)$ dx

$\int \frac{2x-3}{x^2-3x+7}dx = \int \frac{1}{u}du$ .... substitute: x^2-3x+7 = u, $(2x-3)$ dx = du

$= \log_e |u| + c$ .... integrate

$= \log_e |x^2-3x+7| + c$ ....substitute for u

8. $\int\left(\cos^3 x+2\right)\sin x dx$ ………………….. let u = cosx , du = – sinxdx

$\int\left(\cos^3 x+2\right)\sin x dx = -\int\left(u^3+2\right)du$ ……substitute $\left(\cos^3 x+2\right)=\left(u^3+2\right)$, sinxdx=–du

$= -\frac{u^4}{4}-2u+c$ …… integrate

$= -\frac{\cos^4 x}{4}-2\cos x+c$ ….. substitute for u

See Exercises 1,2,3 and 4

Specific types

Some substitutions are more complicated and it may be necessary to manipulate the integral before substitution.

Examples

Find

1. $\int x\sqrt{x+1}dx$ ……… *let u =* $x+1$*,* *du = dx.*

$\int x\sqrt{x+1}dx = \int xu^{\frac{1}{2}}du$ …….*substitute x+1 = u, dx = du*

After substituting for $x+1$ and *dx* the '*x*' term remains. To overcome this problem transpose *u =* $x+1$, to make *x* the subject then substitute for *x*.

$\int x\sqrt{x+1}dx = \int(u-1)\sqrt{u}du$ … *substitute:* $x+1$ *= u, dx = du, x =* $u-1$,

$= \int(u^{\frac{3}{2}}-u^{\frac{1}{2}})du$

$= \frac{2}{5}u^{\frac{5}{2}}-\frac{2}{3}u^{\frac{3}{2}}+c$ ………. *integrate*

$= \frac{2}{5}(x+1)^{\frac{5}{2}}-\frac{2}{3}(x+1)^{\frac{3}{2}}+c$ …….. *substitute for u*

2. $\int\frac{6x+7}{x^2+4x+13}dx$

In this example the numerator is not a multiple of the derivative of the denominator.
The derivative of the denominator is 2*x* + 4.
We need a multiple of 2*x* + 4 in the numerator.
To do this write 6*x* +7 as 3 (2*x* + 4) – 5.

$\therefore \int\frac{6x+7}{x^2+4x+13}dx = \int\frac{3(2x+4)-5}{x^2+4x+13}dx$

$= 3\int\frac{(2x+4)}{x^2+4x+13}dx-\int\frac{5}{x^2+4x+13}dx$

$= 3\int\frac{(2x+4)}{x^2+4x+13}dx-\int\frac{5}{(x+2)^2+9}dx$ complete the square on the denominator

Integrate $3\int \frac{(2x+4)}{x^2+4x+13}dx$ as in example 7 above

$$3\int \frac{2x+4}{x^2+4x+13}dx \qquad \text{let } u = x^2+4x+13 \ \ du = (2x+4)\,dx$$

$$3\int \frac{2x+4}{x^2+4x+13}dx = \int \frac{1}{u}du$$

$$= \log_e |u| + c$$

$$= \log_e \left|x^2-3x+7\right| + c$$

$\int \frac{5}{(x+2)^2+9}dx \ = \ \frac{5}{3}\arctan\frac{(x+2)}{3} + c$ using integration tables.

$$\therefore \int \frac{6x+7}{x^2+4x+13}dx = \log_e \left|x^2-3x+7\right| + \frac{5}{3}\arctan\frac{(x+2)}{3} + c$$

3. $\int \cos^3 x dx$

Rewrite the integral as $\int \cos^2 x \cos x dx = \int (1-\sin^2 x)\cos x dx$

$\int (1-\sin^2 x)\cos x dx$ may be found using the substitution u = sinx

$\int (1-\sin^2 x)\cos x dx$ ……………………..let u = sinx , du = cosxdx

$\int (1-\sin^2 x)\cos x dx = \int (1-u^2)du$ .... substitute sinx =u, cosxdx = du

$= u - \frac{u^3}{3} + c$ ……….integrate

$= \sin x - \frac{\sin^3 x}{3} + c$ …..substitute for u

See Exercise 5

Exercises

Exercise 1

Find:

(a) $\int (5x+1)^4 dx$ (b) $\int 3x^2\left(x^3+2\right)^5 dx$ (c) $\int \frac{2}{(x+3)^3}dx$

(d) $\int (2x+1)\left(x^2+x+3\right)^3 dx$ (e) $\int 6x^2\sqrt{x^3+3}dx$

Exercise 2

Find:

(a) $\int \cos 8x dx$ (b) $\int \sin(3x+2)dx$ (c) $\int \sec^2 x \tan x dx$

(d) $\int 2\cos x \sin^2 x dx$ (e) $\int x^2 \sin(\pi - x^3)dx$

Exercise 3

Find:

(a) $\int \frac{1}{x+5}dx$ (b) $\int \frac{4}{3x+5}dx$ (c) $\int \frac{2x}{2+x^2}dx$

(d) $\int \frac{x}{x^2+5}dx$ (e) $\int \frac{2x^2}{x^3-3}dx$ (f) $\int \frac{\cos x}{2\sin x+3}dx$

(g) $\int \frac{2x-3}{x^2-3x-9}dx$ (h) $\int \frac{2\log_e x}{x}dx$

Exercise 4

Find:

(a) $\int e^{3x}dx$ (b) $\int e^{x+4}dx$ (c) $\int e^{2-5x}dx$

(d) $\int 6x^2e^{x^3}dx$ (e) $\int \cos x.e^{2\sin x}dx$ (f) $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$

(g) $\int \left(3-\sin^2 x\right)\cos x dx$

Exercise 5

Find:

(a) $\int \frac{x}{\sqrt{(x-2)}}dx$ (b) $\int \frac{4x-3}{x^2-4x+20}dx$ (c) $\int x^3\sqrt{1-x^2}dx$

(d) $\int \sin^3 x dx$

Answers

1.

(a) $\frac{1}{25}(5x+1)^5 + c$ (b) $\frac{1}{6}(x^3+2)^6 + c$ (c) $\frac{-1}{(x+3)^2} + c$

(d) $\frac{1}{4}(x^2+x+3)^4 + c$ (e) $\frac{4}{3}\left(x^3+3\right)^{\frac{3}{2}} + c$

2.

(a) $\frac{1}{8}\sin 8x + c$ (b) $-\frac{1}{3}\cos(3x+2) + c$ (c) $\frac{\tan^2 x}{2} + c$

(d) $\frac{2}{3}\sin^3 x + c$ (e) $\frac{\cos(\pi - x^3)}{3} + c$

3.

(a) $\ln(x+5) + c$ (b) $\frac{4}{3}\ln(3x+5) + c$ (c) $\ln(2+x^2)+c$

(d) $\frac{1}{2}\ln(x^2+5) + c$ (e) $\frac{2}{3}\ln(x^3-3) + c$ (f) $\frac{1}{2}\ln(2\sin x+3) + c$

(g) $\ln(x^2-3x-9) + c$ (h) $(\ln x)^2 + c$

4.

(a) $\frac{1}{3}e^{3x} + c$ (b) $e^{x+4} + c$ (c) $-\frac{1}{5}e^{2-5x} + c$

(d) $2e^{x^3} + c$ (e) $\frac{1}{2}e^{2\sin x} + c$ (f) $2\sin\sqrt{x}+c$

(g) $3\sin x - \frac{\sin^3 x}{3} + c$

5.

(a) $\frac{2}{3}(x-2)^{\frac{3}{2}} + 4(x-2)^{\frac{1}{2}} + c$ (b) $2\ln(x^2-4x+20) + \frac{5}{4}\arctan\frac{x-2}{4} + c$

(c) $\frac{\left(1-x^2\right)^{\frac{5}{2}}}{5} - \frac{\left(1-x^2\right)^{\frac{3}{2}}}{3} + c$ (d) $-\cos x + \frac{1}{3}\cos^3 x + c$