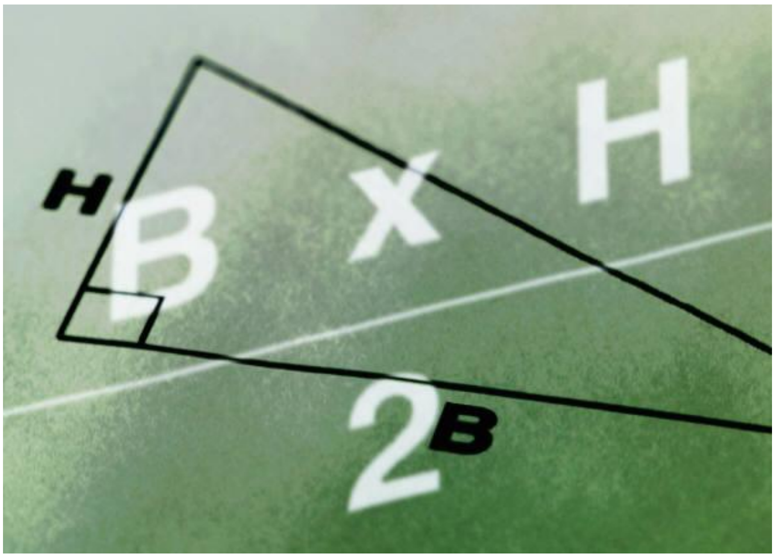
STUDY KNOWHOW PROGRAM
STUDY AND LEARNING CENTRE

RMIT UNIVERSITY

Complex Numbers



CONTENTS

COMPLEX NUMBERS

Real and complex numbers

Equations such as $x+1=7$, $3x=10$ and $x^2-7=0$ can all be solved within the real number system.

But there is no real number which satisfies $x^2+1=0$. To obtain solutions to this and other similar equations the complex numbers were developed.

The imaginary number i is defined such that $i^2=-1$

i.e.

$$\boxed{i = \sqrt{-1}}$$

NB: $i^2=-1$,
$i^3=-i$,
$i^4=1$,
$i^5=i$, etc.

> A number z of the form $z=x+yi$, where x and y are real numbers is called a complex number

- x is called the real part of z, denoted by $Re\{z\}$, and
- y is called the imaginary part of z, denoted by $Im\{z\}$

Examples

1. If $z=5-3i$ then $Re\{z\}=5$ and $Im\{z\}=-3$
2. If $z=\sqrt{3}i$ then $Re\{z\}=0$ and $Im\{z\}=\sqrt{3}$

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal:

i.e.

$$\boxed{a+bi = c+di \quad \text{if and only if} \quad a=c \text{ and } b=d}$$

Example

If $z_1=x-\frac{i}{3}$, $z_2=\sqrt{2}+yi$ and $z_1=z_2$ find the values of x and y.

$$Re\{z_1\}=Re\{z_2\} \Rightarrow x=\sqrt{2}$$

and $Im\{z_1\}=Im\{z_2\} \Rightarrow y=-\frac{1}{3}$

$\therefore x=\sqrt{2}$ and $y=-\frac{1}{3}$

Addition and Subtraction of Complex Numbers

To add or subtract complex numbers we add or subtract the real and imaginary parts separately:

$$(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i$$

Examples

1. $(2 + 3i) + (4 - i) = (2 + 4) + (3 - 1)i$
   $= 6 + 2i$

2. If $z_1 = 1 - i$ and $z_2 = 3 - 5i$ find $z_1 - z_2$

$z_1 - z_2 = (1 - i) - (3 - 5i)$
$= (1 - 3) + (-1 - (-5))i$
$z_1 - z_2 = -2 + 4i$

See Exercise 1

Multiplication of Complex Numbers

If $z_1 = a + bi$ and $z_2 = c + di$ are two complex numbers then

$kz_1 = k(a + bi)$
$= ka + kbi$
and
$z_1 z_2 = (a + bi)(c + di)$
$= ac + adi + bci + bdi^2$
$= (ac - bd) + (ad + bc)i$ [since $i^2 = -1$]

Examples

1. Expand and simplify $i(3 + 4i)$
$i(3 + 4i) = 3i + 4i^2$
$i(3 + 4i) = -4 + 3i$

2. If $z_1 = 1 - i$ and $z_2 = 3 - 5i$ find $z_1 z_2$
$z_1 z_2 = (1 - i)(3 - 5i)$
$= 3 - 3i - 5i + 5i^2$
$z_1 z_2 = 3 - 8i - 5$
$= -2 - 8i$

See Exercise 2

Complex Conjugates

A pair of complex numbers of the form $a + bi$ and $a - bi$ are called complex conjugates.

If $z = x + yi$ then the conjugate of z is denoted by $\bar{z} = x - yi$

Eg: $2 + 3i$ and $2 - 3i$ are a conjugate pair
$1 - i$ and $1 + i$ are a conjugate pair
$-4i$ and $4i$ are a conjugate pair

NB:

The product of a conjugate pair of complex numbers is a real number

$$z\bar{z} = (x+yi)(x-yi) = x^2 + y^2$$

Properties of Conjugates:

If z_1 and z_2 represent two conjugate numbers then

(i) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
(ii) $\overline{z_1 z_2} = \bar{z}_1\bar{z}_2$
(iii) $\bar{\bar{z}} = z$

Examples

If $z = 2 - i$ and $w = -3 + 4i$ find

1. $\bar{z}$
2. $\bar{z} - \bar{w}$
3. $\overline{z + w}$

Solutions

1. $\bar{z} = 2 + i$
2. $\bar{z} - \bar{w} = (2+i) - (-3-4i)$
$= 2 + 3 + i + 4i$
$\bar{z} - \bar{w} = 5 + 5i$
3. $\overline{z + w} = \overline{2 - i + (-3+4i)}$
$= \overline{-1 + 3i}$
$= -1 - 3i$

See Exercise 3

Division of complex numbers

If $z_1 = a + bi$ and $z_2 = c + di$, then $\frac{z_1}{z_2} = \frac{a+bi}{c+di}$.

To express $\frac{z_1}{z_2}$ in the form $x + yi$ we make use of the conjugate to change the denominator into a real number.

Examples

1. Express $\frac{2-i}{1+3i}$ in the form $x + yi$

$\frac{2-i}{1+3i} = \frac{2-i}{1+3i} \times \frac{1-3i}{1-3i}$
$= \frac{2-i-6i-3}{1+9}$
$= \frac{-1-7i}{10}$
$= -\frac{1}{10} - \frac{7}{10}i$

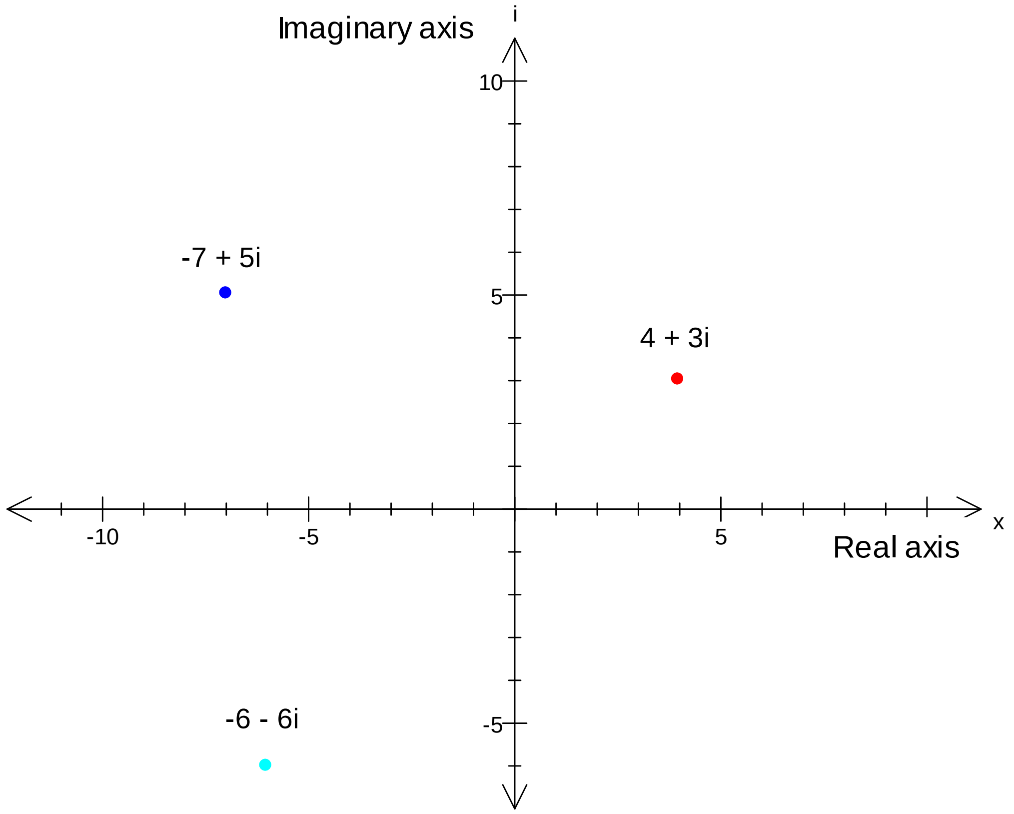
Operations on fractions involving complex numbers follow the same rules as algebraic fractions

2. Simplify $\frac{i}{1-4i}+\frac{2}{3+i}$

$$\frac{i}{1-4i}+\frac{2}{3+i}=\frac{i}{1-4i}\times\frac{1+4i}{1+4i}+\frac{2}{3+i}\times\frac{3-i}{3-i}$$
$$=\frac{i-4}{1+16}+\frac{6-2i}{9+1}$$
$$=\frac{i-4}{17}+\frac{6-2i}{10}$$
$$=\frac{i-4}{17}\times\frac{10}{10}+\frac{6-2i}{10}\times\frac{17}{17}$$
$$=\frac{10i-40}{170}+\frac{102-34i}{170}$$
$$=\frac{10i-40+102-34i}{170}$$
$$+\frac{2}{3+i}=\frac{62-24i}{170}$$
$$+\frac{2}{3+i}=\frac{2(31-12i)}{170}$$
$$+\frac{2}{3+i}=\frac{31-12i}{85}$$

See Exercise 4

An Argand Diagram is a geometrical representation of the set of complex numbers. The complex number $z=x+yi$ can be plotted as a point represented by the ordered pair (x,y) on the complex number plane:



See Exercise 5

Exercises

Exercise 1

1. Express the following in terms of i in simplest surd form

   (a) $\sqrt{-9}$ (b) $\sqrt{-2}$ (c) $\sqrt{-5} \times \sqrt{3}$

   (d) $\sqrt{-5} \times \sqrt{10}$ (e) $\sqrt{-6} \times \sqrt{12}$

2. Evaluate

   (a) i^4 (b) i^9 (c) $i^7 - i^{11}$ (d) $i^5 + i^6 - i^7$ (e) $2i - i^6 + 2i^7$

3. State the value of $Re\{z\}$ and $Im\{z\}$ for these complex numbers:

   (a) $2 + 7i$ (b) $10 - i$ (c) $\pi + 3i$ (d) $\frac{i}{6}$ (e) -8

4. Find the values of x and y

   (a) $x + yi = 4 + 9i$ (b) $x + yi = 3 - i$

   (c) $x + yi = 23$ (d) $x + yi = -\sqrt{2}i$

   (e) $x + i = -5 + yi$

Exercise 2

1. Expand and simplify

   (a) $i(3 - 2i)$ (b) $2i^3(1 - 5i)$ (c) $(8 - 3i)(2 - 5i)$

   (d) $(4 - 3i)^2$ (e) $(3 + 2i)(3 - 2i)$

2. If z_1 = -1 + 3i and z_2= 2 - i find each of the following

   (a) $z_1 z_2$ (b) $2z_1 - z_2$ (c) $(z_1 - z_2)^2$

3. Find the value of x and y if $(x + yi)(2 - 3i) = -13i$

Exercise 3

1. Find the conjugate of each of the following complex numbers:

   (a) $4 + 9i$ (b) $-3 - 15i$ (c) $\sqrt{3} - 4i$

2. Find the conjugate of (2 – i)(4 + 7i)

3. If z = 2 - i and w = 1 + 2i express the following in the form x + yi:

   (a) $\bar{z}$ (b) $\overline{z + w}$ (c) $\bar{z} + \bar{w}$ (d) $\overline{zw}$ (e) $\overline{\bar{z} - \bar{w}}$

Exercise 4

1. Express the following in the form x + yi

   (a) $\frac{4-9i}{3}$ (b) $\frac{1}{3-i}$ (c) $\frac{5+i}{2-7i}$

2. Simplify $\dfrac{2}{1-i} + \dfrac{3+i}{i}$

3. If w = -1 + 6i express $\dfrac{w+1}{w-i}$ in the form $x + yi$

Exercise 5

If z = 2 – 3i and w = 1 + 4i, illustrate on an Argand diagram

1. z 2. w 3. $z + w$ 4. $\overline{z + w}$ 5. $2z - w$

Answers

Exercise 1

1. (a) 3i (b) $\sqrt{2}i$ (c) $\sqrt{15}i$ (d) $5\sqrt{2}i$ (e) $6\sqrt{2}i$

2. (a) 1 (b) i (c) 0 (d) 2i -1 (e) 1

3. (a) $Re\{z\} = 2,\ Im\{z\} = 7$ (b) $Re\{z\} = 10,\ Im\{z\} = -1$
   (c) $Re\{z\} = \pi,\ Im\{z\} = 3$ (d) $Re\{z\} = 0,\ Im\{z\} = \frac{1}{6}$
   (e) $Re\{z\} = -8,\ Im\{z\} = 0$

4. (a) x = 4, y = 9 (b) x = 3, y = -1
   (c) x = 23, y = 0 (d) x = 0, $y = -\sqrt{2}$
   (e) x = -5, y = 1

Exercise 2

1 (a) 2 + 3i (b) -10 - 2i (c) 1 – 46i (d) 7 – 24i (e) 13
2. (a) 1 + 7i (b) -4 + 7i (c) -7 – 24i
3 x = 3, y = -2

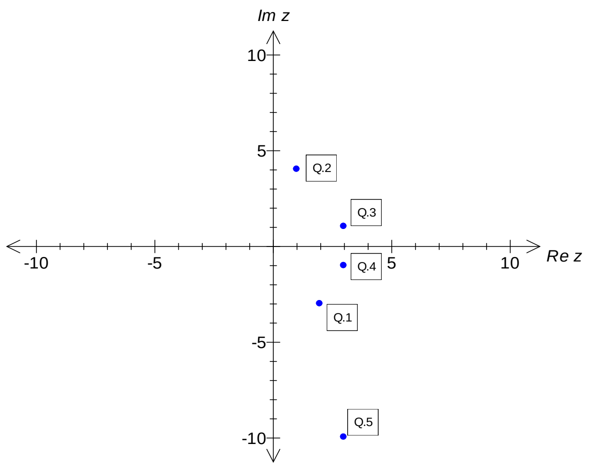
Exercise 3

1 (a) 4 – 9i (b) -3 + 15i (c) $\sqrt{3} + 4i$
2. 15 – 10i
3. (a) 2 + i (b) 3 – i (c) 3 – i
   (d) 4 – 3i (e) 1 – 3i

Exercise 4

1. (a) $\frac{4}{3} - 3i$ (b) $\frac{3}{10} + \frac{1}{10}i$ (c) $\frac{3}{53} + \frac{37}{53}i$
2. 2 – 2i
3. $\frac{15}{13} - \frac{3}{13}i$

Exercise 5

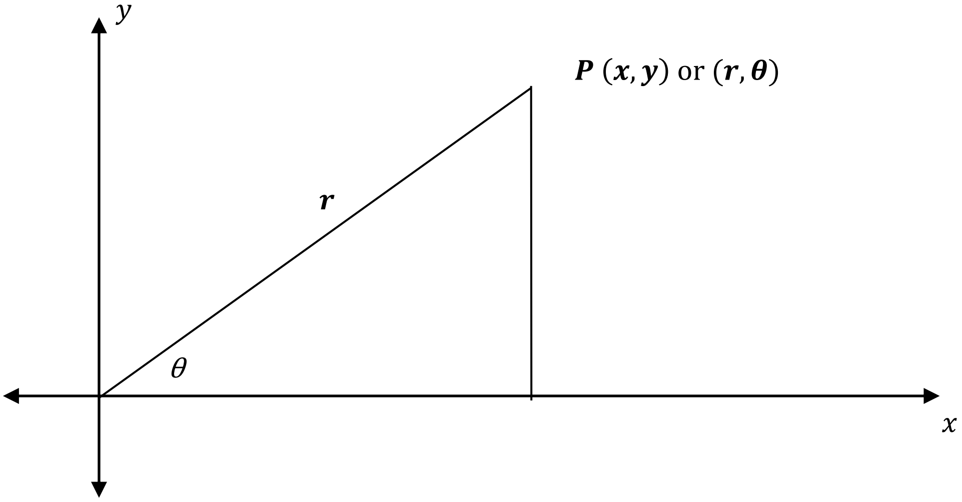


POLAR FORM OF A COMPLEX NUMBER

Rectangular and Polar Form

When a complex number is expressed in the form $z = x + yi$ it is said to be in rectangular form.

But a point P with Cartesian coordinates (x, y) can also be represented by the polar coordinates (r, θ) where r is the distance of the point P from the origin and θ is the angle that $\overrightarrow{OP}$ makes with the positive x-axis. Note that all angles are expressed in radians unless there is a degrees symbol °.



NB: $x = r\cos\theta$ and $y = r\sin\theta$
and $x^2 + y^2 = r^2$ or $r = \sqrt{x^2 + y^2}$

To express a complex number z in polar form:

$$\begin{aligned} z &= x + yi \\ &= r\cos\theta + r\sin\theta\, i \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$

which we abbreviate to $z = rcis\theta$

So, the polar form of the complex number z is

$$z = rcis\theta$$

where $r = \sqrt{x^2 + y^2}$
and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Modulus of z

The modulus of z, $|z|$ is the distance of the point z from the origin.

$$\text{mod}\, z = |z| = |x + yi| = \sqrt{x^2 + y^2} = r$$

The argument and Argument of z

The argument of z, $\arg z$, is the angle measured from the positive direction of the x-axis to $\overrightarrow{OP}$

If $\arg z = \theta$ then $\sin\theta = \frac{y}{|z|}$ and $\cos\theta = \frac{x}{|z|}$ and $\tan\theta = \frac{y}{x}$

An infinite number of arguments of z exist

e.g. If $z = i$ then $\arg z = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}, \ldots$

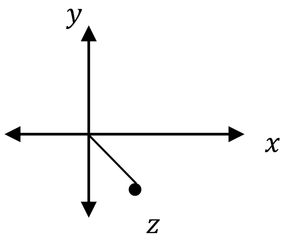
We define the Argument of z:

$$Arg\ z = \theta \text{ where } -\pi < \theta \leq \pi$$

Examples

1. Express $z = 1 - i$ in polar form

$x = 1, y = -1$ [NB: z is in the 4th quadrant]



$$r = |z| = \sqrt{x^2 + y^2}$$
$$= \sqrt{1+1} = \sqrt{2}$$

$$\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1$$

$$\theta = \tan^{-1}(-1) = \frac{-\pi}{4}$$ [since z is in the 4th quadrant]

$$z = rcis\theta$$
$$\therefore z = \sqrt{2}cis\left(\frac{-\pi}{4}\right)$$

2. Express $z = 2cis\left(\frac{4\pi}{3}\right)$ in rectangular form i.e in the form $z = x + yi$

$$2cis\left(\frac{4\pi}{3}\right) = 2\left[\cos\left(\frac{4\pi}{3}\right) + \sin\left(\frac{4\pi}{3}\right)i\right]$$

$$= 2\left(-\frac{1}{2}\right) + 2\left(-\frac{\sqrt{3}}{2}\right)i$$

$$= -1 - \sqrt{3}i$$

See Exercise 1

Operations on Complex Numbers in Polar Form

Addition and Subtraction

Complex numbers in polar form are best converted to the form $x + yi$ before addition or subtraction

Multiplication and Division

If $z_1 = r_1 cis\theta_1$ and $z_2 = r_2 cis\theta_2$ then it can be shown using trigonometric identities that

$$z_1 z_2 = r_1 r_2 cis(\theta_1 + \theta_2)$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} cis(\theta_1 - \theta_2)$$

Exercises

1. If $z_1 = 2cis\left(\frac{\pi}{4}\right)$ and $z_2 = 3cis\left(\frac{5\pi}{6}\right)$ find $z_1 z_2$ in polar form, where $-\pi < \theta \leq \pi$

$$\begin{aligned} z_1 z_2 &= 2cis\left(\frac{\pi}{4}\right) \times 3cis\left(\frac{5\pi}{6}\right) \\ &= (2 \times 3)cis\left(\frac{\pi}{4} + \frac{5\pi}{6}\right) \\ &= 6cis\left(\frac{13\pi}{12}\right) \\ &= 6cis\left(\frac{13\pi}{12} - 2\pi\right) \quad [\text{so that } -\pi < \theta \leq \pi] \\ &= 6cis\left(\frac{-11\pi}{12}\right) \end{aligned}$$

2. If $u = 1 + 3i$ and $v = 2 - i$ find $\frac{u}{v}$ in polar form with $-\pi < \theta \leq \pi$. Two approaches are possible:

$u = 1 + 3i$ i.e $x = 1, y = 3$

$r = \sqrt{1^2 + 3^2} = \sqrt{10}$

$\theta = \tan^{-1}\left(\frac{3}{1}\right) = 1.25$

$\therefore u = \sqrt{10}cis(1.25)$

$v = 2 - 1$ i.e $x = 2, y = -1$

$r = \sqrt{2^2 + (-1)^2} = \sqrt{5}$

$\theta = \tan^{-1}\left(\frac{-1}{2}\right) = -0.46$

$\therefore v = \sqrt{5}cis(-0.46)$

$$\begin{aligned} \text{Then } \frac{u}{v} &= \frac{\sqrt{10}cis(1.25)}{\sqrt{5}cis(-0.46)} \\ &= \frac{\sqrt{10}}{\sqrt{5}}cis(1.25 + 0.46) \\ &= \sqrt{2}cis(1.71) \end{aligned}$$

$$\begin{aligned} \frac{u}{v} &= \frac{1+3i}{2-i} \\ &= \frac{1+3i}{2-i} \times \frac{2+i}{2+i} \\ &= \frac{-1+7i}{4+1} \\ &= \frac{-1+7i}{5} \\ &= -\frac{1}{5} + \frac{7}{5}i \end{aligned}$$

$\therefore\ x = -\frac{1}{5} = -0.2,\ y = \frac{7}{5} = 1.4$

$r = \sqrt{(-0.2)^2 + 1.4^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1.4}{-0.2}\right) = 1.71$

$\frac{u}{v} = \sqrt{2}cis(1.71)$

See Exercise 2

Exercises

Exercise 1

1. Find the polar form (in radians) of the following complex numbers:
   (a) $z = -1 + i$ (b) $z = -\sqrt{3} + i$
   (c) $z = -3i$ (d) $z = -2 - 4i$

2. Express each of the following complex numbers in rectangular form
   (a) $3cis\left(\frac{\pi}{4}\right)$ (b) $\sqrt{7}cis(\pi)$
   (c) $8cis\left(\frac{\pi}{2}\right)$ (d) $10cis(0.41)$

3. If $z = 2 + i$ and $w = 1 - 4i$ find each of the following in polar form using radians where appropriate:
   (a) $|z|$ (b) $|w|$ (c) $Arg|z|$ (d) $|\overline{w}|$ (e) $Arg(zw)$ (f) zw

Exercise 2

1. Simplify

(a) $4cis\left(\frac{\pi}{3}\right) \times 3cis\left(\frac{\pi}{4}\right)$ (b) $\dfrac{3cis\left(\frac{5\pi}{6}\right)}{12cis\left(\frac{\pi}{6}\right)}$

2. If $u = 6cis\left(\frac{3\pi}{4}\right)$ and $v = 4cis\left(-\frac{\pi}{4}\right)$ express $\dfrac{u}{v}$ in polar form
3. If $z = 1 - \sqrt{3i}$, find $\bar{z}$ and express both z and $\bar{z}$ in polar form using radians.

Answers

Exercise 1

1. (a) $\sqrt{2}cis\left(\frac{3\pi}{4}\right)$ (b) $2cis\left(\frac{5\pi}{6}\right)$
   (c) $3cis\left(-\frac{\pi}{2}\right)$ (d) $\sqrt{20}cis(-2.03)$
2. (a) $\frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}i$ (b) $-\sqrt{7}$
   (c) 8i (d) 9.2 + 4i
3. (a) $\sqrt{5}$ (b) $\sqrt{17}$ (c) 0.46
   (d) $\sqrt{17}$ (e) -0.86 (f) $9.22cis(-0.86)$

Exercise 2

1. (a) $12cis\left(\frac{7\pi}{12}\right)$ (b) $\frac{1}{4}cis\left(\frac{2\pi}{3}\right)$
2. (a) $\frac{3}{2}cis(\pi)$
3. $z = 2cis\left(-\frac{\pi}{3}\right)$ $\bar{z} = 2cis\left(\frac{\pi}{3}\right)$

DE MOIVRE'S THEOREM

Integral Powers of Complex Numbers

De Moivre's theorem states that:

$$(cis\theta)^n = cis(n\theta)$$

We make use of this result to calculate an integral power of a complex number:

If $z = rcis\theta$

then

$$z^n = r^n cis(n\theta)$$

Examples

1. Express $(1 - i)^6$ in the form $x + yi$

$$\begin{aligned} (1-i)^6 &= \left[\sqrt{2}cis\left(\frac{-\pi}{4}\right)\right]^6 && \text{[change to polar form]} \\ &= \sqrt{2}^6 cis\left(6 \times \frac{-\pi}{4}\right) && \text{[by De Moivre's theorem]} \\ &= 8cis\left(\frac{-3\pi}{2}\right) \\ &= 8i \end{aligned}$$

2. Simplify $\frac{(\sqrt{3}-i)^6}{(1+i)^8}$ and give the answer in rectangular form

$$\begin{aligned} \sqrt{3} - i &= 2cis\left(-\frac{\pi}{6}\right) && \text{[change to polar form]} \\ \left(\sqrt{3} - i\right)^6 &= 64cis(-\pi) && \text{[by De Moivre's theorem]} \end{aligned}$$

and

$$\begin{aligned} 1 + i &= \sqrt{2}cis\left(\frac{\pi}{4}\right) && \text{[change to polar form]} \\ (1+i)^8 &= 16cis(2\pi) && \text{[by De Moivre's theorem]} \\ \therefore \frac{(\sqrt{3}-i)^6}{(1+i)^8} &= \frac{64cis(-\pi)}{16cis(2\pi)} \\ &= \frac{64}{16}cis(-\pi - 2\pi) && \text{[by De Moivre's theorem]} \\ &= 4cis(-3\pi) \\ &= -4 \end{aligned}$$

See Exercise 1

Roots of a Complex Number

$z^n = rcis\theta$ will have n solutions of the form

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} cis\left(\frac{\theta + 2\pi k}{n}\right), \qquad k = 0,1,\ldots,n-1$$

Example

Solve $z^4 = 1 - \sqrt{3}i$

$$1 - \sqrt{3}i = 2cis\left(-\tfrac{\pi}{3}\right) \qquad \text{[change to polar form]}$$

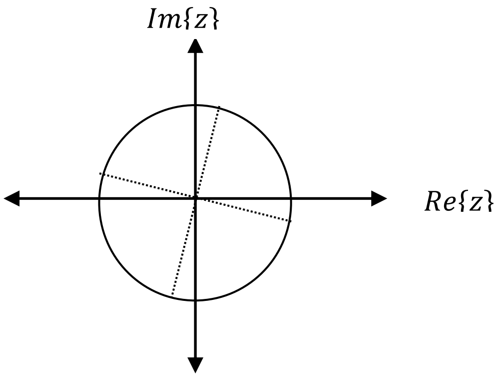
then $z^4 = 2cis\left(-\frac{\pi}{3}\right),\ 2cis\left(-\frac{\pi}{3} + 2\pi\right),\ 2cis\left(-\frac{\pi}{3} + 4\pi\right),\ 2cis\left(-\frac{\pi}{3} + 6\pi\right)$

[as four solutions are required]

ie $z^4 = 2cis\left(-\frac{\pi}{3}\right),\ 2cis\left(\frac{5\pi}{3}\right),\ 2cis\left(\frac{11\pi}{3}\right),\ 2cis\left(\frac{17\pi}{3}\right)$

$\therefore z = 2^{\frac{1}{4}}cis\left(-\frac{\pi}{12}\right),\ 2^{\frac{1}{4}}cis\left(\frac{5\pi}{12}\right),\ 2^{\frac{1}{4}}cis\left(\frac{11\pi}{12}\right),\ 2^{\frac{1}{4}}cis\left(\frac{17\pi}{12}\right)$

The solutions may be represented graphically:



NB: The solutions of $z^n = rcis\theta$ lie on a circle with centre the origin and radius $r^{\frac{1}{n}}$ and they divide the circle into arcs of equal length. The symmetrical nature of the solutions can be used to find all solutions if one is known.

See Exercise 2

Exercises

Exercise 1

1. Evaluate giving your answers in polar form with $-\pi < \theta \leq \pi$

   (a) $\left(\sqrt{3}+i\right)^3$ (b) $(1-i)^5$ (c) $\left(-2\sqrt{3}+2i\right)^2$

2. Simplify each of the following giving the answer in polar form

   (a) $(1+i)^4(2-2i)^3$ (b) $\dfrac{\left(2-2\sqrt{3}i\right)^4}{(-1+i)^6}$

Exercise 2

1. Solve giving the answers in polar form with $-\pi < \theta \leq \pi$

   (a) $z^3 = -1$ (b) $z^4 = 16i$ (c) $z^3 = \sqrt{6} - \sqrt{2}i$

2. If $\sqrt{3}+i$ is one solution of $z^3 = 8i$ use a diagram to find the other solutions in rectangular form.

Answers

Exercise 1

1. (a) $8cis\left(\frac{\pi}{2}\right)$ (b) $2^{\frac{5}{2}}cis\left(\frac{3\pi}{4}\right)$ (c) $16cis\left(\frac{-\pi}{3}\right)$

2. (a) $2^{\frac{13}{2}}cis\left(\frac{\pi}{4}\right)$ (b) $32cis\left(\frac{\pi}{6}\right)$

Exercise 2

1. (a) $cis\left(\frac{\pi}{3}\right),\ cis(\pi), cis\left(\frac{-\pi}{3}\right)$

   (b) $2cis\left(\frac{\pi}{8}\right),\ 2cis\left(\frac{5\pi}{8}\right), 2cis\left(\frac{-7\pi}{8}\right), 2cis\left(\frac{-3\pi}{8}\right)$

   (c) $\sqrt{2}cis\left(\frac{-\pi}{18}\right),\ \sqrt{2}cis\left(\frac{11\pi}{18}\right), \sqrt{2}cis\left(\frac{-13\pi}{18}\right)$

2. $-\sqrt{3}+i$ and $-2i$