TR1.6: CIRCULAR FUNCTIONS

Definition of circular functions

The trigonometric ratios that have been defined in right-angled triangles can be extended to angles greater than 90^0.

This is done using a circle with a radius of 1unit, called a **unit circle.**

The centre of the unit circle is at the point (0,0) on the x–y graph.

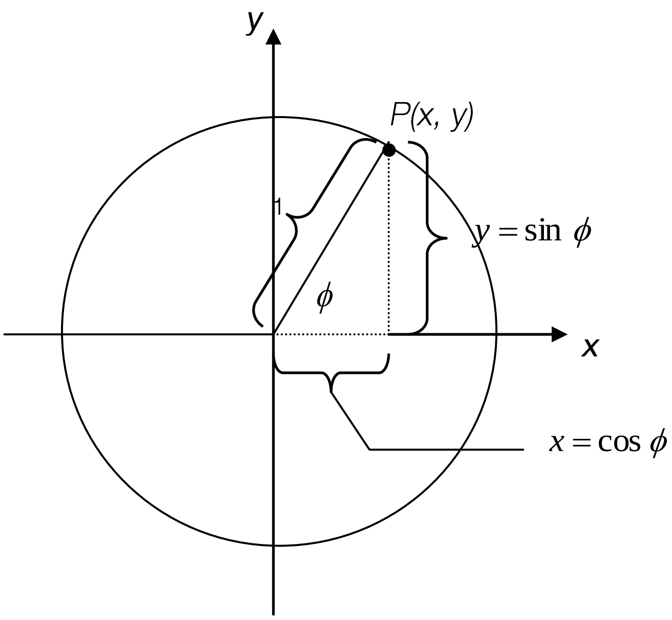
P(x,y) is any point on the circle and can be described by relating the Cartesian coordinates ***x*** and ***y*** to the angle ϕ.

In the triangle OPQ

$$\cos(\phi) = \frac{OQ}{OP} = \frac{x}{1} \qquad \sin(\phi) = \frac{QP}{OP} = \frac{y}{1}$$

$$\cos(\phi) = x \qquad \sin(\phi) = y$$

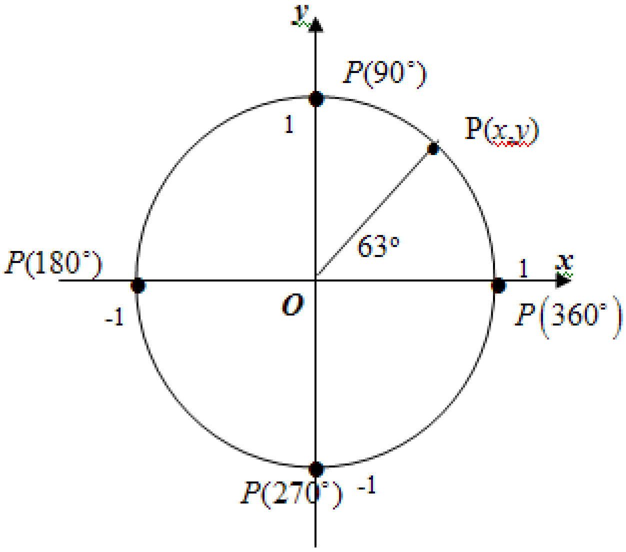
The ***x*** coordinate of *P*,	$\mathbf{x = \cos\phi}$
The ***y*** coordinate of *P*,	$\mathbf{y = \sin\phi}$



The cos and sin of angles greater than 90° are now defined as the x and y coordinates respectively of the point P.

The tangent of any angle is given by $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Examples



The cartesian coordinates for the intercepts of the unit circle on the ***x*** and ***y*** axes are straightforward to determine, since the unit circle has a radius of 1 unit.

For other angles use the calculator.

1. Moving anticlockwise through an angle of 180°, from the positive direction of the *x*- axis the position *P* (180°) is (-1, 0).

2.The coordinates of the point on the unit circle that makes an angle of 63° with the positive direction of the ***x***-axis are

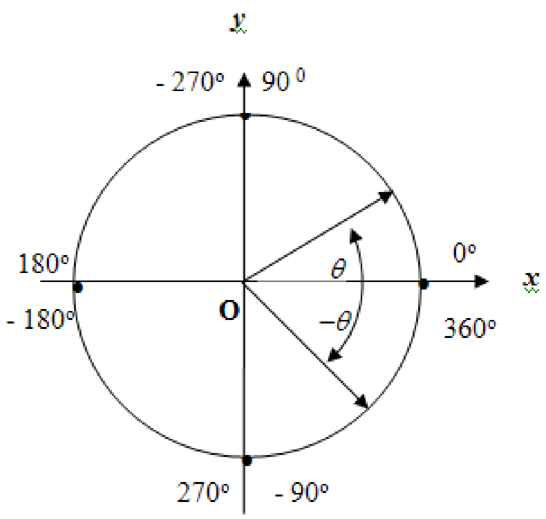
$x = \cos 63^o$ = 0.454 rounded to three decimal places

$y = \sin 63^o$ = 0.891. rounded to three decimal places

Positive and negative angles

Angles measured *anticlockwise* from the positive direction of the *x* – axis are *positive* angles.

Angles measured *clockwise* from the positive direction of the *x* – axis are *negative* angles.



See Exercise 1

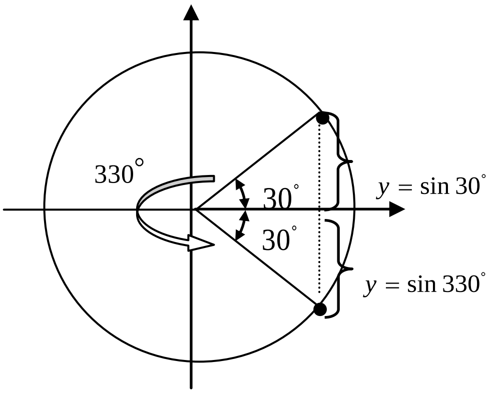
Signs of trigonometric functions

When trying to find exact values for trigonometric functions, plotting the point on a unit circle often simplifies the process.
This is especially helpful when the angles are greater than 90°.

Example

Evaluate $\sin 330^\circ$

Since there are 360° in a full revolution, then turning through an angle of 330° will leave a remainder of 30° radians to complete the circle. $330^\circ = (360-30)^\circ$



From this diagram we can see that the magnitude or size of $\sin 330^\circ$ and $\sin 30^\circ$ are the same.

However the point corresponding to an angle of 330° is below the *x*-axis. This means that the value of $\sin 330^\circ$ will be negative, while $\sin 30^\circ$ is positive

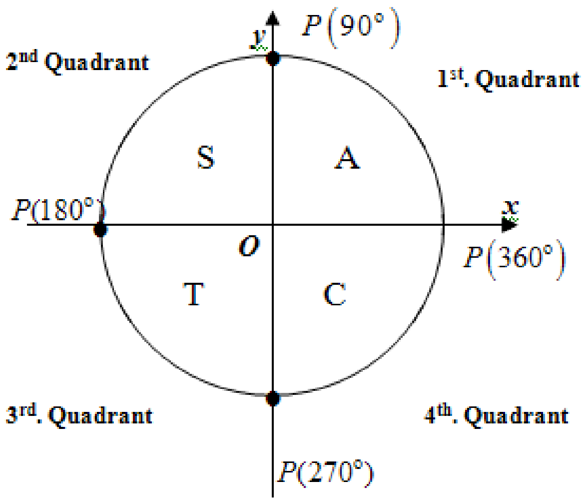
From the 'special' triangles that we looked at in right triangle trigonometry, we know that $\sin 30^\circ$ is equal to $\frac{1}{2}$.

Therefore $\sin 330^\circ$ is equal to $-\frac{1}{2}$.

A general rule that is very useful is $\sin\left(360^\circ - \theta\right) = -\sin\theta$

This leads us to a number of relationships between the trigonometric ratios in the four quadrants of the unit circle

The *unit circle* is divided into *quadrants* as shown below.



In the first quadrant of the unit circle *all* values of sin, cos and tan are positive (***x*** and ***y*** are both

The signs of the trigonometric functions sin, cos and tan can be summarised as follows:

1^{st} quadrant $\Rightarrow$ $0 < \theta < 90^\circ$ $\Rightarrow$ All are positive. (*x* and *y* both positive)
2^{nd} quadrant $\Rightarrow$ $90^\circ < \theta < 180^\circ$ $\Rightarrow$ Sin is positive. (*y* positive, *x* negative)
3^{rd} quadrant $\Rightarrow$ $180^\circ < \theta < 270^\circ$ $\Rightarrow$ Tan is positive.(*x* negative, *y* negative)
4^{th} quadrant $\Rightarrow$ $270^\circ < \theta < 360^\circ$ $\Rightarrow$ Cos is positive.(*x* positive, *y* negative)

1. Sin 50° and $\sin 135^\circ$ are positive because $\sin\theta$ is positive in the 1^{st}. and 2^{nd}. quadrants. (50° is in the 1^{st}.. quadrant and 135^o is in the2nd..quadrant).
2. Cos 135° is negative because $\cos\theta$ is negative in the 2^{nd}. quadrant.
3. Tan 135° and $\tan 335^\circ$ are negative because $\tan\theta$ is negative in the 2^{nd}.and 4^{th}. quadrants.

There are many relationships between the quadrants, shown in the following diagram:

Quadrant 2

By symmetry

$\sin(180^\circ - \theta) = b = \sin\theta$

$\cos(180^\circ - \theta) = -a = -\cos\theta$

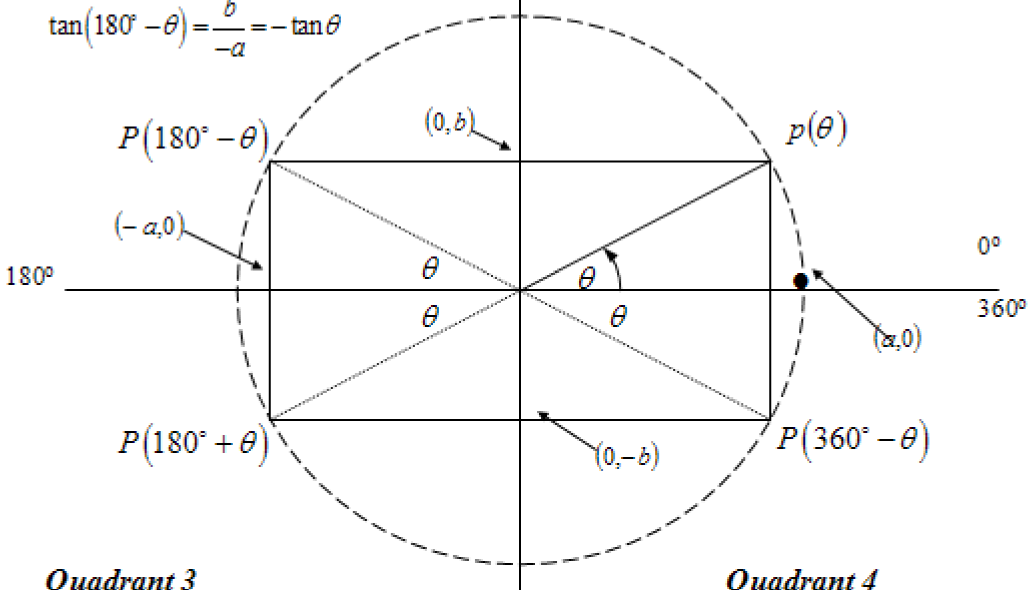
$\tan(180^\circ - \theta) = \dfrac{b}{-a} = -\tan\theta$

Quadrant 1

$\sin(\theta + n(360^\circ)) = \sin\theta$

$\cos(\theta + n(360^\circ)) = \cos\theta$

$\tan(\theta + n(360^\circ)) = \tan\theta$



Quadrant 3

$\sin(180^\circ + \theta) = -b = -\sin\theta$

$\cos(180^\circ + \theta) = -a = -\cos\theta$

$\tan(180^\circ + \theta) = \dfrac{-b}{-a} = \tan\theta$

Quadrant 4

$\sin(360^\circ - \theta) = -b = -\sin\theta$

$\cos(360^\circ - \theta) = a = \cos\theta$

$\tan(360^\circ - \theta) = \dfrac{-b}{a} = -\tan\theta$

Also from the above diagram
$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$
$$\tan(-\theta) = -\tan\theta$$

Note: Although the diagram above has an angle θ between 0 and 360^0, these relationships are true for <u>all</u> values of θ.

Examples

$$\sin(225^\circ) = \sin(180^\circ + 45^\circ) = -\sin(45^\circ) = -\frac{\sqrt{2}}{2}$$

$$\tan(135^\circ) = \tan(180^\circ - 45^\circ) = -\tan(45^\circ) = -1$$

$$\sin(390^\circ) = \sin(360^\circ + 30^\circ) = \sin(30^\circ) = 0.5$$

$$\sin(-60^\circ) = -\sin(60^\circ) = -\frac{\sqrt{3}}{2}$$

$$\cos(-60^\circ) = \cos(60^\circ) = 0.5$$

See Exercise 2

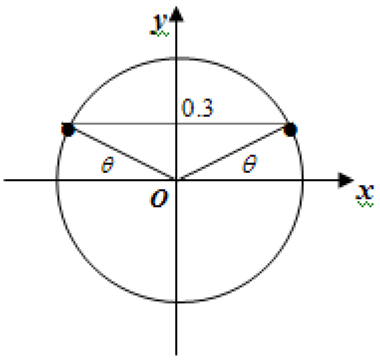
Finding angles

Often the value of the trigonometric function is given and the corresponding angles, within a given domain, are required.

Example

1. Given $\sin\theta = 0.3$, find all values of θ in the domain $0 \geq \theta \leq 360^\circ$

When solving these types of questions it is useful to draw a simple diagram.



$\sin\theta$ is positive in the 1st. and 2nd. quadrants.

There is a solution in the 1st. and 2nd. quadrants.

Solving for θ in the first quadrant

$\sin\theta = 0.3$

$\theta = \sin^{-1} 0.3$

$\theta = 17.46^\circ$

Using symmetry the angle in the 2nd. quadrant is

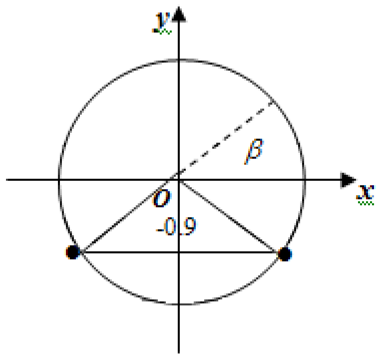
$180^\circ - \theta = 180^\circ - 17.46$

$= 162.54^\circ$

The two solutions for $\sin\theta = 0.3$ are:

$\theta = 17.46^\circ$ and $\theta = 162.54^\circ$

2. For $\sin \theta^o = -0.9$, find all the values of θ in the domain $0 \le \theta \le 360^\circ$



$\sin \theta$ is negative in the 3rd.st. and 4th. quadrants.

There is a solution in the 3rd. and 4th. quadrants

In this question neither of the possible points on the unit circle are in the first quadrant. When this happens it is easier to first let $\sin \beta = 0.9$

$$\sin \beta = 0.9$$

$$\beta = \sin^{-1} 0.9$$

$$\beta = 64.16^\circ$$

From symmetry the angles in the 3rd. and 4th. quadrants are

$180^\circ + \beta^\circ$ and $360^\circ - \beta^\circ$ respectively.

So $\theta = 180^\circ + 64.16^\circ$ and $\theta = 360^\circ - 64.16^\circ$

The required angles are:

$\theta = 244.16^o$ and $\theta = 295.84^o$

See Exercise 3

Exercises

1. What are the coordinates for points on the unit circle that make the following angles with the positive x-axis?

1. 30^o	2. 125^o	3. -60^o
4. 270^o	5. -180^o	6. 720^o

2. Find the exact value for:

1. $\sin 330^\circ$	2. $\cos 210^\circ$	3. $\sin(-30^\circ)$
4. $\cos 90^\circ$	5. $\tan 300^\circ$	6. $\cos 180^\circ$
7. $\sin(-120^\circ)$	8. $\cos 315^\circ$	9.

3. Answer the following:

   1. If $\sin \theta = 0.25$ find θ for $0^\circ \le \theta \le 360^\circ$
   2. If $\tan \theta = 0.8$ find θ for $0^\circ \le \theta \le 360^\circ$
   3. If $\cos \theta = 0.4$ find θ for $-180^\circ \le \theta \le 180^\circ$
   4. If $\cos \theta = -0.4$ find θ for $-180^\circ \le \theta \le 180^\circ$
   5. If $\tan \theta = -1.5$ find θ for $0^\circ \le \theta \le 360^\circ$
   6. If $\cos \theta = -0.3$ find θ for $0^\circ \le \theta \le 360^\circ$

Answers

Exercise 1

7. (0.87,0.5)
8. (-0.56,0.82)
9. (0.5,- 0.87)
10. (0,-1)
11. (– 1,0)
12. (1,0)

Exercise 2

1. -0.5
2. $-\frac{\sqrt{3}}{2}$
3. -0.5
4. 0
5. $-\sqrt{3}$
6. -1
7. $-\frac{\sqrt{3}}{2}$
8. $\frac{\sqrt{2}}{2}$

Exercise 3

1. 14.5°, 165.5°
2. 38.7°, 218.7°
3. 66.4°, -66.4°
4. 113.6°, - 113.6°
5. 123.7°, 303.7°
6. 107.5°, 252.5°