Supplementary Material to:

On the Dynamics of BOLT: An Articulated, Closed Kinematic Chain Planar Monopod

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I. Introduction

This supplementary material is intended to support the paper [1]. Interested readers are advised to follow the supplementary material alongside the paper, as the material is not self -contained. Note that the equations that appear in the accompanying paper are highlighted in red.

II. Kinematic simplification

Refer to Fig. 1 and Fig. 7 in the manuscript to follow along this section. The base link of the six bar mechanism is fixed at angle, θ. All the other link angles with respect to the x-axis are represented by θ_i where i denotes the respective body. $i \in \{1,2 \dots 5\}$. Note, that $\theta_1$is the cranks shaft angle that is known and controlled.

In reference to the first loop utilizing the kinematic loop closure, the following relation is arrived at

$$\vec{R}_1 + \vec{R}_2 - \vec{R}_3 - \vec{R}_4 = 0 \tag{1}$$

Further expanding the above equation utilizing the Euler's formula

$$l_c e^{i\theta} + l_f e^{i\theta_4} - l_2 e^{i\theta_3} - l_1 e^{i\theta_1} = 0 \tag{2}$$

$$l_c \cos(\theta) + l_f \cos(\theta_4) - l_2 \cos(\theta_2) - l_1 \cos(\theta_1) + i\big(l_c \sin(\theta) + l_f \sin(\theta_4) - l_2 \sin(\theta_2) - l_1 \sin(\theta_1)\big) = 0$$

MATLAB's *fsolve* can then be employed to solve for the unknown angles, θ_2 and θ_3. Similarly the second loop equations can be formulated to calculate the remaining unknown angles θ_4 and θ_5. The important takeaway from this analysis reiterates what is intuitive: that all the angles are a function of θ_1. This implies that loop closure equations need not be solved every cycle of θ_1 to calculate the position of the foot. Rather, a look up table can be implemented and simple open kinematic chain equation can provide with the foot position. The inverse and forward kinematics are hence in the simplified form for CKCs.

III. Equations of motion

Refer to Fig. 7 of the manuscript to follow along this section. The equation of motion during flight-phase in dependent generalized coordinates is given by:

$$\begin{cases} \boldsymbol{H}'(\boldsymbol{q}_d)\ddot{\boldsymbol{q}} + \boldsymbol{C}'(\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d)\dot{\boldsymbol{q}}_d + \boldsymbol{g}'(\boldsymbol{q}_d) \\ \phi(\boldsymbol{q}_d) = 0 \end{cases} \tag{3}$$

Where,

$$H(q_d) = \text{zeros}\ (7)$$

$$\begin{aligned} H_{1,1}(q_d) = {} & I_1 + I_2 + I_3 + l_1^2 m_2 + l_{cm,2}^2 m_2 \\ & + l_{cm,3}^2 m_3 + l_s^2 m_3 + 2 l_1 l_{cm,3} m_3 \cos(q_2 + q_3) \\ & + 2 l_1 l_{cm,2} m_2 \cos(q_2) + 2 l_1 l_s m_3 \cos(q_2) \\ & + 2 l_{cm,3} l_s m_3 \cos(q_3) \end{aligned}$$

$$\begin{aligned} H_{1,2}(q_d) = {} & -0.5\ (m_3(2 l_s \sin(q_1 + q_2) \\ & + 2 l_1 \sin(q_1) + 2 l_{cm,3} \sin(q_1 + q_2 + q_{+3}))) \\ & - 0.5 (m_2\ \ (2 l_{cm,2} \sin(q_1 + q_2) + 2 l_1 \sin(q_1))) \end{aligned}$$

$$\begin{aligned} H_{1,3}(q_d) = {} & 0.5\ (m_3(2 l_s \cos(q_1 + q_2) \\ & + 2 l_1 \cos(q_1) + 2 l_{cm,3} \cos(q_1 + q_2 + q_{+3}))) \\ & + 0.5 (m_2\ \ (2 l_{cm,2} \cos(q_1 + q_2) + 2 l_1 \cos(q_1))) \end{aligned}$$

$$\begin{aligned} H_{1,4}(q_d) = {} & I_2 + I_3 + l_{cm,2}^2 m_2 + l_{cm,3}^2 m_3 + l_s^2 m_3 \\ & + l_1 l_{cm,3} m_3 + \cos(q_2 + q_3) \\ & + l_1 l_{cm,2} m_2 \cos(q_2) + l_1 l_s m_3 \cos(q_2) \\ & + 2 l_{cm,3} l_s m_3 \cos(q_3) \end{aligned}$$

$$\begin{aligned} H_{1,5}(q_d) = {} & I_3 + l_{cm,3}^2 m_3 + l_1 l_{cm,3} m_3 \cos(q_2 + q_3) \\ & + l_{cm,3} l_s m_3 \cos(q_3) \end{aligned}$$

$$\begin{aligned} H_{2,1}(q_d) = {} & -0.5\ (m_3(2 l_s \sin(q_1 + q_2) \\ & + 2 l_1 \sin(q_1) + 2 l_{cm,3} \sin(q_1 + q_2 + q_{+3}))) \\ & - 0.5 (m_2\ \ (2 l_{cm,2} \sin(q_1 + q_2) + 2 l_1 \sin(q_1))) \end{aligned}$$

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$H_{2,2}(q_d) = m_1 + m_2 + m_3 + m_4 + m_5 + m_b$

$H_{2,4}(q_d) = -0.5\ (m_3(2l_s \sin(q_1 + q_2)$
$+2l_{cm,3}\sin(q_1 + q_2 + q_{+3})))$
$-\ l_{cm,2}m_2\ \ \sin(q_1 + q_2)$

$H_{2,5}(q_d) = -l_{cm,3}m_3 \sin(q_1 + q_2 + q_3)$

$H_{2,6}(q_d) = -0.5\ (m_5(2l_{cm,5} \sin(q_4 + q_5)$
$+2l_4\sin(q_4))) - l_{cm,4}m_4\ \ \sin(q_4)$

$H_{2,7}(q_d) = -l_{cm,5}m_5 \sin(q_4 + q_5)$

$H_{3,1}(q_d) = 0.5(m_3(2l_s \cos(q_1 + q_2) + 2l_1 \cos(q_1))$
$+\ 2l_{cm,3}\cos(q_1 + q_2 + q_3))) + 0.5(m\_2\ (2l_{cm,2} \cos(q_1 +$
$q_2) + 2l_1\cos(q_1)))$

$H_{3,3}(q_d) = m_1 + m_2 + m_3 + m_4 + m_5$

$H_{3,4}(q_d) = 0.5\ (m_3(2l_s \cos(q_1 + q_2)$
$+2l_{cm,3}\cos(q_1 + q_2 + q_{+3})))$
$+\ l_{cm,2}\ \mathrm{m}_2\cos(q_1 + q_2)$

$H_{3,5}(q_d) = l_{cm,3}m_3\cos(q_1 + q_2 + q_3)$

$H_{3,6}(q_d) = 0.5\ (m_5(2l_{cm,5} \cos(q_4 + q_5)$
$+2l_4\cos(q_4))) + l_{cm,4}\ \mathrm{m}_4\cos(q_4)$

$H_{3,7}(q_d) = l_{cm,5}m_5\cos(q_4 + q_5)$

$H_{4,1}(q_d) = I_2 + I_3 + l_{cm,2}^2m_3 + l_s^2m_3$
$+l_1l_{cm,3}m_3 \cos(q_2 + q_3) + l_1l_{cm,2}m_2 \cos(q_2)$
$+\ l_1l_sm_3 \cos(q_2) + 2l_{cm,3}l_sm_3 \cos(q_3)$

$H_{4,2}(q_d) = -0.5(m_3(2l_s \sin(q_1 + q_2)$
$+2l_{cm,3}\sin(q_1 + q_2 + q_3))) - l_{cm,2}m_2\sin(q_1$
$+\ q_2)$

$H_{4,3}(q_d) = 0.5(m_3(2l_s \cos(q_1 + q_2)$
$+2l_{cm,3}\cos(q_1 + q_2 + q_3))) + l_{cm,2}m_2\cos(q_1$
$+\ q_2)$

$H_{4,4}(q_d) = I_2 + I_3 + l_{cm,2}^2m_2 + l_{cm,3}^2m_3 + l_s^2m_3$
$+2l_{cm,3}l_sm_3\cos(q_3)$

$H_{4,5}(q_d) = I_3 + l_{cm,3}^2m_3 + l_{cm,3}l_sm_3\cos(q_3)$

$H_{5,1}(q_d) = I_3 + l_{cm,3}^2m_3 + l_1l_{cm,3}m_3 \cos(q_2 + q_3)$
$+l_{cm,3}l_sm_3cos(q_3)$

$H_{5,2}(q_d) = -l_{cm,3}m_3\sin(q_1 + q_2 + q_3)$

$H_{5,3}(q_d) = l_{cm,3}m_3\cos(q_1 + q_2 + q_3)$
$+\ l_1l_{cm,2}m_2\cos(q_2)$

$H_{5,4}(q_d) = I_3 + l_{cm,3}^2m_3 + l_{cm,3}l_sm_3\cos(q_3)$

$H_{5,5}(q_d) = I_3 + l_{cm,3}^2m_3$

$H_{6,2}(q_d) = -0.5\left(m_5\big(2l_{cm,5} \sin(q_4 + q_5) + 2l_4 \sin(q_4)\big)\right)$
$-\ l_{cm,4}m_4\sin(q_4)$

$H_{6,3}(q_d) = 0.5\left(m_5\big(2l_{cm,5} \cos(q_4 + q_5) + 2l_4 \cos(q_4)\big)\right)$
$+\ l_{cm,4}m_4\cos(q_4)$

$H_{6,6}(q_d) = I_4 + I_5 + l_4^2m_5 + l_{cm,4}^2m_4 + l_{cm,5}^2m_5$
$+\ 2l_4l_{cm,5}m_5\cos(q_5)$

$H_{6,7}(q_d) = I_5 + l_{cm,5}^2m_5 + l_4l_{cm,5}m_5\cos(q_5)$

$H_{7,2}(q_d) = -l_{cm,5}m_5\sin(q_4 + q_5)$

$H_{7,3}(q_d) = l_{cm,5}m_5\cos(q_4 + q_5)$

$H_{7,6}(q_d) = I_5 + l_{cm,5}^2m_5 + l_4l_{cm,5}m_5\cos(q_5)$

$H_{7,7}(q_d) = I_5 + l_{cm,5}^2m_5$

$C(q_d) = \mathrm{zeros}(7)$

$C_{1,1}(q_d) = -\dot{q}_3(l_1l_{cm,3}m_3 \sin(q_2 + q_3)$
$+l_{cm,3}l_sm_3\sin(q_3))$
$-\ \dot{q}_2(l_1l_{cm,3}m_3 \sin(q_2 + q_3)$
$+\ l_1l_{cm,2}m_2 \sin(q_2) + l_1l_sm_3\sin(q_2))$

$C_{1,4}(q_d) = -\dot{q}_3(l_1l_{cm,3}m_3 \sin(q_2 + q_3)$
$+l_{cm,3}l_sm_3\sin(q_3))$
$-\ \dot{q}_1(l_1l_{cm,3}m_3 \sin(q_2 + q_3)$
$+\ l_1l_{cm,2}m_2 \sin(q_2) + l_1l_sm_3\sin(q_2))$
$-\ \dot{q}_2(l_1l_{cm,3}m_3 \sin(q_2 + q_3)$
$+\ l_1l_{cm,2}m_2 \sin(q_2) + l_1l_sm_3\sin(q_2))$

$C_{1,5}(q_d) = -\dot{q}_1(l_1l_{cm,3}\ m_3 \sin(q_2 + q_3)$
$+l_{cm,3}\ l_sm_3\sin(q_3))$
$-\ \dot{q}_2(l_1l_{cm,3}m_3 \sin(q_2 + q_3)$
$+\ l_{cm,3}l_sm_3\sin(q_3))$
$-\ \dot{q}_3(l_1l_{cm,3}m_3 \sin(q_2 + q_3)$
$+\ l_{cm,3}\ l_sm_3\sin(q_3))$

$$C_{2,1}(q_d) = -\dot{q}_1(0.5(m_3(2l_s\cos(q_1+q_2) +2l_1\cos(q_1) + 2l_{cm,3}\cos(q_1+q_2+q_3))) + 0.5(m_2(2l_{cm,2}\cos(q_1+q_2) + 2l_s\cos(q_1)))) - \dot{q}_2(0.5(m_3(2l_s\cos(q_1+q_2) + 2l_{cm,3}\cos(q_1 + q_2+q_3))) + l_{cm,2}m_2\cos(q_1+q_2)) - \dot{q}_3 l_{cm,3}m_3\cos(q_1+q_2+q_3)$$

$$C_{2,4}(q_d) = -\dot{q}_1(0.5(m_3(2l_s\cos(q_1+q_2) +2l_{cm,3}\cos(q_1+q_2+q_3))) + l_{cm,2}m_2\cos(q_1 + q_2)) - \dot{q}_2(0.5((m_3(2l_s\cos(q_1+q_2) + 2l_{cm,3}\cos(q_1+q_2+q_3))) + l_{cm,2}m_2\cos(q_1 + q_2) - \dot{q}_3 l_{cm,3}m_3\cos(q_1+q_2+q_3)$$

$$C_{2,5}(q_d) = -\dot{q}_1 l_{cm,3}\, m_3\cos(q_1+q_2+q_3) -\dot{q}_2\, l_{cm,3}m_3\cos(q_1+q_2+q_3) - \dot{q}_3 l_{cm,3}m_3\cos(q_1+q_2+q_3)$$

$$C_{2,6}(q_d) = -\dot{q}_4(0.5(m_5(2l_{cm,5}\cos(q_4+q_5)) +2l_4\cos(q_4))) + l_{cm,4}\, m_4\cos(q_4)) - \dot{q}_5 l_{cm,5}\, m_5\cos(q_4+q_5)$$

$$C_{2,7}(q_d) = -\dot{q}_4 l_{cm,5}m_5\cos(q_4+q_5) -\dot{q}_5 l_{cm,5}m_5\cos(q_4+q_5)$$

$$C_{3,1}(q_d) = -\dot{q}_2(0.5(m_3(2l_s\sin(q_1+q_2) +2l_{cm,3}\sin(q_1+q_2+q_3))) + l_{cm,2}m_2\sin(q_1+q_2)) - \dot{q}_1(0.5(m_3\,(2l_s\sin(q_1+q_2) + 2l_1\sin(q_1) + 2\,l_{cm,3}\sin(q_1+q_2+q_3))) + 0.5(m_2(2l_{cm,2}\sin(q_1+q_2) + 2l_1 sin(q_1))\,)) - \dot{q}_3 l_{cm,3}m_3\sin(q_1+q_2+q_3)$$

$$C_{3,2}(q_d) = 0$$

$$C_{3,3}(q_d) = 0$$

$$C_{3,4}(q_d) = -\dot{q}_1(0.5(m_3(2l_s\sin(q_1+q_2) +2l_{cm,3}\sin(q_1+q_2+q_3))) + l_{cm,2}m_2\sin(q_1 + q_2)) - \dot{q}_2(0.5((m_3(2l_s\sin(q_1+q_2) + 2l_{cm,3}\sin(q_1+q_2+q_3))) + l_{cm,2}m_2\sin(q_1 + q_2) - \dot{q}_3 l_{cm,3}m_3\sin(q_1+q_2+q_3)$$

$$C_{3,5}(q_d) = -\dot{q}_1 l_{cm,3}m_3\sin(q_1+q_2+q_3) -\dot{q}_2 l_{cm,3}m_3\sin(q_1+q_2+q_3) - \dot{q}_3 l_{cm,3}m_3\sin(q_1+q_2+q_3)$$

$$C_{3,6}(q_d) = -\dot{q}_4(0.5(m_5(2l_{cm,5}\sin(q_4+q_5) +2l_4\sin(q_4))) + l_{cm,4}\, m_4\sin(q_4) - \dot{q}_5 l_{cm,5}m_5\sin(q_4+q_5)$$

$$C_{3,7}(q_d) = -\dot{q}_4 l_{cm,5}m_5\sin(q_4+q_5) -\dot{q}_5 l_{cm,5}m_5\sin(q_4+q_5)$$

$$C_{4,1}(q_d) = \dot{q}_1(l_1 l_{cm,3}m_3\sin(q_2+q_3) + l_1 l_{cm,2}m_2\sin(q_2)) +l_1 l_s m_3\sin(q_2)) - \dot{q}_3 l_{cm,3}l_s m_3\sin(q_3)$$

$$C_{4,4}(q_d) = -\dot{q}_3 l_{cm,3}l_s m_3\sin(q_3)$$

$$C_{4,5}(q_d) = -\dot{q}_1 l_{cm,3}l_s m_3\sin(q_3) - \dot{q}_2 l_{cm,3}l_s m_3\sin(q_3) -\dot{q}_3 l_{cm,3}l_s m_3\sin(q_3)$$

$$C_{5,1}(q_d) = \dot{q}_1\big(l_1 l_{cm,3}m_3\sin(q_2+q_3) + l_{cm,3}l_s m_3\sin(q_3)\big) + \dot{q}_2 l_{cm,3}l_s m_3\sin(q_3)$$

$$C_{5,4}(q_d) = \dot{q}_1 l_{cm,3}l_s m_3\sin(q_3) + \dot{q}_2 l_{cm,3}l_s\, m_3\sin(q_3)$$

$$C_{6,6}(q_d) = -\dot{q}_5 l_4 l_{cm,5}m_5\sin(q_5)$$

$$C_{6,7}(q_d) = -\dot{q}_4 l_4 l_{cm,5}m_5\sin(q_5) - \dot{q}_5 l_4 l_{cm,5}m_5\sin(q_5)$$

$$C_{7,6}(q_d) = \dot{q}_4 l_4 l_{cm,5}m_5\sin(q_5)$$

$$g(q_d) = \text{zeros}(7{,}1)$$

$$g_1(q_d) = gm_3(l_s\cos(q_1+q_2) + l_1\cos(q_1) +l_{cm,3}\cos(q_1+q_2+q_3)) + gm_2(l_{cm,2}\cos(q_1+q_2) + l_1\cos(q_1))$$

$$g_3(q_d) = g(m + m_1 + m_2 + m_3)$$

$$g_4(q_d) = gm_3(l_s\cos(q_1+q_2) +l_{cm,3}\cos(q_1+q_2+q_3)) + gm_2 l_{cm,2}\cos(q_1+q_2)$$

$$g_5(q_d) = gl_{cm,3}m_3\cos(q_1+q_2+q_3)$$

$$g_6(q_d) = g\mathrm{m}_5\big(\mathrm{l}_{\mathrm{cm},5}\cos(\mathrm{q}_4+\mathrm{q}_5) + \mathrm{l}_4\cos(\mathrm{q}_4)\big) +\mathrm{gl}_{\mathrm{cm},4}\mathrm{m}_4\cos(q_4)$$

$$g_7(q_d) = gl_{cm,5}\mathrm{m}_5\cos(q_4+q_3)$$

$$\phi_1(q_d) = l_2\cos(q_1+q_2) + l_1\cos(q_1)$$

$$-l_f\cos(q_4)-l_c\cos(\theta)$$

$$\phi_2(q_d)=l_2\sin(q_1+q_2)+l_1\sin(q_1)$$
$$-l_f\sin(q_4)-l_c\sin(\theta)$$

$$\phi_3(q_d)=l_s\cos(q_1+q_2)-l_t\cos(q_4+q_5)$$
$$+l_1\cos(q_1)-l_4\cos(q_4)-l_c\cos(\theta)+l_3\cos(q_1$$
$$+q_2+q_3)$$

$$\phi_4(q_d)=l_s\sin(q_1+q_2)-l_t\sin(q_4+q_5)$$
$$+l_1\sin(q_1)-l_4\sin(q_4)-l_c\sin(\theta)+l_3\sin(q_1$$
$$+q_2+q_3)$$

IV. SINGULARLY PERTURBED EQUATIONS OF MOTION

The singularly perturbed dynamics are given in (4). However, the analytical solution for $\boldsymbol{H(q,z)}$, $\boldsymbol{C(q,z)}$ and $\boldsymbol{g(q,z)}$ is non-existent. We have to suffice with the numerical solution. Here we show the steps to numerically calculate the corresponding matrices.

$$\begin{bmatrix}\boldsymbol{H}(\boldsymbol{q},\boldsymbol{z}) & 0_{n_q\times 1}\\ 0_{n_z\times 1} & \boldsymbol{J}_z(\boldsymbol{q},\boldsymbol{z})\end{bmatrix}\begin{bmatrix}\ddot{q}\\ \dot{z}\end{bmatrix}=\begin{bmatrix}-\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}},z,\dot{\boldsymbol{z}})\dot{\boldsymbol{q}}-\boldsymbol{g}(\boldsymbol{q},\boldsymbol{z})+\boldsymbol{B\tau}\\ -\frac{1}{\varepsilon}\phi(\boldsymbol{q},\boldsymbol{z})-J_q(\boldsymbol{q},\boldsymbol{z})\dot{\boldsymbol{q}}\end{bmatrix} \quad (4)$$

The first bundle of equations in (2) is arrived at by the following steps.

$$\Gamma(q,z)=zeros(7)$$

$$\Gamma_{1,1}(q,z)=-l_2\sin(q_1+q_2)-l_1\sin(q_1)$$
$$\Gamma_{1,4}(q,z)=-l_2\sin(q_1+q_2)$$
$$\Gamma_{1,6}(q,z)=l_f\sin(q_4)$$

$$\Gamma_{2,1}(q,z)=l_2\cos(q_1+q_2)+l_1\cos(q_1)$$
$$\Gamma_{2,4}(q,z)=l_2cos(q_1+q_2)$$
$$\Gamma_{2,6}(q,z)=-l_f\cos(q_4)$$

$$\Gamma_{3,1}(q,z)=-l_s\sin(q_1+q_2)-l_1\sin(q_1)$$
$$-l_3\sin(q_1+q_2+q_3)$$
$$\Gamma_{3,4}(q,z)=-l_s\sin(q_1+q_2)-l_3\sin(q_1+q_2+q_3)$$
$$\Gamma_{3,5}(q,z)=-l_3\sin(q_1+q_2+q_3)$$
$$\Gamma_{3,6}(q,z)=l_t\sin(q_4+q_5)+l_4sin(q_4)$$
$$\Gamma_{3,7}(q,z)=l_t\sin(q_4+q_5)$$

$$\Gamma_{4,1}(q,z)=l_s\cos(\mathrm{q}_1+\mathrm{q}_2)+l_1\cos(q_1)$$
$$+l_3\cos(q_1+q_2+q_3)$$
$$\Gamma_{4,4}(q,z)=l_s\cos(q_1+q_2)+l_3\cos(q_1+q_2+q_3)$$
$$\Gamma_{4,5}(q,z)=l_3\cos(q_1+q_2+q_3)$$
$$\Gamma_{4,6}(q,z)=-l_t\cos(q_4+q_5)-l_4cos(q_4)$$
$$\Gamma_{4,7}(q,z)=-l_t\cos(q_4+q_5)$$

$$\Gamma_{5,1}(q,z)=1$$
$$\Gamma_{6,2}(q,z)=1$$
$$\Gamma_{7,3}(q,z)=1$$

$$\Gamma(\dot{q},\dot{z})=zeros(7)$$

$$\Gamma_{1,1}(\dot{q},\dot{z})=-\dot{q}_1l_1\cos(q_1)-l_2\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2)$$
$$\Gamma_{1,4}(\dot{q},\dot{z})=-l_2\cos(q_1+q_2)\,(\dot{q}_1+\dot{q}_2)$$
$$\Gamma_{1,6}(\dot{q},\dot{z})=\dot{q}_4l_f\cos(q_4)$$

$$\Gamma_{2,1}(\dot{q},\dot{z})=-\dot{q}_1l_1\sin(q_1)-l_2\sin(q_1+q_2)(\dot{q}_1+\dot{q}_2)$$
$$\Gamma_{2,4}(\dot{q},\dot{z})=-l_2sin(q_1+q_2)(\dot{q}_1+\dot{q}_2)$$
$$\Gamma_{2,6}(\dot{q},\dot{z})=\dot{q}_4l_f\sin(q_4)$$

$$\Gamma_{3,1}(\dot{q},\dot{z})=-\dot{q}_1l_1\cos(q_1)$$
$$-l_3\cos(q_1+q_2+q_3)(\dot{q}_1+\dot{q}_2+\dot{q}_3)$$
$$-l_s\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2)$$
$$\Gamma_{3,4}(\dot{q},\dot{z})=-l_3\cos(q_1+q_2+q_3)\,(\dot{q}_1+\dot{q}_2+\dot{q}_3)$$
$$-l_s\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2)$$
$$\Gamma_{3,5}(\dot{q},\dot{z})=-l_3\cos(q_1+q_2+q_3)\,(\dot{q}_1+\dot{q}_2+\dot{q}_3)$$
$$\Gamma_{3,6}(\dot{q},\dot{z})=\dot{q}_4l_4\cos(q_4)+l_t\cos(q_4+q_5)$$
$$(\dot{q}_4+\dot{q}_5)$$
$$\Gamma_{3,7}(\dot{q},\dot{z})=l_t\cos(q_4+q_5)(\dot{q}_4+\dot{q}_5)$$

$$\Gamma_{4,1}(\dot{q},\dot{z})=\dot{q}_1l_1\sin(\mathrm{q}_1)-l_3\sin(q_1+q_2+q_3)$$
$$(\dot{q}_1+\dot{q}_2+\dot{q}_3)-l_s\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2)$$
$$\Gamma_{4,4}(\dot{q},\dot{z})=-l_3\sin(q_1+q_2+q_3)\,(\dot{q}_1+\dot{q}_2+\dot{q}_3)$$
$$-l_s\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2)$$
$$\Gamma_{4,5}(\dot{q},\dot{z})=-l_3\sin(q_1+q_2+q_3)(\dot{q}_1+\dot{q}_2+\dot{q}_3)$$
$$\Gamma_{4,6}(\dot{q},\dot{z})=\dot{q}_4l_4\sin(q_4)+l_tsin(q_4+q_5)(\dot{q}_4+\dot{q}_5)$$
$$\Gamma_{4,7}(\dot{q},\dot{z})=l_t\sin(q_4+q_5)(\dot{q}_4+\dot{q}_5)$$

$$\rho(\boldsymbol{q}_d)=\Gamma^{-1}(\boldsymbol{q},\boldsymbol{z})\begin{bmatrix}0\\ I_{n_q\times n_q}\end{bmatrix} \quad (5)$$

Further,

$$\dot{\rho}(\boldsymbol{q}_d) = \Gamma^{-1}(q,z) * \Gamma^{-1}(\dot{q},\dot{z}) * \rho \qquad (6)$$

This leads to,

$$H(\boldsymbol{q},\boldsymbol{z}) = \rho' * H(\boldsymbol{q}_\mathrm{d}) * \rho \qquad (7)$$

$$C(\boldsymbol{q},\boldsymbol{z}) = \rho' * C(\boldsymbol{q}_\mathrm{d}) * \rho + \rho' * H(\boldsymbol{q}_\mathrm{d}) * \dot{\rho} \qquad (8)$$

$$g(\boldsymbol{q},\boldsymbol{z}) = \rho' * g(\boldsymbol{q}_\mathrm{d}) \qquad (9)$$

Finally, the second bundle of equations results in:

$$J_{q(1)}(q,z) = -l_2 \sin(q_1 + q_2) - l_1 \sin(q_1)$$
$$J_{q(2)}(q,z) = l_2 \cos(q_1 + q_2) + l_1 \cos(q_1)$$
$$J_{q(3)}(q,z) = -l_s \sin(q_1 + q_2) - l_1 \sin(q_1) - l_3 \sin(q_1 + q_2 + q_3)$$
$$J_{q(4)}(q,z) = l_s \cos(q_1 + q_2) + l_1 \cos(q_1) + l_3 \cos(q_1 + q_2 + q_3)$$

$$J_z(q,z) = zeros(4)$$

$$J_{z(1,1)}(q,z) = -l_2 \sin(q_1 + q_2)$$
$$J_{z(1,3)}(q,z) = l_f \sin(q_4)$$

$$J_{z(2,1)}(q,z) = l_2 \cos(q_1 + q_2)$$
$$J_{z(2,3)}(q,z) = -l_f \cos(q_4)$$

$$J_{z(3,1)}(q,z) = -l_s \sin(q_1 + q_2) - l_3 \sin(q_1 + q_2 + q_3)$$
$$J_{z(3,2)}(q,z) = -l_3 \sin(q_1 + q_2 + q_3)$$
$$J_{z(3,3)}(q,z) = l_t \sin(q_4 + q_5) + l_4 \sin(q_4)$$
$$J_{z(3,4)}(q,z) = l_t \sin(q_4 + q_5)$$

$$J_{z(4,1)}(q,z) = l_s \cos(q_1 + q_2) + l_3 \cos(q_1 + q_2 + q_3)$$
$$J_{z(4,2)}(q,z) = l_3 \cos(q_1 + q_2 + q_3)$$
$$J_{z(4,3)}(q,z) = -l_t \cos(q_4 + q_5) - l_4 \cos(q_4)$$
$$J_{z(4,4)}(q,z) = -l_t \cos(q_4 + q_5)$$

The same process applies for arriving at the SPF stance phase dynamics.

REFERENCES

[1] V. Kamidi and P. Ben-Tzvi, "On the dynamics of BOLT: An Articulated, Closed Kinematic Chain Planar Monopod," *in IEEE/ASME Transactions of Mechatronics, submitted*.