Proof of the claim in Sec. III-A

(Derivation of a closed form approximation for the phase offset that maximizes the forward speed of a snake robot)

Liljebäck et al. [5] showed that δ_{opt}, the δ that maximizes the speed of a snake robot (in a simplified dynamic model) during serpentine locomotion is given by the δ that maximizes

$$k_\delta = \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} c_{ij} \sin\left((j-i)\delta\right), \tag{1}$$

where c_{ij} is the ijth entry of $\mathbf{C} = \mathbf{A}\overline{\mathbf{D}} \in \mathbb{R}^{(N-1)\times(N-1)}$, with $\overline{\mathbf{D}} = \mathbf{D}^{\mathrm{T}}\left(\mathbf{D}\mathbf{D}^{\mathrm{T}}\right)^{-1} \in \mathbb{R}^{N\times(N-1)}$ and $\mathbf{A} = \begin{bmatrix} 1 & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ \mathbf{0} & & 1 & 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 1 & -1 & & \mathbf{0} \\ & \ddots & \ddots & \\ \mathbf{0} & & 1 & -1 \end{bmatrix} \in \mathbb{R}^{(N-1)\times N}$, the so called *addition* and *difference* matrices.

We obtain a closed form approximation of δ as a function of N by deriving some properties of $\mathbf{C} = \mathbf{A}\mathbf{D}^{\mathrm{T}}\left(\mathbf{D}\mathbf{D}^{\mathrm{T}}\right)^{-1}$. First of all, we define $n = N-1$, to clean up the notation. We also define the matrix $\mathbf{B} = \left(\mathbf{D}\mathbf{D}^{\mathrm{T}}\right)^{-1} \in \mathbb{R}^{n\times n}$, with ijth element b_{ij}. Note that since a matrix times its transpose is symmetric, and the inverse of a symmetric matrix is also symmetric, $\mathbf{B}$ is symmetric. We will need the following:

Prop. 1. The sum of the kth diagonal of $\mathbf{B} \in \mathbb{R}^{n\times n}$ is $b_k \equiv \dfrac{T_k}{n+1}$, where T_k is the kth tetrahedral number $T_k = \dfrac{1}{6}k(k+1)(k+2)$.

Note that by the kth diagonal, we mean counting from the bottom left or top right corner of a matrix and moving toward the main diagonal. For example, a $\mathbf{B}$ matrix of dimension n=4, with each of the diagonals marked with its own bracket style, could be represented as: $\mathbf{B}_4 = \dfrac{1}{5}\begin{bmatrix} \langle 4\rangle & \{3\} & [2] & (1) \\ \{3\} & \langle 6\rangle & \{4\} & [2] \\ [2] & \{4\} & \langle 6\rangle & \{3\} \\ (1) & [2] & \{3\} & \langle 4\rangle \end{bmatrix}$. To prove Prop. 1, we will need to show three things:

1.1) The lower left element and the top right element of $\mathbf{B}$ are both $\dfrac{1}{(n+1)}$, and the rows and columns increases sequentially in units of $\dfrac{1}{(n+1)}$ as we move left along the top row, right along the bottom row, up along the leftmost column, or down along the rightmost column, as shown in Fig. 1 for the case n = 4.

1.2) The first n diagonals (starting at the bottom left corner) of a **B** matrix of dimension n are precisely the diagonals of all of its lower dimensional counterparts, scaled by $\frac{1}{(n+1)}$.

1.3) The sum of the main diagonal of $\mathbf{B}$ is $b_n = \frac{T_n}{n+1}$.

To prove 1.1., note that for the first column of **B**, $-b_{n-1,1} + 2b_{n,1} = 0 \Rightarrow b_{n-1,1} = 2b_{n,1}$ and $-b_{n-2,1} + 2b_{n-1,1} - b_{n,1} = 0 \Rightarrow b_{n-1,1} = 3b_{n,1}$. That the first column increases sequentially by factors of $b_{n,1}$ from the nth element to the first can be proved by induction on the ith element from the end, with base case $i = 2$ (We've already shown cases $i = 2$ and $i = 3$, and case $i = 0$ is true by definition). Namely, suppose $b_{n-i,1} = (i+1)b_{n,1}$ for some $i \geq 2$. Then, $-b_{n-i-1,1} + 2b_{n-i,1} - b_{n-i+1,1} = 0$ $\Rightarrow b_{n-i-1,1} = 2(i+1)b_{n,1} - ib_{n,1} = (i+2)b_{n,1} \Rightarrow b_{n-i-1,1} - b_{n-i,1} = b_{n,1}$, which proves the claim. Reaching the first element, we have $2b_{1,1} - b_{2,1} = 1 \Rightarrow 2nb_{n,1} - (n-1)b_{n,1} = 1 \Rightarrow b_{n,1} = \frac{1}{(n+1)}$. Applying the same argument to the last column of **B** shows that its elements increase in units of $\frac{1}{(n+1)}$ from the top right corner to the lower right corner. Thus, since the matrix is symmetric, the top and bottom rows increase in the same way: right to left along the top row and left to right along the bottom row.

For 1.2., notice that the ith element on the kth diagonal, $\left(n-(k-1)+i, k+i\right)$ (where $k \in [1,n]$ and $i \in [1,k]$), is recursively generated by the two elements directly under it according to the rule $-b_{n-(k-1)+i,k+i} + 2b_{n-(k-1)+i+1,k+i} - b_{n-(k-1)+i+2,k+i} = 0$ until we get to the second to bottom row, where $b_{n-(k-1)+i,k+i} = 2b_{n-(k-1)+i+1,k+i}$. Thus, the first n diagonals of **B** are generated from the bottom row in the same manner for a **B** matrix of any dimension, and differ only by the bottom row's scaling factor of $\frac{1}{(n+1)}$.

Now for 1.3. Here $\mathbf{DD}^{\mathrm{T}} = \begin{bmatrix} 2 & -1 & & \mathbf{0} \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ \mathbf{0} & & -1 & 2 \end{bmatrix}$ is the so called "second difference matrix," in which, by definition, the sum of the kth lower diagonal, moving from the bottom left toward the main diagonal, is $2\sum_{i=1}^{k} b_{i+(n-k),i} - \left(\sum_{i=1}^{k-1} b_{i+(n-k),(i-1)} + \sum_{i=1}^{k-1} b_{i+(n-k),(i+1)} \right)$. Since $\mathbf{DD}^{\mathrm{T}}\mathbf{B} = \mathbf{I} \in \mathbb{R}^{n \times n}$, $\mathrm{tr}\left(\mathbf{DD}^{\mathrm{T}}\mathbf{B}\right) = n$, and since $\left(\mathbf{DD}^{\mathrm{T}}\right)^{-1}$ is symmetric, we can write $\sum_{i=1}^{n} b_{i,i} - \sum_{i=1}^{n-1} b_{i,i-1} = \frac{n}{2}$ for the main diagonal. So, the difference of the sums of the inmost two diagonals is related to the dimension of the matrix. Define $f(k)$ as the sum of the kth diagonal in an n-dimensional **B** matrix, moving from the outside in. Then, for the case $k = n$, the difference of the sums of the inner two adjacent diagonals is given by the recursion relation $f(n)-f(n-1)=\frac{n}{2}=\frac{n}{2}(n+1)\frac{1}{n+1}$, which we recognize as $\frac{T_n}{n+1}$.

We can now prove by inverse induction on n that the kth diagonal of a **B** matrix of dimension n sums to $\frac{T_k}{n+1}$. This is clearly the case for $n=1$ or $n=2$. For $n=m\geq 2$, the sum of the main diagonal is $\frac{T_m}{m+1}$, and the sums of the outer diagonals are all given by $\frac{T_k}{m+1}$ by 1.3, provided that the claim holds for all $n<m$, which is true by the induction hypothesis. This proves the lemma.

Now, multiplying $\left(\mathbf{DD}^{\mathrm{T}}\right)^{-1}$ from the left by the transpose difference matrix $\mathbf{D}^{\mathrm{T}}$ has the effect of subtracting adjacent diagonals, from the top right to the bottom left. In other words, the sum of the kth diagonal $\sum_{i=1}^{n} b_{i+k,i}$ for some $k\in[0,n-1]$ becomes $\sum_{i=1}^{n} b_{i,j-k}-\sum_{i=1}^{n} b_{i+1,j-k}$. Thus, the sum of the kth diagonal in the upper triangle of $\overline{\mathbf{D}}=\mathbf{D}^{\mathrm{T}}\left(\mathbf{DD}^{\mathrm{T}}\right)^{-1}$ becomes $\frac{T_k-T_{k-1}}{n+1}$ and the lower triangle is just the negative of this. But T_k-T_{k-1} is just the kth triangular number $t_k=\frac{1}{2}k(k+1)$. So to summarize, the sum of the kth diagonal of a $\overline{\mathbf{D}}$ matrix of dimension $(n–1)\times n$ is $\overline{d}_k\equiv\pm\frac{t_k}{n+1}$, where the plus sign corresponds to the diagonals in $\overline{\mathbf{D}}$'s upper triangle and the minus sign to those in the lower triangle.

We now show that the diagonals of $\mathbf{C}=\mathbf{A}\overline{\mathbf{D}}$ sum to $c_k\equiv\pm\frac{2t_k}{n+1}$, except for the main diagonal, where the plus sign again corresponds to the diagonals in the upper triangle and the minus sign to those in the lower triangle. We will do this by looking separately at the upper and lower triangles of $\mathbf{N}\equiv\mathbf{AM}$, where **M** is an arbitrary matrix of dimension $(n–1)\times n$. First note that left-side multiplication by the **A** matrix adds each row of **M** to the one above it, and the bottom row is cut off. But in the lower triangle of **M**, moving an element up by one row also moves it up by one diagonal, so an alternate way of viewing left-multiplication by **A** is that each diagonal is added to the one above it for $k\leq n-1$, and the element belonging to the first row of **M** is removed. But as we have already shown, the element of the kth diagonal of the lower triangle of **M** that belongs to **M**'s bottom row is just k. So, denoting the sum of the kth diagonal in the lower triangles of **N** and **M** as n^k and m_k, respectively, $n_k=m_k+m_{k+1}-k$. But the recursion relation for the triangular sequence is just $f(k)=f(k-1)+k$, so $n_k=2m_k$. The situation for the upper triangle is exactly the same, except that the element which is removed from the kth diagonal of **M** is the one that belongs to **M**'s top row, and as we have shown, the element of the kth diagonal of the upper triangle of **M** that belongs to **M**'s top row is k. Thus, the sum of the kth diagonal of **N** is twice the sum of the kth diagonal of **M**, except for the middle diagonal. We summarize these facts in the following result:

Prop. 2. The kth diagonal of $\mathbf{C}$ sums to $c_k = \pm \frac{2t_k}{n+1}$.

Given this fact about $\mathbf{C}$, we can now simplify the double sum $k_\delta = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \sin\left((j-i)\delta\right)$. Since the definition of the kth diagonal is that $j–i = k$, for some $0 < k < N–1$, the sine term is unchanged along each diagonal, and its sign flips exactly when $\mathbf{C}$'s sign flips. Thus, we can change k_δ to a sum over the diagonals: $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \sin\left((j-i)\delta\right) = 2\sum_{k=1}^{n} c_k \sin\left((n-k)\delta\right)$ $= 2\sum_{k=1}^{n} \frac{2k(k+1)/2}{n+1} \sin\left((n-k)\delta\right) = \left(\frac{2}{n+1}\right) \sum_{k=1}^{n} k(k+1) \sin\left((n-k)\delta\right)$. Plugging in $N–1$ for n, we arrive at the following:

Prop. 3.
$$k_\delta = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} c_{ij} \sin\left((j-i)\delta\right) = \frac{2}{N} \sum_{k=1}^{N-1} k(k+1) \sin\left((N-1-k)\delta\right) \tag{2}$$

We can expand the summation:

$$k_\delta = \frac{1}{4N}\left[\csc^3\left(\frac{\delta}{2}\right)\right]\left[(N-2)(N+1)\cos\left(\frac{1}{2}\delta\right) - N(N-1)\cos\left(\frac{3}{2}\delta\right) + 2\cos\left(\frac{1}{2}\delta(1-2N)\right)\right]$$

Taking a derivative with respect to δ, and setting the result equal to zero,

$$\left[1+N(N-1)\right]\cos\delta + (N-2)\cos(\delta N) - (N+1)\left[N-2+\cos\left(\delta(N-1)\right)\right] = 0$$

Taylor expanding at $\delta = 0$ and solving,

$$\delta_{\text{opt}} \approx \sqrt{\frac{30}{3N^2-3N+2}} \tag{3}$$

Taking an additional term in the Taylor series,

$$\delta_{\text{opt}} \approx \operatorname{Re}\sqrt{2}\sqrt{\frac{42N^2-42N+\sqrt{14}\left(-24N^4+48N^3-96N^2+72N-34\right)^{\frac{1}{2}}+28}{5N^4-10N^3+13N^2-8N+3}} \tag{4}$$

This last expression approximates the exact solution to within 1.6 percent error for all $N \geq 3$, and falls below 0.1 percent error for all $N \geq 40$.

Note: It is easy to verify that $\delta = \frac{\pi}{N-1}$ maximizes $\sum_{k=1}^{N-1} k \sin\left((N-1-k)\delta\right)$ and is also close to optimal for $k_\delta = \sum_{k=1}^{N-1} k(k+1)\sin\left((N-1-k)\delta\right)$, so we can use this expression as a less unwieldy alternative to the Taylor expansion:

$$\delta_{\text{opt}} \approx \frac{\pi}{N-1} \tag{5}$$

This last expression provides us with a physical interpretation of δ_{opt}: the optimal joint phase offset is approximately the value required for the first and last joint to be π out of phase. In other words, a snake driving its joints at δ_{opt} attempts to form a discrete approximation to a single period of the serpenoid curve. Figure 1 compares the three closed-form approximations for δ_{opt} in (3), (4), and (5) with numerical solutions of the two exact expressions in (1) and (2).

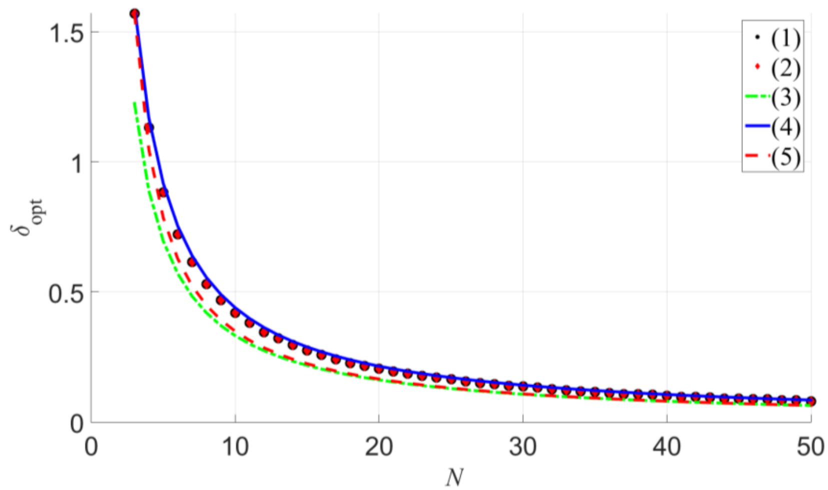

Fig. 1. Comparison of the three approximate expressions for δ_{opt}, alongside the numerical solutions, for N = 3 to 50.

References

[1] P. Liljebäck, K. Pettersen, Ø. Stavdahl, and J. Gravdahl, *Snake Robots: Modeling, Mechatronics, and Control.* London, UK: Springer, 2013.