3. Prove that a graph is bipartite if and only if does not contain any odd-length cycles.

<u>Prove:</u>		<u>Condition:</u>
Graph is bipartite	$\longrightarrow$	No odd length cycles in the graph

<u>Other assumptions:</u>

1. Every edge connects a vertex of U to one in W.

Information

$U \subset V$

$W \subset V$

$U \cap W = \emptyset$ $\rightarrow$ From here we assume that vertices can only be in U or in W. Not in both.

$U \cup W = V$

2. Each edge has endpoints in differing sets.

$U \longrightarrow W — U — W — U$

We assume that the Graph G is a connected graph, since a graph is bipartite if and only if all of its connected components are bipartite.

Therefore we can proof this only for connected components. Let's consider a origin in our connected graph, and check the distances.

origin U	— W	— U	— W	— U	
0	1	2	3	4	← distance from origin

Therefore:

$$U = \{ v \in V(G) \mid d(v_1, \text{origin}) \text{ is even} \}$$

$$W = \{ v \in V(G) \mid d(v_1, \text{origin}) \text{ is odd} \}$$

Since in the previous information we have than $U \cap W = \emptyset$, we can confirm that this partition is correct, since the distance of a vertex to the origin can not be even and odd at the same time.

Also this partition is correct taking into account the information of $U \cup W = V$, since we can only have even or odd distances, and nothing else.

With all of this information let's prove that a Graph is bipartite if and only if doesn't not contains cycles of odd length:

PROOF BY CONTRADICTION

Suppose for the sake of contradiction that exists two vertices V_1, V_2 and both one in the same set (U or W), such that V_1, V_2 are joined by an edge of $E(G)$, that would demonstrate that our partition is incorrect and therefore the graph is not bipartite.

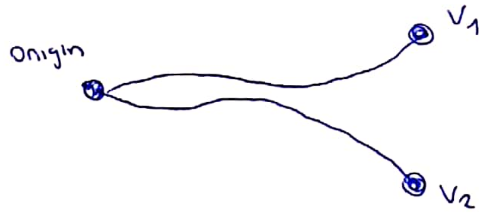
We know that both of them are not the same vertex, since they are joined by an edge. Let's assume that $V_1 =$ origin, then $d(V_1, \text{origin} = 0)$.

Since 0 it's even, that would mean that $d(V_2, \text{origin} = \text{even})$, since they both pertain to same set. This also means that $d(V_2, V_1 = \text{even})$, since $V_1 = \text{origin}$.

However we assumed that V_1 and V_2 are joined by an edge, so $d(V_1, V_2) = 1$

1 is odd, so we demonstrated that $V_1 \neq$ origin or $V_2 \neq$ origin, which let us with the next graph

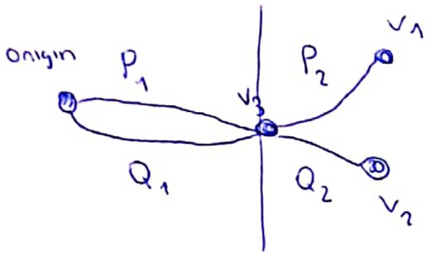
$$V_1 \neq V_2 \neq \text{origin}$$



G is a connected graph, so exists a shortest path from origin to V_1 (P) and from origin to V_2 (Q)

P and Q are both even or both odd, since V_1 and V_2 belong to the same set which is defined by the distance to origin

We know that V_1 and V_2 have at least one vertex in common, since they both start at origin, but we could have another intermediate node, such as that:



However, since we assumed P and Q to be the shortest's paths from origin to V_1 and from origin to V_2, we can assume that

$$P_1 = Q_1$$

(You will go through the shortest one, since V_3 is a node in common)

Therefore we have this

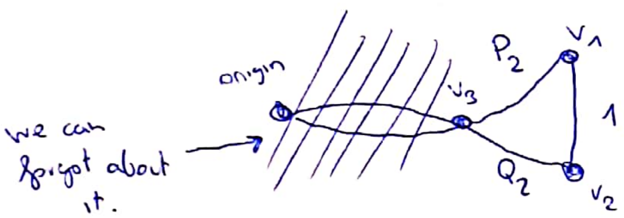
$$|P| = |P_1| + |P_2|$$
$$|Q| = |Q_1| + |Q_2|$$

Since $P_1 = Q_1$, and P and Q are both even or both odd, that means that P_2 and Q_2 are both even or both odd.

↓

$|P_2|$ and $|Q_2|$ are both even or both odd.

But we assumed that V_1 and V_2 were connected, so:



← let's consider this cycle

$$\underbrace{P_2 + Q_2}_{} + 1$$

(any number) $2K$ → always even

$2K + 1$ → always odd.

If we assumed ~~that two elements of the same~~ odd length cycles, that would mean that we have elements from the same set connected.

Therefore we demonstrated by proof of contradiction that if we have an odd length cycle, we have two connected components from the same set, and therefore G is not bipartite.

Prove:		Condition:
No odd length cycles in the graph	$\longrightarrow$	Graph is bipartite

PROOF OF CONTRADICTION:

Let's assume that we have a cycle, which is odd. (C)

$$C = (v_1, v_2, \ldots, v_n, v_1)$$

Since it's and odd cycle, ~~[illegible]~~, the number of edges is odd,
~~and therefore taking into account the theory of connectivity, vertices = edges + 1,~~
~~which leave our cycle with a [illegible] even number of vertices.~~
~~If without loss of generality, if v_1 is even~~
and therefore v_n is odd.

Let's sort this sets and check for a contradiction.

$$\underset{\text{odd}}{v_1} \in U, \quad \underset{\text{even}}{v_2} \in W, \quad \underset{\text{odd}}{v_3} \in U \ldots$$

Therefore

$$\text{odd} \to U, \quad \text{even} \to W$$

but $n = \text{odd}$ and $1 = \text{odd}$. We have a contradiction because if we join them because of the odd length cycle, they will belong to same set, and a bipartite graph can not connect two elements of the same set

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