Final Experience

Seth Bertlshofer
Alexis Tyler
Dustin Ginos
Kevin Burgon

December 15, 2015

Sub-Experience One: The Mad Mail Carrier

Suppose Sue is a Mail Carrier who is crazy. He likes to ensure that non of the n houses on his delivery route get the mail they are supposed to. Your goal, should you choose to accept it, for this sub-experience is to determine the number of ways Sue can deliver mail so that no one gets their mail in two ways. One method is to use the Principle of Inclusion/Exclusion (PIE). The other method is to use an exponential generating function to solve a recurrence which you'll develop. Put D_n equal to the number of ways Sue can distribute mail to n houses so that none of them gets the correct mail.

Sub-Experience One, Part One: PIE Approach

Use the PIE to determine D_n.

Solution:

Sub-Experience Two, Part One: PIE Approach

The formula you obtain above should involve a truncated power series for e^{-1}. Show that $D_n = \lfloor \frac{n!}{e} + \frac{1}{2} \rfloor, for\ n > 0$. (For $n = 0$, the formula doesn't work: $D_0 = 1$, but the formula gives 0.)

Solution:

Sub-Experience Three, Part One: PIE Approach

SE 1.3.1. Prove the recurrence $D_n = (n-1)D_{n-1} + (n-1)D_{n-2},\ for\ n \geq 2,\ and\ D_0 = 1, D_1 = 0.$
SE 1.3.2. Deduce, from the above recurrence $D_n = nD_{n-1} + (-1)^n,\ for\ n \geq 1,\ and\ D_0 = 1.$
SE 1.3.3. Use an exponential generating function to solve the recurrence from part 1.3.2

Solution:

Sub-Experience Two: Developing the EGF Lobe

Here are a few counting problems, the first of which is particularly easy to solve with an exponential generating function. The answer is very simple and so it should have an elegant combinatorial proof.

1. Using exponential generating function, determine the number of n-digit quaternary sequences built from {0,1,2,3} with an even number of 0's and an odd number of 1's.

2. *Not using* exponential generating functions, please solve the counting problem above.

3. Solve the recurrence from the pizza-cutting problem using exponential generating functions; that is, find a closed formula for $P(n)$ where, for $n \geq 1, P(n) = P(n-1) + n$ with $P(0) = 1$. Of course, you know the solution should be $P(n) = \frac{1}{2}n^2 + \frac{1}{2}n + 1$.

Solution:

Sub-Experience Three: Catalan Numbers

Roll up your sleeves and prepare to get down and dirty with generating functions. The goal of the following sequence of problems is to prove that the sequence $(b_n)_n \geq 0$ generated by the recurrence relation

$$b_n = b_0 b_{n-1} + b_1 b_{n-2} + ... + b_{n-1} b_0\ with\ b_0 = 1$$

gives the answer to the following counting questions, and that $\binom{b_n = \frac{1}{n+1} 2n}{n}$. These numbers are called the *Catalan Numbers* and are nearly as ubiquitous as the Fibonacci Numbers.

How many binary trees on n vertices are there? Follow the steps below to produce a closed formula for

b_n using a generating function.

3.0 Denote by b_n the number of binary trees on vertices. Please prove that $b_n = b_0 b_{n-1} + b_1 b_{n-2} + ... + b_{n-1} b_0$ with $b_0 = 1$.
3.1 Let $g(x) = \sum_{n=0}^{\infty} b_n x^n$ be the generating function for the sequence $b_n n \geq 0$ of the number of binary trees on n vertices. Show that $g(x) = 1 + x(g(x))^2$.
3.2 Use the quadratic formula to show that $g(x) = 1 - \frac{\sqrt{1-4x}}{2x}$. Note that you will have two options to use for $g(x)$, one of which you must rule out.
3.3 Recall *The Binomial Theorem : For any real number r,* $(1+z)^r = \sum_{k=0}^{\infty} \binom{r}{k} z^k$, where

$$\binom{r}{k} = \left\{ \begin{matrix} \frac{r(r-1)(r-2)...(r-k+1)}{k!} & , & r \in \mathbb{R},\ 0 \leq k \in \mathbb{Z} \\ 0, & , & k < 0 \end{matrix} \right\}$$

Use the Binomial Theorem to obtain $g(x) = \frac{1}{2x}(1 - \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k}(-4x)^k)$.
3.4 Manipulate $g(x) = \frac{1}{2x}(1 - \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k}(-4x)^k)$ into $g(x) = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{k}(-1)^n 2^{2n+1} x^n$. The reason for doing this is so that x^n's coefficient can be read (remember that b_n is defined to be$[x^n]g(x)$.) The following steps are suggested, but not required; follow them only if you have no idea of your own. Begin with $g(x) = \frac{1}{2x}(1 - \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k}(-4x)^k)$.

- Pull out the first term on the summation, and simplify
- Substitute $n+1$ for k; This will allow the summation to begin at zero.
- Pull the negative that resulted from the first manipulation in this list into the summation.
- Bring $\frac{1}{2x}$ into the summation.

3.5 Verify that $\binom{\frac{1}{2}}{n+1} = \frac{1}{n+1} \binom{\frac{n-1}{2}}{n} (-1)^n \frac{1}{2}$
3.6 Verify that $\binom{2n}{n} \frac{1}{2^{2n}} = \binom{\frac{n-1}{2}}{n}$.
3.7 Finally, coerce $g(x)$ into $\sum_{n \geq 0} \binom{2n}{n} \frac{x^n}{n+1}$.

Solution:

Sub-Experience Four: Non-Standard Dice

A 6-sided die labeled with the integers 1,2,3,4,5,6 will be called a *standard die.* The goal for this part of the Midterm Experience is to determine all ways to label a pair of dice with positive integers so that the probabilities of rolling the usual sums 2,3,...,12 are the same, but the labels are non-standard.

Step 1. Let $p(x) = x + x^2 + x^3 + x^4 + x^5 + x^6$, and explain why $(p(x))^2$ is the generating function for the probabilities of outcomes in rolling a pair of standard dice.

Step 2. Let $A = (a_1, a_2, a_3, a_4, a_5, a_6)$ and $B = (b_1, b_2, b_3, b_4, b_5, b_6)$ be two lists of positive integers. Put $p_A(x) = x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6}$ and $p_B(x) = x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6}$. Explain why finding a_is and b_is such that $p_A(x)p_B(x) = (p(x))^2$ is relevant this part of the Experience.

Step 3. Factor $p(x)$ into irreducible polynomials and use this factorization to help solve for the a_is and b_is. Specifically, the factorization will force the form of $p_A(x)$ to be something like $p_1(x)^q p_2(x)^r p_3(x)^s p_4(x)^t$, where $0 \leq q, r, s, t \leq 2$ and $p_i(x)$, for $1 \leq i \leq 4$, is a factor of $p(x)$. In your solution to this step, you must motivate why you take this step.

Step 4. Begin to reduce the possibilities for q,r,s, and t by using information from $p_A(1)$ and $p_A(0)$.

Note that, on one hand $p_A(1) = 1^{a_1} + 1^{a_2} + 1^{a_3} + 1^{a_4} + 1^{a_5} + 1^{a_6} = 6$ (*since* $a_i > 0$), and on the other hand we have $p_A(1) = p_1(1)^q p_2(1)^r p_3(1)^s p_4(1)^t$. Similarly, there are two ways to view $p_A(0)$.

Step 5. List all possible ways to label a pair of dice so that the probabilities of obtaining the sums 2,3,4,5,6,7,8,9,10,11,12 are $\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}$, respectively. One such way will be the standard way. In your solution for this step, explain why you have proved that the labels you have found are the only possible ones that give the desired probabilities for roll-outcomes.

Solution:

Step 1:
Say we have ax^n, where n represents the number rolled on a standard die and a represents the number of ways you can roll n, then the probability of rolling a standard die, $p(x)$, can be represented as $p(x) = x + x^2 + x^3 + x^4 + x^5 + x^6$.
Now when we have two standard dice, n represents the sum of the number when the dice are rolled, and a still represents the number of ways you can roll n. Now we have:

$$x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}$$

which is equal to:

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)(x + x^2 + x^3 + x^4 + x^5 + x^6) = p(x)p(x) = (p(x))^2$$

Step 2:
Assume there are two dice $A+B$ labeled with positive integers that give you the same probabilities as regular dice. Then

$$A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$$

$$B = \{b_1, b_2, b_3, b_4, b_5, b_6\}$$

If we use the same method of notation as in step 1 then

$$p_A(x) = x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6}$$

$$p_B(x) = x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6}$$

Since we are finding all the ways to label a pair of non-standard dice such that the probabilities of rolling the usual sums is the same (represented by $(p(x))^2$). Then, we need to find a_is and b_is such that $p_A(x)p_B(x) = (p(x))^2$.

Step 3:
To help solve for the a_is and b_is we can factor $p(x)$ into irreducible polynomials.

$$p(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6) = x(x+1)(x^2+x+1)(x^2-x+1)$$

we can also see that

$$(p(x))^2 = x^2(x+1)^2(x^2+x+1)^2(x^2-x+1)^2$$

This forces the form of $p_A(x)$ to be $x^q(x+1)^r(x^2+x+1)^s(x^2-x+1)^t$ where $0 \leq q, r, s, t \leq 2$.

Step 4:
To reduce the possibilities for q,r,s, and t. Let's use the info $p_A(1)$ and $p_A(0)$.
$p_A(1)$ will help restrict our possibilities for r and s because $p_A(1) = 1^{a_1} + 1^{a_1} + 1^{a_1} + 1^{a_1} + 1^{a_1} + 1^{a_1} = 6 = (1)^q(1+1)^r(1^2+1+1)^s + (1^2-1+1)^t = 1^q 2^r 3^s 1^t$. From this we see that r and s must equal 1.

Now $p_A(0) = 0 = 0^q 1^r 1^s 1^t$, so q cannot equal 0.

If we let q=2 then $x^2(x+1)(x^2+x+1)(x^2-x+1)^0 = x^5 2x^4 2x^3 x^2$ which gives us the set of labels {5,4,4,3,3,2} However, we want labels whose sums are the same as two standard die and since $a_i, b_i > 0$ then the smallest number on b_i could only be a 1 therefore making the smallest sum of $a_i + b_i = 3$, when we need it to be 2. So our only option is to let $a = 1$. So we know that $q = 1,\ r = 1,\ and\ s = 1$ so now to find t which can either be 0,1, or 2.

Step 5:
If we let $t = 0$ then $x(x+1)(x^2+x+1)(x^2-x+1)^0 = x^4+x^3+x^3+x^2+x^2+x$ which gives us the labels {4,3,3,2,2,1}

If $t = 1$ then:

$$x(x+1)(x^2+x+1)(x^2-x+1)^1 = x^6+x^5+x^4+x^3+x^2+x$$

which gives us the labels {6,5,4,3,2,1} which is a regular die.
Finally if $t = 2$ then

$$x(x+1)(x^2+x+1)(x^2-x+1)^2 = x^8+x^6+x^5+x^4+x^3+x$$

which gives us the labels {8,6,5,4,3,1}.

So the labels {4,3,3,2,2,1} and {8,6,5,4,3,1} are the only non-standard labels with positive integers such that the probabilities of rolling the usual sums 2,3,...,12 are the same as standard dice.
To verify this we can also see that

$$(x^4+x^3+x^3+x^2+x^2+x)(x^8+x^6+x^5+x^4+x^3+x) = x^2+2x^3+3x^4+4x^5+5x^6+6x^7+5x^8+4x^9+3x^{10}+2x^{11}+x^{12}$$

Sub-Experience Five: Eul-ing the GF Machine
Recall that a *partition of a positive integer n* is a nondecreasing sequence $\vec{\lambda} = (\lambda_1, \lambda_2, ...)$ with $\sum_{i\geq 1} \lambda_i = n$. If the sequence $\vec{\lambda}$ has k entries, then we call it a *partition of n into k parts*, and we call the λ_is the *parts*. The following problems may each be solved in many ways, but all may be solved by examining an appropriately constructed generating function (not necessarily the same for function for each problem). Note that each problem requests a number, but a proof that the number you produce is correct should be given.
Suppose $S = \{a_1, a_1, ..., a_k\}$ is a set of positive integers sharing no divisor; that is, $a_1, a_2, ...a_k$ are *relatively prime*. The **Frobenius number** of S is the largest number that cannot be expressed as a linear combination, with integer coefficients, of numbers from S. If a_1 and a_2 are relatively prime, their Frobenius number can be computed easily; it is $a_1a_2 - a_1 - a_2$. But if a_1, a_2, and a_3 are relatively prime, their Frobenius number is not known in general; that is, there is no formula for it — it must be computed *ad hoc*.

1. Determine the number of partitions of 20 into at most 5 parts. In other terms count the number of $\vec{\lambda}$s with no more than 5 entries that sum to 20. For example (1,19),(1,1,1,1,16),(5,5,5,5),(2,3,4,5,6), and (20) are partitions of 20 to be included in the count. These are not: (1,1,1,1,1,15), (2,2,2,2,2,2,2,2,2,2), (3,3,3,3,3,5).
2. Determine the number of partitions of 30 into parts of size less than or equal to 5. Here, a partition of 30 is not counted if any $\lambda_i > 5$. So these partitions of 30 are counted: (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,5,5), (1,1,2,2,2,3,4,5,5,5), (5,5,5,5,5,5); while these are not: (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,10), (6,6,6,6,6),
3. Determine $p(20)$, the number of partitions of 20.
4. Determine the Frobenius number for the set {7,9,11}.

Solution:

Sub-Experience Six: The Twelve-Fold Way

Suppose you have a set N of n objects which may be labeled or not. You also have a set X of x containers which may be labeled or not. Please determine formula for the number of distributions of the objects in N into the containers of X as the possibilities range over N's objects labeled or not, and X's containers labeled or not, and where the following restrictions on containers hold: (1) no restrictions, (2) each container may hold at most one object, and lastly (3) no container may be empty. The table below catalogs all the possibilities. So the task is to determine a formula which is the number of distributions of N into X satisfying the corresponding entry's constraints. Another perspective on this problem is as follows. Count the number of distinct functions f:N $\to X$; the restrictions on the containers translate correspondingly to (1) no restriction on f, (2) f is one-to-one, (3) f is onto.

Twelve Formula

N	X	unrestricted	at most one	nonempty
labeled	labeled	1.	2.	3.
unlabeled	labeled	4.	5.	6.
labeled	unlabeled	7.	8.	9.
unlabeled	unlabeled	10.	11.	12.

Please: No resources, outside of class notes and the help of others in class, are allowed on this part of the experience.

Solution: