

## Desc. Experience 1

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### Sub-Experience One: The Mad Mail Carrier

Suppose Sue is a Mail Carrier who is crazy. He likes to ensure that non of the  $n$  houses on his delivery route get the mail they are supposed to. Your goal, should you choose to accept it, for this sub-experience is to determine the number of ways Sue can deliver mail so that no one gets their mail in two ways. One method is to use the Principle of Inclusion/Exclusion (PIE). The other method is to use an exponential generating function to solve a recurrence which you'll develop. Put  $D_n$  equal to the number of ways Sue can distribute mail to  $n$  houses so that none of them gets the correct mail.

**Sub-Experience One, Part One: PIE Approach** Use the PIE to determine  $D_n$ .

**Solution:**

**Sub-Experience Two, Part One: PIE Approach** The formula you obtain above should involve a truncated power series for  $e^{-1}$ . Show that  $D_n = \lfloor \frac{n!}{e} + \frac{1}{2} \rfloor$ , for  $n > 0$ . (For  $n = 0$ , the formula doesn't work:  $D_0 = 1$ , but the formula gives 0.)

**Solution:**

**Sub-Experience Three, Part One: PIE Approach SE 1.3.1.** Prove the recurrence  $D_n = (n - 1)D_{n-1} + (n - 1)D_{n-2}$ , for  $n \geq 2$ , and  $D_0 = 1, D_1 = 0$ .

**SE 1.3.2.** Deduce, from the above recurrence  $D_n = nD_{n-1} + (-1)^n$ , for  $n \geq 1$ , and  $D_0 = 1$ .

**SE 1.3.3.** Use an exponential generating function to solve the recurrence from part 1.3.2

**Solution:**

### Sub-Experience Two: Stirling Numbers of the Second Kind

Recall that a partition of a set  $X$  into  $k$  blocks is a set  $\Pi = \{B_1, \dots, B_k\}$  where the  $B_i$ s are disjoint nonempty subsets of  $X$  whose union is  $X$ . Define  $\binom{n}{k}$  to be the number of partitions of a set with  $n$  elements into  $k$  blocks.

**2.1** Use the Principle of Inclusion/Exclusion to create a formula for  $\binom{n}{k}$ . Please verify that the formula you find actually works.

**2.2** Use exponential generating functions to create a formula for  $\binom{n}{k}$ . Please verify that the formula you find actually works. It may appear different than the found in part 2.1, so be careful.

**Solution:**

### Sub-Experience Three: Catalan Numbers

Roll up your sleeves and prepare to get down and dirty with generating functions. The goal of the following sequence of problems is to prove that the sequence  $(b_n)_n \geq 0$  generated by the recurrence relation

$$b_n = b_0 b_{n-1} + b_1 b_{n-2} + \dots + b_{n-1} b_0 \text{ with } b_0 = 1$$

gives the answer to the following counting questions, and that  $\binom{n}{n+1} 2^n$ . These numbers are called the *Catalan Numbers* and are nearly as ubiquitous as the Fibonacci Numbers.

How many binary trees on  $n$  vertices are there? Follow the steps below to produce a closed formula for  $b_n$  using a generating function.

**3.0** Denote by  $b_n$  the number of binary trees on vertices. Please prove that  $b_n = b_0 b_{n-1} + b_1 b_{n-2} + \dots + b_{n-1} b_0$  with  $b_0 = 1$ .

**3.1** Let  $g(x) = \sum_{n=0}^{\infty} b_n x^n$  be the generating function for the sequence  $b_n, n \geq 0$  of the number of binary trees on  $n$  vertices. Show that  $g(x) = 1 + x(g(x))^2$ .

**3.2** Use the quadratic formula to show that  $g(x) = 1 - \frac{\sqrt{1-4x}}{2x}$ . Note that you will have two options to use for  $g(x)$ , one of which you must rule out.

3.3 Recall *The Binomial Theorem* : For any real number  $r$ ,  $(1+z)^r = \sum_{k=0}^{\infty} \binom{r}{k} z^k$ , where

$$\binom{r}{k} = \begin{cases} \frac{r(r-1)(r-2)\dots(r-k+1)}{k!} & r \in \mathbb{R}, 0 \leq k \in \mathbb{Z} \\ 0, & k < 0 \end{cases}$$

Use the Binomial Theorem to obtain  $g(x) = \frac{1}{2x}(1 - \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k}(-4x)^k)$ .

3.4 Manipulate  $g(x) = \frac{1}{2x}(1 - \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k}(-4x)^k)$  into  $g(x) = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n}(-1)^n 2^{2n+1} x^n$ . The reason for doing this is so that  $x^n$ 's coefficient can be read (remember that  $b_n$  is defined to be  $[x^n]g(x)$ .) The following steps are suggested, but not required; follow them only if you have no idea of your own. Begin with  $g(x) = \frac{1}{2x}(1 - \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k}(-4x)^k)$ .

- Pull out the first term on the summation, and simplify
- Substitute  $n+1$  for  $k$ ; This will allow the summation to begin at zero.
- Pull the negative that resulted from the first manipulation in this list into the summation.
- Bring  $\frac{1}{2x}$  into the summation.

3.5 Verify that  $\binom{\frac{1}{2}}{n+1} = \frac{1}{n+1} \binom{\frac{n-1}{2}}{n} (-1)^{n+1} \frac{1}{2}$

3.6 Verify that  $\binom{2n}{n} \frac{1}{2^{2n}} = \binom{\frac{n-1}{2}}{n}$ .

3.7 Finally, coerce  $g(x)$  into  $\sum_{n \geq 0} \binom{2n}{n} \frac{x^n}{n+1}$ .

**Solution:**

#### Sub-Experience Four: Non-Standard Dice

A 6-sided die labeled with the integers 1,2,3,4,5,6 will be called a *standard die*. The goal for this part of the Midterm Experience is to determine all ways to label a pair of dice with positive integers so that the probabilities of rolling the usual sums 2,3,...,12 are the same, but the labels are non-standard.

**Step 1.** Let  $p(x) = x + x^2 + x^3 + x^4 + x^5 + x^6$ , and explain why  $(p(x))^2$  is the generating function for the probabilities of outcomes in rolling a pair of standard dice.

**Step 2.** Let  $A = (a_1, a_2, a_3, a_4, a_5, a_6)$  and  $B = (b_1, b_2, b_3, b_4, b_5, b_6)$  be two lists of positive integers. Put  $p_A(x) = x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6}$  and  $p_B(x) = x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6}$ . Explain why finding  $a_i$ s and  $b_i$ s such that  $p_A(x)p_B(x) = (p(x))^2$  is relevant this part of the Experience.

**Step 3.** Factor  $p(x)$  into irreducible polynomials and use this factorization to help solve for the  $a_i$ s and  $b_i$ s. Specifically, the factorization will force the form of  $p_A(x)$  to be something like  $p_1(x)^q p_2(x)^r p_3(x)^s p_4(x)^t$ , where  $0 \leq q, r, s, t \leq 2$  and  $p_i(x)$ , for  $1 \leq i \leq 4$ , is a factor of  $p(x)$ . In your solution to this step, you must motivate why you take this step.

**Step 4.** Begin to reduce the possibilities for  $q, r, s$ , and  $t$  by using information from  $p_A(1)$  and  $p_A(0)$ . Note that, on one hand  $p_A(1) = 1^{a_1} + 1^{a_2} + 1^{a_3} + 1^{a_4} + 1^{a_5} + 1^{a_6} = 6$  (since  $a_i > 0$ ), and on the other hand we have  $p_A(1) = p_1(1)^q p_2(1)^r p_3(1)^s p_4(1)^t$ . Similarly, there are two ways to view  $p_A(0)$ .

**Step 5.** List all possible ways to label a pair of dice so that the probabilities of obtaining the sums 2,3,4,5,6,7,8,9,10,11,12 are  $\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}$ , respectively. One such way will be the standard way. In your solution for this step, explain why you have proved that the labels you have found are the only possible ones that give the desired probabilities for roll-outcomes.

**Solution:**

#### Sub-Experience Five: Eul-ing the GF Machine

Recall that a *partition of a positive integer  $n$*  is a nondecreasing sequence  $\vec{\lambda} = (\lambda_1, \lambda_2, \dots)$  with  $\sum_{i \geq 1} \lambda_i = n$ .

Suppose  $S = \{a_1, a_1, \dots, a_k\}$  is a set of positive integers sharing no divisor; that is,  $a_1, a_2, \dots, a_k$  are *relatively prime*. The **Frobenius number** of  $S$  is the largest number that cannot be expressed as a linear combination, with integer coefficients, of numbers from  $S$ . If  $a_1$  and  $a_2$  are relatively prime, their Frobenius number can be computed easily; it is  $a_1 a_2 - a_1 - a_2$ . But if  $a_1, a_2$ , and  $a_3$  are relatively prime, their Frobenius number is not known in general; that is, there is no formula for it — it must be computed *ad hoc*.

1. Determine the number of partitions of 20 into at most 5 parts. In other terms count the number of  $\vec{\lambda}$ s with no more than 5 entries that sum to 20. For example  $(1,19), (1,1,1,1,16), (5,5,5,5), (2,3,4,5,6)$ , and  $(20)$  are partitions of 20 to be included in the count. These are not:  $(1,1,1,1,1,15), (2,2,2,2,2,2,2,2,2), (3,3,3,3,3,5)$ .
2. Determine the number of partitions of 30 into parts of size less than or equal to 5. Here, a partition of 30 is not counted if any  $\lambda_i > 5$ . So these partitions of 30 are counted:  $(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,5,5), (1,1,2,2,2,3,4,5,5,5), (5,5,5,5,5,5)$ ; while these are not:  $(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,10), (6,6,6,6,6)$ .
3. Determine  $p(20)$ , the number of partitions of 20.
4. Determine the Frobenius number for the set  $\{7,9,11\}$ .

### Sub-Experience Six: The Twelve-Fold Way

Suppose you have a set  $N$  of  $n$  objects which may be labeled or not. You also have a set  $X$  of  $x$  containers which may be labeled or not. Please determine formula for the number of distributions of the objects in  $N$  into the containers of  $X$  as the possibilities range over  $N$ 's objects labeled or not, and  $X$ 's containers labeled or not, and where the following restrictions on containers hold: (1) no restrictions, (2) each container may hold at most one object, and lastly (3) no container may be empty. The table below catalogs all the possibilities. So the task is to determine a formula which is the number of distributions of  $N$  into  $X$  satisfying the corresponding entry's constraints. Another perspective on this problem is as follows. Count the number of distinct functions  $f:N \rightarrow X$ ; the restrictions on the containers translate correspondingly to (1) no restriction on  $f$ , (2)  $f$  is one-to-one, (3)  $f$  is onto.

N	X	unrestricted	at most one	nonempty
labeled	labeled	1.	2.	3.
unlabeled	labeled	4.	5.	6.
labeled	unlabeled	7.8.9.		
unlabeled	unlabeled	10.	11.	12.

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