Cohen Forcing

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some date

work in progress

Contents

0 Blurb

These notes are mainly based on lectures for the Part III course "Forcing and the Continuum Hypothesis" by Benedikt Löwe at the University of Cambridge in 2025. There are prettier sets of notes, available at

https://zeramorphic.uk/gh/maths-compiled/iii/forcing/build/main.pdf

and

https://danielnaylor.uk/notes/III/Lent/FC/FC.pdf,

lest you think this is the best place to learn the material from.

1 A Countable Transitive Model of ZFC?

So you wish to prove that the continuum hypothesis (CH), the assertion that $2^{\aleph_0} = \aleph_1$ cannot be proven from the axioms of ZFC set theory. You took a class in model theory and you learned that one can show this by showing that $\mathsf{ZFC} + \neg\mathsf{CH}$ is consistent. The typical approach is to exhibit a model of $\mathsf{ZFC} + \neg\mathsf{CH}$. Let us try to do that.

1.1 Wishful Thinking

Stop. You realise that a model of $\mathsf{ZFC} + \neg\mathsf{CH}$ will, in particular, also be a model of ZFC. Darn. The whole point of ZFC set theory was that it was supposed to be able to formalise (most of)[1] the mathematics we do in our day-to-day lives. A certain pesky Kurt Gödel prevents us from explicitly exhibiting such a model.

But who is going to stop us from simply running off with the assumption that ZFC is consistent?[2] If we do this, then we can hope to obtain a theorem and proof of the following form.

Theorem
If ZFC *is consistent, then* $\mathsf{ZFC} + \neg\mathsf{CH}$ *is consistent.*

Proof. Let M be a model of ZFC. [$\mathfrak{Black\ magic}$]. Therefore we have obtained a model N of $\mathsf{ZFC} + \neg\mathsf{CH}$. □

In such a proof as above, N will presumably be created from M. But at this point, we have no information about M, other than that it models ZFC. Ideally, we would like M to be a *countable transitive* model of ZFC. By "transitive", we mean transitive with respect to the membership relation $\in$. That is, if $x \in y \in M$, then $x \in M$.

To get a countable model of ZFC is fairly easy. Recall the Löwenheim–Skolem theorem, asserting that theories in first-order logic are unable to control the cardinalities of their infinite models.

Theorem 1.1.1 (The Löwenheim–Skolem Theorem)
Let T be a consistent $\mathcal{L}$-theory. Suppose there exists an infinite model of T. Then for all cardinals $\kappa \geq |\mathcal{L}| + \aleph_0$, there exists a model of T of cardinality κ.

A special case of this theorem, which also arises from the proof of Gödel's completeness theorem via Henkin terms, is that if a language $\mathcal{L}$ is countable, then a consistent $\mathcal{L}$-theory T has an infinite model if and only if it has a countably infinite model.

In particular, as any model of ZFC must be infinite, there must also exist a countable model of ZFC. So how do we get transitivity?

1.2 Only a Sith Deals in Absolutes

Why do we even want to get a countable transitive model of ZFC in the first place?

We want a countable model so we are able to access things from *outside* of the model. This lets us adjoin new elements, such as real numbers, to the model, because there are uncountably many real numbers out in the metatheory. This is not unlike a field extension.

Transitive models are desirable due to the fact that they make a lot of formulas "absolute".

[1] Category theory jumpscare.

[2] If you are an amused reader from the future with the knowledge that ZFC is inconsistent, how is the climate doing? Thought so. Focus on your own problems.

Definition 1.2.1
Let M and N be $\mathcal{L}$-structures with $M \subseteq N$ and let $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}$-formula with n free variables. We say that $\varphi(x_1, \ldots, x_n)$ is absolute between M and N if, for all $a_1, \ldots, a_n \in M$,

$$M \models \varphi(a_1, \ldots, a_n) \text{ if and only if } N \models \varphi(a_1, \ldots, a_n).$$

Replacing "if and only if" in the definition above with "if" gives us the notion of downwards absoluteness, whereas replacing "if and only if" with "only if" gives us the notion of upwards absoluteness. Formulas which are absolute between structures M and N, with $M \subseteq N$, are great because they really do let us view N as a certain kind of extension of M.

Atomic formulas are always absolute between M and N whenever M is a substructure of N. This is pretty much by definition: an $\mathcal{L}$-structure M is a substructure of an $\mathcal{L}$-structure N if the domain of M is a subset of the domain of N and the inclusion function $\iota \colon M \to N$ is an injective homomorphism of $\mathcal{L}$-structures. Consequently, propositional connectives of atomic formulas are also always absolute between substructures.

But things become icky when we introduce quantifiers. We cannot simply "add" things to a model and expect it to preserve the truth of formulas in the original structure.

Example 1.2.2
Consider the language $\mathcal{L}_\in$ of set theory, which only consists of one relation symbol $\in$, and has no constant symbols or function symbols. So the only atomic formulas are of the form $x = y$ or $x \in y$, for variables x and y. All the "interesting" formulas are not going to be propositional connectives of atomic formulas.

Let M be a model of ZFC and let $\varnothing$ be the empty set in M. We extend M by letting $N := M \cup \{*\}$ declaring $* \in^N \varnothing$. Then

$$M \models \forall z.(z \notin \varnothing),$$

but

$$N \not\models \forall z.(z \notin \varnothing).$$

Thus even very simple formulas, such as $\varphi_\varnothing(x) := \forall z.(z \notin x)$ asserting that x is empty, are not absolute between M and N. □

Of particular interest are formulas which are absolute between some model and the ambient universe in the metatheory.

Definition 1.2.3
Let M be an $\mathcal{L}_\in$-structure. An $\mathcal{L}_\in$-formula $\varphi(x_1, \ldots, x_n)$ is said to be absolute for M if, for all $a_1, \ldots, a_n \in M$,

$$M \models \varphi(a_1, \ldots, a_n) \text{ if and only if } \varphi(a_1, \ldots, a_n) \text{ is true.}$$

If our metatheory is ZFC set theory, then the above "$\varphi(a_1, \ldots, a_n)$ is true" is interpreted as ZFC $\vdash \varphi(a_1, \ldots, a_n)$. As before, we similarly have the notions of a formula being upwards absolute and downwards absolute for an $\mathcal{L}_\in$-structure M.

Definition 1.2.4
An $\mathcal{L}_\in$-structure M is said to be transitive if, for all $x \in M$ and $y \in x$, we have $y \in M$.

Phrased differently, a transitive $\mathcal{L}_\in$-structure M is one such that for all $x \in M$ we have $x \subseteq M$.

The idea is that if M is a *transitive* model of ZFC, then lots of formulas are absolute for M. Consequently, if we have a transitive models M and N of ZFC with $M \subseteq N$, then we can really view N as a particularly neat extension of M.

Recall that Δ_0 is the smallest class of all $\mathcal{L}_\in$-formulas containing the atomic formulas and is closed under propositional connectives and bounded quantification. By bounded quantification, we mean quantifiers of the form $\forall x \in y.\varphi$ or $\exists x \in y.\varphi$.

Lemma 1.2.5
If φ is a Δ_0 formula in $\mathcal{L}_\in$, then φ is absolute for M for any transitive $\mathcal{L}_\in$-strucutre M.

Proof. By induction on φ. $\square$

This is particularly neat because lots of familiar expressions in set theory are expressible as Δ_0 formulas.

Example 1.2.6
All of the following are expressible as Δ_0 formulas.

- $x = y$
- $x \in y$
- $x \subseteq y$
- $z = \{x\}$
- $z = \{x, y\}$
- $z = \langle x, y \rangle \coloneqq \{\{x\}, \{x, y\}\}$
- $z = \varnothing$
- $z = x \cup y$
- $z = x \cap y$
- $z = x \setminus y$
- $z = x \cup \{x\}$
- z is transitive
- $z = \bigcup x$
- z is an ordered pair
- $z = x \times y$
- z is a relation
- $z = \mathrm{dom}(R)$ and R is a relation
- $z = \mathrm{ran}(R)$ and R is a relation
- f is a function
- f is an injective function
- f is a surjective function
- f is a bijective function
- α is an ordinal
- α is a successor ordinal
- α is a limit ordinal
- $x = \omega$, where ω is the first countable ordinal

- $n \in \omega$. $\square$

Have I convinced you that transitive $\mathcal{L}_\in$-structures are great yet? If not, then check this out. In attempting to build a countable transitive model of ZFC, simply ensuring that our structure is transitive will yield several axioms of ZFC.

Lemma 1.2.7
If M is a transitive $\mathcal{L}_\in$-structure, then

$$M \models \texttt{Extensionality} + \texttt{Foundation}.$$

If, furthermore, for all $x, y \in M$ we have $\{x, y\} \in M$ and $\bigcup x \in M$, then

$$M \models \texttt{Extensionality} + \texttt{Foundation} + \texttt{Pairing} + \texttt{Union}.$$

Proof. Just do it. $\square$

Woohoo. Only infinitely many more axioms to go...

We have seen that many formulas are absolute for transitive $\mathcal{L}_\in$-structures. There are, however, formulas which are *not* absolute. For instance, the following formulas are not absolute:

- $x = \mathcal{P}(y)$
- $\mathcal{F} = y^x$, that is, $\mathcal{F}$ is the set of all functions from x to y
- κ is a cardinal
- $|X| = |Y|$
- $\beta = \operatorname{cf}(\alpha)$
- α is a regular cardinal.

The non-absoluteness of these formulas will become apparent in the development of later subsections. For now, note that the formula

$$\kappa \text{ is a cardinal}$$

is *downwards* absolute for any transitive $\mathcal{L}_\in$-structure, eventhough it will *not* be *upwards* absolute.

1.3 Light at the End of the Tunnel

So we begin our mission in trying to get a countable transitive model of ZFC.

First, we recall the Tarski–Vaught test, which gives us information about $\mathcal{L}$-embeddings, which are injective $\mathcal{L}$-homomorphisms between $\mathcal{L}$-structures.

Lemma 1.3.1 (The Tarski–Vaught Test)
Let M and N be $\mathcal{L}$-structures and let $i\colon M \to N$ be an $\mathcal{L}$-embedding. Let Φ be a collection of $\mathcal{L}$-formulas which is closed under subformulas. Then the following are equivalent:

(1) for all $\varphi(x_1, \ldots, x_k) \in \Phi$ and all $a_1, \ldots, a_k \in M$,

$$M \models \varphi(a_1, \ldots, a_k) \textit{ if and only if } N \models \varphi(i(a_1), \ldots, i(a_k));$$

(2) for all formulas $\varphi(x, y_1, \ldots, y_k) \in \Phi$ and for all $a_1, \ldots, a_k \in M$, if there exists $n \in N$ such that

$$N \models \varphi(n, i(a_1), \ldots, i(a_k)),$$

then there exists $m \in M$ such that

$$N \models \varphi(i(m), i(a_1), \ldots, i(a_k)).$$

Proof. By induction on the complexity of the formulas in Φ. $\square$

In particular if Φ above is the class of *all* $\mathcal{L}$-formulas, then Lemma 1.3.1 provides a characterisation for when an $\mathcal{L}$-embedding is actually an *elementary* $\mathcal{L}$-embedding.

We call property (2) in Lemma 1.3.1 the Tarski–Vaught criterion. Notice that this makes no reference to the truth of φ in M.

Let us specialise the Tarski–Vaught test to the language $\mathcal{L}_\in$ of set theory and to the case when the embedding $i\colon M \to N$ is actually an inclusion. The formulation of the Tarski–Vaught test which we are particularly interested in is as follows.

Lemma 1.3.2 (The Tarski–Vaught Test for $\mathcal{L}_\in$)
Let M and N be $\mathcal{L}_\in$-structures with $M \subseteq N$. Let Φ be a collection of $\mathcal{L}_\in$-formulas which is closed under subformulas. Then the following are equivalent.

(1) all formulas in Φ are absolute between M and N;

(2) for all formulas $\varphi(x, y_1, \ldots, y_k) \in \Phi$ and for all $a_1, \ldots, a_k \in M$, if there exists $n \in N$ such that

$$N \models \varphi(n, a_1, \ldots, a_k),$$

then there exists $m \in M$ such that

$$N \models \varphi(m, a_1, \ldots, a_k).$$

The Tarski–Vaught test, though very easy to prove, implies a number of really surprising results.

Definition 1.3.3
A hierarchy is a class of sets $\{Z_\alpha\}_{\alpha\in\mathrm{Ord}}$ such that:

- *each Z_α is a transitive set;*
- $\mathrm{Ord} \cap Z_\alpha = \alpha$ *for each ordinal α;*
- *if $\alpha < \beta$ then $Z_\alpha \subseteq Z_\beta$;*
- *if λ is a limit ordinal then $Z_\lambda = \bigcup_{\alpha<\lambda} Z_\alpha$.*

Given a hierarchy of sets $\{Z_\alpha\}_{\alpha\in\mathrm{Ord}}$, we can define the class $Z := \bigcup_{\alpha\in\mathrm{Ord}} Z_\alpha$.

A particular example of a hierarchy is the von Neumann hierarchy $\{V_\alpha\}_{\alpha\in\mathrm{Ord}}$, where $V_0 := \varnothing$ and $V_{\alpha+1} := \mathcal{P}(V_\alpha)$ for each ordinal α.

Theorem 1.3.4 (The Lévy Reflection Theorem)
Let $\{Z_\alpha\}_{\alpha\in\mathrm{Ord}}$ be a hierarchy and let φ be an $\mathcal{L}_\in$-formula. Then, for all ordinals α, there exists an ordinal $\theta > \alpha$ such that φ is absolute between Z_θ and Z.

Proof. Let Φ be the collection of all subformulas of φ. Note that Φ is a finite set. Define

$$\theta_0 := \alpha + 1.$$

Now, for $i < \omega$, for a formula $\psi(y, x_1, \ldots, x_n) \in \Phi$, and for $\bar{p} = (p_1, \ldots, p_n) \in Z^n$, define

$$o(\psi, \bar{p}) := \min(\{\alpha \in \mathrm{Ord} : \text{there exists } z \in Z_\alpha \text{ such that } Z \models \psi(z, p_1, \ldots, p_n)\}$$

with the convention $\min \varnothing := 0$. Then define

$$o(\bar{p}) := \max_{\psi\in\Phi} o(\psi, \bar{p}).$$

With this, we can define

$$\theta_{i+1} := \max\left\{\theta_i + 1,\ \sup\left\{o(\bar{p}) \,:\, \bar{p} \in \bigcup_{k<\omega} Z^k_{\theta_i}\right\}\right\}.$$

Then, defining $\theta := \sup_{i<\omega} \theta_i$, the Tarski–Vaught test implies that φ is absolute between Z_θ and Z. $\square$

We will use this to show that ZFC comes remarkably close to proving its own consistency. In fact, we will come remarkably close to getting a countable transitive model of ZFC.

Theorem 1.3.5
Let $T \subsetneq$ ZFC be a finite collection of axioms of ZFC. Then

$$\mathsf{ZFC} \vdash \text{“there exists a countable transitive } \mathcal{L}_\in\text{-structure } \tilde{M} \text{ with } \tilde{M} \models T\text{”}.$$

Proof. Without loss of generality, we may assume that T includes the axiom of extensionality. As T is finite, we can create the $\mathcal{L}_\in$-sentence $\varphi := \bigwedge_{\psi\in T} \psi$. The Lévy reflection theorem (Theorem 1.3.4) yields an ordinal α such that φ is absolute for V_α. As φ is a conjunction of axioms of ZFC, and our metatheory is ZFC, we get that $V_\alpha \models \varphi$. In particular, V_α is a model of T.

Now, we can use the (downwards) Löwenheim–Skolem theorem to obtain a countable elementary substructure M of V_α. Let us spell out the details for completeness.

Suppose that $\bar{p} \in V_\alpha^n$ and $\psi(y, x_1, \ldots, x_n)$ is an $\mathcal{L}_\in$-formula. If $V_\alpha \models \exists y.\psi(y, \bar{p})$, then choose $w(\psi, \bar{p}) \in V_\alpha$ to be such that

$$V_\alpha \models \psi(w(\psi, \bar{p}), \bar{p}).$$

If $V_\alpha \models \neg\exists y.\psi(y, \bar{p})$, then we simply let $w(\psi, \bar{p}) := \varnothing$. In either case, $w(\psi, \bar{p}) \in V_\alpha$.

With these, inductively construct

- $M_0 := \varnothing$,
- $M_{i+1} := \{\, w(\psi, \bar{p}) \,:\, \psi(y, x_1, \ldots, x_n) \text{ is an } \mathcal{L}_\in\text{-formula, } \bar{p} \in M_i^n, \text{ and } n < \omega \,\}$,
- $M := \bigcup_{i<\omega} M_i$.

Then M is countable, by construction, and M is an elementary substructure of V_α, by the Tarski–Vaught test (Lemma 1.3.2). So we have obtained a countable model M of T.

We then perform Mostowski collapse on M to obtain a transitive model $\tilde{M}$ which is $\mathcal{L}_\in$-isomorphic to M. This $\tilde{M}$ is a countable transitive model of T. $\square$

In particular, for any finite $T \subseteq$ ZFC, we have that ZFC $\vdash \mathrm{Con}(T)$, where $\mathrm{Con}(T)$ is the assertion that the theory T is consistent.

Be careful! The above does *not* say that

$$\mathsf{ZFC} \vdash \text{“for every finite } T \subseteq \mathsf{ZFC}\text{, we have } \mathrm{Con}(T)\text{”}.$$

This would immediately imply that ZFC $\vdash$ Con(ZFC), contradicting Gödel's second incompleteness theorem.

We would *really* like to run the argument of the theorem above with T being all the infinitely-many axioms of ZFC. But the difficulty in trying using the Lévy reflection theorem for this case is that $\bigwedge_{\psi\in T} \psi$ will not be an $\mathcal{L}_\in$-formula if T is not finite.

In fact, not only does the argument not run through if we replaced T with all of ZFC, Gödel's incompleteness theorem outright destroys any hope of doing so!

1.4 ... But It Is the Light of an Oncoming Train

Recall that our baseline assumptions in the metatheory is ZFC together with the assumption that ZFC is consistent. From that, the hope was to get a countable *transitive* model of ZFC.

Proposition 1.4.1
Suppose that ZFC *is consistent. Then*

$$\mathsf{ZFC} + \mathrm{Con}(\mathsf{ZFC}) \nvdash \text{“there exists a transitive model of } \mathsf{ZFC}\text{”}.$$

Proof. The formula $\mathrm{Con}(T)$, asserting the consistency of a theory T, is a Δ_0 formula and is thus absolute for any transitive model. So if M is a transitive model of ZFC, then $M \models \mathsf{ZFC} + \mathrm{Con}(\mathsf{ZFC})$.

So, if

$$\mathsf{ZFC} + \mathrm{Con}(\mathsf{ZFC}) \vdash \text{“there exists a transitive model of } \mathsf{ZFC}\text{”},$$

then

$$\mathsf{ZFC} + \mathrm{Con}(\mathsf{ZFC}) \vdash \mathrm{Con}(\mathsf{ZFC} + \mathrm{Con}(\mathsf{ZFC})),$$

contradicting Gödel's second incompleteness theorem. □

Bugger.

1.5 ... But We Are a Bigger Train

But, again, who's going to stop us? We could perform the arguments of Section 1.3 as a Platonist and obtain a countable transitive model of ZFC. While an infinite conjunction of formulas is not a formula, making the Lévy reflection argument not follow through, we all know what an infinite conjunction of formulas means. This won't be an argument in ZFC, so the assertion of the existence of such a model will be an additional assumption in our metatheory.

Strictly speaking, we do not need to do this. We can instead perform all the arguments in the metatheory as follows.

Theorem
If ZFC *is consistent, then* $\mathsf{ZFC} + \neg\mathsf{CH}$ *is consistent.*

Proof. Suppose $\mathsf{ZFC} \vdash \mathsf{CH}$. Let T be the (finite) set of all ZFC axioms which appears in such a proof. Then there exists a countable transitive model M of T. [$\mathfrak{Witchcraft}$]. We thus obtain a model N of $T + \neg\mathsf{CH}$. □

This lets us keep ZFC as our metatheory. But being able to say “Let M be a countable transitive model of ZFC” is a lot more convenient than saying “Let M be a countable transitive model of a large enough finite fragment of ZFC for which we are supposing that we can prove CH from”.

2 Generic Extensions

2.1 If I Had a Penny for Every Time I Saw the Definition of a Filter...

... I'd have six pennies at the time of writing. But that still feels like a large number.

Definition 2.1.1
A forcing notion / forcing poset is a partially ordered set $(\mathbb{P}, \preceq)$ *which has a maximum element, denoted* $\mathbb{1}_{\mathbb{P}}$. *When the context is clear, we simply write* $\mathbb{1}$ *for* $\mathbb{1}_{\mathbb{P}}$.

Elements of a forcing notion $\mathbb{P}$ *are called conditions. Let* $p, q \in \mathbb{P}$. *We say that p is stronger than q / q is weaker than p if* $p \preceq q$. *We say that* p *and* q *are compatible if there exists* $r \in \mathbb{P}$ *such that* $p \succeq r \preceq q$. *We say* p *and* q *are incompatible*[3], *and write* $p \perp q$, *if* p *and* q *are not compatible.*

As a side remark, it is rather annoying to note that set theorists around the world are split on using forcing notions with a *maximum* element versus a *minimum* element, developing the theory with inequalities all pointing in opposite directions. They are, of course, dual to the other, with no real advantage of one over the other.

Definition 2.1.2
Let $(\mathbb{P}, \preceq)$ *be a forcing notion.*

- *An antichain*[4] *in* $\mathbb{P}$ *is a subset* $A \subseteq \mathbb{P}$ *such that*

  $$p \text{ and } q \text{ are incompatible, for any } p, q \in \mathbb{P}.$$

- *A set* $D \subseteq \mathbb{P}$ *is said to be dense in* $\mathbb{P}$ *if for all* $p \in \mathbb{P}$ *there exists* $d \in D$ *such that* $d \preceq p$.

- *For* $p \in \mathbb{P}$, *we say that* D *is dense below p if for all* $q \preceq p$ *there exists* $d \in D$ *such that* $d \preceq q$.

- *A filter on* $\mathbb{P}$ *is a subset* $F \subseteq \mathbb{P}$ *such that all of the following three properties hold:*
  - *for all* $p, q \in F$ *there exists* $r \in F$ *such that* $p \succeq r \preceq q$;
  - *for all* $p \in F$ *there exists* $q \in F$ *such that* $q \succeq p$;
  - $\mathbb{1}_{\mathbb{P}} \in F$.

- *Let* $\mathcal{D}$ *be a collection of dense sets in* $\mathbb{P}$. *A filter* $G \subseteq \mathbb{P}$ *is said to be* $\mathcal{D}$*-generic if for every* $D \in \mathcal{D}$, *we have*
  $$D \cap G \neq \varnothing.$$

- *If* M *is a countable transitive model of* ZFC *and* $(\mathbb{P}, \preceq) \in M$ *is a forcing notion, then we say that a filter* $G \subseteq \mathbb{P}$ *is* $\mathbb{P}$*-generic over* M *if, for every dense* $D \subseteq \mathbb{P}$ *with* $D \in M$, *we have*
  $$D \cap G \neq \varnothing.$$

Theorem 2.1.3
Let $(\mathbb{P}, \preceq)$ *be a forcing notion and let* $\mathcal{D}$ *be a countable collection of dense subsets of* $\mathbb{P}$. *Then there exists a* $\mathcal{D}$*-generic filter.*

[3]Note that this is different from the notion of incompa*r*ible elements in a partial order, where we say that p and q are incomparible if both $p \npreceq q$ and $q \npreceq p$.

[4]This is different from the usual order-theoretic definition of an antichain, which is a collection of pairwise incompa*r*ible elements.

Proof. Enumerate $\mathcal{D} = \{D_0, D_1, D_2, D_3, \dots\}$. Choose $p_0 \in D_0$, and choose $p_{n+1} \in D_{n+1}$ with $p_{n+1} \preceq p_n$. Then

$$\bigcup_{n\in\mathbb{N}} \{\, q \in \mathbb{P} : q \succeq p_n \,\}$$

is a $\mathcal{D}$-generic filter on $\mathbb{P}$. $\square$

In particular, if M is any countable transitive model of ZFC and $(\mathbb{P}, \preceq)$ is any forcing notion, then there exists a $\mathbb{P}$-generic filter G over M. Note that, most of the time, $G \notin M$.[5]

Example 2.1.4
Let X and Y be sets, and consider the poset $\mathrm{Fn}(X,Y)$ of all finite partial functions $f\colon X \to Y$ ordered by

$$f \preceq g \quad \text{if and only if} \quad f \supseteq g,$$

with the empty function as the maximum element of the forcing notion. We have that f is stronger than g if and only if f extends g. One can intuitively think of this as f giving us "more information" than g, or that f has "less[6] possible extensions" than g.

Any filter $F \subseteq \mathrm{Fn}(X,Y)$ will be a set of compatible finite partial functions from X to Y. Intuitively, each $f \in F$ gives "partial information" for the partial function $\bigcup F$ from X to Y.

For $x \in X$, let

$$D_x \coloneqq \{\, f \in \mathrm{Fn}(X,Y) : x \in \mathrm{dom}(f) \,\}.$$

Note that each D_x is dense in $\mathrm{Fn}(X,Y)$, so the collection $\mathcal{D} \coloneqq \{\, D_x : x \in X \,\}$ is a collection of dense subsets of $\mathrm{Fn}(X,Y)$. Then, for any $\mathcal{D}$-generic filter G, we see that $\bigcup G$ is a total function from X to Y, i.e. $\mathrm{dom}(\bigcup G) = X$ (though we may not necessarily have $\mathrm{ran}(\bigcup G) = Y$). $\square$

Definition 2.1.5
The Cohen forcing notion is the poset $\mathbb{C} \coloneqq \mathrm{Fn}(\omega, 2)$ ordered by

$$f \preceq g \quad \textit{if and only if} \quad f \supseteq g.$$

Let M be a countable transitive model of ZFC. *Let G be a $\mathbb{C}$-generic filter over M. We call the function $\bigcup G$ a Cohen real.*

If there is ever any evidence of the disconnect of set theory from the rest of mathematics, it is the use of the symbol $\mathbb{C}$ for the Cohen forcing notion.

Note that a Cohen real is a total function from ω to 2. Furthermore, if M is a countable transitive model of ZFC and G is a $\mathbb{C}$-generic filter over M, then

$$\bigcup G \notin M.$$

Indeed, for any function $f\colon \omega \to 2$, define the set

$$N_f \coloneqq \{\, p \in \mathbb{C} : p(n) \neq f(n) \text{ for some } n \in \mathrm{dom}(p) \,\},$$

which is dense in $\mathbb{C}$. Then, as

$$G \cap N_f \neq \varnothing, \quad \text{for all } f \in 2^\omega \cap M,$$

we conclude that $\bigcup G \notin M$. So Cohen reals can be viewed as real numbers "outside" some fixed countable transitive model of ZFC.

More generally for cardinal κ, let $\mathrm{Add}(\omega, \kappa)$ be the set of all finite functions $\omega \times \kappa \to 2$, ordered under reverse inclusion. Note that we can identify $\mathbb{C}$ with $\mathrm{Add}(\omega, 1)$. If M is a countable transitive model of ZFC and G is an $\mathrm{Add}(\omega, \kappa^M)$-generic filter over M, then $\bigcup G$ can be identified with an injection from κ^M into $\mathcal{P}(\omega)$.

[5] A sufficient condition for $G \notin M$ is when $(\mathbb{P}, \preceq)$ is separative, i.e. for any $p \in \mathbb{P}$ there exist $q, r \preceq p$ such that $q \perp r$.

[6] *"Fewer."* — Stannis Baratheon.

2.2 O M[G]

For this subsection, fix a countable transitive model M of ZFC, fix a forcing notion $(\mathbb{P}, \preceq) \in M$, and fix a filter G which is $\mathbb{P}$-generic over M.

We already remarked that, in many casees, $G \notin M$. We are going to build a "smallest" (in some appropriate sense) model of ZFC which extends M and contains G. In effect, we will have "adjoined"[7] G to M.

Definition 2.2.1
A set τ is called a <u>$\mathbb{P}$-name</u> *if every $z \in \tau$ is of the form $z = (\sigma, p)$ for some $\mathbb{P}$-name σ and some condition $p \in \mathbb{P}$.*

This definition appeals to the well-foundedness of the relation $\in$; the empty set is a $\mathbb{P}$-name. We can build the class $\mathrm{Name}^{\mathbb{P}}$ of $\mathbb{P}$-names from the bottom-up as follows:

$$\begin{aligned} \mathrm{Name}^{\mathbb{P}}_0 &:= \varnothing, \\ \mathrm{Name}^{\mathbb{P}}_{\alpha+1} &:= \mathcal{P}(\mathrm{Name}^{\mathbb{P}}_\alpha \times \mathbb{P}), && \text{for all ordinals } \alpha, \\ \mathrm{Name}^{\mathbb{P}}_\lambda &:= \bigcup_{\alpha<\lambda} \mathrm{Name}^{\mathbb{P}}_\alpha, && \text{for all limit ordinals } \lambda, \\ \mathrm{Name}^{\mathbb{P}} &:= \bigcup_{\alpha \in \mathrm{Ord}} \mathrm{Name}^{\mathbb{P}}_\alpha. \end{aligned}$$

The class $\mathrm{Name}^{\mathbb{P}}$ may also be denoted by $V^{\mathbb{P}}$ in the literature, to denote that these are the $\mathbb{P}$-names in V.

> *"To force is to name names"*
>
> — Karagila (2023, Section 2.1).

Note that the formula "x is a $\mathbb{P}$-name" is absolute for transitive models. So

$$M^{\mathbb{P}} := \{\, \tau \in M : M \models \text{“}\tau \text{ is a } \mathbb{P}\text{-name”} \,\} = \mathrm{Name}^{\mathbb{P}} \cap M.$$

Definition 2.2.2
Let τ be a $\mathbb{P}$-name. We define the <u>interpretation of τ by G</u> *to be*

$$\mathrm{val}(\tau, G) := \{\, \mathrm{val}(\sigma, G) \;:\; (\sigma, p) \in \tau \textit{ for some } p \in G \,\}.$$

Again, this is a definition by recursion; $\mathrm{val}(\varnothing, G) = \varnothing$.

If N is a transitive model of ZFC with $\tau, G \in N$, then the formula "$z = \mathrm{val}(\tau, G)$" is absolute for N.

Intuitively, a $\mathbb{P}$-name $\tau \in M^{\mathbb{P}}$ can be thought of as "instructions" in M for creating a new set $\mathrm{val}(\tau, G)$. The $\mathbb{P}$-generic filter G can then be thought of as a machine that actually creates the set $\mathrm{val}(\tau, G)$ when it reads the instruction τ. This $\mathrm{val}(\tau, G)$ is a set which may or may not live inside M. But even if $\mathrm{val}(\tau, G) \notin M$, the model M still has "some idea" of what $\mathrm{val}(\tau, G)$ is. After all, $\mathrm{val}(\tau, G)$ was built from τ which lives in M.

Suppose we have $\mathbb{P}$-names τ and σ and a condition $p \in \mathbb{P}$ with $(\sigma, p) \in \tau$. The "machine" G creates new sets

$$\mathrm{val}(\sigma, G) \quad \text{and} \quad \mathrm{val}(\tau, G).$$

When do we have $\mathrm{val}(\sigma, G) \in \mathrm{val}(\tau, G)$? By definition, this holds if G declares p as "being large", i.e. if $p \in G$. So, intuitively, the closer p is to $\mathbb{1}_{\mathbb{P}}$ (that is, the higher up p is in $\mathbb{P}$), the "more

[7] I wrote adjoin*ed* and not adjoin*t*. The category-pilled can despair and the category-phobic can breathe a sigh of relief.

likely" it is that $\mathrm{val}(\sigma, G) \in \mathrm{val}(\tau, G)$. Indeed, if $p = \mathbb{1}_{\mathbb{P}}$, then we *must* have $\mathrm{val}(\sigma, G) \in \mathrm{val}(\tau, G)$, simply because $\mathbb{1}_{\mathbb{P}} \in G$ by definition of G being a filter.

The remarkable thing is that this machine $\tau \mapsto \mathrm{val}(\tau, G)$ will, at the very least, produce an entire copy of M! Indeed for any $x \in M$, define the canonical $\mathbb{P}$-name for x to be

$$\check{x} := \{ (\check{y}, \mathbb{1}_{\mathbb{P}}) : y \in x \} \in M^{\mathbb{P}}.$$

Everything is defined by recursion on $\in$, in case you haven't noticed; $\check{\varnothing} = \varnothing$, and so $\mathrm{val}(\check{\varnothing}, G) = \varnothing$. Then, by $\in$-induction, we have

$$\mathrm{val}(\check{x}, G) = x \quad \text{for all } x \in M.$$

But our machine will also be able to create G! Let the canonical $\mathbb{P}$-name for G be

$$\Gamma := \{ (\check{p}, p) : p \in \mathbb{P} \},$$

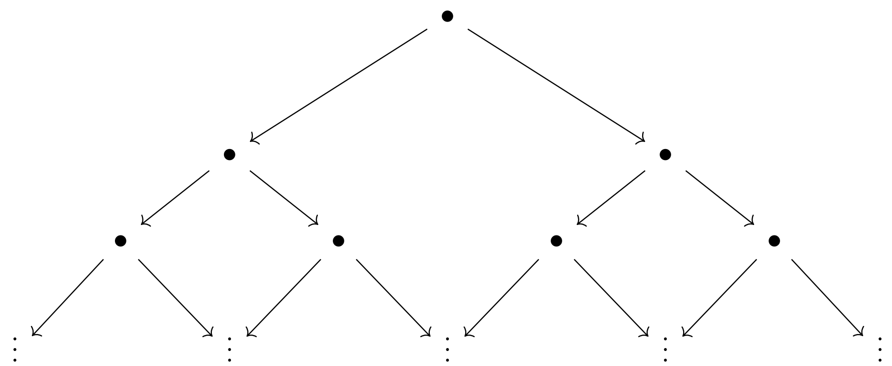
where $\check{p}$ is the canonical name for $p \in \mathbb{P}$. Then $\Gamma \in M^{\mathbb{P}}$ and $\mathrm{val}(\Gamma, G) = G$.

This is particularly curious. The model M essentially contains a bunch of instructions (the $\mathbb{P}$-names in M) for producing a bigger version of itself! Even if the countable transitive model M does not contain G, the model M can still get glimpses of G through its canonical $\mathbb{P}$-name Γ, and the same can be said for other sets $\mathrm{val}(\tau, G)$ which do not live in M. I like to picture a person sitting in their house and staring at a blueprint of a house extension.

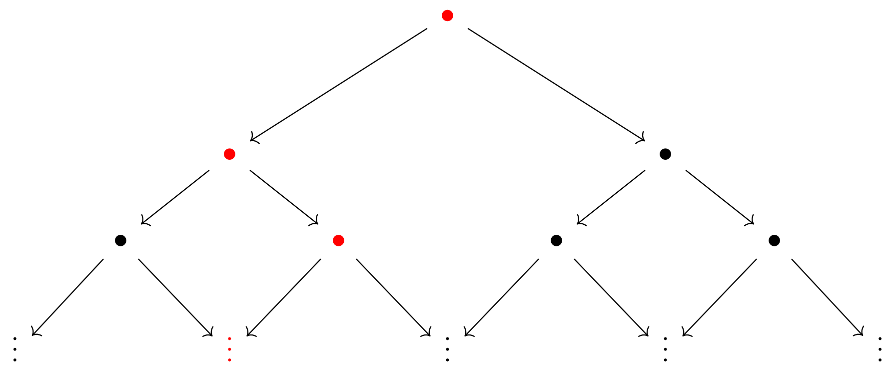
Definition 2.2.3
The generic extension of M by G is

$$M[G] := \{ \mathrm{val}(\tau, G) \,:\, \tau \in M^{\mathbb{P}} \}.$$

Let us recall the Cohen forcing notion $\mathbb{C} = \mathrm{Fn}(\omega, 2)$ as an example. We can view $\mathbb{C}$ as a collection of finite fragments of an infinite binary tree



A filter on $\mathbb{C}$ will give rise to an infinite branch in this tree. A filter G which is $\mathbb{C}$-generic over M will give rise to a function $\bigcup G \notin M$, which can be viewed as an infinite branch in the tree which is not in M.



The generic extension $M[G]$ will then satisfy $G \in M[G]$, and so (once we have shown that $M[G]$ is transitive and satisfies the axiom of union) we have $\bigcup G \in M[G]$. This "adjoins" the Cohen real $\bigcup G$ to M in a very "controlled" way.

As strongly hinted by the narration so far, this $M[G]$ is the model of ZFC we are looking for.

2.3 We Love Black Boxes

Fix a countable transitive model M of ZFC and a forcing notion $(\mathbb{P}, \preceq) \in M$.

Definition 2.3.1 (The Semantic Forcing Relation)
Let $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}_\in$-formula, let $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$ be $\mathbb{P}$-names, and let $p \in \mathbb{P}$. We say that p forces $\varphi(\tau_1, \ldots, \tau_n)$, and write $p \Vdash_{M,\mathbb{P}} \varphi(\tau_1, \ldots, \tau_n)$, if and only if

$$M[G] \models \varphi\big(\operatorname{val}(\tau_1, G),\ \ldots,\ \operatorname{val}(\tau_n, G)\big) \quad \text{for all } \mathbb{P}\text{-generic filters } G \text{ over } M \text{ with } p \in G.$$

When the context is clear, we often omit the subscript "$M, \mathbb{P}$" from the forcing relation and simply write $p \Vdash \varphi(\tau_1, \ldots, \tau_n)$.

Lemma 2.3.2
Let $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}_\in$-formula, let $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$ be $\mathbb{P}$-names, and let $p, q \in \mathbb{P}$ be such that $p \preceq q$ and $q \Vdash \varphi(\tau_1, \ldots, \tau_n)$. Then $p \Vdash \varphi(\tau_1, \ldots, \tau_n)$.

Proof. By definition of $\Vdash$, recalling that G is a filter. □

The following two theorems are at the very heart of the theory of forcing.

Theorem 2.3.3 (The Definability Theorem)
The relation "$p \Vdash_{M,\mathbb{P}} \varphi$" is absolute for all transitive models containing M.

Theorem 2.3.4 (The Forcing Theorem)
Let G be a $\mathbb{P}$-generic filter over M, let $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}_\in$-formula, and let $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$ be $\mathbb{P}$-names. Then the following are equivalent:

- $M[G] \models \varphi\big(\operatorname{val}(\tau_1, G),\ \ldots,\ \operatorname{val}(\tau_n, G)\big)$;
- *there exists $p \in G$ such that $p \Vdash_{M,\mathbb{P}} \varphi(\tau_1, \ldots, \tau_n)$.*

Their proofs are rather involved, using the *syntactic* forcing relation, and we will delay doing them for a moment. In practice, we pretty much don't care about the syntactic forcing relation, which will turn out to be equivalent to the semantic one anyway. If you feeling liberal enough, you are free to accept the above two theorems as black boxes and not read Section 2.5 later.

The following is quite a crucial result.

Lemma 2.3.5 (The Mixing Lemma / The Existential Completeness Lemma)
Let $\varphi(x, x_1, \ldots x_n)$ be an $\mathcal{L}_\in$-formula, let $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$ be $\mathbb{P}$-names, and let $p \in \mathbb{P}$. Then the following are equivalent:

- $p \Vdash \exists x. \varphi(x, \tau_1, \ldots, \tau_n)$
- *there exists $\tau \in M^{\mathbb{P}}$ such that $p \Vdash \varphi(\tau, \tau_1, \ldots, \tau_n)$.*

Proof. Suppose that ... □

2.4 Proving the Axioms

Fix a countable transitive model M of ZFC, a forcing notion $(\mathbb{P}, \preceq) \in M$, and a filter G which is $\mathbb{P}$-generic over M.

Theorem 2.4.1
All of the following hold:

- $M \subseteq M[G]$*;*
- $G \in M[G]$*;*
- $M \subseteq M[G] \subseteq N$ *for any transitive model* N *of* ZFC *with* $M \subseteq N$ *and* $G \in N$*;*
- $\mathrm{Ord} \cap M = \mathrm{Ord} \cap M[G]$*;*
- $M[G]$ *is a countable transitive model of* ZFC*;*

Proof. That $M \subseteq M[G]$ follows from defining $\check{x} \coloneqq \{ (\check{y}, \mathbb{1}_{\mathbb{P}}) : y \in x \}$ and showing that, for all $x \in M$,

$$\check{x} \in M^{\mathbb{P}} \quad \text{and} \quad \mathrm{val}(\check{x}, G) = x.$$

That $G \in M[G]$ follows from defining the $\mathbb{P}$-name $\Gamma \coloneqq \{ (\check{p}, p) : p \in \mathbb{P} \}$ and showing that

$$\Gamma \in M^{\mathbb{P}} \quad \text{and} \quad \mathrm{val}(\Gamma, G) = \Gamma.$$

The set $M[G]$ is countable because M is countable. The set $M[G]$ is transitive because if $\tau \in M$ is a $\mathbb{P}$-name and

$$x \in \mathrm{val}(\tau, G) \in M[G],$$

then there exist a condition $p \in G$ and a $\mathbb{P}$-name σ (a priori not known to be in M) such that $x = \mathrm{val}(\sigma, G)$ with $(\sigma, p) \in \tau$. The transitivity of M then yields $\sigma \in M$, giving $x = \mathrm{val}(\sigma, G) \in M[G]$.

Now we show that $M[G]$ is the minimal transitive model extending M and containing G. Let N be a transitive model of ZFC with $M \subseteq N$ and $G \in N$. Then, for each $\tau \in M^{\mathbb{P}}$, we have that $\tau \in N$. So, since $G \in N$, we must also have that $\mathrm{val}(\tau, G) = (\mathrm{val}(\tau, G))^N \in N$, by the absoluteness of the formula "$z = \mathrm{val}(\tau, G)$". Thus $M[G] \subseteq N$.

Next, we show that $\mathrm{Ord} \cap M = \mathrm{Ord} \cap M[G]$. It is easy to see that $\mathrm{Ord} \cap M \subseteq \mathrm{Ord} \cap M[G]$, because $M \subseteq M[G]$. Now, for any $\tau \in M^{\mathbb{P}}$, we can show by induction that

$$\mathrm{rank}(\mathrm{val}(\tau, G)) \leq \mathrm{rank}(\tau).$$

Hence $\mathrm{Ord} \cap M[G] \subseteq \mathrm{Ord} \cap M$.

Let us now begin proving that $M[G] \models \mathsf{ZFC}$.

As $M[G]$ has now been established to be transitive, we immediately get

$$M[G] \models \texttt{Extensionality} + \texttt{Foundation}.$$

Furthermore, $\varnothing, \omega \in M$. So, since $M \subseteq M[G]$, we also have

$$M[G] \models \texttt{Empty Set} + \texttt{Infinity}.$$

Now, for $\mathbb{P}$-names $\tau, \sigma \in M^{\mathbb{P}}$, we have that

$$\{ \mathrm{val}(\tau, G), \mathrm{val}(\sigma, G) \} = \mathrm{val}\left(\{ (\tau, \mathbb{1}), (\sigma, \mathbb{1}) \},\ G \right),$$

giving

$$M[G] \models \texttt{Pairing}.$$

Rather similarly, though requiring more effort, for a $\mathbb{P}$-name $\tau \in M^{\mathbb{P}}$, we have

$$\bigcup \operatorname{val}(\tau, G) = \operatorname{val}\Bigg(\Big\{(\rho, r) \in M^{\mathbb{P}} \times G \ : (\sigma, p) \in \tau,\ (\rho, q) \in \sigma,\ r \preceq p,\ \text{and } r \preceq q \\ \text{for some } \sigma, \rho \in M^{\mathbb{P}} \text{ and some } p, q \in G\Big\}, \\ G\Bigg).$$

Thus

$$M[G] \models \texttt{Union}.$$

[proof of power set]
[proof of separation]
[proof of replacement]
Let $\operatorname{val}(\tau, G) \in M[G]$. Write $\operatorname{dom}(\tau) := \{\, \sigma \in M^{\mathbb{P}} : \exists p \in \mathbb{P}.(\sigma, p) \in \tau \,\}$. Since $M \models \texttt{Choice}$, there is an injection $i\colon \operatorname{dom}(\tau) \to \alpha$ in M, for some ordinal $\alpha \in M$. In $M[G]$, define the function $i^*\colon \operatorname{val}(\tau, G) \to \alpha$ by

$$i^*(y) := \min\{\, i(\sigma) : \sigma \in \operatorname{dom}(\tau) \text{ and } y = \operatorname{val}(\sigma, G) \,\}, \quad \text{for each } y \in \operatorname{val}(\tau, G).$$

Then i^* is an injection in $M[G]$ of $\operatorname{val}(\tau, G)$ into the ordinal $\alpha \in M$. Thus

$$M[G] \models \texttt{Choice}.$$

Therefore

$$M[G] \models \mathsf{ZFC}. \qquad \square$$

2.5 Try to Not Gouge Your Eyes Out

We now have to repay the technical debt and prove the definability theorem (Theorem 2.3.3) and the forcing theorem (Theorem 2.3.4)

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