The Cantor Space and the Baire Space

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Definition (The Cantor Space)
The Cantor space is the space $2^{\mathbb{N}} = \{0,1\}^{\mathbb{N}}$ equipped with the product topology, where $2 = \{0,1\}$ is given the discrete topology. We also give $2^{\mathbb{N}}$ the measure induced by the uniform measure on $\{0,1\}$, i.e. for $A_1, A_2, A_3, \ldots \subseteq \{0,1\}$ with $A_n = \{0,1\}$ for all but finitely many n, we declare the set $\prod_n A_n$ to have measure $\prod_n \frac{|A_n|}{2}$.

Definition (The Baire Space)
The Baire space is the space $\mathbb{N}^{\mathbb{N}}$ with the product topology, with the discrete topology on $\mathbb{N}$.

The Cantor space $2^{\mathbb{N}}$ and the Baire space $\mathbb{N}^{\mathbb{N}}$ are to be thought of as alternative representations of the real line $\mathbb{R}$. They are not isomorphic to $\mathbb{R}$, perhaps except in the category Set, but they are "close enough" to being isomorphic. More specifically:

- The space $2^{\mathbb{N}}$ is homeomorphic to the usual Cantor set in $\mathbb{R}$
- The space $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers.
- The measure given on $2^{\mathbb{N}}$ is equivalent to the Lebesgue measure on the closed interval $[0,1]$ where we write each number $x \in [0,1]$ as its binary expansion.

The fact that $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$ have "similar" characteristics to $\mathbb{R}$ allows us to answer questions about $\mathbb{R}$ by instead answering them in $2^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$.

We are mainly interested in cardinal characteristics of the continuum, and the inequalities between them which are provable in ZFC set theory. The first four cardinals we are interested in are $\mathrm{non}(\mathcal{M})$, $\mathrm{cov}(\mathcal{M})$, $\mathrm{non}(\mathcal{L})$, and $\mathrm{cov}(\mathcal{L})$.

Definition (The Uniformity and Covering Numbers of Meagre Sets and Lebesgue Null Sets)
The cardinal $\mathrm{non}(\mathcal{M})$ is the minimum cardinality of any subset of $\mathbb{R}$ which is not meagre. The cardinal $\mathrm{cov}(\mathcal{M})$ is the minimum cardinality of any family of meagre subsets of $\mathbb{R}$ which cover all of $\mathbb{R}$.
The cardinals $\mathrm{non}(\mathcal{L})$ and $\mathrm{cov}(\mathcal{L})$ are defined similarly with "Lebesgue measure zero" in place of "meagre" above.

It is clear that the four cardinals $\mathrm{non}(\mathcal{M})$, $\mathrm{cov}(\mathcal{M})$, $\mathrm{non}(\mathcal{L})$, and $\mathrm{cov}(\mathcal{L})$ lie weakly between $\aleph_1$ and $2^{\aleph_0}$. We now want to show that

$$\mathrm{cov}(\mathcal{L}) \leq \mathrm{non}(\mathcal{M}) \quad \text{and} \quad \mathrm{cov}(\mathcal{M}) \leq \mathrm{non}(\mathcal{L}).$$

To do this, we establish a theorem showing how $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$ behave as an alternative representation for $\mathbb{R}$.

Theorem

The values of the cardinals $\mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M})$ do not change when we replace $\mathbb{R}$ with $[0,1]$, $2^{\mathbb{N}}$, or $\mathbb{N}^{\mathbb{N}}$ in the previous definition.

The values of the cardinals $\mathrm{non}(\mathcal{L})$ and $\mathrm{cov}(\mathcal{L})$ do not change when we replace $\mathbb{R}$ with $[0,1]$ or $2^{\mathbb{N}}$ in the previous definition.

Proof (Sketch)

There exist a null-set-preserving homeomorphism from $\mathbb{R}$ to $(0,1)$, a null-set-preserving homeomorphism from $[0,1]$ to $2^{\mathbb{N}} \setminus \{\text{sequences which end in an infinite string of 1's}\}$, and a homeomorphism from $(0,1) \setminus \mathbb{Q}$ to $\mathbb{N}^{\mathbb{N}}$. So we have "isomorphisms" between $\mathbb{R}$, $[0,1]$, $2^{\mathbb{N}}$, and $\mathbb{N}^{\mathbb{N}}$, modulo a countable set. Since countable sets are Lebesgue null and meagre, we conclude that the values of $\mathrm{non}(\mathcal{M})$, $\mathrm{cov}(\mathcal{M})$, $\mathrm{non}(\mathcal{L})$, and $\mathrm{cov}(\mathcal{L})$ do not change when we replace $\mathbb{R}$ with $[0,1]$, $2^{\mathbb{N}}$, or $\mathbb{N}^{\mathbb{N}}$. □

Now we establish the desired inequalities $\mathrm{cov}(\mathcal{L}) \leq \mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M}) \leq \mathrm{non}(\mathcal{L})$.

Theorem (Rothberger)

$\mathrm{cov}(\mathcal{L}) \leq \mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M}) \leq \mathrm{non}(\mathcal{L})$.

Proof

First, we show that $\mathbb{R}$ can be decomposed into a Lebesgue null set and a meagre set.

Lemma

There exist $A, B \subseteq \mathbb{R}$ with $A \cup B = \mathbb{R}$, and A is Lebesgue null and B is meagre.

Proof

Let $q_1, q_2, q_3, \ldots$ enumerate the rationals. Then

$$A := \limsup_{n \to \infty} \left(q_n - \frac{1}{2^n}, q_n + \frac{1}{2^n}\right) \quad \text{and} \quad B := \mathbb{R} \setminus A$$

works. □

Returning to the proof of the theorem, we use the lemma to decompose $2^{\mathbb{N}}$ into a measure-zero set A and a meagre set B. Then for any non-meagre set $X \subseteq 2^{\mathbb{N}}$, the family $\{x + A : x \in X\}$ of measure-zero sets covers $2^{\mathbb{N}}$, where addition is understood to be addition of functions modulo 2. Indeed, if there exists some $z \in 2^{\mathbb{N}} \setminus \bigcup_{x \in X}(x+A)$, then we claim that $z + X$ is disjoint from A, so $z + X \subseteq B$, meaning X is meagre. To see this, if $z + x \in A$ then $z \in -x + A = x + A$, contrary to the assumption $z \in 2^{\mathbb{N}} \setminus \bigcup_{x \in X}(x+A)$. This, together with the previous theorem, yields $\mathrm{cov}(\mathcal{L}) \leq \mathrm{non}(\mathcal{M})$. Similarly, $\mathrm{cov}(\mathcal{M}) \leq \mathrm{non}(\mathcal{L})$. $\square$

The proof above showed how it was convenient to work in $2^{\mathbb{N}}$ instead of $\mathbb{R}$, as we were able to say that $x = -x$ for $x \in 2^{\mathbb{N}}$.

$$\begin{tikzcd}
 & \mathrm{cov}(\mathcal{L}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[dr] & \\
\aleph_1 \arrow[ur] \arrow[dr] & & & 2^{\aleph_0} \\
 & \mathrm{cov}(\mathcal{M}) \arrow[r] & \mathrm{non}(\mathcal{L}) \arrow[ur] &
\end{tikzcd}$$

The diagram above shows the inequalities established so far, where an arrow $\kappa \to \lambda$ represents the inequality $\kappa \leq \lambda$ in ZFC.

Definition (The Unbounding Number)
For funtions $f, g : \mathbb{N} \to \mathbb{N}$, we write $f \leq^* g$ if and only if there exists some $N \in \mathbb{N}$ such that $f(n) \leq g(n)$ for all $n \geq N$.
The cardinal $\mathfrak{b}$ is the minimum cardinality of any family $\mathcal{B} \subseteq \mathbb{N}^{\mathbb{N}}$ for which there does not exist any $g \in \mathbb{N}^{\mathbb{N}}$ such that $f \leq^* g$ for all $f \in \mathcal{B}$.

This cardinal $\mathfrak{b}$ is defined from $\mathbb{N}^{\mathbb{N}}$. We will use the fact that $\mathrm{non}(\mathcal{M})$ can also be defined on $\mathbb{N}^{\mathbb{N}}$ (instead of $\mathbb{R}$) to establish $\mathfrak{b} \leq \mathrm{non}(\mathcal{M})$.

Theorem
$\mathfrak{b} \leq \mathrm{non}(\mathcal{M})$.

Proof
For $f \in \mathbb{N}^{\mathbb{N}}$, let $C^*_f := \{g \in \mathbb{N}^{\mathbb{N}} : g \leq^* f\}$. Then for $f, g \in \mathbb{N}^{\mathbb{N}}$, we have

$$f \leq^* g \quad \text{if and only if} \quad C^*_f \subseteq C^*_g.$$

We claim that, for any $f \in \mathbb{N}^{\mathbb{N}}$, the set C_f^* is meagre in $\mathbb{N}^{\mathbb{N}}$. Note that this will imply that $\mathfrak{b} \leq \mathrm{non}(\mathcal{M})$.
To see the claim, we first characterise compact sets in $\mathbb{N}^{\mathbb{N}}$.

Lemma

Let $K \subseteq \mathbb{N}^{\mathbb{N}}$. Then K is compact if and only if K is closed and K is contained in an infinite product of finite subsets of $\mathbb{N}$.

Proof

If K is compact then K is closed since $\mathbb{N}^{\mathbb{N}}$ is Hausdorff. Furthermore we can continuously project K onto any of its coordinates, and this image must be compact in $\mathbb{N}$. So K is contained in an infinite product of finite subsets of $\mathbb{N}$.
The converse follows from the product of infinitely many compact sets also being compact. $\square$

In particular, compact sets in $\mathbb{N}^{\mathbb{N}}$ are nowhere dense. Now, for $f \in \mathbb{N}^{\mathbb{N}}$, we have

$$C_f^* = \bigcup_{N \in \mathbb{N}} \bigcup_{M \in \mathbb{N}} \{ g \in \mathbb{N}^{\mathbb{N}} : \ g(n) \leq M \text{ for } n \leq N, \text{ and } g(n) \leq f(n) \text{ for } n > N \}.$$

So C_f^* is a countable union of compact sets, meaning C_f^* is meagre. $\square$

$$\begin{tikzcd}
 & \mathrm{cov}(\mathcal{L}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[dr] & \\
\aleph_1 \arrow[ur] \arrow[r] \arrow[dr] & \mathfrak{b} \arrow[ur] & & 2^{\aleph_0} \\
 & \mathrm{cov}(\mathcal{M}) \arrow[r] & \mathrm{non}(\mathcal{L}) \arrow[ur] &
\end{tikzcd}$$

We can also use $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$ in consistency arguments for questions originating from $\mathbb{R}$. For example, in Paul Cohen's original model for the negation of the continuum hypothesis, he appends strictly more than $\aleph_1$-many (in the model) functions $\mathbb{N} \to 2$ to the model. Furthermore, by considering the topology on $2^{\mathbb{N}}$, and since $\mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M})$ are unchanged when considering them defined on $2^{\mathbb{N}}$, one can show, using the Cohen model, that

$$\aleph_1 = \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}$$

is consistent with ZFC.
As another example, the Laver model identifies $\mathbb{R}$ with $\mathbb{N}^{\mathbb{N}}$ to show that

$$\aleph_1 = \mathrm{cov}(\mathcal{L}) = \mathrm{non}(\mathcal{L}) < \mathfrak{b} = \aleph_2 = 2^{\aleph_0}$$

is consistent with ZFC.

Viewing $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$ as alternative representations of $\mathbb{R}$ is also allows us to see particular subsets of $\mathbb{R}$ from a different point of view.

$\boxed{\text{Proposition}}$ (Lusin-Souslin)
Let X be a Polish space, and let $A \subseteq X$ be Borel. Then there exists a closed set $F \subseteq \mathbb{N}^{\mathbb{N}}$ and a continuous bijection $f : F \to A$.

$\boxed{\text{Proposition}}$
Let X be a non-empty perfect Polish space. Then there exists a topological embedding of $2^{\mathbb{N}}$ into X.

$\boxed{\text{Proposition}}$
If X is an uncountable Polish space, then X is Borel isomorphic to $2^{\mathbb{N}}$.