Just How Many Real Numbers Are There?

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Contents

0 The Opposite of an Advertisement

For the experienced set theorists and logicians who just want to see what this document is about and click away afterwards, everything in here builds up to the following theorem.

Theorem
Suppose that ZFC *is consistent and let* κ *be any cardinal of uncountable cofinality. Then* ZFC + "$2^{\aleph_0} = \kappa$" *is also consistent.*

These notes are mainly based on lectures for the Part III Forcing and the Continuum Hypothesis course by Benedikt Löwe at the University of Cambridge in 2025. There are prettier sets of notes of these lectures, typed up by Sky Wilshaw and available at

https://zeramorphic.uk/gh/maths-compiled/iii/forcing/build/main.pdf

for a rendition of the course lectured by Richard Matthews which ran in the academic year 2023–2024, and typed up by Daniel Naylor and available at

https://danielnaylor.uk/notes/III/Lent/FC/FC.pdf

for a rendition of the course lectured by Benedikt Löwe which ran in the academic year 2024–2025.

There are also plenty of well-written texts to learn the material present in this piece. Kunen (1980, Chapters IV–VII) is perhaps "the" book to learn the material from and is about as formal as one can get in terms of the presentation of the arguments. Halbeisen (2017, Chapter 15 and Chapter 16) and Weaver (2014, Chapters 6–13) are gentler introductions. Shelah (1982, Chapter I) is another very complete introduction to forcing. Džamonja (2020, Chapters 6–9) covers the "must-know" material about forcing and has many (deliberate) black boxes. Karagila (2017, Section 8) and Karagila (2023, Sections 1–3) are my personal favourite. They are quick and to the point. People who prefer watching YouTube videos can try having a look at Hart and Löwe (2021). Jech (2003, Chapters 12–14) is perhaps better used as a reference text than a place to learn the material for the first time, and uses a Boolean algebras approach to forcing as opposed to the preorders approach we will be using. Chow (2009) is useful for getting a high-level view of forcing through the Boolean algebras approach. Finally, I cannot resist mentioning Shelah and Strüngmann (2021, Section 2). Just check it out.

I have stated all the brilliant references above lest you think this is the best place to learn the material from.

1 A Countable Transitive Model of ZFC?

So you wish to prove that the continuum hypothesis (CH), the assertion that $2^{\aleph_0} = \aleph_1$, can neither be proven nor disproven from the axioms of ZFC set theory. You took a class in model theory and you learned that one can show this by showing that both ZFC + CH and ZFC + ¬CH are consistent. The typical approach is to exhibit models of ZFC + CH and ZFC + ¬CH. Let us try to do that.

1.1 Wishful Thinking

Stop. You realise that models of ZFC + CH or ZFC + ¬CH will, in particular, also be models of ZFC. Darn. The whole point of ZFC set theory was that it was supposed to be able to formalise (most of)[1] the mathematics we do in our day-to-day lives. A certain pesky Kurt Gödel prevents us from explicitly exhibiting such a model.

But who is going to stop us from simply running off with the assumption that ZFC is consistent?[2] If we do this, then we can hope to obtain a theorem and proof of the following form.

Theorem
If ZFC *is consistent, then* ZFC + CH *and* ZFC + ¬CH *are also consistent.*

Proof. Let $\mathbf{M}$ be a model of ZFC. [Black magic]. Therefore we have obtained a model $\mathbf{N}$ of ZFC+CH and a model $\mathbf{N}'$ of ZFC + ¬CH. □

In such a proof as above, $\mathbf{N}$ and $\mathbf{N}'$ will presumably be some modification of $\mathbf{M}$. But at this point, we have no information about $\mathbf{M}$, other than that it models ZFC. Ideally, we would like M to be a *countable transitive* model of ZFC, for reasons which will become apparent later. By "transitive", we mean transitive with respect to the membership relation $\in$. That is, if $x \in y \in \mathbf{M}$, then $x \in \mathbf{M}$.

To get a countable model of ZFC is fairly easy. Recall the Löwenheim–Skolem theorem, asserting that theories in first-order logic are unable to control the cardinalities of their infinite models.

Theorem 1.1.1 (The Löwenheim–Skolem Theorem)
Let T be a consistent first-order $\mathcal{L}$-theory. Suppose there exists an infinite model of T. Then for all cardinals $\kappa \geq |\mathcal{L}| + \aleph_0$, there exists a model of T of cardinality κ.

A special case of this theorem, which also arises from the proof of Gödel's completeness theorem via Henkin terms, is that if a language $\mathcal{L}$ is countable, then a consistent $\mathcal{L}$-theory T has an infinite model if and only if it has a countably infinite model.

In particular, as any model of ZFC must be infinite, there must also exist a countable model of ZFC. So how do we get transitivity?

1.2 Only a Sith Deals in Absolutes

Why do we even want to get a countable transitive model of ZFC in the first place?

We want a countable model so we are able to access things from *outside* of the model. This lets us adjoin new elements, such as real numbers, to the model, because there are uncountably many real numbers out in the metatheory. This is not unlike a field extension.

Transitive models are desirable due to the fact that they make a lot of formulas "absolute".

[1] Category theory jumpscare.

[2] If you are an amused reader from the future with the knowledge that ZFC is inconsistent, how is the climate doing? Thought so. Focus on your own problems.

Definition 1.2.1
Let $\mathbf{M}$ and $\mathbf{N}$ be $\mathcal{L}$-structures with $\mathbf{M} \subseteq \mathbf{N}$ and let $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}$-formula with n free variables $x_1, \ldots, x_n$. We say that $\varphi(x_1, \ldots, x_n)$ is absolute between $\mathbf{M}$ and $\mathbf{N}$ if, for all $a_1, \ldots, a_n \in \mathbf{M}$,

$$\mathbf{M} \models \varphi(a_1, \ldots, a_n) \text{ if and only if } \mathbf{N} \models \varphi(a_1, \ldots, a_n).$$

Replacing "if and only if" in the definition above with "if" gives us the notion of downwards absoluteness, whereas replacing "if and only if" with "only if" gives us the notion of upwards absoluteness. Formulas which are absolute between structures $\mathbf{M}$ and $\mathbf{N}$, with $\mathbf{M} \subseteq \mathbf{N}$, are great because they really do let us view $\mathbf{N}$ as a certain kind of extension of $\mathbf{M}$.

Atomic formulas are always absolute between $\mathbf{M}$ and $\mathbf{N}$ whenever $\mathbf{M}$ is a substructure of $\mathbf{N}$. This is pretty much by definition: an $\mathcal{L}$-structure $\mathbf{M}$ is a substructure of an $\mathcal{L}$-structure $\mathbf{N}$ if the domain of $\mathbf{M}$ is a subset of the domain of $\mathbf{N}$ and the inclusion function $\iota\colon \mathbf{M} \to \mathbf{N}$ is an injective homomorphism of $\mathcal{L}$-structures. Consequently, propositional connectives of atomic formulas are also always absolute between substructures.

But things become icky when we introduce quantifiers. We cannot simply "add" things to a model and expect it to preserve the truth of formulas in the original structure.

Example 1.2.2
Consider the language $\mathcal{L}_\in$ of set theory, which only consists of one relation symbol $\in$, and has no constant symbols or function symbols. So the only atomic formulas are of the form $x = y$ or $x \in y$, for variables x and y. All the "interesting" formulas are not going to be propositional connectives of atomic formulas.

Let $\mathbf{M}$ be a model of ZFC and let $\varnothing$ be the empty set in $\mathbf{M}$. We extend $\mathbf{M}$ by letting $\mathbf{N} := \mathbf{M} \cup \{*\}$ declaring $* \in^{\mathbf{N}} \varnothing$. Then

$$\mathbf{M} \models \forall z.(z \notin \varnothing),$$

but

$$\mathbf{N} \not\models \forall z.(z \notin \varnothing).$$

Thus even a very simple formula such as $\varphi_\varnothing(x) := \forall z.(z \notin x)$ asserting that x is empty, is not absolute between $\mathbf{M}$ and $\mathbf{N}$. $\square$

Of particular interest are formulas which are absolute between some model and the ambient universe in the metatheory.

Definition 1.2.3
Let $\mathbf{M}$ be an $\mathcal{L}_\in$-structure. An $\mathcal{L}_\in$-formula $\varphi(x_1, \ldots, x_n)$ is said to be absolute for $\mathbf{M}$ if, for all $a_1, \ldots, a_n \in \mathbf{M}$,

$$\mathbf{M} \models \varphi(a_1, \ldots, a_n) \text{ if and only if } \varphi(a_1, \ldots, a_n) \text{ is true.}$$

It is a bit difficult to formally state what "$\varphi(a_1, \ldots, a_n)$ is true" means, because there is no truth predicate (cf. Tarski's undefinability of truth). For many formulas of interest, one can interpret this simply "provable in ZFC" by maintaining ZFC as our metatheory. As before, we similarly have the notions of a formula being upwards absolute and downwards absolute for an $\mathcal{L}_\in$-structure $\mathbf{M}$.

Definition 1.2.4
An $\mathcal{L}_\in$-structure $\mathbf{M}$ is said to be transitive if, for all $x \in \mathbf{M}$ and $y \in x$, we have $y \in \mathbf{M}$.

Phrased differently, a transitive $\mathcal{L}_\in$-structure $\mathbf{M}$ is one such that for all $x \in \mathbf{M}$ we have $x \subseteq \mathbf{M}$.

The idea is that if $\mathbf{M}$ is a *transitive* model of ZFC, then lots of formulas are absolute for $\mathbf{M}$. Consequently, if we have a transitive models $\mathbf{M}$ and $\mathbf{N}$ of ZFC with $\mathbf{M} \subseteq \mathbf{N}$, then we can really view $\mathbf{N}$ as a particularly neat extension of $\mathbf{M}$.

Recall that Δ_0 is the smallest class of all $\mathcal{L}_\in$-formulas containing the atomic formulas and is closed under propositional connectives and bounded quantification. By bounded quantification, we mean quantifiers of the form $\forall x \in y.\varphi$ or $\exists x \in y.\varphi$.

Lemma 1.2.5
If φ is a Δ_0 formula in $\mathcal{L}_\in$, then φ is absolute for $\mathbf{M}$ for any transitive $\mathcal{L}_\in$-strucutre $\mathbf{M}$.

Proof. By induction on the complexity of φ. $\square$

This is particularly neat because lots of familiar expressions in set theory are expressible as Δ_0 formulas.

Example 1.2.6
All of the following are expressible in ZFC as Δ_0 formulas.

- $x = y$
- $x \in y$
- $x \subseteq y$
- $z = \{x\}$
- $z = \{x, y\}$
- $z = \langle x, y\rangle \coloneqq \{\{x\}, \{x, y\}\}$
- $z = \varnothing$
- $z = x \cup y$
- $z = x \cap y$
- $z = x \setminus y$
- $z = x \cup \{x\}$
- z is transitive
- $z = \bigcup x$
- z is an ordered pair
- $z = x \times y$
- z is a relation
- $z = \mathrm{dom}(R)$ and R is a relation
- $z = \mathrm{ran}(R)$ and R is a relation
- f is a function
- f is an injective function
- f is a surjective function
- f is a bijective function

- α is an ordinal
- α is a successor ordinal
- α is a limit ordinal
- $x = \omega$, where ω is the first countable ordinal
- $n \in \omega$. $\square$

We can go one step further. A Π_1 formula is a formula which is provably equivalent to a formula of the form $\forall x_1. \cdots .\forall x_n.\varphi$, where φ is a Δ_0 formula. Similarly, a formula is said to be Σ_1 if it is provably equivalent to a formula of the form $\exists x_1. \cdots .\exists x_n.\varphi$, where φ is a Δ_0 formula. A formula is said to be a Δ_1 formula if it is both a Π_1 formula and a Σ_1 formula.

Observe that Π_1 formulas are downwards absolute between any two structures $\mathbf{M}$ and $\mathbf{N}$ with $\mathbf{M} \subseteq \mathbf{N}$, whereas Σ_1 formulas are upwards absolute between any two strucutres $\mathbf{M}$ and $\mathbf{N}$ with $\mathbf{M} \subseteq \mathbf{N}$. Consequently, Δ_1 formulas are also absolute between any two structures $\mathbf{M}$ and $\mathbf{N}$ with $\mathbf{M} \subseteq \mathbf{N}$.

Example 1.2.7
All of the following are expressible in ZFC as Δ_1 formulas.

- x is a finite set
- E is a well-founded relation on X
- $\gamma = \alpha + \beta$, where α and β are ordinals
- $\gamma = \alpha \cdot \beta$, where α and β are ordinals
- $\alpha = \operatorname{rank}(x)$
- t is the transitive closure of x. $\square$

If F, G, and H are three class functions, we can define another class function R by transfinite recursion as follows:

$$\begin{aligned} R(0,\bar{x}) &\coloneqq F(\bar{x}), \\ R(\alpha+1,\bar{x}) &\coloneqq G(\alpha, R(\alpha,\bar{x}),\bar{x}), && \text{for all ordinals } \alpha, \\ R(\lambda,\bar{x}) &\coloneqq H(\lambda, \{R(\alpha,\bar{x}) : \alpha < \lambda\}, \bar{x}), && \text{for all limit ordinals } \lambda. \end{aligned}$$

Let $T \subsetneq \mathsf{ZFC}$ be a large enough finite fragment of ZFC which is sufficiently strong to prove the existence and absoluteness of all the relevant operations we will use, including F, G, and H. Then the operation R defined above by transfinite recursion is also absolute.

Furthermore, because we can encode formulas as finite strings of natural numbers, the formula "$X \models \varphi$" can also be defined by the usual Tarski recursion and is thus also absolute.

Have I convinced you that transitive $\mathcal{L}_\in$-structures are great yet? If not, then check this out. In attempting to build a countable transitive model of ZFC, simply ensuring that our structure is transitive will yield several axioms of ZFC.

Lemma 1.2.8
If $\mathbf{M}$ is a transitive $\mathcal{L}_\in$-structure, then

$$\mathbf{M} \models \texttt{Extensionality} + \texttt{Foundation}.$$

If, furthermore, for all $x, y \in \mathbf{M}$ we have $\{x, y\} \in \mathbf{M}$ and $\bigcup x \in \mathbf{M}$, then

$$\mathbf{M} \models \texttt{Extensionality} + \texttt{Foundation} + \texttt{Pairing} + \texttt{Union}.$$

Proof. Just do it. □

Woohoo. Only infinitely many more axioms to go to get a transitive model of ZFC...

We have seen that many formulas are absolute for transitive $\mathcal{L}_\in$-structures. There are, however, formulas which are *not* absolute. For instance, the following formulas are not absolute:

- $x = \mathcal{P}(y)$
- $\mathcal{F} = y^x$, that is, $\mathcal{F}$ is the set of all functions from x to y
- κ is a cardinal
- $|X| = |Y|$
- $\beta = \mathrm{cf}(\alpha)$
- α is a regular cardinal.

The non-absoluteness of these formulas will become apparent in the development of later subsections. For now, note that the formula

$$\kappa \text{ is a cardinal}$$

is *downwards* absolute for any transitive $\mathcal{L}_\in$-structure, eventhough it will *not* be *upwards* absolute.

1.3 Light at the End of the Tunnel

So we begin our mission in trying to get a countable transitive model of ZFC.

First, we recall the Tarski–Vaught test, which gives us information about $\mathcal{L}$-embeddings, which are injective $\mathcal{L}$-homomorphisms between $\mathcal{L}$-structures.

Lemma 1.3.1 (The Tarski–Vaught Test)
Let $\mathbf{M}$ and $\mathbf{N}$ be $\mathcal{L}$-structures and let $i\colon \mathbf{M} \to \mathbf{N}$ be an $\mathcal{L}$-embedding. Let Φ be a collection of $\mathcal{L}$-formulas which is closed under subformulas. Then the following are equivalent:

(1) for all $\varphi(x_1, \dots, x_k) \in \Phi$ and all $a_1, \dots, a_k \in \mathbf{M}$,

$$\mathbf{M} \models \varphi(a_1, \dots, a_k) \text{ if and only if } \mathbf{N} \models \varphi(i(a_1), \dots, i(a_k));$$

(2) for all formulas $\varphi(x, y_1, \dots, y_k) \in \Phi$ and for all $a_1, \dots, a_k \in \mathbf{M}$, if there exists $n \in \mathbf{N}$ such that

$$\mathbf{N} \models \varphi(n, i(a_1), \dots, i(a_k)),$$

then there exists $m \in \mathbf{M}$ such that

$$\mathbf{N} \models \varphi(i(m), i(a_1), \dots, i(a_k)).$$

Proof. By induction on the complexity of the formulas in Φ. □

In particular if Φ above is the class of *all* $\mathcal{L}$-formulas, then the above Tarski–Vaught test (Lemma 1.3.1) provides a characterisation for when an $\mathcal{L}$-embedding is actually an *elementary* $\mathcal{L}$-embedding.

We call property (2) in Lemma 1.3.1 the Tarski–Vaught criterion. Notice that this makes no reference to the truth of φ in $\mathbf{M}$.

Let us specialise the Tarski–Vaught test to the language $\mathcal{L}_\in$ of set theory and to the case when the embedding $i\colon \mathbf{M} \to \mathbf{N}$ is actually an inclusion of sets. The formulation of the Tarski–Vaught test which we are particularly interested in is as follows.

Lemma 1.3.2 (The Tarski–Vaught Test for $\mathcal{L}_\in$)
Let $\mathbf{M}$ and $\mathbf{N}$ be $\mathcal{L}_\in$-structures with $\mathbf{M} \subseteq \mathbf{N}$. Let Φ be a collection of $\mathcal{L}_\in$-formulas which is closed under subformulas. Then the following are equivalent.

(1) all formulas in Φ are absolute between $\mathbf{M}$ and $\mathbf{N}$;

(2) for all formulas $\varphi(x, y_1, \ldots, y_k) \in \Phi$ and for all $a_1, \ldots, a_k \in \mathbf{M}$, if there exists $n \in \mathbf{N}$ such that

$$\mathbf{N} \models \varphi(n, a_1, \ldots, a_k),$$

then there exists $m \in \mathbf{M}$ such that

$$\mathbf{N} \models \varphi(m, a_1, \ldots, a_k).$$

The Tarski–Vaught test, though very easy to prove, implies a number of really surprising results.

Definition 1.3.3
A hierarchy is a class of sets $\{Z_\alpha\}_{\alpha \in \mathrm{Ord}}$ such that:

- *each Z_α is a transitive set;*
- $\mathrm{Ord} \cap Z_\alpha = \alpha$ *for each ordinal α;*
- *if $\alpha < \beta$ then $Z_\alpha \subseteq Z_\beta$;*
- *if λ is a limit ordinal then $Z_\lambda = \bigcup_{\alpha < \lambda} Z_\alpha$.*

Given a hierarchy of sets $\{Z_\alpha\}_{\alpha \in \mathrm{Ord}}$, we can define the class $Z := \bigcup_{\alpha \in \mathrm{Ord}} Z_\alpha$.

A particular example of a hierarchy is the von Neumann hierarchy $\{\mathbf{V}_\alpha\}_{\alpha \in \mathrm{Ord}}$, where $\mathbf{V}_0 := \varnothing$ and $\mathbf{V}_{\alpha+1} := \mathcal{P}(\mathbf{V}_\alpha)$ for each ordinal α.

Theorem 1.3.4 (The Lévy Reflection Theorem)
Let $\{Z_\alpha\}_{\alpha \in \mathrm{Ord}}$ be a hierarchy and let φ be an $\mathcal{L}_\in$-formula. Then, for all ordinals α, there exists an ordinal $\theta > \alpha$ such that φ is absolute between Z_θ and Z.

Proof. Let Φ be the collection of all subformulas of φ. Note that Φ is a finite set. Define

$$\theta_0 := \alpha + 1.$$

Now, for $i < \omega$, for a formula $\psi(y, x_1, \ldots, x_n) \in \Phi$, and for $\bar{p} = (p_1, \ldots, p_n) \in Z^n$, define

$$o(\psi, \bar{p}) := \min(\{\alpha \in \mathrm{Ord} : \text{there exists } z \in Z_\alpha \text{ such that } Z \models \psi(z, p_1, \ldots, p_n)\})$$

with the convention $\min \varnothing := 0$. Then define

$$o(\bar{p}) := \max_{\psi \in \Phi} o(\psi, \bar{p}).$$

With this, we can define

$$\theta_{i+1} := \max\left\{\theta_i + 1, \sup\left\{o(\bar{p}) : \bar{p} \in \bigcup_{k<\omega} Z_{\theta_i}^k\right\}\right\}.$$

Then, defining $\theta := \sup_{i<\omega} \theta_i$, the Tarski–Vaught test implies that φ is absolute between Z_θ and Z. $\square$

We will use this to show that ZFC comes remarkably close to proving its own consistency. In fact, we will come very close to getting a countable transitive model of ZFC. But before that, let us remark on some metamathematical subtleties. The Lévy reflection theorem (Theorem 1.3.4) is really a theorem schema, rather than a theorem. For every $\mathcal{L}_\in$-formula φ, we have an instance of the Lévy reflection theorem for φ which is provable in ZFC. We cannot package the entire schema within ZFC because of the requirement that we decide the formulas "$Z \models \psi(z, p_1, \dots, p_n)$". As Z is not necessarily a set, we cannot translate this into a model-theoretic argument. So we really do appeal to the fact that there are only finitely many formulas appearing in Φ.

Now let us almost get a countable transitive model of ZFC. The phrasing of the following theorem is meant to indicate that we also have a theorem schema in ZFC, rather than simply a theorem in ZFC.

Theorem 1.3.5
Let $T \subsetneq \mathsf{ZFC}$ be a finite collection of axioms of ZFC. Then

$$\mathsf{ZFC} \vdash \text{"there exists a countable transitive } \mathcal{L}_\in\text{-structure } \tilde{M} \text{ with } \tilde{M} \models T\text{"}.$$

Proof. Without loss of generality, we may assume that T includes the axiom of extensionality. As T is finite, we can create the $\mathcal{L}_\in$-sentence $\varphi := \bigwedge_{\psi \in T} \psi$. The Lévy reflection theorem (Theorem 1.3.4) yields an ordinal α such that φ is absolute for $\mathbf{V}_\alpha$. As φ is a conjunction of axioms of ZFC, and our metatheory is ZFC, we get that $\mathbf{V}_\alpha \models \varphi$. In particular, $\mathbf{V}_\alpha$ is a model of T.

Now, we can use the (downwards) Löwenheim–Skolem theorem to obtain a countable elementary substructure $\mathbf{M}$ of $\mathbf{V}_\alpha$. Let us spell out the details for completeness.

Suppose that $\bar{p} \in \mathbf{V}_\alpha^n$ and $\psi(y, x_1, \dots, x_n)$ is an $\mathcal{L}_\in$-formula. If $\mathbf{V}_\alpha \models \exists y.\psi(y, \bar{p})$, then choose $w_{\psi,\bar{p}} \in \mathbf{V}_\alpha$ to be such that

$$\mathbf{V}_\alpha \models \psi(w_{\psi,\bar{p}}, \bar{p}).$$

If $\mathbf{V}_\alpha \models \neg\exists y.\psi(y, \bar{p})$, then we simply let $w_{\psi,\bar{p}} := \varnothing$. In either case, $w_{\psi,\bar{p}} \in \mathbf{V}_\alpha$.

With these, inductively construct

- $\mathbf{M}_0 := \varnothing$,
- $\mathbf{M}_{i+1} := \{\, w_{\psi,\bar{p}} : \psi(y, x_1, \dots, x_n) \text{ is an } \mathcal{L}_\in\text{-formula, } \bar{p} \in \mathbf{M}_i^n, \text{ and } n < \omega \,\}$,
- $\mathbf{M} := \bigcup_{i<\omega} M_i$.

Then $\mathbf{M}$ is countable, by construction, and $\mathbf{M}$ is an elementary substructure of $\mathbf{V}_\alpha$, by the Tarski–Vaught test (Lemma 1.3.2). So we have obtained a countable model $\mathbf{M}$ of T.

We then perform Mostowski collapse on $\mathbf{M}$ to obtain a transitive model $\tilde{\mathbf{M}}$ which is $\mathcal{L}_\in$-isomorphic to $\mathbf{M}$. This $\tilde{\mathbf{M}}$ is a countable transitive model of T. □

We remark that the countable transitive model $\tilde{\mathbf{M}}$ in the proof of Theorem 1.3.5 above is *not* necessarily an elementary substructure of $\mathbf{V}_\alpha$, eventhough it is isomorphic to an elementary substructure $\mathbf{M}$ of $\mathbf{V}_\alpha$. It is, however, elementarily equivalent to $\mathbf{V}_\alpha$.

In particular, Theorem 1.3.5 implies that

$$\text{for any finite } T \subseteq \mathsf{ZFC}, \text{ we have that } \mathsf{ZFC} \vdash \mathrm{Con}(T),$$

where $\mathrm{Con}(T)$ is the assertion that the theory T is consistent. But be careful! This does *not* say that

$$\mathsf{ZFC} \vdash \text{"for every finite } T \subseteq \mathsf{ZFC}, \text{ we have } \mathrm{Con}(T)\text{"}.$$

The latter would immediately imply that $\mathsf{ZFC} \vdash \mathrm{Con}(\mathsf{ZFC})$, contradicting Gödel's second incompleteness theorem.

We would *really* like to run the argument of the theorem above with T being all the infinitely-many axioms of ZFC. But the difficulty in trying using the Lévy reflection theorem for this case is that $\bigwedge_{\psi \in T} \psi$ will not be an $\mathcal{L}_\in$-formula if T is not finite.

In fact, not only does the argument not run through if we replaced T with all of ZFC, Gödel's incompleteness theorem outright destroys any hope of doing so!

1.4 … But it is the Light of an Oncoming Train

Recall that our baseline assumptions in the metatheory is ZFC together with the assumption that ZFC is consistent. From that, the hope was to get a countable *transitive* model of ZFC.

Proposition 1.4.1
Suppose that ZFC *is consistent. Then*

$$\mathsf{ZFC} + \mathrm{Con}(\mathsf{ZFC}) \nvdash \text{“there exists a transitive model of } \mathsf{ZFC}\text{”.}$$

Proof. The formula $\mathrm{Con}(T)$, asserting the consistency of a theory T, is a Δ_0 formula and is thus absolute for any transitive model. So if $\mathbf{M}$ is a transitive model of ZFC, then $\mathbf{M} \models \mathsf{ZFC} + \mathrm{Con}(\mathsf{ZFC})$.

So, if

$$\mathsf{ZFC} + \mathrm{Con}(\mathsf{ZFC}) \vdash \text{“there exists a transitive model of } \mathsf{ZFC}\text{”},$$

then

$$\mathsf{ZFC} + \mathrm{Con}(\mathsf{ZFC}) \vdash \mathrm{Con}(\mathsf{ZFC} + \mathrm{Con}(\mathsf{ZFC})),$$

contradicting Gödel's second incompleteness theorem. ☐

Bugger.

1.5 … But We Are a Bigger Train

But, again, who's going to stop us? We could perform the arguments of Section 1.3 as a Platonist and obtain a countable transitive model of ZFC. While an infinite conjunction of formulas is not a formula, making the Lévy reflection argument not follow through, we all know what an infinite conjunction of formulas means. This won't be an argument in ZFC, so the assertion of the existence of such a model will be an additional assumption in our metatheory.

Strictly speaking, we do not need to do this. We can instead perform all the arguments in the metatheory as follows.

Theorem
If ZFC *is consistent, then* $\mathsf{ZFC} + \neg\mathsf{CH}$ *is also consistent.*

Proof. Suppose $\mathsf{ZFC} \vdash \mathsf{CH}$. Let T be the (finite) set of all ZFC axioms which appears in such a proof and adjoint it with all of the axioms we need to perform our consistency proofs. Then there exists a countable transitive model $\mathbf{M}$ of T. [Witchcraft]. We thus obtain a model $\mathbf{N}$ of $T + \neg\mathsf{CH}$, which is a contradiction. ☐

This lets us keep ZFC as our metatheory. But being able to say “Let M be a countable transitive model of ZFC” is a lot more convenient than saying “Let M be a countable transitive model of a large enough finite fragment of ZFC for which we are supposing that we can prove CH from and adjoin it with all of the following axioms we need for our consistency argument”.

2 The Constructible Universe

We first establish the consistency of $\mathsf{ZFC} + \mathsf{CH}$ given the consistency of ZFC, and we will do so via the constructible universe. This is historically accurate: Gödel (1938) first proved the consistency of $\mathsf{ZFC}+\mathsf{CH}$ in this way. This also makes sense technically: the consistency argument from the constructible universe is just a lot easier than with the technique of forcing introduced by Cohen (1963).

The idea is as follows: starting from $\varnothing$, we will construct a *minimal* transitive model of ZFC, adding only the sets we are absolutely obliged to add to maintain a model of ZFC. What we mean by the word "minimal" will be made precise later. The construction will be *so* special that not only do we end up with a model of CH, we end up with a model of GCH, which is the assertion that $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$ for all ordinals α.

2.1 A Petite Model

Definition 2.1.1 (The Definable Power Set Operation)
Let A be a set. A subset $B \subseteq A$ is said to be <u>definable with parameters in A</u> if there exists an $\mathcal{L}_\in$-formula $\varphi(x_1, \ldots, x_n, y)$ and parameters $a_1, \ldots, a_n \in A$ such that

$$B = \{\, b \in A : A \models \text{“}\varphi(a_1, \ldots, a_n, b)\text{”} \}.$$

The <u>definable power set of A</u> is then

$$\operatorname{Def}(A) \coloneqq \{\, B \in \mathcal{P}(A) : B \textit{ is definable with parameters in } A \,\}$$

Note that $A \mapsto \operatorname{Def}(A)$ is an absolute operation.

Definition 2.1.2 (The Constructible Universe)
The <u>constructible universe</u> is the following hierarchy:

- $\mathbf{L}_0 \coloneqq \varnothing$*;*
- $\mathbf{L}_{\alpha+1} \coloneqq \operatorname{Def}(\mathbf{L}_\alpha)$ *for all ordinals* α*;*
- $\mathbf{L}_\lambda \coloneqq \bigcup_{\alpha<\lambda} \mathbf{L}_\alpha$ *for all limit ordinals* λ*;*
- $\mathbf{L} \coloneqq \bigcup_{\alpha\in\mathrm{Ord}} \mathbf{L}_\alpha$.

By the closure of absoluteness under transfinite recursion, the $\mathbf{L}$ hierarchy is absolute for transitive models of finite fragments of ZFC which are strong enough to prove its existence. More explicitly, if $T \subseteq \mathsf{ZFC}$ is strong enough to prove that the $\mathbf{L}$ hierarchy exists and $\mathbf{M}$ is a transitive model of T, then

$$\text{for all ordinals } \alpha \in \mathrm{Ord} \cap \mathbf{M}, \text{ if } X \in \mathbf{M} \text{ is such that } \mathbf{M} \models \text{“}X = \mathbf{L}_\alpha\text{”, then } X = \mathbf{L}_\alpha.$$

In this case, we have $\bigcup_{\alpha\in\mathrm{Ord}\cap\mathbf{M}} \mathbf{L}_\alpha \subseteq \mathbf{M}$. Once we have shown that $\mathbf{L} \models \mathsf{ZFC}$, it will be in this sense that $\mathbf{L}$ is a "minimal" model of ZFC.

Let us get some quick observations out of the way. Firstly, $\mathbf{L}$ is transitive. This is quite clear: whenever X is transitive then $\operatorname{Def}(X)$ is also transitive. In particular, any $X \in \mathbf{L}$ appears as either the empty set or the definable subset of some $Y \in \mathbf{L}$ with parameters in Y.

Whenever A is a finite set, we have $\operatorname{Def}(A) = \mathcal{P}(A)$. Thus $\mathbf{L}_n = \mathbf{V}_n$ for all $n < \omega$, and consequently we also have $\mathbf{L}_\omega = \mathbf{V}_\omega$.

We can also prove by induction that $|\mathbf{L}_\alpha| = |\alpha|$ for all ordinals $\alpha \geq \omega$. Indeed, the observation prior to this tells us that $|\mathbf{L}_\omega| = |\mathbf{V}_\omega| = |\omega|$. Suppose inductively that, for $\alpha \geq \omega$, we have $|\mathbf{L}_\alpha| = |\alpha|$. Then $|\operatorname{Def}(\mathbf{L}_\alpha)| \geq |\alpha|$ as all the singleton subsets of $\mathbf{L}_\alpha$ are definable with parameters in $\mathbf{L}_\alpha$. On the other hand, there are only $|\alpha|$-many n-tuples $(\varphi(x_1, \ldots, x_n, y), a_1, \ldots, a_n)$ where

$\varphi(x_1, \ldots, x_n)$ is an $\mathcal{L}_\in$-formula and $a_1, \ldots, a_n \in \mathbf{L}_\alpha$. Thus $|\operatorname{Def}(\mathbf{L}_\alpha)| \leq |\alpha|$. Finally, suppose that $\lambda \geq \omega$ is a limit ordinal and inductively suppose that $|\mathbf{L}_\alpha| = |\alpha|$ for all $\alpha < \lambda$. Then we clearly have $|\bigcup_{\alpha<\lambda} \mathbf{L}_\alpha| = |\lambda|$.

In particular, $\mathbf{L}_{\omega+1} \subsetneq \mathbf{V}_{\omega+1}$ because $|\mathbf{L}_{\omega+1}| = |\omega+1| = \aleph_0$ whereas $|\mathbf{V}_{\omega+1}| = 2^{\aleph_0}$. This observation, however, does not imply that $\mathbf{L} \subsetneq \mathbf{V}$. It may just be the case that $\mathbf{L}$ grows "a lot slower" than $\mathbf{V}$ but still eventually captures all the elements of $\mathbf{V}$. The axiom of constructibility is the assertion that

$$\mathbf{V} = \mathbf{L},$$

or in other words,

$$\forall x. \exists \alpha. x \in \mathbf{L}_\alpha.$$

In time, we will prove that this axiom is consistent with ZFC.

2.2 Proving the Axioms

We will now show that $\mathbf{L} \models \mathsf{ZF}$.

Lemma 2.2.1
$\mathbf{L} \models$ `Extensionality` + `Foundation`.

Proof. This is simply because $\mathbf{L}$ is transitive. □

Lemma 2.2.2
$\mathbf{L} \models$ `Empty Set` + `Infinity`.

Proof. The axiom of the empty set follows from $\varnothing \in \mathbf{L}_1$ and the transitivity of $\mathbf{L}$. Also, the formula $x \in \omega$ can be expressed elementarily (as a Δ_0 formula, in fact). So we have $\omega \in \mathbf{L}_{\omega+1}$. □

Lemma 2.2.3
$\mathbf{L} \models$ `Pairing`.

Proof. For $x, y \in \mathbf{L}$, let α be an ordinal such that $x, y \in \mathbf{L}_\alpha$. Then

$$\{x, y\} = \{\, z \in \mathbf{L}_\alpha : \mathbf{L}_\alpha \models \text{“}z = x \text{ or } z = y\text{”} \,\} \in \mathbf{L}_{\alpha+1}.$$ □

Lemma 2.2.4
$\mathbf{L} \models$ `Union`.

Proof. For $X \in \mathbf{L}$, let α be an ordinal such that $X \in \mathbf{L}_\alpha$. Then

$$\bigcup X = \{\, z \in \mathbf{L}_\alpha : \mathbf{L}_\alpha \models \text{“}\exists y.(z \in y \in X)\text{”} \,\} \in \mathbf{L}_{\alpha+1}.$$ □

Lemma 2.2.5
$\mathbf{L} \models$ `Power Set`.

Proof. For any $X \in \mathbf{L}$, let α be an ordinal such that $\mathcal{P}(X) \cap \mathbf{L} \subseteq \mathbf{L}_\alpha$, which is possible by the axiom schema of replacement in the metatheory. Then

$$\mathcal{P}^{\mathbf{L}}(X) = \{\, Y \in \mathbf{L}_\alpha : Y \subseteq X \,\} \in \mathbf{L}_{\alpha+1}.$$ □

Lemma 2.2.6
$\mathbf{L} \models$ `Replacement Schema`.

Proof. Fix an $\mathcal{L}_\in$-formula $\varphi(x, y)$ and a set $z \in \mathbf{L}$, and suppose that

$$\mathbf{L} \models \text{“}\forall x \in z.\exists! y.\varphi(x, y)\text{”}.$$

We aim to show that the set

$$Y := \{\, y \in \mathbf{L} : \mathbf{L} \models \text{“}\exists x \in z.\varphi(x, y)\text{”}\,\}$$

is in $\mathbf{L}$.

Let α be an ordinal such that $z \in \mathbf{L}_\alpha$. For each $y \in Y$, let $\beta_y := \min\{\, \delta \in \mathrm{Ord} : y \in \mathbf{L}_\delta \,\}$. Choose an ordinal γ which is strictly bigger than α and all the β_y's. Then $Y \subseteq \mathbf{L}_\gamma$. Now, by the Lévy reflection theorem (Theorem 1.3.4), there exists some $\zeta > \gamma$ such that the formula “$\exists x \in z.\varphi(x, y)$” is absolute between $\mathbf{L}_\zeta$ and $\mathbf{L}$. Therefore

$$Y = \{\, y \in \mathbf{L}_\zeta : \mathbf{L}_\zeta \models \text{“}\exists x \in z.\varphi(x, y)\text{”}\,\} \in \mathbf{L}_{\zeta+1}.$$ □

Lemma 2.2.7
$\mathbf{L} \models$ `Separation Schema`.

Proof. This is simply because

$$\mathbf{L} \models \texttt{Extensionality} + \texttt{Empty Set} + \texttt{Power Set} + \texttt{Replacement Schema},$$

as all these axioms together imply the axiom schema of separation.

Alternatively, one can modify the proof of $\mathbf{L} \models$ `Replacement Schema` (Lemma 2.2.6) to directly obtain a proof of $\mathbf{L} \models$ `Separation Schema`. □

2.3 The Constructible Universe is Extremely Pro-Choice

We have so far shown that $\mathbf{L} \models \mathsf{ZF}$. We will now extend this to show that $\mathbf{L} \models \mathsf{ZFC}$.

Lemma 2.3.1
$\mathbf{L} \models$ `Choice`.

Proof (Sketch). For each ordinal α, we will define a bijection $\pi_\alpha \colon \mathbf{L}_\alpha \to \eta_\alpha$, where η_α is some ordinal. This will be done in such a way so that whenever $\beta < \alpha$, we will have

$$\pi_\beta = \pi_\alpha|_{\mathbf{L}_\beta}.$$

The bijection $\pi_0 \colon \mathbf{L}_0 \to 0$ is just the empty function.

Suppose $\pi_\alpha \colon \mathbf{L}_\alpha \to \eta_\alpha$ has been defined. Let Fml denote the set of all $\mathcal{L}_\in$-formulas. Choose some fied well-order of Fml. Then we can get a well order of the set $\mathrm{Fml} \times \mathbf{L}_\alpha^{<\omega}$ in with the order-type of some ordinal η'_α. Then for any $x \in \mathbf{L}_{\alpha+1}$, we define

$$\pi'_{\alpha+1}(x) := \begin{cases} \pi_\alpha(x) & \text{if } x \in \mathbf{L}_\alpha, \\ & \text{if } \xi \text{ is the least ordinal in } \eta'_\alpha \text{ corresponding to} \\ \eta_\alpha + \xi & \quad \text{the least } (\varphi, \bar{p}) \in \mathrm{Fml} \times \mathbf{L}_\alpha^{<\omega} \text{ such that} \\ & \quad x = \{\, w \in \mathbf{L}_\alpha \;:\; \mathbf{L}_\alpha \models \text{“}\varphi(\bar{p}, w)\text{”}\,\}, \end{cases}$$

and then collapse down the injection $\pi'_{\alpha+1} \colon \mathbf{L}_{\alpha+1} \to \eta_\alpha + \eta'_\alpha$ to a bijection $\pi_{\alpha+1} \colon \mathbf{L}_{\alpha+1} \to \eta_{\alpha+1}$ for some ordinal η_α.

If λ is a limit ordinal and $\pi_\alpha \colon \mathbf{L}_\alpha \to \eta_\alpha$ has already been defined for all $\alpha < \lambda$, just take

$$\pi_\lambda := \bigcup_{\alpha < \lambda} \pi_\alpha.$$ □

In the proof of $\mathbf{L} \models$ `Choice` above (Lemma 2.3.1), we have actually shown that $\mathbf{L}$ satisfies a very strong version of the axiom of choice, the axiom of global choice:

there exists an absolutely definable bijective operation from $\mathbf{L}$ to Ord.

This gives us a *global* well-ordering of $\mathbf{L}$, and this is provable within $\mathbf{L}$.

2.4 Obvious! Wait. Not Obvious. Wait! Obvious!

Lemma 2.4.1
$\mathbf{L} \models$ "$\mathbf{V} = \mathbf{L}$".

Proof. This *may* seem obvious, but it is not totally immediate as saying $\mathbf{L} = \mathbf{L}$. This is because $\mathbf{L}$ may have its own version of the constructible hierarchy inside it, so we actually have to prove that $\mathbf{L} = \mathbf{L}^{\mathbf{L}}$.

But the absoluteness of the definable power set operation $X \mapsto \mathrm{Def}(X)$ gives us the absoluteness of the operation $\alpha \mapsto \mathbf{L}_\alpha$, and so we do indeed have $\mathbf{L} = \mathbf{L}^{\mathbf{L}}$. □

Collecting everything together, we have the following.

Theorem 2.4.2
$\mathbf{L} \models \mathsf{ZFC} +$ "$\mathbf{V} = \mathbf{L}$".

Proof. All of the previous 9 lemmas. □

Let us now show that if ZFC is consistent then so is $\mathsf{ZFC}+$"$\mathbf{V} = \mathbf{L}$". First, it is tempting to argue as follows:

> *Proof.* Let $\mathbf{M}$ be a model of ZFC. Then $\mathbf{L}^{\mathbf{M}}$ is a model of $\mathsf{ZFC}+$"$\mathbf{V} = \mathbf{L}$". □

This does not quite work. The second sentence does not (at least, a priori)[3] follow from the first sentene. We merely have that

$$\mathbf{M} \models \text{``}\mathbf{L}^{\mathbf{M}} \text{ is a model of } \mathsf{ZFC} + \text{``}\mathbf{V} = \mathbf{L}\text{'' ''},$$

but there is no reason to believe we can preserve entailment when we "pull" this model $\mathbf{L}^{\mathbf{M}}$ out into the metatheory, as $\mathbf{M}$ is not transitive.

Instead, we have the following syntactic proof.

Theorem 2.4.3
If ZFC *is consistent, then* $\mathsf{ZFC} +$ "$\mathbf{V} = \mathbf{L}$" *is consistent.*

Proof. If there is a proof of $\perp$ in $\mathsf{ZFC}+$"$\mathbf{V} = \mathbf{L}$", then we can relativise the entire proof to $\mathbf{L}$ and obtain a proof of $\perp$ in ZFC. □

As a consequence, if $\mathsf{ZFC}+$"$\mathbf{V} = \mathbf{L}$" $\vdash \varphi$ for some $\mathcal{L}_\in$-sentence φ, then the consistency of ZFC implies the consistency of $\mathsf{ZFC} + \varphi$.

When we want to prove that $\mathbf{L} \models \varphi$, for some $\mathcal{L}_\in$-sentence φ, it is often easier to show that $\mathsf{ZFC}+$"$\mathbf{V} = \mathbf{L}$" $\vdash \varphi$.

2.5 Checkpoint

We will now show that if ZFC is consistent then so is $\mathsf{ZFC} + \mathsf{GCH}$, and we will do so by showing that $\mathbf{L} \models \mathsf{GCH}$.

Definition 2.5.1
Let T_L be a sufficiently large finite fragment of ZFC *which is enough to prove the existence and absoluteness of the* $\mathbf{L}$ *hierarchy. The* <u>*condensation sentence*</u> *is*

$$\sigma \equiv \left(\bigwedge_{\varphi \in T_{\mathbf{L}}} \varphi \right) \wedge \text{``there is no largest ordinal''} \wedge \text{``}\mathbf{V} = \mathbf{L}\text{''}.$$

[3] I surmise that the second sentence is not true in general, but I do not know for sure.

Throughout this subsection, we will use the symbol σ to denote the condensation sentence. Observe that $\mathbf{L} \models \sigma$.

Lemma 2.5.2
Let σ be the condensation sentence. If $\mathbf{M}$ is a transitive $\mathcal{L}_{\in}$-structure and $\mathbf{M} \models \sigma$, then

$$\mathbf{M} = \mathbf{L}_\alpha \quad \textit{for some ordinal } \alpha.$$

Proof. Let $\lambda := \mathrm{Ord} \cap \mathbf{M}$. We claim that

$$\mathbf{M} = \mathbf{L}_\lambda.$$

Since $\mathbf{M} \models$ "there is no largest ordinal", this λ is in fact a limit ordinal. So the claim above becomes

$$\mathbf{M} = \bigcup_{\alpha<\lambda} \mathbf{L}_\alpha = \bigcup_{\alpha\in\mathrm{Ord}\cap\mathbf{M}} \mathbf{L}_\alpha$$

by definition of the $\mathbf{L}$ hierarchy. Now, the operation $\alpha \mapsto \mathbf{L}_\alpha$ is absolute for $\mathbf{M}$, so the assumption $\mathbf{M} \models$ "$\mathbf{V} = \mathbf{L}$" gives

$$\mathbf{M} \subseteq \bigcup_{\alpha\in\mathrm{Ord}\cap\mathbf{M}} \mathbf{L}_\alpha.$$

But also, for any $\alpha \in \mathrm{Ord} \cap \mathbf{M}$, we have $\mathbf{M} \supseteq \mathbf{L}_\alpha$ since $\mathbf{M} \models T_{\mathbf{L}}$, where $T_{\mathbf{L}}$ is a large enough finite fragment of ZFC which proves the existence and absoluteness of the $\mathbf{L}$ hierarchy. So

$$\mathbf{M} \supseteq \bigcup_{\alpha\in\mathrm{Ord}\cap\mathbf{M}} \mathbf{L}_\alpha. \qquad \square.$$

Lemma 2.5.3
Let κ be an infinite cardinal. If $X \subseteq \kappa$ and $X \in \mathbf{L}$, then $X \in \mathbf{L}_{\kappa^+}$.

Proof. Let T_X be the transitive closure of $\{X\}$, i.e. let T_X be the smallest set with $T_X \supseteq \{X\}$ such that for all x, if $x \in T_X$ then $x \subseteq T_X$. In particular, $X \in T_X$. As $X \in \mathbf{L}$, we also have that $T_X \in \mathbf{L}$. Let γ be a large enough ordinal such that

$$X \in T_X \in \mathbf{L}_\gamma.$$

By the Lévy reflection theorem (Theorem 1.3.4), there is an ordinal $\theta > \gamma$ such that $\mathbf{L}_\theta \models \sigma$, where σ is the condensation sentence. (Spelling out the details, the Lévy reflection theorem (Theorem 1.3.4) says that σ is absolute between $\mathbf{L}_\theta$ and $\mathbf{L}$, and we already know that $\mathbf{L} \models \sigma$.) At this point, we have

$$X \in T_X \in \mathbf{L}_\gamma \subseteq \mathbf{L}_\theta.$$

Now, $\mathbf{L}_\theta$ is a transitive and extensional set, and $T_X \subseteq \mathbf{L}_\theta$. So, by the Tarski–Vaught test, the Löwenheim–Skolem theorem, and the Mostowski collapse theorem (essentially, an argument as in the proof of Theorem 1.3.5), there is a transitive model $\mathbf{N}$ which is elementarily equivalent to $\mathbf{L}_\theta$ satisfying

$$T_X \subseteq \mathbf{N} \quad \text{and} \quad |\mathbf{N}| = |T_X| \leq \kappa.$$

In particular, $\mathbf{N} \models \sigma$ since $\mathbf{N}$ is elementarily equivalent to $\mathbf{L}_\theta$. So, by the previous lemma (Lemma 2.5.2), we have that $\mathbf{N} = \mathbf{L}_\alpha$ for some ordinal α.[4] But also, $|\alpha| = |\mathbf{L}_\alpha| = |\mathbf{N}| \leq \kappa$, and so $\alpha < \kappa^+$. Therefore

$$X \in T_X \subseteq \mathbf{L}_\alpha \subseteq \mathbf{L}_{\kappa^+}. \qquad \square$$

[4] At this point, it may seem like we are chasing our own tail. We started with an $\mathbf{L}_\theta$ with $X \in \mathbf{L}_\theta$ and $\mathbf{L}_\theta \models \sigma$, and now we obtained an $\mathbf{L}_\alpha$ with $X \in \mathbf{L}_\alpha$ and $\mathbf{L}_\alpha \models \sigma$. It seems like we have done a lot of work to get back to right where we started. But we actually made progress: we had no idea how large θ was, but the next sentence in the proof will give an upper bound for α.

Theorem 2.5.4
$\mathbf{L} \models \mathsf{GCH}$.

Proof. We will show that $\mathsf{ZFC} + \text{“}\mathbf{V} = \mathbf{L}\text{”} \vdash \mathsf{GCH}$.

Assume that $\mathbf{V} = \mathbf{L}$. Let κ be an infinite cardinal. Elementary set theory gives $\kappa^+ \leq 2^\kappa$. Now, we have just established that every subset of κ is in $\mathbf{L}_{\kappa^+}$. Hence $\mathcal{P}(\kappa) \subseteq \mathbf{L}_{\kappa^+}$. So we have

$$2^\kappa = |\mathcal{P}(\kappa)| \leq |\mathbf{L}_{\kappa^+}| = |\kappa^+|.$$

Therefore $2^\kappa = \kappa^+$. □

That's it! We have proven that, for as long as ZFC is consistent, then so is $\mathsf{ZFC} + \mathsf{GCH}$! As often is the case in mathematics, when trying to see if a result is true, if we find an example of the result holding and struggle to come up with a counterexample, we would conjecture that the result is true.

This was not the case for Kurt Gödel. The proof of the consistency of $\mathsf{ZFC} + \mathsf{GCH}$ required constructing such a special model of ZFC: the *minimal* possible transitive model. Why should any of our results hold if we instead looked for a “fatter” universe?

3 Generic Extensions

Perhaps well-known to the reader is that Cohen (1963) and Cohen (1964) introduced the method of forcing to establish the consistency of the negation of the continuum hypothesis from ZFC. We shall embark on this journey which very few mathematicians have taken.

3.1 If I Had a Penny for Every Time I Saw the Definition of a Filter...

... I'd have six pennies at the time of writing. But that still feels like a large number.

Definition 3.1.1
A forcing notion is a preordered[5] *set* $(\mathbb{P}, \preceq)$ *which has an element, denoted* $\mathbb{1}_{\mathbb{P}}$*, satisfying*

$$x \preceq \mathbb{1}_{\mathbb{P}} \quad \text{for all } x \in \mathbb{P}.$$

When the context is clear, we simply write $\mathbb{1}$ *for* $\mathbb{1}_{\mathbb{P}}$*. Elements of a forcing notion* $\mathbb{P}$ *are called (forcing) conditions, and the condition* $\mathbb{1}_{\mathbb{P}}$ *is called the weakest condition.*

Let $p, q \in \mathbb{P}$*. We say that p is stronger than q / q is weaker than p if* $p \preceq q$*. We say that p and q are compatible if there exists* $r \in \mathbb{P}$ *such that* $p \succeq r \preceq q$*. We say p and q are incompatible*[6]*, and write* $p \perp q$*, if p and q are not compatible.*

As a side remark, it is rather annoying to note that set theorists around the world are split on using forcing notions with a *maximum* element versus a *minimum* element, developing the theory with inequalities all pointing in the opposite directions. They are, of course, dual to the other, with no real advantage of one over the other.

The reader may groan at the word "preorder". Everyone loves partial orders, but nobody likes preorders. Imagine having $x \preceq y$ and $y \preceq x$ but $x \neq y$. Diabolical. We will, however, basically pretend as if our forcing notions are partially ordered sets with a maximum element. Every application that we will care about in this piece will be a partially ordered set with a maximum element. It is only when one learns even more forcing (such as iterated forcing) that one encounters forcing notions which are preorders but may not be partial orders.

Alright, that's enough about conventions.

Definition 3.1.2
Let $(\mathbb{P}, \preceq, \mathbb{1}_{\mathbb{P}})$ *be a forcing notion.*

- *An antichain*[7] *in* $\mathbb{P}$ *is a subset* $A \subseteq \mathbb{P}$ *such that*

  $$p \text{ and } q \text{ are incompatible, for any } p, q \in \mathbb{P}.$$

  An antichain A in $\mathbb{P}$ *is said to be a maximal antichain in* $\mathbb{P}$ *if for any other antichain* A' *in* $\mathbb{P}$ *with* $A \subseteq A'$*, we have* $A = A'$*.*

- *A set* $D \subseteq \mathbb{P}$ *is said to be dense in* $\mathbb{P}$ *if for all* $p \in \mathbb{P}$ *there exists* $d \in D$ *such that* $d \preceq p$*.*

- *For* $p \in \mathbb{P}$*, a set* $D \subseteq \mathbb{P}$ *is said to be dense below p if for all* $q \preceq p$ *there exists* $d \in D$ *such that* $d \preceq q$*.*

- *A filter on* $\mathbb{P}$ *is a subset* $F \subseteq \mathbb{P}$ *such that all of the following three properties hold:*

  - *for all* $p, q \in F$ *there exists* $r \in F$ *such that* $p \succeq r \preceq q$*;*

[5] A preorder is a reflexive and transitive binary relation. In particular, an antisymmetric preorder is a partial order.

[6] Note that this is different from the notion of incompar*a*ble elements in a preorder, where we say that p and q are incomparable if both $p \not\preceq q$ and $q \not\preceq p$.

[7] This is different from the usual order-theoretic definition of an antichain, which is a collection of pairwise incompar*a*ble elements.

- *for all $p \in F$ there exists $q \in F$ such that $q \succeq p$;*
- $\mathbb{1}_{\mathbb{P}} \in F$.

- *Let $\mathcal{D}$ be a collection of dense sets in $\mathbb{P}$. A filter $G \subseteq \mathbb{P}$ is said to be* <u>$\mathcal{D}$-generic</u> *if for every $D \in \mathcal{D}$, we have*
$$D \cap G \neq \varnothing.$$

- *If $\mathbf{M}$ is a countable transitive model of* ZFC *and $(\mathbb{P}, \preceq, \mathbb{1}_{\mathbb{P}}) \in M$ is a forcing notion, then we say that a filter $G \subseteq \mathbb{P}$ is* <u>$\mathbb{P}$-generic over $\mathbf{M}$</u> *if, for every dense $D \subseteq \mathbb{P}$ with $D \in \mathbf{M}$, we have*
$$D \cap G \neq \varnothing.$$

Theorem 3.1.3 (The Rasiowa–Sikorski Lemma)
Let $(\mathbb{P}, \preceq)$ be a forcing notion and let $\mathcal{D}$ be a countable collection of dense subsets of $\mathbb{P}$. Then there exists a $\mathcal{D}$-generic filter.

Proof. Enumerate $\mathcal{D} = \{D_0, D_1, D_2, D_3, \dots\}$. Choose $p_0 \in D_0$, and choose $p_{n+1} \in D_{n+1}$ with $p_{n+1} \preceq p_n$. Then
$$\bigcup_{n \in \mathbb{N}} \{\, q \in \mathbb{P} : q \succeq p_n \,\}$$
is a $\mathcal{D}$-generic filter on $\mathbb{P}$. □

In particular, if $\mathbf{M}$ is any countable transitive model of ZFC and $(\mathbb{P}, \preceq)$ is any forcing notion, then there exists a $\mathbb{P}$-generic filter G over $\mathbf{M}$. Note that, most of the time, $G \notin \mathbf{M}$.[8]

Example 3.1.4
Let X and Y be sets, and consider the poset $\operatorname{Fin}(X, Y)$ of all finite partial functions $f \colon X \to Y$ ordered by
$$f \preceq g \quad \text{if and only if} \quad f \supseteq g,$$
with the empty function as the maximum element of the forcing notion. We have that f is stronger than g if and only if f extends g. One can intuitively think of this as f giving us "more information" than g, or that f has "less[9] possible extensions" than g.

Any filter $F \subseteq \operatorname{Fin}(X, Y)$ will be a set of compatible finite partial functions from X to Y. Intuitively, each $f \in F$ gives "partial information" for the partial function $\bigcup F$ from X to Y.

For $x \in X$, let
$$D_x \coloneqq \{\, f \in \operatorname{Fin}(X, Y) : x \in \operatorname{dom}(f) \,\}.$$
Note that each D_x is dense in $\operatorname{Fin}(X, Y)$, so the collection $\mathcal{D} \coloneqq \{\, D_x : x \in X \,\}$ is a collection of dense subsets of $\operatorname{Fin}(X, Y)$. Then, for any $\mathcal{D}$-generic filter G, we see that $\bigcup G$ is a total function from X to Y, i.e. $\operatorname{dom}(\bigcup G) = X$ (though we may not necessarily have $\operatorname{ran}(\bigcup G) = Y$). □

Definition 3.1.5
The <u>*Cohen forcing notion*</u> *is the poset $\mathbb{C} \coloneqq \operatorname{Fin}(\omega, 2)$ ordered by*
$$f \preceq g \quad \textit{if and only if} \quad f \supseteq g.$$
Let $\mathbf{M}$ be a countable transitive model of ZFC*. Let G be a $\mathbb{C}$-generic filter over $\mathbf{M}$. We call the function $\bigcup G$ a* <u>*Cohen real*</u>*.*

[8] A sufficient condition for $G \notin \mathbf{M}$ is when $(\mathbb{P}, \preceq)$ is <u>separative</u>, i.e. for any $p \in \mathbb{P}$ there exist $q, r \preceq p$ such that $q \perp r$.

[9] *"Fewer."* — Stannis Baratheon.

If there is ever any evidence of the disconnect of set theory from the rest of mathematics, it is the use of the symbol $\mathbb{C}$ for the Cohen forcing notion.

Note that a Cohen real is a total function from ω to 2. Furthermore, if $\mathbf{M}$ is a countable transitive model of ZFC and G is a $\mathbb{C}$-generic filter over $\mathbf{M}$, then

$$\bigcup G \notin \mathbf{M}.$$

Indeed, for any function $f\colon \omega \to 2$, define the set

$$N_f \coloneqq \{\, p \in \mathbb{C} : p(n) \neq f(n) \text{ for some } n \in \mathrm{dom}(p)\,\},$$

which is dense in $\mathbb{C}$. Then, as

$$G \cap N_f \neq \varnothing, \quad \text{for all } f \in 2^\omega \cap \mathbf{M},$$

we conclude that $\bigcup G \notin \mathbf{M}$. So Cohen reals can be viewed as real numbers "outside" some fixed countable transitive model of ZFC.

Here is a particularly useful characterisation of generic filters over a model $\mathbf{M}$.

Lemma 3.1.6
Let $\mathbf{M}$ be a countable transitive model of ZFC, let $(\mathbb{P}, \preceq, \mathbb{1}_{\mathbb{P}}) \in \mathbf{M}$ be a forcing notion, and let $G \subseteq \mathbb{P}$. Then the following are equivalent:

- *G is a $\mathbb{P}$-generic filter over $\mathbf{M}$, i.e. for every $D \in \mathbf{M}$ which is dense in $\mathbb{P}$, we have $G \cap D \neq \varnothing$;*
- *for every $A \in \mathbf{M}$ which is a maximal antichain in $\mathbb{P}$, we have $G \cap A \neq \varnothing$;*
- *for every $B \in \mathbf{M}$ with $B \subseteq \mathbb{P}$ satisfying the property*

  "for all $p \in \mathbb{P}$ there exists $b \in B$ which is compatible with p",

  we have $G \cap B \neq \varnothing$.

Furthermore, if G is a $\mathbb{P}$-generic filter over $\mathbf{M}$ this case, then all of the following hold:

- *for every $S \subseteq \mathbb{P}$, we either have that $G \cap S \neq \varnothing$ or there exists some $p \in G$ which is incompatible with all the conditions in S;*
- *for all $p \in G$ and every $D \in \mathbf{M}$ which is dense below $\mathbb{P}$, we have $G \cap D \neq \varnothing$.*

Proof. Just do it. □

3.2 O M[G]

"To force is to name names"

— Karagila (2023, Section 2.1).

For this subsection, fix a countable transitive model $\mathbf{M}$ of ZFC, fix a forcing notion $(\mathbb{P}, \preceq, \mathbb{1}_{\mathbb{P}}) \in \mathbf{M}$, and fix a filter G which is $\mathbb{P}$-generic over $\mathbf{M}$.

We already remarked that, in many casees, $G \notin \mathbf{M}$. We are going to build a "smallest" (in some appropriate sense) model of ZFC which extends $\mathbf{M}$ and contains G. In effect, we will have "adjoined"[10] G to $\mathbf{M}$.

[10] I wrote adjoin*ed* and not adjoin*t*. The category-pilled can despair and the category-phobic can breathe a sigh of relief.

Definition 3.2.1
A set $\dot{x}$ is called a <u>$\mathbb{P}$-name</u> *if every $z \in \dot{x}$ is of the form $z = (\dot{y}, p)$ for some $\mathbb{P}$-name $\dot{y}$ and some condition $p \in \mathbb{P}$.*

We will always denote names with a dot or a check on top.

This definition appeals to the well-foundedness of the relation $\in$; the empty set is a $\mathbb{P}$-name. We can build the class $\mathrm{Name}^{\mathbb{P}}$ of $\mathbb{P}$-names from the bottom-up as follows:

$$\begin{aligned}
\mathrm{Name}^{\mathbb{P}}_0 &:= \varnothing, \\
\mathrm{Name}^{\mathbb{P}}_{\alpha+1} &:= \mathcal{P}(\mathrm{Name}^{\mathbb{P}}_\alpha \times \mathbb{P}), && \text{for all ordinals } \alpha, \\
\mathrm{Name}^{\mathbb{P}}_\lambda &:= \bigcup_{\alpha<\lambda} \mathrm{Name}^{\mathbb{P}}_\alpha, && \text{for all limit ordinals } \lambda, \\
\mathrm{Name}^{\mathbb{P}} &:= \bigcup_{\alpha\in\mathrm{Ord}} \mathrm{Name}^{\mathbb{P}}_\alpha.
\end{aligned}$$

The class $\mathrm{Name}^{\mathbb{P}}$ may also be denoted by $\mathbf{V}^{\mathbb{P}}$ in the literature, to denote that these are the $\mathbb{P}$-names in $\mathbf{V}$.

For a $\mathbb{P}$-name $\dot{x}$, we define

$$\mathrm{dom}(\dot{x}) := \{\, \dot{y} \in \mathrm{Name}^{\mathbb{P}} : \exists p \in \mathbb{P}.((\dot{y}, p) \in \dot{x}) \,\}$$

for the set of $\mathbb{P}$-names which appear in $\dot{x}$. This is simply a particular instance of writing $\mathrm{dom}(R) := \{\, x \in A : \exists y \in B.(xRy) \,\}$ for the domain of a binary relation $R \subseteq A \times B$.

Note that the formula "x is a $\mathbb{P}$-name" is absolute for transitive models. We define

$$\mathbf{M}^{\mathbb{P}} := \{\, \dot{x} \in \mathbf{M} : \mathbf{M} \models \text{“}\dot{x} \text{ is a } \mathbb{P}\text{-name”} \,\} = \mathrm{Name}^{\mathbb{P}} \cap \mathbf{M}.$$

Definition 3.2.2
Let $\dot{x}$ be a $\mathbb{P}$-name. We define the <u>evaluation of $\dot{x}$ by G</u> *to be*

$$\dot{x}^G := \{\, \dot{y}^G \;:\; (\dot{y}, p) \in \dot{x} \textit{ for some } p \in G \,\}.$$

Again, this is a definition by recursion; $\varnothing^G = \varnothing$.

If $\mathbf{N}$ is a transitive model of ZFC with $\dot{x}, G \in \mathbf{N}$, then the formula "$z = \dot{x}^G$" is absolute for $\mathbf{N}$.

Intuitively, a $\mathbb{P}$-name $\dot{x} \in \mathbf{M}^{\mathbb{P}}$ can be thought of as "instructions" in M for creating a new set $\dot{x}^G$. The $\mathbb{P}$-generic filter G can then be thought of as a machine that actually creates the set $\dot{x}^G$ when it reads the instruction $\dot{x}$. This $\dot{x}^G$ is a set which may or may not live inside $\mathbf{M}$. But even if $\dot{x}^G \notin \mathbf{M}$, the model $\mathbf{M}$ still has "some idea" of what $\dot{x}$ is. After all, $\dot{x}^G$ was built from $\dot{x}$ which lives in $\mathbf{M}$.

Suppose we have $\mathbb{P}$-names $\dot{x}$ and $\dot{y}$ and a condition $p \in \mathbb{P}$ with $(\dot{y}, p) \in \dot{x}$. The "machine" G creates new sets

$$\dot{y}^G \quad \text{and} \quad \dot{x}^G.$$

When do we have $\dot{y}^G \in \dot{x}^G$? By definition, this holds if G declares p as "being large", i.e. if $p \in G$. So, intuitively, the closer p is to $\mathbb{1}_{\mathbb{P}}$ (that is, the higher up p is in $\mathbb{P}$), the "more likely" it is that $\dot{y}^G \in \dot{x}^G$. Indeed, if $p = \mathbb{1}_{\mathbb{P}}$, then we *must* have $\dot{y}^G \in \dot{x}^G$, simply because $\mathbb{1}_{\mathbb{P}} \in G$ by definition of G being a filter.

The remarkable thing is that this machine $\dot{x} \mapsto \dot{x}^G$ will, at the very least, produce an entire copy of $\mathbf{M}$!

Definition 3.2.3
For $x \in \mathbf{M}$, the <u>canonical $\mathbb{P}$-name for x</u> *is*

$$\check{x} := \{\, (\check{y}, \mathbb{1}_{\mathbb{P}}) : y \in x \,\} \in M^{\mathbb{P}}.$$

Everything is defined by recursion on $\in$, in case you haven't caught on; $\check{\varnothing} = \varnothing$, and so $\check{\varnothing}^G = \varnothing$. Then, by $\in$-induction, we can show that

$$\check{x}^G = x \quad \text{for all } x \in M,$$

regardless of which $\mathbb{P}$-generic filter G we picked.

Our machine $\dot{x} \mapsto \dot{x}^G$ will also be able to create G!

Definition 3.2.4
Let the canonical $\mathbb{P}$-name for G be

$$\Gamma \coloneqq \{\, (\check{p}, p) : p \in \mathbb{P} \,\},$$

where $\check{p}$ is the canonical name for $p \in \mathbb{P}$.

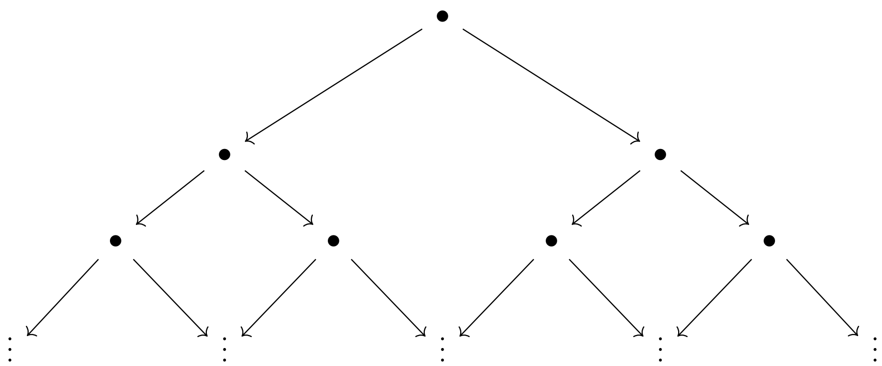
Then $\Gamma \in \mathbf{M}^{\mathbb{P}}$ and $\Gamma^G = G$.

This is particularly curious. The model $\mathbf{M}$ essentially contains a bunch of instructions (the $\mathbb{P}$-names in $\mathbf{M}$) for producing a bigger version of itself! Even if the countable transitive model $\mathbf{M}$ does not contain G, the model $\mathbf{M}$ can still get glimpses of G through its canonical $\mathbb{P}$-name Γ, and the same can be said for other sets $\dot{x}^G$ which do not live in $\mathbf{M}$. I like to this as a person sitting in their house and staring at a blueprint of a house extension.

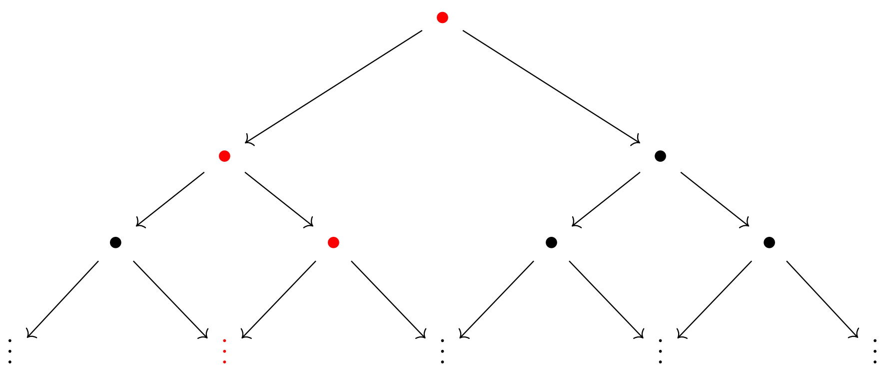
Definition 3.2.5
The generic extension of $\mathbf{M}$ by G is

$$\mathbf{M}[G] \coloneqq \left\{ \dot{x}^G : \dot{x} \in \mathbf{M}^{\mathbb{P}} \right\}.$$

Let us recall the Cohen forcing notion $\mathbb{C} = \mathrm{Fin}(\omega, 2)$ as an example. We can view $\mathbb{C}$ as a collection of finite parts of the infinite binary tree below.



A filter on $\mathbb{C}$ will give rise to an infinite branch in this tree. A filter G which is $\mathbb{C}$-generic over $\mathbf{M}$ will give rise to a function $\bigcup G \notin \mathbf{M}$, which can be viewed as an infinite branch in the tree which is not in $\mathbf{M}$.



The generic extension $\mathbf{M}[G]$ will then satisfy $G \in M[G]$. Once we have shown that $\mathbf{M}[G]$ is transitive and satisfies the axiom of union, we will have $\bigcup G \in M[G]$. This "adjoins" the Cohen real $\bigcup G$ to $\mathbf{M}$ in a very controlled way: $\mathbf{M}$ basically knew everything about this Cohen real except how to construct it.

As strongly hinted by the narration so far, this $\mathbf{M}[G]$ is the model of ZFC we are looking for.

3.3 Some Black Boxes

The next subsection can be quite a pain to read, so we will summarise the main results here for the reader who really could not care about their proofs. As usual, fix a countable transitive model $\mathbf{M}$ of ZFC. Also fix a forcing notion $(\mathbb{P}, \preceq, \mathbb{1}_{\mathbb{P}}) \in \mathbf{M}$.

The generic model theorem (Theorem 3.3.1 below) states that $\mathbf{M}[G]$ is a countable transitive model of ZFC, containing G as an element, which is slightly "fatter" than $\mathbf{M}$ but not "taller" than $\mathbf{M}$. It further states that it is the smallest such model.

Theorem 3.3.1 (The Generic Model Theorem)
For any $\mathbb{P}$-generic filter G over $\mathbf{M}$, all of the following hold.

- $\mathbf{M}[G]$ *is a countable transitive model of* ZFC*;*
- $\mathbf{M} \subseteq \mathbf{M}[G]$ *and* $G \in \mathbf{M}[G]$*;*
- *if* $\mathbf{N}$ *is another transitive model of* ZFC *with* $\mathbf{M} \subseteq \mathbf{N}$ *and* $G \in \mathbf{N}$*, then* $\mathbf{M}[G] \subseteq \mathbf{N}$*;*
- $\mathrm{Ord} \cap \mathbf{M} = \mathrm{Ord} \cap \mathbf{M}[G]$.

Proof. The countability of $\mathbf{M}[G]$ follows from the countability of $\mathbf{M}$. For the transitivity of $\mathbf{M}[G]$, suppose that $y \in \dot{x}^G \in \mathbf{M}[G]$ for some $\dot{x} \in \mathbf{M}^{\mathbb{P}}$. Then by definition of evaluation by G (Definition 3.2.2), there exists some $\dot{y} \in \mathrm{dom}(\dot{x})$ and some $p \in G$ such that $(\dot{y}, p) \in \dot{x}$ and $y = \dot{y}^G$. As $\mathbf{M}$ is transitive, we have that $\dot{y} \in \mathbf{M}$. Therefore $y = \dot{y}^G \in \mathbf{M}[G]$ by definition of $\mathbf{M}[G]$ (Definition 3.2.5).

We have already established that $\mathbf{M} \subseteq \mathbf{M}[G]$ and $G \in \mathbf{M}[G]$ in our discussion involving canonical names.

If $\mathbf{N}$ is another transitive model of ZFC with $\mathbf{M} \subseteq \mathbf{N}$ then, in particular, $\mathbf{M}^{\mathbb{P}} \subseteq \mathbf{N}$. If we further have that $G \in \mathbf{N}$ then $\mathbf{N}$ can perform the operation of evaluating all the $\mathbb{P}$-names in $\mathbf{M}$ by G, and by the absoluteness of the formula "$y = \dot{x}^G$". Thus $\mathbf{M}[G] \subseteq \mathbf{N}$.

Clearly $\mathrm{Ord} \cap \mathbf{M} \subseteq \mathrm{Ord} \cap \mathbf{M}[G]$, simply because $\mathbf{M} \subseteq \mathbf{M}[G]$. Now, for any $\dot{x} \in \mathbf{M}^{\mathbb{P}}$, one can show by induction that

$$\mathrm{rank}(\dot{x}^G) \leq \mathrm{rank}(\dot{x}),$$

where the rank is taken in the metatheory. Therefore $\mathrm{Ord} \cap \mathbf{M}[G] \subseteq \mathrm{Ord} \cap \mathbf{M}$.

We defer the proof that $\mathbf{M}[G] \models$ ZFC to Section 3.4. □

The truth of formulas in $\mathbf{M}[G]$ is connected to the forcing notion in the following way.

Definition 3.3.2 (The (Semantic) Forcing Relation)
Let $p \in \mathbb{P}$, let $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}_\in$-formula with n free variables[11], and let $\dot{x}_1, \ldots, \dot{x}_n \in \mathbf{M}^{\mathbb{P}}$ be $\mathbb{P}$-names in $\mathbf{M}$. We say that <u>p forces $\varphi(\dot{x}_1, \ldots, \dot{x}_n)$</u>, and write $p \Vdash \varphi(\dot{x}_1, \ldots, \dot{x}_n)$, if and only if

$$\mathbf{M}[G] \models \varphi(\dot{x}_1^G, \ldots, \dot{x}_n^G) \quad \textit{for all } \mathbb{P}\textit{-generic filters } G \textit{ over } \mathbf{M} \textit{ with } p \in G.$$

The following Theorem 3.3.3 is one of the most important properties about the forcing relation: formulas which are true in the generic extension must have been forced by some condition in the generic filter.

[11] This φ could also be an $\mathcal{L}_\in$-sentence; it could have no free variables.

Theorem 3.3.3 (The (Semantic) Forcing Theorem)
Let G be a $\mathbb{P}$-generic filter over $\mathbf{M}$. Let $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}_\in$-formula with n free variables $x_1, \ldots, x_n$ and let $\dot{x}_1, \ldots, \dot{x}_n \in \mathbf{M}^\mathbb{P}$. Then the following are equivalent:

- $\mathbf{M}[G] \models \varphi(\dot{x}_1^G, \ldots, \dot{x}_n^G)$;
- *there exists $p \in G$ such that $p \Vdash \varphi(\dot{x}_1, \ldots, \dot{x}_n)$.*

Proof. See Section 3.4. □

Theorem 3.3.4 (The Definability Theorem)
The relation $\Vdash$ is absolutely definable within $\mathbf{M}$, i.e. there is a relation $\Vdash^*$ which is definable in $\mathbf{M}$, and absolute for transitive models containing $\mathbf{M}$, such that for all $p \in \mathbb{P}$, all $\mathcal{L}_\in$-formulas $\varphi(x_1, \ldots, x_n)$ with n free variables, and all $\mathbb{P}$-names $\dot{x}_1, \ldots, \dot{x}_n \in \mathbf{M}$, we have that

$$p \Vdash \varphi(\dot{x}_1, \ldots, \dot{x}_n) \quad \textit{if and only if} \quad \mathbf{M} \models \text{“}p \Vdash^* \varphi(\dot{x}_1, \ldots, \dot{x}_n)\text{”}.$$

Proof. See Section 3.4. □

This last definability theorem (Theorem 3.3.4) is also very important: it allows us to talk about generic extensions of $\mathbf{M}$ from within $\mathbf{M}$ itself. In combination with the forcing theorem (Theorem 3.3.3), formulas which are true in the generic extension must have been forced by some condition in the generic filter, and $\mathbf{M}$ would have known about this potential truth.

The following very simple result shows why we say that a condition $p \in \mathbb{P}$ is “stronger” than another condition $q \in \mathbb{P}$ whenever $p \preceq q$.

Theorem 3.3.5
Let $p \in \mathbb{P}$ be stronger than $q \in P$, let $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}_\in$-formula with n free variables, and let $\dot{x}_1, \ldots, \dot{x}_n \in \mathbf{M}^\mathbb{P}$.

$$\textit{If } q \Vdash \varphi(\dot{x}_1, \ldots, \dot{x}_n), \textit{ then } p \Vdash \varphi(\dot{x}_1, \ldots, \dot{x}_n).$$

Proof. Any $\mathbb{P}$-generic filter over $\mathbf{M}$ containing p must contain q, since $p \preceq q$. □

The following result is another simple but immensely useful; it (essentially) allows us to work independently below independent conditions.

Theorem 3.3.6 (The Mixing Lemma)
Let $A \in \mathbf{M}$ be an antichain in $\mathbb{P}$. For each $p \in A$, let $\dot{x}_p \in \mathbf{M}$ be a $\mathbb{P}$-name. Then there exists a $\mathbb{P}$-name $\dot{y} \in \mathbf{M}$ such that

$$p \Vdash \text{“}\dot{y} = \dot{x}_p\text{”} \quad \textit{for all } p \in A.$$

Proof. Working in $\mathbf{M}$, we let

$$\dot{y} \coloneqq \bigcup_{p \in A} \{\, (\dot{z}, q) \in \operatorname{dom}(\dot{x}_p) \times \mathbb{P} : q \text{ is stronger than } p \text{ and } q \Vdash \text{“}\dot{z} \in \dot{x}_p\text{”} \,\}.$$

By the definability theorem (Theorem 3.3.4), we do indeed have that $\dot{y} \in \mathbf{M}^\mathbb{P}$.

Fix any $p \in A$ and let us show that $p \Vdash \text{“}\,\dot{y} = \dot{x}_p\text{”}$. By definition of $\Vdash$ (Definition 3.3.2), for any $\mathbb{P}$-generic filter G over $\mathbf{M}$ with $p \in G$, we want to show that

$$\mathbf{M}[G] \models \text{“}\dot{y}^G = \dot{x}_p^G\text{”}.$$

We will use the fact that $\mathbf{M}[G] \models$ `Extensionality` (by Theorem 3.3.1).

First, we show that $\dot{y}^G \subseteq \dot{x}_p^G$. For any $\dot{z}^G \in \dot{y}^G$, the definition of evaluation by G (Definition 3.2.2) tells us that there exists $q \in G$ such that $(\dot{z}, q) \in \dot{y}$. By definition of $\dot{y}$, this q is stronger than some $p' \in A$ and satisfies

$$q \Vdash \text{“}\dot{z} \in \dot{x}_{p'}\text{”}.$$

As G is a filter and $q \in G$, we have $p' \in G$. But now $p, p' \in G \cap A$. As G is a filter and A is an antichain, this implies that $p = p'$. Therefore

$$q \Vdash \text{“}\dot{z} \in \dot{x}_p\text{”}$$

and so $\dot{z}^G \in \dot{x}_p^G$.

Now we show that $\dot{y}^G \supseteq \dot{x}_p^G$. For any $\dot{z}^G \in \dot{x}_p^G$, the forcing theorem (Theorem 3.3.3) tells us that there exists $q \in G$ such that

$$q \Vdash \text{“}\dot{z} \in \dot{x}_p\text{”}.$$

Let $r \in G$ be a common extension of both p and q. Then

$$r \Vdash \text{“}\dot{z} \in \dot{x}_p\text{”},$$

since r is stronger than q (Theorem 3.3.5). As r is also stronger than $p \in A$, we have that $(\dot{z}, r) \in \dot{y}$. Therefore $\dot{z}^G \in \dot{y}^G$, because $r \in G$. ☐

Here is another very useful result that helps us deal with existential quantifiers.

Theorem 3.3.7 (The Existential Completeness Lemma)
Let $p \in \mathbb{P}$, let $\varphi(y, x_1, \ldots, x_n)$ be an $\mathcal{L}_\in$-formula with $n+1$ free variables, and let $\dot{x}_1, \ldots, \dot{x}_n \in \mathbf{M}$ be $\mathbb{P}$-names. Then the following are equivalent:

- *$p \Vdash$ “$\exists y.\varphi(y, \dot{x}_1, \ldots, \dot{x}_n)$”;*
- *there exists a $\mathbb{P}$-name $\dot{y} \in \mathbf{M}$ such that*

$$p \Vdash \varphi(\dot{y}, \dot{x}_1, \ldots, \dot{x}_n).$$

Proof. The proof of the forward direction is deferred to Section 3.4, and the proof of the converse direction is an easy application of the forcing theorem (Theorem 3.3.3). ☐

3.4 Try to Not Gouge Your Eyes Out

You can skip this section if you are willing to take the black-boxed results in Section 3.3 for granted. Honestly, for your sanity, I recommend that you do.

We will begin by completely abandoning the forcing relation $\Vdash$ as defined in Definition 3.3.2. We will instead define it (syntactically) in $\mathbf{M}$. Provided we then prove that it is equivalent to the original (semantic) forcing relation in Definition 3.3.2, we will achieve the definability theorem (Theorem 3.3.4). Just to emphasise this again so that it doesn't get lost in the middle of a big chunk of text,

> IN THIS SUBSECTION, WE COMPLETELY ABANDON THE PREVIOUS DEFINITION OF $\Vdash$ IN Definition 3.3.2.

So let us completely revamp the forcing relation. Again, fix a countable transitive model $\mathbf{M}$ of ZFC and a forcing notion $(\mathbb{P}, \preceq, \mathbb{1}_\mathbb{P}) \in \mathbf{M}$.

Recall that we can work with just the following six logical symbols: $\bot$, $=$, $\in$, $\wedge$, $\neg$, $\exists$. All other logical connectives can be expressed semantically through a combination of these six symbols:

- $\varphi \vee \psi$ can be expressed as $\neg(\neg\varphi \wedge \neg\psi)$;
- $\varphi \to \psi$ can be expressed as $\neg\varphi \vee \psi$, or equivalently, $\neg(\varphi \wedge \neg\psi)$;
- $\forall x.\varphi(x)$ can be expressed as $\neg(\exists x.(\neg\varphi(x)))$.

Alright. Prepare yourself. Sit down. Take a sip of water. Breathe in. Breathe out.

Definition 3.4.1 (The (Syntactic) Forcing Relation)
Let $p \in \mathbb{P}$, let $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}_\in$-formula with n free variables, and let $\dot{x}_1, \ldots, \dot{x}_n \in \mathbf{M}^\mathbb{P}$ be $\mathbb{P}$-names in $\mathbf{M}$. We define the relation $p \Vdash \varphi(\dot{x}_1, \ldots, \dot{x}_n)$, and say that $\underline{p \text{ forces } \varphi(\dot{x}_1, \ldots, \dot{x}_n)}$, inductively as follows.

- $p \nVdash \bot$.
- *For $\dot{x}_1, \dot{x}_2 \in \mathbf{M}^\mathbb{P}$, we declare*
$$p \Vdash \text{“}\dot{x}_1 = \dot{x}_2\text{”}$$
if and only if both of the following hold:
  (1) for each $(\dot{y}_1, p_1) \in \dot{x}_1$, the set
$$\Big\{ q \preceq p \ : \ \text{if } q \preceq p_1, \text{ then there exists } (\dot{y}_2, p_2) \in \dot{x}_2 \text{ such that } q \preceq p_2 \text{ and } q \Vdash \text{“}\dot{y}_1 = \dot{y}_2\text{”} \Big\}$$
  is dense below p;
  (2) for each $(\dot{y}_2, p_2) \in \dot{x}_2$, the set
$$\Big\{ q \preceq p \ : \ \text{if } q \preceq p_2, \text{ then there exists } (\dot{y}_1, p_1) \in \dot{x}_1 \text{ such that } q \preceq p_1 \text{ and } q \Vdash \text{“}\dot{y}_1 = \dot{y}_2\text{”} \Big\}$$
  is dense below p.
- *For $\dot{x}_1, \dot{x}_2 \in \mathbf{M}^\mathbb{P}$, we declare*
$$p \Vdash \text{“}\dot{x}_1 \in \dot{x}_2\text{”}$$
if and only if the set
$$\Big\{ q \preceq p \ : \ \text{there exists } (\dot{y}_2, p_2) \in \dot{x}_2 \text{ such that } q \preceq p_2 \text{ and } q \Vdash \text{“}\dot{x}_1 = \dot{y}_2\text{”} \Big\}$$
is dense below p.
- *For $\mathcal{L}_\in$-formulas $\varphi(x_1, \ldots, x_n)$ and $\psi(y_1, \ldots, y_k)$ and for $\dot{x}_1, \ldots, \dot{x}_n, \dot{y}_1, \ldots, \dot{y}_k \in \mathbf{M}^\mathbb{P}$, we declare*
$$p \Vdash \text{“}\varphi(\dot{x}_1, \ldots, \dot{x}_n) \wedge \psi(\dot{y}_1, \ldots, \dot{y}_k)\text{”}$$
if and only if
$$p \Vdash \varphi(\dot{x}_1, \ldots, \dot{x}_n) \quad \text{and} \quad p \Vdash \psi(\dot{y}_1, \ldots, \dot{y}_k).$$
- *For an $\mathcal{L}_\in$-formula $\varphi(x_1, \ldots, x_n)$ and for $\dot{x}_1, \ldots, \dot{x}_n \in \mathbf{M}^\mathbb{P}$, we declare*
$$p \Vdash \text{“}\neg\varphi(\dot{x}_1, \ldots, \dot{x}_n)\text{”}$$
if and only if
$$q \nVdash \text{“}\varphi(\dot{x}_1, \ldots, \dot{x}_n)\text{”} \quad \text{for all } q \preceq p.$$
- *For an $\mathcal{L}_\in$-formula $\varphi(y, x_1, \ldots, x_n)$ and for $\dot{x}_1, \ldots, \dot{x}_n \in \mathbf{M}^\mathbb{P}$, we declare*
$$p \Vdash \text{“}\exists y.\varphi(y, \dot{x}_1, \ldots, \dot{x}_n)\text{”}$$
if and only if the set
$$\Big\{ q \preceq p \ : \ \text{there exists } \dot{y} \in \mathbf{M}^\mathbb{P} \text{ such that } q \Vdash \varphi(\dot{y}, \dot{x}_1, \ldots, \dot{x}_n) \Big\}$$
is dense below p.

I apologise for all that. Do worry, it doesn't get better.

We have, at the very least, achieved in laying out a (really messed up) definition of the forcing relation which is definable in $\mathbf{M}$ and is absolute. Strictly speaking, the forcing relation is dependent on both $\mathbf{M}$ and $\mathbb{P}$, and so we should really write $p \Vdash_{\mathbf{M},\mathbb{P}} \varphi(\dot{x}_1, \ldots, \dot{x}_n)$, but I think we can all agree that the notaion is already ridiculous enough provided it is clear from context which $\mathbf{M}$ and $\mathbb{P}$ we are working with.

Speaking of "the notation is already ridiculous enough", from now onwards we shall surpress the $\mathbb{P}$-names and free variables which appear in our formulas, unless we absolutely need to use them. So we will often simply write $p \Vdash \varphi$. In the case that we have more than a singular φ on the right-hand side I will make the effort to put in quotation marks such as $p \Vdash$ "$\varphi \wedge \psi$" so that it is unambiguous when the formulas end and when the rest of the narration begins. I may miss a few though, so use your best judgement to figure out what I mean.

First, here are a few mechanical results.

Lemma 3.4.2
For any $p \in \mathbb{P}$ and any formulas φ, ψ, all of the following hold.

- *The following are equivalent:*
  (1) $p \Vdash \varphi$;
  (2) $r \Vdash \varphi$ for all $r \preceq p$;
  (3) the set $\{ r \in \mathbb{P} : r \Vdash \varphi \}$ is dense below p.
- *$p \Vdash \varphi$ if and only if $p \Vdash \neg\neg\varphi$.*
- *The set $\{ q \in \mathbb{P} \ : \ q \Vdash \varphi \quad \text{or} \quad q \Vdash \neg\varphi \}$ is dense in $\mathbb{P}$.*
- *$p \nVdash$ "$\varphi \wedge \neg\varphi$".*
- *$p \Vdash$ "$\varphi \vee \neg\varphi$".*
- *$p \Vdash$ "$\varphi \vee \psi$" if and only if the set*
  $$\left\{ q \preceq p \ : \ q \Vdash \varphi \text{ or } q \Vdash \psi \right\}$$
  is dense below p.
- *If $p \Vdash$ "$\varphi \to \psi$" and $p \Vdash \varphi$ then $p \Vdash \psi$.*
- *$p \Vdash$ "$\forall x.\varphi(x)$" if and only if*
  $$p \Vdash \varphi(\dot{x}) \quad \text{for all } \dot{x} \in \mathbf{M}^{\mathbb{P}}.$$
- *If $\dot{x}, \dot{y} \in \mathbf{M}^{\mathbb{P}}$ are such that $p \Vdash$ "$\dot{x} = \dot{y}$" and $p \Vdash \varphi(\dot{x})$, then $p \Vdash \varphi(\dot{y})$.*

Proof. Just do it. □

Now we prove the following Theorem 3.4.3 which is the syntactic version of the semantic forcing theorem (Theorem 3.3.3), with the new syntactic forcing relation defined in Definition 3.4.1. Trust me on this one, unless you absolutely have to, please do not read the proof of the following syntactic forcing theorem (Theorem 3.4.3). It's not particularly difficult — it's just a proof by induction — but it's really not worth it.

Theorem 3.4.3 (The Syntactic Forcing Theorem)
Let G be a $\mathbb{P}$-generic filter over $\mathbf{M}$. Then for any formula φ, the following are equivalent:

- $\mathbf{M}[G] \models \varphi$;
- *there exists* $p \in G$ *such that* $p \Vdash \varphi$.

I'll give you one last chance to skip past this proof. Click here: Theorem 3.4.4.

Proof of Theorem 3.4.3. pain.[12]

We induct on the complexity of φ.

First[13] comes the base case $\varphi \equiv$ "$\dot{x}_1 = \dot{x}_2$". We will do this by an induction on the ranks of the $\mathbb{P}$-names.

Assume that there exists $p \in G$ such that $p \Vdash$ "$\dot{x}_1 = \dot{x}_2$". We want to show that $\dot{x}_1^G = \dot{x}_2^G$. Let us start by showing that $\dot{x}_1^G \subseteq \dot{x}_2^G$.

Fix any $\dot{y}_1^G \in \dot{x}_1^G$. Then there exists $p_1 \in G$ such that $(\dot{y}_1, p_1) \in \dot{x}_1$. By definition of the forcing relation (Definition 3.4.1), the set

$$D_1 := \Big\{ q \preceq p \ : \ \text{if } q \preceq p_1, \text{ then there exists } (\dot{y}_2, p_2) \in \dot{x}_2 \text{ such that } q \preceq p_2 \text{ and } q \Vdash \text{“} \dot{y}_1 = \dot{y}_2 \text{”} \Big\}$$

is dense below p. As $p \in G$ and $p_1 \in G$, there is a common extension $q \in G$ of both p and p_1. Then D_1 is also dense below q (Lemma 3.1.6). So we find some $r \in G \cap D_1$ which is stronger than q.

Since $r \in D_1$ and $r \preceq p_1$, there exists $(\dot{y}_2, p_2) \in \dot{x}_2$ such that

$$r \preceq p_2 \quad \text{and} \quad r \Vdash \text{“}\dot{y}_1 = \dot{y}_2\text{”}.$$

By the induction hypothesis applied to $\dot{y}_1$ and $\dot{y}_2$, and since $r \in G$, we have that $\dot{y}_1^G = \dot{y}_2^G$. Finally since $r \in G$ and $r \preceq p_2$, we have that $p_2 \in G$ and hence $\dot{y}_2^G \in \dot{x}_2^G$. Therefore $\dot{y}_1^G \in \dot{x}_2^G$. Consequently $\dot{x}_1^G \subseteq \dot{x}_2^G$.

The reverse inclusion $\dot{x}_1^G \supseteq \dot{x}_2^G$ is similar.

Now we prove the converse for the base case $\varphi \equiv$ "$\dot{x}_1 = \dot{x}_2$". Again, we induct on the ranks of the $\mathbb{P}$-names.

Assume that $\dot{x}_1^G = \dot{x}_2^G$. We want to show that there exists some $r \in G$ with $r \Vdash$ "$\dot{x}_1 = \dot{x}_2$". For $r \in \mathbb{P}$, define the $\mathcal{L}_\in$-formulas

$$\Phi_r^1 \equiv \text{“}\exists (\dot{y}_1, p_1) \in \dot{x}_1.\Big(r \preceq p_1 \text{ and } \forall (\dot{y}_2, p_2) \in \dot{x}_2.\forall q \in \mathbb{P}.\big((q \preceq p_2 \text{ and } q \Vdash \text{“}\dot{y}_1 = \dot{y}_2\text{”}) \to q \perp r\big)\Big)\text{”}$$

$$\Phi_r^2 \equiv \text{“}\exists (\dot{y}_2, p_2) \in \dot{x}_2.\Big(r \preceq p_2 \text{ and } \forall (\dot{y}_1, p_1) \in \dot{x}_1.\forall q \in \mathbb{P}.\big((q \preceq p_1 \text{ and } q \Vdash \text{“}\dot{y}_1 = \dot{y}_2\text{”}) \to q \perp r\big)\Big)\text{”}.$$

Then define the set

$$D := \Big\{ r \in \mathbb{P} \ : \ r \Vdash \text{“}\dot{x}_1 = \dot{x}_2\text{”} \quad \text{or} \quad \Phi_r^1 \quad \text{or} \quad \Phi_r^2 \Big\}.$$

Let us prove that D is dense in $\mathbb{P}$. Fix any $p \in \mathbb{P}$. If $p \Vdash$ "$\dot{x}_1 = \dot{x}_2$" then $p \in D$ and we are done. So suppose that $p \not\Vdash$ "$\dot{x}_1 = \dot{x}_2$". Then one of the sets as specified in the $=$ clause in Definition 3.4.1 is not dense below p. As the other case is similar, let us just deal with the case where there is some $(\dot{y}_1, p_1) \in \dot{x}_1$ for which the set

$$D_1 := \Big\{ q \preceq p \ : \ \text{if } q \preceq p_1, \text{ then there exists } (\dot{y}_2, p_2) \in \dot{x}_2 \text{ such that } q \preceq p_2 \text{ and } q \Vdash \text{“} \dot{y}_1 = \dot{y}_2 \text{”} \Big\}$$

[12] French for "bread".

[13] And worst.

is not dense below p. Then there exists some $r \preceq p$ such that

$$\forall q \preceq r.\Big(q \preceq p_1 \text{ and } \forall(\dot{y}_2, p_2) \in \dot{x}_2.\big(\neg(q \preceq p_2 \text{ and } q \Vdash \text{“}\dot{y}_1 = \dot{y}_2\text{”})\big)\Big)$$

holds. In particular, this gives $r \preceq p_1$. Then Φ^1_r holds, so $r \in D$. So we have that $r \preceq p$ and $r \in D$. Therefore D is dense in $\mathbb{P}$.

Next, let us prove that if $r \in G$ then neither Φ^1_r nor Φ^2_r hold. As the other case is similar, we shall only show that Φ^1_r cannot hold. Suppose, for a contradiction, that Φ^1_r holds. So there exists some $(\dot{y}_1, p_1) \in \dot{x}_1$ such that $r \preceq p_1$ and

$$\forall(\dot{y}_2, p_2) \in \dot{x}_2.\forall q \in \mathbb{P}.\big((q \preceq p_2 \text{ and } q \Vdash \text{“}\dot{y}_1 = \dot{y}_2\text{”}) \rightarrow q \perp r\big).$$

Since we assumed that $r \in G$, we obtain $p_1 \in G$. So $\dot{y}_1^G \in \dot{x}_1^G$. Now, recall that we assumed from the very beginning of this converse proof that $\dot{x}_1^G = \dot{x}_2^G$. So there exists $(\dot{y}_2, p_2) \in \dot{x}_2$, with $p_2 \in G$, such that $\dot{y}_1^G = \dot{y}_2^G$. Applying the induction hypothesis to $\dot{y}_1$ and $\dot{y}_2$, there exists some $q_0 \in G$ such that $q_0 \Vdash$ “$\dot{y}_1 = \dot{y}_2$”. Now find some $q \in G$ which is stronger than both q_0 and p_2. Then we also have that $q \Vdash$ “$\dot{y}_1 = \dot{y}_2$”. So, as specified by Φ^1_r, the conditions q and r must be incompatible. But $q, r \in G$. This is a contradiction.

Putting it all together, as D is dense, there exists some $r \in G \cap D$. As $r \in D$, we have

$$r \Vdash \text{“}\dot{x}_1 = \dot{x}_2\text{”} \quad \text{or} \quad \Phi^1_r \quad \text{or} \quad \Phi^2_r.$$

But also, as $r \in G$, we must have $r \Vdash$ “$\dot{x}_1 = \dot{x}_2$”.

We have thus completed the proof of the syntactic forcing theorem in the base case where $\varphi \equiv$ “$\dot{x}_1 = \dot{x}_2$”.

Now let us prove it in the case where $\varphi \equiv$ “$\dot{x}_1 \in \dot{x}_2$”. This will be a direct proof now that we have obtained the proof of the $=$ case.

Let us assume that there exists some $p \in G$ such that $p \Vdash$ “$\dot{x}_1 \in \dot{x}_2$”. Then the set

$$D := \Big\{\, q \preceq p \;:\; \text{there exists } (\dot{y}_2, p_2) \in \dot{x}_2 \text{ such that } q \preceq p_2 \text{ and } q \Vdash \text{“ }\dot{x}_1 = \dot{y}_2\text{”} \,\Big\}$$

is dense below p, by definition of the $\in$ clause in Definition 3.4.1. So $G \cap D \neq \varnothing$. Choose some $q \in G \cap D$. As $q \in D$, there exists some $(\dot{y}_2, p_2) \in \dot{x}_2$ such that

$$q \preceq p_2 \quad \text{and} \quad q \Vdash \text{“}\dot{x}_1 = \dot{y}_2\text{”}.$$

As $q \in G$, and as we have already proven the syntactic forcing theorem for the $=$ clause, we obtain $\dot{x}_1^G = \dot{y}_2^G$. But also, as $q \in G$ and $q \preceq p_2$, we have $p_2 \in G$. Hence $\dot{y}_2^G \in \dot{x}_2^G$. Therefore we obtain $\dot{x}_1^G \in \dot{x}_2^G$.

For the other implication, suppose that $\dot{x}_1^G \in \dot{x}_2^G$. Then there exists some $(\dot{y}_2, p_2) \in \dot{x}_2$, with $p_2 \in G$, such that $\dot{x}_1^G = \dot{y}_2^G$. By the syntactic forcing theorem for $=$, there exists some $r \in G$ with $r \Vdash$ “$\dot{x}_1 = \dot{y}_2$”. Choose some $p \in G$ which is a common extension of both p_2 and r. Then the set

$$\Big\{\, q \preceq p \;:\; \text{there exists } (\dot{z}_2, q_2) \in \dot{x}_2 \text{ such that } q \preceq q_2 \text{ and } q \Vdash \text{“ }\dot{x}_1 = \dot{z}_2\text{”} \,\Big\}$$

is dense below p, and so $p \Vdash$ “$\dot{x}_1 \in \dot{x}_2$” by the $\in$ clause of Definition 3.4.1.

This completes the proof of the syntactic forcing theorem for $\in$.

Next, suppose inductively that the syntactic forcing theorem holds for a formula φ. We will show that the syntactic forcing theorem also holds for $\neg\varphi$.

Suppose that some $p \in G$ satisfies $p \Vdash$ “$\neg\varphi$”. By definition (Definition 3.4.1),

$$q \not\Vdash \varphi \quad \text{for all } q \preceq p.$$

Suppose, for a contradiction, that $\mathbf{M}[G] \not\models \neg\varphi$. Then $\mathbf{M}[G] \models \varphi$. So by the induction hypothesis applied to φ, there exists some $r \in G$ with $r \Vdash \varphi$. Then choose a common extension $q \in G$ of both p and r. Then we get the contradiction $q \Vdash$ "$\varphi \wedge \neg\varphi$" because $p \Vdash \neg\varphi$ whereas $r \Vdash \varphi$.

Conversely, suppose that $\mathbf{M}[G] \models \neg\varphi$. Define the set

$$D := \{\, p \in \mathbb{P} \ : \ p \Vdash \varphi \text{ or } p \Vdash \neg\varphi \,\}.$$

By Lemma 3.4.2, this set D is dense in $\mathbb{P}$. So there exists some $p \in G \cap D$. By the induction hypothesis, $p \nVdash \varphi$. Therefore we must have $p \Vdash \neg\varphi$.

This completes the proof of the syntactic forcing theorem for $\neg$.

Now we suppose inductively that the syntactic forcing theorem holds for formulas φ and ψ. We will show that the syntactic forcing theorem also holds for $\varphi \wedge \psi$.

If $p \Vdash \varphi \wedge \psi$ for some $p \in G$ then $p \Vdash \varphi$ and $p \Vdash \psi$, so by the inductive hypothesis $\mathbf{M}[G] \models$ "$\varphi \wedge \psi$".

Conversely, if $\mathbf{M}[G] \models$ "$\varphi \wedge \psi$" then the inductive hypothesis yields the existence of some $p, q \in G$ such that $p \Vdash \varphi$ and $q \Vdash \psi$. Then choose some $r \in G$ which is stronger than both p and q. Then this r satisfies $r \Vdash$ "$\varphi \wedge \psi$".

This completes the syntactic forcing theorem for $\wedge$.

Finally, we suppose inductively that the syntactic forcing theorem holds for a formula $\varphi(x)$ whenever x is substituted with a $\mathbb{P}$-name.

If $p \Vdash \exists x.\varphi(x)$ for some $p \in G$ then the set

$$D := \Big\{\, q \preceq p \ : \ \text{there exists } \dot{y} \in \mathbf{M}^{\mathbb{P}} \text{ such that } q \Vdash \varphi(\dot{y}) \,\Big\}$$

is dense below p. So there exists $q \in G \cap D$ and there exists a $\mathbb{P}$-name $\dot{y} \in \mathbf{M}$ such that

$$q \Vdash \varphi(\dot{y}).$$

Then, by the inductive hypothesis, as $q \preceq p$, we have that $\mathbf{M}[G] \models \varphi(\dot{y}^G)$, and so $\mathbf{M}[G] \models \exists x.\varphi(x)$.

Conversely, if $\mathbf{M}[G] \models \exists x.\varphi(x)$ then there exists some $\mathbb{P}$-name $\dot{y} \in \mathbf{M}$ such that $\mathbf{M}[G] \models \varphi(\dot{y}^G)$. So, by the inductive hypothesis, there exists some $p \in G$ with $p \Vdash \varphi(\dot{y})$. Then the set

$$\Big\{\, q \preceq p \ : \ \text{there exists } \dot{z} \in \mathbf{M}^{\mathbb{P}} \text{ such that } q \Vdash \varphi(\dot{z}) \,\Big\}$$

is dense below p. So $p \Vdash \exists x.\varphi(x)$. □

Whew. Good job if you made it through all of that. The worst is over. Though this doesn't mean the rest of this subsection is pleasant. The great thing is that the proof of the syntactic forcing theorem (Theorem 3.4.3) is one of those proofs which you do once and *never* again.[14]

We can now connect our syntactic forcing relation defined in Definition 3.4.1 to the semantic forcing relation defined in Definition 3.3.2 to establish the definability theorem (Theorem 3.3.4).

Theorem 3.4.4 (The Definability Theorem)
Let $p \in \mathbb{P}$ and let φ be a formula. Then the following are equivalent:

- *$p \Vdash \varphi$;*
- *for any $\mathbb{P}$-generic filter G over $\mathbf{M}$ with $p \in G$, we have $\mathbf{M}[G] \models \varphi$.*

Proof. The forward direction is simply the syntactic forcing theorem (Theorem 3.4.3). So we just need to prove the converse.

Suppose, for a contradiction, that $p \nVdash \varphi$. Then (by Lemma 3.4.2 and Definition 3.3.2) there exists $q \preceq p$ such that $q \Vdash \neg\varphi$. But then any $\mathbb{P}$-generic filter G over $\mathbf{M}$ with $q \in G$ will also have $p \in G$. By hypothesis, we have $\mathbf{M}[G] \models \varphi$ since $p \in G$. But by the syntactic forcing theorem (Theorem 3.4.3), we have $\mathbf{M}[G] \models \neg\varphi$ since $q \in G$. This is a contradiction. □

[14] Unless you have to study it for an exam. My condolences if that is the case.

As a corollary, we can establish the semantic version of the forcing theorem (Theorem 3.3.3).

Theorem 3.4.5 (The Semantic Forcing Theorem)
Let G be a $\mathbb{P}$-generic filter over $\mathbf{M}$ and let φ be a formula. Then the following are equivalent:

- $\mathbf{M}[G] \models \varphi$;
- *there exists $p \in G$ such that for any $\mathbb{P}$-generic filter H over $\mathbf{M}$ with $p \in H$, we have $\mathbf{M}[H] \models \varphi$.*

Proof. This is an immediate consequence of the syntactic forcing theorem (Theorem 3.4.3) and the definability theorem (Theorem 3.4.4). □

Let us now prove the existential completeness lemma (Theorem 3.3.7).

Theorem 3.4.6 (The Existential Completeness Lemma)
Let $p \in \mathbb{P}$ and let φ be a formula. Then

$$p \Vdash \text{“}\exists x.\varphi(x)\text{”} \quad \textit{if and only if} \quad \textit{there exists } \dot{x} \in \mathbf{M}^{\mathbb{P}} \textit{ with } p \Vdash \varphi(\dot{x}).$$

Proof. The reverse direction is an easy application of the forcing theorem (Theorem 3.4.3).

Let us prove the forward direction. Suppose that $p \Vdash$ “ $\exists x.\varphi(x)$”. Working in $\mathbf{M}$, define

$$C := \{\, q \preceq p \ : \ \text{there exists } \dot{y} \in \mathbf{M}^{\mathbb{P}} \text{ such that } q \Vdash \varphi(\dot{y}) \,\}$$

and let $A \subseteq C$ be an antichain which is maximal in C (note that this A may not be a maximal antichain in $\mathbb{P}$).

For each $q \in A$, choose some $\dot{y}_q \in \mathbf{M}^{\mathbb{P}}$ such that $q \Vdash \varphi(\dot{y}_q)$. By the mixing lemma (Theorem 3.3.6), there exists a $\mathbb{P}$-name $\dot{x} \in \mathbf{M}$ such that

$$q \Vdash \text{“}\dot{x} = \dot{y}_q\text{”} \quad \text{for all } q \in A.$$

So we obtain $q \Vdash \varphi(\dot{x})$ for all $q \in A$.

We claim that $p \Vdash \varphi(\dot{x})$. Suppose, for a contradiction, that $p \nVdash \varphi(\dot{x})$. Then (by Lemma 3.4.2 and Definition 3.4.1), there exists some condition $r \preceq p$ such that $r \Vdash \neg\varphi(\dot{x})$. As $p \Vdash$ “ $\exists x.\varphi(x)$”, the set C is dense below p, by the $\exists$ clause in the definition of $\Vdash$ (Definition 3.4.1). So there exists $q_0 \in C$ with $q_0 \preceq r$. Then

$$q_0 \Vdash \neg\varphi(\dot{x}).$$

Hence this q_0 is incompatible with every condition in A. But then $A \cup \{q_0\}$ is an antichain in C, contradicting the maximality of A in C. □

We finish off this section by finishing the proof of the generic model theorem (Theorem 3.3.1) by showing that the generic extension satisfies ZFC. Several of these will be proof sketches by just naming the sets that the axioms claim exist.

Lemma 3.4.7
Let G be a $\mathbb{P}$-generic filter over $\mathbf{M}$. Then $\mathbf{M}[G] \models \texttt{Extensionality} + \texttt{Foundation}$.

Proof. This is simply because $\mathbf{M}[G]$ is transitive. □

Lemma 3.4.8
Let G be a $\mathbb{P}$-generic filter over $\mathbf{M}$. Then $\mathbf{M}[G] \models \texttt{Empty Set} + \texttt{Infinity}$.

Proof. This is because $\mathbf{M} \subseteq \mathbf{M}[G]$. In particular, $\varnothing, \omega \in \mathbf{M}[G]$. □

Lemma 3.4.9
Let G be a $\mathbb{P}$-generic filter over $\mathbf{M}$. Then $\mathbf{M}[G] \models \texttt{Pairing}$.

Proof. Let $\dot{x}, \dot{y} \in \mathbf{M}^{\mathbb{P}}$. Define

$$\dot{p} := \{(\dot{x}, \mathbb{1}_{\mathbb{P}}), (\dot{y}, \mathbb{1}_{\mathbb{P}})\} \in \mathbf{M}^{\mathbb{P}}.$$

Then $\dot{p}^G = \{\dot{x}^G, \dot{y}^G\}$. $\square$

Lemma 3.4.10
Let G be a $\mathbb{P}$-generic filter over $\mathbf{M}$. Then $\mathbf{M}[G] \models$ `Union`.

Proof. Let $\dot{x} \in \mathbf{M}^{\mathbb{P}}$. Working in $\mathbf{M}$, define

$$\dot{u} := \Big\{ (\dot{z}, r) \in \mathbf{M}^{\mathbb{P}} \times \mathbb{P} \ : \ \text{there exist } \dot{x}, \dot{y} \in \mathbf{M}^{\mathbb{P}} \text{ and there exist } p, q \in \mathbb{P} \text{ such that } (\dot{z}, q) \in (\dot{y}, p) \in \dot{x} \text{ and } r \text{ is stronger than both } p \text{ and } q \Big\}$$

Then $\dot{u}^G = \bigcup (\dot{x}^G)$. $\square$

Lemma 3.4.11
Let G be a $\mathbb{P}$-generic filter over $\mathbf{M}$. Then $\mathbf{M}[G] \models$ `Separation Schema`.

Proof. Let $\varphi(x)$ be an $\mathcal{L}_{\in}$-formula and let $\dot{A} \in \mathbf{M}^{\mathbb{P}}$. Define the $\mathbb{P}$-name

$$\dot{B} := \{ (\dot{x}, p) \in \operatorname{dom}(\dot{A}) \times \mathbb{P} \ : \ p \Vdash \text{“}\dot{x} \in \dot{A} \text{ and } \varphi(\dot{x})\text{”} \} \in \mathbf{M}.$$

Then $\dot{B}^G = \{ x \in \dot{A}^G \ : \ \varphi(x) \}$. $\square$

Lemma 3.4.12
Let G be a $\mathbb{P}$-generic filter over $\mathbf{M}$. Then $\mathbf{M}[G] \models$ `Power Set`.

Proof. Let $\dot{X} \in \mathbf{M}^{\mathbb{P}}$. Define the $\mathbb{P}$-name

$$\dot{P} := \mathcal{P}^{\mathbf{M}}(\operatorname{dom}(\dot{X}) \times \mathbb{P}) \times \{\mathbb{1}_{\mathbb{P}}\} \in \mathbf{M}$$

Then $\{ S \in \mathbf{M}[G] : S \subseteq \dot{X}^G \} \subseteq \dot{P}^G$. Indeed, if $\dot{S} \in \mathbf{M}$ is a $\mathbb{P}$-name for which $\dot{S}^G \subseteq \dot{X}^G$, then we can define

$$\dot{T} := \{ (\dot{x}, p) \ : \ \dot{x} \in \operatorname{dom}(\dot{X}) \text{ and } p \Vdash \text{“}\dot{x} \in \dot{S}\text{”} \} \in \operatorname{dom}(\dot{P}),$$

and we see that $\dot{S}^G = \dot{T}^G$.

So we can use the fact that $\mathbf{M}[G] \models$ `Separation Schema` (Lemma 3.4.11) to get that $\{ S \subseteq \dot{X}^G : S \in \mathbf{M}[G] \} \in \mathbf{M}[G]$. $\square$

Lemma 3.4.13
Let G be a $\mathbb{P}$-generic filter over $\mathbf{M}$. Then $\mathbf{M}[G] \models$ `Replacement Schema`.

Proof. Let $\varphi(x, y)$ be an $\mathcal{L}_{\in}$-formula, let $\dot{X} \in \mathbf{M}^{\mathbb{P}}$, and suppose that

$$\forall x \in \dot{X}^G . \exists! y . \varphi(x, y).$$

We work in $\mathbf{M}$. Choose an ordinal α large enough so that $\operatorname{dom}(\dot{X}) \subseteq \mathbf{V}_\alpha$ (that is, $\mathbf{M}$'s interpretation of $\mathbf{V}_\alpha$).

Define another $\mathcal{L}_{\in}$-formula ψ as follows:

$$\psi(\dot{x}, p) \equiv \exists \dot{y} . (\dot{y} \in \mathbf{M}^{\mathbb{P}} \text{ and } p \Vdash \varphi(\dot{x}, \dot{y})).$$

By the Lévy reflection theorem (Theorem 1.3.4), there is an ordinal $\theta > \alpha$ such that ψ is absolute between $\mathbf{V}_\theta$ and $\mathbf{V}$.

Define $\dot{Y} := \{ (\dot{y}, \mathbb{1}_{\mathbb{P}}) : \dot{y} \in \mathbf{V}_\theta \}$. Then

$$\mathbf{M}[G] \models \text{“}\forall x \in \dot{X}^G . \exists y \in \dot{Y}^G . \varphi(x, y)\text{”},$$

and so we once again use that $\mathbf{M}[G] \models$ `Separation` (Lemma 3.4.11) to cut down $\dot{Y}^G$ to the image of $\dot{X}^G$ under the class function φ. $\square$

Lemma 3.4.14
Let G be a $\mathbb{P}$-generic filter over $\mathbf{M}$. Then $\mathbf{M}[G] \models$ `Choice`.

Proof. Let $\dot{x} \in \mathbf{M}^{\mathbb{P}}$. In $\mathbf{M}$, there is an injection $i\colon \operatorname{dom}(\dot{x}) \to \alpha$ for some ordinal $\alpha \in \mathbf{M}$.
In $\mathbf{M}[G]$, define the function $i_*\colon \dot{x}^G \to \alpha$ by

$$i_*(y) \coloneqq \min\{\, i(\dot{y}) \;:\; \dot{y} \in \operatorname{dom}(\dot{x}) \text{ and } \dot{y}^G = y \,\} \quad \text{for each } y \in \dot{x}^G.$$

Then i_* is an injection, in $\mathbf{M}[G]$, from $\dot{x}^G$ into α. ☐

Collecting it all together, we get a model of ZFC.

Theorem 3.4.15
Let G be a $\mathbb{P}$-generic filter over $\mathbf{M}$. Then $\mathbf{M}[G] \models \mathsf{ZFC}$.

Proof. All of previous 8 lemmas. ☐

Congratulations on making it to the end of this subsection. Or maybe you skipped straight to this sentence idk.

4 Cohen Forcing

We have done all the preparation and are now ready to begin forcing $\neg\mathsf{CH}$. Throughout, we will fix a countable transitive model $\mathbf{M}$ of ZFC.

4.1 Collapsing a Cardinal Is A Cardinal Sin

First, we need some preliminary results on cardinal preservation. We have learned how to go from a model $\mathbf{M}$ to a generic extension $\mathbf{M}[G]$. We have learned that the formula "κ is a cardinal" is, in general, not absolute between transitive models. We already know it is downwards absolute, but the upwards absoluteness is very often lost. It would thus be nice to know when we can indeed get upwards absoluteness, and thus the preservation of cardinals when moving from $\mathbf{M}$ to its generic extension.

Definition 4.1.1
A forcing notion $(\mathbb{P}, \preceq, \mathbb{1}_{\mathbb{P}}) \in \mathbf{M}$ is said to preserve cardinals if for every $\mathbb{P}$-generic filter G over $\mathbf{M}$, the formula "κ is a cardinal" is absolute between $\mathbf{M}$ and $\mathbf{M}[G]$.

One common class of cardinal-preserving forcing notions are those in which every antichain is countable[15].

Definition 4.1.2
A forcing notion $(\mathbb{P}, \preceq, \mathbb{1}_{\mathbb{P}})$ satisfies the countable chain condition (c.c.c.) if every antichain in $\mathbb{P}$ is countable.

It is *really* annoying that the countable chain condition does not assert that every chain is countable, but rather that every *anti*chain is countable. This is because if one approaches the theory of forcing via Boolean algebras, then the assertion that every chain is countable is equivalent to the assertion that every antichain is countable. This will not be the case for us; we will never speak of countable chains in this piece so that there should be no confusion when the term "c.c.c." is used.

When we have a forcing notion $(\mathbb{P}, \preceq, \mathbb{1}_{\mathbb{P}}) \in \mathbf{M}$ and we say that "$\mathbb{P}$ satisfies c.c.c.", we mean that

$$\mathbf{M} \models \text{“}\mathbb{P} \text{ satisfies c.c.c.”}.$$

Of course, because $\mathbf{M}$ is a *countable* transitive model, *every* antichain in *every* forcing notion in $\mathbf{M}$ will be countable in the metatheory. So this property of satisfying c.c.c. is only interesting when interpreted in $\mathbf{M}$.

Before we can show that c.c.c. forcing notions preserve cardinals, we show that c.c.c. forcing notions allow us to "approximate" any function in the generic extension.

Lemma 4.1.3
Let $(\mathbb{P}, \preceq, \mathbb{1}_{\mathbb{P}}) \in \mathbf{M}$ be a forcing notion satisfying c.c.c., let G be a $\mathbb{P}$-generic filter over $\mathbf{M}$, let $X, Y \in \mathbf{M}$, and let $f\colon X \to Y$ be a function in $\mathbf{M}[G]$. Then there exists a function $F\colon X \to \mathcal{P}^{\mathbf{M}}(Y)$ in $\mathbf{M}$ such that

$$\text{for all } x \in X, \text{ we have } f(x) \in F(x) \subseteq Y \text{ and } \mathbf{M} \models \text{“}F(x) \text{ is countable”}.$$

Proof. Let $\dot{f}$ be a $\mathbb{P}$-name in $\mathbf{M}$ such that $\dot{f}^G = f$. Working in $\mathbf{M}$, we define $F\colon X \to \mathcal{P}^{\mathbf{M}}(Y)$ by

$$F(x) := \{\, y \in Y : \exists p \in \mathbb{P}.(p \Vdash \text{“}\dot{f}(\check{x}) = \check{y}\text{”})\,\} \quad \text{for all } x \in X.$$

This function $F\colon X \to \mathcal{P}^{\mathbf{M}}(Y)$ lives in $\mathbf{M}$ due to the definability of the forcing relation in $\mathbf{M}$ (Theorem 3.3.4). By the forcing theorem (Theorem 3.3.3), we have that

$$f(x) \in F(x) \quad \text{for all } x \in X.$$

[15]By countable, we mean either finite or countably infinite.

Now fix $x \in X$. We will now show that $\mathbf{M} \models$ "$F(x)$ is countable". Working again in $\mathbf{M}$, for each $y \in F(x)$, choose $p_y \in \mathbb{P}$ such that $p_y \Vdash$ "$\dot{f}(\check{x}) = \check{y}$". Observe that, if we have two distinct $y_1, y_2 \in F(x)$, then p_{y_1} and p_{y_2} must be incompatible, otherwise their common extension r would satisfy

$$r \Vdash \text{"}\check{y}_1 \neq \check{y}_2 \text{ and } \dot{f}(\check{x}) = \check{y}_1 \text{ and } \dot{f}(\check{x}) = \check{y}_2\text{"},$$

which is a contradiction. In particular, the mapping $y \mapsto p_y$ is injective. As $\mathbb{P}$ satisfies c.c.c., and as the set $\{p_y\}_{y \in F(x)}$ is an antichain in $\mathbb{P}$, this set $\{p_y\}_{y \in F(x)}$ must be countable. Therefore $F(x)$ must also be countable. □

Armed with this, we can now prove that c.c.c. forcing notions preserve cardinals.

Theorem 4.1.4
All c.c.c. forcing notions in $\mathbf{M}$ *preserve cardinals.*

Proof. Let $(\mathbb{P}, \preceq, \mathbb{1}_{\mathbb{P}}) \in \mathbf{M}$ be a forcing notion which satisfies c.c.c. and let G be a $\mathbb{P}$-generic filter over $\mathbf{M}$.

Suppose, for a contradiction, that there exists $\kappa \in \mathbf{M}$ such that

$$\mathbf{M} \models \text{"}\kappa \text{ is a cardinal"} \quad \text{but} \quad \mathbf{M}[G] \models \text{"}\kappa \text{ is not a cardinal"}.$$

Then, in $\mathbf{M}[G]$, there is a surjection $f \colon \lambda \to \kappa$ for some ordinal $\lambda < \kappa$ with $\lambda \geq \omega$.

Applying the previous lemma (Lemma 4.1.3), there exists a function $F \colon \lambda \to \mathcal{P}^{\mathbf{M}}(\kappa)$ in $\mathbf{M}$ such that

$$\text{for all } \alpha < \lambda, \text{ we have } f(\alpha) \in F(\alpha) \subseteq \kappa \text{ and } \mathbf{M} \models \text{"}F(\alpha) \text{ is countable"}.$$

Define

$$R \coloneqq \bigcup_{\alpha < \lambda} F(\alpha),$$

noting that $R \in \mathbf{M}$. Then

$$\mathbf{M} \models \text{"}|R| \leq |\lambda| \cdot \aleph_0 = |\lambda| < \kappa\text{"}$$

since κ is a cardinal in $\mathbf{M}$.

But also, since $f(\alpha) \in F(\alpha)$ for all $\alpha < \lambda$ and since $f \colon \lambda \to \kappa$ is surjective, we get that $R = \kappa$. Combining this with the calculation above, we obtain $\mathbf{M} \models$ "$|\kappa| < \kappa$", contradicting the assumption that κ is a cardinal in $\mathbf{M}$. □

4.2 Adding a Shit Ton of Reals

Before we force $\neg\mathsf{CH}$, we will need to make use of the following combinatorial result.

Lemma 4.2.1 (The Δ-System Lemma)
Let S be an uncountable collection of finite sets. Then there exists an uncountable $D \subseteq S$ and there exists a finite set R such that

$$A \cap B = R \quad \textit{for any two distinct } A, B \in D.$$

Proof. Without loss of generality, we may assume that all the finite sets in W have the same cardinality $n < \omega$. The proof then proceeds by induction on n. □

Now recall that $\mathrm{Fin}(X, Y)$ denotes the forcing notion consisting of all the finite partial functions from X to Y, ordered by

$$f \text{ is stronger than } g \quad \text{if and only if} \quad f \text{ extends } g.$$

Under the very weak assumption of Y being countable, this forcing notion $\mathrm{Fin}(X, Y)$ becomes a c.c.c. forcing notion!

Lemma 4.2.2
Let $X, Y \in \mathbf{M}$, with $\mathbf{M} \models$ "Y is countable". Then $\mathrm{Fin}(X,Y)$ is a c.c.c. forcing notion in $\mathbf{M}$.

Proof. That $\mathrm{Fin}(X,Y) \in \mathbf{M}$ is easy: all the elements of $\mathrm{Fin}(X,Y)$ are finite partial functions which we can simply write down.

We now work in $\mathbf{M}$ to show that $\mathrm{Fin}(X,Y)$ satisfies c.c.c. Suppose that $A \subseteq \mathrm{Fin}(X,Y)$ is uncountable. We want to show that A is not an antichain in $\mathrm{Fin}(X,Y)$. Let

$$S := \{\, \mathrm{dom}(p) : p \in A \,\}$$

be the set of all domains of the finite partial functions which appear in A. As A is uncountable and Y is countable, this set S must be uncountable. So, by the Δ-system lemma (Lemma 4.2.1), there exists an uncountable $D \subseteq S$ and a finite set R such that

$$B \cap C = R \quad \text{for any two distinct } B, C \in D.$$

There are only $|Y|^{|R|} \leq \max\{\aleph_0, |Y|\}$ functions from R to Y. So there must be uncountably many functions in D are pairwise compatible. $\square$

We are now ready to force $\neg\mathsf{CH}$.

Theorem 4.2.3
Let $\kappa \in \mathbf{M}$ be such that $\mathbf{M} \models$ "κ is an uncountable cardinal". Define the forcing notion $\mathbb{P} := \mathrm{Fin}(\omega \times \kappa, 2) \in \mathbf{M}$ and let G be a $\mathbb{P}$-generic filter over $\mathbf{M}$. Then

$$\mathbf{M}[G] \models \text{“}2^{\aleph_0} \geq \kappa\text{”}.$$

Proof. The forcing notion $\mathrm{Fin}(\omega \times \kappa, 2)$ satisfies c.c.c. (Lemma 4.2.2), so it preserves cardinals (Theorem 4.1.4). In particular, κ remains a cardinal in $\mathbf{M}[G]$ and "maintains its cardinal value".

Let $f := \bigcup G$, so that f is a total function in $\mathbf{M}[G]$ from $\omega \times \kappa$ to 2. For any two distinct $\alpha, \beta < \kappa$, the set

$$\{\, p \in \mathrm{Fin}(\omega \times \kappa, 2) : \exists n \in \omega.((n,\alpha),(n,\beta) \in \mathrm{dom}(p) \text{ and } p(n,\alpha) \neq p(n,\beta)) \,\} \in \mathbf{M}$$

is dense in $\mathrm{Fin}(\omega \times \kappa, 2)$, and so G intersects this set. Thus, for any two distinct $\alpha, \beta < \kappa$ there exists $n \in \omega$ such that $f(n,\alpha) \neq f(n,\beta)$.

By currying, we can identify f with a function $g\colon \kappa \to 2^\omega$ in $\mathbf{M}[G]$. The observation above then tells us that this g is injective. The conclusion follows. $\square$

Therefore, by forcing with the forcing notions $\mathrm{Fin}(\omega \times \kappa, 2)$ with κ an uncountable cardinal, we can force $2^{\aleph_0}$ to be arbitrarily large.

4.3 As Many, or as Few, as You Want!

We have successfully made the set $\mathbb{R}$ bigger than any uncountable cardinal we like. But which values are actually attainable? There are some cardinals which are immediately not attainable due to elementary arguments in ZFC.

Proposition 4.3.1
$2^{\aleph_0}$ has uncountable cofinality.

Proof. This follows from Kőnig's theorem, which states that if $\kappa \geq \aleph_0$ is a cardinal then

$$\kappa^{\mathrm{cf}(\kappa)} > \kappa.$$

In the case of $\kappa = 2^{\aleph_0}$, this says that

$$2^{\aleph_0 \cdot \mathrm{cf}\left(2^{\aleph_0}\right)} = \left(2^{\aleph_0}\right)^{\mathrm{cf}\left(2^{\aleph_0}\right)} > 2^{\aleph_0},$$

and so $\mathrm{cf}\left(2^{\aleph_0}\right) > \aleph_0$. $\square$

So we cannot make $2^{\aleph_0}$ equal to any cardinal with countable cofinality. A particular instantiation of this result is that $\mathsf{ZFC} \vdash$ "$2^{\aleph_0} \neq \aleph_\omega$". This restriction on $\operatorname{cf}\left(2^{\aleph_0}\right) > \aleph_0$ miraculously turns out to be the *only* restriction!

We have managed to lower bound the value of $2^{\aleph_0}$ by any uncountable cardinal. To show which values can be attained, we need to establish upper bounds. To do so, we will show that every subset of ω can be realised as the evaluation by a generic filter of certain kinds of "nice names", and we will then establish an upper bound on the number of those nice names.

Definition 4.3.2
Let $(\mathbb{P}, \preceq, \mathbb{1}_{\mathbb{P}}) \in \mathbf{M}$ be a forcing notion. A <u>*nice $\mathbb{P}$-name for a subset of ω*</u> *is a $\mathbb{P}$-name of the form*

$$\dot{X}_{\mathscr{A}} = \{\, (\check{n}, p) : n \in \omega \text{ and } p \in A_n \,\},$$

where $\mathscr{A} = (A_n)_{n\in\omega}$ is an ω-sequence of antichains in $\mathbb{P}$.

We now show that any subset of ω in the generic extension is the evaluation of some nice name. Note that the result below does not assume *anything* about $\mathbb{P}$ other than that it is a forcing notion.

Theorem 4.3.3
Let $(\mathbb{P}, \preceq, \mathbb{1}_{\mathbb{P}}) \in \mathbf{M}$ be a forcing notion and let G be a $\mathbb{P}$-generic filter over $\mathbf{M}$. Then for any $X \in \mathbf{M}[G]$ with $X \subseteq \omega$, there is a nice $\mathbb{P}$-name $\dot{X}_{\mathscr{A}}$ such that $\dot{X}_{\mathscr{A}}^G = X$.

Proof. First fix some $\mathbb{P}$-name $\dot{X} \in \mathbf{M}$ such that $\dot{X}^G = X$.

Working in $\mathbf{M}$, for each $n \in \omega$ we let

$$C_n := \{\, p \in \mathbb{P} : p \Vdash \text{“}\check{n} \in \dot{X}\text{”} \,\}$$

and then we choose an antichain $A_n \subseteq C_n$ which is maximal in C_n (this A_n may not necessarily be maximal in $\mathbb{P}$). Then $\mathscr{A} := (A_n)_{n\in\omega} \in \mathbf{M}$, by the definability theorem (Theorem 3.3.4), and so we can define the $\mathbb{P}$-name

$$\dot{X}_{\mathscr{A}} := \{\, (\check{n}, p) : n \in \omega \text{ and } p \in A_n \,\}$$

which is a nice $\mathbb{P}$-name in $\mathbf{M}$ for a subset of ω. We claim that

$$\dot{X}_{\mathscr{A}}^G = \dot{X}^G = X.$$

First we check that $\dot{X}_{\mathscr{A}}^G \subseteq \dot{X}^G$. If $n \in \dot{X}_{\mathscr{A}}^G$ then by the definition of evaluation by G (Definition 3.2.2) there exists some $p \in G$ such that $(\check{n}, p) \in \dot{X}_{\mathscr{A}}$, and so $p \in A_n$ by definition of $\dot{X}_{\mathscr{A}}$. As $A_n \subseteq C_n$, this means that

$$p \Vdash \text{“}\check{n} \in \dot{X}\text{”}.$$

Thus $n = \check{n}^G \in \dot{X}^G$.

Now we check that $\dot{X}_{\mathscr{A}}^G \supseteq \dot{X}^G$. If $n \in \dot{X}^G$ then by the forcing theorem (Theorem 3.3.3) there exists some $p \in G$ with

$$p \Vdash \text{“}\check{n} \in \dot{X}\text{”}.$$

We claim that $G \cap A_n \neq \varnothing$. Indeed, if $G \cap A_n = \varnothing$ then there exists some $q \in G$ which is incompatible with all of the conditions in A_n (Lemma 3.1.6). Taking some common extension $r \in G$ of both p and q, we get that $r \in C_n$ and r is incompatible with all the conditions in A_n. But this contradicts A_n being an antichain which is maximal in C_n.

So there exists some $q \in G \cap A_n$. This q satisfies $(\check{n}, q) \in \dot{X}_{\mathscr{A}}$, and so $n = \check{n}^G \in \dot{X}_{\mathscr{A}}^G$. $\square$

Therefore, if $\mathbb{P}$ is a c.c.c. forcing notion and G is a $\mathbb{P}$-generic filter over $\mathbf{M}$, the number of subsets of ω in $\mathbf{M}[G]$ is bounded above by the number of nice $\mathbb{P}$-names in $\mathbf{M}$ for subsets of ω. If $\mathbf{M} \models$ "$\mathbb{P}$ satisfies c.c.c. and $|\mathbb{P}| = \kappa$", then in $\mathbf{M}$ there are at most $\kappa^{\aleph_0}$-many antichains, and hence at most $\left(\kappa^{\aleph_0}\right)^{\aleph_0} = \kappa^{\aleph_0}$-many ω-sequences of antichains, and thus at most $\kappa^{\aleph_0}$-many nice $\mathbb{P}$-names for subsets of ω.

As our final result, we show that $2^{\aleph_0}$ really can be anything it ought to be. The result below assumes that $\mathbf{M} \models \mathsf{GCH}$, which we can get by assuming that $\mathbf{M} \models \mathsf{ZFC} +$ "$\mathbf{V} = \mathbf{L}$".

Theorem 4.3.4
Suppose that $\mathbf{M} \models \mathsf{GCH}$. *Let* $\kappa \in \mathbf{M}$ *be such that*

$$\mathbf{M} \models \text{"}\kappa \textit{ is an uncountable cardinal with } \mathrm{cf}(\kappa) > \aleph_0\text{"}.$$

Consider the forcing notion $\mathbb{P} := \mathrm{Fin}(\omega \times \kappa, 2) \in \mathbf{M}$ *and let* G *be a* $\mathbb{P}$*-generic filter over* $\mathbf{M}$. *Then*

$$\mathbf{M}[G] \models \text{"}2^{\aleph_0} = \kappa\text{"}.$$

Proof. We already know $\mathbb{P}$ satisfies c.c.c. and that $\mathbf{M}[G] \models$ "$2^{\aleph_0} \geq \kappa$".

To show that $\mathbf{M}[G] \models$ "$2^{\aleph_0} \leq \kappa$", as $\mathbf{M} \models$ "$|\mathbb{P}| = \kappa$", we just need to show that there are at most κ-many nice $\mathbb{P}$-names in $\mathbf{M}$ for subsets of ω. From the discussion above, it suffices to show that $\mathbf{M} \models$ "$\kappa^{\aleph_0} = \kappa$".

We work in $\mathbf{M}$. Since $\mathrm{cf}(\kappa) > \aleph_0$, denoting κ^ω for the set of all functions from ω to κ, we have

$$\kappa^\omega = \bigcup_{\omega \leq \alpha < \kappa} \alpha^\omega,$$

where α^ω is the set of all functions from ω to α. Now, for all $\omega \leq \alpha < \kappa$, we have that

$$|\alpha^\omega| = |\alpha|^{\aleph_0} \leq \left(2^{|\alpha|}\right)^{\aleph_0} = 2^{|\alpha| \cdot \aleph_0} = 2^{|\alpha|} = |\alpha|^+ \leq \kappa$$

by GCH. Therefore $\kappa^{\aleph_0} = \kappa$. $\square$

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