

Solutions to exercises in Bart Jacobs’s book “Introduction to Coalgebra: Towards Mathematics of States and Observation”

Ryan Tay

some date very far into the future, if ever

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These are my solutions to all the labelled exercises in [Jacobs \(2017\)](#). This document does not stand on its own; it is meant to supplement the book.

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1 Motivation

1.1 Naturalness of Coalgebraic Representations

Exercise 1.1.1

1. Prove that the composition operation $;$ as defined for coalgebras $S \rightarrow \{\perp\} \cup S$ is associative, i.e. satisfies $s_1 ; (s_2 ; s_3) = (s_1 ; s_2) ; s_3$, for all statements $s_1, s_2, s_3 : S \rightarrow \{\perp\} \cup S$.

Define a statement **skip**: $S \rightarrow \{\perp\} \cup S$ which is a unit for composition $;$ i.e. which satisfies $(\text{skip} ; s) = s = (s ; \text{skip})$, for all $s : S \rightarrow \{\perp\} \cup S$.

2. Do the same for $;$ defined on coalgebras $S \rightarrow \{\perp\} \cup S \cup (S \times E)$.

(In both cases, statements with an associative composition operation and a unit element form a monoid.)

Solution.

1. Recall that the composition operation $;$ was defined as follows:

$$s ; t := \lambda x \in S. \begin{cases} \perp, & \text{if } s(x) = \perp, \\ t(x'), & \text{if } s(x) = x' \in S, \end{cases}$$

for coalgebras $s, t : S \rightarrow \{\perp\} \cup S$. Fix any three coalgebras $s_1, s_2, s_3 : S \rightarrow \{\perp\} \cup S$. Then

$$\begin{aligned} s_1 ; (s_2 ; s_3) &= \lambda x \in S. \begin{cases} \perp, & \text{if } s_1(x) = \perp, \\ (s_2 ; s_3)(x'), & \text{if } s_1(x) = x' \in S, \end{cases} \\ &= \lambda x \in S. \begin{cases} \perp, & \text{if either } s_1(x) = \perp, \text{ or both } s_1(x) = x' \in S \text{ and } s_2(x') = \perp, \\ s_3(x''), & \text{if } s_1(x) = x' \in S \text{ and } s_2(x') = x'' \in S, \end{cases} \\ &= \lambda x \in S. \begin{cases} \perp, & \text{if } (s_1 ; s_2)(x) = \perp, \\ s_3(x''), & \text{if } (s_1 ; s_2)(x) = x'' \in S, \end{cases} \\ &= (s_1 ; s_2) ; s_3. \end{aligned}$$

So the composition operation $;$ is associative.

The coalgebra **skip**: $S \rightarrow \{\perp\} \cup S$ defined by $\text{skip}(x) := x$, for all $x \in S$, satisfies $(\text{skip} ; s) = s = (s ; \text{skip})$ for all coalgebras $s : S \rightarrow \{\perp\} \cup S$.

2. Now we consider the composition operation $;$ defined as follows:

$$s ; t := \lambda x \in S. \begin{cases} \perp, & \text{if } s(x) = \perp, \\ t(x'), & \text{if } s(x) = x' \in S, \\ (x', e), & \text{if } s(x) = (x', e) \in S \times E, \end{cases}$$

for coalgebras $s, t : S \rightarrow \{\perp\} \cup S \cup (S \times E)$. Fix any three coalgebras $s_1, s_2, s_3 : \{\perp\} \cup S \cup (S \times E)$. Then

$$s_1 ; (s_2 ; s_3) = \lambda x \in S. \begin{cases} \perp, & \text{if } s_1(x) = \perp, \\ (s_2 ; s_3)(x'), & \text{if } s_1(x) = x' \in S, \\ (x', e), & \text{if } s_1(x) = (x', e) \in S \times E, \end{cases}$$

$$\begin{aligned}
&= \lambda x \in S. \begin{cases} \perp, & \text{if either } s_1(x) = \perp, \text{ or both } s_1(x) = x' \in S \text{ and } s_2(x') = \perp, \\ s_3(x''), & \text{if } s_1(x) = x' \in S \text{ and } s_2(x') = x'' \in S, \\ (x'', e), & \text{if } s_1(x) = x' \in S \text{ and } s_2(x') = (x'', e) \in S \times E, \\ (x', e), & \text{if } s_1(x) = (x', e) \in S \times E, \end{cases} \\
&= \lambda x \in S. \begin{cases} \perp, & \text{if } (s_1; s_2)(x) = \perp, \\ s_3(x''), & \text{if } (s_1; s_2)(x) = x'' \in S, \\ (x'', e), & \text{if } (s_1; s_2)(x) = (x'', e) \in S \times E, \end{cases} \\
&= (s_1; s_2); s_3.
\end{aligned}$$

So this composition operation $;$ is also associative.

Now define the coalgebra $\text{skip}: S \rightarrow \{\perp\} \cup S \cup (S \times E)$ by $\text{skip}(x) := x$, for all $x \in S$. Then we have $(\text{skip}; s) = s = (s; \text{skip})$ for all coalgebras $s: S \rightarrow \{\perp\} \cup S \cup (S \times E)$. \square

Exercise 1.1.2

Define also a composition monoid $(\text{skip}, ;)$ for coalgebras $S \rightarrow \mathcal{P}(S)$.

Solution. For coalgebras $s, t: S \rightarrow \mathcal{P}(S)$, define

$$s; t := \lambda x \in S. \left(\bigcup_{y \in s(x)} t(y) \right).$$

Then, for coalgebras $s_1, s_2, s_3: S \rightarrow \mathcal{P}(S)$, we have

$$\begin{aligned}
s_1; (s_2; s_3) &= \lambda x \in S. \left(\bigcup_{y \in s_1(x)} (s_2; s_3)(y) \right) \\
&= \lambda x \in S. \left(\bigcup_{y \in s_1(x)} \bigcup_{z \in s_2(y)} s_3(z) \right) \\
&= \lambda x \in S. \left(\bigcup_{z \in (s_1; s_2)(x)} s_3(z) \right) \\
&= (s_1; s_2); s_3.
\end{aligned}$$

Furthermore, defining $\text{skip}: S \rightarrow \mathcal{P}(S)$ by $\text{skip}(x) := \{x\}$ for all $x \in S$, we have

$$\begin{aligned}
(\text{skip}; s) &= \lambda x \in S. \left(\bigcup_{y \in \text{skip}(x)} s(y) \right) \\
&= \lambda x \in S. \left(\bigcup_{y \in \{x\}} s(y) \right) \\
&= \lambda x \in S. s(x) \\
&= s
\end{aligned}$$

and

$$\begin{aligned}
(s; \text{skip}) &= \lambda x \in S. \left(\bigcup_{y \in s(x)} \text{skip}(y) \right) \\
&= \lambda x \in S. \left(\bigcup_{y \in s(x)} \{y\} \right) \\
&= \lambda x \in S. s(x) \\
&= s.
\end{aligned}$$

□

1.2 The Power of Coinduction

Exercise 1.2.1

Compute the `nextdec`-behaviour of $\frac{1}{7} \in [0, 1)$ as in Example 1.2.2.

Solution. We first recall all of the following functions.

1. The final coalgebra `next`: $\{0, \dots, 9\}^\infty \rightarrow \{\perp\} \cup (\{0, \dots, 9\} \times \{0, \dots, 9\}^\infty)$ is defined by

$$\text{next}(\sigma) := \begin{cases} \perp, & \text{if } \sigma \text{ is the empty sequence,} \\ (d, \sigma'), & \text{if } \sigma \text{ has head } d \in \{0, \dots, 9\} \text{ and tail } \sigma' \in \{0, \dots, 9\}^\infty, \text{ i.e. } \sigma = d \cdot \sigma', \end{cases}$$

for all (finite or infinite) sequences $\sigma \in \{0, \dots, 9\}^\infty$.

2. The coalgebra `nextdec`: $[0, 1) \rightarrow \{\perp\} \cup (\{0, \dots, 9\} \times [0, 1))$ is defined by

$$\text{nextdec}(r) := \begin{cases} \perp, & \text{if } r = 0, \\ (d, 10r - d), & \text{if } d \leq 10r < d + 1 \text{ and } d \in \{0, \dots, 9\}, \end{cases}$$

for all $r \in [0, 1)$.

3. The function $\text{beh}_{\text{nextdec}}: [0, 1) \rightarrow \{0, \dots, 9\}^\infty$ is the unique function making

$$\begin{array}{ccc}
\{\perp\} \cup (\{0, \dots, 9\} \times [0, 1)) & \xrightarrow{\text{id}_{\{\perp\}} \cup (\text{id}_{\{0, \dots, 9\}} \times \text{beh}_{\text{nextdec}})} & \{\perp\} \cup (\{0, \dots, 9\} \times \{0, \dots, 9\}^\infty) \\
\uparrow \text{nextdec} & & \uparrow \cong \text{next} \\
[0, 1) & \xrightarrow{\exists! \text{beh}_{\text{nextdec}}} & \{0, \dots, 9\}^\infty
\end{array}$$

commute.

We wish to compute $\text{beh}_{\text{nextdec}}(\frac{1}{7})$. We see that

$$\begin{aligned}
\text{beh}_{\text{nextdec}}\left(\frac{1}{7}\right) &= \text{next}^{-1} \left(\left(\text{id}_{\{\perp\}} \cup (\text{id}_{\{0, \dots, 9\}} \times \text{beh}_{\text{nextdec}}) \right) \left(\text{nextdec}\left(\frac{1}{7}\right) \right) \right) \\
&= \text{next}^{-1} \left(\left(\text{id}_{\{\perp\}} \cup (\text{id}_{\{0, \dots, 9\}} \times \text{beh}_{\text{nextdec}}) \right) \left(\left(1, \frac{3}{7} \right) \right) \right)
\end{aligned}$$

$$\begin{aligned}
&= \text{next}^{-1} \left(\left(1, \text{beh}_{\text{nextdec}} \left(\frac{3}{7} \right) \right) \right) \\
&= 1 \cdot \text{beh}_{\text{nextdec}} \left(\frac{3}{7} \right).
\end{aligned}$$

Continuing in this fashion,

$$\begin{aligned}
\text{beh}_{\text{nextdec}} \left(\frac{1}{7} \right) &= 1 \cdot \text{beh}_{\text{nextdec}} \left(\frac{3}{7} \right) \\
&= 1 \cdot \left(4 \cdot \text{beh}_{\text{nextdec}} \left(\frac{2}{7} \right) \right) \\
&= 1 \cdot \left(4 \cdot \left(2 \cdot \text{beh}_{\text{nextdec}} \left(\frac{6}{7} \right) \right) \right) \\
&= 1 \cdot \left(4 \cdot \left(2 \cdot \left(8 \cdot \text{beh}_{\text{nextdec}} \left(\frac{4}{7} \right) \right) \right) \right) \\
&= 1 \cdot \left(4 \cdot \left(2 \cdot \left(8 \cdot \left(5 \cdot \text{beh}_{\text{nextdec}} \left(\frac{5}{7} \right) \right) \right) \right) \right) \\
&= 1 \cdot \left(4 \cdot \left(2 \cdot \left(8 \cdot \left(5 \cdot \left(7 \cdot \text{beh}_{\text{nextdec}} \left(\frac{1}{7} \right) \right) \right) \right) \right) \right).
\end{aligned}$$

Therefore $\text{beh}_{\text{nextdec}} \left(\frac{1}{7} \right) = \langle 1, 4, 2, 8, 5, 7, 1, 4, 2, 8, 5, 7, 1, 4, 2, 8, 5, 7, \dots \rangle$.

□

Exercise 1.2.2

Formulate appropriate rules for the function **odds**: $A^\infty \rightarrow A^\infty$ in analogy with the rules (1.7) for **evens**.

Solution. We recall that, for a sequence $\sigma := \langle a_0, a_1, a_2, a_3, \dots \rangle \in A^\infty$, the function **odds** satisfies $\text{odds}(\sigma) = \langle a_1, a_3, a_5, \dots \rangle$, and analogously if σ is a finite sequence. The rules we want **odds** to satisfy are:

$$\frac{\sigma \not\rightarrow}{\text{odds}(\sigma) \not\rightarrow}$$

i.e. **odds** should send the empty sequence to the empty sequence;

$$\frac{\sigma \xrightarrow{a} \sigma' \quad \sigma' \not\rightarrow}{\text{odds}(\sigma) \not\rightarrow}$$

i.e. **odds** should send a singleton sequence $\langle a \rangle$ to the empty sequence; and

$$\frac{\sigma \xrightarrow{a} \sigma' \quad \sigma' \xrightarrow{a'} \sigma''}{\text{odds}(\sigma) \xrightarrow{a'} \text{odds}(\sigma')}$$

i.e. if $\sigma = a \cdot a' \cdot \sigma' \in A^\infty$, where $a, a' \in A$, then $\text{odds}(\sigma) = a' \cdot \text{odds}(\sigma')$.

□

Exercise 1.2.3

Use coinduction to define the empty sequence $\langle \rangle \in A^\infty$ as a map $\{\perp\} \rightarrow A^\infty$.

Fix an element $a \in A$, and similarly define the infinite sequence $\vec{a}: \{\perp\} \rightarrow A^\infty$ consisting of only a s.

Solution. We recall that the final coalgebra $\text{next}: A^\infty \rightarrow \{\perp\} \cup (A \times A^\infty)$ is defined by

$$\text{next}(\sigma) := \begin{cases} \perp, & \text{if } \sigma \text{ is the empty sequence,} \\ (a, \sigma'), & \text{if } \sigma \text{ has head } a \in A \text{ and tail } \sigma' \in A^\infty, \text{ i.e. } \sigma = a \cdot \sigma', \end{cases}$$

for all (finite or infinite) sequences $\sigma \in A^\infty$.

For the coalgebra $\kappa_1: \{\perp\} \rightarrow \{\perp\} \cup (A \times \{\perp\})$ defined by $\kappa_1(\perp) := \perp$, the unique function $\text{beh}_{\kappa_1}: \{\perp\} \rightarrow A^\infty$ making

$$\begin{array}{ccc} \{\perp\} \cup (A \times \{\perp\}) & \xrightarrow{\text{id}_{\{\perp\}} \cup (\text{id}_A \times \text{beh}_{\kappa_1})} & \{\perp\} \cup (A \times A^\infty) \\ \uparrow \kappa_1 & & \uparrow \cong \text{next} \\ \{\perp\} & \xrightarrow{\exists! \text{beh}_{\kappa_1}} & A^\infty \end{array}$$

commute satisfies $\text{beh}_{\kappa_1}(\perp) = \langle \rangle$.

For the coalgebra $c_a: \{\perp\} \rightarrow \{\perp\} \cup (A \times \{\perp\})$ defined by $c_a(\perp) := (a, \perp)$, the unique function $\text{beh}_{c_a}: \{\perp\} \rightarrow A^\infty$ making

$$\begin{array}{ccc} \{\perp\} \cup (A \times \{\perp\}) & \xrightarrow{\text{id}_{\{\perp\}} \cup (\text{id}_A \times \text{beh}_{c_a})} & \{\perp\} \cup (A \times A^\infty) \\ \uparrow c_a & & \uparrow \cong \text{next} \\ \{\perp\} & \xrightarrow{\exists! \text{beh}_{c_a}} & A^\infty \end{array}$$

commute satisfies $\text{beh}_{c_a}(\perp) = \vec{a} = \langle a, a, a, \dots \rangle$. □

Exercise 1.2.4

Compute the outcome of $\text{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle)$.

Solution. Recall that we defined the coalgebra $m: A^\infty \times A^\infty \rightarrow \{\perp\} \cup (A \times (A^\infty \times A^\infty))$ by

$$m(\sigma, \tau) := \begin{cases} \perp, & \text{if } \sigma \not\rightarrow \text{ and } \tau \not\rightarrow, \\ (a, (\sigma, \tau')), & \text{if } \sigma \not\rightarrow \text{ and } \tau \xrightarrow{a} \tau', \\ (a, (\tau, \sigma')), & \text{if } \sigma \xrightarrow{a} \sigma', \end{cases}$$

for all $\sigma, \tau \in A^\infty$, and that $\text{merge}: A^\infty \times A^\infty \rightarrow A^\infty$ is the unique function making

$$\begin{array}{ccc} \{\perp\} \cup (A \times (A^\infty \times A^\infty)) & \xrightarrow{\text{id}_{\{\perp\}} \cup (\text{id}_A \times \text{merge})} & \{\perp\} \cup (A \times A^\infty) \\ \uparrow m & & \uparrow \cong \text{next} \\ A^\infty \times A^\infty & \xrightarrow{\exists! \text{merge}} & A^\infty \end{array}$$

commute. Then

$$\text{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) = \text{next}^{-1} \left((\text{id}_{\{\perp\}} \cup (\text{id}_A \times \text{merge})) (m(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle)) \right)$$

$$\begin{aligned}
&= \text{next}^{-1} \left((\text{id}_{\{\perp\}} \cup (\text{id}_A \times \text{merge})) ((a_0, (\langle b_0, b_1, b_2, b_3 \rangle, \langle a_1, a_2 \rangle))) \right) \\
&= \text{next}^{-1} \left((a_0, \text{merge}(\langle b_0, b_1, b_2, b_3 \rangle, \langle a_1, a_2 \rangle)) \right) \\
&= a_0 \cdot \text{merge}(\langle b_0, b_1, b_2, b_3 \rangle, \langle a_1, a_2 \rangle),
\end{aligned}$$

and so on. Eventually, we obtain $\text{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) = \langle a_0, b_0, a_1, b_1, a_2, b_2, b_3 \rangle$. \square

Exercise 1.2.5

Is the merge operation associative, i.e. is $\text{merge}(\sigma, \text{merge}(\tau, \rho))$ the same as $\text{merge}(\text{merge}(\sigma, \tau), \rho)$? Give a proof or a counterexample. Is there a neutral element for merge?

Solution. The merge operation is not associative:

$$\begin{aligned}
\text{merge}(\langle a \rangle, \text{merge}(\langle b \rangle, \langle c \rangle)) &= \text{merge}(\langle a \rangle, \langle b, c \rangle) \\
&= \langle a, b, c \rangle,
\end{aligned}$$

whereas

$$\begin{aligned}
\text{merge}(\text{merge}(\langle a \rangle, \langle b \rangle), \langle c \rangle) &= \text{merge}(\langle a, b \rangle, \langle c \rangle) \\
&= \langle a, c, b \rangle,
\end{aligned}$$

for all $a, b, c \in A$.

The neutral element for merge is the empty sequence: for any $\sigma \in A^\infty$, we have $\text{merge}(\sigma, \langle \rangle) = \text{merge}(\langle \rangle, \sigma) = \sigma$. \square

Exercise 1.2.6

Show how to define an alternative merge function which alternately takes two elements from its argument sequences.

Solution. We will define a coalgebra $m_2: A^\infty \times A^\infty \rightarrow \{\perp\} \cup (A \times (A^\infty \times A^\infty))$ so that the desired merge function is the unique function $\text{merge}_2: A^\infty \times A^\infty \rightarrow A^\infty$ making

$$\begin{array}{ccc}
\{\perp\} \cup (A \times (A^\infty \times A^\infty)) & \xrightarrow{\text{id}_{\{\perp\}} \cup (\text{id}_A \times \text{merge}_2)} & \{\perp\} \cup (A \times A^\infty) \\
\uparrow m_2 & & \uparrow \cong \text{next} \\
A^\infty \times A^\infty & \xrightarrow{\exists! \text{merge}_2} & A^\infty
\end{array}$$

commute. As a motivating example, the desired merge of two infinite streams $\langle a_0, a_1, \dots \rangle$ and $\langle b_0, b_1, \dots \rangle$ should be

$$\text{merge}_2(\langle a_0, a_1, a_2, a_3, \dots \rangle, \langle b_0, b_1, b_2, b_3, \dots \rangle) = \langle a_0, a_1, b_0, b_1, a_2, a_3, b_2, b_3, \dots \rangle.$$

As the diagram above commutes, we would require

$$\text{merge}_2(m_2(\langle a_0, a_1, a_2, a_3, \dots \rangle, \langle b_0, b_1, b_2, b_3, \dots \rangle)) = (a_0, \langle a_1, b_0, b_1, a_2, a_3, b_2, b_3, \dots \rangle)$$

and so m_2 should be defined to satisfy

$$m_2(\langle a_0, a_1, a_2, a_3, \dots \rangle, \langle b_0, b_1, b_2, b_3, \dots \rangle) = (a_0, (\langle a_1, b_0, a_3, b_2, \dots \rangle, \langle b_1, a_2, b_3, a_4, \dots \rangle))$$

Dealing with edge cases separately leads us to the following definition: we define the coalgebra $m_2: A^\infty \times A^\infty \rightarrow \{\perp\} \cup (A \times (A^\infty \times A^\infty))$ as follows.

1. The function m_2 sends the pair $(\langle \rangle, \langle \rangle)$ to \perp , i.e.

$$m_2(\langle \rangle, \langle \rangle) := \perp.$$

2. If $\tau \in A^\infty$ is a non-empty sequence, say $\tau \xrightarrow{a} \tau'$ for some $\tau' \in A^\infty$ and $a \in A$, then

$$m_2(\langle \rangle, \tau) := (a, (\langle \rangle, \tau')).$$

3. If $\sigma = \langle a \rangle$ for some $a \in A$, then

$$m_2(\langle a \rangle, \tau) := (a, (\langle \rangle, \tau))$$

for all $\tau \in A^\infty$.

4. If $\sigma \in A^\infty$ has at least length 2, say $\sigma \xrightarrow{a} \sigma' \xrightarrow{a'} \sigma''$ for some $\sigma', \sigma'' \in A^\infty$ and $a, a' \in A$, then

$$m_2(\sigma, \tau) := \left(a, \left(\text{merge}(\text{odds}(\sigma), \text{evens}(\tau)), \text{merge}(\text{odds}(\tau), \text{evens}(\sigma'')) \right) \right)$$

for all $\tau \in A^\infty$.

Now let $\text{merge}_2: A^\infty \times A^\infty \rightarrow A^\infty$ be the unique function which makes

$$\begin{array}{ccc} \{\perp\} \cup (A \times (A^\infty \times A^\infty)) & \xrightarrow{\text{id}_{\{\perp\}} \cup (\text{id}_A \times \text{merge}_2)} & \{\perp\} \cup (A \times A^\infty) \\ \uparrow m_2 & & \uparrow \cong \text{next} \\ A^\infty \times A^\infty & \xrightarrow{\exists! \text{merge}_2} & A^\infty \end{array}$$

commute. Fix any $\sigma, \tau \in A^\infty$. We argue by cases on (σ, τ) that this function merge_2 is the desired merge function.

1. If $\sigma = \tau = \langle \rangle$, then $\text{merge}_2(\langle \rangle, \langle \rangle) = \langle \rangle$.
2. If $\sigma = \langle \rangle$ and τ is a non-empty sequence, say $\tau = a \cdot \tau'$ for some $a \in A$ and $\tau' \in A^\infty$, then

$$\text{merge}_2(\langle \rangle, \tau) = a \cdot \text{merge}_2(\langle \rangle, \tau').$$

Thus $\text{merge}_2(\langle \rangle, \tau) = \tau$.

3. If $\sigma = \langle a \rangle$ for some $a \in A$, then

$$\begin{aligned} \text{merge}_2(\langle a \rangle, \tau) &= a \cdot \text{merge}_2(\langle \rangle, \tau) \\ &= a \cdot \tau. \end{aligned}$$

4. If $\sigma = a \cdot a' \cdot \sigma''$ for some $a, a' \in A$ and $\sigma'' \in A^\infty$, then

$$\begin{aligned} \text{merge}_2(\sigma, \tau) &= a \cdot \text{merge}_2\left(\text{merge}(\text{odds}(\sigma), \text{evens}(\tau)), \text{merge}(\text{odds}(\tau), \text{evens}(\sigma''))\right) \\ &= a \cdot \text{merge}_2\left(\text{merge}(a' \cdot \text{odds}(\sigma''), \text{evens}(\tau)), \text{merge}(\text{odds}(\tau), \text{evens}(\sigma''))\right) \\ &= a \cdot a' \cdot \text{merge}_2\left(\text{merge}(\text{odds}(\text{merge}(a' \cdot \text{odds}(\sigma''), \text{evens}(\tau))), \right. \\ &\quad \left. \text{evens}(\text{merge}(\text{odds}(\tau), \text{evens}(\sigma'')))\right), \end{aligned}$$

$$\begin{aligned}
& \text{merge}(\text{odds}(\text{merge}(\text{odds}(\tau), \text{evens}(\sigma''))), \\
& \quad \text{odds}(\text{merge}(\text{evens}(\tau), \text{odds}(\sigma'')))) \\
&= a \cdot a' \cdot \text{merge}_2\left(\text{merge}(\text{evens}(\tau), \text{odds}(\tau)), \text{merge}(\text{evens}(\sigma''), \text{odds}(\sigma''))\right) \\
&= a \cdot a' \cdot \text{merge}_2(\tau, \sigma''),
\end{aligned}$$

as desired. \square

Exercise 1.2.7

1. Define three functions $\text{ex}_i: A^\infty \rightarrow A^\infty$, for $i = 0, 1, 2$, which extract the elements at positions $3n + i$.
2. Define $\text{merge3}: A^\infty \times A^\infty \times A^\infty \rightarrow A^\infty$ satisfying the equation $\text{merge3}(\text{ex}_0(\sigma), \text{ex}_1(\sigma), \text{ex}_2(\sigma)) = \sigma$, for all $\sigma \in A^\infty$.

Solution.

1. Define $c_0, c_1, c_2: A^\infty \rightarrow \{\perp\} \cup (A \times A^\infty)$ as follows:

$$\begin{aligned}
c_0(\sigma) &:= \begin{cases} \perp, & \text{if } \sigma = \langle \rangle, \\ (a, \langle \rangle) & \text{if } \sigma = \langle a \rangle \text{ or } \sigma = \langle a, a' \rangle \text{ for some } a, a' \in A, \\ (a, \sigma'''), & \text{if } \sigma \xrightarrow{a} \sigma' \xrightarrow{a'} \sigma'' \xrightarrow{a''} \sigma''', \end{cases} \\
c_1(\sigma) &:= \begin{cases} \perp, & \text{if } \sigma = \langle \rangle \text{ or } \sigma = \langle a \rangle \text{ for some } a \in A, \\ (a', \langle \rangle) & \text{if } \sigma = \langle a, a' \rangle \text{ for some } a, a' \in A, \\ (a', \sigma'''), & \text{if } \sigma \xrightarrow{a} \sigma' \xrightarrow{a'} \sigma'' \xrightarrow{a''} \sigma''', \end{cases} \\
c_2(\sigma) &:= \begin{cases} \perp, & \text{if } \sigma = \langle \rangle, \text{ or } \sigma = \langle a \rangle, \text{ or } \sigma = \langle a, a' \rangle \text{ for some } a, a' \in A, \\ (a'', \sigma'''), & \text{if } \sigma \xrightarrow{a} \sigma' \xrightarrow{a'} \sigma'' \xrightarrow{a''} \sigma'''. \end{cases}
\end{aligned}$$

Then, for $i \in \{0, 1, 2\}$, the function $\text{ex}_i: A^\infty \rightarrow A^\infty$ is the unique function making

$$\begin{array}{ccc}
\{\perp\} \cup (A \times A^\infty) & \xrightarrow{\text{id}_{\{\perp\}} \cup (\text{id}_A \times \text{ex}_i)} & \{\perp\} \cup (A \times A^\infty) \\
\uparrow c_i & & \uparrow \cong \text{next} \\
A^\infty & \xrightarrow{\exists! \text{ex}_i} & A^\infty
\end{array}$$

commute.

2. Define the coalgebra $m_3: A^\infty \times A^\infty \times A^\infty \rightarrow \{\perp\} \cup (A \times (A^\infty \times A^\infty \times A^\infty))$ by

$$m_3(\sigma, \tau, \rho) := \begin{cases} \perp, & \text{if } \sigma = \tau = \rho = \langle \rangle, \\ (a, (\langle \rangle, \langle \rangle, \rho')), & \text{if } \sigma = \tau = \langle \rangle \text{ and } \rho \xrightarrow{a} \rho' \text{ for some } a \in A \text{ and } \rho' \in A^\infty, \\ (a, (\langle \rangle, \rho, \tau')), & \text{if } \sigma = \langle \rangle \text{ and } \tau \xrightarrow{a} \tau' \text{ for some } a \in A \text{ and } \tau' \in A^\infty, \\ (a, (\tau, \rho, \sigma')), & \text{if } \sigma \xrightarrow{a} \sigma' \text{ for some } a \in A \text{ and } \sigma' \in A^\infty. \end{cases}$$

Then we let $\text{merge3}: A^\infty \times A^\infty \times A^\infty \rightarrow A^\infty$ be the unique function making

$$\begin{array}{ccc} \{\perp\} \cup (A \times (A^\infty \times A^\infty \times A^\infty)) & \xrightarrow{\text{id}_{\{\perp\}} \cup (\text{id}_A \times \text{merge3})} & \{\perp\} \cup (A \times A^\infty) \\ \uparrow m_3 & & \uparrow \cong \text{next} \\ A^\infty \times A^\infty \times A^\infty & \xrightarrow{\exists! \text{merge3}} & A^\infty \end{array}$$

commute.

Let us prove that $\text{merge3}(\text{ex}_0(\sigma), \text{ex}_1(\sigma), \text{ex}_2(\sigma)) = \sigma$ for all $\sigma \in A^\infty$, by coinduction. Consider the function $f: A^\infty \rightarrow A^\infty \times A^\infty \times A^\infty$ defined by $f(\sigma) := (\text{ex}_0(\sigma), \text{ex}_1(\sigma), \text{ex}_2(\sigma))$ for all $\sigma \in A^\infty$. We wish to show that $\text{merge3} \circ f = \text{id}_{A^\infty}$.

$$\begin{array}{ccccc} & & \{\perp\} \cup (A \times (A^\infty \times A^\infty \times A^\infty)) & \xrightarrow{\text{id}_{\{\perp\}} \cup (\text{id}_A \times \text{merge3})} & \{\perp\} \cup (A \times A^\infty) \\ & \nearrow \text{id}_{\{\perp\}} \cup (\text{id}_A \times f) & \uparrow m_3 & & \uparrow \cong \text{next} \\ \{\perp\} \cup (A \times A^\infty) & & & & \\ \uparrow \text{next} \cong & & & & \\ A^\infty & \xrightarrow{f} & A^\infty \times A^\infty \times A^\infty & \xrightarrow{\text{merge3}} & A^\infty \end{array}$$

Let us first show that the left square commutes. It certainly commutes when we chase the empty sequence: $(m_3 \circ f)(\langle \rangle) = \perp = ((\text{id}_{\{\perp\}} \cup (\text{id}_A \times f)) \circ \text{next})(\langle \rangle)$. If $\sigma \in A^\infty$ is a non-empty sequence, say $\sigma \xrightarrow{a} \sigma'$ for some $a \in A$ and $\sigma' \in A^\infty$, then we have

$$\begin{aligned} (m_3 \circ f)(\sigma) &= m_3(\text{ex}_0(\sigma), \text{ex}_1(\sigma), \text{ex}_2(\sigma)) \\ &= (a, (\text{ex}_1(\sigma), \text{ex}_2(\sigma), \text{ex}_0(\sigma'))) \\ &= (a, (\text{ex}_0(\sigma'), \text{ex}_1(\sigma'), \text{ex}_2(\sigma'))) \\ &= ((\text{id}_{\{\perp\}} \cup (\text{id}_A \times f)) \circ \text{next})(\sigma), \end{aligned}$$

where the second-to-last equality can also be proven by coinduction. Therefore the outer square commutes, and so

$$\text{next} \circ (\text{merge3} \circ f) = ((\text{id}_{\{\perp\}} \cup (\text{id}_A \times (\text{merge3} \circ f))) \circ \text{next}.$$

The finality of the coalgebra $\text{next}: A^\infty \rightarrow \{\perp\} \cup (A \times A^\infty)$ now yields $\text{merge3} \circ f = \text{id}_{A^\infty}$. \square

Exercise 1.2.8

Consider the sequential composition function $\text{comp}: A^\infty \times A^\infty \rightarrow A^\infty$ for sequences, described by the three rules:

$$\begin{array}{c} \frac{\sigma \not\rightarrow \quad \tau \not\rightarrow}{\text{comp}(\sigma, \tau) \not\rightarrow} \qquad \frac{\sigma \not\rightarrow \quad \tau \xrightarrow{a} \tau'}{\text{comp}(\sigma, \tau) \xrightarrow{a} \text{comp}(\sigma, \tau')} \\ \frac{\sigma \xrightarrow{a} \sigma'}{\text{comp}(\sigma, \tau) \xrightarrow{a} \text{comp}(\sigma', \tau)} \end{array}.$$

1. Show by coinduction that the empty sequence $\langle \rangle = \text{next}^{-1}(\perp) \in A^\infty$ is a unit element for **comp**, i.e. that $\text{comp}(\langle \rangle, \sigma) = \sigma = \text{comp}(\sigma, \langle \rangle)$.
2. Prove also by coinduction that **comp** is associative, and thus that sequences carry a monoid structure.

Solution.

1. Let $f: A^\infty \rightarrow A^\infty$ be defined by $f(\sigma) := \text{comp}(\langle \rangle, \sigma)$. We will show that the diagram

$$\begin{array}{ccc}
 \{\perp\} \cup (A \times A^\infty) & \xrightarrow{\text{id}_{\{\perp\}} \cup (\text{id}_A \times f)} & \{\perp\} \cup (A \times A^\infty) \\
 \uparrow \text{next} \cong & & \cong \uparrow \text{next} \\
 A^\infty & \xrightarrow{f} & A^\infty
 \end{array}$$

commutes, which would yield $f = \text{id}_{A^\infty}$ by the finality of the coalgebra **next**.

First, we chase the empty sequence from the bottom left. We see that

$$\begin{aligned}
 (\text{next} \circ f)(\langle \rangle) &= \text{next}(\text{comp}(\langle \rangle, \langle \rangle)) \\
 &= \text{next}(\langle \rangle) \\
 &= \perp,
 \end{aligned}$$

the first rule for **comp**, and

$$\begin{aligned}
 ((\text{id}_{\{\perp\}} \cup (\text{id}_A \times f)) \circ \text{next})(\langle \rangle) &= (\text{id}_{\{\perp\}} \cup (\text{id}_A \times f))(\perp) \\
 &= \perp.
 \end{aligned}$$

Now if $\sigma \in A^\infty$ is a non-empty sequence, say $\sigma \xrightarrow{a} \sigma'$ for some $a \in A$ and $\sigma' \in A^\infty$, we see that

$$\begin{aligned}
 (\text{next} \circ f)(\sigma) &= \text{next}(\text{comp}(\langle \rangle, a \cdot \sigma')) \\
 &= (a, \text{comp}(\langle \rangle, \sigma')) \\
 &= (a, f(\sigma')),
 \end{aligned}$$

by the second rule for **comp** and the definition of f , and

$$\begin{aligned}
 ((\text{id}_{\{\perp\}} \cup (\text{id}_A \times f)) \circ \text{next})(\sigma) &= ((\text{id}_{\{\perp\}} \cup (\text{id}_A \times f))((a, \sigma'))) \\
 &= (a, f(\sigma')).
 \end{aligned}$$

Thus $\text{next} \circ f = (\text{id}_{\{\perp\}} \cup (\text{id}_A \times f)) \circ \text{next}$. This proves that $\text{comp}(\langle \rangle, \sigma) = \sigma$ for all $\sigma \in A^\infty$.

We now show the other equality, that $\text{comp}(\sigma, \langle \rangle) = \sigma$ for all $\sigma \in A^\infty$, we will show that the function $g: A^\infty \rightarrow A^\infty$ defined by $g(\sigma) := \text{comp}(\sigma, \langle \rangle)$ for all $\sigma \in A^\infty$ also satisfies

$$\text{next} \circ g = (\text{id}_{\{\perp\}} \cup (\text{id}_A \times g)) \circ \text{next}.$$

That $(\text{next} \circ g)(\perp) = ((\text{id}_{\{\perp\}} \cup (\text{id}_A \times g)) \circ \text{next})(\perp)$ is the same as with f . Now if $\sigma \in A^\infty$ is such that $\sigma \xrightarrow{a} \sigma'$ for some $a \in A$ and $\sigma' \in A^\infty$, we see that

$$(\text{next} \circ g)(\sigma) = \text{next}(\text{comp}(a \cdot \sigma', \langle \rangle))$$

$$\begin{aligned}
&= (a, \text{comp}(\sigma', \langle \rangle)) \\
&= (a, g(\sigma')),
\end{aligned}$$

by the third rule for **comp** and the definition of g , and

$$\begin{aligned}
((\text{id}_{\{\perp\}} \cup (\text{id}_A \times g)) \circ \text{next})(\sigma) &= ((\text{id}_{\{\perp\}} \cup (\text{id}_A \times g))((a, \sigma'))) \\
&= (a, g(\sigma')).
\end{aligned}$$

Therefore $g = \text{id}_{A^\infty}$, i.e. $\text{comp}(\sigma, \langle \rangle) = \sigma$ for all $\sigma \in A^\infty$.

2. We will define a coalgebra $c: A^\infty \times A^\infty \times A^\infty \rightarrow \{\perp\} \cup (A \times (A^\infty \times A^\infty \times A^\infty))$ such that the functions $h, k: A^\infty \times A^\infty \times A^\infty \rightarrow A^\infty$ given by

$$\begin{aligned}
h(\sigma, \tau, \rho) &:= \text{comp}(\sigma, \text{comp}(\tau, \rho)) \quad \text{and} \\
k(\sigma, \tau, \rho) &:= \text{comp}(\text{comp}(\sigma, \tau), \rho),
\end{aligned}$$

for all $\sigma, \tau, \rho \in A^\infty$, are both coalgebra homomorphisms from c to **next**.

$$\begin{array}{ccc}
& \xrightarrow{\text{id}_{\{\perp\}} \cup (\text{id}_A \times h)} & \\
\{\perp\} \cup (A \times (A^\infty \times A^\infty \times A^\infty)) & & \{\perp\} \cup (A \times A^\infty) \\
& \xleftarrow{\text{id}_{\{\perp\}} \cup (\text{id}_A \times k)} & \\
\uparrow c & & \uparrow \cong \text{next} \\
A^\infty \times A^\infty \times A^\infty & \xrightarrow{h} & A^\infty \\
& \xleftarrow{k} &
\end{array}$$

The finality of **next** would then yield $h = k$.

Define $c: A^\infty \times A^\infty \times A^\infty \rightarrow \{\perp\} \cup (A \times (A^\infty \times A^\infty \times A^\infty))$ by

$$c(\sigma, \tau, \rho) := \begin{cases} \perp, & \text{if } \sigma = \tau = \rho = \langle \rangle, \\ (a, (\langle \rangle, \langle \rangle, \rho')), & \text{if } \sigma = \tau = \langle \rangle \text{ and } \rho = a \cdot \rho' \text{ for some } a \in A \text{ and } \rho' \in A^\infty, \\ (a, (\langle \rangle, \tau', \rho)), & \text{if } \sigma = \langle \rangle \text{ and } \tau = a \cdot \tau' \text{ for some } a \in A \text{ and } \tau' \in A^\infty, \\ (a, (\sigma', \tau, \rho)), & \text{if } \sigma = a \cdot \sigma' \text{ for some } a \in A \text{ and } \sigma' \in A^\infty. \end{cases}$$

Using the rules for **comp**, it is now elementary to check that h and k make their respective diagrams commute. \square

Exercise 1.2.9

Consider two sets A, B with a function $f: A \rightarrow B$ between them. Use finality to define a function $f^\infty: A^\infty \rightarrow B^\infty$ that applies f element-wise. Use uniqueness to show that this mapping $f \mapsto f^\infty$ is ‘functorial’ in the sense that $(\text{id}_A)^\infty = \text{id}_{A^\infty}$ and $(g \circ f)^\infty = g^\infty \circ f^\infty$.

Solution. For a (non-empty) set B , let $\text{next}_B: B^\infty \rightarrow \{\perp\} \cup (B \times B^\infty)$ denote the final coalgebra defined by

$$\text{next}(\sigma) := \begin{cases} \perp, & \text{if } \sigma \text{ is the empty sequence,} \\ (b, \sigma'), & \text{if } \sigma \text{ has head } b \in B \text{ and tail } \sigma' \in B^\infty, \text{ i.e. } \sigma = b \cdot \sigma', \end{cases}$$

for all $\sigma \in B^\infty$. For a function $f: A \rightarrow B$, define a coalgebra $c_f: A^\infty \rightarrow \{\perp\} \cup (B \times A^\infty)$ by

$$c_f(\sigma) := \begin{cases} \perp, & \text{if } \sigma = \langle \rangle, \\ (f(a), \sigma'), & \text{if } \sigma = a \cdot \sigma' \text{ for some } a \in A \text{ and } \sigma' \in A^\infty, \end{cases}$$

for all $\sigma \in A^\infty$. Let $f^\infty: A^\infty \rightarrow B^\infty$ be the unique function making

$$\begin{array}{ccc} \{\perp\} \cup (B \times A^\infty) & \xrightarrow{\text{id}_{\{\perp\}} \cup (\text{id}_B \times f^\infty)} & \{\perp\} \cup (B \times B^\infty) \\ \uparrow c_f & & \uparrow \cong \text{next}_B \\ A^\infty & \xrightarrow{\exists! f^\infty} & B^\infty \end{array}$$

commute. Then $f(\langle a_0, a_1, a_2, a_3, \dots \rangle) = \langle f(a_0), f(a_1), f(a_2), f(a_3), \dots \rangle$ for all $a_0, a_1, a_2, a_3, \dots \in A$, and analogously for finite sequences.

We see that $c_{\text{id}_A} = \text{next}_A$. So $(\text{id}_A)^\infty = \text{id}_{A^\infty}$ by finality of next_A . Furthermore, for functions $f: A \rightarrow B$ and $g: B \rightarrow C$, we see that

$$\begin{array}{ccc} \{\perp\} \cup (C \times A^\infty) & \xrightarrow{\text{id}_{\{\perp\}} \cup (\text{id}_C \times f^\infty)} & \{\perp\} \cup (C \times B^\infty) \\ \uparrow c_{g \circ f} & & \uparrow c_g \\ A^\infty & \xrightarrow{f^\infty} & B^\infty \end{array}$$

commutes. Consequently, the outer square in the diagram

$$\begin{array}{ccccc} \{\perp\} \cup (C \times A^\infty) & \xrightarrow{\text{id}_{\{\perp\}} \cup (\text{id}_C \times f^\infty)} & \{\perp\} \cup (C \times B^\infty) & \xrightarrow{\text{id}_{\{\perp\}} \cup (\text{id}_C \times g^\infty)} & \{\perp\} \cup (C \times C^\infty) \\ \uparrow c_{g \circ f} & & \uparrow c_g & & \uparrow \cong \text{next}_C \\ A^\infty & \xrightarrow{f^\infty} & B^\infty & \xrightarrow{g^\infty} & C^\infty \end{array}$$

commutes, i.e.

$$\text{next}_C \circ (g^\infty \circ f^\infty) = (\text{id}_{\{\perp\}} \cup (\text{id}_C \times (g^\infty \circ f^\infty))) \circ c_{g \circ f}.$$

The finality of next_C then yields $(g \circ f)^\infty = g^\infty \circ f^\infty$. □

Exercise 1.2.10

Use finality to define a map $\text{st}: A^\infty \times B \rightarrow (A \times B)^\infty$ that maps a sequence $\sigma \in A^\infty$ and an element $b \in B$ to a new sequence in $(A \times B)^\infty$ by adding this b at every position in σ . (This is an example of a ‘strength’ map; see [Exercise 2.5.4](#).)

Solution. Define a coalgebra $c: A^\infty \times B \rightarrow \{\perp\} \cup ((A \times B) \times (A^\infty \times B))$ as follows:

$$c(\sigma, b) := \begin{cases} \perp, & \text{if } \sigma = \langle \rangle, \\ ((a, b), (\sigma', b)), & \text{if } \sigma = a \cdot \sigma' \text{ for some } a \in A \text{ and } \sigma' \in A^\infty, \end{cases}$$

for all $\sigma \in A^\infty$ and $b \in B$. The unique function $\text{st}: A^\infty \times B \rightarrow (A \times B)^\infty$ making

$$\begin{array}{ccc} \{\perp\} \cup ((A \times B) \times (A^\infty \times B)) & \xrightarrow{\text{id}_{\{\perp\}} \cup (\text{id}_{A \times B} \times \text{st})} & \{\perp\} \cup ((A \times B) \times (A \times B)^\infty) \\ \uparrow c & & \uparrow \cong \text{next} \\ A^\infty \times B & \xrightarrow{\exists! \text{st}} & (A \times B)^\infty \end{array}$$

commute will satisfy $\text{st}(\langle a_0, a_1, a_2, \dots \rangle, b) = \langle (a_0, b), (a_1, b), (a_2, b), \dots \rangle$ for all $a_0, a_1, a_2, a_3, \dots \in A$ and $b \in B$, and analogously for finite sequences in A^∞ . \square

1.3 Generality of Temporal Logic of Coalgebras

Exercise 1.3.1

The *nexttime* operator \circ introduced in (1.9) is the so-called **weak** *nexttime*. There is an associated **strong** *nexttime*, given by $\neg \circ \neg$. Note the difference between weak and strong *nexttime* for sequences.

Solution. Recall that, for a sequence coalgebra $c: S \rightarrow \{\perp\} \cup (A \times S)$ and a predicate $P \subseteq S$, we have

$$(\circ P)(x) \quad \text{if and only if} \quad c(x) = \perp \text{ or } c(x) \in A \times P,$$

for all $x \in S$. So,

$$(\circ \neg P)(x) \quad \text{if and only if} \quad c(x) = \perp \text{ or } c(x) \in A \times (S \setminus P),$$

and thus

$$(\neg \circ \neg P)(x) \quad \text{if and only if} \quad c(x) \neq \perp \text{ and } c(x) \notin A \times (S \setminus P).$$

Since the codomain of c is $\{\perp\} \cup (A \times S)$, and since $P \subseteq S$, we can equivalently write this as

$$(\neg \circ \neg P)(x) \quad \text{if and only if} \quad c(x) \in A \times P. \quad \square$$

Exercise 1.3.2

Prove that the ‘truth’ predicate that always holds is a (sequence) invariant. And if P_1 and P_2 are invariants, then so is the intersection $P_1 \cap P_2$. Finally, if P is an invariant, then so is $\circ P$.

Solution. Fix a sequence coalgebra $c: S \rightarrow \{\perp\} \cup (A \times S)$. The truth predicate is the set S itself. Then, for all $x \in S$,

$$(\circ S)(x) \quad \text{if and only if} \quad c(x) = \perp \text{ or } c(x) \in A \times S.$$

Since the codomain of c is $\{\perp\} \cup (A \times S)$, this means that $\circ S = S$, and so S is an invariant.

Now suppose that P_1 and P_2 are invariant, i.e. $P_1 \subseteq \circ P_1$ and $P_2 \subseteq \circ P_2$. Then, for all $x \in S$,

$$\begin{aligned} (\circ(P_1 \cap P_2))(x) & \quad \text{if and only if} \quad c(x) = \perp \text{ or } c(x) \in A \times (P_1 \cap P_2) \\ & \quad \text{if and only if} \quad c(x) = \perp \text{ or } c(x) \in (A \times P_1) \cap (A \times P_2) \\ & \quad \text{if and only if} \quad (c(x) = \perp \text{ or } c(x) \in A \times P_1) \text{ and } (c(x) = \perp \text{ or } c(x) \in A \times P_2) \\ & \quad \text{if and only if} \quad (\circ P_1)(x) \text{ and } (\circ P_2)(x). \end{aligned}$$

Hence $P_1 \cap P_2 \subseteq (\circ P_1) \cap (\circ P_2) = \circ(P_1 \cap P_2)$, and so $P_1 \cap P_2$ is also invariant.

Finally, suppose that P is invariant, i.e. $P \subseteq \circ P$. We aim to show that $\circ P \subseteq \circ \circ P$. Suppose $x \in S$ is such that $(\circ P)(x)$ holds. Then either $c(x) = \perp$ or $c(x) \in A \times P \subseteq A \times \circ P$. Therefore $(\circ \circ P)(x)$ holds. \square

Exercise 1.3.3

1. Show that \Box is an interior operator, i.e. satisfies: $\Box P \subseteq P$, $\Box P \subseteq \Box \Box P$, and $P \subseteq Q \implies \Box P \subseteq \Box Q$.
2. Prove that a predicate P is invariant if and only if $P = \Box P$.

Solution. Fix a sequence coalgebra $c: S \rightarrow \{\perp\} \cup (A \times S)$. Recall that the henceforth operator \Box is defined on predicates $P \subseteq S$ as follows: for all $x \in S$,

$$(\Box P)(x) \text{ if and only if } \text{there exists an invariant } Q \subseteq S \text{ with } x \in Q \subseteq P.$$

In other words, $\Box P$ is the union of all invariants contained in P .

1. If $x \in \Box P$, then there is an invariant $Q \subseteq S$ with $x \in Q \subseteq P$. So $x \in P$ too. Also, Q is an invariant with $x \in Q \subseteq \Box P$. So $x \in \Box \Box P$ as well. Thus $\Box P \subseteq P$ and $\Box P \subseteq \Box \Box P$.

Now suppose $P \subseteq Q \subseteq S$. Then, for any $x \in \Box P$, there is an invariant $R \subseteq S$ with $x \in R \subseteq P \subseteq Q$. So $x \in \Box Q$ as well. Therefore $\Box P \subseteq \Box Q$.

2. For the forward direction, suppose that P is invariant. By definition, $\Box P$ is the union of all invariants contained within P . As P is assumed to be an invariant, we must have $\Box P = P$.

For the converse direction, suppose that $\Box P = P$. We need to show that P is an invariant, i.e. $P \subseteq \circ P$. For any $x \in P = \Box P$, there exists an invariant $Q \subseteq S$ with $x \in Q \subseteq P$. As Q is an invariant, either $c(x) = \perp$ or $c(x) \in A \times Q \subseteq A \times P$. Hence we also have $x \in \circ P$. Therefore $P \subseteq \circ P$, meaning P is an invariant. \square

Exercise 1.3.4

Recall the finite behaviour predicate $\Diamond((-) \nrightarrow)$ from Example 1.3.4.1 and show that it is an invariant: $\Diamond((-) \nrightarrow) \subseteq \circ \Diamond((-) \nrightarrow)$. Hint: For an invariant Q , consider the predicate $Q' = (\neg((-) \nrightarrow) \cap (\circ Q))$.

Solution. Fix a sequence coalgebra $c: S \rightarrow \{\perp\} \cup (A \times S)$. Recall that, for a predicate $P \subseteq S$ and $x \in S$,

$$(\Diamond P)(x) \text{ if and only if } \text{for all invariants } Q \subseteq S, \text{ we have } \neg Q(x) \text{ or } Q \not\subseteq \neg P.$$

That is, $\Diamond P = \neg \Box \neg P$.

Suppose $x \in S$ is such that $\Diamond(x \nrightarrow)$ holds. We need to show that $\circ \Diamond(x \nrightarrow)$ holds, i.e. if $x \xrightarrow{a} x'$ for some $(a, x') \in A \times S$, then $\Diamond(x' \nrightarrow)$ also holds. Fix any invariant $Q \subseteq S$ with $Q \subseteq \neg((-) \nrightarrow)$. We need to show that $\neg Q(x')$.

Following the hint, we consider the predicate

$$Q' := \neg((-) \nrightarrow) \cap (\circ Q).$$

Observe that Q' is an invariant: if $y \in S$ satisfies $Q'(y)$, then there is some $(b, y') \in A \times S$ such that $y \xrightarrow{b} y'$ and $Q(y')$ hold. Then, since $Q \subseteq \neg((-) \nrightarrow)$ and Q is an invariant, we conclude that $Q'(y')$ also holds. So $Q' \subseteq \circ Q'$.

Hence if $Q(x')$ holds, then $Q'(x)$ holds too, contradicting the assumption that $\Diamond(x \nrightarrow)$. \square

Exercise 1.3.5

Let (A, \leq) be a complete lattice, i.e. a poset in which each subset $U \subseteq A$ has a join $\bigvee U \in A$. It is well known that each subset $U \subseteq A$ then also has a meet $\bigwedge U \in A$, given by $\bigwedge U = \bigvee \{a \in A \mid \forall b \in U. a \leq b\}$.

Let $f: A \rightarrow A$ be a monotone function: $a \leq b$ implies $f(a) \leq f(b)$. Recall, e.g. from [Davey and Priestley \(1990, Chapter 4\)](#) that such a monotone f has both a least fixed point $\mu f \in A$ and a greatest fixed point $\nu f \in A$ given by the formulas:

$$\mu f = \bigwedge \{a \in A \mid f(a) \leq a\}, \quad \nu f = \bigvee \{a \in A \mid a \leq f(a)\}.$$

Now let $c: S \rightarrow \{\perp\} \cup (A \times A)$ be an arbitrary sequence coalgebra, with associated nexttime operator \circ .

1. Prove that \circ is a monotone function $\mathcal{P}(S) \rightarrow \mathcal{P}(S)$, i.e. that $P \subseteq Q$ implies $\circ P \subseteq \circ Q$, for all $P, Q \subseteq S$.
2. Check that $\Box P \in \mathcal{P}(S)$ is the greatest fixed point of the function $\mathcal{P}(S) \rightarrow \mathcal{P}(S)$ given by $U \mapsto P \cap \circ U$.
3. Define for $P, Q \subseteq S$ a new predicate $P \mathcal{U} Q \subseteq S$, for ‘ P until Q ’ as the least fixed point of $U \mapsto Q \cup (P \cap \neg \circ \neg U)$. Check that ‘until’ is indeed a good name for $P \mathcal{U} Q$, since it can be described explicitly as

$$\begin{aligned} P \mathcal{U} Q = \{x \in S \mid \exists n \in \mathbb{N}. \exists x_0, x_1, \dots, x_n \in S. \\ x_0 = x \wedge (\forall i < n. \exists a. x_i \xrightarrow{a} x_{i+1}) \wedge Q(x_n) \\ \wedge \forall i < n. P(x_i)\}. \end{aligned}$$

Hint: Don’t use the fixed point definition μ , but first show that this subset is a fixed point, and then that it is contained in an arbitrary fixed point.

(The fixed point definitions that we described above are standard in temporal logic; see e.g. [Emerson \(1990, 3.24–3.25\)](#). The above operation \mathcal{U} is what is called the ‘strong’ until. The ‘weak one’ does not have the negations \neg in its fixed-point description in point 3.)

Solution.

1. For subsets $P, Q \in \mathcal{P}(S)$ with $P \subseteq Q$, and for $x \in S$ such that $(\circ P)(x)$ holds, we have

$$c(x) = \perp \text{ or } c(x) \in A \times P.$$

From the assumption that $P \subseteq Q$, it follows that

$$c(x) = \perp \text{ or } c(x) \in A \times Q,$$

or equivalently, $(\circ Q)(x)$.

2. Fix $P \in \mathcal{P}(S)$ and define $f_P: \mathcal{P}(S) \rightarrow \mathcal{P}(S)$ by $f_P(U) := P \cap \circ U$ for all $U \in \mathcal{P}(S)$. Then the greatest fixed point of f_P is

$$\nu(f_P) := \bigcup_{\substack{U \in \mathcal{P}(S), \\ U \subseteq f_P(U)}} U = \bigcup_{\substack{U \in \mathcal{P}(S), \\ U \subseteq P \cap \circ U}} U = \Box P.$$

3. Fix $P, Q \in \mathcal{P}(S)$, and define $f_{P,Q}: \mathcal{P}(S) \rightarrow \mathcal{P}(S)$ by

$$f_{P,Q}(U) := Q \cup (P \cap \neg \circ \neg U)$$

for all $U \in \mathcal{P}(S)$. Recall, from [Exercise 1.3.1](#), that

$$\neg \circ \neg U = \{ x \in S : c(x) \in A \times U \}.$$

We wish to show that the set

$$\begin{aligned} U_{P,Q} := Q \cup \Big\{ x \in S : & \text{ there exist } n \in \mathbb{Z}_{>0}, x_0, \dots, x_n \in S \text{ and } a_0, \dots, a_{n-1} \in A \\ & \text{ such that } x = x_0 \xrightarrow{a_0} \dots \xrightarrow{a_{n-1}} x_n \text{ and} \\ & P(x_0), \dots, P(x_{n-1}), \text{ and } Q(x_n) \text{ all hold} \Big\} \end{aligned}$$

is the least fixed point of $f_{P,Q}$.

First, observe that

$$\begin{aligned} f_{P,Q}(U_{P,Q}) &= Q \cup (P \cap \neg \circ \neg U_{P,Q}) \\ &= Q \cup (P \cap \{ x \in S : c(x) \in A \times U_{P,Q} \}) \\ &= Q \cup \{ x \in S : P(x) \text{ and } c(x) \in A \times U_{P,Q} \} \\ &= U_{P,Q}, \end{aligned}$$

so that $U_{P,Q}$ is indeed a fixed point of $f_{P,Q}$.

Now we show that $U_{P,Q}$ is the least fixed point of $f_{P,Q}$. Fix some $B \subseteq S$ with $f_{P,Q}(B) = B$, i.e.

$$Q \cup \{ x \in S : P(x) \text{ and } c(x) \in A \times B \} = B.$$

Then we get $U_{P,Q} \subseteq B$ by induction on the length of finite sequences $x_0, \dots, x_n \in S$ and $a_0, \dots, a_{n-1} \in A$ satisfying $x_0 \xrightarrow{a_0} \dots \xrightarrow{a_{n-1}} x_n$, and $P(x_0) \wedge \dots \wedge P(x_{n-1}) \wedge Q(x_n)$. \square

1.4 Abstractness of Coalgebraic Notions

Exercise 1.4.1

Let $(M, +, 0)$ be a monoid, considered as a category. Check that a functor $F: M \rightarrow \mathbf{Sets}$ can be identified with a **monoid action**: a set X together with a function $\mu: X \times M \rightarrow X$ with $\mu(x, 0) = x$ and $\mu(x, m_1 + m_2) = \mu(\mu(x, m_2), m_1)$.

Solution. Suppose we are given functor $F: M \rightarrow \mathbf{Sets}$. This F sends the unique object $\star \in \mathbf{Obj}(M)$ to a set $F(\star) \in \mathbf{Obj}(\mathbf{Sets})$, and sends each $m \in \mathbf{Arr}(M)$ to a function $Fm: F(\star) \rightarrow F(\star)$. The functoriality of F requires that $F(0) = \text{id}_{F(\star)}$ and $F(m_1 + m_2) = F(m_1) \circ F(m_2)$ for all $m_1, m_2 \in \mathbf{Arr}(M)$. We then define a function $\mu_F: F(\star) \times \mathbf{Arr}(M) \rightarrow F(\star)$ by $\mu_F(x, m) := F(m)(x)$ for all $(x, m) \in F(\star) \times M$.

The equality $\mu_F(x, 0) = x$ for all $x \in F(\star)$ follows the equality $F(0) = \text{id}_{F(\star)}$, while the equality $\mu_F(x, m_1 + m_2) = \mu_F(\mu_F(x, m_2), m_1)$ for all $x \in X$ and $m_1, m_2 \in \mathbf{Arr}(M)$ follows from the equality $F(m_1 + m_2) = F(m_1) \circ F(m_2)$.

Now suppose we are given also given a set X and a function $\mu: X \times \mathbf{Arr}(M) \rightarrow X$ with $\mu(x, 0) = x$ and $\mu(x, m_1 + m_2) = \mu(\mu(x, m_2), m_1)$ for all $x \in X$ and $m, m_1, m_2 \in \mathbf{Arr}(M)$. We then define a functor $F_\mu: M \rightarrow \mathbf{Sets}$ by $F_\mu(\star) := X$, for the unique object $\star \in \mathbf{Obj}(M)$, and $F_\mu(m) := \mu(-, m)$ for each $m \in \mathbf{Arr}(M)$. That F_μ is actually a functor follows from the assumptions on μ .

We then have $F_{\mu_F} = F$ and $\mu_{F_\mu} = \mu$. \square

Exercise 1.4.2

Check in detail that the opposite \mathbb{C}^{op} and the product $\mathbb{C} \times \mathbb{D}$ are indeed categories.

Solution. Let \mathbb{C} and \mathbb{D} be categories.

We defined $\text{Obj}(\mathbb{C}^{\text{op}}) := \text{Obj}(\mathbb{C})$. For $X, Y \in \text{Obj}(\mathbb{C})$, write $\text{hom}_{\mathbb{C}}(X, Y)$ for the set of all morphisms with domain X and codomain Y . We then defined $\text{hom}_{\mathbb{C}^{\text{op}}}(X, Y) := \text{hom}_{\mathbb{C}}(Y, X)$, and we defined a composition $X \xleftarrow{f} Y \xleftarrow{g} Z$ in \mathbb{C}^{op} to be the composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ in \mathbb{C} . The associativity and identity laws for composition in \mathbb{C}^{op} follow from those for \mathbb{C} .

We defined $\text{Obj}(\mathbb{C} \times \mathbb{D}) := \text{Obj}(\mathbb{C}) \times \text{Obj}(\mathbb{D})$. For $X, X' \in \text{Obj}(\mathbb{C})$ and $Y, Y' \in \text{Obj}(\mathbb{D})$, we let $\text{hom}_{\mathbb{C} \times \mathbb{D}}((X, Y), (X', Y')) := \text{hom}_{\mathbb{C}}(X, X') \times \text{hom}_{\mathbb{D}}(Y, Y')$. A composition $(X, Y) \xrightarrow{(f, g)} (X', Y') \xrightarrow{(f', g')} (X'', Y'')$ in $\mathbb{C} \times \mathbb{D}$ is defined to be the composition $(X, Y) \xrightarrow{(f'f, g'g)} (X'', Y'')$. For an object (X, Y) in $\mathbb{C} \times \mathbb{D}$, the identity morphism $\text{id}_{(X, Y)}$ is the pair $(\text{id}_X, \text{id}_Y)$. The associativity and identity laws for composition in $\mathbb{C} \times \mathbb{D}$ follow from those for \mathbb{C} and \mathbb{D} . \square

Exercise 1.4.3

Assume an arbitrary category \mathbb{C} with an object $I \in \mathbb{C}$. We form a new category \mathbb{C}/I , the so-called **slice category** over I , with

objects maps $f: X \rightarrow I$ with codomain I in \mathbb{C}
morphisms from $X \xrightarrow{f} I$ to $Y \xrightarrow{g} I$ are morphisms $h: X \rightarrow Y$ in \mathbb{C} for which the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ & \searrow f & \swarrow g \\ & I & \end{array}$$

1. Describe identities and composition in \mathbb{C}/I , and verify that \mathbb{C}/I is a category.
2. Check that taking domains yields a functor $\text{dom}: \mathbb{C}/I \rightarrow \mathbb{C}$.
3. Verify that for $\mathbb{C} = \mathbf{Sets}$, a map $f: X \rightarrow I$ may be identified with an I -indexed family of sets $(X_i)_{i \in I}$, namely where $X_i = f^{-1}(i)$. What do morphisms in \mathbb{C}/I correspond to, in terms of such indexed families?

Solution.

1. The identities and composition in \mathbb{C}/I are simply the identities and composition in \mathbb{C} . So the fact that \mathbb{C}/I is a category follows from \mathbb{C} being a category.
2. We define $\text{dom}: \mathbb{C}/I \rightarrow \mathbb{C}$ as follows: for a morphism h from $X \xrightarrow{f} I$ to $Y \xrightarrow{g} I$ in \mathbb{C}/I , we simply define $\text{dom}(h) := h$. This immediately makes dom a functor from \mathbb{C}/I to \mathbb{C} .
3. The claimed identification is obvious. Now fix a morphism h from $X \xrightarrow{f} I$ to $Y \xrightarrow{g} I$ in \mathbf{Sets}/I , so that the diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ & \searrow f & \swarrow g \\ & I & \end{array}$$

in \mathbf{Sets} commutes. This requires that $g(h(x)) = f(x)$ for all $x \in X$. Identifying $X_i := f^{-1}(i)$ and $Y_i := g^{-1}(i)$ for all $i \in I$, we can identify h with a family of functions $(h_i)_{i \in I}$ such that $h_i(x) \in Y_i$ for all $x \in X_i$, for all $i \in I$. \square

Exercise 1.4.4

Recall that for an arbitrary set A we write A^* for the set of finite sequences $\langle a_0, \dots, a_n \rangle$ of elements $a_i \in A$.

1. Check that A^* carries a monoid structure given by concatenation of sequences, with the empty sequence $\langle \rangle$ as a neutral element.
2. Check that the assignment $A \mapsto A^*$ yields a functor $\mathbf{Sets} \rightarrow \mathbf{Mon}$ by mapping a function $f: A \rightarrow B$ between sets to the function $f^*: A^* \rightarrow B^*$ given by $\langle a_0, \dots, a_n \rangle \mapsto \langle f(a_0), \dots, f(a_n) \rangle$. (Be aware of what needs to be checked: f^* must be a monoid homomorphism, and $(-)^*$ must preserve composition of functions and identity functions.)
3. Prove that A^* is the **free monoid on A** : there is the singleton-sequence insertion map $\eta: A \rightarrow A^*$ which is universal among all mappings of A into a monoid. The latter means that for each monoid $(M, 0, +)$ and function $f: A \rightarrow M$ there is a unique monoid homomorphism $g: A^* \rightarrow M$ with $g \circ \eta = f$.

Solution.

1. Concatenation is associative because all the sequences under consideration are finite.
2. That $(-)^*$ preserves composition and identity functions is obvious, so we just check that for a function $f: A \rightarrow B$, the map $f^*: A^* \rightarrow B^*$ is a monoid homomorphism. Fix finite sequences $\langle a_0, \dots, a_n \rangle, \langle a'_0, \dots, a'_k \rangle \in A^*$. Then

$$\begin{aligned} f(\langle a_0, \dots, a_n \rangle \cdot \langle a'_0, \dots, a'_k \rangle) &= f(\langle a_0, \dots, a_n, a'_0, \dots, a'_k \rangle) \\ &= \langle f(a_0), \dots, f(a_n), f(a'_0), \dots, f(a'_k) \rangle \\ &= \langle f(a_0), \dots, f(a_n) \rangle \cdot \langle f(a'_0), \dots, f(a'_k) \rangle \\ &= f(\langle a_0, \dots, a_n \rangle) \cdot \langle a'_0, \dots, a'_k \rangle \end{aligned}$$

and $f(\langle \rangle) = \langle \rangle$. So f^* is a monoid homomorphism.

3. Define $\eta: A \rightarrow A^*$ by $\eta(a) := \langle a \rangle$ for all $a \in A$. Fix a monoid $(M, 0, +)$ and a function $f: A \rightarrow M$. Define $g: A^* \rightarrow M$ by

$$\begin{aligned} g(\langle \rangle) &:= 0 \\ g(\langle a_0, \dots, a_n \rangle) &:= f(a_0) + \dots + f(a_n) \end{aligned}$$

for all $\langle a_0, \dots, a_n \rangle \in A^*$. This g is clearly a monoid homomorphism, using the associativity of $+$ in M . Observe that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta} & A^* \\ & \searrow f & \downarrow g \\ & & M \end{array}$$

in \mathbf{Sets} commutes: we have $f(a) = g(\eta(a))$ for all $a \in A$. Now suppose that there is another monoid homomorphism $h: A^* \rightarrow M$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta} & A^* \\ & \searrow f & \downarrow h \\ & & M \end{array}$$

in **Sets** commutes. As $h: A^* \rightarrow M$ is a monoid homomorphism and $f = h\eta$, we require that $h(\langle \rangle) = 0$ and

$$\begin{aligned} h(\langle a_0, \dots, a_n \rangle) &= h(\langle a_0 \rangle \cdot \dots \cdot \langle a_n \rangle) \\ &= h(\langle a_0 \rangle) + \dots + h(\langle a_n \rangle) \\ &= h(\eta(a_0)) + \dots + h(\eta(a_n)) \\ &= f(a_0) + \dots + f(a_n) \\ &= g(\langle a_0, \dots, a_n \rangle), \end{aligned}$$

for all $\langle a_0, \dots, a_n \rangle \in A^*$. Therefore $h = g$. \square

Exercise 1.4.5

Recall from (1.3) the statements with exceptions of the form $S \rightarrow \{\perp\} \cup S \cup (S \times E)$.

1. Prove that the assignment $X \mapsto \{\perp\} \cup X \cup (X \times E)$ is functorial, so that the statements are a coalgebra for this functor.
2. Show that all the operations $\text{at}_1, \dots, \text{at}_n, \text{meth}_1, \dots, \text{meth}_m$ of a class as in (1.10) can also be described as a single coalgebra, namely of the functor:

$$X \mapsto D_1 \times \dots \times D_n \times \underbrace{(\{\perp\} \cup X \cup (X \times E)) \times \dots \times (\{\perp\} \cup X \cup (X \times E))}_{m \text{ times}}.$$

Solution.

1. Let $F: \mathbf{Sets} \rightarrow \mathbf{Sets}$ denote this assignment $F(X) := \{\perp\} \cup X \cup (X \times E)$ where all unions are disjoint unions. We define F on morphisms as follows: for functions $f: X \rightarrow Y$, we define $F(f): F(X) \rightarrow F(Y)$ to be the function

$$F(f)(x) := \begin{cases} \perp, & \text{if } x = \perp, \\ f(x), & \text{if } x \in X, \\ (f(x'), e), & \text{if } x = (x', e) \text{ for some } (x', e) \in X \times E. \end{cases}$$

Then $F(\text{id}_X) = \text{id}_{F(X)}$ and $F(gf) = F(g)F(f)$ for all sets X and functions $X \xrightarrow{f} Y \xrightarrow{g} Z$.

2. The functor's definition on morphisms is similar in style with the previous part. \square

Exercise 1.4.6

Recall the nexttime operator \circ for a sequence coalgebra $c: S \rightarrow \mathbf{Seq}(S) = \{\perp\} \cup (A \times S)$ from the previous section. *Exercise 1.3.5.1* says that it forms a monotone function $\mathcal{P}(S) \rightarrow \mathcal{P}(S)$ — with respect to the inclusion order — and thus a functor. Check that invariants are precisely \circ -coalgebras!

Solution. The \circ -coalgebras are simply a subsets $U \subseteq S$ such that $U \subseteq \circ U$. These are precisely what invariants are. \square

2 Coalgebras of Polynomial Functors

2.1 Constructions on Sets

Exercise 2.1.1

Verify in detail the bijective correspondences (2.2), (2.6), (2.11) and (2.16).

Solution. Fix sets X, Y, Z . Following the notation of Equations (2.1), we associate a pair of functions $f: Z \rightarrow X$ and $g: Z \rightarrow Y$ to the function $\langle f, g \rangle: Z \rightarrow X \times Y$ given by $\langle f, g \rangle(z) := \langle f(z), g(z) \rangle$ for all $z \in Z$. Furthermore, we associate to any function $h: Z \rightarrow X \times Y$ a pair the functions $\pi_1 h: Z \rightarrow X$ and $\pi_2 h: Z \rightarrow Y$, where π_1 and π_2 are the relevant projections. Then $\langle \pi_1 h, \pi_2 h \rangle = h$ and $(\pi_1 \langle f, g \rangle, \pi_2 \langle f, g \rangle) = (f, g)$. This establishes the bijective correspondence (2.2).

Continue fixing sets X, Y, Z . Suppose, without loss of generality, that X and Y are disjoint, so that we may use $X \cup Y$ in place of $X + Y$. Following the notation of Equations (2.5), we associate a pair of functions $f: X \rightarrow Z$ and $g: Y \rightarrow Z$ to the function $[f, g]: X + Y \rightarrow Z$ given by

$$[f, g](w) := \begin{cases} f(w), & \text{if } w \in X, \\ g(w), & \text{if } w \in Y \end{cases}$$

for all $w \in X + Y$. Furthermore, to any function $h: X + Y \rightarrow Z$, we associate the pair of functions $h\kappa_1: X \rightarrow Z$ and $h\kappa_2: Y \rightarrow Z$, where κ_1 and κ_2 are the relevant coprojections. Then $[h\kappa_1, h\kappa_2] = h$ and $([f, g]\kappa_1, [f, g]\kappa_2) = (f, g)$. This establishes the bijective correspondence (2.6).

Continue fixing sets X, Y, Z . Following the notations of Equations (2.10), we associate a function $f: Z \times X \rightarrow Y$ to the function $\Lambda(f): Z \rightarrow Y^X$ given by $\Lambda(f)(z) := f(z, -)$ for all $z \in Z$. Furthermore, to each function $g: Z \rightarrow Y^X$, we associate the function $U(g): Z \times X \rightarrow Y$ given by $U(g)(z, x) := g(z)(x)$ for all $(z, x) \in Z \times X$. Then $\Lambda(U(g)) = g$ and $U(\Lambda(f)) = f$. So we have established the bijective correspondence (2.11).

Finally, fix sets X and Y . To each function $f: X \rightarrow \mathcal{P}(Y)$, we associate the relation

$$\text{rel}(f) := \{ (y, x) \in Y \times X : y \in f(x) \}.$$

Also, to each relation $R \subseteq Y \times X$, we associate the function $\text{char}(R): X \rightarrow \mathcal{P}(Y)$ given by

$$\text{char}(R)(x) := \{ y \in Y : R(y, x) \}$$

for all $y \in Y$. Then $\text{rel}(\text{char}(R)) = R$ and $\text{char}(\text{rel}(f)) = f$. We thus obtain the bijective correspondence (2.16). \square

Exercise 2.1.2

Consider a poset (D, \leq) as a category. Check that the product of two elements $d, e \in D$, if it exists, is the meet $d \wedge e$. And a coproduct of d, e , if it exists, is the join $d \vee e$.

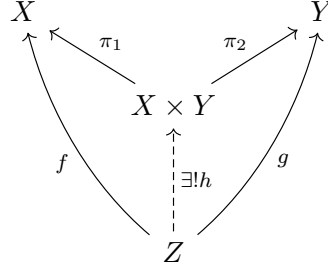
Similarly, show that a final object is a top element \top (with $d \leq \top$, for all $d \in D$) and that an initial object is a bottom element \perp (with $\perp \leq d$, for all $d \in D$).

Solution. These follow immediately, as in a poset (D, \leq) , we have one (and only one) morphism $x \rightarrow y$ if and only if $x \leq y$, for $x, y \in D$, and that the only isomorphisms are identity morphisms. \square

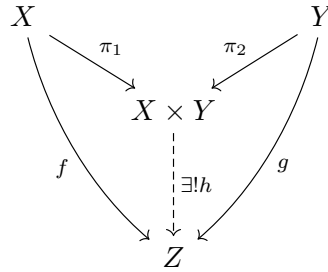
Exercise 2.1.3

Check that a product in a category \mathbb{C} is the same as a coproduct in a category \mathbb{C}^{op} .

Solution. Fix $X, Y, Z \in \mathbf{Obj}(\mathbb{C})$, and suppose the product $X \times Y$ exists in \mathbb{C} . For a pair of morphisms $f: Z \rightarrow X$ and $g: Z \rightarrow Y$, we have the following diagram



in \mathbb{C} commuting. Thus we have the following diagram



in \mathbb{C}^{op} commuting. This makes $X \times Y$ the coproduct of X and Y in \mathbb{C}^{op} , with coprojections π_1 and π_2 . Similarly, coproducts in \mathbb{C}^{op} correspond to products in \mathbb{C} . \square

Exercise 2.1.4

Fix a set A and prove that assignments $X \mapsto A \times X$, $X \mapsto A + X$ and $X \mapsto X^A$ are functorial and give rise to functors $\mathbf{Sets} \rightarrow \mathbf{Sets}$.

Solution. Define $F, G, H: \mathbf{Sets} \rightarrow \mathbf{Sets}$ as follows. For a set X ,

$$\begin{aligned}
 FX &:= A \times X, \\
 GX &:= A + X, \text{ and} \\
 HX &:= X^A.
 \end{aligned}$$

For a function $f: X \rightarrow Y$, define the functions $Ff: A \times X \rightarrow A \times Y$, $Gf: A + X \rightarrow A + Y$, and $Hf: X^A \rightarrow Y^A$ as follows:

$$\begin{aligned}
 (Ff)(a, x) &:= (a, f(x)), & \text{for all } (a, x) \in A \times X, \\
 (Gf)(w) &:= \begin{cases} w, & \text{if } w \in A, \\ f(w), & \text{if } w \in X, \end{cases} & \text{for all } w \in A + X, \\
 (Hf)(h) &:= fh, & \text{for all functions } h: A \rightarrow X,
 \end{aligned}$$

where we have assumed, without loss of generality, that A and X are disjoint so that $X + A$ is treated as $X \cup A$.

Then, for any set X ,

$$\begin{aligned}
 (F\text{id}_X)(a, x) &= (a, \text{id}_X(x)) \\
 &= (a, x), & \text{for all } (a, x) \in A \times X,
 \end{aligned}$$

$$\begin{aligned}
(\text{Gid}_X)(w) &= \begin{cases} w, & \text{if } w \in A, \\ \text{id}_X(w), & \text{if } w \in X, \end{cases} \\
&= w, & \text{for all } w \in A + X, \\
(\text{Hid}_X)(h) &= \text{id}_X h \\
&= h, & \text{for all functions } h: A \rightarrow X,
\end{aligned}$$

so $\text{Fid}_X = \text{id}_{FX}$, $\text{Gid}_X = \text{id}_{GX}$, and $\text{Hid}_X = \text{id}_{HX}$. Now, for functions $X \xrightarrow{f} Y \xrightarrow{g} Z$,

$$\begin{aligned}
(F(gf))(a, x) &= (a, g(f(x))) \\
&= (Fg)(a, f(x)) \\
&= (Fg \circ Ff)(a, x), & \text{for all } (a, x) \in A \times X, \\
(G(gf))(w) &= \begin{cases} w, & \text{if } w \in A, \\ g(f(w)), & \text{if } w \in X, \end{cases} \\
&= (Gg \circ Gf)(w), & \text{for all } w \in A + X, \\
(H(gf))(h) &= \lambda a \in A. (g(f(h(a)))) \\
&= (Hg)(fh) \\
&= (Hg \circ Hf)(h), & \text{for all functions } h: A \rightarrow X,
\end{aligned}$$

so $F(gf) = (Fg)(Ff)$, $G(gf) = (Gg)(Gf)$, and $H(gf) = (Hg)(Hf)$. Thus F , G , and H are functors from **Sets** to **Sets**. \square

Exercise 2.1.5

Prove that the category **PoSets** of partially ordered sets and monotone functions is a BiCCC. The definitions on the underlying sets X of a poset (X, \leq) are like for ordinary sets but should be equipped with appropriate orders.

Solution. The category **PoSets** has a terminal object, namely the singleton poset. Furthermore, given two posets (X_1, \leq_1) and (X_2, \leq_2) , we can define a partial ordering $\leq_{1 \times 2}$ on the product $X_1 \times X_2$ by

$$(x_1, x_2) \leq_{1 \times 2} (x'_1, x'_2) \quad \text{if and only if} \quad x_1 \leq x'_1 \text{ and } x_2 \leq x'_2$$

for all $(x_1, x_2), (x'_1, x'_2) \in X_1 \times X_2$. This poset $(X_1 \times X_2, \leq_{1 \times 2})$ has the universal property of the product: given another poset (X_3, \leq_3) and a pair of monotone functions $f: (X_3, \leq_3) \rightarrow (X_1, \leq_1)$ and $g: (X_3, \leq_3) \rightarrow (X_2, \leq_2)$, we have the diagram

$$\begin{array}{ccccc}
(X_1, \leq_1) & & & & (X_2, \leq_2) \\
& \nwarrow \pi_1 & & \nearrow \pi_2 & \\
& & (X_1 \times X_2, \leq_{1 \times 2}) & & \\
& \nwarrow f & \uparrow \exists! h & \nearrow g & \\
& & (X_3, \leq_3) & &
\end{array}$$

in **PoSets** commuting, where π_1 and π_2 are the relevant projections (which are indeed monotone). The unique monotone function h is given by $h(x_3) := (f(x_3), g(x_3))$ for all $x_3 \in X_3$. Therefore the category **PoSets** has finite products.

The category **PoSets** also has an initial object: the empty poset. Now, given two posets (X_1, \leq_1) and (X_2, \leq_2) , we can define a partial ordering \leq_{1+2} on the coproduct $X_1 + X_2$ by

$$w \leq_{1+2} w' \quad \text{if and only if} \quad (w, w' \in X_1 \text{ and } w \leq_1 w') \text{ or } (w, w' \in X_2 \text{ and } w \leq_2 w')$$

for all $w, w' \in X_1 + X_2$, where we have assumed without loss of generality that X_1 and X_2 are disjoint. Then, given any other poset (X_3, \leq_3) and a pair of monotone functions $f: (X_1, \leq_1) \rightarrow (X_3, \leq_3)$ and $g: (X_2, \leq_2) \rightarrow (X_3, \leq_3)$, we have the diagram

$$\begin{array}{ccc} & (X_3, \leq_3) & \\ f \nearrow & \uparrow \exists! h & \nwarrow g \\ & (X_1 + X_2, \leq_{1+2}) & \\ \kappa_1 \nearrow & & \nwarrow \kappa_2 \\ (X_1, \leq_1) & & (X_2, \leq_2) \end{array}$$

in **PoSets** commuting, where κ_1 and κ_2 are the relevant coprojections (which are also monotone). The unique monotone function h is given by

$$h(w) := \begin{cases} f(w), & \text{if } w \in X_1, \\ g(w), & \text{if } w \in X_2, \end{cases}$$

for all $w \in X_1 + X_2$. Therefore **PoSets** also has finite coproducts.

Now we show that **PoSets** also has exponents. Fix any two posets (X_1, \leq_1) and (X_2, \leq_2) . We define a partial ordering $\leq_2^{X_1}$ on the set $X_2^{X_1}$ as follows:

$$f \leq_2^{X_1} g \quad \text{if and only if} \quad f(x) \leq_2 g(x) \text{ for all } x \in X_1.$$

for all functions $f, g: X_1 \rightarrow X_2$. Then, for any poset (X_3, \leq_3) and monotone function $f: (X_3, \leq_3) \rightarrow (X_2, \leq_2)$, we have the diagram

$$\begin{array}{ccccc} & (X_2^{X_1}, \leq_2^{X_1}) & & (X_1, \leq_1) & \\ & \uparrow \exists! g & & \uparrow \text{id}_{(X_1, \leq_1)} & \\ & (X_3, \leq_3) & & (X_1, \leq_1) & \\ & \uparrow \pi_1 & & \uparrow p_2 & \\ & (X_2^{X_1} \times X_1, \leq_{2^1 \times 1}) & \xrightarrow{\text{ev}} & (X_2, \leq_2) & \\ & \uparrow g \times \text{id}_{(X_1, \leq_1)} & & \uparrow \pi_2 & \\ & (X_3 \times X_1, \leq_{3 \times 1}) & & & \end{array}$$

(Note: The diagram also includes a curved arrow from (X_3, \leq_3) to (X_2, \leq_2) labeled f , and a curved arrow from $(X_3 \times X_1, \leq_{3 \times 1})$ to $(X_2^{X_1}, \leq_2^{X_1})$ labeled p_1 .)

in **PoSets** commuting, where $\text{ev}(h, x_1) := h(x_1)$ for all $(h, x_1) \in X_2^{X_1} \times X_1$, and π_1 , π_2 , p_1 , and p_2 are the relevant projections. The unique monotone function g is given by $g(x_3) := \lambda x_1 \in X_1. f(x_3, x_1)$. Therefore **PoSets** also has exponents. \square

Exercise 2.1.6

Consider the category **Mon** of monoids with monoid homomorphisms between them.

1. Check that the singleton monoid 1 is both an initial and a final object in **Mon**; this is called a zero object.
2. Given two monoids $(M_1, +_1, 0_1)$ and $(M_2, +_2, 0_2)$, one defines a product monoid $M_1 \times M_2$ with componentwise addition $(x, y) + (x', y') = (x +_1 x', y +_2 y')$ and unit $(0_1, 0_2)$. Prove that $M_1 \times M_2$ is again a monoid, which forms a product in the category **Mon** with the standard projection maps $M_1 \xleftarrow{\pi_1} M_1 \times M_2 \xrightarrow{\pi_2} M_2$.
3. Note that there are also coprojections $M_1 \xrightarrow{\kappa_1} M_1 \times M_2 \xleftarrow{\kappa_2} M_2$, given by $\kappa_1(x) = (x, 0_2)$ and $\kappa_2(y) = (0_1, y)$, which are monoid homomorphisms and which makes $M_1 \times M_2$ at the same time the coproduct of M_1 and M_2 in **Mon** (and hence a biproduct). Hint: Define the cotuple $[f, g]$ as $x \mapsto f(x) + g(x)$.

Solution.

1. Any monoid homomorphism $f: (M_1, +_1, 0_1) \rightarrow (M_2, +_2, 0_2)$ must satisfy $f(0_1) = 0_2$, so the singleton monoid is initial in **Mon**. It is also the final in **Mon** because the constant map to the unit is a monoid homomorphism.
2. Fix $(m_1, m_2), (m'_1, m'_2), (m''_1, m''_2) \in M_1 \times M_2$. Then, using the associativity of $+_1$ and $+_2$,

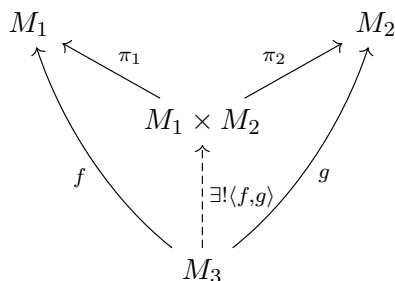
$$\begin{aligned} (m_1, m_2) + ((m'_1, m'_2) + (m''_1, m''_2)) &= (m_1, m_2) + (m'_1 +_1 m''_1, m'_2 +_2 m''_2) \\ &= (m_1 +_1 m'_1 +_1 m''_1, m_2 +_2 m'_2 +_2 m''_2) \\ &= (m_1 +_1 m'_1, m_2 +_2 m'_2) + (m''_1, m''_2). \end{aligned}$$

Furthermore,

$$\begin{aligned}(m_1, m_2) + (0_1, 0_2) &= (m_1 + 1 \ 0_1, m_2 + 2 \ 0_2) \\ &= (m_1, m_2)\end{aligned}$$

and, similarly, $(0_1, 0_2) + (m_1, m_2) = (m_1, m_2)$. So $(M_1 \times M_2, +, (0_1, 0_2))$ is a monoid.

We now show that $M_1 \times M_2$ really is the categorical product of M_1 and M_2 in **Mon**. Fix any other monoid $(M_3, +_3, 0_3)$ and a pair of monoid homomorphisms $f: M_3 \rightarrow M_1$ and $g: M_3 \rightarrow M_2$. We need the diagram



in **Mon** to commute. Indeed, we must have $\langle f, g \rangle(m_3) = (f(m_3), g(m_3))$ for all $m_3 \in M_3$. The fact that $\langle f, g \rangle: M_3 \rightarrow M_1 \times M_2$ is a monoid homomorphism follows from f and g being monoid homomorphisms.

3. Fix any monoid $(M_3, +_3, 0_3)$ and a pair of monoid homomorphisms $f: M_1 \rightarrow M_3$ and $g: M_2 \rightarrow M_3$. We need the diagram

$$\begin{array}{ccc}
 & M_3 & \\
 f \nearrow & \uparrow \exists! [f, g] & \nwarrow g \\
 M_1 & M_1 \times M_2 & M_2 \\
 \searrow \kappa_1 & & \swarrow \kappa_2
 \end{array}$$

in **Mon** to commute. This time we define $[f, g]: M_1 \times M_2 \rightarrow M_3$ by

$$[f, g](m_1, m_2) := f(m_1) +_3 g(m_2)$$

for all $(m_1, m_2) \in M_1 \times M_2$. That $[f, g]$ is a monoid homomorphism follows from f and g being monoid homomorphisms. Then

$$\begin{aligned}
 ([f, g] \circ \kappa_1)(m_1) &= [f, g](m_1, 0_1) \\
 &= f(m_1) +_3 g(0_1) \\
 &= f(m_1)
 \end{aligned}$$

for all $m_1 \in M_1$. Similarly, $([f, g] \circ \kappa_2) = g$.

Now suppose there is another monoid homomorphism $h: M_1 \times M_2 \rightarrow M_3$ satisfying

$$h\kappa_1 = f \quad \text{and} \quad h\kappa_2 = g.$$

Then, for any $(m_1, m_2) \in M_1 \times M_2$,

$$\begin{aligned}
 h(m_1, m_2) &= h(m_1, 0_2) +_3 h(0_1, m_2) \\
 &= h(\kappa_1(m_1)) +_3 h(\kappa_2(m_2)) \\
 &= f(m_1) +_3 g(m_2) \\
 &= [f, g](m_1, m_2).
 \end{aligned}$$

Therefore $[f, g]$ is the unique monoid homomorphism making the diagram above commute. \square

Exercise 2.1.7

Show that in **Sets** products distribute over coproducts, in the sense that the canonical maps

$$(X \times Y) + (X \times Z) \xrightarrow{[\text{id}_X \times \kappa_1, \text{id}_X \times \kappa_2]} X \times (Y + Z)$$

$$0 \xrightarrow{!} X \times 0$$

are isomorphisms. Categories in which this is the case are called **distributive**; see *Cockett (1993)* for more information on distributive categories in general and see *Gumma, Hughes, and Schröder (2003)* for an investigation of such distributivities in categories of coalgebras.

Solution. #??

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Exercise 2.1.8

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Solution. #??



Exercise 2.1.9

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Solution. #??



Exercise 2.1.10

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Solution. #??



Exercise 2.1.11

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Solution. #??



Exercise 2.1.12

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Solution. #??



Exercise 2.1.13

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Solution. #??



Exercise 2.1.14

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Solution. #??



2.2 Polynomial Functors and Their Coalgebras

Exercise 2.2.1

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Solution. #??



Exercise 2.2.2

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Solution. #??



Exercise 2.2.3

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Solution. #??



Exercise 2.2.4

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Solution. #??



Exercise 2.2.5

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Solution. #??

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Exercise 2.2.6

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Solution. #??

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Exercise 2.2.7

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Solution. #??

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Exercise 2.2.8

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Exercise 2.2.9

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Solution. #??

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Exercise 2.2.10

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Solution. #??

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Exercise 2.2.11

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Solution. #??

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Exercise 2.2.12

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Solution. #??

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2.3 Final Coalgebras**Exercise 2.3.1**

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Solution. #??

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Exercise 2.3.2

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Solution. #??

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Exercise 2.3.3

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Solution. #??

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Exercise 2.3.4

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Solution. #??☐**Exercise 2.3.5**

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Solution. #??☐**Exercise 2.3.6**

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Solution. #??☐**Exercise 2.3.7**

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Solution. #??☐**Exercise 2.3.8**

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Solution. #??☐**2.4 Algebras****Exercise 2.4.1**

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Solution. #??☐**Exercise 2.4.2**

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Solution. #??☐**Exercise 2.4.3**

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Solution. #??☐**Exercise 2.4.4**

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Solution. #??☐**Exercise 2.4.5**

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Exercise 2.4.7

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Solution. #??

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Exercise 2.4.8

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Exercise 2.4.9

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Solution. #??

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Exercise 2.4.10

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Solution. #??

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2.5 Adjunctions, Cofree Coalgebras, Behaviour-Realisation**Exercise 2.5.1**

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Solution. #??

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Exercise 2.5.2

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Solution. #??

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Exercise 2.5.3

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Solution. #??

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Exercise 2.5.4

This exercise describes ‘strength’ for endofunctors on **Sets**. In general, this is a useful notion in the theory of datatypes (Cockett and Spencer, 1992), (Cockett and Spencer, 1995) and of computations (Moggi, 1991); see Section 5.2 for a systemic description.

Let $F: \mathbf{Sets} \rightarrow \mathbf{Sets}$ be an arbitrary functor. Consider for sets X, Y the strength map

$$F(X) \times Y \xrightarrow{\text{st}_{X,Y}} F(X \times Y)$$

$$(u, y) \longmapsto F(\lambda x \in X. (x, y))(u)$$

1. Prove that this yields a natural transformation $F(-) \times (-) \xRightarrow{\text{st}} F((-) \times (-))$, where both the domain and codomain are functors $\mathbf{Sets} \times \mathbf{Sets} \rightarrow \mathbf{Sets}$.
2. Describe this strength map for the list functor $(-)^*$ and for the powerset functor \mathcal{P} .

Solution. #??

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Exercise 2.5.5

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Solution. #??



Exercise 2.5.6

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Solution. #??



Exercise 2.5.7

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Solution. #??



Exercise 2.5.8

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Solution. #??



Exercise 2.5.9

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Solution. #??



Exercise 2.5.10

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Solution. #??



Exercise 2.5.11

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Solution. #??



Exercise 2.5.12

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Solution. #??



Exercise 2.5.13

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Solution. #??



Exercise 2.5.14

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Solution. #??



Exercise 2.5.15

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Solution. #??



Exercise 2.5.16

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Solution. #??



Exercise 2.5.17

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Solution. #??



3 Bisimulations

3.1 Relation Lifting, Bisimulations and Congruences

Exercise 3.1.1

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Solution. #??

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Exercise 3.1.2

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Solution. #??

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Exercise 3.1.3

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Solution. #??

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Exercise 3.1.4

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Solution. #??

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Exercise 3.1.5

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Solution. #??

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Exercise 3.1.6

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Solution. #??

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3.2 Properties of Bisimulations

Exercise 3.2.1

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Solution. #??

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Exercise 3.2.2

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Solution. #??

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Exercise 3.2.3

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Solution. #??

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Exercise 3.2.4

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Exercise 3.2.5

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Solution. #??

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Exercise 3.2.6

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Solution. #??

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Exercise 3.2.7

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Solution. #??

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3.3 Bisimulations as Spans and Cospans**Exercise 3.3.1**

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Solution. #??

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Exercise 3.3.2

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Solution. #??

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Exercise 3.3.3

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Solution. #??

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Exercise 3.3.4

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Solution. #??

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3.4 Bisimulations and the Coinduction Proof Principle**Exercise 3.4.1**

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Solution. #??

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Exercise 3.4.2

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Solution. #??

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Exercise 3.4.3

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Solution. #??

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Exercise 3.4.4

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Solution. #??



Exercise 3.4.5

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Solution. #??



Exercise 3.4.6

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Solution. #??



Exercise 3.4.7

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Solution. #??



3.5 Process Semantics

Exercise 3.5.1

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Solution. #??



Exercise 3.5.2

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Solution. #??



Exercise 3.5.3

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Solution. #??



Exercise 3.5.4

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Solution. #??



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