# Solutions to exercises in Bart Jacobs's book "Introduction to Coalgebra: Towards Mathematics of States and Observation"

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# some date very far into the future, if ever

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These are my solutions to all the labelled exercises in Jacobs (2017). This document does not stand on its own; it is meant to supplement the book.

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# 1 Motivation

# 1.1 Naturalness of Coalgebraic Representations

#### Exercise 1.1.1

1. Prove that the composition operation; as defined for coalgebras  $S \to \{\bot\} \cup S$  is associative, i.e. satisfies  $s_1$ ;  $(s_2; s_3) = (s_1; s_2)$ ;  $s_3$ , for all statements  $s_1, s_2, s_3 : S \to \{\bot\} \cup S$ .

Define a statement skip:  $S \to \{\bot\} \cup S$  which is a unit for composition; i.e. which satisfies  $(\text{skip}; s) = s = (s; \text{skip}), \text{ for all } s : S \to \{\bot\} \cup S.$ 

2. Do the same for; defined on coalgebras  $S \to \{\bot\} \cup S \cup (S \times E)$ .

(In both cases, statements with an associative composition operation and a unit element form a monoid.)

Solution.

1. Recall that the composition operation; was defined as follows:

$$s; t := \lambda x \in S.$$
 
$$\begin{cases} \bot, & \text{if } s(x) = \bot, \\ t(x') & \text{if } s(x) = x' \in S, \end{cases}$$

for coalgebras  $s, t: S \to \{\bot\} \cup S$ . Fix any three coalgebras  $s_1, s_2, s_3: S \to \{\bot\} \cup S$ . Then

$$s_{1}; (s_{2}; s_{3}) = \lambda x \in S. \begin{cases} \bot, & \text{if } s_{1}(x) = \bot, \\ (s_{2}; s_{3})(x'), & \text{if } s_{1}(x) = x' \in S, \end{cases}$$

$$= \lambda x \in S. \begin{cases} \bot, & \text{if either } s_{1}(x) = \bot, \text{ or both } s_{1}(x) = x' \in S \text{ and } s_{2}(x') = \bot, \\ s_{3}(x''), & \text{if } s_{1}(x) = x' \in S \text{ and } s_{2}(x') = x'' \in S, \end{cases}$$

$$= \lambda x \in S. \begin{cases} \bot, & \text{if } (s_{1}; s_{2})(x) = \bot, \\ s_{3}(x''), & \text{if } (s_{1}; s_{2})(x) = x'' \in S, \end{cases}$$

$$= (s_{1}; s_{2}); s_{3}.$$

So the composition operation; is associative.

The coalgebra  $\mathsf{skip} \colon S \to \{\bot\} \cup S$  defined by  $\mathsf{skip}(x) \coloneqq x$ , for all  $x \in S$ , satisfies  $(\mathsf{skip} \, ; \, s) = s = (s \, ; \, \mathsf{skip})$  for all coalgebras  $s \colon S \to \{\bot\} \cup S$ .

2. Now we consider the composition operation; defined as follows:

$$s; t \coloneqq \lambda x \in S. \begin{cases} \bot, & \text{if } s(x) = \bot, \\ t(x'), & \text{if } s(x) = x' \in S, \\ (x', e), & \text{if } s(x) = (x', e) \in S \times E, \end{cases}$$

for coalgebras  $s, t: S \to \{\bot\} \cup S \cup (S \times E)$ . Fix any three coalgebras  $s_1, s_2, s_3: \{\bot\} \cup S \cup (S \times E)$ . Then

$$s_1; (s_2; s_3) = \lambda x \in S.$$

$$\begin{cases} \bot, & \text{if } s_1(x) = \bot, \\ (s_2; s_3)(x'), & \text{if } s_1(x) = x' \in S, \\ (x', e), & \text{if } s_1(x) = (x', e) \in S \times E, \end{cases}$$

$$= \lambda x \in S. \begin{cases} \bot, & \text{if either } s_1(x) = \bot, \text{ or both } s_1(x) = x' \in S \text{ and } s_2(x') = \bot, \\ s_3(x''), & \text{if } s_1(x) = x' \in S \text{ and } s_2(x') = x'' \in S, \\ (x'', e), & \text{if } s_1(x) = x' \in S \text{ and } s_2(x') = (x'', e) \in S \times E, \\ (x', e), & \text{if } s_1(x) = (x', e) \in S \times E, \end{cases}$$

$$= \lambda x \in S. \begin{cases} \bot, & \text{if } (s_1; s_2)(x) = \bot, \\ s_3(x''), & \text{if } (s_1; s_2)(x) = x'' \in S, \\ (x'', e), & \text{if } (s_1; s_2)(x) = (x'', e) \in S \times E, \end{cases}$$

$$= (s_1; s_2); s_3.$$

So this composition operation; is also associative.

Now define the coalgebra  $\mathsf{skip} \colon S \to \{\bot\} \cup S \cup (S \times E)$  by  $\mathsf{skip}(x) \coloneqq x$ , for all  $x \in S$ . Then we have  $(\mathsf{skip} \colon s) = s = (s \colon \mathsf{skip})$  for all coalgebras  $s \colon S \to \{\bot\} \cup S \cup (S \times E)$ .

# Exercise 1.1.2

Define also a composition monoid (skip, ;) for coalgebras  $S \to \mathcal{P}(S)$ .

Solution. For coalgebras  $s, t: S \to \mathcal{P}(S)$ , define

$$s; t := \lambda x \in S. \left( \bigcup_{y \in s(x)} t(y) \right).$$

Then, for coalgebras  $s_1, s_2, s_3 \colon S \to \mathcal{P}(S)$ , we have

$$s_1; (s_2; s_3) = \lambda x \in S. \left( \bigcup_{y \in s_1(x)} (s_2; s_3)(y) \right)$$

$$= \lambda x \in S. \left( \bigcup_{y \in s_1(x)} \bigcup_{z \in s_2(y)} s_3(z) \right)$$

$$= \lambda x \in S. \left( \bigcup_{z \in (s_1; s_2)(x)} s_3(z) \right)$$

$$= (s_1; s_2); s_3.$$

Furthermore, defining skip:  $S \to \mathcal{P}(S)$  by  $\mathsf{skip}(x) \coloneqq \{x\}$  for all  $x \in S$ , we have

$$\begin{aligned} (\mathsf{skip}\,;\,s) &= \lambda x \in S. \left( \bigcup_{y \in \mathsf{skip}(x)} s(y) \right) \\ &= \lambda x \in S. \left( \bigcup_{y \in \{x\}} s(y) \right) \\ &= \lambda x \in S. s(x) \\ &= s \end{aligned}$$

and

$$(s\,;\,\mathsf{skip}) = \lambda x \in S. \left(\bigcup_{y \in s(x)} \mathsf{skip}(y)\right)$$

$$= \lambda x \in S. \left(\bigcup_{y \in s(x)} \{y\}\right)$$

$$= \lambda x \in S.s(x)$$

$$= s.$$

### 1.2 The Power of Coinduction

#### Exercise 1.2.1

Compute the nextdec-behaviour of  $\frac{1}{7} \in [0,1)$  as in Example 1.2.2.

Solution. We first recall all of the following functions.

1. The final coalgebra  $\mathsf{next} \colon \{0,\dots,9\}^\infty \to \{\bot\} \cup \big(\{0,\dots,9\} \times \{0,\dots,9\}^\infty\big)$  is defined by

$$\mathsf{next}(\sigma) \coloneqq \begin{cases} \bot, & \text{if } \sigma \text{ is the empty sequence,} \\ (d,\sigma'), & \text{if } \sigma \text{ has head } d \in \{0,\dots,9\} \text{ and tail } \sigma' \in \{0,\dots,9\}^\infty, \text{ i.e. } \sigma = d \cdot \sigma', \end{cases}$$

for all (finite or infinite) sequences  $\sigma \in \{0, \dots, 9\}^{\infty}$ .

2. The coalgebra nextdec:  $[0,1) \to \{\bot\} \cup (\{0,\ldots,9\} \times [0,1))$  is defined by

$$\mathsf{nextdec}(r) \coloneqq \begin{cases} \bot, & \text{if } r = 0, \\ (d, 10r - d), & \text{if } d \leq 10r < d + 1 \text{ and } d \in \{0, \dots, 9\}, \end{cases}$$

for all  $r \in [0, 1)$ .

3. The function beh\_{\sf nextdec} \colon [0,1) \to \{0,\dots,9\}^\infty is the unique function making

$$\{\bot\} \cup \left(\{0,\ldots,9\} \times [0,1)\right) \xrightarrow{\mathrm{id}_{\{\bot\}} \cup \left(\mathrm{id}_{\{0,\ldots,9\}} \times \mathrm{beh}_{\mathsf{nextdec}}\right)} \\ = \bigcap_{\mathsf{next}} \mathsf{nextdec}$$

$$[0,1) \xrightarrow{\exists \mathrm{lbeh}_{\mathsf{nextdec}}} \{\bot\} \cup \left(\{0,\ldots,9\} \times \{0,\ldots,9\}^\infty\right)$$

commute.

We wish to compute beh<sub>nextdec</sub>  $\left(\frac{1}{7}\right)$ . We see that

$$\begin{split} \operatorname{beh_{nextdec}} \left( \frac{1}{7} \right) &= \operatorname{next}^{-1} \left( \left( \operatorname{id}_{\{\bot\}} \ \cup \ \left( \operatorname{id}_{\{0,\ldots,9\}} \times \operatorname{beh_{nextdec}} \right) \right) \left( \operatorname{nextdec} \left( \frac{1}{7} \right) \right) \right) \\ &= \operatorname{next}^{-1} \left( \left( \operatorname{id}_{\{\bot\}} \ \cup \ \left( \operatorname{id}_{\{0,\ldots,9\}} \times \operatorname{beh_{nextdec}} \right) \right) \left( \left( 1, \, \frac{3}{7} \right) \right) \right) \end{split}$$

$$\begin{split} &= \mathsf{next}^{-1} \bigg( \left( 1, \, \mathrm{beh}_{\mathsf{nextdec}} \bigg( \frac{3}{7} \bigg) \right) \bigg) \\ &= 1 \cdot \mathrm{beh}_{\mathsf{nextdec}} \bigg( \frac{3}{7} \bigg). \end{split}$$

Continuing in this fashion,

$$\begin{split} \operatorname{beh_{nextdec}}\left(\frac{1}{7}\right) &= 1 \cdot \operatorname{beh_{nextdec}}\left(\frac{3}{7}\right) \\ &= 1 \cdot \left(4 \cdot \operatorname{beh_{nextdec}}\left(\frac{2}{7}\right)\right) \\ &= 1 \cdot \left(4 \cdot \left(2 \cdot \operatorname{beh_{nextdec}}\left(\frac{6}{7}\right)\right)\right) \\ &= 1 \cdot \left(4 \cdot \left(2 \cdot \left(8 \cdot \operatorname{beh_{nextdec}}\left(\frac{4}{7}\right)\right)\right)\right) \\ &= 1 \cdot \left(4 \cdot \left(2 \cdot \left(8 \cdot \left(5 \cdot \operatorname{beh_{nextdec}}\left(\frac{5}{7}\right)\right)\right)\right)\right) \\ &= 1 \cdot \left(4 \cdot \left(2 \cdot \left(8 \cdot \left(5 \cdot \left(7 \cdot \operatorname{beh_{nextdec}}\left(\frac{1}{7}\right)\right)\right)\right)\right)\right). \end{split}$$

Therefore beh<sub>nextdec</sub> $\left(\frac{1}{7}\right) = \langle 1, 4, 2, 8, 5, 7, 1, 4, 2, 8, 5, 7, 1, 4, 2, 8, 5, 7, \dots \rangle$ .

#### Exercise 1.2.2

Formulate appropriate rules for the function odds:  $A^{\infty} \to A^{\infty}$  in analogy with the rules (1.7) for evens.

Solution. We recall that, for a sequence  $\sigma := \langle a_0, a_1, a_2, a_3, \dots \rangle \in A^{\infty}$ , the function odds satisfies odds $(\sigma) = \langle a_1, a_3, a_5, \dots \rangle$ , and analogously if  $\sigma$  is a finite sequence. The rules we want odds to satisfy are:

$$\frac{\sigma \not\rightarrow}{\mathsf{odds}(\sigma) \not\rightarrow}$$

i.e. odds should send the empty sequence to the empty sequence;

$$\frac{\sigma \xrightarrow{a} \sigma' \qquad \sigma' \not\rightarrow}{\operatorname{odds}(\sigma) \not\rightarrow}$$

i.e. odds should send a singleton sequence  $\langle a \rangle$  to the empty sequence; and

$$\frac{\sigma \xrightarrow{a} \sigma' \qquad \sigma' \xrightarrow{a'} \sigma''}{\operatorname{odds}(\sigma) \xrightarrow{a'} \operatorname{odds}(\sigma')}$$

i.e. if  $\sigma = a \cdot a' \cdot \sigma' \in A^{\infty}$ , where  $a, a' \in A$ , then  $\mathsf{odds}(\sigma) = a' \cdot \mathsf{odds}(\sigma')$ .

#### Exercise 1.2.3

Use coinduction to define the empty sequence  $\langle \rangle \in A^{\infty}$  as a map  $\{\bot\} \to A^{\infty}$ .

Fix an element  $a \in A$ , and similarly define the infinite sequence  $\overrightarrow{a} : \{\bot\} \to A^{\infty}$  consisting of only as.

Solution. We recall that the final coalgebra next:  $A^{\infty} \to \{\bot\} \cup (A \times A^{\infty})$  is defined by

$$\mathsf{next}(\sigma) \coloneqq \begin{cases} \bot, & \text{if } \sigma \text{ is the empty sequence,} \\ (a,\sigma'), & \text{if } \sigma \text{ has head } a \in A \text{ and tail } \sigma' \in A^\infty, \text{ i.e. } \sigma = a \cdot \sigma', \end{cases}$$

for all (finite or infinite) sequences  $\sigma \in A^{\infty}$ .

For the coalgebra  $\kappa_1 : \{\bot\} \to \{\bot\} \cup (A \times \{\bot\})$  defined by  $\kappa_1(\bot) := \bot$ , the unique function  $\operatorname{beh}_{\kappa_1} : \{\bot\} \to A^{\infty}$  making

$$\{\bot\} \cup (A \times \{\bot\}) \xrightarrow{\operatorname{id}_{\{\bot\}} \cup (\operatorname{id}_A \times \operatorname{beh}_{\kappa_1})} \{\bot\} \cup (A \times A^{\infty})$$

$$\kappa_1 \qquad \qquad \cong \uparrow \operatorname{next}$$

$$\{\bot\} \xrightarrow{\exists ! \operatorname{beh}_{\kappa_1}} A^{\infty}$$

commute satisfies beh\_ $\kappa_1(\bot) = \langle \rangle$ .

For the coalgebra  $c_a : \{\bot\} \to \{\bot\} \cup (A \times \{\bot\})$  defined by  $c_a(\bot) := (a, \bot)$ , the unique function beh<sub>ca</sub>:  $\{\bot\} \to A^{\infty}$  making

commute satisfies  $beh_{c_a}(\bot) = \overrightarrow{d} = \langle a, a, a, \ldots \rangle$ .

#### Exercise 1.2.4

Compute the outcome of merge( $\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle$ ).

Solution. Recall that we defined the coalgebra  $m: A^{\infty} \times A^{\infty} \to \{\bot\} \cup (A \times (A^{\infty} \times A^{\infty}))$  by

$$m(\sigma,\tau) := \begin{cases} \bot, & \text{if } \sigma \not\to \text{ and } \tau \not\to, \\ (a,(\sigma,\tau')), & \text{if } \sigma \not\to \text{ and } \tau \xrightarrow{a} \tau', \\ (a,(\tau,\sigma')), & \text{if } \sigma \xrightarrow{a} \sigma', \end{cases}$$

for all  $\sigma, \tau \in A^{\infty}$ , and that merge:  $A^{\infty} \times A^{\infty} \to A^{\infty}$  is the unique function making

$$\{\bot\} \cup (A \times (A^{\infty} \times A^{\infty})) \xrightarrow{\operatorname{id}_{\{\bot\}}} \cup (\operatorname{id}_{A} \times \operatorname{merge}) \\ \downarrow \\ \downarrow \\ M \qquad \qquad \cong \\ \uparrow \operatorname{next} \\ A^{\infty} \times A^{\infty} \xrightarrow{\exists ! \operatorname{merge}} A^{\infty}$$

commute. Then

$$\mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) = \mathsf{next}^{-1} \Big( \big( \mathrm{id}_{\{\bot\}} \ \cup \ (\mathrm{id}_A \times \mathsf{merge}) \big) \big( m(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) \big) \Big) \Big) + \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) \Big) \Big) + \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) \Big) \Big) \Big) + \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) \Big) \Big) \Big) \Big( \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) \Big) \Big) \Big) \Big( \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) \Big) \Big) \Big) \Big( \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) \Big) \Big) \Big) \Big( \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) \Big) \Big) \Big) \Big( \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) \Big) \Big) \Big) \Big( \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) \Big) \Big) \Big) \Big( \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) \Big) \Big) \Big) \Big( \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) \Big) \Big) \Big) \Big( \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) \Big) \Big) \Big) \Big( \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) \Big) \Big) \Big) \Big( \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) \Big) \Big) \Big) \Big( \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) \Big) \Big) \Big) \Big( \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) \Big) \Big) \Big) \Big( \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) \Big) \Big) \Big) \Big( \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) \Big) \Big) \Big) \Big( \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) \Big) \Big) \Big) \Big( \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) \Big) \Big) \Big) \Big( \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle a_0, b_1, b_2, b_3 \rangle) \Big) \Big) \Big) \Big) \Big( \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle a_0, b_1, b_2, b_3 \rangle) \Big) \Big) \Big) \Big) \Big( \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle a_0, a_1, a_2 \rangle, \langle a_0, a_1, a_2 \rangle, \langle a_0, a_1, a_2 \rangle \Big) \Big) \Big) \Big) \Big) \Big( \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle a_0, a_1, a_2 \rangle, \langle a_0, a_1, a_2 \rangle \Big) \Big) \Big) \Big) \Big( \mathrm{id}_A \times \mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle$$

$$\begin{split} &= \mathsf{next}^{-1} \Big( \big( \mathrm{id}_{\{\bot\}} \ \cup \ (\mathrm{id}_A \times \mathsf{merge}) \big) \big( (a_0, (\langle b_0, b_1, b_2, b_3 \rangle, \langle a_1, a_2 \rangle)) \big) \Big) \\ &= \mathsf{next}^{-1} \Big( \big( a_0, \mathsf{merge}(\langle b_0, b_1, b_2, b_3 \rangle, \langle a_1, a_2 \rangle) \big) \Big) \\ &= a_0 \cdot \mathsf{merge}(\langle b_0, b_1, b_2, b_3 \rangle, \langle a_1, a_2 \rangle), \end{split}$$

and so on. Eventually, we obtain  $\mathsf{merge}(\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2, b_3 \rangle) = \langle a_0, b_0, a_1, b_1, a_2, b_2, b_3 \rangle.$ 

#### Exercise 1.2.5

Is the merge operation associative, i.e. is  $merge(\sigma, merge(\tau, \rho))$  the same as  $merge(merge(\sigma, \tau), \rho)$ ? Give a proof or a counterexample. Is there a neutral element for merge?

Solution. The merge operation is not associative:

$$\begin{split} \mathsf{merge}(\langle a \rangle, \mathsf{merge}(\langle b \rangle, \langle c \rangle) &= \mathsf{merge}(\langle a \rangle, \langle b, c \rangle) \\ &= \langle a, b, c \rangle, \end{split}$$

whereas

$$\begin{aligned} \mathsf{merge}(\mathsf{merge}(\langle a \rangle, \langle b \rangle), \langle c \rangle) &= \mathsf{merge}(\langle a, b \rangle, \langle c \rangle) \\ &= \langle a, c, b \rangle, \end{aligned}$$

for all  $a, b, c \in A$ .

The neutral element for merge is the empty sequence: for any  $\sigma \in A^{\infty}$ , we have  $\mathsf{merge}(\sigma, \langle \rangle) = \mathsf{merge}(\langle \rangle, \sigma) = \sigma$ .

#### Exercise 1.2.6

Show how to define an alternative merge function which alternatingly takes two elements from its argument sequences.

Solution. We will define a coalgebra  $m_2: A^{\infty} \times A^{\infty} \to \{\bot\} \cup (A \times (A^{\infty} \times A^{\infty}))$  so that the desired merge function is the unique function  $\operatorname{merge}_2: A^{\infty} \times A^{\infty} \to A^{\infty}$  making

commute. As a motivating example, the desired merge of two infinite streams  $\langle a_0, a_1, \dots \rangle$  and  $\langle b_0, b_1, \dots \rangle$  should be

$$\mathsf{merge}_2(\langle a_0, a_1, a_2, a_3, \dots \rangle, \langle b_0, b_1, b_2, b_3, \dots \rangle) = \langle a_0, a_1, b_0, b_1, a_2, a_3, b_2, b_3, \dots \rangle.$$

As the diagram above commutes, we would require

$$\mathsf{merge}_2\big(m_2(\langle a_0, a_1, a_2, a_3, \dots \rangle, \langle b_0, b_1, b_2, b_3, \dots \rangle)\big) = \big(a_0, \langle a_1, b_0, b_1, a_2, a_3, b_2, b_3, \dots \rangle\big)$$

and so  $m_2$  should be defined to satisfy

$$m_2(\langle a_0, a_1, a_2, a_3, \dots \rangle, \langle b_0, b_1, b_2, b_3, \dots \rangle) = (a_0, (\langle a_1, b_0, a_3, b_2, \dots \rangle, \langle b_1, a_2, b_3, a_4, \dots))$$

Dealing with edge cases separately leads us to the following definition: we define the coalgebra  $m_2: A^{\infty} \times A^{\infty} \to \{\bot\} \cup (A \times (A^{\infty} \times A^{\infty}))$  as follows.

1. The function  $m_2$  sends the pair  $(\langle \rangle, \langle \rangle)$  to  $\perp$ , i.e.

$$m_2(\langle \rangle, \langle \rangle) \coloneqq \bot.$$

2. If  $\tau \in A^{\infty}$  is a non-empty sequence, say  $\tau \xrightarrow{a} \tau'$  for some  $\tau' \in A^{\infty}$  and  $a \in A$ , then

$$m_2(\langle \rangle, \tau) := (a, (\langle \rangle, \tau')).$$

3. If  $\sigma = \langle a \rangle$  for some  $a \in A$ , then

$$m_2(\langle a \rangle, \tau) \coloneqq (a, (\langle \rangle, \tau))$$

for all  $\tau \in A^{\infty}$ .

4. If  $\sigma \in A^{\infty}$  has at least length 2, say  $\sigma \xrightarrow{a} \sigma' \xrightarrow{a'} \sigma''$  for some  $\sigma', \sigma'' \in A^{\infty}$  and  $a, a' \in A$ , then

$$m_2(\sigma,\tau) \coloneqq \Big(a, \big(\mathsf{merge}(\mathsf{odds}(\sigma), \mathsf{evens}(\tau)), \mathsf{merge}(\mathsf{odds}(\tau), \mathsf{evens}(\sigma''))\big)\Big)$$

for all  $\tau \in A^{\infty}$ .

Now let  $\mathsf{merge}_2 \colon A^\infty \times A^\infty \to A^\infty$  be the unique function which makes

commute. Fix any  $\sigma, \tau \in A^{\infty}$ . We argue by cases on  $(\sigma, \tau)$  that this function  $\mathsf{merge}_2$  is the desired merge function.

- 1. If  $\sigma = \tau = \langle \rangle$ , then  $\mathsf{merge}_2(\langle \rangle, \langle \rangle) = \langle \rangle$ .
- 2. If  $\sigma = \langle \rangle$  and  $\tau$  is a non-empty sequence, say  $\tau = a \cdot \tau'$  for some  $a \in A$  and  $\tau' \in A^{\infty}$ , then

$$\mathsf{merge}_2(\langle \rangle, \tau) = a \cdot \mathsf{merge}_2(\langle \rangle, \tau').$$

Thus  $merge_2(\langle \rangle, \tau) = \tau$ .

3. If  $\sigma = \langle a \rangle$  for some  $a \in A$ , then

$$\begin{aligned} \operatorname{merge}_2(\langle a \rangle, \tau) &= a \cdot \operatorname{merge}_2(\langle \rangle, \tau) \\ &= a \cdot \tau. \end{aligned}$$

4. If  $\sigma = a \cdot a' \cdot \sigma''$  for some  $a, a' \in A$  and  $\sigma'' \in A^{\infty}$ , then

$$\begin{split} \mathsf{merge}\big(\mathsf{odds}(\mathsf{merge}(\mathsf{odds}(\tau),\mathsf{evens}(\sigma''))),\\ \mathsf{odds}(\mathsf{merge}(\mathsf{evens}(\tau),\mathsf{odds}(\sigma'')))\big) \Big) \\ &= a \cdot a' \cdot \mathsf{merge}_2\Big(\mathsf{merge}\big(\mathsf{evens}(\tau),\mathsf{odds}(\tau)\big),\mathsf{merge}\big(\mathsf{evens}(\sigma''),\mathsf{odds}(\sigma'')\big)\Big) \\ &= a \cdot a' \cdot \mathsf{merge}_2(\tau,\sigma''), \end{split}$$

as desired.

#### Exercise 1.2.7

- 1. Define three functions  $ex_i: A^{\infty} \to A^{\infty}$ , for i = 0, 1, 2, which extract the elements at positions 3n + i.
- 2. Define merge3:  $A^{\infty} \times A^{\infty} \times A^{\infty} \to A^{\infty}$  satisfying the equation merge3(ex<sub>0</sub>( $\sigma$ ), ex<sub>1</sub>( $\sigma$ ), ex<sub>2</sub>( $\sigma$ )) =  $\sigma$ , for all  $\sigma \in A^{\infty}$ .

Solution.

1. Define  $c_0, c_1, c_2 \colon A^{\infty} \to \{\bot\} \cup (A \times A^{\infty})$  as follows:

$$c_{0}(\sigma) := \begin{cases} \bot, & \text{if } \sigma = \langle \rangle, \\ (a, \langle \rangle) & \text{if } \sigma = \langle a \rangle \text{ or } \sigma = \langle a, a' \rangle \text{ for some } a, a' \in A, \\ (a, \sigma'''), & \text{if } \sigma \xrightarrow{a} \sigma' \xrightarrow{a'} \sigma'' \xrightarrow{a''} \sigma''', \end{cases}$$

$$c_{1}(\sigma) := \begin{cases} \bot, & \text{if } \sigma = \langle \rangle \text{ or } \sigma = \langle a \rangle \text{ for some } a \in A, \\ (a', \langle \rangle) & \text{if } \sigma = \langle a, a' \rangle \text{ for some } a, a' \in A, \\ (a', \sigma'''), & \text{if } \sigma \xrightarrow{a} \sigma' \xrightarrow{a'} \sigma'' \xrightarrow{a''} \sigma''', \end{cases}$$

$$c_{2}(\sigma) := \begin{cases} \bot, & \text{if } \sigma = \langle \rangle, \text{ or } \sigma = \langle a \rangle, \text{ or } \sigma = \langle a, a' \rangle \text{ for some } a, a' \in A, \\ (a'', \sigma'''), & \text{if } \sigma \xrightarrow{a} \sigma' \xrightarrow{a'} \sigma'' \xrightarrow{a''} \sigma'''. \end{cases}$$

Then, for  $i \in \{0, 1, 2\}$ , the function  $ex_i : A^{\infty} \to A^{\infty}$  is the unique function making

commute.

2. Define the coalgebra  $m_3: A^{\infty} \times A^{\infty} \times A^{\infty} \to \{\bot\} \cup (A \times (A^{\infty} \times A^{\infty} \times A^{\infty}))$  by

$$m_{3}(\sigma, \tau, \rho) := \begin{cases} \bot, & \text{if } \sigma = \tau = \rho = \langle \rangle, \\ \left(a, (\langle \rangle, \langle \rangle, \rho')\right), & \text{if } \sigma = \tau = \langle \rangle \text{ and } \rho \xrightarrow{a} \rho' \text{ for some } a \in A \text{ and } \rho' \in A^{\infty}, \\ \left(a, (\langle \rangle, \rho, \tau')\right), & \text{if } \sigma = \langle \rangle \text{ and } \tau \xrightarrow{a} \tau' \text{ for some } a \in A \text{ and } \tau' \in A^{\infty}, \\ \left(a, (\tau, \rho, \sigma')\right), & \text{if } \sigma \xrightarrow{a} \sigma' \text{ for some } a \in A \text{ and } \sigma' \in A^{\infty}. \end{cases}$$

Then we let merge3:  $A^{\infty} \times A^{\infty} \times A^{\infty} \to A^{\infty}$  be the unique function making

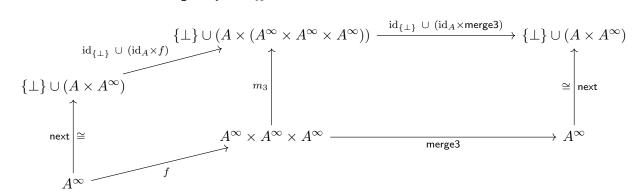
$$\{\bot\} \cup (A \times (A^{\infty} \times A^{\infty} \times A^{\infty})) \xrightarrow{\operatorname{id}_{\{\bot\}} \cup (\operatorname{id}_{A} \times \operatorname{merge3})} \{\bot\} \cup (A \times A^{\infty})$$

$$= \bigcap_{n \in X} \operatorname{next}$$

$$A^{\infty} \times A^{\infty} \times A^{\infty} \xrightarrow{\exists ! \operatorname{merge3}}$$

commute.

Let us prove that  $\operatorname{merge3}(\operatorname{ex}_0(\sigma),\operatorname{ex}_1(\sigma),\operatorname{ex}_2(\sigma))=\sigma$  for all  $\sigma\in A^\infty$ , by coinduction. Consider the function  $f\colon A^\infty\to A^\infty\times A^\infty\times A^\infty$  defined by  $f(\sigma)\coloneqq (\operatorname{ex}_0(\sigma),\operatorname{ex}_1(\sigma),\operatorname{ex}_2(\sigma))$  for all  $\sigma\in A^\infty$ . We wish to show that  $\operatorname{merge3}\circ f=\operatorname{id}_{A^\infty}$ .



Let us first show that the left square commutes. It certainly commutes when we chase the empty sequence:  $(m_3 \circ f)(\langle \rangle) = \bot = ((\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times f)) \circ \mathsf{next})(\langle \rangle)$ . If  $\sigma \in A^{\infty}$  is a non-empty sequence, say  $\sigma \xrightarrow{a} \sigma'$  for some  $a \in A$  and  $\sigma' \in A^{\infty}$ , then we have

$$(m_3 \circ f)(\sigma) = m_3(\mathsf{ex}_0(\sigma), \mathsf{ex}_1(\sigma), \mathsf{ex}_2(\sigma))$$

$$= (a, (\mathsf{ex}_1(\sigma), \mathsf{ex}_2(\sigma), \mathsf{ex}_0(\sigma')))$$

$$= (a, (\mathsf{ex}_0(\sigma'), \mathsf{ex}_1(\sigma'), \mathsf{ex}_2(\sigma')))$$

$$= ((\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times f)) \circ \mathsf{next})(\sigma)$$

where the second-to-last equality can also be proven by coinduction. Therefore the outer square commutes, and so

$$\mathsf{next} \circ (\mathsf{merge3} \circ f) = \left( \left( \mathrm{id}_{\{\bot\}} \cup \left( \mathrm{id}_A \times (\mathsf{merge3} \circ f) \right) \right) \circ \mathsf{next}.$$

The finality of the coalgebra next:  $A^{\infty} \to \{\bot\} \cup (A \times A^{\infty})$  now yields merge $3 \circ f = \mathrm{id}_{A^{\infty}}$ .

#### Exercise 1.2.8

Consider the sequential composition function comp:  $A^{\infty} \times A^{\infty} \to A^{\infty}$  for sequences, described by the three rules:

$$\begin{array}{c|c} \sigma \not \rightarrow & \tau \not \rightarrow \\ \hline \mathsf{comp}(\sigma,\tau) \not \rightarrow & & & & & & & & & & & & \\ \hline & \sigma \not \rightarrow & \tau \xrightarrow{a} \tau' \\ \hline & & & & & & & \\ \hline & \sigma \xrightarrow{a} \sigma' \\ \hline & \mathsf{comp}(\sigma,\tau) \xrightarrow{a} \mathsf{comp}(\sigma',\tau) \end{array} .$$

- 1. Show by coinduction that the empty sequence  $\langle \rangle = \mathsf{next}^{-1}(\bot) \in A^{\infty}$  is a unit element for comp, i.e. that  $\mathsf{comp}(\langle \rangle, \sigma) = \sigma = \mathsf{comp}(\sigma, \langle \rangle)$ .
- 2. Prove also by coinduction that comp is associative, and thus that sequences carry a monoid structure.

Solution.

1. Let  $f: A^{\infty} \to A^{\infty}$  be defined by  $f(\sigma) := \mathsf{comp}(\langle \rangle, \sigma)$ . We will show that the diagram

$$\{\bot\} \cup (A \times A^{\infty}) \xrightarrow{\operatorname{id}_{\{\bot\}} \ \cup \ (\operatorname{id}_A \times f)} } \{\bot\} \cup (A \times A^{\infty})$$

$$\stackrel{\operatorname{next}}{\cong} \qquad \qquad \stackrel{\cong}{\cong} \operatorname{next}$$

$$A^{\infty} \xrightarrow{\qquad \qquad \qquad \qquad \qquad \qquad \qquad } A^{\infty}$$

commutes, which would yield  $f = id_{A^{\infty}}$  by the finality of the coalgebra next.

First, we chase the empty sequence from the bottom left. We see that

$$\begin{aligned} (\mathsf{next} \circ f)(\langle \rangle) &= \mathsf{next}(\mathsf{comp}(\langle \rangle, \langle \rangle)) \\ &= \mathsf{next}(\langle \rangle) \\ &= \bot, \end{aligned}$$

the first rule for comp, and

$$((\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times f)) \circ \mathsf{next}) (\langle \rangle) = (\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times f)) (\bot)$$

$$= \bot$$

Now if  $\sigma \in A^{\infty}$  is a non-empty sequence, say  $\sigma \xrightarrow{a} \sigma'$  for some  $a \in A$  and  $\sigma' \in A^{\infty}$ , we see that

$$\begin{split} (\mathsf{next} \circ f)(\sigma) &= \mathsf{next}(\mathsf{comp}(\langle \rangle, a \cdot \sigma')) \\ &= (a, \mathsf{comp}(\langle \rangle, \sigma')) \\ &= (a, f(\sigma')), \end{split}$$

by the second rule for comp and the definition of f, and

$$\begin{split} \big( \big( \mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times f) \big) \circ \mathsf{next} \big) (\sigma) &= \big( \big( \mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times f) \big) \big( (a, \sigma') \big) \\ &= (a, f(\sigma')). \end{split}$$

Thus  $\operatorname{next} \circ f = (\operatorname{id}_{\{\bot\}} \cup (\operatorname{id}_A \times f)) \circ \operatorname{next}$ . This proves that  $\operatorname{comp}(\langle \rangle, \sigma) = \sigma$  for all  $\sigma \in A^{\infty}$ .

We now show the other equality, that  $\mathsf{comp}(\sigma, \langle \rangle) = \sigma$  for all  $\sigma \in A^{\infty}$ , we will show that the function  $g \colon A^{\infty} \to A^{\infty}$  defined by  $g(\sigma) \coloneqq \mathsf{comp}(\sigma, \langle \rangle)$  for all  $\sigma \in A^{\infty}$  also satisfies

$$\mathsf{next} \circ g = \left( \mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times g) \right) \circ \mathsf{next}.$$

That  $(\mathsf{next} \circ g)(\bot) = ((\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times g)) \circ \mathsf{next})(\bot)$  is the same as with f. Now if  $\sigma \in A^{\infty}$  is such that  $\sigma \xrightarrow{a} \sigma'$  for some  $a \in A$  and  $\sigma' \in A^{\infty}$ , we see that

$$(\mathsf{next} \circ g)(\sigma) = \mathsf{next}(\mathsf{comp}(a \cdot \sigma', \langle \rangle))$$

$$= (a, comp(\sigma', \langle \rangle))$$
$$= (a, g(\sigma')),$$

by the third rule for comp and the definition of g, and

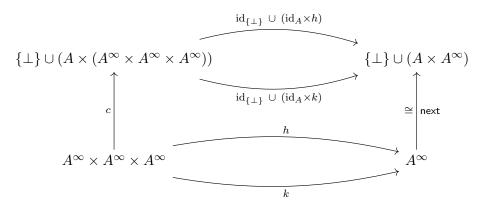
$$((\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times g)) \circ \mathsf{next})(\sigma) = ((\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times g)))((a, \sigma'))$$
$$= (a, g(\sigma')).$$

Therefore  $g = \mathrm{id}_{A^{\infty}}$ , i.e.  $\mathsf{comp}(\sigma, \langle \rangle) = \sigma$  for all  $\sigma \in A^{\infty}$ .

2. We will define a coalgebra  $c: A^{\infty} \times A^{\infty} \times A^{\infty} \to \{\bot\} \cup (A \times (A^{\infty} \times A^{\infty} \times A^{\infty}))$  such that the functions  $h, k: A^{\infty} \times A^{\infty} \times A^{\infty} \to A^{\infty}$  given by

$$h(\sigma, \tau, \rho) := \mathsf{comp}(\sigma, \mathsf{comp}(\tau, \rho))$$
 and  $k(\sigma, \tau, \rho) := \mathsf{comp}(\mathsf{comp}(\sigma, \tau), \rho),$ 

for all  $\sigma, \tau, \rho \in A^{\infty}$ , are both coalgebra homomorphisms from c to next.



The finality of next would then yield h = k.

Define 
$$c: A^{\infty} \times A^{\infty} \times A^{\infty} \to \{\bot\} \cup (A \times (A^{\infty} \times A^{\infty} \times A^{\infty}))$$
 by

$$c(\sigma,\tau,\rho) \coloneqq \begin{cases} \bot, & \text{if } \sigma = \tau = \rho = \langle \rangle, \\ \left(a,(\langle \rangle,\langle \rangle,\rho'), & \text{if } \sigma = \tau = \langle \rangle \text{ and } \rho = a \cdot \rho' \text{ for some } a \in A \text{ and } \rho' \in A^{\infty}, \\ \left(a,(\langle \rangle,\tau',\rho)\right), & \text{if } \sigma = \langle \rangle \text{ and } \tau = a \cdot \tau' \text{ for some } a \in A \text{ and } \tau' \in A^{\infty}, \\ \left(a,(\sigma',\tau,\rho)\right), & \text{if } \sigma = a \cdot \sigma' \text{ for some } a \in A \text{ and } \sigma' \in A^{\infty}. \end{cases}$$

Using the rules for comp, it is now elementary to check that h and k make their respective diagrams commute.

## Exercise 1.2.9

Consider two sets A, B with a function  $f: A \to B$  between them. Use finality to define a function  $f^{\infty}: A^{\infty} \to B^{\infty}$  that applies f element-wise. Use uniqueness to show that this mapping  $f \mapsto f^{\infty}$  is 'functorial' in the sense that  $(\mathrm{id}_A)^{\infty} = \mathrm{id}_{A^{\infty}}$  and  $(g \circ f)^{\infty} = g^{\infty} \circ f^{\infty}$ .

Solution. For a (non-empty) set B, let  $\mathsf{next}_B \colon B^\infty \to \{\bot\} \cup (B \times B^\infty)$  denote the final coalgebra defined by

$$\mathsf{next}(\sigma) \coloneqq \begin{cases} \bot, & \text{if } \sigma \text{ is the empty sequence,} \\ (b, \sigma'), & \text{if } \sigma \text{ has head } b \in B \text{ and tail } \sigma' \in B^\infty, \text{ i.e. } \sigma = b \cdot \sigma', \end{cases}$$

for all  $\sigma \in B^{\infty}$ . For a function  $f: A \to B$ , define a coalgebra  $c_f: A^{\infty} \to \{\bot\} \cup (B \times A^{\infty})$  by

$$c_f(\sigma) \coloneqq \begin{cases} \bot, & \text{if } \sigma = \langle \rangle, \\ (f(a), \sigma'), & \text{if } \sigma = a \cdot \sigma' \text{ for some } a \in A \text{ and } \sigma' \in A^{\infty}, \end{cases}$$

for all  $\sigma \in A^{\infty}$ . Let  $f^{\infty} : A^{\infty} \to B^{\infty}$  be the unique function making

$$\{\bot\} \cup (B \times A^{\infty}) \xrightarrow{\operatorname{id}_{\{\bot\}} \ \cup \ (\operatorname{id}_B \times f^{\infty})} \qquad \{\bot\} \cup (B \times B^{\infty})$$

$$\downarrow c_f \qquad \qquad \cong \qquad \qquad = \downarrow \operatorname{next}_B$$

$$\downarrow A^{\infty} \qquad \qquad \exists ! f^{\infty} \qquad \qquad B^{\infty}$$

commute. Then  $f(\langle a_0, a_1, a_2, a_3, \dots \rangle) = \langle f(a_0), f(a_1), f(a_2), f(a_3), \dots \rangle$  for all  $a_0, a_1, a_2, a_3, \dots \in A$ , and analogously for finite sequences.

We see that  $c_{\mathrm{id}_A} = \mathsf{next}_A$ . So  $(\mathrm{id}_A)^{\infty} = \mathrm{id}_{A^{\infty}}$  by finality of  $\mathsf{next}_A$ . Furthermore, for functions  $f \colon A \to B$  and  $g \colon B \to C$ , we see that

$$\{\bot\} \cup (C \times A^{\infty}) \xrightarrow{\operatorname{id}_{\{\bot\}} \cup (\operatorname{id}_{C} \times f^{\infty})} \{\bot\} \cup (C \times B^{\infty})$$

$$\downarrow c_{g \circ f} \qquad \qquad \downarrow c_{g} \qquad$$

commutes. Consequently, the outer square in the diagram

$$\{\bot\} \cup (C \times A^{\infty}) \xrightarrow{\operatorname{id}_{\{\bot\}} \ \cup \ (\operatorname{id}_{C} \times f^{\infty})} \to \{\bot\} \cup (C \times B^{\infty}) \xrightarrow{\operatorname{id}_{\{\bot\}} \ \cup \ (\operatorname{id}_{C} \times g^{\infty})} \to \{\bot\} \cup (C \times C^{\infty})$$

$$\downarrow c_{g \circ f} \qquad \qquad \downarrow c_{g} \qquad \qquad \downarrow$$

commutes, i.e.

$$\mathsf{next}_C \circ (g^\infty \circ f^\infty) = \left( \mathrm{id}_{\{\bot\}} \cup \left( \mathrm{id}_C \times (g^\infty \circ f^\infty) \right) \right) \circ c_{g \circ f}.$$

The finality of  $\operatorname{next}_C$  then yields  $(g \circ f)^{\infty} = g^{\infty} \circ f^{\infty}$ .

## Exercise 1.2.10

Use finality to define a map st:  $A^{\infty} \times B \to (A \times B)^{\infty}$  that maps a sequence  $\sigma \in A^{\infty}$  and an element  $b \in B$  to a new sequence in  $(A \times B)^{\infty}$  by adding this b at every position in  $\sigma$ . (This is an example of a 'strength' map; see Exercise 2.5.4.

Solution. Define a coalgebra  $c: A^{\infty} \times B \to \{\bot\} \cup ((A \times B) \times (A^{\infty} \times B))$  as follows:

$$c(\sigma,b) \coloneqq \begin{cases} \bot, & \text{if } \sigma = \langle \rangle, \\ \big((a,b),(\sigma',b)\big), & \text{if } \sigma = a \cdot \sigma' \text{ for some } a \in A \text{ and } \sigma' \in A^{\infty}, \end{cases}$$

for all  $\sigma \in A^{\infty}$  and  $b \in B$ . The unique function st:  $A^{\infty} \times B \to (A \times B)^{\infty}$  making

commute will satisfy  $\operatorname{st}(\langle a_0, a_1, a_2, \dots \rangle, b) = \langle (a_0, b), (a_1, b), (a_2, b), \dots \rangle$  for all  $a_0, a_1, a_2, a_3, \dots \in A$  and  $b \in B$ , and analogously for finite sequences in  $A^{\infty}$ .

## 1.3 Generality of Temporal Logic of Coalgebras

## Exercise 1.3.1

The nexttime operator  $\circ$  introduced in (1.9) is the so-called **weak** nexttime. There is an associated **strong** nexttime, given by  $\neg \circ \neg$ . Note the difference between weak and strong nexttime for sequences.

Solution. Recall that, for a sequence coalgebra  $c: S \to \{\bot\} \cup (A \times S)$  and a predicate  $P \subseteq S$ , we have

$$(\bigcirc P)(x)$$
 if and only if  $c(x) = \bot$  or  $c(x) \in A \times P$ ,

for all  $x \in S$ . So,

$$(\bigcirc \neg P)(x)$$
 if and only if  $c(x) = \bot$  or  $c(x) \in A \times (S \setminus P)$ ,

and thus

$$(\neg \bigcirc \neg P)(x)$$
 if and only if  $c(x) \neq \bot$  and  $c(x) \notin A \times (S \setminus P)$ .

Since the codomain of c is  $\{\bot\} \cup (A \times S)$ , and since  $P \subseteq S$ , we can equivalently write this as

$$(\neg \bigcirc \neg P)(x)$$
 if and only if  $c(x) \in A \times P$ .

#### Exercise 1.3.2

Prove that the 'truth' predicate that always holds is a (sequence) invariant. And if  $P_1$  and  $P_2$  are invariants, then so is the intersection  $P_1 \cap P_2$ . Finally, if P is an invariant, then so is  $\circ P$ .

Solution. Fix a sequence coalgebra  $c: S \to \{\bot\} \cup (A \times S)$ . The truth predicate is the set S itself. Then, for all  $x \in S$ ,

$$(\bigcirc S)(x)$$
 if and only if  $c(x) = \bot$  or  $c(x) \in A \times S$ .

Since the codomain of c is  $\{\bot\} \cup (A \times S)$ , this means that  $\bigcirc S = S$ , and so S is an invariant. Now suppose that  $P_1$  and  $P_2$  and invariant, i.e.  $P_1 \subseteq \bigcirc P_1$  and  $P_2 \subseteq \bigcirc P_2$ . Then, for all  $x \in S$ ,

$$(\bigcirc(P_1 \cap P_2))(x)$$
 if and only if  $c(x) = \bot$  or  $c(x) \in A \times (P_1 \cap P_2)$   
if and only if  $c(x) = \bot$  or  $c(x) \in (A \times P_1) \cap (A \times P_2)$   
if and only if  $(c(x) = \bot$  or  $c(x) \in A \times P_1)$  and  $(c(x) = \bot$  or  $c(x) \in A \times P_2)$   
if and only if  $(\bigcirc P_1)(x)$  and  $(\bigcirc P_2)(x)$ .

Hence  $P_1 \cap P_2 \subseteq (\bigcirc P_1) \cap (\bigcirc P_2) = \bigcirc (P_1 \cap P_2)$ , and so  $P_1 \cap P_2$  is also invariant.

Finally, suppose that P is invariant, i.e.  $P \subseteq \bigcirc P$ . We aim to show that  $\bigcirc P \subseteq \bigcirc \bigcirc P$ . Suppose  $x \in S$  is such that  $(\bigcirc P)(x)$  holds. Then either  $c(x) = \bot$  or  $c(x) \in A \times P \subseteq A \times \bigcirc P$ . Therefore  $(\bigcirc \bigcirc P)(x)$  holds.

#### Exercise 1.3.3

- 1. Show that  $\square$  is an interior operator, i.e. satisfies:  $\square P \subseteq P$ ,  $\square P \subseteq \square \square P$ , and  $P \subseteq Q \Longrightarrow \square P \subseteq \square Q$ .
- 2. Prove that a predicate P is invariant if and only if  $P = \Box P$ .

Solution. Fix a sequence coalgebra  $c: S \to \{\bot\} \cup (A \times S)$ . Recall that the henceforth operator  $\square$  is defined on predicates  $P \subseteq S$  as follows: for all  $x \in S$ ,

$$(\Box P)(x)$$
 if and only if there exists an invariant  $Q \subseteq S$  with  $x \in Q \subseteq P$ .

In other words,  $\Box P$  is the union of all invariants contained in P.

1. If  $x \in \Box P$ , then there is an invariant  $Q \subseteq S$  with  $x \in Q \subseteq P$ . So  $x \in P$  too. Also, Q is an invariant with  $x \in Q \subseteq \Box P$ . So  $x \in \Box \Box P$  as well. Thus  $\Box P \subseteq P$  and  $\Box P \subseteq \Box \Box P$ .

Now suppose  $P \subseteq Q \subseteq S$ . Then, for any  $x \in \Box P$ , there is an invariant  $R \subseteq S$  with  $x \in R \subseteq P \subseteq Q$ . So  $x \in \Box Q$  as well. Therefore  $\Box P \subseteq \Box Q$ .

2. For the forward direction, suppose that P is invariant. By definition,  $\Box P$  is the union of all invariants contained within P. As P is assumed to be an invariant, we must have  $\Box P = P$ .

For the converse direction, suppose that  $\Box P = P$ . We need to show that P is an invariant, i.e.  $P \subseteq \bigcirc P$ . For any  $x \in P = \Box P$ , there exists an invariant  $Q \subseteq S$  with  $x \in Q \subseteq P$ . As Q is an invariant, either  $c(x) = \bot$  or  $c(x) \in A \times Q \subseteq A \times P$ . Hence we also have  $x \in \bigcirc P$ . Therefore  $P \subseteq \bigcirc P$ , meaning P is an invariant.

## Exercise 1.3.4

Recall the finite behaviour predicate  $\Diamond((-) \not\to)$  from Example 1.3.4.1 and show that it is an invariant:  $\Diamond((-) \not\to) \subseteq \Diamond((-) \not\to)$ . Hint: For an invariant Q, consider the predicate  $Q' = (\neg(-) \not\to) \cap (\bigcirc Q)$ .

Solution. Fix a sequence coalgebra  $c: S \to \{\bot\} \cup (A \times S)$ . Recall that, for a predicate  $P \subseteq S$  and  $x \in S$ ,

$$(\lozenge P)(x)$$
 if and only if for all invariants  $Q \subseteq S$ , we have  $\neg Q(x)$  or  $Q \not\subseteq \neg P$ .

That is,  $\Diamond P = \neg \Box \neg P$ .

Suppose  $x \in S$  is such that  $\Diamond(x \not\to)$  holds. We need to show that  $\Diamond(x \not\to)$  holds, i.e. if  $x \xrightarrow{a} x'$  for some  $(a, x') \in A \times S$ , then  $\Diamond(x' \not\to)$  also holds. Fix any invariant  $Q \subseteq S$  with  $Q \subseteq \neg((-) \not\to)$ . We need to show that  $\neg Q(x')$ .

Following the hint, we consider the predicate

$$Q' \coloneqq \neg \big( (-) \not\to \big) \cap (\bigcirc Q).$$

Observe that Q' is an invariant: if  $y \in S$  satisfies Q'(y), then there is some  $(b, y') \in A \times S$  such that  $y \xrightarrow{b} y'$  and Q(y') hold. Then, since  $Q \subseteq \neg((-) \not\rightarrow)$  and Q is an invariant, we conclude that Q'(y') also holds. So  $Q' \subseteq \bigcirc Q'$ .

Hence if Q(x') holds, then Q'(x) holds too, contradicting the assumption that  $\Diamond(x \not\to)$ .

#### Exercise 1.3.5

Let  $(A, \leq)$  be a complete lattice, i.e. a poset in which each subset  $U \subseteq A$  has a join  $\bigvee U \in A$ . It is well known that each subset  $U \subseteq A$  then also has a meet  $\bigwedge U \in A$ , given by  $\bigwedge U = \bigvee \{ a \in A \mid \forall b \in U.a \leq b \}$ .

Let  $f: A \to A$  be a monotone function:  $a \leq b$  implies  $f(a) \leq f(b)$ . Recall, e.g. from Davey and Priestley (1990, Chapter 4) that such a monotone f has both a least fixed point  $\mu f \in A$  and a greatest fixed point  $\nu f \in A$  given by the formulas:

$$\mu f = \bigwedge \{ a \in A \mid f(a) \le a \}, \qquad \nu f = \bigvee \{ a \in A \mid a \le f(a) \}.$$

Now let  $c: S \to \{\bot\} \cup (A \times A)$  be an arbitrary sequence coalgebra, with associated nexttime operator  $\bigcirc$ .

- 1. Prove that  $\bigcirc$  is a monotone function  $\mathcal{P}(S) \to \mathcal{P}(S)$ , i.e. that  $P \subseteq Q$  implies  $\bigcirc P \subseteq \bigcirc Q$ , for all  $P,Q \subseteq S$ .
- 2. Check that  $\Box P \in \mathcal{P}(S)$  is the greatest fixed point of the function  $\mathcal{P}(S) \to \mathcal{P}(S)$  given by  $U \mapsto P \cap \bigcirc U$ .
- 3. Define for  $P,Q \subseteq S$  a new predicate  $P \ \mathcal{U} \ Q \subseteq S$ , for 'P until Q' as the least fixed point of  $U \mapsto Q \cup (P \cap \neg \bigcirc \neg U)$ . Check that 'until' is indeed a good name for  $P \ \mathcal{U} \ Q$ , since it can be described explicitly as

$$P \ \mathcal{U} \ Q = \{ x \in S \mid \exists n \in \mathbb{N}. \exists x_0, x_1, \dots, x_n \in S.$$
$$x_0 = x \land (\forall i < n. \exists a. x_i \xrightarrow{a} x_{i+1}) \land Q(x_n)$$
$$\land \forall i < n. P(x_i) \}.$$

Hint: Don't use the fixed point definition  $\mu$ , but first show that this subset is a fixed point, and then that it is contained in an arbitrary fixed point.

(The fixed point definitions that we described above are standard in temporal logic; see e.g. Emerson (1990, 3.24–3.25). The above operation  $\mathcal{U}$  is what is called the 'strong' until. The 'weak one' does not have the negations  $\neg$  in its fixed-point description in point 3.)

Solution.

1. For subsets  $P, Q \in \mathcal{P}(S)$  with  $P \subseteq Q$ , and for  $x \in S$  such that  $(\bigcirc P)(x)$  holds, we have

$$c(x) = \bot \text{ or } c(x) \in A \times P.$$

From the assumption that  $P \subseteq Q$ , it follows that

$$c(x) = \bot \text{ or } c(x) \in A \times Q,$$

or equivalently,  $(\bigcirc Q)(x)$ .

2. Fix  $P \in \mathcal{P}(S)$  and define  $f_P \colon \mathcal{P}(S) \to \mathcal{P}(S)$  by  $f_P(U) := P \cap OU$  for all  $U \in \mathcal{P}(S)$ . Then the greatest fixed point of  $f_P$  is

$$\nu(f_P) := \bigcup_{\substack{U \in \mathcal{P}(S), \\ U \subset f_P(U)}} U = \bigcup_{\substack{U \in \mathcal{P}(S), \\ U \subseteq P \cap OU}} U = \square P.$$

3. Fix  $P, Q \in \mathcal{P}(S)$ , and define  $f_{P,Q} \colon \mathcal{P}(S) \to \mathcal{P}(S)$  by

$$f_{PO}(U) := Q \cup (P \cap \neg \bigcirc \neg U)$$

for all  $U \in \mathcal{P}(S)$ . Recall, from Exercise 1.3.1, that

$$\neg \bigcirc \neg U = \{ x \in S : c(x) \in A \times U \}.$$

We wish to show that the set

$$U_{P,Q} := Q \cup \left\{ x \in S : \text{ there exist } n \in \mathbb{Z}_{>0}, \ x_0, \dots, x_n \in S \text{ and } a_0, \dots, a_{n-1} \in A \right\}$$
  
such that  $x = x_0 \xrightarrow{a_0} \cdots \xrightarrow{a_{n-1}} x_n$  and  $P(x_0), \dots, P(x_{n-1}), \text{ and } Q(x_n) \text{ all hold } \right\}$ 

is the least fixed point of  $f_{P,Q}$ .

First, observe that

$$f_{P,Q}(U_{P,Q}) = Q \cup (P \cap \neg \bigcirc \neg U_{P,Q})$$

$$= Q \cup (P \cap \{x \in S : c(x) \in A \times U_{P,Q} \})$$

$$= Q \cup \{x \in S : P(x) \text{ and } c(x) \in A \times U_{P,Q} \}$$

$$= U_{P,Q},$$

so that  $U_{P,Q}$  is indeed a fixed point of  $f_{P,Q}$ .

Now we show that  $U_{P,Q}$  is the least fixed point of  $f_{P,Q}$ . Fix some  $B \subseteq S$  with  $f_{P,Q}(B) = B$ , i.e.

$$Q \cup \{x \in S : P(x) \text{ and } c(x) \in A \times B\} = B.$$

Then we get  $U_{P,Q} \subseteq B$  by induction on the length of finite sequences  $x_0, \ldots x_n \in S$  and  $a_0, \ldots, a_{n-1} \in A$  satisfying  $x_0 \xrightarrow{a_0} \cdots \xrightarrow{a_{n-1}} x_n$ , and  $P(x_0) \wedge \cdots \wedge P(x_{n-1}) \wedge Q(x_n)$ .

#### 1.4 Abstractness of Coalgebraic Notions

#### Exercise 1.4.1

Let (M, +, 0) be a monoid, considered as a category. Check that a functor  $F: M \to \mathbf{Sets}$  can be identified with a **monoid action**: a set X together with a function  $\mu: X \times M \to X$  with  $\mu(x, 0) = x$  and  $\mu(x, m_1 + m_2) = \mu(\mu(x, m_2), m_1)$ .

Solution. Suppose we are given functor  $F: M \to X$ . This F sends the unique object  $\star \in \mathsf{Obj}(M)$  to a set  $F(\star) \in \mathsf{Obj}(\mathbf{Sets})$ , and sends each  $m \in \mathsf{Arr}(M)$  to a function  $Fm: F(\star) \to F(\star)$ . The functoriality of F requires that  $F(0) = \mathrm{id}_{F(\star)}$  and  $F(m_1 + m_2) = F(m_1) \circ F(m_2)$  for all  $m_1, m_2 \in \mathsf{Arr}(M)$ . We then define a function  $\mu_F: F(\star) \times \mathsf{Arr}(M) \to F(\star)$  by  $\mu_F(x, m) := F(m)(x)$  for all  $(x, m) \in F(\star) \times M$ .

The equality  $\mu_F(x,0) = x$  for all  $x \in F(\star)$  follows the equality  $F(0) = \mathrm{id}_{F(\star)}$ , while the equality  $\mu_F(x,m_1+m_2) = \mu_F(\mu_F(x,m_2),m_1)$  for all  $x \in X$  and  $m_1,m_2 \in \mathsf{Arr}(M)$  follows from the equality  $F(m_1+m_2) = F(m_1) \circ F(m_2)$ .

Now suppose we are given also given a set X and a function  $\mu: X \times \mathsf{Arr}(M) \to X$  with  $\mu(x,0) = x$  and  $\mu(x,m_1+m_2) = \mu(\mu(x,m_2),m_1)$  for all  $x \in X$  and  $m,m_1,m_2 \in \mathsf{Arr}(M)$ . We then define a functor  $F_{\mu} \colon M \to \mathbf{Sets}$  by  $F_{\mu}(\star) \coloneqq X$ , for the unique object  $\star \in \mathsf{Obj}(M)$ , and  $F_{\mu}(m) \coloneqq \mu(-,m)$  for each  $m \in \mathsf{Arr}(M)$ . That  $F_{\mu}$  is actually a functor follows from the assumptions on  $\mu$ .

We then have 
$$F_{\mu_F} = F$$
 and  $\mu_{F_{\mu}} = \mu$ .

## Exercise 1.4.2

Check in detail that the opposite  $\mathbb{C}^{op}$  and the product  $\mathbb{C} \times \mathbb{D}$  are indeed categories.

Solution. Let  $\mathbb C$  and  $\mathbb D$  be categories.

We defined  $\operatorname{Obj}(\mathbb{C}^{\operatorname{op}}) := \operatorname{Obj}(\mathbb{C})$ . For  $X, Y \in \operatorname{Obj}(\mathbb{C})$ , write  $\operatorname{hom}_{\mathbb{C}}(X, Y)$  for the set of all morphisms with domain X and codomain Y. We then defined  $\operatorname{hom}_{\mathbb{C}^{\operatorname{op}}}(X, Y) := \operatorname{hom}_{\mathbb{C}}(Y, X)$ , and we defined a composition  $X \stackrel{f}{\leftarrow} Y \stackrel{g}{\leftarrow} Z$  in  $\mathbb{C}^{\operatorname{op}}$  to be the composition  $X \stackrel{f}{\rightarrow} Y \stackrel{g}{\rightarrow} Z$  in  $\mathbb{C}$ . The associativity and identity laws for composition in  $\mathbb{C}^{\operatorname{op}}$  follow from those for  $\mathbb{C}$ .

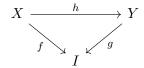
We defined  $\operatorname{Obj}(\mathbb{C} \times \mathbb{D}) := \operatorname{Obj}(\mathbb{C}) \times \operatorname{Obj}(\mathbb{D})$ . For  $X, X' \in \operatorname{Obj}(\mathbb{C})$  and  $Y, Y' \in \operatorname{Obj}(\mathbb{D})$ , we let  $\operatorname{hom}_{\mathbb{C} \times \mathbb{D}}((X,Y),(X',Y')) := \operatorname{hom}_{\mathbb{C}}(X,X') \times \operatorname{hom}_{\mathbb{D}}(Y,Y')$ . A composition  $(X,Y) \xrightarrow{(f,g)} (X',Y') \xrightarrow{(f',g')} (X'',Y'')$  in  $\mathbb{C} \times \mathbb{D}$  is defined to be the composition  $(X,Y) \xrightarrow{(f'f,g'g)} (X'',Y'')$ . For an object (X,Y) in  $\mathbb{C} \times \mathbb{D}$ , the identity morphism  $\operatorname{id}_{(X,Y)}$  is the pair  $(\operatorname{id}_X,\operatorname{id}_Y)$ . The associativity and identity laws for composition in  $\mathbb{C} \times \mathbb{D}$  follow from those for  $\mathbb{C}$  and  $\mathbb{D}$ .

#### Exercise 1.4.3

Assume an arbitrary category  $\mathbb{C}$  with an object  $I \in \mathbb{C}$ . We form a new category  $\mathbb{C}/I$ , the so-called **slice** category over I, with

**objects**  $maps \ f \colon X \to I \ with \ codomain \ I \ in \ \mathbb{C}$ 

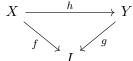
**morphisms** from  $X \xrightarrow{f} I$  to  $Y \xrightarrow{g} I$  are morphisms  $h: X \to Y$  in  $\mathbb{C}$  for which the following diagram commutes:



- 1. Describe identities and composition in  $\mathbb{C}/I$ , and verify that  $\mathbb{C}/I$  is a category.
- 2. Check that taking domains yields a functor dom:  $\mathbb{C}/I \to \mathbb{C}$ .
- 3. Verify that for  $\mathbb{C} = \mathbf{Sets}$ , a map  $f: X \to I$  may be identified with an I-indexed family of sets  $(X_i)_{i \in I}$ , namely where  $X_i = f^{-1}(i)$ . What do morphisms in  $\mathbb{C}/I$  correspond to, in terms of such indexed families?

Solution.

- 1. The identities and composition in  $\mathbb{C}/I$  are simply the identities and composition in  $\mathbb{C}$ . So the fact that  $\mathbb{C}/I$  is a category follows from  $\mathbb{C}$  being a category.
- 2. We define dom:  $\mathbb{C}/I \to \mathbb{C}$  as follows: for a morphism h from  $X \xrightarrow{f} I$  to  $Y \xrightarrow{g} I$  in  $\mathbb{C}/I$ , we simply define dom(h) := h. This immediately makes dom a functor from  $\mathbb{C}/I$  to  $\mathbb{C}$ .
- 3. The claimed identification is obvious. Now fix a morphism h from  $X \xrightarrow{f} I$  to  $Y \xrightarrow{g} I$  in  $\mathbf{Sets}/I$ , so that the diagram



in **Sets** commutes. This requires that g(h(x)) = f(x) for all  $x \in X$ . Identifying  $X_i := f^{-1}(i)$  and  $Y_i := g^{-1}(i)$  for all  $i \in I$ , we can identify h with a family of functions  $(h_i)_{i \in I}$  such that  $h_i(x) \in Y_i$  for all  $x \in X_i$ , for all  $i \in I$ .

#### Exercise 1.4.4

Recall that for an arbitrary set A we write  $A^*$  for the set of finite sequencees  $\langle a_0, \ldots, a_n \rangle$  of elements  $a_i \in A$ .

- 1. Check that  $A^*$  carries a monoid structure given by concatenation of sequences, with the empty sequence  $\langle \rangle$  as a neutral element.
- 2. Check that the assignment  $A \mapsto A^*$  yields a functor  $\mathbf{Sets} \to \mathbf{Mon}$  by mapping a function  $f : A \to B$  between sets to the function  $f^* : A^* \to B^*$  given by  $\langle a_0, \ldots, a_n \rangle \mapsto \langle f(a_0), \ldots, f(a_n) \rangle$ . (Be aware of what needs to be checked:  $f^*$  must be a monoid homomorphism, and  $(-)^*$  must preserve composition of functions and identity functions.)
- 3. Prove that  $A^*$  is the **free monoid on** A: there is the singleton-sequence insertion map  $\eta \colon A \to A^*$  which is universal among all mappings of A into a monoid. The latter means that for each monoid (M,0,+) and function  $f \colon A \to M$  there is a unique monoid homomorphism  $g \colon A^* \to M$  with  $g \circ \eta = f$ .

Solution.

- 1. Concatenation is associative because all the sequences under consideration are finite.
- 2. That  $(-)^*$  preserves composition and identity functions is obvious, so we just check that for a function  $f: A \to B$ , the map  $f^*: A^* \to B^*$  is a monoid homomorphism. Fix finite sequences  $\langle a_0, \ldots, a_n \rangle, \langle a'_0, \ldots, a'_k \rangle \in A^*$ . Then

$$f(\langle a_0, \dots, a_n \rangle \cdot \langle a'_0, \dots, a'_k \rangle) = f(\langle a_0, \dots, a_n, a'_0, \dots, a'_k \rangle)$$

$$= \langle f(a_0), \dots, f(a_n), f(a'_0), \dots, f(a'_k) \rangle$$

$$= \langle f(a_0), \dots, f(a_n) \rangle \cdot \langle f(a'_0), \dots, f(a'_k) \rangle$$

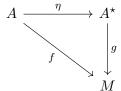
$$= f(\langle a_0, \dots, a_n \rangle) \cdot \langle a'_0, \dots a'_k \rangle)$$

and  $f(\langle \rangle) = \langle \rangle$ . So  $f^*$  is a monoid homomorphism.

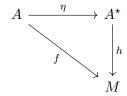
3. Define  $\eta: A \to A^*$  by  $\eta(a) := \langle a \rangle$  for all  $a \in A$ . Fix a monoid (M, 0, +) and a function  $f: A \to M$ . Define  $g: A^* \to M$  by

$$g(\langle \rangle) := 0$$
  
$$g(\langle a_0, \dots, a_n \rangle) := f(a_0) + \dots + f(a_n)$$

for all  $\langle a_0, \ldots, a_n \rangle \in A^*$ . This g is clearly a mononid homomorphism, using the associativity of + in M. Observe that the diagram



in **Sets** commutes: we have  $f(a) = g(\eta(a))$  for all  $a \in A$ . Now suppose that there is another monoid homomorphism  $h: A^* \to M$  such that the diagram



in **Sets** commutes. As  $h: A^* \to M$  is a monoid homomorphism and  $f = h\eta$ , we require that  $h(\langle \rangle) = 0$  and

$$h(\langle a_0, \dots, a_n \rangle) = h(\langle a_0 \rangle \cdot \dots \cdot \langle a_n \rangle)$$

$$= h(\langle a_0 \rangle) + \dots + h(\langle a_n \rangle)$$

$$= h(\eta(a_0)) + \dots + h(\eta(a_n))$$

$$= f(a_0) + \dots + f(a_n)$$

$$= g(\langle a_0, \dots, a_n \rangle),$$

for all  $\langle a_0, \ldots, a_n \rangle \in A^*$ . Therefore h = g.

#### Exercise 1.4.5

Recall from (1.3) the statements with exceptions of the form  $S \to \{\bot\} \cup S \cup (S \times E)$ .

- 1. Prove that the assignment  $X \mapsto \{\bot\} \cup X \cup (X \times E)$  is functorial, so that the statements are a coalgebra for this functor.
- 2. Show that all the operations  $\mathsf{at}_1, \ldots, \mathsf{at}_n, \mathsf{meth}_1, \ldots, \mathsf{meth}_m$  of a class as in (1.10) can also be described as a single coalgebra, namely of the functor:

$$X \mapsto D_1 \times \cdots \times D_n \times \underbrace{(\{\bot\} \cup X \cup (X \times E)) \times \cdots \times (\{\bot\} \cup X \cup (X \times E))}_{m \ times}.$$

Solution.

1. Let  $F: \mathbf{Sets} \to \mathbf{Sets}$  denote this assignment  $F(X) := \{\bot\} \cup X \cup (X \times E)$  where all unions are disjoint unions. We define F on morphisms as follows: for functions  $f: X \to Y$ , we define  $F(f): F(X) \to F(Y)$  to be the function

$$F(f)(x) \coloneqq \begin{cases} \bot, & \text{if } x = \bot, \\ f(x), & \text{if } x \in X, \\ (f(x'), e), & \text{if } x = (x', e) \text{ for some } (x', e) \in X \times E. \end{cases}$$

Then  $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$  and F(gf) = F(g)F(f) for all sets X and functions  $X \xrightarrow{f} Y \xrightarrow{g} Z$ .

2. The functor's definition on morphisms is similar in style with the previous part.

#### Exercise 1.4.6

Recall the nexttime operator  $\circ$  for a sequence coalgebra  $c: S \to \mathsf{Seq}(S) = \{\bot\} \cup (A \times S)$  from the previous section. Exercise 1.3.5.1 says that it forms a monotone function  $\mathcal{P}(S) \to \mathcal{P}(S)$  — with respect to the inclusion order — and thus a functor. Check that invariants are precisely  $\circ$ -coalgebras!

Solution. The  $\bigcirc$ -coalgebras are simply a subsets  $U \subseteq S$  such that  $U \subseteq \bigcirc U$ . These are precisely what invariants are.  $\Box$ 

# 2 Coalgebras of Polynomial Functors

#### 2.1 Constructions on Sets

#### Exercise 2.1.1

Verify in detail the bijective correspondences (2.2), (2.6), (2.11) and (2.16).

Solution. Fix sets X, Y, Z. Following the notation of Equations (2.1), we associate a pair of functions  $f: Z \to X$  and  $g: Z \to Y$  to the function  $\langle f, g \rangle \colon Z \to X \times Y$  given by  $\langle f, g \rangle (z) \coloneqq \langle f(z), g(z) \rangle$  for all  $z \in Z$ . Furthermore, we associate to any function  $h: Z \to X \times Y$  a pair the functions  $\pi_1 h: Z \to X$  and  $\pi_2 h: Z \to Y$ , where  $\pi_1$  and  $\pi_2$  are the relevant projections. Then  $\langle \pi_1 h, \pi_2 h \rangle = h$  and  $\langle \pi_1 \langle f, g \rangle, \pi_2 \langle f, g \rangle \rangle = \langle f, g \rangle$ . This establishes the bijective correspondence (2.2).

Continue fixing sets X, Y, Z. Suppose, without loss of generality, that X and Y are disjoint, so that we may use  $X \cup Y$  in place of X + Y. Following the notation of Equations (2.5), we associate a pair of functions  $f: X \to Z$  and  $g: Y \to Z$  to the function  $[f, g]: X + Y \to Z$  given by

$$[f,g](w) := \begin{cases} f(w), & \text{if } w \in X, \\ g(w), & \text{if } w \in Y \end{cases}$$

for all  $w \in X + Y$ . Furthermore, to any function  $h: X + Y \to Z$ , we associate the pair of functions  $h\kappa_1: X \to Z$  and  $g\kappa_2: Y \to Z$ , where  $\kappa_1$  and  $\kappa_2$  are the relevant coprojections. Then  $[h\kappa_1, h\kappa_2] = h$  and  $([f, g]\kappa_1, [f, g]\kappa_2) = (f, g)$ . This establishes the bijective correspondence (2.6).

Continue fixing sets X, Y, Z. Following the notations of Equations (2.10), we associate a function  $f: Z \times X \to Y$  to the function  $\Lambda(f): Z \to Y^X$  given by  $\Lambda(f)(z) := f(z, -)$  for all  $z \in Z$ . Furthermore, to each function  $g: Z \to Y^X$ , we associate the function  $U(g): Z \times X \to Y$  given by U(g)(z, x) := g(z)(x) for all  $(z, x) \in Z \times X$ . Then  $\Lambda(U(g)) = g$  and  $U(\Lambda(f)) = f$ . So we have established the bijective correspondence (2.11).

Finally, fix sets X and Y. To each function  $f: X \to \mathcal{P}(Y)$ , we associate the relation

$$\operatorname{rel}(f) \coloneqq \{ (y, x) \in Y \times X : y \in f(x) \}.$$

Also, to each relation  $R \subseteq Y \times X$ , we associate the function  $\operatorname{char}(R) \colon X \to \mathcal{P}(Y)$  given by

$$\mathsf{char}(R)(x) \coloneqq \{ \ y \in Y \ : \ R(y,x) \ \}$$

for all  $y \in Y$ . Then rel(char(R)) = R and char(rel(f)) = f. We thus obtain the bijective correspondence (2.16).

#### Exercise 2.1.2

Consider a poset  $(D, \leq)$  as a category. Check that the product of two elements  $d, e \in D$ , if it exists, is the meet  $d \land e$ . And a coproduct of d, e, if it exists, is the join  $d \lor e$ .

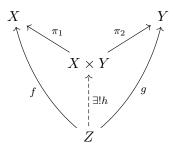
Similarly, show that a final object is a top element  $\top$  (with  $d \leq \top$ , for all  $d \in D$ ) and that an initial object is a bottom element  $\bot$  (with  $\bot \leq d$ , for all  $d \in D$ ).

Solution. These follow immediately, as in a poset  $(D, \leq)$ , we have one (and only one) morphism  $x \to y$  if and only if  $x \leq y$ , for  $x, y \in D$ , and that the only isomorphisms are identity morphisms.

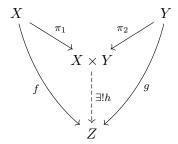
## Exercise 2.1.3

Check that a product in a category  $\mathbb{C}$  is the same as a coproduct in a category  $\mathbb{C}^{op}$ .

Solution. Fix  $X, Y, Z \in \mathsf{Obj}(\mathbb{C})$ , and suppose the product  $X \times Y$  exists in  $\mathbb{C}$ . For a pair of morphisms  $f \colon Z \to X$  and  $g \colon Z \to Y$ , we have the following diagram



in  $\mathbb{C}$  commuting. Thus we have the following diagram



in  $\mathbb{C}^{op}$  commuting. This makes  $X \times Y$  the coproduct of X and Y in  $\mathbb{C}^{op}$ , with coprojections  $\pi_1$  and  $\pi_2$ . Similarly, coproducts in  $\mathbb{C}^{op}$  correspond to products in  $\mathbb{C}$ .

#### Exercise 2.1.4

Fix a set A and prove that assignments  $X \mapsto A \times X$ ,  $X \mapsto A + X$  and  $X \mapsto X^A$  are functorial and give rise to functors **Sets**  $\rightarrow$  **Sets**.

Solution. Define  $F, G, H : \mathbf{Sets} \to \mathbf{Sets}$  as follows. For a set X,

$$FX := A \times X$$
,  
 $GX := A + X$ , and  
 $HX := X^A$ .

For a function  $f: X \to Y$ , define the functions  $Ff: A \times X \to A \times Y$ ,  $Gf: A + X \to A + Y$ , and  $Hf: X^A \to Y^A$  as follows:

$$(Ff)(a,x) \coloneqq (a,f(x)), \qquad \text{for all } (a,x) \in A \times X,$$
 
$$(Gf)(w) \coloneqq \begin{cases} w, & \text{if } w \in A, \\ f(w), & \text{if } w \in X, \end{cases} \text{ for all } w \in A + X,$$
 
$$(Hf)(h) \coloneqq fh, \qquad \text{for all functions } h \colon A \to X,$$

where we have assumed, without loss of generality, that A and X are disjoint so that X + A is treated as  $X \cup A$ .

Then, for any set X,

$$(Fid_X)(a, x) = (a, id_X(x))$$
  
=  $(a, x)$ , for all  $(a, x) \in A \times X$ ,

$$(G\mathrm{id}_X)(w) = \begin{cases} w, & \text{if } w \in A, \\ \mathrm{id}_X(w), & \text{if } w \in X, \end{cases}$$

$$= w, & \text{for all } w \in A + X,$$

$$(H\mathrm{id}_X)(h) = \mathrm{id}_X h$$

$$= h, & \text{for all functions } h \colon A \to X,$$

so  $Fid_X = id_{FX}$ ,  $Gid_X = id_{GX}$ , and  $Hid_X = id_{HX}$ . Now, for functions  $X \xrightarrow{f} Y \xrightarrow{g} Z$ ,

$$(F(gf))(a,x) = (a,g(f(x)))$$

$$= (Fg)(a,f(x))$$

$$= (Fg \circ Ff)(a,x), \qquad \text{for all } (a,x) \in A \times X,$$

$$(G(gf))(w) = \begin{cases} w, & \text{if } w \in A, \\ g(f(w)), & \text{if } w \in X, \end{cases}$$

$$= (Gg \circ Gf)(w), \qquad \text{for all } w \in A + X,$$

$$(H(gf))(h) = \lambda a \in A.(g(f(h(a)))),$$

$$= (Hg)(fh)$$

$$= (Hg \circ Hf)(h), \qquad \text{for all functions } h \colon A \to X,$$

so F(gf) = (Fg)(Ff), G(gf) = (Gg)(Gf), and H(gf) = (Hg)(Hf). Thus F, G, and H are functors from **Sets** to **Sets**.

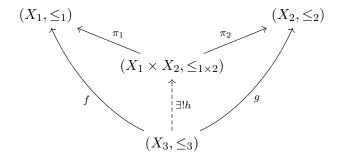
#### Exercise 2.1.5

Prove that the category **PoSets** of partially ordered sets and monotone functions is a BiCCC. The definitions on the underlying sets X of a poset  $(X, \leq)$  are like for ordinary sets but should be equipped with appropriate orders.

Solution. The category **PoSets** has a terminal object, namely the singleton poset. Furthermore, given two posets  $(X_1, \leq_1)$  and  $(X_2, \leq_2)$ , we can define a partial ordering  $\leq_{1\times 2}$  on the product  $X_1 \times X_2$  by

$$(x_1, x_2) \leq_{1 \times 2} (x_1', x_2')$$
 if and only if  $x_1 \leq x_1'$  and  $x_2 \leq x_2'$ 

for all  $(x_1, x_2), (x'_1, x'_2) \in X_1 \times X_2$ . This poset  $(X_1 \times X_2, \leq_{1 \times 2})$  has the universal property of the product: given another poset  $(X_3, \leq_3)$  and a pair of monotone functions  $f: (X_3, \leq_3) \to (X_1, \leq_1)$  and  $g: (X_3, \leq_3) \to (X_2, \leq_2)$ , we have the diagram

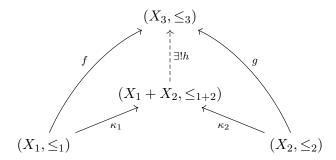


in **PoSets** commuting, where  $\pi_1$  and  $\pi_2$  are the relevant projections (which are indeed monotone). The unique monotone function h is given by  $h(x_3) := (f(x_3), g(x_3))$  for all  $x_3 \in X_3$ . Therefore the category **PoSets** has finite products.

The category **PoSets** also has an initial object: the empty poset. Now, given two posets  $(X_1, \leq_1)$  and  $(X_2, \leq_2)$ , we can define a partial ordering  $\leq_{1+2}$  on the coproduct  $X_1 + X_2$  by

$$w \leq_{1+2} w'$$
 if and only if  $(w, w' \in X_1 \text{ and } w \leq_1 w')$  or  $(w, w' \in X_2 \text{ and } w \leq_2 w')$ 

for all  $w, w' \in X_1 + X_2$ , where we have assumed without loss of generality that  $X_1$  and  $X_2$  are disjoint so that  $X_1 + X_2$  may be identified with  $X_1 \cup X_2$ . Then, given any other poset  $(X_3, \leq_3)$  and a pair of monotone functions  $f: (X_1, \leq_1) \to (X_3, \leq_3)$  and  $g: (X_2, \leq_2) \to (X_3, \leq_3)$ , we have the diagram



in **PoSets** commuting, where  $\kappa_1$  and  $\kappa_2$  are the relevant coprojections (which are also monotone). The unique monotone function h is given by

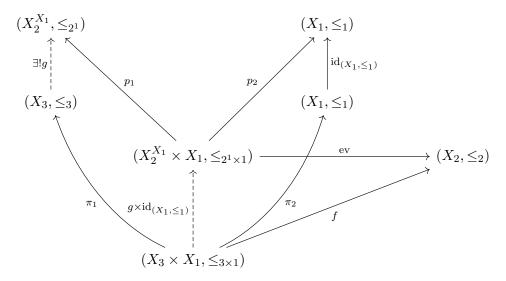
$$h(w) := \begin{cases} f(w), & \text{if } w \in X_1, \\ g(w), & \text{if } w \in X_2, \end{cases}$$

for all  $w \in X_1 + X_2$ . Therefore **PoSets** also has finite coproducts.

Now we show that **PoSets** also has exponents. Fix any two posets  $(X_1, \leq_1)$  and  $(X_2, \leq_2)$ . We define a partial ordering  $\leq_{2^1}$  on the set  $X_2^{X_1}$  as follows:

$$f \leq_{2^1} g$$
 if and only if  $f(x) \leq_2 g(x)$  for all  $x \in X_1$ .

for all functions  $f, g: X_1 \to X_2$ . Then, for any poset  $(X_3, \leq_3)$  and monotone function  $f: (X_3, \leq_3) \to (X_2, \leq_2)$ , we have the diagram



in **PoSets** commuting, where  $\operatorname{ev}(h, x_1) \coloneqq h(x_1)$  for all  $(h, x_1) \in X_2^{X_1} \times X_1$ , and  $\pi_1, \pi_2, p_1$ , and  $p_2$  are the relevant projections. The unique monotone function g is given by  $g(x_3) \coloneqq \lambda x_1 \in X_1. f(x_3, x_1)$ . Therefore **PoSets** also has exponents.

#### Exercise 2.1.6

Consider the category Mon of monoids with monoid homomorphisms between them.

- 1. Check that the singleton monoid 1 is both an initial and a final object in **Mon**; this is called a zero object.
- 2. Given two monoids  $(M_1, +_1, 0_1)$  and  $(M_2, +_2, 0_2)$ , one defines a product monoid  $M_1 \times M_2$  with componentwise addition  $(x, y) + (x', y') = (x +_1, x', y +_2 y')$  and unit  $(0_1, 0_2)$ . Prove that  $M_1 \times M_2$  is again a monoid, which forms a product in the category **Mon** with the standard projection maps  $M_1 \stackrel{\pi_1}{\longleftarrow} M_1 \times M_2 \stackrel{\pi_2}{\longrightarrow} M_2$ .
- 3. Note that there are also coprojections  $M_1 \xrightarrow{\kappa_1} M_1 \times M_2 \xleftarrow{\kappa_2} M_2$ , given by  $\kappa_1(x) = (x, 0_2)$  and  $\kappa_2(y) = (0_1, y)$ , which are monoid homomorphisms and which makes  $M_1 \times M_2$  at the same time the coproduct of  $M_1$  and  $M_2$  in **Mon** (and hence a biproduct). Hint: Define the cotuple [f, g] as  $x \mapsto f(x) + g(x)$ .

Solution.

- 1. Any monoid homomorphism  $f:(M_1, +_1, 0_1) \to (M_2, +_2, 0_2)$  must satisfy  $f(0_1) = 0_2$ , so the singleton monoid is initial in **Mon**. It is also the final in **Mon** because the constant map to the unit is a monoid homomorphism.
- 2. Fix  $(m_1, m_2), (m'_1, m'_2), (m''_1, m''_2) \in M_1 \times M_2$ . Then, using the associativity of  $+_1$  and  $+_2$ ,

$$(m_1, m_2) + ((m'_1, m'_2) + (m''_1, m''_2)) = (m_1, m_2) + (m'_1 +_1 m''_1, m'_2 +_2 m''_2)$$

$$= (m_1 +_1 m'_1 +_1 m''_1, m_2 +_2 m'_2 +_2 m''_2)$$

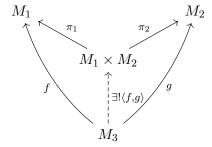
$$= (m_1 +_1 m'_1, m_2 +_2 m'_2) + (m''_1, m''_2).$$

Furthermore,

$$(m_1, m_2) + (0_1, 0_2) = (m_1 +_1 0_1, m_2 +_2, 0_2)$$
  
=  $(m_1, m_2)$ 

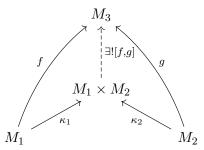
and, similarly,  $(0_1, 0_2) + (m_1, m_2) = (m_1, m_2)$ . So  $(M_1 \times M_2, +, (0_1, 0_2))$  is a monoid.

We now show that  $M_1 \times M_2$  really is the categorical product of  $M_1$  and  $M_2$  in **Mon**. Fix any other monoid  $(M_3, +_3, 0_3)$  and a pair of monoid homomorphisms  $f: M_3 \to M_1$  and  $g: M_3 \to M_2$ . We need the diagram



in **Mon** to commute. Indeed, we must have  $\langle f, g \rangle (m_3) = (f(m_3), g(m_3))$  for all  $m_3 \in M_3$ . The fact that  $\langle f, g \rangle \colon M_3 \to M_1 \times M_2$  is a monoid homomorphism follows from f and g being monoid homomorphisms.

3. Fix any monoid  $(M_3, +_3, 0_3)$  and a pair of monoid homomorphisms  $f: M_1 \to M_3$  and  $g: M_2 \to M_3$ . We need the diagram



in **Mon** to commute. This time we define  $[f,g]: M_1 \times M_2 \to M_3$  by

$$[f,g](m_1,m_2) := f(m_1) +_3 g(m_2)$$

for all  $(m_1, m_2) \in M_1 \times M_2$ . That [f, g] is a monoid homomorphism follows from f and g being monoid homomorphisms. Then

$$([f,g] \circ \kappa_1)(m_1) = [f,g](m_1, 0_1)$$
$$= f(m_1) +_3 g(0_1)$$
$$= f(m_1)$$

for all  $m_1 \in M_1$ . Similarly,  $([f, g] \circ \kappa_2) = g$ .

Now suppose there is another monoid homomorphism  $h: M_1 \times M_2 \to M_3$  satisfying

$$h\kappa_1 = f$$
 and  $h\kappa_2 = g$ .

Then, for any  $(m_1, m_2) \in M_1 \times M_2$ ,

$$h(m_1, m_2) = h(m_1, 0_2) +_3 h(0_1, m_2)$$

$$= h(\kappa_1(m_1)) +_3 h(\kappa_2(m_2))$$

$$= f(m_1) +_3 g(m_2)$$

$$= [f, g](m_1, m_2).$$

Therefore [f,g] is the unique monoid homomorphism making the diagram above commute.

#### Exercise 2.1.7

Show that in **Sets** products distribute over coproducts, in the sense that the canonical maps

$$(X \times Y) + (X \times Z) \xrightarrow{[\mathrm{id}_X \times \kappa_1, \mathrm{id}_X \times \kappa_2]} X \times (Y + Z)$$

$$0 \xrightarrow{!} X \times 0$$

are isomorphisms. Categories in which this is the case are called **distributive**; see Cockett (1993) for more information on distributive categories in general and see Gumma, Hughes, and Schröder (2003) for an investigation of such distributivities in categories of coalgebras.

Solution. In **Sets**, the initial object 0 is the empty set. Consequently, for any set X, the unique map  $0 \stackrel{!}{\to} X \times 0$  is an isomorphism (in fact, ! is the identity morphism on 0) since  $X \times 0 = 0$ .

Now fix sets X, Y, and Z, and let  $Y \xrightarrow{\kappa_1} Y + Z$  and  $Z \xrightarrow{\kappa_2} Y + Z$  denote the appropriate coprojections. We may assume, without loss of generality, that Y and Z are disjoint, so that we may write  $Y \cup Z$  in place of Y + Z, and have  $\kappa_1 \colon Y \to Y \cup Z$  and  $\kappa_2 \colon Z \to Y \cup Z$  be the appropriate inclusion functions.

The function  $[id_X \times \kappa_1, id_X \times \kappa_2]: (X \times Y) + (X \times Z) \to X \times (Y + Z)$  is then given by

$$[\mathrm{id}_X \times \kappa_1, \mathrm{id}_X \times \kappa_2](x, w) = (x, w)$$

for all  $(x, w) \in (X \times Y) + (X \times Z)$ . This is clearly a bijection.

#### Exercise 2.1.8

1. Consider a category with finite products  $(\times, 1)$ . Prove that there are isomorphisms:

$$X \times Y \cong Y \times X$$
,  $(X \times Y) \times Z \cong X \times (Y \times Z)$ ,  $1 \times X \cong X$ .

2. Similarly, show that in a category with finite coproducts (+,0) one has

$$X + Y \cong Y + X$$
,  $(X + Y) + Z \cong X + (Y + Z)$ ,  $0 + X \cong X$ .

(This means that both the finite product and coproduct structure in a category yield so-called symmetric monoidal structure. See Mac Lane (1978) or Borceaux (1994) for more information.)

3. Next, assume that our category also has exponents. Prove that

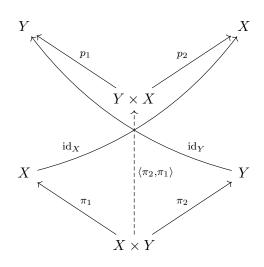
$$X^0 \cong 1, \qquad X^1 \cong X, \qquad 1^X \cong 1.$$

And also that

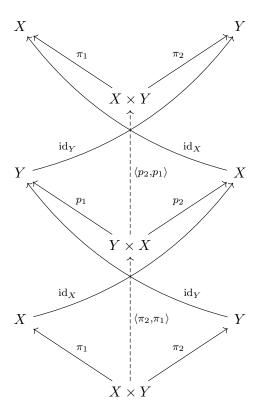
$$Z^{X+Y} \cong Z^X \times Z^Y, \qquad Z^{X\times Y} \cong (Z^Y)^X, \qquad (X \times Y)^Z \cong X^Z \times Y^Z.$$

Solution.

1. Let  $\mathbb{C}$  be a category with finite products. Fix  $X, Y \in \mathsf{Obj}(\mathbb{C})$ . Let  $X \stackrel{\pi_1}{\longleftarrow} X \times Y \xrightarrow{\pi_2} Y$  and  $Y \stackrel{p_1}{\longleftarrow} Y \times X \xrightarrow{p_2} X$  be the relevant projections. We have the diagram



in  $\mathbb C$  commuting. We claim that the unique induced morphism  $X\times Y\xrightarrow{\langle\pi_2,\pi_1\rangle} X\times Y$  is an isomorphism. Of course, its inverse would be the similarly obtained morphism  $Y\times X\xrightarrow{\langle p_2,p_1\rangle} Y\times X$ . Indeed, looking at the diagram

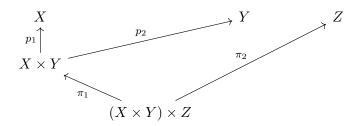


in  $\mathbb{C}$ , we see that

$$\pi_1 \circ \langle p_2, p_1 \rangle \circ \langle \pi_2, \pi_1 \rangle = p_2 \circ \langle \pi_2, \pi_1 \rangle$$
$$= \pi_1$$

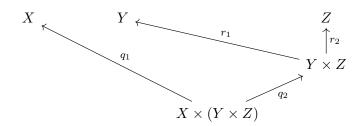
and, similarly,  $\pi_2 \circ \langle p_2, p_1 \rangle \circ \langle \pi_2, \pi_1 \rangle = \pi_2$ . Consequently,  $\langle p_2, p_1 \rangle \circ \langle \pi_2, \pi_1 \rangle = \mathrm{id}_{X \times Y}$ . Similarly, we obtain  $\langle \pi_2, \pi_1 \rangle \circ \langle p_2, p_1 \rangle = \mathrm{id}_{Y \times X}$ . Therefore we have an isomorphism  $X \times Y \xrightarrow{\langle \pi_2, \pi_1 \rangle} Y \times X$ .

Now fix  $X, Y, Z \in \mathsf{Obj}(\mathbb{C})$ . Consider the products  $X \times Y$  and  $(X \times Y) \times Z$  as in the diagram

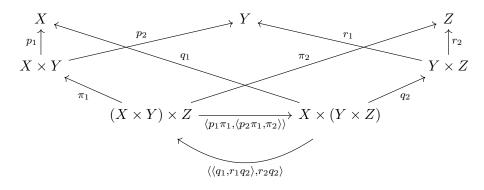


in  $\mathbb{C}$ . These come with associated projections  $X \stackrel{p_1}{\longleftarrow} X \times Y \stackrel{p_2}{\longrightarrow} Y$  and  $X \times Y \stackrel{\pi_1}{\longleftarrow} (X \times Y) \times Z \stackrel{\pi_2}{\longrightarrow} Z$ . We also have projections  $X \stackrel{q_1}{\longleftarrow} X \times (Y \times Z) \stackrel{q_2}{\longrightarrow} Y \times Z$  and  $Y \stackrel{r_1}{\longleftarrow} Y \times Z \stackrel{r_2}{\longrightarrow} Z$ , as depicted in

the diagram



in  $\mathbb{C}$ . From these, we obtain the induced morphisms  $(X \times Y) \times Z \xrightarrow{\langle p_1 \pi_1, \langle p_2 \pi_1, \pi_2 \rangle \rangle} X \times (Y \times Z)$  and  $X \times (Y \times Z) \xrightarrow{\langle \langle q_1, r_1 q_2 \rangle, r_2 q_2 \rangle} (X \times Y) \times Z$ .

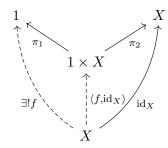


Then

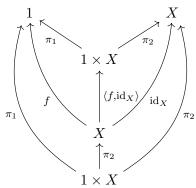
$$\begin{aligned} p_1 \circ \pi_1 \circ \Big( \langle \langle q_1, r_1 q_2 \rangle, r_2 q_2 \rangle \circ \langle p_1 \pi_1, \langle p_2 \pi_1, \pi_2 \rangle \rangle \Big) &= p_1 \circ \Big( \pi_1 \circ \langle \langle q_1, r_1 q_2 \rangle, r_2 q_2 \rangle \Big) \circ \langle p_1 \pi_1, \langle p_2 \pi_1, \pi_2 \rangle \rangle \\ &= p_1 \circ \langle q_1, r_1 q_2 \rangle \circ \langle p_1 \pi_1, \langle p_2 \pi_1, \pi_2 \rangle \rangle \\ &= q_1 \circ \langle p_1 \pi_1, \langle p_2 \pi_1, \pi_2 \rangle \rangle \\ &= p_1 \pi_1, \\ p_2 \circ \pi_1 \circ \Big( \langle \langle q_1, r_1 q_2 \rangle, r_2 q_2 \rangle \circ \langle p_1 \pi_1, \langle p_2 \pi_1, \pi_2 \rangle \rangle \Big) &= p_2 \circ \Big( \pi_1 \circ \langle \langle q_1, r_1 q_2 \rangle, r_2 q_2 \rangle \Big) \circ \langle p_1 \pi_1, \langle p_2 \pi_1, \pi_2 \rangle \rangle \\ &= p_2 \circ \langle q_1, r_1 q_2 \rangle \circ \langle p_1 \pi_1, \langle p_2 \pi_1, \pi_2 \rangle \rangle \\ &= r_1 \circ q_2 \circ \langle p_1 \pi_1, \langle p_2 \pi_1, \pi_2 \rangle \rangle \\ &= r_1 \circ \langle p_2 \pi_1, \pi_2 \rangle \\ &= p_2 \pi_1, \text{ and} \\ \pi_2 \circ \Big( \langle \langle q_1, r_1 q_2 \rangle, r_2 q_2 \rangle \circ \langle p_1 \pi_1, \langle p_2 \pi_1, \pi_2 \rangle \rangle \Big) &= \Big( \pi_2 \circ \langle \langle q_1, r_1 q_2 \rangle, r_2 q_2 \rangle \Big) \circ \langle p_1 \pi_1, \langle p_2 \pi_1, \pi_2 \rangle \rangle \\ &= r_2 \circ q_2 \circ \langle p_1 \pi_1, \langle p_2 \pi_1, \pi_2 \rangle \rangle \\ &= r_2 \circ \langle p_2 \pi_1, \pi_2 \rangle \\ &= r_2 \circ \langle p_2 \pi_1, \pi_2 \rangle \\ &= r_2 \circ \langle p_2 \pi_1, \pi_2 \rangle \\ &= r_2 \circ \langle p_2 \pi_1, \pi_2 \rangle \\ &= r_2 \circ \langle p_2 \pi_1, \pi_2 \rangle \end{aligned}$$

Thus  $\langle \langle q_1, r_1 q_2 \rangle, r_2 q_2 \rangle \circ \langle p_1 \pi_1, \langle p_2 \pi_1, \pi_2 \rangle \rangle = \mathrm{id}_{(X \times Y) \times Z}$ . Via a similar calculation, we also obtain  $\langle p_1 \pi_1, \langle p_2 \pi_1, \pi_2 \rangle \rangle \circ \langle \langle q_1, r_1 q_2 \rangle, r_2 q_2 \rangle = \mathrm{id}_{X \times (Y \times Z)}$ . Therefore, we have an isomorphism  $(X \times Y) \times Z \xrightarrow{\langle p_1 \pi_1, \langle p_2 \pi_1, \pi_2 \rangle \rangle} X \times (Y \times Z)$ .

Now fix  $X \in \mathsf{Obj}(\mathbb{C})$  and let 1 denote the terminal object in  $\mathbb{C}$ . We have the diagram



in  $\mathbb{C}$ , where  $1 \stackrel{\pi_1}{\longleftarrow} 1 \times X \stackrel{\pi_2}{\longrightarrow} X$  are the relevant projections, and  $X \stackrel{f}{\longrightarrow} 1$  is the unique morphism from X to 1. As the diagram commutes, we have  $\pi_2 \circ \langle f, \mathrm{id}_X \rangle = \mathrm{id}_X$ . Furthermore,  $\pi_1 \circ (\langle f, \mathrm{id}_X \rangle \circ \pi_2) = \pi_1$  because 1 is the terminal object and we already have the morphism  $1 \times X \stackrel{\pi_1}{\longrightarrow} 1$ . Moreover,  $\pi_2 \circ (\langle f, \mathrm{id}_X \rangle \circ \pi_2) = \pi_2$ .



Thus  $\langle f, \mathrm{id}_X \rangle \circ \pi_2 = \mathrm{id}_{1 \times X}$ . Therefore we have an isomorphism  $1 \times X \xrightarrow{\pi_2} X$ .

- 2. This is dual to Exercise 2.1.8.1: coproducts in  $\mathbb{C}$  coincide with products in  $\mathbb{C}^{op}$ ; the initial object in  $\mathbb{C}$  is the terminal object in  $\mathbb{C}^{op}$ ; and isomorphisms in  $\mathbb{C}$  are precisely isomorphisms in  $\mathbb{C}^{op}$ .
- 3. Now suppose that the category  $\mathbb{C}$  has all finite products, has all finite coproducts, and has exponents, i.e.  $\mathbb{C}$  is a bicartesian closed category. Denote the initial and terminal objects of  $\mathbb{C}$  by 0 and 1 respectively.

Fix 
$$X \in \mathsf{Obj}(\mathbb{C})$$
.
#??

#### Exercise 2.1.9

Show that the finite powerset also forms a functor  $\mathcal{P}_{fin} \colon \mathbf{Sets} \to \mathbf{Sets}$ .

Solution. The proof that  $\mathcal{P}_{\text{fin}}$ : **Sets**  $\to$  **Sets** is a functor is identical to the proof that the usual power set operation  $\mathcal{P}$ : **Sets**  $\to$  **Sets** is a functor. Given a function  $f: X \to Y$ , the function  $\mathcal{P}_{\text{fin}}f: \mathcal{P}_{\text{fin}}X \to \mathcal{P}_{\text{fin}}Y$  sends finite subsets  $A \subseteq X$  to their image under f. That is, for finite subsets  $A \subseteq X$ , we define

$$(\mathcal{P}_{\operatorname{fin}} f)(A) := \{ f(x) : x \in A \}$$

which is indeed a finite set.

It is clear that  $\mathcal{P}_{\text{fin}} \text{id}_X = \text{id}_{\mathcal{P}_{\text{fin}} X}$  for all sets X. Now given functions  $f: X \to Y$  and  $g: Y \to Z$ ,

$$(\mathcal{P}_{fin}(gf))(A) = \{ g(f(x)) : x \in A \}$$

$$= \{ g(y) : y \in (\mathcal{P}_{fin}f)(A) \}$$

$$= (\mathcal{P}_{fin}g)((\mathcal{P}_{fin}f)(A))$$

$$= (\mathcal{P}_{fin}g \circ \mathcal{P}_{fin}f)(A)$$

for all finite subsets  $A \subseteq X$ . Thus  $\mathcal{P}_{fin}(gf) = (\mathcal{P}_{fin}g)(\mathcal{P}_{fin}f)$ .

#### Exercise 2.1.10

Check that

$$\mathcal{P}(0) \cong 1, \qquad \mathcal{P}(X+Y) \cong \mathcal{P}(X) \times \mathcal{P}(Y).$$

And similarly for the finite powerset  $\mathcal{P}_{fin}$  instead of  $\mathcal{P}$ . This property says that  $\mathcal{P}$  and  $\mathcal{P}_{fin}$  are 'additive'; see Coumans and Jacobs (2013).

Solution. Let 0 and 1 respectively denote the initial and terminal objects in **Sets**. Then  $\mathcal{P}(0) = \mathcal{P}_{\text{fin}}(0) = \mathcal{P}(\emptyset) = \{\emptyset\} \cong 1$ .

Now fix sets X and Y and suppose, without loss of generality, that X and Y are disjoint so that we can write  $X + Y = X \cup Y$ . Then, we have a bijection  $f : \mathcal{P}(X + Y) \to \mathcal{P}X \times \mathcal{P}Y$  defined by

$$f(A) := (\{ z \in A : z \in X \}, \{ z \in A : z \in Y \})$$

for all  $A \subseteq X + Y$ . This is indeed a bijection as it has inverse  $f^{-1}: \mathcal{P}X \times \mathcal{P}Y \to \mathcal{P}(X + Y)$  defined by

$$f^{-1}(A,B) := A \cup B.$$

The proof that  $\mathcal{P}_{fin}(X+Y) \cong \mathcal{P}_{fin}X \times \mathcal{P}_{fin}Y$  is similar.

## Exercise 2.1.11

Notice that a power set  $\mathcal{P}(X)$  can also be understood as exponent  $2^X$ , where  $2 = \{0, 1\}$ . Check that the exponent functoriality gives rise to the contravariant powerset  $\mathbf{Sets}^{\mathrm{op}} \to \mathbf{Sets}$ .

Solution. The identification of  $\mathcal{P}(X)$  with  $2^X$  is via the isomorphism  $\alpha_X \colon \mathcal{P}(X) \to 2^X$  defined by

$$\alpha_X(A) := \lambda x \in X. \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A, \end{cases}$$

for all  $A \subseteq X$ .

Fix a function  $f: X \to Y$ . The function  $2^f: 2^Y \to 2^X$  is given by

$$(2^f)(k) := \lambda x \in X.k(f(x)),$$

for all functions  $k \colon Y \to 2$ . We then see that  $\alpha_X^{-1} \circ 2^f \circ \alpha_Y \colon \mathcal{P}(Y) \to \mathcal{P}(X)$  satisfies

$$(\alpha_X^{-1} \circ 2^f \circ \alpha_Y)(B) = (\alpha_X^{-1} \circ 2^f) \left( \lambda y \in Y : \begin{cases} 1, & \text{if } y \in B, \\ 0, & \text{if } y \notin B \end{cases} \right)$$
$$= \alpha_X^{-1} \left( \lambda x \in X : \begin{cases} 1, & \text{if } f(x) \in B, \\ 0, & \text{if } f(x) \notin B \end{cases} \right)$$
$$= \left\{ x \in X : f(x) \in B \right\}$$

for all  $B \subseteq Y$ . This is precisely how the contravariant power set functor is defined on morphisms.  $\square$ 

#### Exercise 2.1.12

Consider a function  $f: X \to Y$ . Prove that

- 1. The direct image  $\mathcal{P}(f) = \bigsqcup_f \colon \mathcal{P}(X) \to \mathcal{P}(Y)$  preserves all joins and that the inverse image  $f^{-1}(-)\colon \mathcal{P}(Y) \to \mathcal{P}(X)$  preserves not only joins but also meets and negation (i.e. all the Boolean structure).
- 2. There is a Galois connection  $\bigsqcup_f(U) \subseteq V \iff U \subseteq f^{-1}(V)$ , as claimed in (2.15).
- 3. There is a product function  $\prod_f \colon \mathcal{P}(X) \to \mathcal{P}(Y)$  given by  $\prod_f (U) = \{ y \in Y \mid \forall x \in X. f(x) = y \Rightarrow x \in U \}$ , with a Galois connection  $f^{-1}(V) \subseteq U \iff V \subseteq \prod_f (U)$ .

Solution.

1. For a collection  $\{A_i\}_{i\in I}$  of subsets of X, we see that

$$(\mathcal{P}f)\left(\bigcup_{i\in I}A_i\right) = \left\{f(x) : x\in\bigcup_{i\in I}A_i\right\}$$
$$=\bigcup_{i\in I}\left\{f(x) : x\in A_i\right\}$$
$$=\bigcup_{i\in I}(\mathcal{P}f)(A_i).$$

So  $\mathcal{P}f$  preserves all joins. Furthermore, for a collection  $\{B_j\}_{j\in J}$  of subsets of X,

$$f^{-1}\left(\bigcup_{j\in J} B_j\right) = \left\{x \in X : f(x) \in \bigcup_{j\in J} B_j\right\}$$
$$= \bigcup_{j\in J} \{x \in X : f(x) \in B_j\}$$
$$= \bigcup_{j\in J} f^{-1}(B_j).$$

So  $f^{-1}(-)$  also preserves all joins. Moreover,

$$f^{-1}\left(\bigcap_{j\in J} B_j\right) = \left\{x \in X : f(x) \in \bigcap_{j\in J} B_j\right\}$$
$$= \bigcap_{j\in J} \{x \in X : f(x) \in B_j\}$$
$$= \bigcap_{j\in J} f^{-1}(B_j).$$

So  $f^{-1}(-)$  preserves all meets. Also, for any subset  $B \subseteq Y$ ,

$$f^{-1}(Y \setminus B) = \{ x \in X : f(x) \in Y \setminus B \}$$
$$= X \setminus \{ x \in X : f(x) \in B \}$$
$$= X \setminus f^{-1}(B).$$

So  $f^{-1}(-)$  preserves all negations.

2. Fix a pair of subsets  $U \subseteq X$  and  $V \subseteq Y$ . Then

$$(\mathcal{P}f)(U) \subseteq V$$
 if and only if  $\{f(x) : x \in U\} \subseteq V$   
if and only if for all  $x \in U$  we have  $f(x) \in V$   
if and only if  $U \subseteq \{x \in X : f(x) \in V\}$   
if and only if  $U \subseteq f^{-1}(V)$ ,

as claimed.

3. Fix a pair of subsets  $U \subseteq X$  and  $V \subseteq Y$ . Then

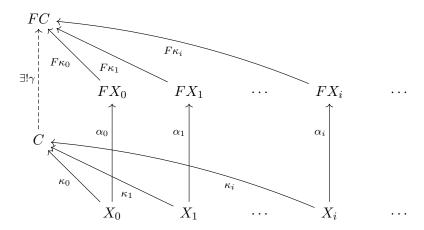
$$f^{-1}(V) \subseteq U$$
 if and only if  $\{x \in X : f(x) \in V\} \subseteq U$  if and only if for all  $x \in X$  with  $f(x) \in V$  we have  $x \in U$  if and only if  $V \subseteq \{y \in Y : \text{for all } x \in X \text{ with } f(x) = y \text{ we have } x \in U\}$  if and only if  $V \subseteq \prod_f (U)$ ,

as desired.

## Exercise 2.1.13

Assume a category  $\mathbb{C}$  has arbitrary, set-indexed coproducts  $\bigsqcup_{i\in I} X_i$ . Demonstrate, as in the proof of Proposition 2.1.5, that the category  $\mathbf{CoAlg}(F)$  of coalgebras of a functor  $F: \mathbb{C} \to \mathbb{C}$  then also has such coproducts.

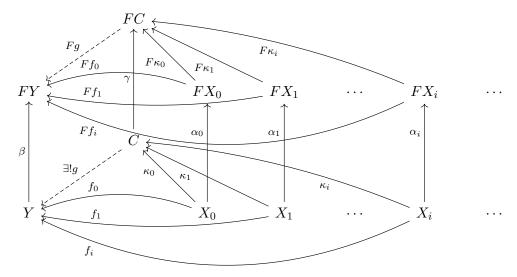
Solution. Let I be a non-empty set and fix an I-indexed tuple  $(X_i \xrightarrow{\alpha_i} FX_i)_{i \in I}$  of F-coalgebras. Let  $C := \bigsqcup_{i \in I} X_i$  be the coproduct of  $(X_i)_{i \in I}$  in  $\mathbb C$  and, for  $i \in I$ , let  $X_i \xrightarrow{\kappa_i} C$  denote the appropriate coprojection. We have the collection of morphisms  $(X_i \xrightarrow{(F\kappa_i)\alpha_i} FC)_{i \in I}$ . So there exists a unique morphism  $\gamma \colon C \to FC$  such that  $\gamma \kappa_i = (F\kappa_i)\alpha_i$  for all  $i \in I$ . That is, the diagram



in  $\mathbb{C}$  commutes. Consequently, we have a collection of homomorphisms of F-coalgebras  $((X_i, \alpha_i) \xrightarrow{\kappa_i} (C, \gamma))_{i \in I}$ .

Now, suppose we are given another F-coalgebra  $Y \xrightarrow{\beta} FY$  and a collection of homomorphisms of F-coalgebras  $((X_i, \alpha_i) \xrightarrow{f_i} (Y, \beta))_{i \in I}$ . Then, as C is the coproduct of  $(X_i)_{i \in I}$  in  $\mathbb{C}$ , there is a unique

morphism  $C \xrightarrow{g} Y$  in  $\mathbb{C}$  such that  $g\kappa_i = f_i$  for all  $i \in I$ .



We now need to verify that g is actually a homomorphism of F-coalgebras from  $(C, \gamma)$  to  $(Y, \beta)$ . We will use the universal property of C as the coproduct in  $\mathbb{C}$ : for all  $i \in I$ , we have

$$\begin{split} \beta g \kappa_i &= \beta f_i, & \text{since } g \kappa_i &= f_i, \\ &= (F f_i) \alpha_i, & \text{since } f_i \text{ is a homomorphism from } (X_i, \alpha_i) \text{ to } (Y, \beta), \\ &= (F g) (F \kappa_i) \alpha_i, & \text{from } g \kappa_i &= f_i \text{ and the functoriality of } F, \\ &= (F g) \gamma \kappa_i, & \text{since } \kappa_i \text{ is a homomorphism from } (X_i, \alpha_i) \text{ to } (C, \gamma). \end{split}$$

Therefore  $\beta g = (Fg)\gamma$ , i.e. g is a homomorphism from  $(C, \gamma)$  to  $(Y, \beta)$ .

# Exercise 2.1.14

For two parallel maps  $f, g: X \to Y$  between objects X, Y in an arbitrary category  $\mathbb{C}$  a **coequaliser**  $q: Y \to Q$  is a map in a diagram

$$X \xrightarrow{g} Y \xrightarrow{q} Q$$

with  $q \circ f = q \circ g$  in a 'universal way': for an arbitrary map  $h: Y \to Z$  with  $h \circ f = h \circ g$  tehre is a unique map  $k: Q \to Z$  with  $k \circ q = h$ .

- 1. An equalier in a category  $\mathbb{C}$  is a coequaliser in  $\mathbb{C}^{op}$ . Formulate explicitly what an equaliser of two parallel maps is.
- 2. Check that in the category **Sets** the set Q can be defined as the quotient Y/R, where  $R \subseteq Y \times Y$  is the least equivalence relation containing all pairs (f(x), g(x)) for  $x \in X$ .
- 3. Returning to the general case, assume a category  $\mathbb{C}$  has coequalisers. Prove that for an arbitrary functor  $F \colon \mathbb{C} \to \mathbb{C}$  the associated category of coalgebras  $\mathbf{CoAlg}(F)$  also has coequalisers, as in  $\mathbb{C}$ : for two homomorphisms  $f, g \colon X \to Y$  between coalgebras  $c \colon X \to F(X)$  and  $d \colon Y \to F(Y)$  there is by universality an induced coalgebra structure  $Q \to F(Q)$  on the coequaliser Q of the underlying maps f, g, yielding a diagram of coalgebras

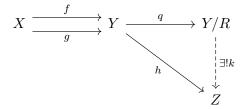
$$\begin{pmatrix} F(X) \\ \uparrow c \\ X \end{pmatrix} \xrightarrow{g} \begin{pmatrix} F(Y) \\ \uparrow d \\ Y \end{pmatrix} \xrightarrow{q} \begin{pmatrix} F(Q) \\ \uparrow \\ Q \end{pmatrix}$$

with the appropriate universal property in  $\mathbf{CoAlg}(F)$ : for each coalgebra  $e: Z \to F(Z)$  with homomorphism  $h: Y \to Z$  satisfying  $h \circ f = h \circ g$  there is a unique homomorphism of coalgebras  $k: Q \to Z$  with  $k \circ g = h$ .

Solution.

- 1. An equaliser of a parallel pair  $X \xrightarrow{f}_{g} Y$  is a morphism  $E \xrightarrow{e} X$  such that both of the following hold:
  - (a) we have fe = ge; and
  - (b) for any morphism  $Z \xrightarrow{h} X$  satisfying fh = gh there exists a unique morphism  $Z \xrightarrow{k} E$  in  $\mathbb{C}$  such that ek = h.
- 2. Fix functions  $f, g: X \to Y$ . Let  $R \subseteq Y \times Y$  be the smallest equivalence relation on Y such that  $\{(f(x), g(x)) : x \in X\} \subseteq R$ , and define  $q: Y \to Y/R$  by q(y) := [y] for all  $y \in Y$ , where [y] denotes the R-equivalence class of  $y \in Y$ .

Fix another function  $h: Y \to Z$  such that hf = hg. We need to show that we have the diagram

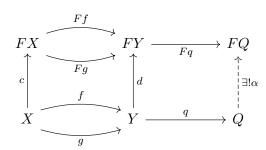


in **Sets** commuting. We define  $k \colon Y/R \to Z$  by  $k([y]) \coloneqq h(y)$  for each R-equivalence class  $[y] \in Y/R$ . Note that this k is well-defined: if  $y, y' \in Y$  are such that yRy' then we can prove by induction on the construction of R (as the reflexive symmetric transitive closure of  $\{(f(x), g(x)) : x \in X\}$ ) that h(y) = h(y'). Then, by construction,  $k \colon Y/R \to Z$  is the unique function satisfying kq = h.

3. Now suppose that  $\mathbb{C}$  has coequalisers. Fix a parallel pair of morphisms  $(X, c) \xrightarrow{f}_{g} (Y, d)$  in  $\mathbf{CoAlg}(F)$ . Let  $Y \xrightarrow{q} Q$  be the coequaliser in  $\mathbb{C}$  of the parallel pair  $X \xrightarrow{f} Y$ . Observe then that

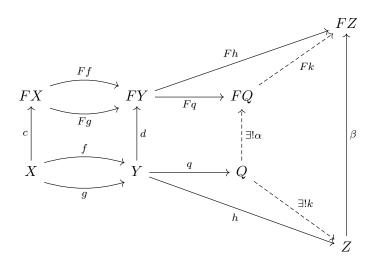
$$(Fq)df = (Fq)(Ff)c$$
, since  $f$  is a homomorphism from  $(X,c)$  to  $(Y,d)$ ,  
 $= F(qf)c$ , by functoriality of  $F$ ,  
 $= F(qg)c$ , since  $qf = qg$ , because  $q$  is the coequaliser of  $f$  and  $g$ ,  
 $= F(q)F(g)c$ , by the functoriality of  $F$ ,  
 $= (Fq)dq$ , since  $q$  is also a homomorphism from  $(X,c)$  to  $(Y,d)$ .

So there must be a unique morphism  $Q \xrightarrow{\alpha} FQ$  in  $\mathbb{C}$  such that  $\alpha q = (Fq)d$ .



So we have an F-coalgebra structure on Q, namely  $Q \xrightarrow{\alpha} FQ$ , and the requirement  $\alpha q = (Fq)d$  says that q is a homomorphism of F-coalgebras from (Y, d) to  $(Q, \alpha)$ .

Now suppose that there is another F-coalgebra  $Z \xrightarrow{\beta} FZ$  and a homomorphism  $(Y,d) \xrightarrow{h} (Z,\beta)$  such that hf = hg. Then there is a unique morphism  $Q \xrightarrow{k} Z$  in  $\mathbb{C}$  such that kq = h.



We now just need to verify that k is a homomorphism from  $(Q, \alpha)$  to  $(Z, \beta)$ , i.e.  $\beta k = (Fk)\alpha$ . We will use the universal property of  $Y \xrightarrow{q} Q$  as the coequaliser of  $X \xrightarrow{f} Y$ : we have

$$\begin{split} \beta kqf &= \beta hf, & \text{since } kq = h, \\ &= (Fh)df, & \text{since } h \text{ is a homomorphism from } (Y,d) \text{ to } (Z,\beta), \\ &= (Fk)(Fq)df, & \text{from } kq = h \text{ and the functoriality of } F, \\ &= (Fk)\alpha qf, & \text{since } q \text{ is a homomorphism from } (Y,d) \text{ to } (Q,\alpha), \\ &= (Fk)\alpha qg, & \text{since } qf = qg \text{ as } q \text{ coequalises } f \text{ and } g, \\ &= \beta hg, & \text{similarly as above,} \\ &= \beta kqg, & \text{since } kq = h. \end{split}$$

The equalities to take away from the calculation above are

$$\beta hf = \beta hq = (\beta k)qf = (\beta k)qq = ((Fk)\alpha)qf = ((Fk)\alpha)qq.$$

By the uniqueness clause in the universal property of coequalisers, we must have  $\beta k = (Fk)\alpha$ .  $\square$ 

# 2.2 Polynomial Functors and Their Coalgebras

#### Exercise 2.2.1

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Solution. 
$$\#$$
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#### Exercise 2.2.2

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Solution. 
$$\#$$
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Exercise 2.2.3 #??	
$Solution. \ \#??$	
Exercise 2.2.4 #??	
Solution. #??	
Exercise 2.2.5 #??	
Solution. #??	
Exercise 2.2.6 #??	
Solution. #??	
Exercise 2.2.7 #??	
Solution. #??	
Exercise 2.2.8 #??	
Solution. #??	
Exercise 2.2.9 #??	
Solution. #??	
Exercise 2.2.10 #??	
Solution. #??	
Exercise 2.2.11 #??	
Solution. #??	
Exercise 2.2.12 #??	
$Solution. \ \#??$	

# Exercise 2.3.1 #?? Solution. #??Exercise 2.3.2 #?? Solution. #??Exercise 2.3.3 #?? Solution. #??Exercise 2.3.4 #?? Solution. #??Exercise 2.3.5 #??Solution. #??Exercise 2.3.6 #?? Solution. #??Exercise 2.3.7 #?? Solution. #??Exercise 2.3.8 #??Solution. #??2.4Algebras Exercise 2.4.1 #?? Solution. #??Exercise 2.4.2 #?? Solution. #??Exercise 2.4.3

Final Coalgebras

2.3

#??

Solution. #??	
Exercise 2.4.4 #??	
Solution. #??	
Exercise 2.4.5 #??	
Solution. #??	
Exercise 2.4.6 #??	
Solution. #??	
Exercise 2.4.7 #??	
Solution. #??	
Exercise 2.4.8 #??	
Solution. #??	
Exercise 2.4.9 #??	
Solution. #??	
Exercise 2.4.10 #??	
Solution. #??	
2.5 Adjunctions, Cofree Coalgebras, Behaviour-Realisation	
Exercise 2.5.1 #??	
$Solution. \ \#??$	
Exercise 2.5.2 #??	
$Solution. \ \#??$	
Exercise 2.5.3 #??	
$Solution. \ \#??$	

#### Exercise 2.5.4

This exercise describes 'strength' for endofunctors on **Sets**. In general, this is a useful notion in the theory of datatypes (Cockett and Spencer, 1992), (Cockett and Spencer, 1995) and of computations (Moggi, 1991); see Section 5.2 for a systemic description.

Let  $F : \mathbf{Sets} \to \mathbf{Sets}$  be an arbitrary functor. Consider for sets X, Y the strength map

$$F(X) \times Y \xrightarrow{\operatorname{st}_{X,Y}} F(X \times Y)$$

$$(u, y) \longmapsto F(\lambda x \in X.(x, y))(u)$$

- 1. Prove that this yields a natural transformation  $F(-)\times(-)\stackrel{\text{st}}{\Rightarrow} F((-)\times(-))$ , where both the domain and codomain are functors  $\mathbf{Sets}\times Sets \to \mathbf{Sets}$ .
- 2. Describe this strength map for the list functor  $(-)^*$  and for the powerset functor  $\mathcal{P}$ .

Solution. #??	
Exercise 2.5.5 #??	
Solution. #??	
Exercise 2.5.6 #??	
Solution. #??	
Exercise 2.5.7 #??	
Solution. #??	
Exercise 2.5.8 #??	
Solution. #??	
Exercise 2.5.9 #??	
Solution. #??	
Exercise 2.5.10 #??	
Solution. #??	
Exercise 2.5.11 #??	
Solution. #??	

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Solution. $\#??$
Exercise 2.5.13 #??
Solution. $\#??$
Exercise 2.5.14 #??
Solution. $\#??$
Exercise 2.5.15 #??
Solution. $\#??$
Exercise 2.5.16 #??
Solution. $\#??$
Exercise 2.5.17 #??
Solution. #??

# 3 Bisimulations

# 3.1 Relation Lifting, Bisimulations and Congruences

Exercise 3.1.1 #??	
Solution. #??	
Exercise 3.1.2 #??	
Solution. #??	
Exercise 3.1.3 #??	
Solution. #??	
Exercise 3.1.4 #??	
Solution. #??	
Exercise 3.1.5 #??	
Solution. #??	
Exercise 3.1.6 #??	
Solution. #??	
3.2 Properties of Bisimulations	
Exercise 3.2.1 #??	
Solution. #??	
Exercise 3.2.2 #??	
Solution. #??	
Exercise 3.2.3 #??	
Solution. #??	
Exercise 3.2.4 #??	
Solution. #??	

Exercise 3.2.5 #??	
Solution. #??	
Exercise 3.2.6 #??	
Solution. #??	
Exercise 3.2.7 #??	
Solution. #??	
3.3 Bisimulations as Spans and Cospans	
Exercise 3.3.1 #??	
Solution. #??	
Exercise 3.3.2 #??	
Solution. #??	
Exercise 3.3.3 #??	
Solution. #??	
Exercise 3.3.4 #??	
Solution. #??	
3.4 Bisimulations and the Coinduction Proof Principle	
Exercise 3.4.1 #??	
Solution. #??	
Exercise 3.4.2 #??	
Solution. #??	
Exercise 3.4.3 #??	
Solution. #??	
Exercise 3.4.4 #??	

Solution. #??	
Exercise 3.4.5 # ? ?	
Solution. #??	
Exercise 3.4.6 # ? ?	
$Solution. \ \#??$	
Exercise 3.4.7 # ? ?	
Solution. #??	
3.5 Process Semantics	
Exercise 3.5.1 #??	
$Solution. \ \#??$	
Exercise 3.5.2 # ? ?	
Solution. #??	
Exercise 3.5.3 # ? ?	
Solution. #??	
Exercise 3.5.4 # ? ?	
$Solution. \ \#??$	

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