Cohen Forcing

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work in progress

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1 A Countable Transitive Model of ZFC?

1.1 A Continuation from Model Theory

So you wish to prove that the continuum hypothesis (CH), the assertion that $2^{\aleph_0} = \aleph_1$, is independent of the axioms of ZFC set theory. You took a class in model theory and you learned that one can show this by showing that $\mathsf{ZFC} + \mathsf{CH}$ and $\mathsf{ZFC} + \neg\mathsf{CH}$ are both consistent. The typical approach is to exhibit models of $\mathsf{ZFC} + \mathsf{CH}$ and $\mathsf{ZFC} + \neg\mathsf{CH}$. So let us try to do that.

Stop. You realise that a model of $\mathsf{ZFC} + \mathsf{CH}$ (or $\mathsf{ZFC} + \neg\mathsf{CH}$) will, in particular, also be a model of ZFC. Darn. The whole point of ZFC set theory was that it was supposed to be able to formalise (most of)[1] the mathematics we do in our day-to-day lives. A certain pesky Kurt Gödel prevents us from explicitly exhibiting such a model.

But who is going to stop us from simply running off with the assumption that ZFC is consistent?[2] If we do this, then we can hope to obtain a theorem and proof of the following form.

Theorem
If ZFC *is consistent, then* $\mathsf{ZFC} + \mathsf{CH}$ *and* $\mathsf{ZFC} + \neg\mathsf{CH}$ *are consistent.*

Proof. Let M be a model of ZFC. [Black magic]. Therefore we have obtained a model M' of $\mathsf{ZFC} + \mathsf{CH}$ and a model M'' of $\mathsf{ZFC} + \neg\mathsf{CH}$. $\square$

In such a proof as above, M' and M'' will presumably be created from M. But at this point, we have no information about M, other than that it models ZFC. Ideally, we would like M to be a *countable transitive* model of ZFC. By "transitive", we mean transitive with respect to the membership relation $\in$. That is, if $x \in y \in M$, then $x \in M$.

To get a countable model of ZFC is fairly easy. Recall the Löwenheim–Skolem theorem, asserting that theories in first-order logic are unable to control the cardinalities of their infinite models.

Theorem 1.1.1 (The Löwenheim–Skolem Theorem)
Let T be a consistent $\mathcal{L}$-theory. Suppose there exists an infinite model of T. Then for all cardinals $\kappa \geq |\mathcal{L}| + \aleph_0$, there exists a model of T of cardinality κ.

A special case of this theorem, which also arises from the proof of Gödel's completeness theorem via Henkin terms, is that if a language $\mathcal{L}$ is countable, then a consistent $\mathcal{L}$-theory T has an infinite model if and only if it has a countably infinite model.

In particular, as any model of ZFC must be infinite, there must also exist a countable model of ZFC. So how do we get transitivity?

1.2 Only a Sith Deals in Absolutes

Why do we even want to get a countable transitive model of ZFC in the first place?

We want a countable model so we are able to access things from *outside* of the model. This lets us adjoin new elements, such as real numbers, to the model, because there are uncountably many real numbers out in the metatheory. This is not unlike a field extension.

Transitive models are desirable due to the fact that they make a lot of formulas "absolute".

Definition 1.2.1
Let M and N be $\mathcal{L}$-structures with $M \subseteq N$ and let $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}$-formula with n free variables. We say that $\varphi(x_1, \ldots, x_n)$ is <u>absolute between M and N</u> if, for all $a_1, \ldots, a_n \in M$,

$$M \models \varphi(a_1, \ldots, a_n) \text{ if and only if } N \models \varphi(a_1, \ldots, a_n).$$

[1] Category theory jumpscare.

[2] If you are an amused reader from the future with the knowledge that ZFC is inconsistent, how is the climate doing? Thought so. Focus on your own problems.

Replacing "if and only if" in the definition above with "if" gives us the notion of downwards absoluteness, whereas replacing "if and only if" with "only if" gives us the notion of upwards absoluteness. Formulas which are absolute between structures M and N, with $M \subseteq N$, are great because they really do let us view N as a certain kind of extension of M.

Atomic formulas are always absolute between M and N whenever M is a substructure of N. This is pretty much by definition: an $\mathcal{L}$-structure M is a substructure of an $\mathcal{L}$-structure N if the domain of M is a subset of the domain of N and the inclusion function $\iota\colon M \to N$ is an injective homomorphism of $\mathcal{L}$-structures. Consequently, propositional connectives of atomic formulas are also always absolute between substructures.

But things become icky when we introduce quantifiers. We cannot simply "add" things to a model and expect it to preserve the truth of formulas in the original structure.

Example 1.2.2

Consider the language $\mathcal{L}_\in$ of set theory, which only consists of one relation symbol $\in$, and has no constant symbols or function symbols. So the only atomic formulas are of the form $x = y$ or $x \in y$, for variables x and y. All the "interesting" formulas are not going to be propositional connectives of atomic formulas.

Let M be a model of ZFC and let $\varnothing$ be the empty set in M. We extend M by letting $N := M \cup \{*\}$ declaring $* \in^N \varnothing$. Then

$$M \models \forall z.(z \notin \varnothing),$$

but

$$N \not\models \forall z.(z \notin \varnothing).$$

Thus even very simple formulas, such as $\varphi_\varnothing(x) := \forall z.(z \notin x)$ asserting that x is empty, are not absolute between M and N. □

Of particular interest are formulas which are absolute between some model and the ambient universe in the metatheory.

Definition 1.2.3

Let M be an $\mathcal{L}_\in$-structure. An $\mathcal{L}_\in$-formula $\varphi(x_1, \ldots, x_n)$ is said to be absolute for M if, for all $a_1, \ldots, a_n \in M$,

$$M \models \varphi(a_1, \ldots, a_n) \text{ if and only if } \varphi(a_1, \ldots, a_n) \text{ is true.}$$

If our metatheory is ZFC set theory, then the above "$\varphi(a_1, \ldots, a_n)$ is true" is interpreted as $\mathsf{ZFC} \vdash \varphi(a_1, \ldots, a_n)$. As before, we similarly have the notions of a formula being upwards absolute and downwards absolute for an $\mathcal{L}_\in$-structure M.

Definition 1.2.4

An $\mathcal{L}_\in$-structure M is said to be transitive if, for all $x \in M$ and $y \in x$, we have $y \in M$.

Phrased differently, a transitive $\mathcal{L}_\in$-structure M is one such that for all $x \in M$ we have $x \subseteq M$.

The idea is that if M is a *transitive* model of ZFC, then lots of formulas are absolute for M. Consequently, if we have a transitive models M and N of ZFC with $M \subseteq N$, then we can really view N as a particularly neat extension of M.

Recall that Δ_0 is the smallest class of all $\mathcal{L}_\in$-formulas containing the atomic formulas and is closed under propositional connectives and bounded quantification. By bounded quantification, we mean quantifiers of the form $\forall x \in y.\varphi$ or $\exists x \in y.\varphi$.

Lemma 1.2.5

If φ is a Δ_0 formula in $\mathcal{L}_\in$, then φ is absolute for M for any transitive $\mathcal{L}_\in$-strucutre M.

Proof. By induction on φ. $\square$

This is particularly neat because lots of familiar expressions in set theory are expressible as Δ_0 formulas.

Example 1.2.6
All of the following are expressible as Δ_0 formulas.

- $x = y$
- $x \in y$
- $x \subseteq y$
- $z = \{x\}$
- $z = \{x, y\}$
- $z = \langle x, y \rangle \coloneqq \{\{x\}, \{x, y\}\}$
- $z = \varnothing$
- $z = x \cup y$
- $z = x \cap y$
- $z = x \setminus y$
- $z = x \cup \{x\}$
- z is transitive
- $z = \bigcup x$
- z is an ordered pair
- $z = x \times y$
- z is a relation
- $z = \mathrm{dom}(R)$ and R is a relation
- $z = \mathrm{ran}(R)$ and R is a relation
- f is a function
- f is an injective function
- f is a surjective function
- f is a bijective function
- α is an ordinal
- α is a successor ordinal
- α is a limit ordinal
- $x = \omega$, where ω is the first countable ordinal
- $n \in \omega$. $\square$

Have I convinced you that transitive $\mathcal{L}_\in$-structures are great yet? If not, then check this out. In attempting to build a countable transitive model of ZFC, simply ensuring that our structure is transitive will yield several axioms of ZFC.

Lemma 1.2.7
If M is a transitive $\mathcal{L}_\in$-structure, then

$$M \models \texttt{Extensionality} + \texttt{Foundation}.$$

If, furthermore, for all $x, y \in M$ we have $\{x, y\} \in M$ and $\bigcup x \in M$, then

$$M \models \texttt{Extensionality} + \texttt{Foundation} + \texttt{Pairing} + \texttt{Union}.$$

Proof. Just do it. □

Woohoo. Only infinitely many more axioms to go...

We have seen that many formulas are absolute for transitive $\mathcal{L}_\in$-structures. There are, however, formulas which are *not* absolute. For instance, the following formulas are not absolute:

- $x = \mathcal{P}(y)$
- $\mathcal{F} = y^x$, that is, $\mathcal{F}$ is the set of all functions from x to y
- κ is a cardinal
- $|X| = |Y|$
- $\beta = \mathrm{cf}(\alpha)$
- α is a regular cardinal.

The non-absoluteness of these formulas will become apparent in the development of later subsections. For now, note that the formula

$$\kappa \text{ is a cardinal}$$

is *downwards* absolute for any transitive $\mathcal{L}_\in$-structure, eventhough it will *not* be *upwards* absolute.

1.3 Light at the End of the Tunnel

So we begin our mission in trying to get a countable transitive model of ZFC.

First, we recall the Tarski–Vaught test, which gives us information about $\mathcal{L}$-embeddings, which are injective $\mathcal{L}$-homomorphisms between $\mathcal{L}$-structures.

Lemma 1.3.1 (The Tarski–Vaught Test)
Let M and N be $\mathcal{L}$-structures and let $i\colon M \to N$ be an $\mathcal{L}$-embedding. Let Φ be a collection of $\mathcal{L}$-formulas which is closed under subformulas. Then the following are equivalent:

(1) for all $\varphi(x_1, \ldots, x_k) \in \Phi$ and all $a_1, \ldots, a_k \in M$,

$$M \models \varphi(a_1, \ldots, a_k) \textit{ if and only if } N \models \varphi(i(a_1), \ldots, i(a_k));$$

(2) for all formulas in Φ of the form $\exists x.\varphi(x, y_1, \ldots, y_k)$, if for all $a_1, \ldots, a_k \in M$ there exists $n \in N$ such that

$$N \models \varphi(n, i(a_1), \ldots, i(a_k)),$$

then there exists $m \in M$ such that

$$N \models \varphi(i(m), i(a_1), \ldots, i(a_k)).$$

Proof. By induction on the complexity of the formulas in Φ. $\square$

In particular if Φ above is the class of *all* $\mathcal{L}$-formulas, then Lemma 1.3.1 provides a characterisation for when an $\mathcal{L}$-embedding is actually an *elementary* $\mathcal{L}$-embedding.

We call property (2) in Lemma 1.3.1 the Tarski–Vaught criterion. Notice that this makes no reference to the truth of φ in M.

Let us specialise the Tarski–Vaught test to the language $\mathcal{L}_\in$ of set theory and to the case when the embedding $i\colon M \to N$ is actually an inclusion. The formulation of the Tarski–Vaught test which we are particularly interested in is as follows.

Lemma 1.3.2 (The Tarski–Vaught Test for $\mathcal{L}_\in$)
Let M and N be $\mathcal{L}_\in$-structures with $M \subseteq N$. Let Φ be a collection of $\mathcal{L}_\in$-formulas which is closed under subformulas. Then the following are equivalent.

(1) all formulas in Φ are absolute between M and N;

(2) for all formulas in Φ of the form $\exists x.\varphi(x, y_1, \dots, y_k)$, if for all $a_1, \dots, a_k \in M$ there exists $n \in N$ such that

$$N \models \varphi(n, a_1, \dots, a_k),$$

then there exists $m \in M$ such that

$$N \models \varphi(m, a_1, \dots, a_k).$$

Theorem 1.3.3 (The Lévy Reflection Theorem)
...

1.4 ... But It Is the Light of an Oncoming Train

Recall that our baseline assumptions in the metatheory is ZFC together with the assumption that ZFC is consistent. From that, the hope was to get a countable *transitive* model of ZFC.

Proposition 1.4.1
If ZFC is consistent, then the sentence $\mathrm{Con}(\mathsf{ZFC})$, asserting the consistency of ZFC, does not imply the sentence "there exists a transitive model of ZFC".

Proof. Suppose that $\mathrm{Con}(\mathsf{ZFC})$ implies "there exists a transitive model of ZFC". The formula $\mathrm{Con}(T)$, asserting the consistency of a theory T, is a Δ_0 formula and is thus absolute for any transitive model. So if M is a transitive model of ZFC, then $M \models \mathsf{ZFC} + \mathrm{Con}(\mathsf{ZFC})$.

Doing the argument above inside ZFC, we obtain

$$\mathsf{ZFC} \vdash \mathrm{Con}(\mathsf{ZFC}) \to \mathrm{Con}(\mathsf{ZFC} + \mathrm{Con}(\mathsf{ZFC})).$$

Phrased differently,

$$\mathsf{ZFC} + \mathrm{Con}(\mathsf{ZFC}) \vdash \mathrm{Con}(\mathsf{ZFC} + \mathrm{Con}(\mathsf{ZFC})),$$

contradicting Gödel's second incompleteness theorem. $\square$

Bugger.

1.5 ... But We Are a Bigger Train

2 Generic Extensions

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