Solutions to exercises in Bart Jacobs's book "Introduction to Coalgebra: Towards Mathematics of States and Observation"

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some date very far into the future, if ever

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These are my solutions to all the labelled exercises in Jacobs (2017). This document does not stand on its own; it is meant to supplement the book.

Contents

1 Motivation

1.1 Naturalness of Coalgebraic Representations

Exercise 1.1.1

1. *Prove that the composition operation ; as defined for coalgebras $S \to \{\perp\} \cup S$ is associative, i.e. satisfies $s_1 \,;\, (s_2 \,;\, s_3) = (s_1 \,;\, s_2) \,;\, s_3$, for all statements $s_1, s_2, s_3 \colon S \to \{\perp\} \cup S$.*

   Define a statement $\mathsf{skip} \colon S \to \{\perp\} \cup S$ which is a unit for composition ; i.e. which satisfies $(\mathsf{skip} \,;\, s) = s = (s \,;\, \mathsf{skip})$, for all $s \colon S \to \{\perp\} \cup S$.

2. *Do the same for ; defined on coalgebras $S \to \{\perp\} \cup S \cup (S \times E)$.*

(In both cases, statements with an associative composition operation and a unit element form a monoid.)

Solution.

1. Recall that the composition operation ; was defined as follows:

$$s \,;\, t \coloneqq \lambda x \in S.\begin{cases} \perp, & \text{if } s(x) = \perp, \\ t(x') & \text{if } s(x) = x' \in S, \end{cases}$$

   for coalgebras $s, t \colon S \to \{\perp\} \cup S$. Fix any three coalgebras $s_1, s_2, s_3 \colon S \to \{\perp\} \cup S$. Then

$$\begin{aligned} s_1 \,;\, (s_2 \,;\, s_3) &= \lambda x \in S.\begin{cases} \perp, & \text{if } s_1(x) = \perp, \\ (s_2 \,;\, s_3)(x'), & \text{if } s_1(x) = x' \in S, \end{cases} \\ &= \lambda x \in S.\begin{cases} \perp, & \text{if either } s_1(x) = \perp, \text{ or both } s_1(x) = x' \in S \text{ and } s_2(x') = \perp, \\ s_3(x''), & \text{if } s_1(x) = x' \in S \text{ and } s_2(x') = x'' \in S, \end{cases} \\ &= \lambda x \in S.\begin{cases} \perp, & \text{if } (s_1 \,;\, s_2)(x) = \perp, \\ s_3(x''), & \text{if } (s_1 \,;\, s_2)(x) = x'' \in S, \end{cases} \\ &= (s_1 \,;\, s_2) \,;\, s_3. \end{aligned}$$

   So the composition operation ; is associative.

   The coalgebra $\mathsf{skip} \colon S \to \{\perp\} \cup S$ defined by $\mathsf{skip}(x) \coloneqq x$, for all $x \in S$, satisfies $(\mathsf{skip} \,;\, s) = s = (s \,;\, \mathsf{skip})$ for all coalgebras $s \colon S \to \{\perp\} \cup S$.

2. Now we consider the composition operation ; defined as follows:

$$s \,;\, t \coloneqq \lambda x \in S.\begin{cases} \perp, & \text{if } s(x) = \perp, \\ t(x'), & \text{if } s(x) = x' \in S, \\ (x', e), & \text{if } s(x) = (x', e) \in S \times E, \end{cases}$$

   for coalgebras $s, t \colon S \to \{\perp\} \cup S \cup (S \times E)$. Fix any three coalgebras $s_1, s_2, s_3 \colon \{\perp\} \cup S \cup (S \times E)$. Then

$$s_1 \,;\, (s_2 \,; s_3) = \lambda x \in S.\begin{cases} \perp, & \text{if } s_1(x) = \perp, \\ (s_2 \,;\, s_3)(x'), & \text{if } s_1(x) = x' \in S, \\ (x', e), & \text{if } s_1(x) = (x', e) \in S \times E, \end{cases}$$

$$
\begin{aligned}
&= \lambda x \in S. \begin{cases} \bot, & \text{if either } s_1(x) = \bot \text{, or both } s_1(x) = x' \in S \text{ and } s_2(x') = \bot, \\ s_3(x''), & \text{if } s_1(x) = x' \in S \text{ and } s_2(x') = x'' \in S, \\ (x'', e), & \text{if } s_1(x) = x' \in S \text{ and } s_2(x') = (x'', e) \in S \times E, \\ (x', e), & \text{if } s_1(x) = (x', e) \in S \times E, \end{cases} \\
&= \lambda x \in S. \begin{cases} \bot, & \text{if } (s_1 \,;\, s_2)(x) = \bot, \\ s_3(x''), & \text{if } (s_1 \,;\, s_2)(x) = x'' \in S, \\ (x'', e), & \text{if } (s_1; s_2)(x) = (x'', e) \in S \times E, \end{cases} \\
&= (s_1 \,;\, s_2) \,;\, s_3.
\end{aligned}
$$

So this composition operation ; is also associative.

Now define the coalgebra $\mathsf{skip}\colon S \to \{\bot\} \cup S \cup (S \times E)$ by $\mathsf{skip}(x) \coloneqq x$, for all $x \in S$. Then we have $(\mathsf{skip}\,;\, s) = s = (s\,;\, \mathsf{skip})$ for all coalgebras $s\colon S \to \{\bot\} \cup S \cup (S \times E)$. $\square$

Exercise 1.1.2
Define also a composition monoid $(\mathsf{skip}, ;)$ *for coalgebras* $S \to \mathcal{P}(S)$*.*

Solution. For coalgebras $s, t\colon S \to \mathcal{P}(S)$, define

$$
s \,;\, t \coloneqq \lambda x \in S. \left(\bigcup_{y \in s(x)} t(y) \right).
$$

Then, for coalgebras $s_1, s_2, s_3\colon S \to \mathcal{P}(S)$, we have

$$
\begin{aligned}
s_1 \,;\, (s_2 \,;\, s_3) &= \lambda x \in S. \left(\bigcup_{y \in s_1(x)} (s_2 \,;\, s_3)(y) \right) \\
&= \lambda x \in S. \left(\bigcup_{y \in s_1(x)} \bigcup_{z \in s_2(y)} s_3(z) \right) \\
&= \lambda x \in S. \left(\bigcup_{z \in (s_1 \,;\, s_2)(x)} s_3(z) \right) \\
&= (s_1 \,;\, s_2) \,;\, s_3.
\end{aligned}
$$

Furthermore, defining $\mathsf{skip}\colon S \to \mathcal{P}(S)$ by $\mathsf{skip}(x) \coloneqq \{x\}$ for all $x \in S$, we have

$$
\begin{aligned}
(\mathsf{skip}\,;\, s) &= \lambda x \in S. \left(\bigcup_{y \in \mathsf{skip}(x)} s(y) \right) \\
&= \lambda x \in S. \left(\bigcup_{y \in \{x\}} s(y) \right) \\
&= \lambda x \in S. s(x) \\
&= s
\end{aligned}
$$

and

$$
\begin{aligned}
(s\,;\,\mathsf{skip}) &= \lambda x \in S. \left(\bigcup_{y \in s(x)} \mathsf{skip}(y) \right) \\
&= \lambda x \in S. \left(\bigcup_{y \in s(x)} \{y\} \right) \\
&= \lambda x \in S. s(x) \\
&= s.
\end{aligned}
$$

□

1.2 The Power of Coinduction

Exercise 1.2.1
Compute the $\mathsf{nextdec}$*-behaviour of* $\frac{1}{7} \in [0,1)$ *as in Example 1.2.2.*

Solution. We first recall all of the following functions.

1. The final coalgebra $\mathsf{next}\colon \{0,\dots,9\}^\infty \to \{\bot\} \cup \big(\{0,\dots,9\} \times \{0,\dots,9\}^\infty\big)$ is defined by

$$
\mathsf{next}(\sigma) := \begin{cases} \bot, & \text{if } \sigma \text{ is the empty sequence,} \\ (d, \sigma'), & \text{if } \sigma \text{ has head } d \in \{0,\dots,9\} \text{ and tail } \sigma' \in \{0,\dots,9\}^\infty, \text{ i.e. } \sigma = d \cdot \sigma', \end{cases}
$$

   for all (finite or infinite) sequences $\sigma \in \{0,\dots,9\}^\infty$.

2. The coalgebra $\mathsf{nextdec}\colon [0,1) \to \{\bot\} \cup \big(\{0,\dots,9\} \times [0,1)\big)$ is defined by

$$
\mathsf{nextdec}(r) := \begin{cases} \bot, & \text{if } r = 0, \\ (d, 10r - d), & \text{if } d \le 10r < d+1 \text{ and } d \in \{0,\dots,9\}, \end{cases}
$$

   for all $r \in [0,1)$.

3. The function $\mathrm{beh}_{\mathsf{nextdec}}\colon [0,1) \to \{0,\dots,9\}^\infty$ is the unique function making

$$
\begin{tikzcd}
\{\bot\} \cup \big(\{0,\dots,9\} \times [0,1)\big) \arrow[r, dashed, "\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_{\{0,\dots,9\}} \times \mathrm{beh}_{\mathsf{nextdec}})"] & \{\bot\} \cup \big(\{0,\dots,9\} \times \{0,\dots,9\}^\infty\big) \\
{[0,1)} \arrow[u, "\mathsf{nextdec}"] \arrow[r, dashed, "\exists!", "\mathrm{beh}_{\mathsf{nextdec}}"'] & \{0,\dots,9\}^\infty \arrow[u, "\cong"', "\mathsf{next}"]
\end{tikzcd}
$$

   commute.

We wish to compute $\mathrm{beh}_{\mathsf{nextdec}}\big(\frac{1}{7}\big)$. We see that

$$
\begin{aligned}
\mathrm{beh}_{\mathsf{nextdec}}\left(\frac{1}{7}\right) &= \mathsf{next}^{-1}\left(\left(\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_{\{0,\dots,9\}} \times \mathrm{beh}_{\mathsf{nextdec}})\right)\left(\mathsf{nextdec}\left(\frac{1}{7}\right)\right)\right) \\
&= \mathsf{next}^{-1}\left(\left(\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_{\{0,\dots,9\}} \times \mathrm{beh}_{\mathsf{nextdec}})\right)\left(\left(1, \frac{3}{7}\right)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathsf{next}^{-1}\left(\left(1,\ \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{3}{7}\right)\right)\right) \\
&= 1 \cdot \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{3}{7}\right).
\end{aligned}
$$

Continuing in this fashion,

$$
\begin{aligned}
\mathrm{beh}_{\mathsf{nextdec}}\left(\frac{1}{7}\right) &= 1 \cdot \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{3}{7}\right) \\
&= 1 \cdot \left(4 \cdot \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{2}{7}\right)\right) \\
&= 1 \cdot \left(4 \cdot \left(2 \cdot \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{6}{7}\right)\right)\right) \\
&= 1 \cdot \left(4 \cdot \left(2 \cdot \left(8 \cdot \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{4}{7}\right)\right)\right)\right) \\
&= 1 \cdot \left(4 \cdot \left(2 \cdot \left(8 \cdot \left(5 \cdot \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{5}{7}\right)\right)\right)\right)\right) \\
&= 1 \cdot \left(4 \cdot \left(2 \cdot \left(8 \cdot \left(5 \cdot \left(7 \cdot \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{1}{7}\right)\right)\right)\right)\right)\right).
\end{aligned}
$$

Therefore $\mathrm{beh}_{\mathsf{nextdec}}\left(\frac{1}{7}\right) = \langle 1, 4, 2, 8, 5, 7, 1, 4, 2, 8, 5, 7, 1, 4, 2, 8, 5, 7, \dots \rangle$.

$\square$

Exercise 1.2.2

Formulate appropriate rules for the function $\mathsf{odds}\colon A^\infty \to A^\infty$ *in analogy with the rules (1.7) for* evens.

Solution. We recall that, for a sequence $\sigma := \langle a_0, a_1, a_2, a_3, \dots \rangle \in A^\infty$, the function odds satisfies $\mathsf{odds}(\sigma) = \langle a_1, a_3, a_5, \dots \rangle$, and analogously if σ is a finite sequence. The rules we want odds to satisfy are:

$$
\frac{\sigma \not\to}{\mathsf{odds}(\sigma) \not\to}
$$

i.e. odds should send the empty sequence to the empty sequence;

$$
\frac{\sigma \xrightarrow{a} \sigma' \qquad \sigma' \not\to}{\mathsf{odds}(\sigma) \not\to}
$$

i.e. odds should send a singleton sequence $\langle a \rangle$ to the empty sequence; and

$$
\frac{\sigma \xrightarrow{a} \sigma' \qquad \sigma' \xrightarrow{a'} \sigma''}{\mathsf{odds}(\sigma) \xrightarrow{a'} \mathsf{odds}(\sigma')}
$$

i.e. if $\sigma = a \cdot a' \cdot \sigma' \in A^\infty$, where $a, a' \in A$, then $\mathsf{odds}(\sigma) = a' \cdot \mathsf{odds}(\sigma')$. $\square$

Exercise 1.2.3

Use coinduction to define the empty sequence $\langle\rangle \in A^\infty$ *as a map* $\{\perp\} \to A^\infty$.

Fix an element $a \in A$, *and similarly define the infinite sequence* $\overrightarrow{a}\colon \{\perp\} \to A^\infty$ *consisting of only* a*s.*

Solution. We recall that the final coalgebra $\mathsf{next}\colon A^\infty \to \{\bot\} \cup \big(A \times A^\infty\big)$ is defined by

$$\mathsf{next}(\sigma) \coloneqq \begin{cases} \bot, & \text{if } \sigma \text{ is the empty sequence,} \\ (a, \sigma'), & \text{if } \sigma \text{ has head } a \in A \text{ and tail } \sigma' \in A^\infty \text{, i.e. } \sigma = a \cdot \sigma', \end{cases}$$

for all (finite or infinite) sequences $\sigma \in A^\infty$.

For the coalgebra $\iota_1\colon \{\bot\} \to \{\bot\} \cup (A \times \{\bot\})$ defined by $\iota_1(\bot) \coloneqq \bot$, the unique function $\mathrm{beh}_{\iota_1}\colon \{\bot\} \to A^\infty$ making

$$\begin{tikzcd}
\{\bot\} \cup (A \times \{\bot\}) \arrow[r, dashed, "\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_A \times \mathrm{beh}_{\iota_1})"] & \{\bot\} \cup (A \times A^\infty) \\
\{\bot\} \arrow[u, "\iota_1"] \arrow[r, dashed, "\exists!", "\mathrm{beh}_{\iota_1}"'] & A^\infty \arrow[u, "\cong"', "\mathsf{next}"]
\end{tikzcd}$$

commute satisfies $\mathrm{beh}_{\iota_1}(\bot) = \langle\rangle$.

For the coalgebra $c_a\colon \{\bot\} \to \{\bot\} \cup (A \times \{\bot\})$ defined by $c_a(\bot) \coloneqq (a, \bot)$, the unique function $\mathrm{beh}_{\mathrm{c_a}}\colon \{\bot\} \to A^\infty$ making

$$\begin{tikzcd}
\{\bot\} \cup (A \times \{\bot\}) \arrow[r, dashed, "\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_A \times \mathrm{beh}_{c_a})"] & \{\bot\} \cup (A \times A^\infty) \\
\{\bot\} \arrow[u, "c_a"] \arrow[r, dashed, "\exists!", "\mathrm{beh}_{c_a}"'] & A^\infty \arrow[u, "\cong"', "\mathsf{next}"]
\end{tikzcd}$$

commute satisfies $\mathrm{beh}_{c_a}(\bot) = \overrightarrow{a} = \langle a, a, a, \dots \rangle$. ◻

Exercise 1.2.4
Compute the outcome of $\mathrm{merge}(\langle a_0, a_1, a_2\rangle, \langle b_0, b_1, b_2, b_3\rangle)$.

Solution. Recall that we defined the coalgebra $m\colon A^\infty \times A^\infty \to \{\bot\} \cup (A \times (A^\infty \times A^\infty))$ by

$$m(\sigma, \tau) \coloneqq \begin{cases} \bot, & \text{if } \sigma \not\to \text{ and } \tau \not\to, \\ (a, (\sigma, \tau')), & \text{if } \sigma \not\to \text{ and } \tau \xrightarrow{a} \tau', \\ (a, (\tau, \sigma')), & \text{if } \sigma \xrightarrow{a} \sigma', \end{cases}$$

for all $\sigma, \tau \in A^\infty$, and that $\mathsf{merge}\colon A^\infty \times A^\infty \to A^\infty$ is the unique function making

$$\begin{tikzcd}
\{\bot\} \cup (A \times (A^\infty \times A^\infty)) \arrow[r, dashed, "\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_A \times \mathsf{merge})"] & \{\bot\} \cup (A \times A^\infty) \\
A^\infty \times A^\infty \arrow[u, "m"] \arrow[r, dashed, "\exists!", "\mathsf{merge}"'] & A^\infty \arrow[u, "\cong"', "\mathsf{next}"]
\end{tikzcd}$$

commute. Then

$$\begin{aligned} \mathsf{merge}(\langle a_0, a_1, a_2\rangle, \langle b_0, b_1, b_2, b_3\rangle) &= \mathsf{next}^{-1}\Big(\big(\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_A \times \mathsf{merge})\big)\big(m(\langle a_0, a_1, a_2\rangle, \langle b_0, b_1, b_2, b_3\rangle)\big)\Big) \\ &= \mathsf{next}^{-1}\Big(\big(\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_A \times \mathsf{merge})\big)\big((a_0, (\langle b_0, b_1, b_2, b_3\rangle, \langle a_1, a_2\rangle))\big)\Big) \end{aligned}$$

$$
\begin{aligned}
&= \mathsf{next}^{-1}\Big(\big(a_0, \mathsf{merge}(\langle b_0, b_1, b_2, b_3\rangle, \langle a_1, a_2\rangle)\big)\Big) \\
&= a_0 \cdot \mathsf{merge}(\langle b_0, b_1, b_2, b_3\rangle, \langle a_1, a_2\rangle),
\end{aligned}
$$

and so on. Eventually, we obtain $\mathsf{merge}(\langle a_0, a_1, a_2\rangle, \langle b_0, b_1, b_2, b_3\rangle) = \langle a_0, b_0, a_1, b_1, a_2, b_2, b_3\rangle$. $\square$

Exercise 1.2.5
Is the merge *operation associative, i.e. is* $\mathsf{merge}(\sigma, \mathsf{merge}(\tau, \rho))$ *the same as* $\mathsf{merge}(\mathsf{merge}(\sigma, \tau), \rho)$*? Give a proof or a counterexample. Is there a neutral element for* merge*?*

Solution. The merge operation is not associative:

$$
\begin{aligned}
\mathsf{merge}(\langle a\rangle, \mathsf{merge}(\langle b\rangle, \langle c\rangle)) &= \mathsf{merge}(\langle a\rangle, \langle b, c\rangle) \\
&= \langle a, b, c\rangle,
\end{aligned}
$$

whereas

$$
\begin{aligned}
\mathsf{merge}(\mathsf{merge}(\langle a\rangle, \langle b\rangle), \langle c\rangle) &= \mathsf{merge}(\langle a, b\rangle, \langle c\rangle) \\
&= \langle a, c, b\rangle,
\end{aligned}
$$

for all $a, b, c \in A$.

The neutral element for merge is the empty sequence: for any $\sigma \in A^\infty$, we have $\mathsf{merge}(\sigma, \langle\rangle) = \mathsf{merge}(\langle\rangle, \sigma) = \sigma$. $\square$

Exercise 1.2.6
Show how to define an alternative merge function which alternatingly takes two elements from its argument sequences.

Solution. We will define a coalgebra $m_2\colon A^\infty \times A^\infty \to \{\bot\} \cup (A \times (A^\infty \times A^\infty))$ so that the desired merge function is the unique function $\mathsf{merge}_2\colon A^\infty \times A^\infty \to A^\infty$ making

$$
\begin{tikzcd}
\{\bot\} \cup (A \times (A^\infty \times A^\infty)) \arrow[r, dashed, "\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_A \times \mathsf{merge}_2)"] & \{\bot\} \cup (A \times A^\infty) \\
A^\infty \times A^\infty \arrow[u, "m_2"] \arrow[r, dashed, "\exists!", "\mathsf{merge}_2"'] & A^\infty \arrow[u, "\cong"', "\mathsf{next}"]
\end{tikzcd}
$$

commute. As a motivating example, the desired merge of two infinite streams $\langle a_0, a_1, \dots\rangle$ and $\langle b_0, b_1, \dots\rangle$ should be

$$
\mathsf{merge}_2(\langle a_0, a_1, a_2, a_3, \dots\rangle, \langle b_0, b_1, b_2, b_3, \dots\rangle) = \langle a_0, a_1, b_0, b_1, a_2, a_3, b_2, b_3, \dots\rangle.
$$

As the diagram above commutes, we would require

$$
\mathsf{merge}_2\big(m_2(\langle a_0, a_1, a_2, a_3, \dots\rangle, \langle b_0, b_1, b_2, b_3, \dots\rangle)\big) = \big(a_0, \langle a_1, b_0, b_1, a_2, a_3, b_2, b_3, \dots\rangle\big)
$$

and so m_2 should be defined to satisfy

$$
m_2(\langle a_0, a_1, a_2, a_3, \dots\rangle, \langle b_0, b_1, b_2, b_3, \dots\rangle) = \big(a_0, (\langle a_1, b_0, a_3, b_2, \dots\rangle, \langle b_1, a_2, b_3, a_4, \dots\rangle)\big)
$$

Dealing with edge cases separately leads us to the following definition: we define the coalgebra $m_2\colon A^\infty \times A^\infty \to \{\bot\} \cup (A \times (A^\infty \times A^\infty))$ as follows.

1. The function m_2 sends the pair $(\langle\rangle, \langle\rangle)$ to $\perp$, i.e.

$$m_2(\langle\rangle, \langle\rangle) \coloneqq \perp.$$

2. If $\tau \in A^\infty$ is a non-empty sequence, say $\tau \xrightarrow{a} \tau'$ for some $\tau' \in A^\infty$ and $a \in A$, then

$$m_2(\langle\rangle, \tau) \coloneqq \big(a, (\langle\rangle, \tau')\big).$$

3. If $\sigma = \langle a \rangle$ for some $a \in A$, then
$$m_2(\langle a \rangle, \tau) \coloneqq \big(a, (\langle\rangle, \tau)\big)$$
for all $\tau \in A^\infty$.

4. If $\sigma \in A^\infty$ has at least length 2, say $\sigma \xrightarrow{a} \sigma' \xrightarrow{a'} \sigma''$ for some $\sigma', \sigma'' \in A^\infty$ and $a, a' \in A$, then

$$m_2(\sigma, \tau) \coloneqq \Big(a, \big(\mathsf{merge}(\mathsf{odds}(\sigma), \mathsf{evens}(\tau)), \mathsf{merge}(\mathsf{odds}(\tau), \mathsf{evens}(\sigma''))\big)\Big)$$

for all $\tau \in A^\infty$.

Now let $\mathsf{merge}_2 \colon A^\infty \times A^\infty \to A^\infty$ be the unique function which makes

$$\begin{tikzcd}
\{\perp\} \cup (A \times (A^\infty \times A^\infty)) \arrow[rr, dashed, "\mathrm{id}_{\{\perp\}} \,\cup\, (\mathrm{id}_A \times \mathsf{merge}_2)"] & & \{\perp\} \cup (A \times A^\infty) \\
A^\infty \times A^\infty \arrow[u, "m_2"] \arrow[rr, dashed, "\exists!", "\mathsf{merge}_2"'] & & A^\infty \arrow[u, "\cong"', "\mathsf{next}"]
\end{tikzcd}$$

commute. Fix any $\sigma, \tau \in A^\infty$. We argue by cases on (σ, τ) that this function merge_2 is the desired merge function.

1. If $\sigma = \tau = \langle\rangle$, then $\mathsf{merge}_2(\langle\rangle, \langle\rangle) = \langle\rangle$.

2. If $\sigma = \langle\rangle$ and τ is a non-empty sequence, say $\tau = a \cdot \tau'$ for some $a \in A$ and $\tau' \in A^\infty$, then

$$\mathsf{merge}_2(\langle\rangle, \tau) = a \cdot \mathsf{merge}_2(\langle\rangle, \tau').$$

Thus $\mathsf{merge}_2(\langle\rangle, \tau) = \tau$.

3. If $\sigma = \langle a \rangle$ for some $a \in A$, then

$$\begin{aligned}
\mathsf{merge}_2(\langle a \rangle, \tau) &= a \cdot \mathsf{merge}_2(\langle\rangle, \tau) \\
&= a \cdot \tau.
\end{aligned}$$

4. If $\sigma = a \cdot a' \cdot \sigma''$ for some $a, a' \in A$ and $\sigma'' \in A^\infty$, then

$$\begin{aligned}
\mathsf{merge}_2(\sigma, \tau) &= a \cdot \mathsf{merge}_2\Big(\mathsf{merge}\big(\mathsf{odds}(\sigma), \mathsf{evens}(\tau)\big), \mathsf{merge}\big(\mathsf{odds}(\tau), \mathsf{evens}(\sigma'')\big)\Big) \\
&= a \cdot \mathsf{merge}_2\Big(\mathsf{merge}\big(a' \cdot \mathsf{odds}(\sigma''), \mathsf{evens}(\tau)\big), \mathsf{merge}\big(\mathsf{odds}(\tau), \mathsf{evens}(\sigma'')\big)\Big) \\
&= a \cdot a' \cdot \mathsf{merge}_2\Big(\mathsf{merge}\big(\mathsf{odds}(\mathsf{merge}(a' \cdot \mathsf{odds}(\sigma''), \mathsf{evens}(\tau))), \\
&\qquad\qquad \mathsf{evens}(\mathsf{merge}(\mathsf{odds}(\tau), \mathsf{evens}(\sigma'')))\big),
\end{aligned}$$

$$
\begin{aligned}
&\mathsf{merge}\Big(\mathsf{odds}(\mathsf{merge}(\mathsf{odds}(\tau), \mathsf{evens}(\sigma''))),\\
&\qquad\qquad \mathsf{odds}(\mathsf{merge}(\mathsf{evens}(\tau), \mathsf{odds}(\sigma''))))\Big)\\
&= a \cdot a' \cdot \mathsf{merge}_2\Big(\mathsf{merge}\big(\mathsf{evens}(\tau), \mathsf{odds}(\tau)\big), \mathsf{merge}\big(\mathsf{evens}(\sigma''), \mathsf{odds}(\sigma'')\big)\Big)\\
&= a \cdot a' \cdot \mathsf{merge}_2(\tau, \sigma''),
\end{aligned}
$$

as desired. □

Exercise 1.2.7

1. *Define three functions* $\mathsf{ex}_i \colon A^\infty \to A^\infty$, *for* $i = 0, 1, 2$, *which extract the elements at positions* $3n + i$.
2. *Define* $\mathsf{merge3} \colon A^\infty \times A^\infty \times A^\infty \to A^\infty$ *satisfying the equation* $\mathsf{merge3}(\mathsf{ex}_0(\sigma), \mathsf{ex}_1(\sigma), \mathsf{ex}_2(\sigma)) = \sigma$, *for all* $\sigma \in A^\infty$.

Solution.

1. Define $c_0, c_1, c_2 \colon A^\infty \to \{\bot\} \cup (A \times A^\infty)$ as follows:

$$
c_0(\sigma) := \begin{cases} \bot, & \text{if } \sigma = \langle\rangle,\\ (a, \langle\rangle) & \text{if } \sigma = \langle a\rangle \text{ or } \sigma = \langle a, a'\rangle \text{ for some } a, a' \in A,\\ (a, \sigma'''), & \text{if } \sigma \xrightarrow{a} \sigma' \xrightarrow{a'} \sigma'' \xrightarrow{a''} \sigma''', \end{cases}
$$

$$
c_1(\sigma) := \begin{cases} \bot, & \text{if } \sigma = \langle\rangle \text{ or } \sigma = \langle a\rangle \text{ for some } a \in A,\\ (a', \langle\rangle) & \text{if } \sigma = \langle a, a'\rangle \text{ for some } a, a' \in A,\\ (a', \sigma'''), & \text{if } \sigma \xrightarrow{a} \sigma' \xrightarrow{a'} \sigma'' \xrightarrow{a''} \sigma''', \end{cases}
$$

$$
c_2(\sigma) := \begin{cases} \bot, & \text{if } \sigma = \langle\rangle, \text{ or } \sigma = \langle a\rangle, \text{ or } \sigma = \langle a, a'\rangle \text{ for some } a, a' \in A,\\ (a'', \sigma'''), & \text{if } \sigma \xrightarrow{a} \sigma' \xrightarrow{a'} \sigma'' \xrightarrow{a''} \sigma'''. \end{cases}
$$

Then, for $i \in \{0, 1, 2\}$, the function $\mathsf{ex}_i \colon A^\infty \to A^\infty$ is the unique function making

$$
\begin{tikzcd}
\{\bot\} \cup (A \times A^\infty) \arrow[r, dashed, "\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_A \times \mathsf{ex}_i)"] & \{\bot\} \cup (A \times A^\infty) \\
A^\infty \arrow[u, "c_i"] \arrow[r, dashed, "\exists!", "\mathsf{ex}_i"'] & A^\infty \arrow[u, "\cong"', "\mathsf{next}"]
\end{tikzcd}
$$

commute.

2. Define the coalgebra $m_3 \colon A^\infty \times A^\infty \times A^\infty \to \{\bot\} \cup (A \times (A^\infty \times A^\infty \times A^\infty))$ by

$$
m_3(\sigma, \tau, \rho) := \begin{cases} \bot, & \text{if } \sigma = \tau = \rho = \langle\rangle,\\ \big(a, (\langle\rangle, \langle\rangle, \rho')\big), & \text{if } \sigma = \tau = \langle\rangle \text{ and } \rho \xrightarrow{a} \rho' \text{ for some } a \in A \text{ and } \rho' \in A^\infty,\\ \big(a, (\langle\rangle, \rho, \tau')\big), & \text{if } \sigma = \langle\rangle \text{ and } \tau \xrightarrow{a} \tau' \text{ for some } a \in A \text{ and } \tau' \in A^\infty,\\ \big(a, (\tau, \rho, \sigma')\big), & \text{if } \sigma \xrightarrow{a} \sigma' \text{ for some } a \in A \text{ and } \sigma' \in A^\infty. \end{cases}
$$

Then we let $\mathsf{merge3}\colon A^\infty \times A^\infty \times A^\infty \to A^\infty$ be the unique function making

$$\begin{tikzcd}
\{\perp\} \cup (A \times (A^\infty \times A^\infty \times A^\infty)) \arrow[r, dashed, "\mathrm{id}_{\{\perp\}} \,\cup\, (\mathrm{id}_A \times \mathsf{merge3})"] & \{\perp\} \cup (A \times A^\infty) \\
A^\infty \times A^\infty \times A^\infty \arrow[u, "m_3"] \arrow[r, dashed, "\exists!", "\mathsf{merge3}"'] & A^\infty \arrow[u, "\cong"', "\mathsf{next}"]
\end{tikzcd}$$

commute.

Let us prove that $\mathsf{merge3}(\mathsf{ex}_0(\sigma), \mathsf{ex}_1(\sigma), \mathsf{ex}_2(\sigma)) = \sigma$ for all $\sigma \in A^\infty$, by coinduction. Consider the function $f\colon A^\infty \to A^\infty \times A^\infty \times A^\infty$ defined by $f(\sigma) := \big(\mathsf{ex}_0(\sigma), \mathsf{ex}_1(\sigma), \mathsf{ex}_2(\sigma)\big)$ for all $\sigma \in A^\infty$. We wish to show that $\mathsf{merge3} \circ f = \mathrm{id}_{A^\infty}$.

$$\begin{tikzcd}
& \{\perp\} \cup (A \times (A^\infty \times A^\infty \times A^\infty)) \arrow[r, "\mathrm{id}_{\{\perp\}} \,\cup\, (\mathrm{id}_A \times \mathsf{merge3})"] & \{\perp\} \cup (A \times A^\infty) \\
\{\perp\} \cup (A \times A^\infty) \arrow[ur, "\mathrm{id}_{\{\perp\}} \,\cup\, (\mathrm{id}_A \times f)"] & & \\
& A^\infty \times A^\infty \times A^\infty \arrow[uu, "m_3"] \arrow[r, "\mathsf{merge3}"'] & A^\infty \arrow[uu, "\cong"', "\mathsf{next}"] \\
A^\infty \arrow[uu, "\mathsf{next}", "\cong"'] \arrow[ur, "f"'] & &
\end{tikzcd}$$

Let us first show that the left square commutes. It certainly commutes when we chase the empty sequence: $(m_3 \circ f)(\langle\rangle) = \perp = \big(\big(\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times f)\big) \circ \mathsf{next}\big)(\langle\rangle)$. If $\sigma \in A^\infty$ is a non-empty sequence, say $\sigma \xrightarrow{a} \sigma'$ for some $a \in A$ and $\sigma' \in A^\infty$, then we have

$$\begin{aligned}
(m_3 \circ f)(\sigma) &= m_3\big(\mathsf{ex}_0(\sigma), \mathsf{ex}_1(\sigma), \mathsf{ex}_2(\sigma)\big) \\
&= \big(a, (\mathsf{ex}_1(\sigma), \mathsf{ex}_2(\sigma), \mathsf{ex}_0(\sigma'))\big) \\
&= \big(a, (\mathsf{ex}_0(\sigma'), \mathsf{ex}_1(\sigma'), \mathsf{ex}_2(\sigma'))\big) \\
&= \big(\big(\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times f)\big) \circ \mathsf{next}\big)(\sigma),
\end{aligned}$$

where the second-to-last equality can also be proven by coinduction. Therefore the outer square commutes, and so

$$\mathsf{next} \circ (\mathsf{merge3} \circ f) = \big(\big(\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times (\mathsf{merge3} \circ f))\big) \circ \mathsf{next}.$$

The finality of the coalgebra $\mathsf{next}\colon A^\infty \to \{\perp\} \cup (A \times A^\infty)$ now yields $\mathsf{merge3} \circ f = \mathrm{id}_{A^\infty}$. $\square$

Exercise 1.2.8

Consider the sequential composition function $\mathsf{comp}\colon A^\infty \times A^\infty \to A^\infty$ *for sequences, described by the three rules:*

$$\frac{\sigma \not\to \qquad \tau \not\to}{\mathsf{comp}(\sigma, \tau) \not\to} \qquad\qquad \frac{\sigma \not\to \qquad \tau \xrightarrow{a} \tau'}{\mathsf{comp}(\sigma, \tau) \xrightarrow{a} \mathsf{comp}(\sigma, \tau')}$$

$$\frac{\sigma \xrightarrow{a} \sigma'}{\mathsf{comp}(\sigma, \tau) \xrightarrow{a} \mathsf{comp}(\sigma', \tau)}\ .$$

1. *Show by coinduction that the empty sequence* $\langle\rangle = \mathsf{next}^{-1}(\perp) \in A^\infty$ *is a unit element for* comp, *i.e. that* $\mathsf{comp}(\langle\rangle, \sigma) = \sigma = \mathsf{comp}(\sigma, \langle\rangle)$.

2. *Prove also by coinduction that* comp *is associative, and thus that sequences carry a monoid structure.*

Solution.

1. Let $f\colon A^\infty \to A^\infty$ be defined by $f(\sigma) \coloneqq \mathsf{comp}(\langle\rangle, \sigma)$. We will show that the diagram

$$\begin{tikzcd}
\{\perp\} \cup (A \times A^\infty) \arrow[rr, "\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times f)"] & & \{\perp\} \cup (A \times A^\infty) \\
A^\infty \arrow[u, "\mathsf{next}", "\cong"'] \arrow[rr, "f"'] & & A^\infty \arrow[u, "\cong", "\mathsf{next}"']
\end{tikzcd}$$

commutes, which would yield $f = \mathrm{id}_{A^\infty}$ by the finality of the coalgebra next.

First, we chase the empty sequence from the bottom left. We see that

$$\begin{aligned}
(\mathsf{next} \circ f)(\langle\rangle) &= \mathsf{next}(\mathsf{comp}(\langle\rangle, \langle\rangle)) \\
&= \mathsf{next}(\langle\rangle) \\
&= \perp,
\end{aligned}$$

the first rule for comp, and

$$\begin{aligned}
\big(\big(\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times f)\big) \circ \mathsf{next}\big)(\langle\rangle) &= \big(\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times f)\big)(\perp) \\
&= \perp.
\end{aligned}$$

Now if $\sigma \in A^\infty$ is a non-empty sequence, say $\sigma \xrightarrow{a} \sigma'$ for some $a \in A$ and $\sigma' \in A^\infty$, we see that

$$\begin{aligned}
(\mathsf{next} \circ f)(\sigma) &= \mathsf{next}(\mathsf{comp}(\langle\rangle, a \cdot \sigma')) \\
&= (a, \mathsf{comp}(\langle\rangle, \sigma')) \\
&= (a, f(\sigma')),
\end{aligned}$$

by the second rule for comp and the definition of f, and

$$\begin{aligned}
\big(\big(\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times f)\big) \circ \mathsf{next}\big)(\sigma) &= \big(\big(\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times f)\big)\big((a, \sigma')\big) \\
&= (a, f(\sigma')).
\end{aligned}$$

Thus $\mathsf{next} \circ f = \big(\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times f)\big) \circ \mathsf{next}$. This proves that $\mathsf{comp}(\langle\rangle, \sigma) = \sigma$ for all $\sigma \in A^\infty$.

We now show the other equality, that $\mathsf{comp}(\sigma, \langle\rangle) = \sigma$ for all $\sigma \in A^\infty$, we will show that the function $g\colon A^\infty \to A^\infty$ defined by $g(\sigma) \coloneqq \mathsf{comp}(\sigma, \langle\rangle)$ for all $\sigma \in A^\infty$ also satisfies

$$\mathsf{next} \circ g = \big(\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times g)\big) \circ \mathsf{next}.$$

That $(\mathsf{next} \circ g)(\perp) = \big(\big(\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times g)\big) \circ \mathsf{next}\big)(\perp)$ is the same as with f. Now if $\sigma \in A^\infty$ is such that $\sigma \xrightarrow{a} \sigma'$ for some $a \in A$ and $\sigma' \in A^\infty$, we see that

$$(\mathsf{next} \circ g)(\sigma) = \mathsf{next}(\mathsf{comp}(a \cdot \sigma', \langle\rangle)$$

$$
\begin{aligned}
&= (a, \mathsf{comp}(\sigma', \langle\rangle)) \\
&= (a, g(\sigma')),
\end{aligned}
$$

by the third rule for comp and the definition of g, and

$$
\begin{aligned}
\big((\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times g)) \circ \mathsf{next}\big)(\sigma) &= \big((\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times g))\big)\big((a, \sigma')\big) \\
&= \big(a, g(\sigma')\big).
\end{aligned}
$$

Therefore $g = \mathrm{id}_{A^\infty}$, i.e. $\mathsf{comp}(\sigma, \langle\rangle) = \sigma$ for all $\sigma \in A^\infty$.

2. We will define a coalgebra $c\colon A^\infty \times A^\infty \times A^\infty \to \{\bot\} \cup (A \times (A^\infty \times A^\infty \times A^\infty))$ such that the functions $h, k\colon A^\infty \times A^\infty \times A^\infty \to A^\infty$ given by

$$
\begin{aligned}
h(\sigma, \tau, \rho) &:= \mathsf{comp}(\sigma, \mathsf{comp}(\tau, \rho)) \qquad \text{and} \\
k(\sigma, \tau, \rho) &:= \mathsf{comp}(\mathsf{comp}(\sigma, \tau), \rho),
\end{aligned}
$$

for all $\sigma, \tau, \rho \in A^\infty$, are both coalgebra homomorphisms from c to next.

$$
\begin{tikzcd}
\{\bot\} \cup (A \times (A^\infty \times A^\infty \times A^\infty)) \arrow[r, bend left, "\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times h)"] \arrow[r, bend right, "\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times k)"'] & \{\bot\} \cup (A \times A^\infty) \\
A^\infty \times A^\infty \times A^\infty \arrow[u, "c"] \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] & A^\infty \arrow[u, "\cong", "\mathsf{next}"']
\end{tikzcd}
$$

The finality of next would then yield $h = k$.

Define $c\colon A^\infty \times A^\infty \times A^\infty \to \{\bot\} \cup (A \times (A^\infty \times A^\infty \times A^\infty))$ by

$$
c(\sigma, \tau, \rho) := \begin{cases}
\bot, & \text{if } \sigma = \tau = \rho = \langle\rangle, \\
\big(a, (\langle\rangle, \langle\rangle, \rho'\big), & \text{if } \sigma = \tau = \langle\rangle \text{ and } \rho = a \cdot \rho' \text{ for some } a \in A \text{ and } \rho' \in A^\infty, \\
\big(a, (\langle\rangle, \tau', \rho)\big), & \text{if } \sigma = \langle\rangle \text{ and } \tau = a \cdot \tau' \text{ for some } a \in A \text{ and } \tau' \in A^\infty, \\
\big(a, (\sigma', \tau, \rho)\big), & \text{if } \sigma = a \cdot \sigma' \text{ for some } a \in A \text{ and } \sigma' \in A^\infty.
\end{cases}
$$

Using the rules for comp, it is now elementary to check that h and k make their respective diagrams commute. □

Exercise 1.2.9

Consider two sets A, B with a function $f\colon A \to B$ between them. Use finality to define a function $f^\infty\colon A^\infty \to B^\infty$ that applies f element-wise. Use uniqueness to show that this mapping $f \mapsto f^\infty$ is 'functorial' in the sense that $(\mathrm{id}_A)^\infty = \mathrm{id}_{A^\infty}$ and $(g \circ f)^\infty = g^\infty \circ f^\infty$.

Solution. For a (non-empty) set B, let $\mathsf{next}_B\colon B^\infty \to \{\bot\} \cup (B \times B^\infty)$ denote the final coalgebra defined by

$$
\mathsf{next}(\sigma) := \begin{cases}
\bot, & \text{if } \sigma \text{ is the empty sequence,} \\
(b, \sigma'), & \text{if } \sigma \text{ has head } b \in B \text{ and tail } \sigma' \in B^\infty, \text{ i.e. } \sigma = b \cdot \sigma',
\end{cases}
$$

for all $\sigma \in B^\infty$. For a function $f\colon A \to B$, define a coalgebra $c_f\colon A^\infty \to \{\bot\} \cup (B \times A^\infty)$ by

$$c_f(\sigma) := \begin{cases} \bot, & \text{if } \sigma = \langle\rangle, \\ (f(a), \sigma'), & \text{if } \sigma = a \cdot \sigma' \text{ for some } a \in A \text{ and } \sigma' \in A^\infty, \end{cases}$$

for all $\sigma \in A^\infty$. Let $f^\infty\colon A^\infty \to B^\infty$ be the unique function making

$$\begin{tikzcd} \{\bot\} \cup (B \times A^\infty) \arrow[r, dashed, "\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_B \times f^\infty)"] & \{\bot\} \cup (B \times B^\infty) \\ A^\infty \arrow[u, "c_f"] \arrow[r, dashed, "\exists!", "f^\infty"'] & B^\infty \arrow[u, "\cong"', "\mathsf{next}_B"] \end{tikzcd}$$

Then $f(\langle a_0, a_1, a_2, a_3, \ldots\rangle) = \langle f(a_0), f(a_1), f(a_2), f(a_3), \ldots\rangle$ for all $a_0, a_1, a_2, a_3, \ldots \in A$, and analogously for finite sequences.

We see that $c_{\mathrm{id}_A} = \mathsf{next}_A$. So $(\mathrm{id}_A)^\infty = \mathrm{id}_{A^\infty}$ by finality of next_A. Furthermore, for functions $f\colon A \to B$ and $g\colon B \to C$, we see that

$$\begin{tikzcd} \{\bot\} \cup (C \times A^\infty) \arrow[r, "\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_C \times f^\infty)"] & \{\bot\} \cup (C \times B^\infty) \\ A^\infty \arrow[u, "c_{g\circ f}"] \arrow[r, "f^\infty"'] & B^\infty \arrow[u, "c_g"'] \end{tikzcd}$$

commutes. Consequently, the outer square in the diagram

$$\begin{tikzcd} \{\bot\} \cup (C \times A^\infty) \arrow[r, "\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_C \times f^\infty)"] & \{\bot\} \cup (C \times B^\infty) \arrow[r, "\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_C \times g^\infty)"] & \{\bot\} \cup (C \times C^\infty) \\ A^\infty \arrow[u, "c_{g\circ f}"] \arrow[r, "f^\infty"'] & B^\infty \arrow[u, "c_g"'] \arrow[r, "g^\infty"'] & C^\infty \arrow[u, "\cong"', "\mathsf{next}_C"] \end{tikzcd}$$

commutes, i.e.

$$\mathsf{next}_C \circ (g^\infty \circ f^\infty) = \big(\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_C \times (g^\infty \circ f^\infty))\big) \circ c_{g\circ f}.$$

The finality of next_C then yields $(g \circ f)^\infty = g^\infty \circ f^\infty$. □

Exercise 1.2.10

Use finality to define a map st: $A^\infty \times B \to (A \times B)^\infty$ *that maps a sequence* $\sigma \in A^\infty$ *and an element* $b \in B$ *to a new sequence in* $(A \times B)^\infty$ *by adding this* b *at every position in* σ. *(This is an example of a 'strength' map; see* *Exercise 2.5.4**.*

Solution. Define a coalgebra $c\colon A^\infty \times B \to \{\bot\} \cup \big((A \times B) \times (A^\infty \times B)\big)$ as follows:

$$c(\sigma, b) := \begin{cases} \bot, & \text{if } \sigma = \langle\rangle, \\ \big((a, b), (\sigma', b)\big), & \text{if } \sigma = a \cdot \sigma' \text{ for some } a \in A \text{ and } \sigma' \in A^\infty, \end{cases}$$

for all $\sigma \in A^\infty$ and $b \in B$. The unique function $\mathrm{st}\colon A^\infty \times B \to (A \times B)^\infty$ making

$$\begin{tikzcd}
\{\bot\} \cup \big((A \times B) \times (A^\infty \times B)\big) \arrow[rr, dashed, "\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_{A\times B} \times \mathrm{st})"] & & \{\bot\} \cup \big((A \times B) \times (A \times B)^\infty\big) \\
A^\infty \times B \arrow[u, "c"] \arrow[rr, dashed, "\exists!", "\mathrm{st}"'] & & (A \times B)^\infty \arrow[u, "\cong"', "\mathsf{next}"]
\end{tikzcd}$$

commute will satisfy $\mathrm{st}(\langle a_0, a_1, a_2, \ldots \rangle, b) = \langle (a_0, b), (a_1, b), (a_2, b), \ldots \rangle$ for all $a_0, a_1, a_2, a_3, \ldots \in A$ and $b \in B$, and analagously for finite sequences in A^∞. □

1.3 Generality of Temporal Logic of Coalgebras

Exercise 1.3.1
#??

Solution. #?? □

Exercise 1.3.2
#??

Solution. #?? □

Exercise 1.3.3
#??

Solution. #?? □

Exercise 1.3.4
#??

Solution. #?? □

Exercise 1.3.5
#??

Solution. #?? □

1.4 Abstractness of Coalgebraic Notions

Exercise 1.4.1
#??

Solution. #?? □

Exercise 1.4.2
#??

Solution. #?? □

Exercise 1.4.3
#??

Solution. #?? □

Exercise 1.4.4
#??

Solution. #?? □

Exercise 1.4.5
#??

Solution. #?? □

Exercise 1.4.6
#??

Solution. #?? □

2 Coalgebras of Polynomial Functors

2.1 Constructions on Sets

Exercise 2.1.1
#??

Solution. #?? □

Exercise 2.1.2
#??

Solution. #?? □

Exercise 2.1.3
#??

Solution. #?? □

Exercise 2.1.4
#??

Solution. #?? □

Exercise 2.1.5
#??

Solution. #?? □

Exercise 2.1.6
#??

Solution. #?? □

Exercise 2.1.7
#??

Solution. #?? □

Exercise 2.1.8
#??

Solution. #?? □

Exercise 2.1.9
#??

Solution. #?? □

Exercise 2.1.10
#??

Solution. #?? □

Exercise 2.1.11
#??

Solution. #?? ☐

Exercise 2.1.12
#??

Solution. #?? ☐

Exercise 2.1.13
#??

Solution. #?? ☐

Exercise 2.1.14
#??

Solution. #?? ☐

2.2 Polynomial Functors and Their Coalgebras

Exercise 2.2.1
#??

Solution. #?? ☐

Exercise 2.2.2
#??

Solution. #?? ☐

Exercise 2.2.3
#??

Solution. #?? ☐

Exercise 2.2.4
#??

Solution. #?? ☐

Exercise 2.2.5
#??

Solution. #?? ☐

Exercise 2.2.6
#??

Solution. #?? ☐

Exercise 2.2.7
#??

Solution. #?? ☐

Exercise 2.2.8
#??

Solution. #?? □

Exercise 2.2.9
#??

Solution. #?? □

Exercise 2.2.10
#??

Solution. #?? □

Exercise 2.2.11
#??

Solution. #?? □

Exercise 2.2.12
#??

Solution. #?? □

2.3 Final Coalgebras

Exercise 2.3.1
#??

Solution. #?? □

Exercise 2.3.2
#??

Solution. #?? □

Exercise 2.3.3
#??

Solution. #?? □

Exercise 2.3.4
#??

Solution. #?? □

Exercise 2.3.5
#??

Solution. #?? □

Exercise 2.3.6
#??

Solution. #?? □

Exercise 2.3.7
#??

Solution. #?? $\square$

Exercise 2.3.8
#??

Solution. #?? $\square$

2.4 Algebras

Exercise 2.4.1
#??

Solution. #?? $\square$

Exercise 2.4.2
#??

Solution. #?? $\square$

Exercise 2.4.3
#??

Solution. #?? $\square$

Exercise 2.4.4
#??

Solution. #?? $\square$

Exercise 2.4.5
#??

Solution. #?? $\square$

Exercise 2.4.6
#??

Solution. #?? $\square$

Exercise 2.4.7
#??

Solution. #?? $\square$

Exercise 2.4.8
#??

Solution. #?? $\square$

Exercise 2.4.9
#??

Solution. #?? $\square$

Exercise 2.4.10
#??

Solution. #?? $\square$

2.5 Adjunctions, Cofree Coalgebras, Behaviour-Realisation

Exercise 2.5.1
#??

Solution. #?? □

Exercise 2.5.2
#??

Solution. #?? □

Exercise 2.5.3
#??

Solution. #?? □

Exercise 2.5.4
This exercise describes 'strength' for endofunctors on **Sets**. *In general, this is a useful notion in the theory of datatypes (Cockett and Spencer, 1992), (Cockett and Spencer, 1995) and of computations (Moggi, 1991); see Section 5.2 for a systemic description.*

Let $F\colon \mathbf{Sets} \to \mathbf{Sets}$ *be an arbitrary functor. Consider for sets* X, Y *the strength map*

$$F(X) \times Y \xrightarrow{\quad \mathrm{st}_{X,Y} \quad} F(X \times Y)$$

$$(u, y) \longmapsto F(\lambda x \in X.(x, y))(u)$$

1. *Prove that this yields a natural transformation* $F(-)\times(-) \stackrel{\mathrm{st}}{\Rightarrow} F((-)\times(-))$, *where both the domain and codomain are functors* $\mathbf{Sets} \times \mathit{Sets} \to \mathbf{Sets}$.

2. *Describe this strength map for the list functor* $(-)^\star$ *and for the powerset functor* $\mathcal{P}$.

Solution. #?? □

Exercise 2.5.5
#??

Solution. #?? □

Exercise 2.5.6
#??

Solution. #?? □

Exercise 2.5.7
#??

Solution. #?? □

Exercise 2.5.8
#??

Solution. #?? □

Exercise 2.5.9
#??

Solution. #?? □

Exercise 2.5.10
#??

Solution. #?? □

Exercise 2.5.11
#??

Solution. #?? □

Exercise 2.5.12
#??

Solution. #?? □

Exercise 2.5.13
#??

Solution. #?? □

Exercise 2.5.14
#??

Solution. #?? □

Exercise 2.5.15
#??

Solution. #?? □

Exercise 2.5.16
#??

Solution. #?? □

Exercise 2.5.17
#??

Solution. #?? □

3 Bisimulations

3.1 Relation Lifting, Bisimulations and Congruences

Exercise 3.1.1
#??

Solution. #?? □

Exercise 3.1.2
#??

Solution. #?? □

Exercise 3.1.3
#??

Solution. #?? □

Exercise 3.1.4
#??

Solution. #?? □

Exercise 3.1.5
#??

Solution. #?? □

Exercise 3.1.6
#??

Solution. #?? □

3.2 Properties of Bisimulations

Exercise 3.2.1
#??

Solution. #?? □

Exercise 3.2.2
#??

Solution. #?? □

Exercise 3.2.3
#??

Solution. #?? □

Exercise 3.2.4
#??

Solution. #?? □

Exercise 3.2.5
#??

Solution. #?? □

Exercise 3.2.6
#??

Solution. #?? □

Exercise 3.2.7
#??

Solution. #?? □

3.3 Bisimulations as Spans and Cospans

Exercise 3.3.1
#??

Solution. #?? □

Exercise 3.3.2
#??

Solution. #?? □

Exercise 3.3.3
#??

Solution. #?? □

Exercise 3.3.4
#??

Solution. #?? □

3.4 Bisimulations and the Coinduction Proof Principle

Exercise 3.4.1
#??

Solution. #?? □

Exercise 3.4.2
#??

Solution. #?? □

Exercise 3.4.3
#??

Solution. #?? □

Exercise 3.4.4
#??

Solution. #?? □

Exercise 3.4.5
#??

Solution. #?? □

Exercise 3.4.6
#??

Solution. #?? □

Exercise 3.4.7
#??

Solution. #?? □

3.5 Process Semantics

Exercise 3.5.1
#??

Solution. #?? □

Exercise 3.5.2
#??

Solution. #?? □

Exercise 3.5.3
#??

Solution. #?? □

Exercise 3.5.4
#??

Solution. #?? □

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