# Contour Integration

MA270 Analysis 3 Revision Lecture

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> One time I boasted, "I can do by other methods any integral anybody else needs contour integration to do."
> So Paul [Olum] puts up this tremendous damn integral he had obtained by starting out with a complex function that he knew the answer to, taking out the real part of it and leaving only the complex part. He had unwrapped it so it was only possible by contour integration! He was always deflating me like that. He was a very smart fellow.
>
> *"Surely You're Joking, Mr. Feynman!"*, by Richard Feynman

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# 1 Theory

Recall that for a continuous path $\gamma\colon [a,b] \to \mathbb{C}$ and a continuous function $f\colon \mathbb{C} \to \mathbb{C}$, we define the contour integral of $f$ along $\gamma$ by

$$\int_\gamma f(z)\,\mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t = \int_a^b \mathbf{Re}(f(\gamma(t))\gamma'(t))\,\mathrm{d}t + i\int_a^b \mathbf{Im}(f(\gamma(t))\gamma'(t))\,\mathrm{d}t.$$

It is perhaps not surprising that the computation of contour integrals from this definition is rather time-consuming and difficult. To aid us in computing contour integrals, we use three big theorems: Cauchy's theorem, the contour deformation theorem, and Cauchy's integral formula.

The first of these theorems says that if we integrate around a region where our integrand is analytic, then the integral is zero.

**Theorem 1.1** (Cauchy's Theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be non-empty, open, and simply connected. Let $f\colon \Omega \to \mathbb{C}$ be analytic, and let $\gamma\colon [a,b] \to \Omega$ be a (piecewise $C^1$)*[1] *continuous loop in $\Omega$. Then*

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

The second theorem says, roughly speaking, says that we can "drag" contours along regions where our integrand is analytic, and this would not change the result of the integral along the contours.

**Theorem 1.2** (Contour Deformation Theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be non-empty, open, and simply connected. Let $f\colon \Omega \to \mathbb{C}$ be analytic, and let $\gamma_1, \gamma_2\colon [a,b] \to \Omega$ be simple regular piecewise $C^1$ paths in $\Omega$ with $\gamma_1(a) = \gamma_2(a)$ and $\gamma_1(b) = \gamma_2(b)$. Then*

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z.$$

The third theorem gives us a formula for dealing with cases when there is a point where our integrand is not analytic, but we want to drag our contour over that point anyway.

**Theorem 1.3** (Cauchy's Integral Formula)**.** *Let $\Omega \subseteq \mathbb{C}$ be non-empty, open, and simply connected, and let $z_0 \in \Omega$. Let $g\colon \Omega \to \mathbb{C}$ be analytic, and let $r > 0$ be small enough so that the closed ball $\overline{B_r(z_0)}$ is contained entirely in $\Omega$. Let $\beta\colon [a,b] \to \mathbb{C}$ be a loop which traces out the circle $\partial B_r(z_0)$ once anticlockwise*[2]*. Then*

$$\int_\beta \frac{g(z)}{z - z_0}\,\mathrm{d}z = 2\pi i \cdot g(z_0).$$

*More generally, for every $n \in \mathbb{Z}_{\geq 0}$ we have*

$$\int_\beta \frac{g(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z = \frac{2\pi i}{n!} \cdot g^{(n)}(z_0).$$

Contour integration involves employing these three theorems to compute integrals.

[1] The condition that $\gamma$ is piecewise $C^1$ is not necessary, however the proofs of Cauchy's theorem without this condition will require more machinery than is developed in the MA270 Analysis 3 module.

[2] An example parametrisation is $\beta\colon [0,2\pi] \to \mathbb{C}$ given by $\beta(t) = z_0 + re^{it}$.

## 2 Examples

Perhaps the simplest example to illustrate contour integration is the evaluation of the integral $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x$. It is fairly easy to show that $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x = \pi$ by using the fact that $\frac{\mathrm{d}}{\mathrm{d}x}\arctan(x)\big|_{x=x_0} = \frac{1}{x_0^2+1}$ for all $x_0 \in \mathbb{R}$. We shall instead use this integral to introduce contour integration. The idea is to shift from the perspective of real integrals representing the area under a curve to the perspective of integrating along a contour in the complex plane.

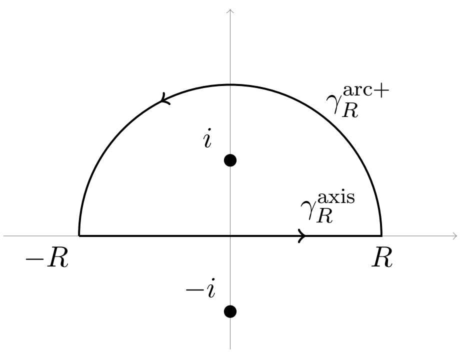
**Example 2.1** (Integrating around one pole)**.** Let

$$I := \int_{-\infty}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x.$$

To simplify notation, let us denote the integrand by $f(z) := \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$. We note that $f$ is analytic on all of $\mathbb{C}$ except at $i$ and $-i$.

For $R > 0$, consider the paths $\gamma_R^{\text{axis}} : [-R, R] \to \mathbb{C}$ and $\gamma_R^{\text{arc}+} : [0, \pi] \to \mathbb{C}$ given by

$$\begin{aligned} \gamma_R^{\text{axis}}(t) &:= t, \\ \gamma_R^{\text{arc}+}(t) &:= Re^{it}. \end{aligned}$$




Note that the choice of parametrisation endows our paths with a specific orientation (from left to right for $\gamma_R^{\text{axis}}$, and anticlockwise for $\gamma_R^{\text{arc}+}$). Our desired integral is then

$$I = \lim_{R\to\infty} \int_{\gamma_R^{\text{axis}}} f(z)\,\mathrm{d}z,$$

since the integral $I$ is absolutely convergent.[3] For $R > 0$, letting $\gamma_R^{+}$ be the concatenation[4] of the path $\gamma_R^{\text{axis}}$ followed by the path $\gamma_R^{\text{arc}+}$, we have

$$\int_{\gamma_R^{\text{axis}}} f(z)\,\mathrm{d}z + \int_{\gamma_R^{\text{arc}+}} f(z)\,\mathrm{d}z = \int_{\gamma_R^{+}} f(z)\,\mathrm{d}z.$$

[3] One can check this by employing an integral version of the Weierstrass $M$-test.

[4] If continuous paths $\rho_1 \colon [a_1, b_1] \to \mathbb{C}$ and $\rho_2 \colon [a_2, b_2] \to \mathbb{C}$ are such that $\rho_1(b_1) = \rho_2(a_2)$, then the function $\rho_3 \colon [a_1, b_1 + b_2 - a_2] \to \mathbb{C}$ by

$$\rho_3(t) := \begin{cases} \rho_1(t) & \text{if } a_1 \leqslant t \leqslant b_1, \\ \rho_2(t + a_2 - b_1) & \text{if } b_1 < t \leqslant b_1 + b_2 - a_2, \end{cases}$$

is a continuous path which acts as a concatenation of $\rho_1$ followed by $\rho_2$.

A quick computation yields

$$\begin{aligned}
\lim_{R\to\infty}\left|\int_{\gamma_R^{\mathrm{arc}+}} f(z)\,\mathrm{d}z\right| &\leqslant \lim_{R\to\infty}\int_{\gamma_R^{\mathrm{arc}+}} |f(z)||\,\mathrm{d}z| \\
&= \lim_{R\to\infty}\int_0^{\pi}\left|\frac{1}{(Re^{it})^2+1}\right|\cdot\left|iRe^{it}\right|\mathrm{d}t \\
&\leqslant \lim_{R\to\infty}\frac{\pi R}{|R^2-1|} \\
&= 0,
\end{aligned}$$

where we used the reverse triangle inequality on the denominator to obtain the second-to-last line. The calculation above yields $\lim_{R\to\infty}\int_{\gamma_R^{\mathrm{arc}+}} f(z)\,\mathrm{d}z = 0$. We thus obtain

$$I = \lim_{R\to\infty}\int_{\gamma_R^{\mathrm{axis}}} f(z)\,\mathrm{d}z = \lim_{R\to\infty}\left(\int_{\gamma_R^{\mathrm{axis}}} f(z)\,\mathrm{d}z + \int_{\gamma_R^{\mathrm{arc}+}} f(z)\,\mathrm{d}z\right) = \lim_{R\to\infty}\int_{\gamma_R^{+}} f(z)\,\mathrm{d}z.$$

By the contour deformation theorem, for any $R>1$, we have

$$\int_{\gamma_R^{+}} f(z) = \int_{\beta} f(z)\,\mathrm{d}z,$$

where $\beta$ traces out the circle $\partial B_1(i)$ once anticlockwise. Cauchy's integral formula then gives us

$$\begin{aligned}
\int_{\beta} f(z)\,\mathrm{d}z &= \int_{\beta}\frac{1}{(z-i)(z+i)}\,\mathrm{d}z \\
&= 2\pi i\cdot\left.\frac{1}{z+i}\right|_{z=i} \\
&= \pi.
\end{aligned}$$

Therefore, $I=\pi$. □

In Example 2.1, it would not have mattered if we chose the semicircle contour to be in the upper-half plane or in the lower-half plane. More specifically, if we instead chose to define $\gamma_R^{\mathrm{arc}-}\colon [0,\pi]\to\mathbb{C}$ by

$$\gamma_R^{\mathrm{arc}-}(t) := Re^{-it}$$

and then defined $\gamma_R^{-}$ to be the concatenation of $\gamma_R^{\mathrm{axis}}$ followed by $\gamma_R^{\mathrm{arc}-}$, we would arrive at the same answer with a near identical argument. This is not true in general. There are cases where the contour we choose *does* matter!

**Example 2.2** (Choosing the correct half of the plane)**.** Let

$$I := \int_{-\infty}^{\infty}\frac{e^{-ix}}{x^2+9}\,\mathrm{d}x.$$

Let the integrand be $f(z) := \frac{e^{-iz}}{z^2+9} = \frac{e^{-iz}}{(z-3i)(z+3i)}$, noting that $f$ has poles at $3i$ and $-3i$. Define the paths $\gamma_R^{\mathrm{axis}} : [-R,R]\to\mathbb{C}$ and $\gamma_R^{\mathrm{arc}-} : [0,\pi]\to\mathbb{C}$ by

$$\begin{aligned}
\gamma_R^{\mathrm{axis}}(t) &:= t, \\
\gamma_R^{\mathrm{arc}-}(t) &:= Re^{-it},
\end{aligned}$$

and define $\gamma_R^-$ to be the concatenation of $\gamma_R^{\text{axis}}$ followed by $\gamma_R^{\text{arc}-}$. Note that $\gamma_R^-$ traces out a semicircle in the *lower*-half of the complex plane.

We have $\lim_{R\to\infty} \int_{\gamma_R^{\text{arc}-}} f(z)\,\mathrm{d}z = 0$, since

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{\gamma_R^{\text{arc}}} f(z)\,\mathrm{d}z\right| &\leqslant \lim_{R\to\infty}\int_0^\pi \left|\frac{e^{-iRe^{-it}}}{(Re^{-it})^2+9}\right|\cdot\left|-iRe^{-it}\right|\mathrm{d}t\\ &= \lim_{R\to\infty}\int_0^\pi \left|\frac{e^{-iR(\cos(-t)+i\sin(-t))}}{(Re^{-it})^2+9}\right|\cdot\left|-iRe^{-it}\right|\mathrm{d}t\\ &\leqslant \lim_{R\to\infty}\int_0^\pi \frac{e^{-R\sin(t)}}{|R^2-9|}\cdot R\,\mathrm{d}t\\ &\leqslant \lim_{R\to\infty}\frac{\pi R}{|R^2-9|}\\ &= 0,\end{aligned}$$

where the second-to-last line follows from the fact that $0 \leqslant e^{-R\sin(t)} \leqslant 1$ for all $t \in [0, \pi]$ and for all $R > 0$. Hence, by the absolute convergence of $I$, we obtain

$$I = \lim_{R\to\infty}\left(\int_{\gamma_R^{\text{axis}}} f(z)\,\mathrm{d}z + \int_{\gamma_R^{\text{arc}-}} f(z)\,\mathrm{d}z\right) = \lim_{R\to\infty}\int_{\gamma_R^-} f(z)\,\mathrm{d}z.$$

The contour deformation theorem and Cauchy's integral formula therefore yield

$$\begin{aligned} I &= \lim_{R\to\infty}\int_{\gamma_R^-} f(z)\,\mathrm{d}z\\ &= -2\pi i\cdot \left.\frac{e^{-iz}}{z-3i}\right|_{z=-3i}\\ &= \frac{\pi}{3e^3}.\end{aligned}$$

Note the minus sign in $-2\pi i$ on the second line. This is introduced because $\gamma_R^-$ traces out the semicircle in the *clockwise* direction. □

*Remark.* Our choice for $\gamma_R^{\text{arc}-}$ forming the arc of a semicircle in the *lower*-half of the complex plane really is crucial here. The argument in Example 2.2 would not have worked if we had instead used the paths $\gamma_R^{\text{arc}+}$ and $\gamma_R^+$ constructed in Example 2.1. Indeed, we would obtain the rather unhelpful inequality

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{\gamma_R^{\text{arc}+}} f(z)\,\mathrm{d}z\right| &\leqslant \lim_{R\to\infty}\int_0^\pi \left|\frac{e^{-iRe^{it}}}{(Re^{it})^2+9}\right|\cdot\left|iRe^{it}\right|\mathrm{d}t\\ &= \lim_{R\to\infty}\int_0^\pi \left|\frac{e^{-iR(\cos(t)+i\sin(t))}}{(Re^{it})^2+9}\right|\cdot\left|iRe^{it}\right|\mathrm{d}t\\ &\leqslant \lim_{R\to\infty}\int_0^\pi \frac{e^{R\sin(t)}}{|R^2-9|}\cdot R\,\mathrm{d}t\\ &= \infty,\end{aligned}$$

from which we cannot deduce that $\lim_{R\to\infty}\int_{\gamma_R^{\text{arc}+}} f(z)\,\mathrm{d}z = 0$. □

We can similarly show that $\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+9}\,\mathrm{d}x = \frac{\pi}{3e^3}$ by constructing a semicircle contour in the *upper*-half of the complex plane. Now recall that, for any $z \in \mathbb{C}$, we have

$$\cos(z) = \frac{e^{iz}+e^{-iz}}{2} = \mathbf{Re}(e^{iz}) \quad\text{and}\quad \sin(z) = \frac{e^{iz}-e^{-iz}}{2i} = \mathbf{Im}(e^{iz}).$$

This allows us to turn any real integral involving sin and cos into a complex integral involving the exponential. As a quick example, we have

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+9}\,\mathrm{d}x = \frac{1}{2}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+9}\,\mathrm{d}x + \int_{-\infty}^{\infty} \frac{e^{-ix}}{x^2+9}\,\mathrm{d}x\right) = \frac{\pi}{3e^3}.$$

Alternatively, we may compute this integral as follows:

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+9}\,\mathrm{d}x = \mathbf{Re}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+9}\,\mathrm{d}x\right) = \frac{\pi}{3e^3}.$$

This example is a little contrived, as the computation of the integral $\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+9}\,\mathrm{d}x$ using contour integration is not much more tedious than computing $\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+9}\,\mathrm{d}x$. Nevertheless, there are cases where complexifying the integrand can actually make things easier to compute.

**Example 2.3** (Complexifying the integrand)**.** Let

$$I := \int_0^{2\pi} e^{\cos(t)} \cos(\sin(t))\,\mathrm{d}t.$$

We first note[5] that

$$I = \mathbf{Re}\left(\int_0^{2\pi} e^{e^{it}}\,\mathrm{d}t\right) = \mathbf{Re}\left(\frac{1}{i}\int_0^{2\pi} \frac{e^{e^{it}}}{e^{it}} \cdot ie^{it}\,\mathrm{d}t\right) = \mathbf{Re}\left(\frac{1}{i}\int_\beta \frac{e^z}{z}\,\mathrm{d}z\right),$$

where $\beta\colon [0, 2\pi] \to \mathbb{C}$ is the loop $\beta(t) := e^{it}$. Cauchy's integral formula then yields

$$\int_\beta \frac{e^z}{z}\,\mathrm{d}z = 2\pi i,$$

from which it follows that

$$I = \mathbf{Re}\left(\frac{1}{i}\int_\beta \frac{e^z}{z}\,\mathrm{d}z\right) = 2\pi. \qquad \square$$

In both Example 2.1 and Example 2.2, the contour $\gamma_R^+$ travelled around just one pole of the integrand $f$, making it easy for us to deform the contour and employ Cauchy's integral formula. What do we do when the contour $\gamma_R^+$ goes around multiple poles of the integrand $f$? The idea is to reduce $\gamma_R^+$ to a contour which goes around each pole individually, and then use Cauchy's integral formula to evaluate the contributions from each pole.

**Example 2.4** (Integrating around multiple poles)**.** Let

$$I := \int_0^{\infty} \frac{\cos(x)}{x^6+1}\,\mathrm{d}x.$$

Observe that the integrand is even. So, letting $f(z) := \frac{e^{iz}}{z^6+1}$, we obtain $2I = \mathbf{Re}(J)$, where

$$J := \int_{-\infty}^{\infty} f(z)\,\mathrm{d}z.$$

Note that $f$ has poles at

$$i,\ -i,\ \frac{\sqrt{3}}{2}+\frac{1}{2}i,\ \frac{\sqrt{3}}{2}-\frac{1}{2}i,\ -\frac{\sqrt{3}}{2}+\frac{1}{2}i,\ \text{and}\ -\frac{\sqrt{3}}{2}-\frac{1}{2}i.$$

[5]We would not be doing analysis if we did not pull a rabbit out of a hat at least once.

For $R > 0$, consider again the paths $\gamma_R^{\text{axis}}$, $\gamma_R^{\text{arc+}}$, and $\gamma_R^+$ as in Example 2.1. That is, $\gamma_R^{\text{axis}} \colon [-R, R] \to C$ and $\gamma_R^{\text{arc+}} \colon [0, \pi] \to \mathbb{C}$ are defined by

$$\begin{aligned} \gamma_R^{\text{axis}}(t) &:= t && \text{for } t \in [-R, R], \\ \gamma_R^{\text{arc+}}(t) &:= Re^{it} && \text{for } t \in [0, \pi], \end{aligned}$$

and $\gamma_R^+$ is the concatenation of $\gamma_R^{\text{axis}}$ followed by $\gamma_R^{\text{arc+}}$. As the integral $J$ is absolutely convergent, we have

$$J = \lim_{R\to\infty} \int_{\gamma_R^{\text{axis}}} f(z)\, \mathrm{d}z.$$

As $\gamma_R^+$ is the concatenation of $\gamma_R^{\text{axis}}$ and $\gamma_R^{\text{arc+}}$, we obtain

$$\int_{\gamma_R^{\text{axis}}} f(z)\, \mathrm{d}z + \int_{\gamma_R^{\text{arc+}}} f(z)\, \mathrm{d}z = \int_{\gamma_R^+} f(z)\, \mathrm{d}z.$$

Observe that $\lim_{R\to\infty} \int_{\gamma_R^{\text{arc+}}} f(z)\, \mathrm{d}z = 0$, because

$$\begin{aligned} \lim_{R\to\infty} \left| \int_{\gamma_R^{\text{arc+}}} f(z)\, \mathrm{d}z \right| &\leqslant \lim_{R\to\infty} \int_{\gamma_R^{\text{arc+}}} |f(z)|\, |\mathrm{d}z| \\ &= \lim_{R\to\infty} \int_0^{\pi} \left| \frac{e^{iRe^{it}}}{(Re^{it})^6 + 1} \right| \cdot \left| iRe^{it} \right| \mathrm{d}t \\ &\leqslant \lim_{R\to\infty} \frac{\pi R}{|R^6 - 1|} \\ &= 0. \end{aligned}$$

It follows that

$$J = \lim_{R\to\infty} \left( \int_{\gamma_R^{\text{axis}}} f(z)\, \mathrm{d}z + \int_{\gamma_R^{\text{arc+}}} f(z)\, \mathrm{d}z \right) = \lim_{R\to\infty} \int_{\gamma_R^+} f(z)\, \mathrm{d}z.$$

For simplicity, we denote the poles of $f$ by

$$z_1 := i,\ z_2 := -i,\ z_3 := \frac{\sqrt{3}}{2} + \frac{1}{2}i,\ z_4 := \frac{\sqrt{3}}{2} - \frac{1}{2}i,\ z_5 := -\frac{\sqrt{3}}{2} + \frac{1}{2}i, \text{ and } z_6 := -\frac{\sqrt{3}}{2} - \frac{1}{2}i.$$

Note that, of these poles, only $z_1$, $z_3$, and $z_5$ lie in the region enclosed by $\gamma_R^+$ for sufficiently large $R > 0$. The contour deformation theorem then gives us

$$\int_{\gamma_R} f(z)\, \mathrm{d}z = \sum_{j\in\{1,3,5\}} \left( \int_{\beta_j} f(z)\, \mathrm{d}z \right),$$

where $\beta_j \colon [0, 2\pi] \to \mathbb{C}$ is given by $\beta_j(t) = z_j + re^{it}$ for some sufficiently small constant $r > 0$ such that the images of the $\beta_j$'s are disjoint. Consequently, Cauchy's integral formula yields

$$J = 2\pi i \sum_{j\in\{1,3,5\}} \left( \left. \frac{e^{iz}}{\prod_{k\neq j}(z - z_k)} \right|_{z=z_j} \right).$$

Therefore

$$I = \mathbf{Re} \left( \pi i \sum_{j\in\{1,3,5\}} \left( \left. \frac{e^{iz}}{\prod_{k\neq j}(z - z_k)} \right|_{z=z_j} \right) \right). \qquad \square$$

If the reader is unsatisfied with the final form of the solution in Example 2.4 and has plenty of spare time to kill, they are welcome to finish the simplification themselves. We must warn that this is unlikely to be a productive use of the reader's time, unless the reader wishes to train their stamina for brute computations.

We shall now explore various different contours that can be constructed other than a circles and semicircles. This will be useful for cases where the usual semicircle contour does not work, due to the existence of poles on the real axis.

**Example 2.5** (Pole at the origin)**.** Let

$$I := \int_{-\infty}^{\infty} \frac{\sin(x)}{x(x^2+1)}\,\mathrm{d}x.$$

First note that

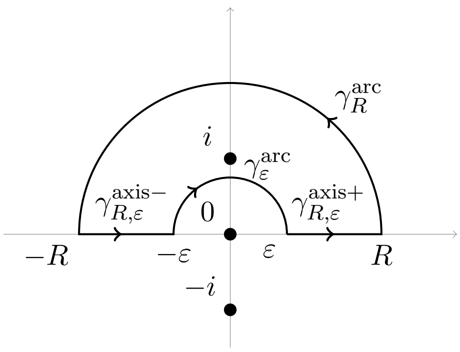
$$I = \mathbf{Im}(J), \quad \text{where } J := \int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+1)}\,\mathrm{d}x,$$

so it suffices to compute this new integral $J$. Let $f(z) := \frac{e^{iz}}{z(z^2+1)}$ denote the integrand of $J$.

We can no longer use the semicircular contour from before, since we cannot integrate over the pole at the origin. Instead, we adjust the contour slightly as follows. For $0 < \varepsilon < 1 < R$, we define the paths $\gamma_{R,\varepsilon}^{\text{axis}+}$, $\gamma_{R,\varepsilon}^{\text{axis}-}$, $\gamma_R^{\text{arc}}$, and $\gamma_\varepsilon^{\text{arc}}$ by:[6]

$$\begin{aligned}
\gamma_{R,\varepsilon}^{\text{axis}+}(t) &:= t && \text{for } t \in [\varepsilon, R],\\
\gamma_R^{\text{arc}}(t) &:= Re^{it} && \text{for } t \in [0, \pi],\\
\gamma_{R,\varepsilon}^{\text{axis}-}(t) &:= t && \text{for } t \in [-R, -\varepsilon],\\
\gamma_\varepsilon^{\text{arc}}(t) &:= \varepsilon e^{-it} && \text{for } t \in [-\pi, 0].
\end{aligned}$$

Then define $\gamma_{R,\varepsilon}$ as the concatenation of the four paths above, in order of appearance, producing the loop in the figure below.




We have[7]

$$J = \lim_{R\to\infty} \lim_{\varepsilon\to 0} \int_{\gamma_{R,\varepsilon}^{\text{axis}+}} f(z)\,\mathrm{d}z + \int_{\gamma_{R,\varepsilon}^{\text{axis}-}} f(z)\,\mathrm{d}z.$$

due to the absolute convergence of $J$. It is routine to check that $\lim_{R\to\infty} \int_{\gamma_R^{\text{arc}}} f(z)\,\mathrm{d}z = 0$. Consequently, the contour deformation theorem and Cauchy's integral formula yield

$$J + \lim_{\varepsilon\to 0} \int_{\gamma_\varepsilon^{\text{arc}}} f(z)\,\mathrm{d}z = 2\pi i \cdot \left.\frac{e^{iz}}{z(z+i)}\right|_{z=i} = -\frac{\pi}{e} i.$$

[6] Apologies for the abuse of notation with $\gamma_R^{\text{arc}}$ and $\gamma_\varepsilon^{\text{arc}}$. Having separate notation to distinguish clockwise versus anticlockwise directions only seemed to clutter things more. The authors of this article would appreciate new notation ideas for these contours. For this example, we will consistently use $R$ for the bigger arc and $\varepsilon$ for the smaller arc.

[7] If the reader is concerned about the order of the limits, we may stipulate that $\varepsilon = \frac{1}{R}$ so we only need to consider the limit as $R \to \infty$.

Now,

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon^{\mathrm{arc}}} f(z)\,\mathrm{d}z &= \lim_{\varepsilon\to 0}\int_{-\pi}^{0} \frac{e^{i\varepsilon e^{-it}}}{\varepsilon e^{-it}\left(\left(\varepsilon e^{-it}\right)^2+1\right)}\cdot\left(-i\varepsilon e^{-it}\right)\mathrm{d}t\\
&= -i\lim_{\varepsilon\to 0}\int_0^{\pi}\frac{e^{i\varepsilon e^{it}}}{\left(\varepsilon e^{it}\right)^2+1}\,\mathrm{d}t\\
&= -i\int_0^{\pi}\lim_{\varepsilon\to 0}\left(\frac{e^{i\varepsilon e^{it}}}{\left(\varepsilon e^{it}\right)^2+1}\right)\mathrm{d}t\\
&= -\pi i.
\end{aligned}$$

We can move the limit into the integral in the second-to-last line above due to the uniform convergence of the integrand. Therefore $J=\left(\pi-\frac{\pi}{e}\right)i$, and hence $I=\pi-\frac{\pi}{e}$. □

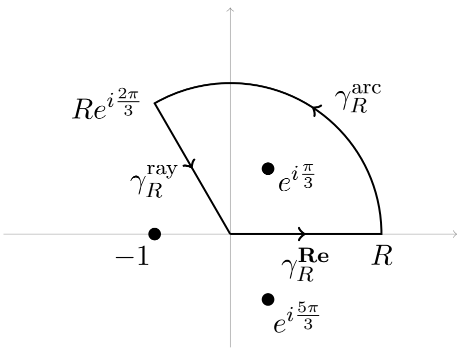
**Example 2.6** (Pie contour)**.** Let

$$I := \int_0^{\infty}\frac{1}{x^3+1}.$$

Denote the integrand by $f(z):=\frac{1}{z^3+1}$. Extending this half-line to the usual semicircle contour does not work, because of the pole of $f$ at $-1$. Instead, for $R>0$, we consider the paths

$$\begin{aligned}
\gamma_R^{\mathbf{Re}}(t) &:= t && \text{for } t\in[0,R],\\
\gamma_R^{\mathrm{arc}}(t) &:= Re^{it} && \text{for } t\in\left[0,\frac{2\pi}{3}\right],\\
\gamma_R^{\mathrm{ray}}(t) &:= -e^{i\frac{2\pi}{3}}t && \text{for } t\in[-R,0],
\end{aligned}$$

and let $\gamma_R$ denote the concatenation of all the three paths above, in order of appearance, producing the loop in the figure below.




Note that

$$I=\lim_{R\to\infty}\int_{\gamma_R^{\mathbf{Re}}} f(z)\,\mathrm{d}z.$$

Now, it is routine to check that $\lim_{R\to\infty}\int_{\gamma_R^{\mathrm{arc}}} f(z)\,\mathrm{d}z=0$. Furthermore,

$$\begin{aligned}
\lim_{R\to\infty}\int_{\gamma_R^{\mathrm{ray}}} f(z)\,\mathrm{d}z &= \lim_{R\to\infty}\int_{-R}^{0}\frac{1}{\left(-e^{\frac{2\pi i}{3}}t\right)^3+1}\cdot\left(-e^{\frac{2\pi i}{3}}\right)\mathrm{d}t\\
&= -e^{\frac{2\pi i}{3}}\lim_{R\to\infty}\int_0^{R}\frac{1}{t^3+1}\,\mathrm{d}t\\
&= -e^{\frac{2\pi i}{3}} I.
\end{aligned}$$

The contour deformation theorem and Cauchy's integral formula give us

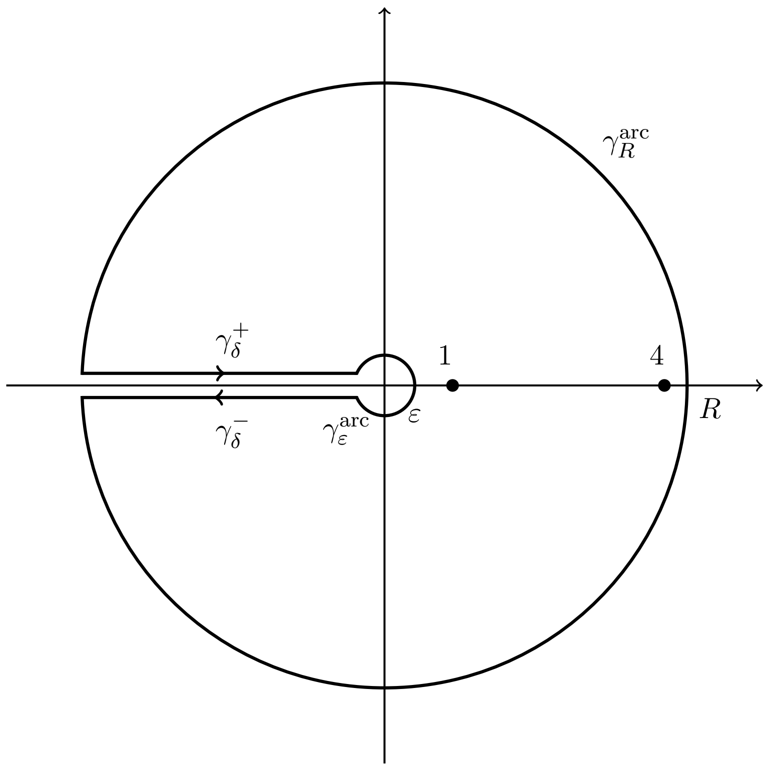
$$\begin{aligned} I - e^{\frac{2\pi i}{3}} I &= \lim_{R\to\infty} \left( \int_{\gamma_R^{\text{axis}}} f(z)\,\mathrm{d}z + \int_{\gamma_R^{\text{arc}}} f(z)\,\mathrm{d}z + \int_{\gamma_R^{\text{ray}}} f(z)\,\mathrm{d}z \right) \\ &= 2\pi i \cdot \left. \frac{1}{(z+1)\left(z - e^{\frac{5\pi i}{3}}\right)} \right|_{z=e^{\pi i/3}} \\ &= \pi \left( \frac{1}{\sqrt{3}} - \frac{1}{3} i \right), \end{aligned}$$

yielding $I = \frac{2\pi}{3\sqrt{3}}$. □

**Example 2.7** (Branch Cut)**.** Let

$$I := \int_{-\infty}^{0} \frac{z^{1/2}}{z^2 - 5z + 4}\,\mathrm{d}z.$$

This does not make sense as a real integral, as the square root of a negative number is not a real number. However, if we allow ourselves to access complex numbers, this is simply a contour integral along the negative real axis with integrand $f(z) := \frac{z^{1/2}}{z^2-5z+4}$. Note that $f$ has poles at 1 and 4. Since $z^{\frac{1}{2}} = e^{\frac{1}{2}\log(z)}$, the function $f$ requires a branch cut to be defined continuously. As is conventional, we take the branch cut along the half-line $(-\infty, 0]$. We cannot integrate $f$ across this branch cut, so we use a contour as in the figure below.




This is known as a *keyhole contour*, and consists of four individual paths: an arc $\gamma_R^{\text{arc}}$ of radius $R > 4$, an arc $\gamma_\varepsilon^{\text{arc}}$ of radius $0 < \varepsilon < 1$, and two parallel line segments $\gamma_\delta^+$ and $\gamma_\delta^-$, joining the two arcs. The perpendicular distance between the lines traced out by $\gamma_\delta^+$ and $\gamma_\delta^-$ is $2\delta$. For the following parametrisations to make sense, we also require that $0 < \delta < \varepsilon$.

$$
\begin{aligned}
\gamma_R^{\mathrm{arc}}(t) &:= Re^{it} && \text{for } t \in \left[-\pi + \sin^{-1}\left(\frac{\delta}{R}\right), \pi - \sin^{-1}\left(\frac{\delta}{R}\right)\right], \\
\gamma_\varepsilon^{\mathrm{arc}}(t) &:= \varepsilon e^{-it} && \text{for } t \in \left[-\pi + \sin^{-1}\left(\frac{\delta}{\varepsilon}\right), \pi - \sin^{-1}\left(\frac{\delta}{\varepsilon}\right)\right], \\
\gamma_\delta^{+}(t) &:= t + i\delta && \text{for } t \in \left[-\sqrt{R^2-\delta^2}, -\sqrt{\varepsilon^2-\delta^2}\right], \\
\gamma_\delta^{-}(t) &:= -t - i\delta && \text{for } t \in \left[\sqrt{\varepsilon^2-\delta^2}, \sqrt{R^2-\delta^2}\right].
\end{aligned}
$$

Let $C$ be the concatenation of these four paths (in the order suggested by the figure above). We have that

$$\int_C f(z)\,\mathrm{d}z = \int_{\gamma_R^{\mathrm{arc}}} f(z)\,\mathrm{d}z + \int_{\gamma_\delta^{+}} f(z)\,\mathrm{d}z + \int_{\gamma_\varepsilon^{\mathrm{arc}}} f(z)\,\mathrm{d}z + \int_{\gamma_\delta^{-}} f(z)\,\mathrm{d}z.$$

The integrals over $\gamma_R^{\mathrm{arc}}$ and $\gamma_\varepsilon^{\mathrm{arc}}$ converge to 0 as $R \to \infty$ and $\varepsilon \to 0$ respectively, and it is routine to check this.

For the integrals over $\gamma_\delta^{+}$ and $\gamma_\delta^{-}$, taking $R \to \infty$ and $\varepsilon, \delta \to 0$ gives

$$\int_{\gamma_\delta^{+}} f(z)\,\mathrm{d}z = \int_{-\sqrt{R^2-\delta^2}}^{-\sqrt{\varepsilon^2-\delta^2}} \frac{(t+i\delta)^{1/2}}{(t+i\delta)^2 - 5(t+i\delta) + 4}\,\mathrm{d}t \longrightarrow \int_{-\infty}^{0} \frac{t^{1/2}}{t^2 - 5t + 4}\,\mathrm{d}t = I,$$

and

$$
\begin{aligned}
\int_{\gamma_\delta^{-}} f(z)\,\mathrm{d}z &= \int_{\sqrt{\varepsilon^2-\delta^2}}^{\sqrt{R^2-\delta^2}} \frac{(-t-i\delta)^{1/2}}{(-t-i\delta)^2 - 5(-t-i\delta) + 4}(-1)\,\mathrm{d}t \longrightarrow \int_0^{\infty} \frac{(-t)^{1/2}e^{i\pi}}{(-t)^2 - 5(-t) + 4}(-1)\,\mathrm{d}t \\
&= e^{i\pi} \int_0^{-\infty} \frac{t^{1/2}}{t^2 - 5t + 4}\,\mathrm{d}t = \int_{-\infty}^{0} \frac{t^{1/2}}{t^2 - 5t + 4}\,\mathrm{d}t = I.
\end{aligned}
$$

For the integral over $\gamma_\delta^{-}$, we introduce a factor of $e^{i\pi} = -1$ when taking $\delta \to 0$. This follows from our choice of principal branch for the complex logarithm.

Putting all this together, we have

$$\int_C f(z)\,\mathrm{d}z = I + I = 2I.$$

The contour $C$ contains two poles, so we proceed as in Example 2.4 to obtain

$$I = \frac{1}{2}\int_C f(z)\,\mathrm{d}z = \frac{1}{2} \cdot 2\pi i \left( \left.\frac{z^{1/2}}{z-1}\right|_{z=4} + \left.\frac{z^{1/2}}{z-4}\right|_{z=1} \right) = \frac{\pi i}{3}. \qquad \square$$

So far, every integrand we have seen had poles of order 1. Consequently, we only used the formula

$$\int_\beta \frac{g(z)}{z - z_0}\,\mathrm{d}z = 2\pi i \cdot g(z_0)$$

in Cauchy's integral formula. We are yet to use the more general formula

$$\int_\beta \frac{g(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z = \frac{2\pi i}{n!} \cdot g^{(n)}(z_0) \qquad \text{for all } n \in \mathbb{Z}_{\geq 0}$$

which appears in Cauchy's integral formula. This is used when we have higher-order poles in our integrand.

**Example 2.8** (Higher-order poles)**.** Let

$$I := \int_{-\infty}^{\infty} \frac{\cos(x)}{(x^2+1)^4}\,\mathrm{d}x.$$

Let $f(z) := \frac{e^{iz}}{(z^2+1)^4} = \frac{e^{iz}}{(z-i)^4(z+i)^4}$, noting that $f$ has poles at $i$ and $-i$. Then

$$I = \mathbf{Re}\left(\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x\right).$$

Take the usual semicircle contour $\gamma_R^+$ of radius $R > 0$ in the upper-half of the complex plane. It is routine to check that

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{R\to\infty} \int_{\gamma_R^+} f(z)\,\mathrm{d}z.$$

Therefore, Cauchy's integral formula gives

$$\begin{aligned} I &= \mathbf{Re}\left(\lim_{R\to\infty} \int_{\gamma_R^+} f(z)\,\mathrm{d}z\right) \\ &= \frac{2\pi i}{3!}\cdot g^{(3)}(i), \qquad \text{where } g(z) := \frac{e^{iz}}{(z+i)^4}, \\ &= \frac{37\pi}{48e}. \end{aligned}$$

□

Finally, there are cases when we can spot when a contour integral vanishes, by virtue of Cauchy's theorem.

**Example 2.9** (Spotting an integral which evaluates to zero)**.** Let

$$I := \int_{-\infty}^{\infty} \frac{1}{z^2 - 3iz - 2}\,\mathrm{d}z.$$

Letting the integrand be $f(z) := \frac{1}{z^2-3iz-2} = \frac{1}{(z-i)(z-2i)}$, we note that $f$ has poles at $i$ and $2i$, and that both of these poles are in the upper-half of the complex plane. For $R > 0$, let $\gamma_R^-$ be the usual semicircle contour of radius $R > 0$ in the lower-half of the complex plane. It is routine to check that

$$I = \lim_{R\to\infty} \int_{\gamma_R^-} f(z)\,\mathrm{d}z.$$

As $f$ is analytic on all of the lower-half of the complex plane, Cauchy's theorem allows us to conclude that $I = 0$. □

There are still many other types of contours which can be chosen for different integrands; these are only a handful of the most popular contours one would consider when choosing to perform contour integration.

Finally, we remark that one very often sees the notations

$$\oint_{\partial B_r(z_0)} f(z)\,\mathrm{d}z \qquad \text{or} \qquad \int_{\partial B_r(z_0)} f(z)\,\mathrm{d}z \qquad \text{or} \qquad \int_{|z-z_0|=r} f(z)\,\mathrm{d}z,$$

among many others, to denote $\int_\beta f(z)\,\mathrm{d}z$, where $\beta\colon [0,2\pi] \to \mathbb{C}$ is the loop $\beta(t) := z_0 + re^{it}$ traversing the circle $\partial B_r(z_0)$ once anticlockwise. Similarly, if $C$ is the image of some simple loop in $\mathbb{C}$, and no direction is specified, then we mean that $C$ is oriented positively (i.e. the interior of the region enclosed by $C$ is to the *left* of the tangent vector tracing out $C$). This allows us to write statements like

$$\oint_C f(z)\,\mathrm{d}z, \qquad \text{where } C \text{ is a regular hexagon centred at } 0,$$

quickly and without ambiguity. In this article, we have made every effort to stick to formal notations without obscuring the details. However, people in the real world rarely write things out to the degree of detail as we have in this article.

This concludes our introduction to contour integration. This is perhaps one of the most powerful integration techniques, allowing us to compute many tricky integrals. Contour integration comes up in the computation and inversion of Fourier and Laplace transforms, and plays a key role in the method of steepest descent for asymptotically expanding integrals. Applications aside, we hope this allows the reader to perform better in integration competitions. If nothing else, this is something you could use to impress others at a party.

> What is the value of the contour integral around all the countries which have won the Eurovision Song Contest? Zero, because the Poles have yet to win.

## 3 Exercises

**Exercise 3.1.** Consider the integral

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} \, \mathrm{d}x,$$

which appears in Example 2.1. Evaluate this integral by setting up a contour which travels around the pole $-i$ (instead of around $i$ as in Example 2.1). You should still obtain $\pi$ as the value of this integral.

**Exercise 3.2.** Consider the integral

$$\int_{-\infty}^{\infty} \frac{1}{z^2-3iz-2},$$

which appears in Example 2.9. Verify that we still obtain a value of 0 if we computed this integral by forming a semicircle contour in the *upper*-half of the complex plane.

Now, for $n \in \mathbb{Z}_{\geq 2}$, let $z_1, \ldots, z_n \in \mathbb{C}$ be distinct with $\mathbf{Im}(z_j) > 0$ for all $j \in \{1, \ldots, n\}$. Show that

$$\sum_{j=1}^{n} \frac{1}{\prod_{k \neq j}(z_j - z_k)} = 0.$$

**Exercise 3.3.** Generalise Example 2.4 in the following way: for each $n \in \mathbb{Z}_{>0}$, evaluate

$$\int_{0}^{\infty} \frac{\cos(x)}{x^{2n}+1} \, \mathrm{d}x.$$

In particular, evaluate

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1} \, \mathrm{d}x.$$

**Exercise 3.4.** Generalise Example 2.6 in the following way: for each $n \in \mathbb{Z}_{\geq 2}$, show that

$$\int_{0}^{\infty} \frac{1}{x^n+1} \, \mathrm{d}x = \frac{\pi/n}{\sin(\pi/n)}.$$

**Exercise 3.5.** Show that

$$\int_{-\infty}^{\infty} \frac{x\sin(x)}{4x^2+9} \, \mathrm{d}x = \frac{\pi}{4e^{3/2}}.$$

**Exercise 3.6.** Show that

$$\int_{-\infty}^{\infty} \sin\left(e^x\right) \mathrm{d}x = \frac{\pi}{2}.$$

**Exercise 3.7.** Show that

$$\int_{0}^{\infty} e^{-x^2} \cos\left(x^2\right) \mathrm{d}x = \frac{\sqrt{\pi\left(1+\sqrt{2}\right)}}{4}.$$

*Hint:* *Consider a pie contour of angle $\frac{\pi}{8}$.*

**Exercise 3.8.** Show that

$$\int_{0}^{\infty} \sin\left(x^2\right) \mathrm{d}x = \sqrt{\frac{\pi}{8}}.$$

This is known as the *Fresnel integral*.

*Hint:* *Try computing this simultaneously with the corresponding integral for cos.*

**Exercise 3.9.** Fix any $r > 1$, and let $f\colon B_r(0) \to \mathbb{C}$ be analytic. By considering the unit circle parametrised by $\beta(t) := e^{it}$, for $t \in [0, 2\pi]$, prove that

$$\int_0^{2\pi} \mathbf{Re}\left(f\left(e^{it}\right)\right) \mathrm{d}t = 2\pi \mathbf{Re}(f(0)) \qquad \text{and} \qquad \int_0^{2\pi} \mathbf{Im}\left(f\left(e^{it}\right)\right) \mathrm{d}t = 2\pi \mathbf{Im}(f(0)).$$

Hence show that

$$\int_0^{2\pi} \exp\left(\frac{\cos(t)+4}{8\cos(t)+17}\right) \cos\left(\frac{\sin(t)}{8\cos(t)+17}\right) \mathrm{d}t = 2\pi e^{1/4}.$$

*Hint: Use the manipulation* $\cos\left(\frac{\sin(t)}{8\cos(t)+17}\right) = \cos\left(\frac{-\sin(t)}{\cos^2(t)+8\cos(t)+16+\sin^2(t)}\right)$.

Motivated by the proofs of the identities above, show that

$$\int_0^{2\pi} \exp\left(\frac{\cos(t)+4}{8\cos(t)+17}\right) \left(\cos\left(\frac{\sin(t)}{8\cos(t)+17}\right)\cos(2t) - \sin\left(\frac{\sin(t)}{8\cos(t)+17}\right)\sin(2t)\right) \mathrm{d}t$$
$$= \frac{9\pi e^{1/4}}{256}.$$

**Exercise 3.10.** Using a keyhole contour, show that

$$\int_0^{\infty} \frac{\log(x)}{x^2+2x+2}\,\mathrm{d}x = \frac{\pi \log(2)}{8}.$$

**Exercise 3.11.** Show that

$$\int_0^1 \sqrt{x(1-x)}\,\mathrm{d}x = \frac{\pi}{8}.$$

**Exercise 3.12.** Show that

$$\int_{-\infty}^{\infty} \frac{1}{(e^x - x)^2 + \pi^2}\,\mathrm{d}x = \frac{1}{1+\Omega},$$

where $\Omega$ is the unique real number satisfying $\Omega e^{\Omega} = 1$.

**Exercise 3.13.** Show that

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{\cosh(x)}\,\mathrm{d}x = \frac{\pi}{\cosh\left(\frac{\pi}{2}\right)}.$$

**Exercise 3.14.** Consider the identity

$$\int_{-1}^{1} \frac{1}{x}\sqrt{\frac{1+x}{1-x}} \log\left(\frac{2x^2+2x+1}{2x^2-2x+1}\right) \mathrm{d}x = 4\pi \operatorname{arccot}(\sqrt{\varphi}), \qquad \text{where } \varphi := \frac{1+\sqrt{5}}{2},$$

which was (in)famously stated by Cleo on Mathematics Stack Exchange. The accepted answer employs contour integration. Read it. Admire it.

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