Cohen Forcing

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1 A Countable Transitive Model of ZFC?

1.1 A Continuation from Model Theory

So you wish to prove that the continuum hypothesis (CH), the assertion that $2^{\aleph_0} = \aleph_1$, is independent of the axioms of ZFC set theory. You take a class in model theory and you learn that one can show this by showing that $\mathsf{ZFC} + \mathsf{CH}$ and $\mathsf{ZFC} + \neg\mathsf{CH}$ are both consistent. The typical approach is to exhibit models of $\mathsf{ZFC} + \mathsf{CH}$ and $\mathsf{ZFC} + \neg\mathsf{CH}$. So let us try to do that.

Stop. You realise that a model of $\mathsf{ZFC} + \mathsf{CH}$ (or $\mathsf{ZFC} + \neg\mathsf{CH}$) will, in particular, also be a model of ZFC. Darn. The whole point of ZFC set theory was that it was supposed to be able to formalise (most of)[1] the mathematics we do in our day-to-day lives. A certain pesky Kurt Gödel prevents us from explicitly exhibiting such a model.

But who is going to stop us from simply running off with the assumption that ZFC is consistent?[2] If we do this, then we can hope to obtain a theorem and proof of the following form.

Theorem
If ZFC *is consistent, then* $\mathsf{ZFC} + \mathsf{CH}$ *and* $\mathsf{ZFC} + \neg\mathsf{CH}$ *are consistent.*

Proof. Let M be a model of ZFC. [Black magic]. Therefore we have obtained a model M' of $\mathsf{ZFC} + \mathsf{CH}$ and a model M'' of $\mathsf{ZFC} + \neg\mathsf{CH}$. $\square$

In such a proof as above, M' and M'' will presumably be created from M. But at this point, we have no information about M, other than that it models ZFC. Our goal is to improve this model M to a *countable transitive* model of ZFC. By "transitive", we mean transitive with respect to the membership relation $\in$. That is, if $x \in y \in M$, then $x \in M$.

To get a countable model of ZFC is fairly easy. Recall the Löwenheim–Skolem theorem, asserting that theories in first-order logic are unable to control the cardinalities of their infinite models.

Theorem (The Löwenheim–Skolem Theorem)
Let T be a consistent $\mathcal{L}$-theory. Suppose there exists an infinite model of T. Then for all cardinals $\kappa \geq |\mathcal{L}| + \aleph_0$, there exists a model of T of cardinality κ.

A special case of this theorem, which also arises from the proof of Gödel's completeness theorem via Henkin terms, is that if a language $\mathcal{L}$ is countable, then a consistent $\mathcal{L}$-theory T has an infinite model if and only if it has a countably infinite model.

In particular, as any model of ZFC must be infinite, there must also exist a countable model of ZFC.

1.2 Only a Sith Deals in Absolutes

Why do we even want to get a countable transitive model of ZFC in the first place?

We want a countable model so we are able to access things from *outside* of the model. This lets adjoin new elements, such as real numbers, to the model, because there are uncountably many real numbers out in the metatheory. This is not unlike a field extension.

Transitive models are desirable due to the fact that they make a lot of formulas "absolute".

Definition
Let M and N be $\mathcal{L}$-structures with $M \subseteq N$ and let $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}$-formula with n free variables. We say that $\varphi(x_1, \ldots, x_n)$ is <u>absolute between M and N</u> if, for all $a_1, \ldots, a_n \in M$,

$$M \models \varphi(a_1, \ldots, a_n) \textit{ if and only if } N \models \varphi(a_1, \ldots, a_n).$$

[1] Category theory jumpscare.

[2] If you are an amused reader from the future with the knowledge that ZFC is inconsistent, how is the climate doing? Thought so. Focus on your own problems.

Replacing "if and only if" in the definition above with "if" gives us the notion of downwards absoluteness, whereas replacing "if and only if" with "only if" gives us the notion of upwards absoluteness. Formulas which are absolute between structures M and N, with $M \subseteq N$, are great because they really do let us view N as a certain kind of extension of M.

Atomic formulas are always absolute between M and N whenever M is a substructure of N. This is pretty much by definition: an $\mathcal{L}$-structure M is a substructure of an $\mathcal{L}$-structure N if the domain of M is a subset of the domain of N and the inclusion function $\iota \colon M \to N$ is an injective homomorphism of $\mathcal{L}$-structures. Consequently, Boolean combinations of atomic formulas are also always absolute between substructures.

But things become icky when we introduce quantifiers. We cannot simply "add" things to a model and expect it to preserve the truth of formulas in the original structure.

Example

Consider the language $\mathcal{L}_\in$ of set theory, which only consists of one relation symbol $\in$, and has no constant symbols or function symbols. So the only atomic formulas are of the form $x = y$ or $x \in y$, for variables x and y. All the "interesting" formulas are not going to be Boolean combinations of atomic formulas.

Let M be a model of ZFC and let $\varnothing$ be the empty set in M. We extend M by letting $N \coloneqq M \cup \{*\}$ declaring $* \in^N \varnothing$. Then

$$M \models \forall z.(z \notin \varnothing),$$

but

$$N \not\models \forall z.(z \notin \varnothing).$$

Thus even very simple formulas, such as $\varphi_\varnothing(x) \coloneqq \forall z.(z \notin x)$ asserting that x is empty, are not absolute between M and N. □

The idea is that if M and N are both *transitive* models of ZFC with $M \subseteq N$, then lots of formulas are absolute between M and N, and so N can be viewed as a particularly neat extension of M.

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