Application of forcing: Suslin's problem

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1 Introduction

1.1 Suslin's problem

A large part of mathematics is about classifying objects up to some natural notion of isomorphism. In group theory, one is often interested in knowing whether there exists a bijective group homomorphism between two given groups. In linear algebra, the corresponding question would be to ask if there exists a bijective linear map between two given vector spaces. In topology, this manifests as a question of whether two topological spaces are homeomorphic (or homotopy equivalent). Order theory is no exception. Two partially ordered sets are said to be *order-isomorphic* if there exists an order-preserving bijection between them.

The set $\mathbb{Q}$ of rational numbers with its usual ordering has an order-theoretic characterisation.

Fact 1.1.1 (Cantor (1895, Section 9))
Any non-empty countable dense unbounded linear order is order-isomorphic to $\mathbb{Q}$.

By *countable*, we mean finite or countably infinite. As a consequence, one obtains the following order-theoretic characterisation of the set $\mathbb{R}$ of real numbers under its usual ordering.[1]

Fact 1.1.2 (Cantor (1895, Section 11))
Any non-empty complete dense unbounded separable linear order is order-isomorphic to $\mathbb{R}$.

Mikhail Yakovlevich Suslin, in Suslin et al. (1920, Problem 3), asked if the characterisation of $\mathbb{R}$ as a non-empty complete dense unbounded separable linear order could be reformulated by using the weaker property

"every collection of non-empty pairwise disjoint open intervals is countable"

in place of the word "separable".[2] Today, this question is known as *Suslin's problem*.

The modern formulation of Suslin's problem is as follows.

Definition 1.1.3
A Suslin line *is a non-empty complete dense unbounded linear order* $\langle L, \prec \rangle$ *satisfying both of the following properties:*

1. *every collection of pairwise disjoint open intervals in* L *is countable;*
2. *the linear order* L *is not separable, i.e. there does not exist a countable dense subset of* L.

Suslin's problem *asks if a Suslin line exists.*

The original formulation of Suslin's problem was published in Suslin et al. (1920, Problem 3).[3] This was a paper consisting of ten problems in set theory, nine of which went on to be solved (Kanamori, 2011, p. 2) while Suslin's problem was shown to be not be solvable.

One often sees the phrase "Suslin's hypothesis (SH)" for the assertion that no Suslin lines exist. Mikhail Suslin, however, did not hypothesise it (Kanamori, 2011, p. 1); this name likely caught on due to the similarity of the nature of Suslin's problem with the continuum hypothesis (CH). As with CH, if ZFC is consistent, then the existence of a Suslin line is independent of ZFC.

[1] Unrelatedly, there is another characterisation of the linear order $\mathbb{R}$ if we further take into account the canonical field structure on $\mathbb{R}$: Huntington (1903, Theorem II') showed that any complete linearly ordered field is isomorphic (simultaneously in the order-theoretic sense and the algebraic sense) to the linearly ordered field $\mathbb{R}$.

[2] Specifically, Suslin et al. (1920, Problem 3) wrote *"Un ensemble ordonné (linéairement) sans sauts ni lacunes et tel que tout ensemble de ses intervalles (constenant plus qu'un élément) n'empiétant pas les uns sur les autres est au plus dénombrable, est-il nécessairement un continu linéaire (ordinaire)?"*, which Igoshin (1996, Section 6) translates to *"Let a (linearly) ordered set without gaps and jumps possess the property that every set of disjoint non-empty intervals (containing more than one element) is at most countable. Will this set necessarily be an (ordinary) linear continuum?"*.

[3] This paper was published in the year after the unfortunate death of Mikhail Suslin in 1919 due to illness (Igoshin, 1996, Section 5).

1.2 Roadmap

The first step to addressing Suslin's problem is to turn it into a problem of combinatorial flavour. Kurepa (1935, Section 12.D), and independently Miller (1943), made the first steps to solving Suslin's problem by turning it into the problem of the existence of *Suslin trees*. This is what we do in Section 1.3, adapting the proofs from Jech (2003, Lemma 9.13 and Lemma 9.14) and Halbeisen (2017, Lemma 20.3).

The independence of the existence of Suslin trees from ZFC will then be approached via the method of forcing, introduced by Cohen (1963) to establish the independence of CH from ZFC. We will lay out the definitions and notations we will use in Section 1.4, but we will assume that the reader is familiar with an introductory level of the theory of forcing as in Kunen (1980, Chapter VII but *not* Chapter VIII). This includes familiarity with preservation of cardinals in the generic extensions by working with forcing notions satisfying countable (anti)chain or closure conditions. However, we will be adopting notation as in Karagila (2023). Most notably, names of forcing notions will be denoted with dots or checks above letters, e.g. $\dot{x}$, $\dot{A}$, $\check{\alpha}$, instead of Greek letters.

Section 2 will be devoted to forcing the existence of Suslin trees. We will give two different forcing proofs of this result. The first proof, in Section 2.1, will be the original consistency proof by Tennenbaum (1968) which uses a forcing notion of finite approximations to a Suslin tree. The second method, in Section 2.2, is to show that a certain combinatorial principle called the *diamond principle* implies the existence of a Suslin tree and that the diamond principle is consistent with ZFC.

To show the consistency of the non-existence of Suslin trees with ZFC will require us to develop the theory of *iterated forcing*. This will be the goal of Section 3. We first see how to kill a Suslin tree in Section 3.1 and then see how to iterate this process any finite number of times in Section 3.2 in a single generic extension. All this will serve as motivation, as well as providing some technical lemmas, for Section 3.3 where we work with *α-stage finite-support iterated forcings* to produce a model in which there does not exist any Suslin trees.

Finally, Section 4.1 discusses the independence of the existence of Suslin trees from $\mathsf{ZFC} + \mathsf{CH}$ as well as $\mathsf{ZFC} + \neg\mathsf{CH}$, and Section 4.2 mentions a couple other possible proofs of the consistency results we have established.

1.3 Suslin trees

Definition 1.3.1
A partially ordered set $\langle T, \prec\rangle$ is a tree *if there is exactly one $\prec$-maximum element in T and, for all $x \in T$, the set $\{\, y \in T : y \succ x \,\}$ is linearly and well ordered by $\succ$, i.e., for all $x \in T$, every non-empty subset of $\{\, y \in T : y \succ x \,\}$ has a maximum element.*

In Section 1.4, we will follow the convention that a forcing notion is a partially ordered set with a *maximum* element. It is for this reason that we define our trees to start from the top and grow downwards, as opposed to trees in real life. Many of our forcing notions in Section 3 will be trees.

We supplement the definition of a tree with a series of standard terms. Fix a tree $\langle T, \prec\rangle$. Elements of T are called *nodes*. The *field* of $\langle T, \prec\rangle$ is the set T of all nodes in the tree. A *successor* of a node $x \in T$ is simply a node $y \prec x$; an *immediate successor* of a node $x \in T$ is a node $y \prec x$ such that there does not exist $z \in T$ with $y \prec z \prec x$. Dually, we have the notions of *predecessors* and *immediate predecessors* of nodes. The *root* of T is the (unique) node in T which has no predecessor; a *leaf* of T is a node in T which has no successor. For an ordinal α, we define the *αth level* of T to be the set

$$\big\{\, x \in T \,:\, \{y \in T : y \succ x\} \text{ has order-type } \alpha \,\big\}.$$

The *height* of T is the least ordinal α such that the αth level of T is empty. A *chain* in T is a subset of T which is linearly ordered by $\prec$. In contrast, an *antichain* in T is a subset $A \subseteq T$ such that, for any two distinct $x, y \in A$, we have $x \not\prec y$ and $y \not\prec x$. A *branch* in T is a maximal linearly $\prec$-ordered subset of T.

With all this terminology, we can state the definition of a Suslin tree. These are particular types of trees which will be used in the reformulation of Suslin's problem which will be used in showing the independence of the existence of Suslin lines from ZFC.

Definition 1.3.2
A Suslin tree *is a tree* $\langle T, \prec \rangle$ *such that all of the following hold:*

1. *the height of* T *is* ω_1*;*
2. *every chain in* T *is countable;*
3. *every antichain in* T *is countable.*

Note that for every ordinal α, the αth level of a Suslin tree is countable, since the αth level of any tree is an antichain in that tree.[4]. In showing that the existence of a Suslin line is equivalent to the existence of a Suslin tree, it is useful to add to add further properties to Suslin trees to obtain *normal Suslin trees.*

Definition 1.3.3
We say that a tree $\langle T, \prec \rangle$ *of height* $\alpha \leq \omega_1$ *is* normal *if all of the following hold:*

1. *every level of* T *is countable;*
2. *every non-leaf node in* T *has* $\aleph_0$*-many immediate successors;*
3. *for all ordinals* $\beta < \gamma < \alpha$ *and for all nodes* x *in the* β*th level of* T*, there exists a node* $y \prec x$ *in the* γ*th level of* T*;*
4. *for all limit ordinals* $\beta < \alpha$*, if* x *and* y *are in the* β*th level of* T *and have the same set of predecessors, i.e.* $\{ z \in T : z \succ x \} = \{ z \in T : z \succ y \}$*, then* $x = y$.

Normal trees are sometimes also called *well-pruned trees*; we shall see in the proof of Theorem 1.3.4 that we can obtain a normal Suslin tree from a Suslin tree by first cutting off the undesired branches. One of the reasons why normal trees are desirable to work with because they allow us to climb to as high of a level as we want regardless of our position in the tree. We will see a use of this in Lemma 3.1.1 later. Combining Definition 1.3.2 and Definition 1.3.3, we see that a normal Suslin tree is precisely a Suslin tree $\langle T, \prec \rangle$ with the following three additional properties:

1. for all ordinals $\alpha < \beta < \omega_1$ and for all nodes x in the αth level of T, there exists a node $y \prec x$ in the βth level of T;
2. for all limit ordinals $\alpha < \omega_1$, if x and y are in the αth level of T and have the same set of predecessors, then $x = y$;
3. every node in T has $\aleph_0$-many immediate successors.

We now show that the existence of a Suslin line is equivalent to the existence of a Suslin tree. This was discovered by Kurepa (1935, Section 12.D) and later independently discovered by Miller (1943). In fact, we further show that the existence of these two objects will also yield the existence of a normal Suslin tree.

[4] An *Aronszajn tree* is a tree of height ω_1 all of whose levels are countable and chains are countable. We see that every Suslin tree is an Aronszajn tree. In contrast to Suslin trees, ZFC proves that there exist Aronszajn trees.

Theorem 1.3.4 (Kurepa (1935, Section 12.D) and Miller (1943))
The following three assertions are equivalent:

1. *there exists a Suslin line;*
2. *there exists a Suslin tree;*
3. *there exists a normal Suslin tree.*

Proof. First, we show that the existence of a Suslin line implies the existence of a Suslin tree. Let $\langle L, \prec\rangle$ be a Suslin line. Choose points $\{a_\alpha\}_{\alpha<\omega_1}$ and $\{b_\alpha\}_{\alpha<\omega_1}$ as follows:

> for an ordinal $\alpha < \omega_1$, since the countable set $C_\alpha \coloneqq \{ a_\beta : \beta < \alpha \} \cup \{ b_\beta : \beta < \alpha \}$ is not dense in L, choose $a_\alpha, b_\alpha \in L$ such that $a_\alpha \prec b_\alpha$ and $C_\alpha \cap [a_\alpha, b_\alpha] = \varnothing$.

Then define

$$T \coloneqq \{ (a_\alpha, b_\alpha) : \alpha < \omega_1 \} \cup \{L\}.$$

That is, T consists of all the open intervals (a_α, b_α), for $\alpha < \omega_1$, together with the entire line L. Order T under the strict inclusion relation $\subsetneq$. We claim that $\langle T, \subsetneq\rangle$ is a Suslin tree.

By construction, this tree will have height at most ω_1. Furthermore, we have chosen our points $\langle a_\alpha, b_\alpha\rangle$ such that if $\alpha \neq \beta$ then either

$$(a_\alpha, b_\alpha) \cap (a_\beta, b_\beta) = \varnothing \quad \text{or} \quad (a_\alpha, b_\alpha) \subsetneq (a_\beta, b_\beta) \quad \text{or} \quad (a_\alpha, b_\alpha) \supsetneq (a_\beta, b_\beta).$$

So an antichain in $\langle T, \subsetneq\rangle$ is a collection of pairwise disjoint open intervals in $\langle L, \prec\rangle$. So every antichain in T will be countable since L is a Suslin line. Finally, if we have an uncountable chain $\langle (a_\alpha, b_\alpha)\rangle_{\alpha<\omega_1}$ in T, then we can arrange it so that $\langle a_\alpha\rangle_{\alpha<\omega_1}$ is a strictly increasing sequence. Then the intervals $\langle a_\alpha, a_{\alpha+1}\rangle_{\alpha<\omega_1}$ is an uncountable collection of pairwise disjoint open intervals in L, contradicting that L is a Suslin line. So every chain in T is countable. Therefore T is a Suslin tree.

Next, we show that the existence of a Suslin tree implies the existence of a normal Suslin tree. Let $\langle T, \prec\rangle$ be a Suslin tree. We build three new Suslin trees T_1, T_2, and T_3 progressively satisfying each of the three additional properties of normal Suslin trees.

1. Define $T_1 \coloneqq \big\{ x \in T : \{ y \in T : y \prec x \} \text{ has cardinality } \aleph_1 \big\}$.

2. Let $T_{1.5} \coloneqq \big\{ x \in T_1 : x \text{ has at least 2 immediate successors} \big\}$. We may assume that $T_{1.5}$ has a (unique) maximum element, otherwise we can add one in. Let x_0 denote the root of $T_{1.5}$. For each $\prec$-interval $(t, x_0] \subseteq T_{1.5}$ of some limit ordinal length α, insert a new node $a_{(t,x_0]}$ in between $(t, x_0]$ and all the nodes in the αth level of $T_{1.5}$ (such as t) which have the interval $(t, x_0]$ as their set of predecessors. Slightly more formally, we declare:

   (a) for all $x \in T_{1.5}$, declare $a_{(t,x_0]} \prec x$ if and only if $t \prec x$;
   (b) for all $y \in T_{1.5}$, declare $y \prec a_{(t,x_0]}$ if and only if, for all $x \succ t$, we have $y \prec x$.

   In particular, $t \prec a_{(t,x_0]} \prec x$ for all $x \succ t$. Then define T_2 to be the resulting tree after adding all these new nodes $a_{(t,x_0]}$ from this construction.

3. Define $T_3 \coloneqq \{x_0\} \cup (T_2 \setminus T_{1.5})$. In words, T_3 consists of the root of T_2 and all the limit levels of T_2.

Then $\langle T_3, \prec\rangle$ is a normal Suslin tree.

Finally, we show that the existence of a normal Suslin tree implies the existence of a Suslin line. Let $\langle T, \prec\rangle$ be a normal Suslin tree. We let $L \coloneqq \{\, B \subseteq T \,:\, B \text{ is a branch in } T \,\}$, recalling that branches are *maximal* chains. Each $x \in T$ has countably infinitely many immediate successors, because T is a normal Suslin tree, so we can fix a bijection

$$\sigma_x \colon \{\, y \prec x \,:\, y \text{ is an immediate successor of } x \,\} \to \mathbb{Q}$$

for each $x \in T$. We define an ordering $\lhd$ on L as follows:

> For any two distinct $B_1, B_2 \in L$, as T is a normal Suslin tree, B_1 and B_2 will first differ at a successor level of T. So let α be the least ordinal such that B_1 and B_2 differ in the $(\alpha+1)$th level of T. The αth level of B_1 and B_2 contain the same element, say x_0. Let the $(\alpha+1)$th level of B_1 and B_2 be $\{x_1\}$ and $\{x_2\}$ respectively. Then declare
>
> $$B_1 \lhd B_2 \text{ if and only if } \sigma_{x_0}(x_1) <_{\mathbb{Q}} \sigma_{x_0}(x_2).$$

In other words, we used the family of bijections $\{\sigma_x\}_{x\in T}$ to define a lexicographical ordering $\lhd$ on L.

This linear order $\langle L, \lhd\rangle$ is evidently a non-empty, dense, and unbounded linear order. If we have a collection of pairwise disjoint open intervals in L, then we can choose a point in each of those intervals and obtain an antichain in T. As every antichain in T is countable, we get that every collection of pairwise disjoint open intervals in L is countable. To see that L is not separable, given any countable collection $\mathscr{B}$ of branches in T, as they are all countable and as T is a normal Suslin tree, there must exist some $\alpha < \omega_1$ which is bigger than the height of all of the branches in $\mathscr{B}$ and some $x \in T$ at height α. Then the collection of all branches containing x is an open interval which does not contain any of the branches in $\mathscr{B}$.

It remains to ensure that our linear order $\langle L, \lhd\rangle$ is complete. If $\langle L, \lhd\rangle$ is complete, then it is a Suslin line and we are done. If not, then we can can form a Suslin line $\langle \tilde{L}, \sqsubset\rangle$, where the elements of $\tilde{L}$ are Dedekind cuts in L, and $\sqsubset$ is the usual ordering of Dedekind cuts. $\square$

1.4 Forcing conventions

In this subsection, we lay out the conventions and notations for forcing which we will use in the rest of this document. Fix a countable transitive model $\mathbf{M}$ of ZFC.

Definition 1.4.1
A forcing notion *is a partially ordered set (that is, a set equipped with a binary relation which is reflexive, antisymmetric, and transitive)* $\langle \mathbb{P}, \preceq\rangle$ *which has a* weakest condition $\mathbb{1}_{\mathbb{P}} \in \mathbb{P}$ *such that* $p \preceq \mathbb{1}_{\mathbb{P}}$ *for all* $p \in \mathbb{P}$.

Elements of a forcing notion $\mathbb{P}$ are called *(forcing) conditions*. When $p, q \in \mathbb{P}$ are such that $p \preceq q$, we say that p *is stronger than* q. For $p, q \in \mathbb{P}$, if there exists $r \in \mathbb{P}$ which is stronger than both p and q, then we say that p and q are *compatible*; if no such r exists, we say that p and q are *incompatible*. If $\mathbb{P} \in \mathbf{M}$ is a forcing notion, we say that a subset $D \subseteq \mathbb{P}$ is *dense* in $\mathbb{P}$ if for every $p \in \mathbb{P}$ there exists $d \in D$ with $d \preceq p$. We say that a subset $A \subseteq \mathbb{P}$ is an *antichain*[5] in $\mathbb{P}$ if every pair of distinct conditions in $\mathbb{P}$ are incompatible. A filter $G \subseteq \mathbb{P}$ is said to be a $\mathbb{P}$-*generic filter over* $\mathbf{M}$ if for every dense subset $D \subseteq \mathbb{P}$ with $D \in \mathbf{M}$, we have that $G \cap D \neq \varnothing$.

[5] This is not the same as the definition of an antichain in a tree concerning distinct pairwise incompa*ra*ble elements. However, if our forcing notion is actually a tree, then these two notions coincide.

When we go into the theory of iterated forcing later in Section 3, our forcing notions will at first be preorders and *not*[6] partial orders. We will, however, be identifying two conditions $p, q \in \mathbb{P}$ with each other if $p \preceq q \preceq p$ and then work with the quotiented preorders to obtain a partial orders.

Definition 1.4.2
For a forcing notion $\mathbb{P}$, a $\mathbb{P}$-name *is a set $\dot{x}$ such that every $z \in \dot{x}$ is of the form $z = \langle \dot{y}, p\rangle$, where $p \in \mathbb{P}$ and $\dot{y}$ is a $\mathbb{P}$-name. For a $\mathbb{P}$-name $\dot{x}$, we write $\operatorname{dom}(\dot{x}) \coloneqq \{\, \dot{y} : \langle \dot{y}, p\rangle \in \dot{x} \text{ for some } p \in \mathbb{P} \,\}$ for the set of $\mathbb{P}$-names which appear in some ordered pair in $\dot{x}$. For any $x \in \mathbf{M}$, the* canonical $\mathbb{P}$-name for x *is*

$$\check{x} \coloneqq \{\, \langle \check{y}, \mathbb{1}_{\mathbb{P}} \rangle : y \in x \,\}.$$

Definition 1.4.3
If $\mathbb{P} \in \mathbf{M}$ is a forcing notion, G is a $\mathbb{P}$-generic filter over $\mathbf{M}$, and $\dot{x} \in \mathbf{M}$ is a $\mathbb{P}$-name, we define the evaluation of $\dot{x}$ under G *to be*

$$\dot{x}^G \coloneqq \{\, \dot{y}^G \ : \ \dot{y} \text{ is a } \mathbb{P}\text{-name and there exists } p \in G \text{ with } \langle \dot{y}, p\rangle \in \dot{x} \,\}$$

We then define $\mathbf{M}[G] \coloneqq \{\, \dot{x}^G : \dot{x} \in \mathbf{M} \text{ is a } \mathbb{P}\text{-name} \,\}$.

Definition 1.4.4
For a forcing notion $\mathbb{P} \in \mathbf{M}$, we define the forcing relation $\Vdash_{\mathbf{M},\mathbb{P}}$ as follows: for a condition $p \in \mathbb{P}$, for $\mathbb{P}$-names $\dot{x}_1, \dots, \dot{x}_n$, and for a formula $\varphi(x_1, \dots, x_n)$, we write

$$p \Vdash_{\mathbf{M},\mathbb{P}} \varphi(\dot{x}_1, \dots, \dot{x}_n)$$

if, for any $\mathbb{P}$-generic filter G over $\mathbf{M}$ with $p \in G$, we have that

$$\mathbf{M}[G] \models \varphi(\dot{x}_1^G, \dots, \dot{x}_n^G).$$

When this happens, we say that p forces *$\varphi(\dot{x}_1, \dots, \dot{x}_n)$.*

The subscript "$\mathbf{M}, \mathbb{P}$" in the forcing relation will dropped when forcing the existence of a Suslin tree in Section 2 because the forcing notion we are working with will be clear from context. As we venture into iterated forcing in Section 3, it will become important to specify which forcing notion we are working with.

There are certain combinatorial properties of forcing notions which, when satisfies, will let us conclude that certain cardinals are preserved when moving to a generic extension.

Definition 1.4.5
Let $\langle \mathbb{P}, \preceq, \mathbb{1}_{\mathbb{P}} \rangle \in \mathbf{M}$ be a forcing notion. We say that $\mathbb{P}$ satisfies the countable chain condition (c.c.c.) *if $\mathbf{M} \models$ "every antichain in $\mathbb{P}$ is countable". We say that $\mathbb{P}$ is* countably closed *if the following holds in $\mathbf{M}$: for all $\alpha < \omega_1$ and for all α-sequences of conditions $\langle p_\eta \rangle_{\eta < \alpha}$ ordered in increasing strength (i.e. if $\eta \leq \eta'$ then $p_{\eta'}$ is stronger than p_η) there exists some $q \in \mathbb{P}$ which is stronger than every p_η.*

[6]This is paraphrased from Kanamori (2003, the "Forcing Preliminaries" section of Chapter 10)

2 Making Suslin trees

Our main interest in this section is the following consistency result by Tennenbaum (1968, Theorem 1): if ZFC is consistent, then ZFC + "there exists a Suslin tree" is also consistent. This was perhaps one of the first[7] applications of forcing after Cohen (1963) discovered it. We will force the existence of Suslin trees in Section 2.1 à la Tennenbaum (1968).

Several years after Stanley Tennenbaum forced the existence of a Suslin tree, Jensen (1972, Theorem 6.2) showed that the existence of a Suslin tree follows from a certain combinatorial principle called the *diamond principle* ($\diamondsuit$). Jensen (1972, Lemma 6.5) then went on to show that ZFC together with the axiom of constructibility $\mathbf{V}=\mathbf{L}$ can prove $\diamondsuit$, establishing the consistency of the existence of Suslin trees without the use of forcing.

The fact that $\mathbf{V}=\mathbf{L}$ proves that there exists a Suslin tree is rather intriguing; one could be forgiven for thinking that $\mathbf{L}$ is a particularly nice model of ZFC and that it is surprising that a pathological object such as a Suslin tree could be made in $\mathbf{L}$. Was this a surprise to set theorists at the time?

> *"I think the answer [to whether Jensen's proof that there is a Suslin tree in* $\mathbf{L}$ *came as a surprise] is Yes. Someone told me that when Solovay learned that Jensen had proved SH false in L, he said "Damn!"*
>
> . . .
>
> *Jensen and I were in California from July 1967 for nine months (for the four-week AMS Summer School at UCLA, and then for two quarters at Stanford (at Dana Scott's invitation)). Then we both returned to Bonn, where I was till August 8th. Whilst in Bonn, Jensen asked me to check a proof, that SH is false in L. I found his proof was correct.*
>
> . . .
>
> *Kunen had invited me to Madison for the academic year 1968–69; whilst I was in Madison, Jensen sent both of us (and many other people, I expect) a manuscript paper formulating diamond and construction a Souslin tree from it. So it was an axiomatizion of his earlier proof.*
>
> . . .
>
> *One evening during the UCLA Summer School, people wrote on a blackboard statements of all the new results they knew about and Dana Scott copied them into a notebook. Back at Stanford he handed me that notebook and asked me if I could turn it into a survey. So I set to work, and circulated the first manuscript draft around Christmas 1967. Various people sent me corrections additions and comments; Solovay was complementary, as was Chang.*
>
> *I then revised the survey, to give the second manuscript draft, and Dana's secretary typed it (about the time I left Stanford to return to Bonn). The title was "A survey of recent results in set theory". [T]hen the trouble started as people kept sending me news of new results, which I tried to include in a new draft. Eventially in 1974 when i was in Vancouver for the ICM, Galvin suggested I should change the title to "A survey of old results in set theory", and Hajman said he would publish it in Periodica Hungarica. So I stopped adding totally new results but listed new results arising from questions in the 1968 version.*
>
> *I see that in the 1968 version, whether V=L decides SH is listed as an open problem (P1229). So Jensen's construction was probably done in late June or early July 1968."* — Mathias (2025).

[7] Tennenbaum (1968, Footnote *) said that the consistency of the existence of Suslin trees with ZFC was discovered in the summer of 1963. For comparison, Cohen (1963) was only submitted for publication in September 1963.

The survey in the quote above is in reference to Mathias (1979). It is, however, worth noting that in a personal communication to the author of this essay, Solovay (2025) said that he *"ha[s] no clear memory on [whether he was surprised by Jensen's proof that* $\mathbf{L}$ *has a Suslin tree]"*.

As a rather fun historical side remark, Hamkins (2015) wrote the following personal recollection: the symbol $\diamondsuit$ for the diamond principle was chosen because Ronald Jensen *"simply need[ed] a new symbol that had not yet been used"*. Löwe (2025b) has the following personal recollection: *"I recall the following story from Jensen himself (he told me the story in the late 1990s): his papers during that period were typed using typewriters with removable maths daisy wheels. There was a special wheel for mathematical symbols which happened to have diamond and box symbols on it that Jensen didn't use for anything else, so he used them for his combinatorial principles."*

In Section 2.2 we present the proof of the existence of a Suslin tree from $\diamondsuit$, and we then go on to force the consistency of $\mathsf{ZFC} + \diamondsuit$.

The proof of the consistency of the existence of Suslin trees we present in Section 2.1 is adapted from lectures by Hart and Löwe (2021, Lecture 15). In Section 2.2, the proof that $\diamondsuit$ implies the existence of Suslin trees is adapted from Jech (2003, Theorem 15.26), and the proof of the consistency of $\diamondsuit$ in Section 2.2 follows lectured material by Kumar (2023, Section 22).

2.1 Forcing with finite trees

As usual, fix a countable transitive model $\mathbf{M}$ of ZFC. We will use the following forcing notion as used by Tennenbaum (1968, Definition 2). In $\mathbf{M}$, let $\mathbb{P}$ be the forcing notion consisting of all finite trees $\langle t, \prec_t \rangle$, with $t \subseteq \omega_1$, such that

$$\text{if } \alpha \prec_t \beta \text{ then } \alpha > \beta.$$

For $\langle t_1, \prec_{t_1} \rangle, \langle t_2, \prec_{t_2} \rangle \in \mathbb{P}$, we stipulate that the condition $\langle t_1, \prec_{t_1} \rangle$ is stronger than the condition $\langle t_1, \prec_{t_1} \rangle$ if and only if

$$t_1 \supseteq t_2 \quad \text{and} \quad \prec_{t_2} = \prec_{t_1} \cap (t_2 \times t_2).$$

Finally, we let G be a $\mathbb{P}$-generic filter over $\mathbf{M}$.

Observe that, for any ordinal $\alpha \in \omega_1^{\mathbf{M}}$, the set $D_\alpha \coloneqq \{ \langle t, \prec_t \rangle \in \mathbb{P} : \alpha \in t \}$ is dense in $\mathbb{P}$, and so G intersects D_α. We can thus take the union over all trees in G to produce a tree T_G which extends all the trees in G, and the field of this T_G will be all of $\omega_1^{\mathbf{M}}$. More formally, define $T_G \coloneqq \langle \omega_1^{\mathbf{M}}, \prec_{T_G} \rangle$ by $\prec_{T_G} \coloneqq \bigcup_{\langle t, \prec_t \rangle \in G} \prec_t$. This $T_G \coloneqq \langle \omega_1^{\mathbf{M}}, \prec_{T_G} \rangle$ is indeed a tree; the well-foundedness of the relation $\prec_{T_G}$ follows from the well-foundedness of the relation $<$ for ordinals.

Our aim is to show that

$$\mathbf{M}[G] \models \text{"}T_G \text{ is a Suslin tree"}.$$

We will make use of the following Lemma 2.1.1, commonly called the *Δ-system lemma.*

Lemma 2.1.1 (The Δ-system lemma; Shanin (1946))
Let W be an uncountable collection of finite sets. Then there exists an uncountable $Z \subseteq W$, called a Δ-system *and a finite set R, called the* root of the Δ-system, *such that*

$$\textit{for any } A, B \in Z, \ \textit{if } A \neq B \ \textit{then } A \cap B = R.$$

We now begin to show that $\mathbf{M}[G] \models$ "T_G is a Suslin tree". First, we show that the forcing notion $\mathbb{P}$ of finite trees satisfies c.c.c. and so we have that all cardinals are preserved when moving from the ground model $\mathbf{M}$ to the generic extension $\mathbf{M}[G]$.

Lemma 2.1.2 (Tennenbaum (1968, Lemma 2))
The forcing notion $\mathbb{P}$ satisfies c.c.c.

Proof. Let $\{\langle t_j, \prec_{t_j}\}_{j\in J}$ be an uncountable collection of trees in $\mathbb{P}$. With the pigeonhole principle, we may assume, without loss of generality, that there exists $n \in \omega$ such that $|t_j| = n$ for all $j \in J$. The Δ-system lemma 2.1.1 lets us further assume that there exists a finite set R such that for any $j_1, j_2 \in J$,

$$\text{if } j_1 \neq j_2 \text{ then } t_{j_1} \cap t_{j_2} = R.$$

Now, for $j \in J$, let $\langle \alpha_{j,i} : 0 \leq i < n\rangle$ be a strictly increasing sequence of the ordinals which appear in the tree t_j. Let $I_j \coloneqq \{\, i \in \{0, \ldots, n-1\} : \alpha_{j,i} \in R \,\}$ be the set of indices for the ordinals in R in this sequence. Let $\lhd_j$ be the tree structure on $\{0, \ldots, n-1\}$ defined by

$$i_1 \lhd_j i_2 \text{ if and only if } \alpha_{j,i_1} \prec_{t_j} \alpha_{j,i_2},$$

for all $i_1, i_2 \in \{0, \ldots, n-1\}$, which is the order induced by t_j on $\{0, \ldots, n-1\}$.

The assignment $J \ni j \mapsto \langle I_j, \lhd_j\rangle \in \mathcal{P}(n) \times \mathcal{P}(n \times n)$ is a mapping from an uncountable set J into the finite set $\mathcal{P}(n) \times \mathcal{P}(n \times n)$. So we may assume, without loss of generality, that there exists $\langle I, \lhd\rangle \in \mathcal{P}(n) \times \mathcal{P}(n \times n)$ such that, for all $j \in J$, we have $\langle I_j, \lhd_j\rangle = \langle I, \lhd\rangle$.

Finally, as $\{t_j\}_{j\in J}$ form an uncountable Δ-system with root R, there must exist uncountably many $j \in J$ such that $\max(R) < \min(t_j \setminus R)$. So, once more, without loss of generality, we may assume that I is an intial segment of $\{0, \ldots, n-1\}$, say $I = \{0, \ldots, k\}$ for some $k < n-1$.

Then for any two distinct $j_1, j_2 \in J$ we have that $(t_{j_1} \setminus R) \cap (t_{j_2} \setminus R) = \varnothing$, and R occurs as an "initial tree" in both the trees $\langle t_{j_1}, \prec_{t_{j_1}}\rangle$ and $\langle t_{j_2}, \prec_{t_{j_2}}\rangle$.

So we can define $s \coloneqq t_{j_1} \cup t_{j_2}$ and $\prec_s \coloneqq \prec_{t_{j_1}} \cup \prec_{t_{j_2}}$. Then $\langle s, \prec_s\rangle$ is a condition in $\mathbb{P}$ which is stronger than both $\langle t_{j_1}, \prec_{t_{j_1}}\rangle$ and $\langle t_{j_2}, \prec_{t_{j_2}}\rangle$.

Therefore $\mathbb{P}$ satisfies c.c.c. □

From this we immediately obtain that the field of T_G in the generic extension is all the countable ordinals in the generic extension.

Lemma 2.1.3 (Tennenbaum (1968, Lemma 3))
Let G and T be defined as above. Then $\mathbf{M}[G] \models$ "the field of T_G is all of ω_1".

Proof. By the genericity of G, we know that the field of T_G is all of $\omega_1^{\mathbf{M}}$. Now, the previous Lemma 2.1.2 showed that the forcing notion $\mathbb{P}$ of finite trees satisfies c.c.c., from which it follows that $\omega_1^{\mathbf{M}} = \omega_1^{\mathbf{M}[G]}$, and so the result follows. □

We now modify the proof of Lemma 2.1.2 to show that, in $\mathbf{M}[G]$, every antichain in T_G is countable.

Lemma 2.1.4 (Tennenbaum (1968, Lemma 6))
Let G be a $\mathbb{P}$-generic filter over $\mathbf{M}$. Then $\mathbf{M}[G] \models$ "every antichain in T_G is countable".

Proof. Suppose that $A \in \mathbf{M}[G]$ is such that $\mathbf{M}[G] \models$ "A is an uncountable antichain in T_G". So there exists $\langle t_0, \prec_{t_0}\rangle \in G$ such that $\langle t_0, \prec_{t_0}\rangle \Vdash$ "$\dot{A}$ is an uncountable antichain in $\dot{T}_G$", where $\dot{A}$ is a $\mathbb{P}$-name with $\dot{A}^G = A$ and $\dot{T}_G$ is a $\mathbb{P}$-name with $\dot{T}_G^G = T_G$.

For each $\beta \in A$, let $\langle t'_\beta, \prec_{t'_\beta}\rangle \in G$ be such that $\langle t'_\beta, \prec_{t'_\beta}\rangle \Vdash$ "$\check{\beta} \in \dot{A}$", and then let $\langle t_\beta, \prec_{t_\beta}\rangle$ be any condition stronger than $\langle t'_\beta, \prec_{t'_\beta}\rangle$ such that $\beta \in t_\beta$. As G is a filter we may assume that $\langle t'_\beta, \prec'_{t_\beta}\rangle$ is stronger than $\langle t_0, \prec_{t_0}\rangle$, so that $\langle t_\beta, \prec_{t_\beta}\rangle$ is also stronger than $\langle t_0, \prec_{t_0}\rangle$.

By the Δ-system lemma 2.1.1, we may assume, without loss of generality, that there exists a finite set R such that

$$t_{\beta_1} \cap t_{\beta_2} = R \qquad \text{for any } \beta_1, \beta_2 \in A \text{ with } \beta_1 \neq \beta_2,$$

that $\beta \notin R$ for all $\beta \in A$, and that there exists $n \in \omega$ such that $|t_\beta| = n$ for all $\beta \in A$.

For each $\beta \in A$, let $\langle \alpha_{\beta,i} : 0 \leq i < n \rangle$ be a strictly increasing sequence of the ordinals which appear in the tree t_β. Let i_β be the index of β in this sequence, so that $\alpha_{\beta,i_\beta} = \beta$. Let $I_\beta \coloneqq \{\, i \in \{0,\ldots,n-1\} : \alpha_{\beta,i} \in R \,\}$ be the set of indices for the ordinals in R in this sequence. Let $\lhd_\beta$ be the tree structure on $\{0,\ldots,n-1\}$ defined by

$$i_1 \lhd_\beta i_2 \text{ if and only if } \alpha_{\beta,i_1} \prec_{t_\beta} \alpha_{\beta,i_2},$$

for all $i_1, i_2 \in \{0,\ldots,n-1\}$, which is the order induced by t_β on $\{0,\ldots,n-1\}$.

Then, as in the proof of Lemma 2.1.2, we may assume without loss of generality that there exists $\langle i, I, \lhd \rangle \in n \times \mathcal{P}(n) \times \mathcal{P}(n \times n)$ such that $\langle i_\beta, I_\beta, \lhd_\beta \rangle = \langle i, I, \lhd \rangle$ for all $\beta \in A$, and that I is an initial segment of $\{0,\ldots,n-1\}$.

Now choose $\beta_1, \beta_2 \in A$ such that

$$\max(R) < \min(t_{\beta_1} \setminus R) \leq \beta_1 \leq \max(t_{\beta_1}) < \min(t_{\beta_2} \setminus R) \leq \beta_2.$$

Then define $s \coloneqq t_{\beta_1} \cup t_{\beta_2}$ and define a tree structure $\prec_s$ on s by

$$\begin{aligned} \alpha \prec_s \alpha' \quad \text{if and only if} \quad & \quad \alpha, \alpha' \in t_{\beta_1} \text{ and } \alpha \prec_{t_{\beta_1}} \alpha', \\ \text{or} \quad & \alpha, \alpha' \in t_{\beta_2} \text{ and } \alpha \prec_{t_{\beta_2}} \alpha', \\ \text{or} \quad & \alpha \in (t_{\beta_2} \setminus R),\ \ \alpha' \in (t_{\beta_1} \setminus R),\ \ \beta_1 \preceq_{t_{\beta_1}} \alpha', \text{ and} \\ & \ \alpha \preceq_{t_{\beta_2}} \gamma \text{ for some } \gamma \in (t_{\beta_2} \setminus R) \text{ with } \beta_2 \preceq_{t_{\beta_2}} \gamma, \end{aligned}$$

for all $\alpha, \alpha' \in s$. Then $\langle s, \prec_s \rangle \in \mathbb{P}$ and is a condition which is stronger than both $\langle t_{\beta_1}, \prec_{t_{\beta_1}} \rangle$ and $\langle t_{\beta_2}, \prec_{t_{\beta_2}} \rangle$, and we also have that $\beta_2 \prec_s \beta_1$. Then $\langle s, \prec_s \rangle \Vdash$ "$\check{\beta}_2 \mathrel{\dot{\prec}_{T_G}} \check{\beta}_1$", where $\dot{\prec}_{T_G}$ is a $\mathbb{P}$-name with $\dot{\prec}_{T_G}^G = \prec_{T_G}$. Therefore $\langle t_0, \prec_{t_0} \rangle$ could not have forced the statement "$\dot{A}$ is an antichain in $\dot{T}_G$". ☐

We now show that, in $\mathbf{M}[G]$, the tree T_G cannot have any uncountable chains because we can always turn a chain into an antichain of the same size.

Lemma 2.1.5 (Tennenbaum (1968, Lemma 7))
Let G be a $\mathbb{P}$-generic filter over $\mathbf{M}$. Then $\mathbf{M}[G] \models$ "every chain in T_G is countable".

Proof. Suppose $C = \{\alpha_\eta\}_{\eta < \omega_1^{\mathbf{M}[G]}} \in \mathbf{M}[G]$ is such that $\mathbf{M}[G] \models$ "C is an uncountable chain in T_G", where $\alpha_\eta < \alpha_{\eta'}$ whenver $\eta < \eta' < \omega_1^{\mathbf{M}[G]}$.

We now work in $\mathbf{M}[G]$. For each $\eta < \omega_1$, let $\langle t_\eta, \prec_{t_\eta} \rangle \in G$ be such that $\langle t_\eta, \prec_{t_\eta} \rangle \Vdash$ "$\check{\alpha}_\eta \in \dot{C}$", where $\dot{C}$ is a $\mathbb{P}$-name with $\dot{C}^G = C$. Then the set

$$\begin{aligned} D \coloneqq \Big\{ \langle s, \prec_s \rangle \in \mathbb{P} \ : \ & \langle s, \prec_s \rangle \text{ is stronger than } \langle t_\eta, \prec_{t_\eta} \rangle, \\ & \alpha_\eta, \alpha_{\eta+1} \in s, \text{ and} \\ & \text{there exists } \beta \in s \text{ such that } \beta \prec_s \alpha_\eta,\ \alpha_{\eta+1} \not\preceq_s \beta, \text{ and } \alpha_{\eta+1} \not\succeq_s \beta \Big\} \end{aligned}$$

is dense below $\langle t_\eta, \prec_{t_\eta} \rangle$, and so G intersects D. So choose $\langle s_\eta, \prec_{s_\eta} \rangle \in G \cap D$ and choose $\beta_\eta \in s_\eta$ such that $\beta_\eta \prec_{s_\eta} \alpha_\eta$, $\alpha_{\eta+1} \not\preceq_{s_\eta} \beta_\eta$, and $\alpha_{\eta+1} \not\succeq_{s_\eta} \beta_\eta$. Then

$$\langle s_\eta, \prec_{s_\eta} \rangle \Vdash \text{“}\check{\beta}_\eta \mathrel{\dot{\prec}_{T_G}} \check{\alpha}_\eta, \text{ and } \check{\alpha}_{\eta+1} \text{ and } \check{\beta}_\eta \text{ are pairwise incomparable in } \dot{T}_G\text{”},$$

where $\dot{T}_G$ and $\dot{\prec}_{T_G}$ are $\mathbb{P}$-names with $\dot{T}_G^G = T_G$ and $\dot{\prec}_{T_G}^G = \prec_{T_G}$ respectively. Then β_η is in the field of T_G and

$$\beta_\eta \prec_{T_G} \alpha_\eta,\ \alpha_{\eta+1} \not\preceq_{T_G} \beta_\eta, \text{ and } \alpha_{\eta+1} \not\succeq_{T_G} \beta_\eta.$$

Therefore the set $\{\beta_\eta\}_{\eta < \omega_1}$ is an uncountable antichain in T_G. But this contradicts Lemma 2.1.4 which asserted that every antichain in T_G is countable. ☐

Packaging all of the work above gives us the following Theorem 2.1.6.

Theorem 2.1.6 (Tennenbaum (1968, Theorem 1))
Let G be a $\mathbb{P}$-generic filter over $\mathbf{M}$. Then $\mathbf{M}[G] \models$ "T_G is a Suslin tree".

Proof. This is because, in $\mathbf{M}[G]$, the field of T_G is uncountable (Lemma 2.1.3), every antichain in T_G is countable (Lemma 2.1.4), and every chain in T_G is countable (Lemma 2.1.5). □

2.2 The diamond principle

There is another way to get the consistency of the existence of a Suslin tree, employing a combinatorial principle called the *diamond principle*.

Definition 2.2.1
Let $C \subseteq \omega_1$. We say that:

1. *C is* closed *if, for all $\alpha < \omega_1$, we have that if $\sup(C \cap \alpha) = \alpha$ then $\alpha \in C$;*
2. *C is* unbounded *if $\sup C = \omega_1$;*
3. *C is a* club set *if C is closed and unbounded.*

We say that a set $\Sigma \subseteq \omega_1$ is stationary *if $\Sigma \cap C \neq \varnothing$ for all club sets $C \subseteq \omega_1$. A $\diamondsuit$-sequence is a sequence $\langle S_\alpha \rangle_{\alpha<\omega_1}$ of sets with $S_\alpha \subseteq \alpha$ such that, for all $A \subseteq \omega_1$, the set $\{\, \alpha < \omega_1 \,:\, A \cap \alpha = S_\alpha \,\}$ is stationary. The* diamond principle*, denoted $\diamondsuit$, asserts that a $\diamondsuit$-sequence exists.*

Theorem 2.2.2 (Jensen (1972, Theorem 6.2))
If $\diamondsuit$ holds, then there exists a Suslin tree.

Proof. Let $\langle S_\alpha \rangle_{\alpha<\omega_1}$ be a $\diamondsuit$-sequence. We construct a Suslin tree $\langle T, \prec \rangle$ by its levels; for $\alpha < \omega_1$, let T_α denote the first α levels of T, i.e.

$$T_\alpha \coloneqq \{\, x \in T \;:\; \{\, y \in T \,:\, y \succ x \,\} \text{ has order-type } < \alpha \,\}.$$

For instance, T_1 will contain just the root of T, and T_ω will contain all the finite levels of T. In fact, we will construct it so that the field of T will be all of ω_1 and that, for any $\alpha < \omega_1$, the field of T_α is a countable ordinal.

Let $T_1 \coloneqq \{0\}$. If, for $\alpha < \omega_1$, the tree T_α has already been defined, then we break the definition of $T_{\alpha+1}$ into three cases depending on whether α is a successor ordinal or a limit ordinal and depending on whether S_α, from our $\diamondsuit$-sequence, is a maximal antichain in T_α. We suppose inductively that T_α is a normal tree whose field is some countable ordinal.

1. If α is a successor ordinal then T_α has a bottom-most level. So we define $T_{\alpha+1}$ by adding $\aleph_0$-many immediate successors to each of the nodes in the bottom-most level of T_α in a way such that the field of $T_{\alpha+1}$ is also a countable ordinal.
2. If α is a limit ordinal and S_α is a maximal antichain in T_α, we will build an end-extension $T_{\alpha+1}$ of T_α so that S_α is still a maximal antichain in $T_{\alpha+1}$ and that the $T_{\alpha+1}$ is a normal tree with some countable ordinal as its field. For every $x \in T_\alpha$, there is some $\xi \in S_\alpha$ such that either $x \preceq \xi$ or $x \succeq \xi$ in T, because S_α is a maximal antichain in T_α. Choose a cofinal branch B_x in T of length α with $x, \xi \in B_x$, which is possible because α has countable cofinality and because T_α is assumed to be a normal tree. We then add a new point at the bottom of each of these branches B_x. We let $T_{\alpha+1}$ be the resulting tree after repeating this process for all $x \in T_\alpha$, adding the new nodes appropriately so that the field of $T_{\alpha+1}$ is some countable ordinal.

3. If α is a limit ordinal and S_α is *not* a maximal antichain in T_α, then we just let $T_{\alpha+1}$ be any normal tree of height $\alpha+1$ which is an end-extension of T_α and such that the field of $T_{\alpha+1}$ is also a countable ordinal.

If $\alpha < \omega_1$ is a limit ordinal and the trees $\langle T_\eta \rangle_{\eta<\alpha}$ have already been defined, then we just let $T_\alpha \coloneqq \bigcup_{\eta<\alpha} T_\eta$.

Finally, we let $T \coloneqq \bigcup_{\alpha<\omega_1} T_\alpha$ and we claim that this T is a normal Suslin tree. That T is a normal tree of height ω_1 with field ω_1 is clear from construction. If we show that every antichain in T is countable, then we get as an immediate consequence that every chain in T is countable since every node in T has $\aleph_0$-many distinct pairwise incomparable immediate successors.

Suppose that $A \subseteq T$ is an antichain in T and, without loss of generality, we may suppose that A is a maximal antichain in T. As $\langle S_\alpha \rangle_{\alpha<\omega_1}$ is a $\diamondsuit$-sequence, the set $\Sigma \coloneqq \{\, \alpha < \omega_1 \;:\; A \cap \alpha = S_\alpha \,\}$ is stationary, i.e. Σ intersects all club sets.

Let

$$C_1 \coloneqq \{\, \alpha < \omega_1 \;:\; A \cap T_\alpha \text{ is a maximal antichain in } T_\alpha \,\}.$$

We claim that C_1 is a club set. Indeed, for any limit ordinal $\alpha < \omega_1$ with $\sup(C_1 \cap \alpha) = \alpha$, if we can extend $A \cap T_\alpha$ to a bigger antichain in T_α then this extension can also be done at some T_β with $\beta < \alpha$. So C_1 is closed. To see that C_1 is unbounded, for any $\alpha_0 < \omega_1$ we inductively construct an increasing sequence $\langle \alpha_n \rangle_{n<\omega}$ of countable ordinals as follows: since there are only countably many ordinals in T_{α_n}, there exists some countable ordinal $\alpha_{n+1} > \alpha_n$ such that every node in T_{α_n} is comparable with some node in $A \cap T_{\alpha_{n+1}}$. Then $\alpha \coloneqq \sup_{n<\omega} \alpha_n \in C_1$.

By construction of the T_α's, the set

$$C_2 \coloneqq \{\, \alpha < \omega_1 \;:\; \alpha \text{ is a limit ordinal and } T_\alpha = \alpha \,\}$$

is also a club set. Therefore, as the set Σ defined above is stationary, there exists some $\alpha \in \Sigma \cap C_1 \cap C_2$. Then S_α is a maximal antichain in T_α, and thus, by construction, also a maximal antichain in $T_{\alpha+1}$. But now S_α is a maximal antichain of bounded height in $T_{\alpha+1}$, so S_α must also be a maximal antichain in the whole of T. Then, as $S_\alpha = A \cap \alpha$ and A is also a maximal antichain in the whole of T, we must have $A \cap \alpha = A$, from which it follows that A is countable. Therefore every antichain in T is countable and hence T is a normal Suslin tree. □

Recalling that Gödel (1938) showed that the consistency of ZFC implies the consistency of ZFC + $\mathbf{V}=\mathbf{L}$, one could show the consistency of $\diamondsuit$ with ZFC by showing that it holds in $\mathbf{L}$. We omit this proof as it is slightly orthogonal to the forcing proofs we are doing in this essay.

Fact 2.2.3 (Jensen (1972, Lemma 6.5))
If $\mathbf{V}=\mathbf{L}$ holds, then $\diamondsuit$ holds.

Proof. Omitted; see Jech (2003, Theorem 13.21). □

We will instead show that $\diamondsuit$ is consistent with ZFC via forcing. First, we will need a preliminary technical lemma about countably closed forcing notions.

Lemma 2.2.4
Let $\langle \mathbb{P}, \preceq, \mathbb{1}_{\mathbb{P}} \rangle \in \mathbf{M}$ be a countably closed forcing notion and let G be a $\mathbb{P}$-generic filter over $\mathbf{M}$. Then any $X \in \mathbf{M}$ and any $\alpha < \omega_1^{\mathbf{M}}$, we have $X^\alpha \cap \mathbf{M} = X^\alpha \cap \mathbf{M}[G]$, where X^α denotes the set of all functions from α to X.

Proof. Clearly $X^\alpha \cap \mathbf{M} \subseteq X^\alpha \cap \mathbf{M}[G]$, so we just need to show the reverse inclusion.

Denote $F \coloneqq X^\alpha \cap \mathbf{M}$. Suppose, for a contradiction, that there exists some function $f\colon \alpha \to X$ in $\mathbf{M}[G]$ such that $f \notin F$. Choose some $p \in G$ such that

$$p \Vdash \text{“}\dot{f} \text{ is a function from } \check{\alpha} \text{ to } \check{X} \text{ but } \dot{f} \notin \check{F}\text{”},$$

where $\dot{f}$ is a $\mathbb{P}$-name with $\dot{f}^G = f$.

We now work in $\mathbf{M}$. For $\eta < \alpha$, we recursively define a sequence of pairs $\langle p_\eta, x_\eta \rangle \in \mathbb{P} \times X$ as follows. For $\beta < \alpha$, assuming $\langle \langle p_\eta, x_\eta \rangle : \eta < \beta \rangle$ has been defined, choose $p_\beta \in \mathbb{P}$ and $x_\beta \in X$ such that p_β is stronger than any condition in $\{p\} \cup \{p_\eta : \eta < \beta\}$ and $p_\beta \Vdash \text{“}\dot{f}(\check{\beta}) = \check{x}_\beta\text{”}$, which is possible due to the countable closure of $\mathbb{P}$.

Then, again using the countable closure of $\mathbb{P}$, we choose some $p_\alpha \in \mathbb{P}$ which is stronger than any p_η, for $\eta < \alpha$. Define the function $g\colon \alpha \to X$ by $g(\eta) \coloneqq x_\eta$. Then $p_\alpha \Vdash \text{“}\dot{f} = \check{g} \in \check{F}\text{”}$. But this contradicts the assumption that p, which is weaker than p_α, forces “$\dot{f} \notin \check{F}$”. $\square$

Now we force $\diamondsuit$. In $\mathbf{M}$, let $\mathbb{P}$ be the forcing notion consisting of all sequences $s\colon \alpha \to \mathcal{P}(\alpha)$, where $\alpha < \omega_1$, such that $s_\eta \subseteq \eta$ for all $\eta < \alpha$. For $s, t \in \mathbb{P}$, we stipulate that

$$s \text{ is stronger than } t \quad \text{if and only if} \quad s \text{ extends } t \text{ as a function, i.e. } s \supseteq t.$$

Finally, we let G be a $\mathbb{P}$-generic filter over $\mathbf{M}$.

Notice that $\mathbb{P}$ is a countably closed forcing notion: the union of a countably many compatible countable sequences is also a countable sequence. In particular, $\omega_1^{\mathbf{M}} = \omega_1^{\mathbf{M}[G]}$. So we can write ω_1 for both of them without ambiguity.

For any $\alpha < \omega_1$, the set $D_\alpha \coloneqq \{\, s \in \mathbb{P} \ :\ \alpha \in \operatorname{dom}(s) \,\}$ is dense in $\mathbb{P}$, and so $\alpha \in \bigcup G$. In $\mathbf{M}[G]$, we define the sequence $S\colon \omega_1 \to \mathcal{P}(\omega_1)$ as follows:

$$S_\alpha \coloneqq \left(\bigcup G\right)(\alpha) \quad \text{for all } \alpha < \omega_1.$$

Theorem 2.2.5
Let G be a $\mathbb{P}$-generic filter over $\mathbf{M}$ and define the sequence $\langle S_\alpha \rangle_{\alpha<\omega_1}$ by $S_\alpha \coloneqq (\bigcup G)(\alpha)$ for each $\alpha < \omega_1$. Then $\mathbf{M}[G] \models$ “$\langle S_\alpha \rangle_{\alpha<\omega_1}$ is a $\diamondsuit$-sequence”.

Proof. Suppose $C, A \in \mathbf{M}[G]$ are such that $\mathbf{M}[G] \models$ “$C, A \subseteq \omega_1$ and C is a club set”. We want to show that there exists some $\alpha \in C$ with $A \cap \alpha = S_\alpha$. So we suppose, for a contradiction, that

$$\mathbf{M}[G] \models \text{“}C, A \subseteq \omega_1,\ C \text{ is a club set, and for all } \alpha \in C \text{ we have } A \cap \alpha \neq S_\alpha\text{”}.$$

Let $\dot{C}$, $\dot{A}$, and $\dot{S}_\alpha$ be $\mathbb{P}$-names with $\dot{C}^G = C$, $\dot{A}^G = A$, and $\dot{S}_\alpha^G = S_\alpha$. Then there exists some $p \in G$ such that

$$p \Vdash \text{“}\dot{C}, \dot{A} \subseteq \check{\omega}_1,\ \dot{C} \text{ is a club set, and for all } \alpha \in \dot{C} \text{ we have } \dot{A} \cap \alpha \neq \dot{S}_\alpha\text{”}.$$

Let $1_A\colon \omega_1 \to \{0,1\}$ denote the indicator function of A, i.e.

$$1_A(\alpha) = \begin{cases} 1 & \text{if } \alpha \in A, \\ 0 & \text{if } \alpha \notin A. \end{cases}$$

Then for every $\alpha < \omega_1$, we have $1_A|_\alpha \in \mathbf{M}$ because $\mathbb{P}$ is countably closed (Lemma 2.2.4). Let $\dot{1}_{\dot{A}}$ be a $\mathbb{P}$-name such that $\dot{1}_{\dot{A}}^G = 1_A$.

We now work in $\mathbf{M}$. For $n < \omega$, we recursively define a sequence of triples $\langle \alpha_n, s_n, g_n \rangle \in \omega_1 \times \mathbb{P} \times \{0,1\}^{<\omega_1}$ as follows.

1. Define $\alpha_0 \coloneqq \omega$. Choose $s_0 \in \mathbb{P}$ stronger than p such that $s_0 \Vdash$ "$\dot{1}_{\dot{A}}|_{\check{\alpha}_0} = \check{g}_0$", where $g_0 \colon \omega \to \{0,1\}$ is some function.

2. Inductively suppose that $\langle \alpha_n, s_n, g_n \rangle$ has been defined for some $n < \omega$. Using the countable closure of $\mathbb{P}$, Lemma 2.2.4, and that $p \Vdash$ "$\dot{C}$ is unbounded in $\check{\omega}_1$", we can choose some ordinal $\alpha_{n+1} < \omega_1$, some condition $s_{n+1} \in \mathbb{P}$, and some function $g_{n+1} \colon \alpha_{n+1} \to \{0,1\}$ such that all of the following hold:

   (a) $\alpha_{n+1} > \max\{\alpha_n, \operatorname{dom}(s_n)\}$;

   (b) $\alpha_n < \operatorname{dom}(s_{n+1})$;

   (c) s_{n+1} is stronger than s_n;

   (d) $s_{n+1} \Vdash$ "$\check{\alpha}_{n+1} \in \dot{C}$ and $\dot{1}_{\dot{A}}|_{\check{\alpha}_{n+1}} = \check{g}_{n+1}$".

Now define $\alpha_* \coloneqq \sup_{n<\omega} \alpha_n$, $s \coloneqq \bigcup_{n<\omega} s_n$, and $g \coloneqq \bigcup_{n<\omega} g_n$, noting that $s \in \mathbb{P}$ is stronger than each s_n and $\operatorname{dom}(s) = \alpha_*$. Furthermore, g is a function from α_* to $\{0,1\}$. By construction, we have $s \Vdash$ "for all $n < \check{\omega}$ we have $\check{\alpha}_n \in \dot{C}$". As s is stronger than p, we have $s \Vdash$ "$\dot{C} \subseteq \check{\omega}_1$ and $\dot{C}$ is a club set". In particular, $s \Vdash$ "$\dot{C}$ is closed". So $s \Vdash$ "$\check{\alpha}_* \in \dot{C}$".

Finally let $A_* \coloneqq \{\, \beta < \alpha_* : g(\beta) = 1 \,\}$ indicated by g. Define $s_* \in \mathbb{P}$ to be the following $(\alpha_* + 1)$-sequence:

$$s_*(\eta) \coloneqq s(\eta) \text{ for } \eta < \alpha_* \quad \text{and} \quad s_*(\alpha_*) \coloneqq A_*.$$

That is, s_* is the α_*-sequence s appended with the set A_*.

Any $\mathbb{P}$-generic filter G containing s_* will, by definition, yield $\dot{S}^G_{\alpha_*} = A_*$. Consequently,

$$s_* \Vdash \text{“}\check{\alpha}_* \in \dot{C} \text{ and } \dot{A} \cap \alpha_* = \check{A}_* = \dot{S}_{\alpha_*}\text{”}.$$

But this contradicts the fact that s_* is stronger than p, which forced the sentence "for all $\alpha \in \dot{C}$ we have $\dot{A} \cap \alpha \neq \dot{S}_\alpha$". □

3 Breaking Suslin trees

Solovay and Tennenbaum (1971) developed the theory of iterated forcing to show the following consistency result: if ZFC is consistent, then ZFC + "there does not exist a Suslin tree" is also consistent.

Historically, Stanley Tennenbaum first figured out how to use a single forcing notion to kill a Suslin tree, which we discuss in detail in Section 3.1. The idea would then be to somehow iterate the process to eventually kill every Suslin tree to produce a model in which there are no Suslin trees.

> *"Tennenbaum had worked out how to kill one Souslin tree but iterating the process was beyond him; Solovay told me that he ... decided to do the iteration theory himself; so that became their joint paper."* — Mathias (2025).

There is also the following account by Robert Solovay himself regarding this event.

> *"For much of the year, Stan was trying to prove that the iteration did not collapse cardinals. And he was considering iterations of lengths 2 and 3. My role was to passively listen to his proofs and spot the errors in them . . . . At one of those meetings one of us (probably Stan) made progress and finally found reasonable conditions under which a two stage iteration did not collapse cardinals. Somehow this got me seriously thinking about the problem and by the time of our next meeting I had a proof of the theorem."* — Robert Solovay, 2006, in Kanamori (2011, Section 3).

Solovay and Tennenbaum (1971) proceeded to establish the consistency of *Martin's axiom* with ZFC, and then used Martin's axiom to prove that there are no Suslin trees.

We will modify this procedure to force the consistency of the non-existence of Suslin trees with ZFC, adapting the proofs in Baumgartner (1983, Sections 1–3), Karagila (2023, Section 6 and Section 7), and Kunen (1980, Chapter VII and Chapter VIII).

3.1 How to kill a Suslin tree

Fix a countable transitive model $\mathbf{M}$ of $\mathsf{ZFC}+\mathsf{GCH}$. The use of GCH will become apparent later when we start performing iterated forcing. For now, we are interested in an observation by Stanley Tennenbaum that forcing with that Suslin tree will "kill" it.

Lemma 3.1.1 (Solovay and Tennenbaum (1971, Section 2.2))
Let $\langle T, \preceq\rangle \in \mathbf{M}$ be such that $\mathbf{M} \models$ "$\langle T, \preceq\rangle$ is a normal Suslin tree". Let G be a $\langle T, \preceq\rangle$-generic filter over $\mathbf{M}$. Then $\mathbf{M}[G] \models$ "$\langle T, \preceq\rangle$ is not a Suslin tree".

Proof. As $\langle T, \preceq\rangle$ is a Suslin tree, it satisifies c.c.c. So $\omega_1^{\mathbf{M}} = \omega_1^{\mathbf{M}[G]}$.

We now work in $\mathbf{M}[G]$. For each ordinal $\alpha < \omega_1^{\mathbf{M}[G]}$, the set

$$D_\alpha \coloneqq \{\, p \in \mathbb{P} : \text{there exists } \beta > \alpha \text{ such that } p \text{ is in the } \beta\text{th level of } \langle T, \preceq\rangle \,\}$$

is dense in $\mathbb{P}$, because $\langle T, \preceq\rangle$ is a *normal* Suslin tree, and so $G \cap D_\alpha \neq \varnothing$. In particular, $\mathbf{M}[G] \models$ "G is uncountable". But also, since T is a tree, this G must be a chain in T. Therefore $\mathbf{M}[G] \models$ "G is an uncountable chain in $\langle T, \preceq\rangle$". $\square$

Suppose $\langle T, \preceq\rangle$ is a normal Suslin tree in $\mathbf{M}$, and that G is a generic filter on $\langle T, \preceq\rangle$ over M. Lemma 3.1.1 above shows that $\langle T, \preceq\rangle$ is no longer a Suslin tree in $\mathbf{M}[G]$. Furthermore, if N is any transitive model of ZFC with $\mathbf{M} \subseteq N$, then $\langle T, \preceq\rangle$ will not be a Suslin tree in N either, for we will either have $N \models$ "G is an uncountable antichain in $\langle T, \preceq\rangle$" or $N \models$ "T is countable", because $\mathbf{M}[G]$ contains bijections between G, T, and $\omega_1^{\mathbf{M}[G]}$.

> *"Once a Suslin tree is killed, it stays dead."* — Solovay and Tennenbaum (1971, Section 2).

3.2 Kill two Suslin trees with one filter

We still keep a countable transitive model $\mathbf{M}$ of $\mathsf{ZFC}+\mathsf{GCH}$. It is tempting to simply iterate the process of killing one Suslin tree (as in Lemma 3.1.1) to kill all the Suslin trees in $\mathbf{M}$, so that we end up with a model of ZFC in which there does not exist any Suslin trees.

This task, however, is not as easy as it seems. For one, new Suslin trees may pop up when we form the generic extensions. So we will have to kill the new trees as well. Another concern is the limit stages of this iterative process; we cannot simply take a union of an increasing ω-sequence of generic extensions and expect to end up with a model of ZFC.

Rather than picking a generic filter G_1 on a Suslin tree $\langle T_1, \prec_1\rangle \in \mathbf{M}$ over $\mathbf{M}$, then picking another generic filter G_2 on another Suslin tree $\langle T_2, \prec_2\rangle \in \mathbf{M}[G_1]$, then picking another generic filter G_3 on another Suslin tree $\langle T_3, \prec_3\rangle \in \mathbf{M}[G_1][G_2]$, and so on, we instead control the entire iteration process from within $\mathbf{M}$ itself and accomplish the entire iteration process in a single generic extension.

The idea of controlling this iteration process from within $\mathbf{M}$ can already be seen in two-stage iterations.

Definition 3.2.1
Let $\mathbb{P} \in \mathbf{M}$ be a forcing notion and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbf{M},\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is a forcing notion". The two-stage iteration of $\mathbb{P}$ and $\dot{\mathbb{Q}}$ *is*

$$\mathbb{P} * \dot{\mathbb{Q}} \coloneqq \{\, \langle p, \dot{q}\rangle \ : \ p \in \mathbb{P} \text{ and } \mathbb{1}_{\mathbb{P}} \Vdash_{\mathbf{M},\mathbb{P}} \text{“}\dot{q} \in \dot{\mathbb{Q}}\text{”}\},$$

and we stipulate an ordering on $\mathbb{P} * \dot{\mathbb{Q}}$ as follows: $\langle p_1, \dot{q}_1\rangle$ is stronger than $\langle p_2, \dot{q}_2\rangle$ if and only if

$$p_1 \text{ is stronger than } p_2, \text{ and } p_1 \Vdash_{\mathbf{M},\mathbb{P}} \text{“}\dot{q}_1 \text{ is stronger than } \dot{q}_2\text{”}.$$

As it stands, the object $\mathbb{P} * \dot{\mathbb{Q}}$ defined in Definition 3.2.1 above may not be a set in $\mathbf{M}$. We can resolve this by identifying a $\mathbb{P}$-name $\dot{q}_1$ with another $\mathbb{P}$-name $\dot{q}_2$ if and only if $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbf{M},\mathbb{P}}$ "$\dot{q}_1 = \dot{q}_2$", and then performing Scott's trick on the equivalence classes for all the names $\dot{q}$ which appear in $\dot{\mathbb{Q}}$ to restrict ourselves to a set of $\mathbb{P}$-names.[8]

The next Theorem 3.2.2 partially[9] shows that forcing with $\mathbb{P} * \dot{\mathbb{Q}}$ is the same as forcing with $\mathbb{P}$ and then forcing with the forcing notion which $\dot{\mathbb{Q}}$ evaluates to.

Theorem 3.2.2
Let $\mathbb{P} \in \mathbf{M}$ be a forcing notion and let $\dot{\mathbb{Q}} \in \mathbf{M}$ be a $\mathbb{P}$-name such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbf{M},\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is a forcing notion". Let K be a $(\mathbb{P}*\dot{\mathbb{Q}})$-generic filter over $\mathbf{M}$. Then the set $G \coloneqq \{\, p \in \mathbb{P} : \text{there exists } \dot{q} \text{ such that } \langle p, \dot{q}\rangle \in K \,\}$ is a $\mathbb{P}$-generic filter over $\mathbf{M}$, the set $H \coloneqq \{\, \dot{q}^G : \text{there exists } p \in \mathbb{P} \text{ such that } \langle p, \dot{q}\rangle \in K \,\}$ is a $\dot{\mathbb{Q}}^G$-generic filter over $\mathbf{M}[G]$, and $\mathbf{M}[K] = \mathbf{M}[G][H]$.

Proof. The proofs that G and H are indeed filters on $\mathbb{P}$ and $\dot{\mathbb{Q}}^G$ respectively are mechanical. We will omit them and only check the genericity of G and H.

First, we show that G is $\mathbb{P}$-generic over $\mathbf{M}$. Let $D \in \mathbf{M}$ be a dense subset of $\mathbb{P}$. Then the set

$$S_D \coloneqq \{\, \langle p, \dot{q}\rangle \in \mathbb{P} * \dot{\mathbb{Q}} \ : \ p \in D \,\} \in \mathbf{M}$$

is a dense subset of $\mathbb{P}*\dot{\mathbb{Q}}$. Indeed, for any $\langle p, \dot{q}\rangle \in \mathbb{P}*\dot{\mathbb{Q}}$, just choose some $p' \in D$ which is stronger than p, and we will have that $\langle p', \dot{q}\rangle \in S_D$ is stronger than $\langle p, \dot{q}\rangle$.

[8] This is one of the many possible resolutions presented by Karagila (2023, the Remark in Section 6.1).

[9] We use the word "partially" because there is a converse to Theorem 3.2.2 which we will not be using. See Baumgartner (1983, Theorem 1.1 (a)) or Kunen (1980, Exercise VIII.J15) for this.

So the $(\mathbb{P} * \dot{\mathbb{Q}})$-generic filter K intersects the set S_D above, meaning there exists $\langle p, \dot{q} \rangle \in K \cap S_D$. This p satisfies $p \in G \cap D$.

Next, we show that H is $\dot{\mathbb{Q}}^G$-generic over $\mathbf{M}[G]$. Let $E \in \mathbf{M}[G]$ be a dense subset of $\dot{\mathbb{Q}}^G$. Let $\dot{E}$ be a $\mathbb{P}$-name for E so that $\dot{E}^G = E$ and let $p_0 \in G$ be such that $p_0 \Vdash_{\mathbf{M},\mathbb{P}}$ "$\dot{E}$ is dense in $\dot{\mathbb{Q}}$". In $\mathbf{M}$, choose some maximal antichain $A \subseteq \mathbb{P}$ such that $p_0 \in A$. Now define the $\mathbb{P}$-name

$$\begin{aligned}\dot{F} :=& \{ \langle \dot{x}, r \rangle : r \Vdash_{\mathbf{M},\mathbb{P}} \text{“}\dot{x} \in \dot{E}\text{”}, r \in \mathbb{P} \text{ is stronger than } p_0, \text{ and } \dot{x} \in \operatorname{dom}(\dot{E}) \} \\ &\cup \bigcup_{p \in A \setminus \{p_0\}} \{ \langle \dot{x}, r \rangle : r \Vdash_{\mathbf{M},\mathbb{P}} \text{“}\dot{x} \in \dot{\mathbb{Q}}\text{”}, r \in \mathbb{P} \text{ is stronger than } p, \text{ and } \dot{x} \in \operatorname{dom}(\dot{\mathbb{Q}}) \}.\end{aligned}$$

Fix any $\mathbb{P}$-generic filter G' over $\mathbf{M}$. We claim that if $G' \cap A = \{p_0\}$ then $\dot{F}^{G'} = \dot{E}^{G'}$. First, we show that $\dot{F}^{G'} \subseteq \dot{E}^{G'}$. If $\dot{x}^{G'} \in \dot{F}^{G'}$, then $\langle \dot{x}, r \rangle \in \dot{F}$ for some $r \in G'$. As G' is a filter and A is an antichain, the definition of $\dot{F}$ tells us that r is stronger than p_0, that $r \Vdash_{\mathbf{M},\mathbb{P}}$ "$\dot{x} \in \dot{E}$", and that $\dot{x} \in \operatorname{dom}(\dot{E})$. Thus $\dot{x}^{G'} \in \dot{E}^{G'}$. Now we show the reverse inclusion $\dot{F}^{G'} \supseteq \dot{E}^{G'}$. If $\dot{x}^{G'} \in \dot{E}^{G'}$ then $r \Vdash_{\mathbf{M},\mathbb{P}}$ "$\dot{x} \in \dot{E}$" for some $r \in G'$. Letting $r' \in G'$ be a common extension of both r and p_0, we see that $\langle \dot{x}, r' \rangle \in \dot{F}$, and so $\dot{x}^{G'} \in \dot{F}^{G'}$. Similarly, if $G' \cap A = \{p\}$ for some $p \in A \setminus \{p_0\}$, then $\dot{F}^{G'} = \dot{\mathbb{Q}}^{G'}$. In both cases, we have that $\mathbf{M}[G'] \models$ "$\dot{F}^{G'}$ is dense in $\dot{\mathbb{Q}}^{G'}$". As G' was arbitrary and A was a maximal antichain in $\mathbb{P}$, we conclude that $\dot{F}^G = E$ and $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbf{M},\mathbb{P}}$ "$\dot{F}$ is dense in $\dot{\mathbb{Q}}$".

Now, the set

$$S_F := \{ \langle p, \dot{q} \rangle \in \mathbb{P} * \dot{\mathbb{Q}} \ : \ p \Vdash_{\mathbf{M},\mathbb{P}} \dot{q} \in \dot{F} \} \in \mathbf{M}$$

is a dense subset of $\mathbb{P} * \dot{\mathbb{Q}}$. Indeed, for any $\langle p, \dot{q} \rangle \in \mathbb{P} * \dot{\mathbb{Q}}$, as $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbf{M},\mathbb{P}}$ "$\dot{F}$ is dense in $\dot{\mathbb{Q}}$", there exists a $\mathbb{P}$-name $\dot{q}' \in \mathbf{M}$ such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbf{M},\mathbb{P}}$ "$\dot{q}'$ is stronger than $\dot{q}$ and $\dot{q}' \in \dot{F}$". Then $\langle p, \dot{q}' \rangle \in S_F$ is stronger than $\langle p, \dot{q} \rangle$.

So there exists $\langle p, \dot{q} \rangle \in K \cap S_F$, and this $\dot{q}$ satisfies $\dot{q}^G \in H \cap \dot{F}^G = H \cap E$.

Finally, $G, H \in \mathbf{M}[K]$, so $\mathbf{M}[G][H] \subseteq \mathbf{M}[K]$. One can also check that

$$K = \{ \langle p, \dot{q} \rangle \in \mathbb{P} * \dot{\mathbb{Q}} \ : \ p \in G \text{ and } \dot{q}^G \in H \}.$$

Therefore $K \in \mathbf{M}[G][H]$, giving $\mathbf{M}[K] \subseteq \mathbf{M}[G][H]$. □

As is often the case, we wish to preserve cardinals when we move to a generic extension. The following Theorem 3.2.3 says that the two-stage iteration of c.c.c. forcing notions still satisfies c.c.c.

Theorem 3.2.3
Let $\mathbb{P} \in \mathbf{M}$ be a c.c.c. forcing notion and let $\dot{\mathbb{Q}} \in \mathbf{M}$ be a $\mathbb{P}$-name such that

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbf{M},\mathbb{P}} \text{“}\dot{\mathbb{Q}} \textit{ is a c.c.c. forcing notion”.}$$

Then $\mathbb{P} * \dot{\mathbb{Q}}$ is also a c.c.c. forcing notion.

Proof. Suppose, for a contradiction, that $\{\langle p_\alpha, \dot{q}_\alpha \rangle : \alpha < \omega_1^{\mathbf{M}}\}$ is an uncountable antichain in $\mathbb{P} * \dot{\mathbb{Q}}$ in $\mathbf{M}$. We flip the ordered pairs in this antichain to define the $\mathbb{P}$-name $\dot{A} := \{ \langle \dot{q}_\alpha, p_\alpha \rangle : \alpha < \omega_1^{\mathbf{M}} \} \in \mathbf{M}$.

Fix any $\mathbb{P}$-generic filter G over $\mathbf{M}$. Then $\dot{A}^G = \{ \dot{q}_\alpha^G : \alpha < \omega_1^{\mathbf{M}} \text{ and } p_\alpha \in G \}$. We claim that $\dot{A}^G$ must be an antichain in $\dot{\mathbb{Q}}^G$. To see this, suppose that for some distinct $\alpha, \beta < \omega_1^{\mathbf{M}}$ with $p_\alpha, p_\beta \in G$ there exists some $\dot{q}^G \in \dot{\mathbb{Q}}^G$ which is stronger than both $\dot{q}_\alpha^G$ and $\dot{q}_\beta^G$. Choose some $p \in G$ which is stronger than both p_α and p_β such that $p \Vdash$ "$\dot{q}$ is stronger than both $\dot{q}_\alpha$ and $\dot{q}_\beta$". Then $\langle p, \dot{q} \rangle$ is stronger than $\langle p_\alpha, \dot{q}_\alpha \rangle$ and $\langle p_\beta, \dot{q}_\beta \rangle$, contradicting that $\langle p_\alpha, \dot{q}_\alpha \rangle$ and $\langle p_\beta, \dot{q}_\beta \rangle$ come from an antichain. Then, as $\dot{\mathbb{Q}}^G$ satisfies c.c.c., we must have $\mathbf{M}[G] \models$ "$\dot{A}^G$ is countable". As G was arbitrary, we obtain $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbf{M},\mathbb{P}}$ "$\dot{A}$ is countable".

Now define the $\mathbb{P}$-name $\dot{B} := \{ \langle \check{\alpha}, p_\alpha \rangle : \alpha < \omega_1^{\mathbf{M}} \} \in \mathbf{M}$, and observe that this will give $p_\alpha \Vdash_{\mathbf{M},\mathbb{P}}$ "$\check{\alpha} \in \dot{B}$" for all $\alpha < \omega_1^{\mathbf{M}}$. But by the argument with $\dot{A}$ above, we also must have that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbf{M},\mathbb{P}}$ "$\dot{B}$ is countable".Then, because $\mathbb{P}$ satisfies c.c.c., there must exist $\alpha < \omega_1^{\mathbf{M}}$ such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbf{M},\mathbb{P}}$ "$\dot{B} \subseteq \check{\alpha}$". In particular, $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbf{M},\mathbb{P}}$ "$\check{\alpha} \notin \dot{B}$", which contradicts the observation above. □

In particular, as Suslin trees satisfy c.c.c., if $\mathbb{P}$ is a normal Suslin tree and $\dot{\mathbb{Q}}$ is forced by $\mathbb{1}_{\mathbb{P}}$ to be a normal Suslin tree, then the previous Theorem 3.2.3 tells us that $\mathbb{P} * \dot{\mathbb{Q}}$ is a c.c.c. forcing notion which helps us kill both the Suslin trees.

We will prove one more technical lemma before we move on to proving the consistency of the non-existence of Suslin trees. Recall that we started with a model $\mathbf{M}$ of $\mathsf{ZFC} + \mathsf{GCH}$. We use GCH to control the size of $2^{\aleph_1}$ in any generic extension as well as to show that our two-stage iterated forcing notion $\mathbb{P} * \dot{\mathbb{Q}}$ will not be too big, provided $\mathbb{P}$ is not too big and $\dot{\mathbb{Q}}$ is forced to not be too big.

Lemma 3.2.4
Let $\mathbb{P} \in \mathbf{M}$ be a c.c.c. forcing notion with $\mathbf{M} \models$ "$|\mathbb{P}| \leq \aleph_2$". Then $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbf{M},\mathbb{P}}$ "$2^{\aleph_1} = \aleph_2$". If, in addition, $\dot{\mathbb{Q}} \in \mathbf{M}$ is a $\mathbb{P}$-name such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbf{M},\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is a forcing notion and $|\dot{\mathbb{Q}}| \leq \aleph_2$", then $\mathbf{M} \models$ "$|\mathbb{P} * \dot{\mathbb{Q}}| \leq \aleph_2$".

Proof. The proof for both statements will involve computing an upper bound for suitable "nice names".

Fix any $\mathbb{P}$-generic filter G over $\mathbf{M}$. We want to show that $\mathbf{M}[G] \models$ "$2^{\aleph_1} = \aleph_2$". Certainly we have $\mathbf{M}[G] \models$ "$2^{\aleph_1} \geq \aleph_2$" since $\mathbf{M}[G] \models \mathsf{ZFC}$.

Suppose $X \in \mathbf{M}[G]$ is such that $\mathbf{M} \models$ "$X \subseteq \omega_1$". As $\mathbb{P}$ satisfies c.c.c., we have $\omega_1^{\mathbf{M}} = \omega_1^{\mathbf{M}[G]}$. Let $\dot{X} \in \mathbf{M}$ be a $\mathbb{P}$-name for X, so that $\dot{X}^G = X$. Working in $\mathbf{M}$, for each $\alpha \in \omega_1$, define

$$C_\alpha := \{\, p \in \mathbb{P} \ : \ p \Vdash_{\mathbf{M},\mathbb{P}} \text{“}\check{\alpha} \in \dot{X}\text{”} \,\},$$

and choose an antichain $A_\alpha \subseteq C_\alpha$ which is maximal in C_α. Let $\mathscr{A} := \langle A_\alpha \rangle_{\alpha < \omega_1}$. Then define

$$\dot{Y}_{\mathscr{A}} := \{\, (\check{\alpha}, p) \ : \ \alpha \in \omega_1 \text{ and } p \in A_\alpha \,\}.$$

We claim that $\dot{Y}_{\mathscr{A}}^G = \dot{X}^G = X$.

Whenever $\check{\alpha}^G \in \dot{X}^G$, there is some $p \in G$ with $p \Vdash$ "$\check{\alpha} \in \dot{X}$". We claim that $G \cap A_\alpha \neq \varnothing$. Indeed, if $G \cap A_\alpha = \varnothing$ then there exists some $q \in G$ which is incompatible with all the conditions in A_α. But then taking a common extension $r \in G$ of both p and q yields an antichain $A_\alpha \cup \{r\} \subseteq C_\alpha$, contradicting the maximality of A_α. So there must exist some $q \in G \cap A_\alpha$. Then $(\check{\alpha}, q) \in \dot{Y}_{\mathscr{A}}$, and so $\check{\alpha}^G \in \dot{Y}^G$.

For the other inclusion, whenever $\check{\alpha}^G \in \dot{Y}_{\mathscr{A}}^G$, there must exist some $p \in G$ such that $(\check{\alpha}, p) \in \dot{Y}_{\mathscr{A}}$. Then, by definition of $\dot{Y}_{\mathscr{A}}$ and A_α, we get that $p \Vdash_{\mathbf{M},\mathbb{P}}$ "$\check{\alpha} \in \dot{X}$". Hence $\check{\alpha}^G \in \dot{X}^G$.

Now, each $\dot{Y}_{\mathscr{A}}$ is uniquely determined by the ω_1-sequence $\langle A_\alpha \rangle_{\alpha<\omega_1}$ of antichains in $\mathbb{P}$. As $\mathbb{P}$ satisfies c.c.c., each A_α is countable. Therefore, in $\mathbf{M}$, since we have GCH, there are at most $(\aleph_2^{\aleph_0})^{\aleph_1} = \aleph_2$-many possibilities for $\mathscr{A}$ and thus at most $\aleph_2$-many such names $\dot{Y}_{\mathscr{A}}$ for subsets of ω_1 in $\mathbf{M}[G]$. Therefore, as $\mathbb{P}$ satisfies c.c.c., we have $\mathbf{M}[G] \models$ "$2^{\aleph_1} \leq \aleph_2$". As G was arbitrary, $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbf{M},\mathbb{P}}$ "$2^{\aleph_1} = \aleph_2$".

Next we show that $\mathbf{M} \models$ "$|\mathbb{P} * \dot{\mathbb{Q}}| \leq \aleph_2$".

Again, we work in $\mathbf{M}$. As $\mathbb{P} * \dot{\mathbb{Q}}$ consists of pairs $\langle p, \dot{q} \rangle$ where $p \in \mathbb{P}$ and $\dot{q}$ is a $\mathbb{P}$-name such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbf{M},\mathbb{P}}$ "$\dot{q} \in \dot{\mathbb{Q}}$", we just need to show that $\aleph_2$ is an upper bound on the number of equivalence classes for such $\mathbb{P}$-names $\dot{q}$, recalling that we identified $\mathbb{P}$-names $\dot{q}_1$ and $\dot{q}_2$ if and only if $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbf{M},\mathbb{P}}$ "$\dot{q}_1 = \dot{q}_2$".

Since $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbf{M},\mathbb{P}}$ "$|\dot{\mathbb{Q}}| \leq \aleph_2$" and $\mathbb{P}$ satisfies c.c.c., we can suppose without loss of generality that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbf{M},\mathbb{P}}$ "$\dot{\mathbb{Q}} \subseteq \check{\omega}_2$".

Suppose that $\dot{q}$ is a $\mathbb{P}$-name with $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbf{M},\mathbb{P}}$ "$\dot{q} \in \dot{\mathbb{Q}}$". For each $\alpha \in \omega_2$, define the set

$$S_\alpha := \{\, p \in \mathbb{P} \ : \ p \Vdash_{\mathbf{M},\mathbb{P}} \text{“}\dot{q} = \check{\alpha}\text{”} \,\},$$

and choose an antichain $B_\alpha \subseteq S_\alpha$ which is maximal in S_α. Let $\mathscr{B} := \langle B_\alpha \rangle_{\alpha<\omega_2}$. Then define

$$\dot{r}_{\mathscr{B}} := \{\, (\check{\beta}, p) \ : \ \beta < \alpha < \omega_2 \text{ and } p \in B_\alpha \,\}.$$

Notice that, whenever $\alpha \neq \beta$ and we have two conditions $p, p' \in \mathbb{P}$ with $p \Vdash_{\mathbf{M},\mathbb{P}}$ "$\dot{q} = \check{\alpha}$" and $p' \Vdash_{\mathbf{M},\mathbb{P}}$ "$\dot{q} = \check{\beta}$", then p and p' must be incompatible. So $\bigcup_{\alpha<\omega_2} B_\alpha$ must be an antichain in $\mathbb{P}$ and thus countable. So there can only be countably $\alpha < \omega_1$ such that $B_\alpha \neq \varnothing$. So there are at most $\left(\aleph_2^{\aleph_0}\right)^{\aleph_0} = \aleph_2$-many possibilities for $\mathscr{B}$, and thus for $\dot{r}_{\mathscr{B}}$, by GCH.

Let G be any $\mathbb{P}$-generic filter over $\mathbf{M}$. Suppose that $\alpha < \omega_2^{\mathbf{M}} = \omega_2^{\mathbf{M}[G]}$ is such that $\mathbf{M}[G] \models$ "$\dot{q}^G = \alpha$". Then there exists some $p \in G$ such that $p \Vdash_{\mathbf{M},\mathbb{P}}$ "$\dot{q} = \check{\alpha}$". By a similar argument with A_α and C_α above, we have $G \cap B_\alpha \neq \varnothing$, so we can choose some $p' \in G \cap B_\alpha$. Now, clearly, $\dot{r}^G_{\mathscr{B}} = \alpha$ since $\bigcup_{\alpha<\omega_2} B_\alpha$ is an antichain. Therefore $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbf{M},\mathbb{P}}$ "$\dot{r}_{\mathscr{B}} = \dot{q}$". Therefore there are $\aleph_2$-many equivalence classes of conditions $\dot{q}$ such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbf{M},\mathbb{P}}$ "$\dot{q} \in \dot{\mathbb{Q}}$". □

3.3 Arboricide

We keep fixing a countable transitive model $\mathbf{M}$ of $\mathsf{ZFC} + \mathsf{GCH}$. Having established two-stage iterations, we now move on to iterations of length α for any ordinal α.

Definition 3.3.1
For an ordinal $\alpha \geq 1$, a finite-support iteration of length α *is a pair of sequences $\langle\langle\mathbb{P}_\eta\rangle_{\eta\leq\alpha}, \langle\dot{\mathbb{Q}}_\eta\rangle_{\eta<\alpha}\rangle$ such that all of the following hold.*

1. *For each $\eta \leq \alpha$, the set $\mathbb{P}_\eta$ is a forcing notion consisting of η-sequences p such that*

   for all $\xi < \eta$, the set $p(\xi)$ is a $\mathbb{P}_\xi$-name.

2. *For each $\eta < \alpha$, the set $\dot{\mathbb{Q}}_\eta$ is a $\mathbb{P}_\eta$-name, and there is a $\mathbb{P}_\eta$-name $\dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\eta}$ satisfying*

   $$\mathbb{1}_{\mathbb{P}_\eta} \Vdash_{\mathbf{M},\mathbb{P}_\eta} \text{“}\dot{\mathbb{Q}}_\eta \textit{ is a forcing notion with weakest condition } \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\eta}\text{”.}$$

3. *$\mathbb{P}_0 = \{\varnothing\}$ is the trivial forcing notion.*

4. *If α is a successor ordinal, say $\alpha = \beta + 1$, then $\langle\langle\mathbb{P}_\eta\rangle_{\eta\leq\beta}, \langle\dot{\mathbb{Q}}_\eta\rangle_{\eta<\beta}\rangle$ is a finite-support iteration of length β with $\mathbb{P}_\beta = \{\, p|_\beta : p \in \mathbb{P}_{\beta+1}\,\}$, and a $(\beta+1)$-sequence p belongs to $\mathbb{P}_{\beta+1}$ if and only if*

   $$p|_\beta \in \mathbb{P}_\beta \quad \textit{and} \quad \mathbb{1}_{\mathbb{P}_\beta} \Vdash_{\mathbf{M},\mathbb{P}_\beta} \text{“}p(\beta) \in \dot{\mathbb{Q}}_\beta\text{”.}$$

   We stipulate an ordering on $\mathbb{P}_{\beta+1}$ as follows: we declare $p \in \mathbb{P}_{\beta+1}$ to be stronger than $q \in \mathbb{P}_{\beta+1}$ if and only if

   $$p|_\beta \textit{ is stronger than } q|_\beta \textit{ in } \mathbb{P}_\beta \quad \textit{and} \quad p|_\beta \Vdash_{\mathbf{M},\mathbb{P}_\beta} \text{“}p(\beta) \textit{ is stronger than } q(\beta) \textit{ in } \dot{\mathbb{Q}}_\beta\text{”.}$$

5. *If α is a limit ordinal, then for every $\eta < \alpha$, the pair of sequences $\langle\langle\mathbb{P}_\xi\rangle_{\xi\leq\eta}, \langle\dot{\mathbb{Q}}_\xi\rangle_{\xi<\eta}\rangle$ is a finite-support iteration of length η with $\mathbb{P}_\eta = \{\, p|_\eta : p \in \mathbb{P}_\alpha\,\}$, and all of the follwing hold.*

   (a) *The α-sequence $\bar{\mathbb{1}}_{\mathbb{P}_\alpha}$ defined by $\bar{\mathbb{1}}_{\mathbb{P}_\alpha}(\eta) := \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\eta}$, for all $\eta < \alpha$, is in $\mathbb{P}_\alpha$.*

   (b) *If $p \in \mathbb{P}_\alpha$, then $p|_\eta \in \mathbb{P}_\eta$ for all $\eta < \alpha$.*

   (c) *If $p \in \mathbb{P}_\alpha$, then the set* $\operatorname{support}(p) := \{\, \eta < \alpha : \mathbb{1}_{\mathbb{P}_\eta} \Vdash_{\mathbf{M},\mathbb{P}_\eta} \text{“}p(\eta) \neq \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\eta}\text{”}\}$ *is finite.*

   (d) *If $p \in \mathbb{P}_\alpha$, then for any $\eta < \alpha$ and for any $q \in \mathbb{P}_\eta$, if q is stronger than $p|_\eta$ in $\mathbb{P}_\eta$, then there exists $r \in \mathbb{P}_\alpha$ such that $r|_\eta = q$ and $r|_{\alpha\setminus\eta} = p|_{\alpha\setminus\eta}$.*

   We stipulate an ordering on $\mathbb{P}_\alpha$ as follows: we declare $p \in \mathbb{P}_\alpha$ to be stronger than $q \in \mathbb{P}_\alpha$ if and only if

   $$\textit{for all } \eta < \alpha, \textit{ we have that } p|_\eta \textit{ is stronger than } q|_\eta \textit{ in } \mathbb{P}_\eta.$$

Observe if $\alpha \geq 1$ is a successor ordinal, say $\alpha = \beta+1$, and $\langle \langle \mathbb{P}_\eta \rangle_{\eta \leq \beta+1}, \langle \dot{\mathbb{Q}}_\eta \rangle_{\eta < \beta+1} \rangle$ is a finite-support iteration system of length $\beta + 1$, then $\mathbb{P}_{\beta+1} \cong \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$. In particular, $\mathbb{P}_1 \cong \dot{\mathbb{Q}}_0^G$ where G is the trivial $\mathbb{P}_0$-generic filter over $\mathbf{M}$.

If we specify appropriate names $\langle \dot{\mathbb{Q}}_\eta \rangle_{\eta < \alpha}$ then we can recover the entire finite-support iteration system $\langle \langle \mathbb{P}_\eta \rangle_{\eta \leq \alpha}, \langle \dot{\mathbb{Q}}_\eta \rangle_{\eta < \alpha} \rangle$. Indeed, if we have defined $\langle \langle \mathbb{P}_\eta \rangle_{\eta \leq \beta}, \langle \dot{\mathbb{Q}}_\eta \rangle_{\eta < \beta} \rangle$, where $\beta < \alpha$, then Definition 3.3.1 tells us precisely what $\mathbb{P}_{\beta+1}$ is along with its ordering; if $\lambda \leq \alpha$ is a limit ordinal and we have defined $\langle \langle \mathbb{P}_\eta \rangle_{\eta < \lambda}, \langle \dot{\mathbb{Q}}_\eta \rangle_{\eta < \lambda} \rangle$, then we insert $\bar{\mathbb{1}}_{\mathbb{P}_\alpha}$ into $\mathbb{P}_\alpha$ and close $\mathbb{P}_\alpha$ under all the other requirements.

We produce an α-stage analogue to Theorem 3.2.2, which stated that two-stage iterated forcing really is doing forcing twice, in the following Theorem 3.3.2.

Theorem 3.3.2
Let $\langle \langle \mathbb{P}_\eta \rangle_{\eta \leq \alpha}, \langle \dot{\mathbb{Q}}_\eta \rangle_{\eta < \alpha} \rangle \in \mathbf{M}$ be a finite-support iteration of length $\alpha \geq 1$, let G_α be a $\mathbb{P}_\alpha$-generic filter over $\mathbf{M}$, and for $\beta < \alpha$ let $G_\beta \coloneqq \{ p|_\beta : p \in G_\alpha \}$. Then G_β is a $\mathbb{P}_\beta$-generic filter over $\mathbf{M}$ and $\mathbf{M}[G_\beta] \subseteq \mathbf{M}[G_\alpha]$.

Proof. Again, the proof that G_β really is a filter on $\mathbb{P}_\beta$ is mechanical; we simply check its genericity.

Suppose that $D \in \mathbf{M}$ is a dense subset of $\mathbb{P}_\beta$. For each $q \in \mathbb{P}_\beta$, we let $\bar{q} \in \mathbb{P}_\alpha$ be the α-sequence satisfying $\bar{q}|_\beta = q$ and $\bar{q}|_{\alpha \setminus \beta} = \bar{\mathbb{1}}_{\mathbb{P}_\alpha}|_{\alpha \setminus \beta}$. Then the set

$$\bar{D} \coloneqq \{ p \in \mathbb{P}_\alpha : \text{there exists some } q \in D \text{ such that } p \text{ is stronger than } \bar{q} \text{ in } \mathbb{P}_\alpha \}$$

is dense in $\mathbb{P}_\alpha$. Indeed, for any $r \in \mathbb{P}_\alpha$, there exists $q \in D$ which is stronger than $r|_\beta$ in $\mathbb{P}_\beta$, and so we let $p \in \mathbb{P}_\alpha$ be the α-sequence satisfying $p|_\beta = q$ and $p|_{\alpha \setminus \beta} = r|_{\alpha \setminus \beta}$. Then $p \in \bar{D}$ and p is stronger than r in $\mathbb{P}_\alpha$.

Thus G_α intersects $\bar{D}$. Let $p \in G_\alpha \cap \bar{D}$. Then $p|_\beta \in G_\beta$, by definition of G_β, and there exists $q \in D$ such that p is stronger than $\bar{q}$ in $\mathbb{P}_\alpha$, by definition of $\bar{D}$. Then $p|_\beta$ is stronger than q in $\mathbb{P}_\beta$, and so $q \in G_\beta$. Thus $q \in G_\beta \cap D$.

Finally, $G_\beta \in \mathbf{M}[G_\alpha]$, so we must have $\mathbf{M}[G_\beta] \subseteq \mathbf{M}[G_\alpha]$. □

Recall that Theorem 3.2.3 and Lemma 3.2.4 stated that two-stage iterations of c.c.c. forcing notions which are not too large will also be a c.c.c. forcing notion which is not too large. For the sake of time and space, we shall simply state and not prove the α-stage versions of these two results in the following Fact 3.3.3.

Fact 3.3.3
Let $\langle \langle \mathbb{P}_\eta \rangle_{\eta \leq \alpha}, \langle \dot{\mathbb{Q}}_\eta \rangle_{\eta < \alpha} \rangle \in \mathbf{M}$ be a finite-support iteration of length $\alpha \geq 1$. Suppose that, for every $\eta < \alpha$,

$$\mathbb{1}_{\mathbb{P}_\eta} \Vdash_{\mathbf{M}, \mathbb{P}_\eta} \text{“}|\dot{\mathbb{Q}}_\eta| \leq \aleph_1 \textit{ and } \dot{\mathbb{Q}}_\eta \textit{ satisfies c.c.c.”}.$$

Then $|\mathbb{P}_\alpha| \leq \aleph_2$ and $\mathbb{P}_\alpha$ satisfies c.c.c.

Proof. Omitted; see Baumgartner (1983, Theorem 2.2 and Corollary 2.3). The proof relies on the assumption that $\mathbf{M} \models \mathsf{GCH}$ and proceeds by induction on α. Note that the successor stages for this result has already been established in Theorem 3.2.3 and Lemma 3.2.4. At limit stages, Fodor's lemma is used. □

The whole point of doing iterated forcing is to show that we will have exhausted all the forcings which we wish to do.

Theorem 3.3.4
Let $\langle \langle \mathbb{P}_\eta \rangle_{\eta \leq \alpha}, \langle \dot{\mathbb{Q}}_\eta \rangle_{\eta < \alpha} \rangle \in \mathbf{M}$ be a finite-support iteration of c.c.c. forcing notions. Let G_α be a $\mathbb{P}_\alpha$-generic filter over $\mathbf{M}$, and let $X \in \mathbf{M}[G_\alpha]$ be such that $X \subseteq \omega_1^{\mathbf{M}}$. Then, there exists an ordinal $\beta < \omega_2^{\mathbf{M}}$ such that $X \in \mathbf{M}[G_\beta]$, where $G_\beta = \{ p|_\beta : p \in G_\alpha \}$.

Proof. Let $\dot{X} \in \mathbf{M}$ be a $\mathbb{P}_\alpha$-name such that $\dot{X}^{G_\alpha} = X$. For each $\gamma < \omega_1^{\mathbf{M}}$, let

$$C_\gamma \coloneqq \{\, p \in \mathbb{P}_\alpha : p \Vdash_{\mathbf{M},\mathbb{P}_\alpha} \text{“}\check{\gamma} \in \dot{X}\text{”} \,\},$$

and choose an antichain $A_\gamma \subseteq C_\gamma$ which is maximal in C_γ.

We claim that

$$X = \{\, \gamma < \omega_1^{\mathbf{M}} : G_\alpha \cap A_\gamma \neq \varnothing \,\}.$$

To see that $X \supseteq \{\, \gamma < \omega_1^{\mathbf{M}} : G_\alpha \cap A_\gamma \neq \varnothing \,\}$, for any $\gamma < \omega_1^{\mathbf{M}}$ for which there exists $p \in G_\alpha \cap A_\gamma$, we have that

$$p \in G_\alpha \quad \text{and} \quad p \Vdash_{\mathbf{M},\mathbb{P}_\alpha} \text{“}\check{\gamma} \in \dot{X}\text{”},$$

and so $\gamma = \check{\gamma}^{G_\alpha} \in \dot{X}^{G_\alpha} = X$. To see the reverse inclusion $X \subseteq \{\, \gamma < \omega_1^{\mathbf{M}} : G_\alpha \cap A_\gamma \neq \varnothing \,\}$, suppose for a contradiction that there exists $\gamma \in X$ with $G_\alpha \cap A_\gamma = \varnothing$. Then there exists $q \in G_\alpha$ which is incompatible with all the conditions in A_γ. Find some $p \in G_\alpha$ such that $p \Vdash_{\mathbf{M},\mathbb{P}_\alpha} \check{\gamma} \in \dot{X}$. Let $r \in G_\alpha$ be a common extension of both p and q. Then we still have $r \Vdash_{\mathbf{M},\mathbb{P}_\alpha}$ “$\check{\gamma} \in \dot{X}$”, and so $r \in C_\gamma$. But this r is a common extension of q and is thus incompatible with all conditions in A_γ. This contradicts the maximality of A_γ in C_γ.

Now, as $\mathbb{P}_\alpha$ satisfies c.c.c., each A_γ is countable. Also, each $p \in A_\gamma$ has finite support. So there exists $\beta < \omega_2^{\mathbf{M}}$ such that, for all $\gamma < \omega_1^{\mathbf{M}}$ and all $p \in A_\gamma$, we have $\operatorname{support}(p) \subseteq \beta$.

Now, for each $\gamma < \omega_1^{\mathbf{M}}$, let $A'_\gamma \coloneqq \{\, p|_\beta : p \in A_\gamma \,\}$. Then we claim that

$$X = \{\, \gamma < \omega_1^{\mathbf{M}} : G_\alpha \cap A_\gamma \neq \varnothing \,\} = \{\, \gamma < \omega_1^{\mathbf{M}} : G_\beta \cap A'_\gamma \neq \varnothing \,\}.$$

That $\{\, \gamma < \omega_1^{\mathbf{M}} : G_\alpha \cap A_\gamma \neq \varnothing \,\} \subseteq \{\, \gamma < \omega_1^{\mathbf{M}} : G_\beta \cap A'_\gamma \neq \varnothing \,\}$ is obvious: if $p \in G_\alpha \cap A_\gamma$ then $p|_\beta \in G_\beta \cap A'_\gamma$. For the reverse inclusion $\{\, \gamma < \omega_1^{\mathbf{M}} : G_\alpha \cap A_\gamma \neq \varnothing \,\} \supseteq \{\, \gamma < \omega_1^{\mathbf{M}} : G_\beta \cap A'_\gamma \neq \varnothing \,\}$, if $p \in G_\beta \cap A'_\gamma$ then there exists $q \in G_\alpha$ and there exists $r \in A_\gamma$ such that $q|_\beta = r|_\beta = p$. But also $\operatorname{support}(r) \subseteq \beta$, so we must have $r|_\beta = p$ and $r|_{\alpha \setminus \beta} = \mathbb{1}_{\mathbb{P}_\alpha}|_{\alpha\setminus\beta}$. Now, q is stronger than r in $\mathbb{P}_\alpha$, and so $r \in G_\alpha$. Thus $r \in G_\alpha \cap A_\gamma$. Therefore $X \in \mathbf{M}[G_\beta]$. ◻

We now begin producing a forcing notion which would give a model of ZFC in which no Suslin trees exist. In $\mathbf{M}$, fix a surjective *bookkeeping* function $f \colon \omega_2^{\mathbf{M}} \to \omega_2^{\mathbf{M}} \times \omega_2^{\mathbf{M}}$ such that for any $\alpha, \beta, \gamma \in \omega_2^{\mathbf{M}}$,

$$\text{if } f(\alpha) = \langle \beta, \gamma \rangle, \text{ then } \beta \leq \alpha,$$

For instance, the Gödel pairing function works as a suitable f.

The rough idea is as follows. We may restrict our attention to just the normal Suslin trees with field ω_1, since $\mathbf{M}$ has the axiom of choice. Also recall that $\mathbf{M} \models \mathsf{GCH}$. So, in $\mathbf{M}$, there are at most $2^{\aleph_1} = \aleph_2$ normal Suslin trees to consider. Let $\mathbf{M}_0 \coloneqq \mathbf{M}$, and let $\{T_{0,\gamma}\}_{\gamma<\omega_2}$ be an enumeration of all the normal Suslin trees in $\mathbf{M}_0$. We force with $T_{f(0)}$ to produce a model $\mathbf{M}_1$. Lemma 3.2.4 tells us that we still have $\mathbf{M}_1 \models$ “$2^{\aleph_1} = \aleph_2$”. So let $\{T_{1,\gamma}\}_{\gamma<\omega_2}$ enumerate all the normal Suslin trees in $\mathbf{M}_1$. We then force with $T_{f(1)}$ to produce a model $\mathbf{M}_2$. And we rinse and repeat.

$$\begin{tikzcd}
\mathbf{M}_0 & T_{0,0} \arrow[r] & T_{0,1} \arrow[d] & T_{0,2} \arrow[d] & T_{0,3} \arrow[d] & \cdots \\
\mathbf{M}_1 & T_{1,0} \arrow[urr] & T_{1,1} \arrow[l] & T_{1,2} \arrow[d] & T_{1,3} \arrow[d] & \cdots \\
\mathbf{M}_2 & T_{2,0} \arrow[uurrr] & T_{2,1} \arrow[l] & T_{2,2} \arrow[l] & T_{2,3} \arrow[d] & \cdots \\
\vdots & & & & \vdots &
\end{tikzcd}$$

The requirement that our bookkeeping function f satisfies "if $f(\alpha) = \langle \beta, \gamma \rangle$ then $\beta \leq \alpha$" is there to ensure that we are always forcing with trees that we have already built.

With the help of Theorem 3.3.4, we will do this in such a way that once we hit stage $\mathbf{M}_{\omega_2}$, every potential normal Suslin tree in $\mathbf{M}_{\omega_2}$ must have appeared as some $T_{\beta,\gamma} \in \mathbf{M}_\beta$, with $\beta < \omega_2$, and so if $f(\alpha) = \langle \beta, \gamma \rangle$ then we would have killed that Suslin tree when producing $\mathbf{M}_{\alpha+1}$. This idea is not dissimilar to the zig-zagging proof of the countability of $\mathbb{N}^2$.

Using finite-support iterations, we will control this entire process from within $\mathbf{M}$ itself. That is, we will produce a finite-support iteration $\langle \langle \mathbb{P}_\eta \rangle_{\eta \leq \omega_2^{\mathbf{M}}}, \langle \dot{S}_\eta \rangle_{\eta < \omega_2^{\mathbf{M}}} \rangle \in \mathbf{M}$ such that forcing with $\mathbb{P}_{\omega_2^{\mathbf{M}}}$ accomplishes the entire killing process in one fell swoop.

Suppose, inductively, that we have defined c.c.c. forcing notions $\langle \mathbb{P}_\eta \rangle_{\eta \leq \alpha} \in \mathbf{M}$ and names $\langle \dot{S}_\eta \rangle_{\eta < \alpha} \in \mathbf{M}$, where $\alpha < \omega_2^{\mathbf{M}}$, such that $\langle \langle \mathbb{P}_\eta \rangle_{\eta \leq \alpha}, \langle \dot{S}_\eta \rangle_{\eta < \alpha} \rangle$ is a finite-support iteration of length α. By Lemma 3.2.4, we know that $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash_{\mathbf{M},\mathbb{P}_\alpha}$ "$2^{\aleph_1} = \aleph_2$". So we can let $\dot{L}_\alpha$ be a $\mathbb{P}_\alpha$-name such that

$$\mathbb{1}_{\mathbb{P}_\beta} \Vdash_{\mathbf{M},\mathbb{P}_\beta} \text{“}\dot{L} \text{ is a function with domain } \omega_2 \text{ enumerating all the normal Suslin trees with field } \check{\omega}_1\text{”.}$$

Let $f(\alpha) = \langle \beta, \gamma \rangle$. For any $\mathbb{P}_\alpha$-generic filter G_α over $\mathbf{M}$, letting $G_\beta \coloneqq \{ p|_\beta : p \in G_\alpha \}$, we know from Theorem 3.3.2 that $(\dot{L}_\beta(\check{\gamma}))^{G_\beta} \in \mathbf{M}[G_\alpha]$. So we let $\dot{S}_\alpha$ be a $\mathbb{P}_\alpha$-name such that, for any $\mathbb{P}_\alpha$-generic filter G_α over $\mathbf{M}$,

$$\begin{gathered}\mathbf{M}[G_\alpha] \models \text{“if } \left(\dot{L}_\beta(\check{\gamma})\right)^{G_\beta} \text{ is a normal Suslin tree, then } \dot{S}_\alpha^{G_\alpha} = \left(\dot{L}_\beta(\check{\gamma})\right)^{G_\beta}; \\ \text{otherwise } \dot{S}_\alpha^{G_\alpha} \text{ is a trivial forcing notion”,}\end{gathered}$$

where $G_\beta \coloneqq \{ p|_\beta : p \in G_\alpha \}$. Then $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash_{\mathbf{M},\mathbb{P}_\alpha}$ "$\dot{S}_\alpha$ is a c.c.c. forcing notion with $|\dot{S}_\alpha| \leq \aleph_1$".

Thus we have a finite-support iteration $\langle \langle \mathbb{P}_\eta \rangle_{\eta \leq \omega_2^{\mathbf{M}}}, \langle \dot{S}_\eta \rangle_{\eta < \omega_2^{\mathbf{M}}} \rangle$ c.c.c. forcing notions, by Fact 3.3.3. We will, finally, show that forcing with $\mathbb{P}_{\omega_2^{\mathbf{M}}}$ will yield a model of ZFC with no Suslin trees.

Let $G_{\omega_2^{\mathbf{M}}}$ be a $\mathbb{P}_{\omega_2^{\mathbf{M}}}$-generic filter over $\mathbf{M}$. Suppose, for a contradiction, that there exists $T \in \mathbf{M}[G_{\omega_2^{\mathbf{M}}}]$ such that $\mathbf{M}[G_{\omega_2^{\mathbf{M}}}] \models$ "T is a normal Suslin tree". Without loss of generality, we may assume that the field of T is a subset of $\omega_1^{\mathbf{M}[G_{\omega_2^{\mathbf{M}}}]} = \omega_1^{\mathbf{M}}$. By Theorem 3.3.4, there exists some $\beta < \omega_2^{\mathbf{M}}$ such that $T \in \mathbf{M}[G_\beta]$, where $G_\beta = \{ p|_\beta : p \in G_{\omega_2^{\mathbf{M}}} \}$. Choose some $\gamma < \omega_2^{\mathbf{M}}$ be such that $\mathbf{M}[G_\beta] \models$ "$T = \dot{T}_{\beta,\gamma}^{G_\beta}$". Using our bookkeeping function f, we then choose some $\alpha < \omega_2^{\mathbf{M}}$ such that $f(\alpha) = \langle \beta, \gamma \rangle$. As T is a normal Suslin tree in $\mathbf{M}[G_{\omega_2^{\mathbf{M}}}]$, it must also be a normal Suslin tree in $\mathbf{M}[G_\beta]$. Therefore $\dot{S}_\alpha^{G_\alpha} = T$. But then T is not a Suslin tree in $\mathbf{M}[G_{\alpha+1}]$.

Packaging all of this together, we obtain the following Theorem 3.3.5.

Theorem 3.3.5 (Solovay and Tennenbaum (1971, Lemma 7.2 and Theorem 7.11))
Let $\mathbb{P}_{\omega_2^{\mathbf{M}}}$ be as above and let $G_{\omega_2^{\mathbf{M}}}$ be a $\mathbb{P}_{\omega_2^{\mathbf{M}}}$-generic filter over $\mathbf{M}$. Then, in $\mathbf{M}[G_{\omega_2^{\mathbf{M}}}]$, there does not exist a Suslin tree.

4 Closing Remarks

4.1 Connections with the continuum hypothesis

The similarity of the statement of the continuum hypothesis (CH) and the statement of Suslin's problem sparks the question if they imply each other at all. The answer to this is no in both directions.

One way to instantly spot that GCH is consistent with the existence of a Suslin tree is that they both hold in $\mathbf{L}$, as discussed in Section 2.2.

Though we did not do it, the forcing notion in Section 2.1 can be used to show that CH is independent from ZFC + "there exists a Suslin tree". Recall that the forcing notion $\mathbb{P}$ of finite trees from Section 2.1 is a c.c.c. forcing notion. Thus, via a very similar calculation as in Lemma 3.2.4, we can show that both of the following:

1. If $\mathbf{M}$ is a countable transitive model of $\mathsf{ZFC} + \mathsf{GCH}$ and G is a $\mathbb{P}$-generic filter over $\mathbf{M}$, then $\mathbf{M}[G]$ is a model of $\mathsf{ZFC} + \mathsf{GCH}$ + "there exists a Suslin tree".

2. If $\mathbf{M}$ is a countable transitive model of $\mathsf{ZFC} + \neg\mathsf{CH}$ and G is a $\mathbb{P}$-generic filter over $\mathbf{M}$, then $\mathbf{M}[G]$ is a model of $\mathsf{ZFC} + \neg\mathsf{CH}$ + "there exists a Suslin tree".

This shows that CH is independent of ZFC + "there exists a Suslin tree".

A modification can be made to the iteration argument in Section 3.3 to yield the original result by Solovay and Tennenbaum (1971, Theorem 7.11), which established the consistency of $\mathsf{ZFC} + \neg\mathsf{CH}$ with *Martin's axiom*. A specific instantiation of their result says it is consistent with ZFC that: for any partially ordered set $\langle \mathbb{P}, \preceq \rangle$ in which every antichain is countable and for any collection $\mathscr{D}$ of dense subsets of $\mathbb{P}$ with $|\mathscr{D}| \leq \aleph_1$, there exists a filter on $\mathbb{P}$ which intersects every dense set in $\mathscr{D}$. In an argument nearly identical to Lemma 3.1.1, we see that no normal Suslin trees can exist in such a model of ZFC. Furthermore, the continuum hypothesis cannot hold in this model. Indeed, if $\{ r_\alpha : \alpha < \omega_1 \}$ is any set of functions from ω to 2, then the set

$$\begin{aligned}\mathscr{D} \coloneqq \{ \{ p : p \text{ is a finite function from } \omega \text{ to 2 and } p \not\subseteq r_\alpha \} \ : \ \alpha \leq \omega_1 \} \\ \cup \{ \{ p : p \text{ is a finite function from } \omega \text{ to 2 and } n \in \operatorname{dom}(p) \} \ : \ n < \omega \}\end{aligned}$$

is a collection of at most $\aleph_1$-many dense subsets of the set $\mathbb{P}$ of finite functions from ω to 2. The filter obtained from Martin's axiom applied to $\mathbb{P}$ and $\mathscr{D}$ would then let us obtain a function from ω to 2 that is not equal to r_α for any $\alpha < \omega_1$.

To obtain a model of $\mathsf{ZFC} + \mathsf{CH}$ in which no Suslin trees exist is more difficult and was achieved by Ronald Jensen (cf. Devlin and Johnsbråten (1974, Chapters VIII–X)) through a very different approach to iterated forcing.

4.2 Other approaches

There are several other ways of obtaining obtaining a model of set theory with and without Suslin trees.

Jech (1967, Theorem 1) used Petr Vopěnka's ∇-models to establish the consistency of the existence of Suslin trees. A modern treatment, via a Boolean-algebras approach to forcing, of this result can be found in Jech (2003, Theorem 15.23). In both cases, the idea was to use a forcing notion consisting of countable normal trees ordered by end-extension.

Shelah (1984, Theorem 1.1) showed that the Cohen forcing notion of adding even *one* Cohen real via forcing will also add a Suslin tree to the generic extension.

In Section 4.1 we mentioned Jensen's approach to yielding a model in which CH holds but there does not exist a Suslin tree. Shelah (1982, Chapter V.6 and Chapter VIII) establishes this result through the method of *proper forcing*.

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