Infinities which may or may not exist

An introduction to cardinal characteristics of the continuum

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Cantor's Discovery of Different Infinities

Theorem (Cantor's Diagonal Argument)

There does not exist a surjection from $\mathbb{N}$ to $\mathbb{R}$. In other words, $|\mathbb{N}| < |\mathbb{R}|$.

Proof. Suppose $x_0, x_1, x_2, \ldots$ is an enumeration of $[0,1]$. Write them in their decimal expansions, each not ending with an infinite string of 9's:

$$\begin{aligned} x_0 &= 0.x_{00}x_{01}x_{02}x_{03}x_{04}\ldots \\ x_1 &= 0.x_{10}x_{11}x_{12}x_{13}x_{14}\ldots \\ x_2 &= 0.x_{20}x_{21}x_{22}x_{23}x_{24}\ldots \\ &\vdots \end{aligned}$$

Then $y := 0.y_0y_1y_2\ldots$, where

$$y_n := \begin{cases} 0 & \text{if } x_{nn} \neq 0, \\ 1 & \text{if } x_{nn} = 0, \end{cases}$$

does not appear in the enumeration. $\square$

Cantor's Discovery of Different Infinities

Cantor's diagonal argument can be generalised to obtain Cantor's theorem.

Theorem (Cantor's Theorem)

For any set X, there is no surjection from X to its power set $\mathscr{P}(X)$.

Proof. Suppose $f : X \to \mathscr{P}(X)$ is a surjection. Consider

$$S \coloneqq \{\, x \in X : x \notin f(x) \,\}.$$

If $f(x) = S$ for some $x \in X$, then $x \in S$ if and only if $x \notin f(x) = S$. $\square$

Consequently,

$$|\mathbb{N}| < |\mathscr{P}(\mathbb{N})| < |\mathscr{P}(\mathscr{P}(\mathbb{N}))| < |\mathscr{P}(\mathscr{P}(\mathscr{P}(\mathbb{N})))| < \cdots$$

Recall that we also have

$$|\mathbb{N}| < |\mathbb{R}| < |\mathscr{P}(\mathbb{R})| < |\mathscr{P}(\mathscr{P}(\mathbb{R}))| < \cdots$$

Cantor's Discovery of Different Infinities

Theorem

We have $|\mathscr{P}(\mathbb{N})| = |\mathbb{R}|$.

Proof. We have injections

$$\mathbb{R} \xhookrightarrow{\text{arctan}} [0,1] \xhookrightarrow{\text{binary expansions}} \mathscr{P}(\mathbb{N}) \xhookrightarrow{\text{binary expansions}} \mathbb{R}. \qquad \square$$

Does there exist a set X with $|\mathbb{N}| < |X| < |\mathbb{R}|$?

- Countable: Integers, Rational Numbers, Algebraic Numbers
- Bijective to $\mathbb{R}$: Irrational Numbers, Transcendental Numbers, The Cantor Set, The Vitali Set

The *continuum hypothesis* (**CH**) asserts that no such set X exists.

Independence

The axioms of **ZFC** set theory were laid out in the hopes of forming a "solid" foundation for mathematics. The hope was that all these axioms are consistent, and that all "true" statements are provable from these axioms.

Then Kurt Gödel published his incompleteness theorems. The continuum hypothesis (**CH**) was subsequently shown to be independent of **ZFC**.

Theorem (Kurt Gödel, 1940)

If **ZFC** is consistent, then **ZFC** + **CH** is consistent.

Theorem (Paul Cohen, 1963)

If **ZFC** is consistent, then **ZFC** + $\neg$**CH** is consistent.

Meagre Subsets of $\mathbb{R}$

Definition (Nowhere Dense Sets)

We say $X \subseteq \mathbb{R}$ is *nowhere dense* if $\mathrm{int}(\mathrm{cl}(X)) = \varnothing$.

Definition (Meagre Sets)

We say $X \subseteq \mathbb{R}$ is *meagre* if X is a countable union of nowhere dense sets.

Definition (The Uniformity Number for Meagre Sets)

We use $\mathbf{non}(\mathcal{M})$ to denote the smallest cardinality of any non-meagre set.

Clearly, $|\mathbb{N}| < \mathbf{non}(\mathcal{M}) \leq |\mathbb{R}|$.

Eventually-Dominating Functions

Definition (Eventual Domination)

Given two functions $f, g\colon \mathbb{N} \to \mathbb{N}$, we say g *eventually dominates* f, written $f \leq^* g$, if there exists some $N \in \mathbb{N}$ such that

$$f(n) \leq g(n) \quad \text{for all } n \geq N.$$

Definition (Unbounding Sets)

A set $\mathcal{B}$ of functions from $\mathbb{N}$ to $\mathbb{N}$ is *unbounded* if there does not exist some function $g\colon \mathbb{N} \to \mathbb{N}$ which eventually dominates every function in $\mathcal{B}$.

Definition (The Unbounding Number)

We use $\mathfrak{b}$ to denote the smallest cardinality of any unbounded set.

Eventually-Dominating Functions

Theorem

We have $|\mathbb{N}| < \mathfrak{b} \leq |\mathbb{R}|$

Proof. Let $\mathcal{B} = \{f_0, f_1, f_2, \dots\}$ be a countable set of functions from $\mathbb{N}$ to $\mathbb{N}$. Write out all their values in a grid:

$$\begin{array}{l|cccccc}
f_0 & & f_0(0) & f_0(1) & f_0(2) & f_0(3) & \dots \\
f_1 & & f_1(0) & f_1(1) & f_1(2) & f_1(3) & \dots \\
f_2 & & f_2(0) & f_2(1) & f_2(2) & f_2(3) & \dots \\
f_3 & & f_3(0) & f_3(1) & f_3(2) & f_3(3) & \dots \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Define $g\colon \mathbb{N} \to \mathbb{N}$ by $g(n) := \max_{j \leq n} f_j(n)$. This function g eventually dominates every function in $\mathcal{B}$.

For the other inequality, $|\{\text{all functions from } \mathbb{N} \text{ to } \mathbb{N}\}| = |\mathscr{P}(\mathbb{N})|$. $\square$

Rearranging Infinite Series

Theorem (Riemann's Rearrangement Theorem)

Suppose a series $\sum_{n\in\mathbb{N}} a_n$ of real numbers converges to $L \in \mathbb{R}$.

- If $\sum_{n\in\mathbb{N}} |a_n|$ converges, then all rearrangements $\sum_{n\in\mathbb{N}} a_{p(n)}$ also converge to L.
- If $\sum_{n\in\mathbb{N}} |a_n|$ diverges, then for any $L' \in \mathbb{R} \cup \{-\infty, \infty\}$ there is a rearrangement $\sum_{n\in\mathbb{N}} a_{p(n)}$ which converges to L'.

Consequently, $\sum |a_n|$ converges if and only if all rearrangements $\sum a_{p(n)}$ converge to the same value as $\sum a_n$. A series $\sum a_n$ is *conditionally convergent* if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Rearranging Infinite Series

Definition (Rearrangement Set)

A set $\mathcal{RR}$ of bijections from $\mathbb{N}$ to $\mathbb{N}$ is a *rearrangement set* if for all conditionally convergent series $\sum a_n$ there exists $p \in \mathcal{RR}$ such that $\sum a_{p(n)}$ diverges or $\sum a_{p(n)} \neq \sum a_n$.

Definition (The Rearrangement Number)

We use $\mathfrak{rr}$ to denote the smallest cardinality of any rearrangement set.

Upper and Lower Bounds for $\mathfrak{rr}$

Theorem (Paul Larson, Andreas Blass, 2015)

We have $\mathfrak{rr} \leq \mathbf{non}(\mathcal{M})$.

Proof. Let $\mathbb{N}^{\mathbb{N}}$ be the set of all functions from $\mathbb{N}$ to $\mathbb{N}$, and $\mathrm{Sym}(\mathbb{N}) \subset \mathbb{N}^{\mathbb{N}}$ be the set of all bijections in $\mathbb{N}^{\mathbb{N}}$. Equipping $\mathbb{N}$ with the discrete topology and $\mathbb{N}^{\mathbb{N}}$ with the product topology, we have $\mathrm{Sym}(\mathbb{N}) \cong \mathbb{N}^{\mathbb{N}} \cong \mathbb{R} \setminus \mathbb{Q}$.
For any conditionally convergent series $\sum a_n$, and for each $k \in \mathbb{N}$, the set

$$U_k^{(a_n)} := \left\{ p \in \mathrm{Sym}(\mathbb{N}) \ : \ \sum_{n=0}^{m} a_{p(n)} \geq k \text{ for some } m \in \mathbb{N} \right\}$$

is open and dense in $\mathrm{Sym}(\mathbb{N})$. So $\bigcap_{k \in \mathbb{N}} U_k^{(a_n)}$ is a dense G_δ set. Therefore any non-meagre set in $\mathrm{Sym}(\mathbb{N})$ is a rearrangement set. $\square$

Upper and Lower Bounds for $\mathfrak{rr}$

Theorem (Joel Hamkins, Will Brian, 2015)

We have $\mathfrak{b} \leq \mathfrak{rr}$.

Proof. Suppose $\mathcal{RR}$ is a rearrangement set with $|\mathcal{RR}| < \mathfrak{b}$. For each $p \in \mathcal{RR}$, define $f_p \colon \mathbb{N} \to \mathbb{N}$ by $f_p(x) := \max_{j \leq p^{-1}(x)} p(j)$. Then there is some $g \colon \mathbb{N} \to \mathbb{N}$ which eventually dominates each f_p. Define $\zeta \colon \mathbb{N} \to \mathbb{N}$ by

$$\zeta(0) := g(0),$$
$$\zeta(k+1) := \zeta(k) + g(\zeta(k)) + 1.$$

Then the series $\sum_{n \in \mathbb{N}} a_n$, where

$$a_n = \begin{cases} \frac{(-1)^k}{k+1} & \text{if } n = \zeta(k) \text{ for some (unique) } k \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

is a counterexample to $\mathcal{RR}$ being a rearrangement set. $\square$

Forcing Strict Inequalities

We thus have

$$|\mathbb{N}| < \mathfrak{b} \leq \mathfrak{rr} \leq \textbf{non}(\mathcal{M}) \leq |\mathbb{R}|.$$

Theorem (as appears in Jech 2003 and Blass 2010)

If **ZFC** is consistent, then **ZFC** cannot prove $\textbf{non}(\mathcal{M}) = |\mathbb{R}|$.

Proof. Fix a countable transitive model **M** of **ZFC** and an uncountable regular cardinal κ. Let $\mathbb{P}$ be the forcing notion of all functions in $\{0,1\}^{\kappa^{\mathbf{M}} \times \mathbb{N}}$ with finite domain, where p is stronger than q if and only if $q \subseteq p$. If $G \subseteq \mathbb{P}$ is a generic filter, then $f := \bigcup G$ is a function with domain $\kappa^{\mathbf{M}} \times \mathbb{N}$. The map

$$\kappa^{\mathbf{M}} \ni \alpha \mapsto f(\alpha, \cdot) \in \{0,1\}^{\mathbb{N}}$$

is injective in $\mathbf{M}[G]$, and $f(\alpha, \cdot) \notin \mathbf{M}$ for all $\alpha \in \kappa^{\mathbf{M}}$. Therefore

$$\mathbf{M}[G] \models \left|\{0,1\}^{\mathbb{N}}\right| \geq \kappa,$$

as $\mathbb{P}$ satisfies c.c.c.

Forcing Strict Inequalities

Theorem (as appears in Jech 2003 and Blass 2010)

If **ZFC** is consistent, then **ZFC** cannot prove $\mathbf{non}(\mathcal{M}) = |\mathbb{R}|$.

Proof (cont'd). So we have $|\mathbb{R}| \geq \kappa$ in the *Cohen model* $\mathbf{M}[G]$. Now we show $\mathbb{R}^{\mathbf{M}}$ is non-meagre in $\mathbb{R}^{\mathbf{M}[G]}$.
We will use the fact that, in **ZFC**, a set $\tilde{Y} \subseteq \{0,1\}^{\mathbb{N}}$ is meagre if and only if there is some $\tilde{x} \in \{0,1\}^{\mathbb{N}}$ and some interval partition $\tilde{\Pi}$ of $\mathbb{N}$ such that for all $\tilde{y} \in \tilde{Y}$ we do not have $\tilde{x}|_J = \tilde{y}|_J$ for infinitely many $J \in \tilde{\Pi}$.
Any real in the Cohen model is in a submodel generated by one Cohen real. Suppose $x \in \{0,1\}^{\mathbb{N}}$ is a real in the forcing extension generated by one Cohen real, and let $\Pi = \{\, [i_n, i_{n+1}) : n \in \mathbb{N} \,\}$ be an interval partition of $\mathbb{N}$. Then construct a real $y \in \{0,1\}^{\mathbb{N}}$ in $\mathbf{M}$ so that for any condition $p \in \mathbb{P}$ and $n \in \mathbb{N}$, the condition p cannot force the formula "y does not agree with x on any interval of Π beyond n". So $\mathbf{M}[G]$ does not satisfy this formula. $\square$

Forcing Strict Inequalities

Recall that

$$|\mathbb{N}| < \mathfrak{b} \leq \mathfrak{rr} \leq \textbf{non}(\mathcal{M}) \leq |\mathbb{R}|,$$

and that $\textbf{non}(\mathcal{M}) < |\mathbb{R}|$ is consistent with **ZFC**.

Theorem (Blass, Brendle, Brian, Hamkins, Hardy, and Larson, 2018)

If **ZFC** is consistent, then **ZFC** cannot prove that $\mathfrak{b} = \mathfrak{rr}$.

Proof. Letting $\textbf{cov}(\mathcal{L})$ be the smallest cardinality of any collection of Lebesgue null sets which covers all of $\mathbb{R}$, we have $\textbf{cov}(\mathcal{L}) \leq \mathfrak{rr}$. Furthermore, $\textbf{cov}(\mathcal{L}) < \mathfrak{b}$ in the Laver model, and $\textbf{cov}(\mathcal{L}) > \mathfrak{b}$ in the random reals model. The lower bounds for $\mathfrak{rr}$ are independent, so they can both be consistently strict. $\square$

Question

Can **ZFC** prove $\mathfrak{rr} = \textbf{non}(\mathcal{M})$?

References

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