Solutions to exercises in Bart Jacobs's book "Introduction to Coalgebra: Towards Mathematics of States and Observation"

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some date very far into the future, if ever

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These are my solutions to all the labelled exercises in Jacobs (2017). This document does not stand on its own; it is meant to supplement the book.

Contents

1 Motivation

1.1 Naturalness of Coalgebraic Representations

Exercise 1.1.1

1. *Prove that the composition operation ; as defined for coalgebras $S \to \{\perp\} \cup S$ is associative, i.e. satisfies $s_1 \,;\, (s_2 \,;\, s_3) = (s_1 \,;\, s_2) \,;\, s_3$, for all statements $s_1, s_2, s_3 \colon S \to \{\perp\} \cup S$.*

   Define a statement $\mathsf{skip}\colon S \to \{\perp\} \cup S$ *which is a unit for composition ; i.e. which satisfies* $(\mathsf{skip} \,;\, s) = s = (s \,;\, \mathsf{skip})$, *for all* $s \colon S \to \{\perp\} \cup S$.

2. *Do the same for ; defined on coalgebras $S \to \{\perp\} \cup S \cup (S \times E)$.*

(In both cases, statements with an associative composition operation and a unit element form a monoid.)

Solution.

1. Recall that the composition operation ; was defined as follows:

$$s \,;\, t \coloneqq \lambda x \in S. \begin{cases} \perp, & \text{if } s(x) = \perp, \\ t(x') & \text{if } s(x) = x' \in S, \end{cases}$$

   for coalgebras $s, t \colon S \to \{\perp\} \cup S$. Fix any three coalgebras $s_1, s_2, s_3 \colon S \to \{\perp\} \cup S$. Then

$$\begin{aligned} s_1 \,;\, (s_2 \,;\, s_3) &= \lambda x \in S. \begin{cases} \perp, & \text{if } s_1(x) = \perp, \\ (s_2 \,;\, s_3)(x'), & \text{if } s_1(x) = x' \in S, \end{cases} \\ &= \lambda x \in S. \begin{cases} \perp, & \text{if either } s_1(x) = \perp, \text{ or both } s_1(x) = x' \in S \text{ and } s_2(x') = \perp, \\ s_3(x''), & \text{if } s_1(x) = x' \in S \text{ and } s_2(x') = x'' \in S, \end{cases} \\ &= \lambda x \in S. \begin{cases} \perp, & \text{if } (s_1 \,;\, s_2)(x) = \perp, \\ s_3(x''), & \text{if } (s_1 \,;\, s_2)(x) = x'' \in S, \end{cases} \\ &= (s_1 \,;\, s_2) \,;\, s_3. \end{aligned}$$

   So the composition operation ; is associative.

   The coalgebra $\mathsf{skip}\colon S \to \{\perp\} \cup S$ defined by $\mathsf{skip}(x) \coloneqq x$, for all $x \in S$, satisfies $(\mathsf{skip} \,;\, s) = s = (s \,;\, \mathsf{skip})$ for all coalgebras $s \colon S \to \{\perp\} \cup S$.

2. Now we consider the composition operation ; defined as follows:

$$s \,;\, t \coloneqq \lambda x \in S. \begin{cases} \perp, & \text{if } s(x) = \perp, \\ t(x'), & \text{if } s(x) = x' \in S, \\ (x', e), & \text{if } s(x) = (x', e) \in S \times E, \end{cases}$$

   for coalgebras $s, t \colon S \to \{\perp\} \cup S \cup (S \times E)$. Fix any three coalgebras $s_1, s_2, s_3 \colon \{\perp\} \cup S \cup (S \times E)$. Then

$$s_1 \,;\, (s_2 \,; s_3) = \lambda x \in S. \begin{cases} \perp, & \text{if } s_1(x) = \perp, \\ (s_2 \,;\, s_3)(x'), & \text{if } s_1(x) = x' \in S, \\ (x', e), & \text{if } s_1(x) = (x', e) \in S \times E, \end{cases}$$

$$
\begin{aligned}
&= \lambda x \in S. \begin{cases} \bot, & \text{if either } s_1(x) = \bot, \text{ or both } s_1(x) = x' \in S \text{ and } s_2(x') = \bot, \\ s_3(x''), & \text{if } s_1(x) = x' \in S \text{ and } s_2(x') = x'' \in S, \\ (x'', e), & \text{if } s_1(x) = x' \in S \text{ and } s_2(x') = (x'', e) \in S \times E, \\ (x', e), & \text{if } s_1(x) = (x', e) \in S \times E, \end{cases} \\
&= \lambda x \in S. \begin{cases} \bot, & \text{if } (s_1 \,;\, s_2)(x) = \bot, \\ s_3(x''), & \text{if } (s_1 \,;\, s_2)(x) = x'' \in S, \\ (x'', e), & \text{if } (s_1; s_2)(x) = (x'', e) \in S \times E, \end{cases} \\
&= (s_1 \,;\, s_2) \,;\, s_3.
\end{aligned}
$$

So this composition operation ; is also associative.

Now define the coalgebra $\mathsf{skip}\colon S \to \{\bot\} \cup S \cup (S \times E)$ by $\mathsf{skip}(x) \coloneqq x$, for all $x \in S$. Then we have $(\mathsf{skip}\,;\, s) = s = (s\,;\, \mathsf{skip})$ for all coalgebras $s\colon S \to \{\bot\} \cup S \cup (S \times E)$. □

Exercise 1.1.2

Define also a composition monoid $(\mathsf{skip}, \,;\,)$ *for coalgebras* $S \to \mathcal{P}(S)$.

Solution. For coalgebras $s, t\colon S \to \mathcal{P}(S)$, define

$$
s\,;\, t \coloneqq \lambda x \in S. \left(\bigcup_{y \in s(x)} t(y) \right).
$$

Then, for coalgebras $s_1, s_2, s_3\colon S \to \mathcal{P}(S)$, we have

$$
\begin{aligned}
s_1 \,;\, (s_2 \,;\, s_3) &= \lambda x \in S. \left(\bigcup_{y \in s_1(x)} (s_2 \,;\, s_3)(y) \right) \\
&= \lambda x \in S. \left(\bigcup_{y \in s_1(x)} \bigcup_{z \in s_2(y)} s_3(z) \right) \\
&= \lambda x \in S. \left(\bigcup_{z \in (s_1 \,;\, s_2)(x)} s_3(z) \right) \\
&= (s_1 \,;\, s_2) \,;\, s_3.
\end{aligned}
$$

Furthermore, defining $\mathsf{skip}\colon S \to \mathcal{P}(S)$ by $\mathsf{skip}(x) \coloneqq \{x\}$ for all $x \in S$, we have

$$
\begin{aligned}
(\mathsf{skip}\,;\, s) &= \lambda x \in S. \left(\bigcup_{y \in \mathsf{skip}(x)} s(y) \right) \\
&= \lambda x \in S. \left(\bigcup_{y \in \{x\}} s(y) \right) \\
&= \lambda x \in S. s(x) \\
&= s
\end{aligned}
$$

and

$$\begin{aligned}(s\,;\,\mathsf{skip}) &= \lambda x \in S.\left(\bigcup_{y \in s(x)} \mathsf{skip}(y)\right)\\ &= \lambda x \in S.\left(\bigcup_{y \in s(x)} \{y\}\right)\\ &= \lambda x \in S.s(x)\\ &= s.\end{aligned}$$

□

1.2 The Power of Coinduction

Exercise 1.2.1
Compute the nextdec*-behaviour of* $\frac{1}{7} \in [0,1)$ *as in Example 1.2.2.*

Solution. We first recall all of the following functions.

1. The final coalgebra $\mathsf{next}\colon \{0,\dots,9\}^\infty \to \{\bot\} \cup \big(\{0,\dots,9\} \times \{0,\dots,9\}^\infty\big)$ is defined by

$$\mathsf{next}(\sigma) := \begin{cases}\bot, & \text{if } \sigma \text{ is the empty sequence,}\\ (d,\sigma'), & \text{if } \sigma \text{ has head } d \in \{0,\dots,9\} \text{ and tail } \sigma' \in \{0,\dots,9\}^\infty\text{, i.e. } \sigma = d\cdot\sigma',\end{cases}$$

   for all (finite or infinite) sequences $\sigma \in \{0,\dots,9\}^\infty$.

2. The coalgebra $\mathsf{nextdec}\colon [0,1) \to \{\bot\} \cup \big(\{0,\dots,9\} \times [0,1)\big)$ is defined by

$$\mathsf{nextdec}(r) := \begin{cases}\bot, & \text{if } r = 0,\\ (d, 10r - d), & \text{if } d \le 10r < d+1 \text{ and } d \in \{0,\dots,9\},\end{cases}$$

   for all $r \in [0,1)$.

3. The function $\mathrm{beh}_{\mathsf{nextdec}}\colon [0,1) \to \{0,\dots,9\}^\infty$ is the unique function making

$$\begin{tikzcd}
\{\bot\} \cup \big(\{0,\dots,9\} \times [0,1)\big) \arrow[r, dashed, "\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_{\{0,\dots,9\}} \times \mathrm{beh}_{\mathsf{nextdec}})"] & \{\bot\} \cup \big(\{0,\dots,9\} \times \{0,\dots,9\}^\infty\big)\\
{[0,1)} \arrow[u, "\mathsf{nextdec}"] \arrow[r, dashed, "\exists!\mathrm{beh}_{\mathsf{nextdec}}"'] & \{0,\dots,9\}^\infty \arrow[u, "\mathsf{next}", "\cong"']
\end{tikzcd}$$

   commute.

We wish to compute $\mathrm{beh}_{\mathsf{nextdec}}\big(\frac{1}{7}\big)$. We see that

$$\begin{aligned}\mathrm{beh}_{\mathsf{nextdec}}\left(\frac{1}{7}\right) &= \mathsf{next}^{-1}\left(\left(\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_{\{0,\dots,9\}} \times \mathrm{beh}_{\mathsf{nextdec}})\right)\left(\mathsf{nextdec}\left(\frac{1}{7}\right)\right)\right)\\ &= \mathsf{next}^{-1}\left(\left(\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_{\{0,\dots,9\}} \times \mathrm{beh}_{\mathsf{nextdec}})\right)\left(\left(1, \frac{3}{7}\right)\right)\right)\end{aligned}$$

$$
\begin{aligned}
&= \mathsf{next}^{-1}\left(\left(1,\ \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{3}{7}\right)\right)\right) \\
&= 1 \cdot \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{3}{7}\right).
\end{aligned}
$$

Continuing in this fashion,

$$
\begin{aligned}
\mathrm{beh}_{\mathsf{nextdec}}\left(\frac{1}{7}\right) &= 1 \cdot \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{3}{7}\right) \\
&= 1 \cdot \left(4 \cdot \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{2}{7}\right)\right) \\
&= 1 \cdot \left(4 \cdot \left(2 \cdot \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{6}{7}\right)\right)\right) \\
&= 1 \cdot \left(4 \cdot \left(2 \cdot \left(8 \cdot \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{4}{7}\right)\right)\right)\right) \\
&= 1 \cdot \left(4 \cdot \left(2 \cdot \left(8 \cdot \left(5 \cdot \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{5}{7}\right)\right)\right)\right)\right) \\
&= 1 \cdot \left(4 \cdot \left(2 \cdot \left(8 \cdot \left(5 \cdot \left(7 \cdot \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{1}{7}\right)\right)\right)\right)\right)\right).
\end{aligned}
$$

Therefore $\mathrm{beh}_{\mathsf{nextdec}}\left(\frac{1}{7}\right) = \langle 1, 4, 2, 8, 5, 7, 1, 4, 2, 8, 5, 7, 1, 4, 2, 8, 5, 7, \dots \rangle$.

□

Exercise 1.2.2
Formulate appropriate rules for the function $\mathsf{odds}\colon A^\infty \to A^\infty$ *in analogy with the rules (1.7) for* evens.

Solution. We recall that, for a sequence $\sigma := \langle a_0, a_1, a_2, a_3, \dots \rangle \in A^\infty$, the function odds satisfies $\mathsf{odds}(\sigma) = \langle a_1, a_3, a_5, \dots \rangle$, and analogously if σ is a finite sequence. The rules we want odds to satisfy are:

$$\frac{\sigma \not\rightarrow}{\mathsf{odds}(\sigma) \not\rightarrow}$$

i.e. odds should send the empty sequence to the empty sequence;

$$\frac{\sigma \xrightarrow{a} \sigma' \qquad \sigma' \not\rightarrow}{\mathsf{odds}(\sigma) \not\rightarrow}$$

i.e. odds should send a singleton sequence $\langle a \rangle$ to the empty sequence; and

$$\frac{\sigma \xrightarrow{a} \sigma' \qquad \sigma' \xrightarrow{a'} \sigma''}{\mathsf{odds}(\sigma) \xrightarrow{a'} \mathsf{odds}(\sigma')}$$

i.e. if $\sigma = a \cdot a' \cdot \sigma' \in A^\infty$, where $a, a' \in A$, then $\mathsf{odds}(\sigma) = a' \cdot \mathsf{odds}(\sigma')$. □

Exercise 1.2.3
Use coinduction to define the empty sequence $\langle\rangle \in A^\infty$ *as a map* $\{\perp\} \to A^\infty$.

Fix an element $a \in A$, *and similarly define the infinite sequence* $\overrightarrow{a}\colon \{\perp\} \to A^\infty$ *consisting of only* a*s.*

Solution. We recall that the final coalgebra $\mathsf{next}\colon A^\infty \to \{\bot\} \cup \big(A \times A^\infty\big)$ is defined by

$$\mathsf{next}(\sigma) \coloneqq \begin{cases} \bot, & \text{if } \sigma \text{ is the empty sequence,} \\ (a, \sigma'), & \text{if } \sigma \text{ has head } a \in A \text{ and tail } \sigma' \in A^\infty\text{, i.e. } \sigma = a \cdot \sigma', \end{cases}$$

for all (finite or infinite) sequences $\sigma \in A^\infty$.

For the coalgebra $\kappa_1\colon \{\bot\} \to \{\bot\} \cup (A \times \{\bot\})$ defined by $\kappa_1(\bot) \coloneqq \bot$, the unique function $\mathrm{beh}_{\kappa_1}\colon \{\bot\} \to A^\infty$ making

$$\begin{tikzcd} \{\bot\} \cup (A \times \{\bot\}) \arrow[r, dashed, "\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_A \times \mathrm{beh}_{\kappa_1})"] & \{\bot\} \cup (A \times A^\infty) \\ \{\bot\} \arrow[u, "\kappa_1"] \arrow[r, dashed, "\exists!\mathrm{beh}_{\kappa_1}"'] & A^\infty \arrow[u, "\cong"', "\mathsf{next}"] \end{tikzcd}$$

commute satisfies $\mathrm{beh}_{\kappa_1}(\bot) = \langle\rangle$.

For the coalgebra $c_a\colon \{\bot\} \to \{\bot\} \cup (A \times \{\bot\})$ defined by $c_a(\bot) \coloneqq (a, \bot)$, the unique function $\mathrm{beh}_{c_a}\colon \{\bot\} \to A^\infty$ making

$$\begin{tikzcd} \{\bot\} \cup (A \times \{\bot\}) \arrow[r, dashed, "\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_A \times \mathrm{beh}_{c_a})"] & \{\bot\} \cup (A \times A^\infty) \\ \{\bot\} \arrow[u, "c_a"] \arrow[r, dashed, "\exists!\mathrm{beh}_{c_a}"'] & A^\infty \arrow[u, "\cong"', "\mathsf{next}"] \end{tikzcd}$$

commute satisfies $\mathrm{beh}_{c_a}(\bot) = \overrightarrow{a} = \langle a, a, a, \dots \rangle$. $\square$

Exercise 1.2.4
Compute the outcome of $\mathrm{merge}(\langle a_0, a_1, a_2\rangle, \langle b_0, b_1, b_2, b_3\rangle)$.

Solution. Recall that we defined the coalgebra $m\colon A^\infty \times A^\infty \to \{\bot\} \cup (A \times (A^\infty \times A^\infty))$ by

$$m(\sigma, \tau) \coloneqq \begin{cases} \bot, & \text{if } \sigma \not\to \text{ and } \tau \not\to, \\ (a, (\sigma, \tau')), & \text{if } \sigma \not\to \text{ and } \tau \xrightarrow{a} \tau', \\ (a, (\tau, \sigma')), & \text{if } \sigma \xrightarrow{a} \sigma', \end{cases}$$

for all $\sigma, \tau \in A^\infty$, and that $\mathsf{merge}\colon A^\infty \times A^\infty \to A^\infty$ is the unique function making

$$\begin{tikzcd} \{\bot\} \cup (A \times (A^\infty \times A^\infty)) \arrow[r, dashed, "\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_A \times \mathsf{merge})"] & \{\bot\} \cup (A \times A^\infty) \\ A^\infty \times A^\infty \arrow[u, "m"] \arrow[r, dashed, "\exists!\mathsf{merge}"'] & A^\infty \arrow[u, "\cong"', "\mathsf{next}"] \end{tikzcd}$$

commute. Then

$$\mathsf{merge}(\langle a_0, a_1, a_2\rangle, \langle b_0, b_1, b_2, b_3\rangle) = \mathsf{next}^{-1}\Big(\big(\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_A \times \mathsf{merge})\big)\big(m(\langle a_0, a_1, a_2\rangle, \langle b_0, b_1, b_2, b_3\rangle)\big)\Big)$$

$$
\begin{aligned}
&= \mathsf{next}^{-1}\Big(\big(\mathrm{id}_{\{\perp\}} \ \cup \ (\mathrm{id}_A \times \mathsf{merge})\big)\big((a_0, (\langle b_0, b_1, b_2, b_3\rangle, \langle a_1, a_2\rangle)))\big)\Big) \\
&= \mathsf{next}^{-1}\Big(\big(a_0, \mathsf{merge}(\langle b_0, b_1, b_2, b_3\rangle, \langle a_1, a_2\rangle)\big)\Big) \\
&= a_0 \cdot \mathsf{merge}(\langle b_0, b_1, b_2, b_3\rangle, \langle a_1, a_2\rangle),
\end{aligned}
$$

and so on. Eventually, we obtain $\mathsf{merge}(\langle a_0, a_1, a_2\rangle, \langle b_0, b_1, b_2, b_3\rangle) = \langle a_0, b_0, a_1, b_1, a_2, b_2, b_3\rangle$. □

Exercise 1.2.5
Is the merge *operation associative, i.e. is* $\mathsf{merge}(\sigma, \mathsf{merge}(\tau, \rho))$ *the same as* $\mathsf{merge}(\mathsf{merge}(\sigma, \tau), \rho)$*? Give a proof or a counterexample. Is there a neutral element for* merge*?*

Solution. The merge operation is not associative:

$$
\begin{aligned}
\mathsf{merge}(\langle a\rangle, \mathsf{merge}(\langle b\rangle, \langle c\rangle)) &= \mathsf{merge}(\langle a\rangle, \langle b, c\rangle) \\
&= \langle a, b, c\rangle,
\end{aligned}
$$

whereas

$$
\begin{aligned}
\mathsf{merge}(\mathsf{merge}(\langle a\rangle, \langle b\rangle), \langle c\rangle) &= \mathsf{merge}(\langle a, b\rangle, \langle c\rangle) \\
&= \langle a, c, b\rangle,
\end{aligned}
$$

for all $a, b, c \in A$.

The neutral element for merge is the empty sequence: for any $\sigma \in A^\infty$, we have $\mathsf{merge}(\sigma, \langle\rangle) = \mathsf{merge}(\langle\rangle, \sigma) = \sigma$. □

Exercise 1.2.6
Show how to define an alternative merge function which alternatingly takes two elements from its argument sequences.

Solution. We will define a coalgebra $m_2 \colon A^\infty \times A^\infty \to \{\perp\} \cup (A \times (A^\infty \times A^\infty))$ so that the desired merge function is the unique function $\mathsf{merge}_2 \colon A^\infty \times A^\infty \to A^\infty$ making

$$
\begin{tikzcd}
\{\perp\} \cup (A \times (A^\infty \times A^\infty)) \arrow[r, dashed, "\mathrm{id}_{\{\perp\}} \ \cup \ (\mathrm{id}_A \times \mathsf{merge}_2)"] & \{\perp\} \cup (A \times A^\infty) \\
A^\infty \times A^\infty \arrow[u, "m_2"] \arrow[r, dashed, "\exists! \mathsf{merge}_2"'] & A^\infty \arrow[u, "\mathsf{next}"', "\cong"]
\end{tikzcd}
$$

commute. As a motivating example, the desired merge of two infinite streams $\langle a_0, a_1, \ldots\rangle$ and $\langle b_0, b_1, \ldots\rangle$ should be

$$\mathsf{merge}_2(\langle a_0, a_1, a_2, a_3, \ldots\rangle, \langle b_0, b_1, b_2, b_3, \ldots\rangle) = \langle a_0, a_1, b_0, b_1, a_2, a_3, b_2, b_3, \ldots\rangle.$$

As the diagram above commutes, we would require

$$\mathsf{merge}_2\big(m_2(\langle a_0, a_1, a_2, a_3, \ldots\rangle, \langle b_0, b_1, b_2, b_3, \ldots\rangle)\big) = \big(a_0, \langle a_1, b_0, b_1, a_2, a_3, b_2, b_3, \ldots\rangle\big)$$

and so m_2 should be defined to satisfy

$$m_2(\langle a_0, a_1, a_2, a_3, \ldots\rangle, \langle b_0, b_1, b_2, b_3, \ldots\rangle) = \big(a_0, (\langle a_1, b_0, a_3, b_2, \ldots\rangle, \langle b_1, a_2, b_3, a_4, \ldots\rangle)\big)$$

Dealing with edge cases separately leads us to the following definition: we define the coalgebra $m_2 \colon A^\infty \times A^\infty \to \{\perp\} \cup (A \times (A^\infty \times A^\infty))$ as follows.

1. The function m_2 sends the pair $(\langle\rangle, \langle\rangle)$ to $\perp$, i.e.

$$m_2(\langle\rangle, \langle\rangle) \coloneqq \perp.$$

2. If $\tau \in A^\infty$ is a non-empty sequence, say $\tau \xrightarrow{a} \tau'$ for some $\tau' \in A^\infty$ and $a \in A$, then

$$m_2(\langle\rangle, \tau) \coloneqq \big(a, (\langle\rangle, \tau')\big).$$

3. If $\sigma = \langle a \rangle$ for some $a \in A$, then
$$m_2(\langle a\rangle, \tau) \coloneqq \big(a, (\langle\rangle, \tau)\big)$$
for all $\tau \in A^\infty$.

4. If $\sigma \in A^\infty$ has at least length 2, say $\sigma \xrightarrow{a} \sigma' \xrightarrow{a'} \sigma''$ for some $\sigma', \sigma'' \in A^\infty$ and $a, a' \in A$, then

$$m_2(\sigma, \tau) \coloneqq \Big(a, \big(\mathsf{merge}(\mathsf{odds}(\sigma), \mathsf{evens}(\tau)), \mathsf{merge}(\mathsf{odds}(\tau), \mathsf{evens}(\sigma''))\big)\Big)$$

for all $\tau \in A^\infty$.

Now let $\mathsf{merge}_2 \colon A^\infty \times A^\infty \to A^\infty$ be the unique function which makes

$$\begin{tikzcd}
\{\perp\} \cup (A \times (A^\infty \times A^\infty)) \arrow[r, dashed, "\mathrm{id}_{\{\perp\}} \,\cup\, (\mathrm{id}_A \times \mathsf{merge}_2)"] & \{\perp\} \cup (A \times A^\infty) \\
A^\infty \times A^\infty \arrow[u, "m_2"] \arrow[r, dashed, "\exists! \mathsf{merge}_2"'] & A^\infty \arrow[u, "\cong"', "\mathsf{next}"]
\end{tikzcd}$$

commute. Fix any $\sigma, \tau \in A^\infty$. We argue by cases on (σ, τ) that this function merge_2 is the desired merge function.

1. If $\sigma = \tau = \langle\rangle$, then $\mathsf{merge}_2(\langle\rangle, \langle\rangle) = \langle\rangle$.

2. If $\sigma = \langle\rangle$ and τ is a non-empty sequence, say $\tau = a \cdot \tau'$ for some $a \in A$ and $\tau' \in A^\infty$, then

$$\mathsf{merge}_2(\langle\rangle, \tau) = a \cdot \mathsf{merge}_2(\langle\rangle, \tau').$$

Thus $\mathsf{merge}_2(\langle\rangle, \tau) = \tau$.

3. If $\sigma = \langle a \rangle$ for some $a \in A$, then

$$\begin{aligned}
\mathsf{merge}_2(\langle a\rangle, \tau) &= a \cdot \mathsf{merge}_2(\langle\rangle, \tau) \\
&= a \cdot \tau.
\end{aligned}$$

4. If $\sigma = a \cdot a' \cdot \sigma''$ for some $a, a' \in A$ and $\sigma'' \in A^\infty$, then

$$\begin{aligned}
\mathsf{merge}_2(\sigma, \tau) &= a \cdot \mathsf{merge}_2\Big(\mathsf{merge}\big(\mathsf{odds}(\sigma), \mathsf{evens}(\tau)\big), \mathsf{merge}\big(\mathsf{odds}(\tau), \mathsf{evens}(\sigma'')\big)\Big) \\
&= a \cdot \mathsf{merge}_2\Big(\mathsf{merge}\big(a' \cdot \mathsf{odds}(\sigma''), \mathsf{evens}(\tau)\big), \mathsf{merge}\big(\mathsf{odds}(\tau), \mathsf{evens}(\sigma'')\big)\Big) \\
&= a \cdot a' \cdot \mathsf{merge}_2\Big(\mathsf{merge}\big(\mathsf{odds}(\mathsf{merge}(a' \cdot \mathsf{odds}(\sigma''), \mathsf{evens}(\tau))), \\
&\qquad\qquad \mathsf{evens}(\mathsf{merge}(\mathsf{odds}(\tau), \mathsf{evens}(\sigma'')))\big),
\end{aligned}$$

$$
\begin{aligned}
&\mathsf{merge}\big(\mathsf{odds}(\mathsf{merge}(\mathsf{odds}(\tau), \mathsf{evens}(\sigma''))),\\
&\qquad \mathsf{odds}(\mathsf{merge}(\mathsf{evens}(\tau), \mathsf{odds}(\sigma''))))\big)\\
&= a \cdot a' \cdot \mathsf{merge}_2\Big(\mathsf{merge}\big(\mathsf{evens}(\tau), \mathsf{odds}(\tau)\big), \mathsf{merge}\big(\mathsf{evens}(\sigma''), \mathsf{odds}(\sigma'')\big)\Big)\\
&= a \cdot a' \cdot \mathsf{merge}_2(\tau, \sigma''),
\end{aligned}
$$

as desired. □

Exercise 1.2.7

1. *Define three functions* $\mathsf{ex}_i \colon A^\infty \to A^\infty$, *for* $i = 0, 1, 2$, *which extract the elements at positions* $3n + i$.
2. *Define* $\mathsf{merge3} \colon A^\infty \times A^\infty \times A^\infty \to A^\infty$ *satisfying the equation* $\mathsf{merge3}(\mathsf{ex}_0(\sigma), \mathsf{ex}_1(\sigma), \mathsf{ex}_2(\sigma)) = \sigma$, *for all* $\sigma \in A^\infty$.

Solution.

1. Define $c_0, c_1, c_2 \colon A^\infty \to \{\perp\} \cup (A \times A^\infty)$ as follows:

$$
c_0(\sigma) := \begin{cases} \perp, & \text{if } \sigma = \langle\rangle,\\ (a, \langle\rangle) & \text{if } \sigma = \langle a\rangle \text{ or } \sigma = \langle a, a'\rangle \text{ for some } a, a' \in A,\\ (a, \sigma'''), & \text{if } \sigma \xrightarrow{a} \sigma' \xrightarrow{a'} \sigma'' \xrightarrow{a''} \sigma''', \end{cases}
$$
$$
c_1(\sigma) := \begin{cases} \perp, & \text{if } \sigma = \langle\rangle \text{ or } \sigma = \langle a\rangle \text{ for some } a \in A,\\ (a', \langle\rangle) & \text{if } \sigma = \langle a, a'\rangle \text{ for some } a, a' \in A,\\ (a', \sigma'''), & \text{if } \sigma \xrightarrow{a} \sigma' \xrightarrow{a'} \sigma'' \xrightarrow{a''} \sigma''', \end{cases}
$$
$$
c_2(\sigma) := \begin{cases} \perp, & \text{if } \sigma = \langle\rangle, \text{ or } \sigma = \langle a\rangle, \text{ or } \sigma = \langle a, a'\rangle \text{ for some } a, a' \in A,\\ (a'', \sigma'''), & \text{if } \sigma \xrightarrow{a} \sigma' \xrightarrow{a'} \sigma'' \xrightarrow{a''} \sigma'''. \end{cases}
$$

Then, for $i \in \{0, 1, 2\}$, the function $\mathsf{ex}_i \colon A^\infty \to A^\infty$ is the unique function making

$$
\begin{tikzcd}
\{\perp\} \cup (A \times A^\infty) \arrow[r, dashed, "\mathrm{id}_{\{\perp\}} \,\cup\, (\mathrm{id}_A \times \mathsf{ex}_i)"] & \{\perp\} \cup (A \times A^\infty) \\
A^\infty \arrow[u, "c_i"] \arrow[r, dashed, "\exists! \mathsf{ex}_i"'] & A^\infty \arrow[u, "\cong"', "\mathsf{next}"]
\end{tikzcd}
$$

commute.

2. Define the coalgebra $m_3 \colon A^\infty \times A^\infty \times A^\infty \to \{\perp\} \cup (A \times (A^\infty \times A^\infty \times A^\infty))$ by

$$
m_3(\sigma, \tau, \rho) := \begin{cases} \perp, & \text{if } \sigma = \tau = \rho = \langle\rangle,\\ \big(a, (\langle\rangle, \langle\rangle, \rho')\big), & \text{if } \sigma = \tau = \langle\rangle \text{ and } \rho \xrightarrow{a} \rho' \text{ for some } a \in A \text{ and } \rho' \in A^\infty,\\ \big(a, (\langle\rangle, \rho, \tau')\big), & \text{if } \sigma = \langle\rangle \text{ and } \tau \xrightarrow{a} \tau' \text{ for some } a \in A \text{ and } \tau' \in A^\infty,\\ \big(a, (\tau, \rho, \sigma')\big), & \text{if } \sigma \xrightarrow{a} \sigma' \text{ for some } a \in A \text{ and } \sigma' \in A^\infty. \end{cases}
$$

Then we let $\mathsf{merge3}\colon A^\infty \times A^\infty \times A^\infty \to A^\infty$ be the unique function making

$$\begin{tikzcd}
\{\bot\} \cup (A \times (A^\infty \times A^\infty \times A^\infty)) \arrow[r, dashed, "\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times \mathsf{merge3})"] & \{\bot\} \cup (A \times A^\infty) \\
A^\infty \times A^\infty \times A^\infty \arrow[u, "m_3"] \arrow[r, dashed, "\exists!\mathsf{merge3}"'] & A^\infty \arrow[u, "\cong"', "\mathsf{next}"]
\end{tikzcd}$$

commute.

Let us prove that $\mathsf{merge3}(\mathsf{ex}_0(\sigma), \mathsf{ex}_1(\sigma), \mathsf{ex}_2(\sigma)) = \sigma$ for all $\sigma \in A^\infty$, by coinduction. Consider the function $f\colon A^\infty \to A^\infty \times A^\infty \times A^\infty$ defined by $f(\sigma) := \big(\mathsf{ex}_0(\sigma), \mathsf{ex}_1(\sigma), \mathsf{ex}_2(\sigma)\big)$ for all $\sigma \in A^\infty$. We wish to show that $\mathsf{merge3} \circ f = \mathrm{id}_{A^\infty}$.

$$\begin{tikzcd}
 & \{\bot\} \cup (A \times (A^\infty \times A^\infty \times A^\infty)) \arrow[r, "\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times \mathsf{merge3})"] & \{\bot\} \cup (A \times A^\infty) \\
\{\bot\} \cup (A \times A^\infty) \arrow[ur, "\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times f)"] & & \\
 & A^\infty \times A^\infty \times A^\infty \arrow[uu, "m_3"] \arrow[r, "\mathsf{merge3}"'] & A^\infty \arrow[uu, "\cong"', "\mathsf{next}"] \\
A^\infty \arrow[uu, "\mathsf{next}", "\cong"'] \arrow[ur, "f"'] & &
\end{tikzcd}$$

Let us first show that the left square commutes. It certainly commutes when we chase the empty sequence: $(m_3 \circ f)(\langle\rangle) = \bot = \big(\big(\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times f)\big) \circ \mathsf{next}\big)(\langle\rangle)$. If $\sigma \in A^\infty$ is a non-empty sequence, say $\sigma \xrightarrow{a} \sigma'$ for some $a \in A$ and $\sigma' \in A^\infty$, then we have

$$\begin{aligned}
(m_3 \circ f)(\sigma) &= m_3\big(\mathsf{ex}_0(\sigma), \mathsf{ex}_1(\sigma), \mathsf{ex}_2(\sigma)\big) \\
&= \big(a, (\mathsf{ex}_1(\sigma), \mathsf{ex}_2(\sigma), \mathsf{ex}_0(\sigma'))\big) \\
&= \big(a, (\mathsf{ex}_0(\sigma'), \mathsf{ex}_1(\sigma'), \mathsf{ex}_2(\sigma'))\big) \\
&= \big(\big(\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times f)\big) \circ \mathsf{next}\big)(\sigma),
\end{aligned}$$

where the second-to-last equality can also be proven by coinduction. Therefore the outer square commutes, and so

$$\mathsf{next} \circ (\mathsf{merge3} \circ f) = \big(\big(\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times (\mathsf{merge3} \circ f))\big) \circ \mathsf{next}.$$

The finality of the coalgebra $\mathsf{next}\colon A^\infty \to \{\bot\} \cup (A \times A^\infty)$ now yields $\mathsf{merge3} \circ f = \mathrm{id}_{A^\infty}$. □

Exercise 1.2.8

Consider the sequential composition function $\mathsf{comp}\colon A^\infty \times A^\infty \to A^\infty$ *for sequences, described by the three rules:*

$$\frac{\sigma \not\to \qquad \tau \not\to}{\mathsf{comp}(\sigma, \tau) \not\to} \qquad\qquad \frac{\sigma \not\to \qquad \tau \xrightarrow{a} \tau'}{\mathsf{comp}(\sigma, \tau) \xrightarrow{a} \mathsf{comp}(\sigma, \tau')}$$

$$\frac{\sigma \xrightarrow{a} \sigma'}{\mathsf{comp}(\sigma, \tau) \xrightarrow{a} \mathsf{comp}(\sigma', \tau)}\,.$$

1. *Show by coinduction that the empty sequence* $\langle\rangle = \mathsf{next}^{-1}(\perp) \in A^\infty$ *is a unit element for* comp, *i.e. that* $\mathsf{comp}(\langle\rangle, \sigma) = \sigma = \mathsf{comp}(\sigma, \langle\rangle)$.

2. *Prove also by coinduction that* comp *is associative, and thus that sequences carry a monoid structure.*

Solution.

1. Let $f\colon A^\infty \to A^\infty$ be defined by $f(\sigma) \coloneqq \mathsf{comp}(\langle\rangle, \sigma)$. We will show that the diagram

$$\begin{tikzcd}
\{\perp\} \cup (A \times A^\infty) \arrow[rr, "\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times f)"] & & \{\perp\} \cup (A \times A^\infty) \\
A^\infty \arrow[u, "\mathsf{next}", "\cong"'] \arrow[rr, "f"'] & & A^\infty \arrow[u, "\cong", "\mathsf{next}"']
\end{tikzcd}$$

commutes, which would yield $f = \mathrm{id}_{A^\infty}$ by the finality of the coalgebra next.

First, we chase the empty sequence from the bottom left. We see that

$$\begin{aligned}
(\mathsf{next} \circ f)(\langle\rangle) &= \mathsf{next}(\mathsf{comp}(\langle\rangle, \langle\rangle)) \\
&= \mathsf{next}(\langle\rangle) \\
&= \perp,
\end{aligned}$$

the first rule for comp, and

$$\begin{aligned}
\big(\big(\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times f)\big) \circ \mathsf{next}\big)(\langle\rangle) &= \big(\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times f)\big)(\perp) \\
&= \perp.
\end{aligned}$$

Now if $\sigma \in A^\infty$ is a non-empty sequence, say $\sigma \xrightarrow{a} \sigma'$ for some $a \in A$ and $\sigma' \in A^\infty$, we see that

$$\begin{aligned}
(\mathsf{next} \circ f)(\sigma) &= \mathsf{next}(\mathsf{comp}(\langle\rangle, a \cdot \sigma')) \\
&= (a, \mathsf{comp}(\langle\rangle, \sigma')) \\
&= (a, f(\sigma')),
\end{aligned}$$

by the second rule for comp and the definition of f, and

$$\begin{aligned}
\big(\big(\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times f)\big) \circ \mathsf{next}\big)(\sigma) &= \big(\big(\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times f)\big)\big((a, \sigma')\big) \\
&= (a, f(\sigma')).
\end{aligned}$$

Thus $\mathsf{next} \circ f = \big(\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times f)\big) \circ \mathsf{next}$. This proves that $\mathsf{comp}(\langle\rangle, \sigma) = \sigma$ for all $\sigma \in A^\infty$.

We now show the other equality, that $\mathsf{comp}(\sigma, \langle\rangle) = \sigma$ for all $\sigma \in A^\infty$, we will show that the function $g\colon A^\infty \to A^\infty$ defined by $g(\sigma) \coloneqq \mathsf{comp}(\sigma, \langle\rangle)$ for all $\sigma \in A^\infty$ also satisfies

$$\mathsf{next} \circ g = \big(\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times g)\big) \circ \mathsf{next}.$$

That $(\mathsf{next} \circ g)(\perp) = \big(\big(\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times g)\big) \circ \mathsf{next}\big)(\perp)$ is the same as with f. Now if $\sigma \in A^\infty$ is such that $\sigma \xrightarrow{a} \sigma'$ for some $a \in A$ and $\sigma' \in A^\infty$, we see that

$$(\mathsf{next} \circ g)(\sigma) = \mathsf{next}(\mathsf{comp}(a \cdot \sigma', \langle\rangle)$$

$$
\begin{aligned}
&= (a, \mathsf{comp}(\sigma', \langle\rangle)) \\
&= (a, g(\sigma')),
\end{aligned}
$$

by the third rule for comp and the definition of g, and

$$
\begin{aligned}
\big((\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times g)\big) \circ \mathsf{next}\big)(\sigma) &= \big((\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times g))\big)\big((a, \sigma')\big) \\
&= \big(a, g(\sigma')\big).
\end{aligned}
$$

Therefore $g = \mathrm{id}_{A^\infty}$, i.e. $\mathsf{comp}(\sigma, \langle\rangle) = \sigma$ for all $\sigma \in A^\infty$.

2. We will define a coalgebra $c\colon A^\infty \times A^\infty \times A^\infty \to \{\perp\} \cup (A \times (A^\infty \times A^\infty \times A^\infty))$ such that the functions $h, k\colon A^\infty \times A^\infty \times A^\infty \to A^\infty$ given by

$$
\begin{aligned}
h(\sigma, \tau, \rho) &\coloneqq \mathsf{comp}(\sigma, \mathsf{comp}(\tau, \rho)) \qquad \text{and} \\
k(\sigma, \tau, \rho) &\coloneqq \mathsf{comp}(\mathsf{comp}(\sigma, \tau), \rho),
\end{aligned}
$$

for all $\sigma, \tau, \rho \in A^\infty$, are both coalgebra homomorphisms from c to next.

$$
\begin{tikzcd}
\{\perp\} \cup (A \times (A^\infty \times A^\infty \times A^\infty)) \arrow[rr, bend left, "\mathrm{id}_{\{\perp\}} \,\cup\, (\mathrm{id}_A \times h)"] \arrow[rr, bend right, "\mathrm{id}_{\{\perp\}} \,\cup\, (\mathrm{id}_A \times k)"'] & & \{\perp\} \cup (A \times A^\infty) \\
A^\infty \times A^\infty \times A^\infty \arrow[u, "c"] \arrow[rr, bend left, "h"] \arrow[rr, bend right, "k"'] & & A^\infty \arrow[u, "\cong"', "\mathsf{next}"]
\end{tikzcd}
$$

The finality of next would then yield $h = k$.

Define $c\colon A^\infty \times A^\infty \times A^\infty \to \{\perp\} \cup (A \times (A^\infty \times A^\infty \times A^\infty))$ by

$$
c(\sigma, \tau, \rho) \coloneqq \begin{cases}
\perp, & \text{if } \sigma = \tau = \rho = \langle\rangle, \\
\big(a, (\langle\rangle, \langle\rangle, \rho'\big), & \text{if } \sigma = \tau = \langle\rangle \text{ and } \rho = a \cdot \rho' \text{ for some } a \in A \text{ and } \rho' \in A^\infty, \\
\big(a, (\langle\rangle, \tau', \rho)\big), & \text{if } \sigma = \langle\rangle \text{ and } \tau = a \cdot \tau' \text{ for some } a \in A \text{ and } \tau' \in A^\infty, \\
\big(a, (\sigma', \tau, \rho)\big), & \text{if } \sigma = a \cdot \sigma' \text{ for some } a \in A \text{ and } \sigma' \in A^\infty.
\end{cases}
$$

Using the rules for comp, it is now elementary to check that h and k make their respective diagrams commute. □

Exercise 1.2.9

Consider two sets A, B with a function $f\colon A \to B$ between them. Use finality to define a function $f^\infty\colon A^\infty \to B^\infty$ that applies f element-wise. Use uniqueness to show that this mapping $f \mapsto f^\infty$ is 'functorial' in the sense that $(\mathrm{id}_A)^\infty = \mathrm{id}_{A^\infty}$ and $(g \circ f)^\infty = g^\infty \circ f^\infty$.

Solution. For a (non-empty) set B, let $\mathsf{next}_B\colon B^\infty \to \{\perp\} \cup (B \times B^\infty)$ denote the final coalgebra defined by

$$
\mathsf{next}(\sigma) \coloneqq \begin{cases}
\perp, & \text{if } \sigma \text{ is the empty sequence}, \\
(b, \sigma'), & \text{if } \sigma \text{ has head } b \in B \text{ and tail } \sigma' \in B^\infty, \text{ i.e. } \sigma = b \cdot \sigma',
\end{cases}
$$

for all $\sigma \in B^\infty$. For a function $f\colon A \to B$, define a coalgebra $c_f\colon A^\infty \to \{\bot\} \cup (B \times A^\infty)$ by

$$c_f(\sigma) \coloneqq \begin{cases} \bot, & \text{if } \sigma = \langle\rangle, \\ (f(a), \sigma'), & \text{if } \sigma = a \cdot \sigma' \text{ for some } a \in A \text{ and } \sigma' \in A^\infty, \end{cases}$$

for all $\sigma \in A^\infty$. Let $f^\infty\colon A^\infty \to B^\infty$ be the unique function making

$$\begin{tikzcd} \{\bot\} \cup (B \times A^\infty) \arrow[r, dashed, "\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_B \times f^\infty)"] & \{\bot\} \cup (B \times B^\infty) \\ A^\infty \arrow[u, "c_f"] \arrow[r, dashed, "\exists! f^\infty"'] & B^\infty \arrow[u, "\cong"', "\mathsf{next}_B"] \end{tikzcd}$$

commute. Then $f(\langle a_0, a_1, a_2, a_3, \dots\rangle) = \langle f(a_0), f(a_1), f(a_2), f(a_3), \dots\rangle$ for all $a_0, a_1, a_2, a_3, \dots \in A$, and analogously for finite sequences.

We see that $c_{\mathrm{id}_A} = \mathsf{next}_A$. So $(\mathrm{id}_A)^\infty = \mathrm{id}_{A^\infty}$ by finality of next_A. Furthermore, for functions $f\colon A \to B$ and $g\colon B \to C$, we see that

$$\begin{tikzcd} \{\bot\} \cup (C \times A^\infty) \arrow[r, "\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_C \times f^\infty)"] & \{\bot\} \cup (C \times B^\infty) \\ A^\infty \arrow[u, "c_{g\circ f}"] \arrow[r, "f^\infty"'] & B^\infty \arrow[u, "c_g"'] \end{tikzcd}$$

commutes. Consequently, the outer square in the diagram

$$\begin{tikzcd} \{\bot\} \cup (C \times A^\infty) \arrow[r, "\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_C \times f^\infty)"] & \{\bot\} \cup (C \times B^\infty) \arrow[r, "\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_C \times g^\infty)"] & \{\bot\} \cup (C \times C^\infty) \\ A^\infty \arrow[u, "c_{g\circ f}"] \arrow[r, "f^\infty"'] & B^\infty \arrow[u, "c_g"'] \arrow[r, "g^\infty"'] & C^\infty \arrow[u, "\cong"', "\mathsf{next}_C"] \end{tikzcd}$$

commutes, i.e.

$$\mathsf{next}_C \circ (g^\infty \circ f^\infty) = \big(\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_C \times (g^\infty \circ f^\infty))\big) \circ c_{g\circ f}.$$

The finality of next_C then yields $(g \circ f)^\infty = g^\infty \circ f^\infty$. □

Exercise 1.2.10

Use finality to define a map $\mathrm{st}\colon A^\infty \times B \to (A \times B)^\infty$ *that maps a sequence* $\sigma \in A^\infty$ *and an element* $b \in B$ *to a new sequence in* $(A \times B)^\infty$ *by adding this* b *at every position in* σ*. (This is an example of a 'strength' map; see Exercise 2.5.4.*

Solution. Define a coalgebra $c\colon A^\infty \times B \to \{\bot\} \cup \big((A \times B) \times (A^\infty \times B)\big)$ as follows:

$$c(\sigma, b) \coloneqq \begin{cases} \bot, & \text{if } \sigma = \langle\rangle, \\ \big((a, b), (\sigma', b)\big), & \text{if } \sigma = a \cdot \sigma' \text{ for some } a \in A \text{ and } \sigma' \in A^\infty, \end{cases}$$

for all $\sigma \in A^\infty$ and $b \in B$. The unique function $\mathrm{st}\colon A^\infty \times B \to (A \times B)^\infty$ making

$$\begin{tikzcd}
\{\bot\} \cup \big((A \times B) \times (A^\infty \times B)\big) \arrow[r, dashed, "\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_{A\times B} \times \mathrm{st})"] & \{\bot\} \cup \big((A \times B) \times (A \times B)^\infty\big) \\
A^\infty \times B \arrow[u, "c"] \arrow[r, dashed, "\exists! \mathrm{st}"'] & (A \times B)^\infty \arrow[u, "\cong", "\mathsf{next}"']
\end{tikzcd}$$

commute will satisfy $\mathrm{st}(\langle a_0, a_1, a_2, \ldots \rangle, b) = \langle (a_0, b), (a_1, b), (a_2, b), \ldots \rangle$ for all $a_0, a_1, a_2, a_3, \ldots \in A$ and $b \in B$, and analagously for finite sequences in A^∞. $\square$

1.3 Generality of Temporal Logic of Coalgebras

Exercise 1.3.1
The nexttime operator $\bigcirc$ introduced in (1.9) is the so-called ***weak*** *nexttime. There is an associated* ***strong*** *nexttime, given by $\neg\bigcirc\neg$. Note the difference between weak and strong nexttime for sequences.*

Solution. Recall that, for a sequence coalgebra $c\colon S \to \{\bot\} \cup (A \times S)$ and a predicate $P \subseteq S$, we have

$$(\bigcirc P)(x) \quad \text{if and only if} \quad c(x) = \bot \text{ or } c(x) \in A \times P,$$

for all $x \in S$. So,

$$(\bigcirc\neg P)(x) \quad \text{if and only if} \quad c(x) = \bot \text{ or } c(x) \in A \times (S \setminus P),$$

and thus

$$(\neg\bigcirc\neg P)(x) \quad \text{if and only if} \quad c(x) \neq \bot \text{ and } c(x) \notin A \times (S \setminus P).$$

Since the codomain of c is $\{\bot\} \cup (A \times S)$, and since $P \subseteq S$, we can equivalently write this as

$$(\neg\bigcirc\neg P)(x) \quad \text{if and only if} \quad c(x) \in A \times P. \qquad \square$$

Exercise 1.3.2
Prove that the 'truth' predicate that always holds is a (sequence) invariant. And if P_1 and P_2 are invariants, then so is the intersection $P_1 \cap P_2$. Finally, if P is an invariant, then so is $\bigcirc P$.

Solution. Fix a sequence coalgebra $c\colon S \to \{\bot\} \cup (A \times S)$. The truth predicate is the set S itself. Then, for all $x \in S$,

$$(\bigcirc S)(x) \quad \text{if and only if} \quad c(x) = \bot \text{ or } c(x) \in A \times S.$$

Since the codomain of c is $\{\bot\} \cup (A \times S)$, this means that $\bigcirc S = S$, and so S is an invariant.

Now suppose that P_1 and P_2 and invariant, i.e. $P_1 \subseteq \bigcirc P_1$ and $P_2 \subseteq \bigcirc P_2$. Then, for all $x \in S$,

$$\begin{aligned}
(\bigcirc(P_1 \cap P_2))(x) \quad & \text{if and only if} \quad c(x) = \bot \text{ or } c(x) \in A \times (P_1 \cap P_2) \\
& \text{if and only if} \quad c(x) = \bot \text{ or } c(x) \in (A \times P_1) \cap (A \times P_2) \\
& \text{if and only if} \quad \big(c(x) = \bot \text{ or } c(x) \in A \times P_1\big) \text{ and } \big(c(x) = \bot \text{ or } c(x) \in A \times P_2)\big) \\
& \text{if and only if} \quad (\bigcirc P_1)(x) \text{ and } (\bigcirc P_2)(x).
\end{aligned}$$

Hence $P_1 \cap P_2 \subseteq (\bigcirc P_1) \cap (\bigcirc P_2) = \bigcirc(P_1 \cap P_2)$, and so $P_1 \cap P_2$ is also invariant.

Finally, suppose that P is invariant, i.e. $P \subseteq \bigcirc P$. We aim to show that $\bigcirc P \subseteq \bigcirc\bigcirc P$. Suppose $x \in S$ is such that $(\bigcirc P)(x)$ holds. Then either $c(x) = \bot$ or $c(x) \in A \times P \subseteq A \times \bigcirc P$. Therefore $(\bigcirc\bigcirc P)(x)$ holds. $\square$

Exercise 1.3.3

1. *Show that $\Box$ is an interior operator, i.e. satisfies: $\Box P \subseteq P$, $\Box P \subseteq \Box\Box P$, and $P \subseteq Q \implies \Box P \subseteq \Box Q$.*

2. *Prove that a predicate P is invariant if and only if $P = \Box P$.*

Solution. Fix a sequence coalgebra $c\colon S \to \{\bot\} \cup (A \times S)$. Recall that the henceforth operator $\Box$ is defined on predicates $P \subseteq S$ as follows: for all $x \in S$,

$$(\Box P)(x) \quad \text{if and only if} \quad \text{there exists an invariant } Q \subseteq S \text{ with } x \in Q \subseteq P.$$

In other words, $\Box P$ is the union of all invariants contained in P.

1. If $x \in \Box P$, then there is an invariant $Q \subseteq S$ with $x \in Q \subseteq P$. So $x \in P$ too. Also, Q is an invariant with $x \in Q \subseteq \Box P$. So $x \in \Box\Box P$ as well. Thus $\Box P \subseteq P$ and $\Box P \subseteq \Box\Box P$.

   Now suppose $P \subseteq Q \subseteq S$. Then, for any $x \in \Box P$, there is an invariant $R \subseteq S$ with $x \in R \subseteq P \subseteq Q$. So $x \in \Box Q$ as well. Therefore $\Box P \subseteq \Box Q$.

2. For the forward direction, suppose that P is invariant. By definition, $\Box P$ is the union of all invariants contained within P. As P is assumed to be an invariant, we must have $\Box P = P$.

   For the converse direction, suppose that $\Box P = P$. We need to show that P is an invariant, i.e. $P \subseteq \bigcirc P$. For any $x \in P = \Box P$, there exists an invariant $Q \subseteq S$ with $x \in Q \subseteq P$. As Q is an invariant, either $c(x) = \bot$ or $c(x) \in A \times Q \subseteq A \times P$. Hence we also have $x \in \bigcirc P$. Therefore $P \subseteq \bigcirc P$, meaning P is an invariant. $\Box$

Exercise 1.3.4
Recall the finite behaviour predicate $\Diamond\big((-) \not\to\big)$ from Example 1.3.4.1 and show that it is an invariant: $\Diamond\big((-) \not\to\big) \subseteq \bigcirc\Diamond\big((-) \not\to\big)$. Hint: For an invariant Q, consider the predicate $Q' = \big(\neg(-) \not\to\big) \cap (\bigcirc Q)$.

Solution. Fix a sequence coalgebra $c\colon S \to \{\bot\} \cup (A \times S)$. Recall that, for a predicate $P \subseteq S$ and $x \in S$,

$$(\Diamond P)(x) \quad \text{if and only if} \quad \text{for all invariants } Q \subseteq S, \text{ we have } \neg Q(x) \text{ or } Q \not\subseteq \neg P.$$

That is, $\Diamond P = \neg\Box\neg P$.

Suppose $x \in S$ is such that $\Diamond\big(x \not\to\big)$ holds. We need to show that $\bigcirc\Diamond\big(x \not\to\big)$ holds, i.e. if $x \xrightarrow{a} x'$ for some $(a, x') \in A \times S$, then $\Diamond\big(x' \not\to\big)$ also holds. Fix any invariant $Q \subseteq S$ with $Q \subseteq \neg\big((-) \not\to\big)$. We need to show that $\neg Q(x')$.

Following the hint, we consider the predicate

$$Q' \coloneqq \neg\big((-) \not\to\big) \cap (\bigcirc Q).$$

Observe that Q' is an invariant: if $y \in S$ satisfies $Q'(y)$, then there is some $(b, y') \in A \times S$ such that $y \xrightarrow{b} y'$ and $Q(y')$ hold. Then, since $Q \subseteq \neg\big((-) \not\to\big)$ and Q is an invariant, we conclude that $Q'(y')$ also holds. So $Q' \subseteq \bigcirc Q'$.

Hence if $Q(x')$ holds, then $Q'(x)$ holds too, contradicting the assumption that $\Diamond\big(x \not\to\big)$. $\Box$

Exercise 1.3.5
Let $(A, \leq)$ be a complete lattice, i.e. a poset in which each subset $U \subseteq A$ has a join $\bigvee U \in A$. It is well known that each subset $U \subseteq A$ then also has a meet $\bigwedge U \in A$, given by $\bigwedge U = \bigvee\{\, a \in A \mid \forall b \in U. a \leq b \,\}$.

Let $f\colon A \to A$ be a monotone function: $a \leq b$ implies $f(a) \leq f(b)$. Recall, e.g. from Davey and Priestley (1990, Chapter 4) that such a monotone f has both a least fixed point $\mu f \in A$ and a greatest fixed point $\nu f \in A$ given by the formulas:

$$\mu f = \bigwedge \{\, a \in A \mid f(a) \leq a \,\}, \qquad \nu f = \bigvee \{\, a \in A \mid a \leq f(a) \,\}.$$

Now let $c\colon S \to \{\perp\} \cup (A \times A)$ be an arbitrary sequence coalgebra, with associated nexttime operator $\bigcirc$.

1. *Prove that $\bigcirc$ is a monotone function $\mathcal{P}(S) \to \mathcal{P}(S)$, i.e. that $P \subseteq Q$ implies $\bigcirc P \subseteq \bigcirc Q$, for all $P, Q \subseteq S$.*

2. *Check that $\Box P \in \mathcal{P}(S)$ is the greatest fixed point of the function $\mathcal{P}(S) \to \mathcal{P}(S)$ given by $U \mapsto P \cap \bigcirc U$.*

3. *Define for $P, Q \subseteq S$ a new predicate $P \;\mathcal{U}\; Q \subseteq S$, for 'P until Q' as the least fixed point of $U \mapsto Q \cup (P \cap \neg\bigcirc\neg U)$. Check that 'until' is indeed a good name for $P \;\mathcal{U}\; Q$, since it can be described explicitly as*

$$\begin{aligned} P \;\mathcal{U}\; Q = \{\, x \in S \mid\, & \exists n \in \mathbb{N}. \exists x_0, x_1, \ldots, x_n \in S. \\ & x_0 = x \wedge (\forall i < n. \exists a. x_i \xrightarrow{a} x_{i+1}) \wedge Q(x_n) \\ & \wedge \forall i < n. P(x_i) \,\}. \end{aligned}$$

   Hint: Don't use the fixed point definition μ, but first show that this subset is a fixed point, and then that it is contained in an arbitrary fixed point.

(The fixed point definitions that we described above are standard in temporal logic; see e.g. Emerson (1990, 3.24–3.25). The above operation $\mathcal{U}$ is what is called the 'strong' until. The 'weak one' does not have the negations $\neg$ in its fixed-point description in point 3.)

Solution.

1. For subsets $P, Q \in \mathcal{P}(S)$ with $P \subseteq Q$, and for $x \in S$ such that $(\bigcirc P)(x)$ holds, we have

$$c(x) = \perp \text{ or } c(x) \in A \times P.$$

   From the assumption that $P \subseteq Q$, it follows that

$$c(x) = \perp \text{ or } c(x) \in A \times Q,$$

   or equivalently, $(\bigcirc Q)(x)$.

2. Fix $P \in \mathcal{P}(S)$ and define $f_P\colon \mathcal{P}(S) \to \mathcal{P}(S)$ by $f_P(U) \coloneqq P \cap \bigcirc U$ for all $U \in \mathcal{P}(S)$. Then the greatest fixed point of f_P is

$$\nu(f_P) \coloneqq \bigcup_{\substack{U \in \mathcal{P}(S), \\ U \subseteq f_P(U)}} U = \bigcup_{\substack{U \in \mathcal{P}(S), \\ U \subseteq P \cap \bigcirc U}} U = \Box P.$$

3. Fix $P, Q \in \mathcal{P}(S)$, and define $f_{P,Q}\colon \mathcal{P}(S) \to \mathcal{P}(S)$ by

$$f_{P,Q}(U) \coloneqq Q \cup (P \cap \neg\bigcirc\neg U)$$

for all $U \in \mathcal{P}(S)$. Recall, from Exercise 1.3.1, that

$$\neg\bigcirc\neg U = \{\, x \in S \;:\; c(x) \in A \times U \,\}.$$

We wish to show that the set

$$U_{P,Q} \coloneqq Q \cup \Big\{\, x \in S \;:\; \begin{aligned}[t] &\text{there exist } n \in \mathbb{Z}_{>0},\ x_0, \dots, x_n \in S \text{ and } a_0, \dots, a_{n-1} \in A \\ &\text{such that } x = x_0 \xrightarrow{a_0} \cdots \xrightarrow{a_{n-1}} x_n \text{ and} \\ &P(x_0), \dots, P(x_{n-1}),\ \text{and } Q(x_n) \text{ all hold} \Big\}\end{aligned}$$

is the least fixed point of $f_{P,Q}$.

First, observe that

$$\begin{aligned} f_{P,Q}(U_{P,Q}) &= Q \cup (P \cap \neg\bigcirc\neg U_{P,Q}) \\ &= Q \cup (P \cap \{\, x \in S \;:\; c(x) \in A \times U_{P,Q} \,\}) \\ &= Q \cup \{\, x \in S \;:\; P(x) \text{ and } c(x) \in A \times U_{P,Q} \,\} \\ &= U_{P,Q}, \end{aligned}$$

so that $U_{P,Q}$ is indeed a fixed point of $f_{P,Q}$.

Now we show that $U_{P,Q}$ is the least fixed point of $f_{P,Q}$. Fix some $B \subseteq S$ with $f_{P,Q}(B) = B$, i.e.

$$Q \cup \{\, x \in S \;:\; P(x) \text{ and } c(x) \in A \times B \,\} = B.$$

Then we get $U_{P,Q} \subseteq B$ by induction on the length of finite sequences $x_0, \dots x_n \in S$ and $a_0, \dots, a_{n-1} \in A$ satisfying $x_0 \xrightarrow{a_0} \cdots \xrightarrow{a_{n-1}} x_n$, and $P(x_0) \wedge \cdots \wedge P(x_{n-1}) \wedge Q(x_n)$. $\square$

1.4 Abstractness of Coalgebraic Notions

Exercise 1.4.1
*Let $(M, +, 0)$ be a monoid, considered as a category. Check that a functor $F\colon M \to$ **Sets** can be identified with a **monoid action**: a set X together with a function $\mu\colon X \times M \to X$ with $\mu(x, 0) = x$ and $\mu(x, m_1 + m_2) = \mu(\mu(x, m_2), m_1)$.*

Solution. Suppose we are given functor $F\colon M \to X$. This F sends the unique object $\star \in \mathsf{Obj}(M)$ to a set $F(\star) \in \mathsf{Obj}(\mathbf{Sets})$, and sends each $m \in \mathsf{Arr}(M)$ to a function $Fm\colon F(\star) \to F(\star)$. The functoriality of F requires that $F(0) = \mathrm{id}_{F(\star)}$ and $F(m_1 + m_2) = F(m_1) \circ F(m_2)$ for all $m_1, m_2 \in \mathsf{Arr}(M)$. We then define a function $\mu_F\colon F(\star) \times \mathsf{Arr}(M) \to F(\star)$ by $\mu_F(x, m) \coloneqq F(m)(x)$ for all $(x, m) \in F(\star) \times M$.

The equality $\mu_F(x, 0) = x$ for all $x \in F(\star)$ follows the equality $F(0) = \mathrm{id}_{F(\star)}$, while the equality $\mu_F(x, m_1 + m_2) = \mu_F(\mu_F(x, m_2), m_1)$ for all $x \in X$ and $m_1, m_2 \in \mathsf{Arr}(M)$ follows from the equality $F(m_1 + m_2) = F(m_1) \circ F(m_2)$.

Now suppose we are given also given a set X and a function $\mu\colon X \times \mathsf{Arr}(M) \to X$ with $\mu(x, 0) = x$ and $\mu(x, m_1 + m_2) = \mu(\mu(x, m_2), m_1)$ for all $x \in X$ and $m, m_1, m_2 \in \mathsf{Arr}(M)$. We then define a functor $F_\mu\colon M \to$ **Sets** by $F_\mu(\star) \coloneqq X$, for the unique object $\star \in \mathsf{Obj}(M)$, and $F_\mu(m) \coloneqq \mu(-, m)$ for each $m \in \mathsf{Arr}(M)$. That F_μ is actually a functor follows from the assumptions on μ.

We then have $F_{\mu_F} = F$ and $\mu_{F_\mu} = \mu$. $\square$

Exercise 1.4.2
Check in detail that the opposite $\mathbb{C}^{\mathrm{op}}$ and the product $\mathbb{C} \times \mathbb{D}$ are indeed categories.

Solution. Let $\mathbb{C}$ and $\mathbb{D}$ be categories.

We defined $\mathsf{Obj}(\mathbb{C}^{\text{op}}) := \mathsf{Obj}(\mathbb{C})$. For $X, Y \in \mathsf{Obj}(\mathbb{C})$, write $\hom_{\mathbb{C}}(X,Y)$ for the set of all morphisms with domain X and codomain Y. We then defined $\hom_{\mathbb{C}^{\text{op}}}(X,Y) := \hom_{\mathbb{C}}(Y,X)$, and we defined a compositition $X \overset{f}{\leftarrow} Y \overset{g}{\leftarrow} Z$ in $\mathbb{C}^{\text{op}}$ to be the composition $X \overset{f}{\to} Y \overset{g}{\to} Z$ in $\mathbb{C}$. The associativity and identity laws for composition in $\mathbb{C}^{\text{op}}$ follow from those for $\mathbb{C}$.

We defined $\mathsf{Obj}(\mathbb{C} \times \mathbb{D}) := \mathsf{Obj}(\mathbb{C}) \times \mathsf{Obj}(\mathbb{D})$. For $X, X' \in \mathsf{Obj}(\mathbb{C})$ and $Y, Y' \in \mathsf{Obj}(\mathbb{D})$, we let $\hom_{\mathbb{C}\times\mathbb{D}}((X,Y),(X',Y')) := \hom_{\mathbb{C}}(X,X') \times \hom_{\mathbb{D}}(Y,Y')$. A composition $(X,Y) \xrightarrow{(f,g)} (X',Y') \xrightarrow{(f',g')} (X'',Y'')$ in $\mathbb{C} \times \mathbb{D}$ is defined to be the composition $(X,Y) \xrightarrow{(f'f,\, g'g)} (X'',Y'')$. For an object (X,Y) in $\mathbb{C} \times \mathbb{D}$, the identity morphism $\mathrm{id}_{(X,Y)}$ is the pair $(\mathrm{id}_X, \mathrm{id}_Y)$. The associativity and identity laws for composition in $\mathbb{C} \times \mathbb{D}$ follow from those for $\mathbb{C}$ and $\mathbb{D}$. $\square$

Exercise 1.4.3

Assume an arbitrary category $\mathbb{C}$ with an object $I \in \mathbb{C}$. We form a new category $\mathbb{C}/I$, the so-called ***slice category*** *over I, with*

objects *maps $f \colon X \to I$ with codomain I in $\mathbb{C}$*

morphisms *from $X \overset{f}{\to} I$ to $Y \overset{g}{\to} I$ are morphisms $h \colon X \to Y$ in $\mathbb{C}$ for which the following diagram commutes:*

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & I & \end{tikzcd}$$

1. *Describe identities and composition in $\mathbb{C}/I$, and verify that $\mathbb{C}/I$ is a category.*
2. *Check that taking domains yields a functor* $\mathrm{dom} \colon \mathbb{C}/I \to \mathbb{C}$.
3. *Verify that for $\mathbb{C} =$* **Sets***, a map $f \colon X \to I$ may be identified with an I-indexed family of sets $(X_i)_{i\in I}$, namely where $X_i = f^{-1}(i)$. What do morphisms in $\mathbb{C}/I$ correspond to, in terms of such indexed families?*

Solution.

1. The identities and composition in $\mathbb{C}/I$ are simply the identities and composition in $\mathbb{C}$. So the fact that $\mathbb{C}/I$ is a category follows from $\mathbb{C}$ being a category.

2. We define $\mathrm{dom} \colon \mathbb{C}/I \to \mathbb{C}$ as follows: for a morphism h from $X \overset{f}{\to} I$ to $Y \overset{g}{\to} I$ in $\mathbb{C}/I$, we simply define $\mathrm{dom}(h) := h$. This immediately makes dom a functor from $\mathbb{C}/I$ to $\mathbb{C}$.

3. The claimed identification is obvious. Now fix a morphism h from $X \overset{f}{\to} I$ to $Y \overset{g}{\to} I$ in $\mathbf{Sets}/I$, so that the diagram

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & I & \end{tikzcd}$$

in **Sets** commutes. This requires that $g(h(x)) = f(x)$ for all $x \in X$. Identifying $X_i := f^{-1}(i)$ and $Y_i := g^{-1}(i)$ for all $i \in I$, we can identify h with a family of functions $(h_i)_{i\in I}$ such that $h_i(x) \in Y_i$ for all $x \in X_i$, for all $i \in I$.

$\square$

Exercise 1.4.4
Recall that for an arbitrary set A we write $A^\star$ for the set of finite sequencees $\langle a_0, \ldots, a_n \rangle$ of elements $a_i \in A$.

1. *Check that $A^\star$ carries a monoid structure given by concatenation of sequences, with the empty sequence $\langle\rangle$ as a neutral element.*
2. *Check that the assignment $A \mapsto A^\star$ yields a functor* **Sets** $\to$ **Mon** *by mapping a function $f\colon A \to B$ between sets to the function $f^\star\colon A^\star \to B^\star$ given by $\langle a_0, \ldots, a_n \rangle \mapsto \langle f(a_0), \ldots, f(a_n) \rangle$. (Be aware of what needs to be checked: $f^\star$ must be a monoid homomorphism, and $(-)^\star$ must preserve composition of functions and identity functions.)*
3. *Prove that $A^\star$ is the* ***free monoid on*** *A: there is the singleton-sequence insertion map $\eta\colon A \to A^\star$ which is universal among all mappings of A into a monoid. The latter means that for each monoid $(M, 0, +)$ and function $f\colon A \to M$ there is a unique monoid homomorphism $g\colon A^\star \to M$ with $g \circ \eta = f$.*

Solution.

1. Concatenation is associative because all the sequences under consideration are finite.
2. That $(-)^\star$ preserves composition and identity functions is obvious, so we just check that for a function $f\colon A \to B$, the map $f^\star\colon A^\star \to B\star$ is a monoid homomorphism. Fix finite sequences $\langle a_0, \ldots, a_n \rangle, \langle a_0', \ldots, a_k' \rangle \in A^\star$. Then

$$\begin{aligned} f(\langle a_0, \ldots, a_n \rangle \cdot \langle a_0', \ldots, a_k' \rangle) &= f(\langle a_0, \ldots, a_n, a_0', \ldots, a_k' \rangle) \\ &= \langle f(a_0), \ldots, f(a_n), f(a_0'), \ldots, f(a_k') \rangle \\ &= \langle f(a_0), \ldots, f(a_n) \rangle \cdot \langle f(a_0'), \ldots, f(a_k') \rangle \\ &= f(\langle a_0, \ldots, a_n \rangle) \cdot \langle a_0', \ldots a_k' \rangle) \end{aligned}$$

   and $f(\langle\rangle) = \langle\rangle$. So $f^\star$ is a monoid homomorphism.

3. Define $\eta\colon A \to A^\star$ by $\eta(a) \coloneqq \langle a \rangle$ for all $a \in A$. Fix a monoid $(M, 0, +)$ and a function $f\colon A \to M$. Define $g\colon A^\star \to M$ by

$$\begin{aligned} g(\langle\rangle) &\coloneqq 0 \\ g(\langle a_0, \ldots, a_n \rangle) &\coloneqq f(a_0) + \cdots + f(a_n) \end{aligned}$$

   for all $\langle a_0, \ldots, a_n \rangle \in A^\star$. This g is clearly a mononid homomorphism, using the associativity of $+$ in M. Observe that the diagram

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, "f"'] & A^\star \arrow[d, "g"] \\ & M \end{tikzcd}$$

   in **Sets** commutes: we have $f(a) = g(\eta(a))$ for all $a \in A$. Now suppose that there is another monoid homomorphism $h\colon A^\star \to M$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, "f"'] & A^\star \arrow[d, "h"] \\ & M \end{tikzcd}$$

in **Sets** commutes. As $h\colon A^\star \to M$ is a monoid homomorphism and $f = h\eta$, we require that $h(\langle\rangle) = 0$ and

$$\begin{aligned} h(\langle a_0, \dots, a_n\rangle) &= h(\langle a_0\rangle \cdot \dots \cdot \langle a_n\rangle) \\ &= h(\langle a_0\rangle) + \cdots + h(\langle a_n\rangle) \\ &= h(\eta(a_0)) + \cdots + h(\eta(a_n)) \\ &= f(a_0) + \cdots + f(a_n) \\ &= g(\langle a_0, \dots, a_n\rangle), \end{aligned}$$

for all $\langle a_0, \dots, a_n\rangle \in A^\star$. Therefore $h = g$. □

Exercise 1.4.5
Recall from (1.3) the statements with exceptions of the form $S \to \{\perp\} \cup S \cup (S \times E)$.

1. *Prove that the assignment $X \mapsto \{\perp\} \cup X \cup (X \times E)$ is functorial, so that the statements are a coalgebra for this functor.*

2. *Show that all the operations $\mathsf{at}_1, \dots, \mathsf{at}_n, \mathsf{meth}_1, \dots, \mathsf{meth}_m$ of a class as in (1.10) can also be described as a single coalgebra, namely of the functor:*

$$X \mapsto D_1 \times \cdots \times D_n \times \underbrace{(\{\perp\} \cup X \cup (X \times E)) \times \cdots \times (\{\perp\} \cup X \cup (X \times E))}_{m \text{ times}}.$$

Solution.

1. Let $F\colon \mathbf{Sets} \to \mathbf{Sets}$ denote this assignment $F(X) \coloneqq \{\perp\} \cup X \cup (X \times E)$ where all unions are disjoint unions. We define F on morphisms as follows: for functions $f\colon X \to Y$, we define $F(f)\colon F(X) \to F(Y)$ to be the function

$$F(f)(x) \coloneqq \begin{cases} \perp, & \text{if } x = \perp, \\ f(x), & \text{if } x \in X, \\ (f(x'), e), & \text{if } x = (x', e) \text{ for some } (x', e) \in X \times E. \end{cases}$$

   Then $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(gf) = F(g)F(f)$ for all sets X and functions $X \xrightarrow{f} Y \xrightarrow{g} Z$.

2. The functor's definition on morphisms is similar in style with the previous part. □

Exercise 1.4.6
Recall the nexttime operator $\bigcirc$ for a sequence coalgebra $c\colon S \to \mathsf{Seq}(S) = \{\perp\} \cup (A \times S)$ from the previous section. Exercise 1.3.5.1 says that it forms a monotone function $\mathcal{P}(S) \to \mathcal{P}(S)$ — with respect to the inclusion order — and thus a functor. Check that invariants are precisely $\bigcirc$-coalgebras!

Solution. The $\bigcirc$-coalgebras are simply a subsets $U \subseteq S$ such that $U \subseteq \bigcirc U$. These are precisely what invariants are. □

2 Coalgebras of Polynomial Functors

2.1 Constructions on Sets

Exercise 2.1.1
Verify in detail the bijective correspondences (2.2), (2.6), (2.11) and (2.16).

Solution. Fix sets X, Y, Z. Following the notation of Equations (2.1), we associate a pair of functions $f\colon Z \to X$ and $g\colon Z \to Y$ to the function $\langle f, g\rangle\colon Z \to X \times Y$ given by $\langle f, g\rangle(z) := \langle f(z), g(z)\rangle$ for all $z \in Z$. Furthermore, we associate to any function $h\colon Z \to X \times Y$ a pair the functions $\pi_1 h\colon Z \to X$ and $\pi_2 h\colon Z \to Y$, where π_1 and π_2 are the relevant projections. Then $\langle \pi_1 h, \pi_2 h\rangle = h$ and $(\pi_1\langle f, g\rangle, \pi_2\langle f, g\rangle) = (f, g)$. This establishes the bijective correspondence (2.2).

Continue fixing sets X, Y, Z. Suppose, without loss of generality, that X and Y are disjoint, so that we may use $X \cup Y$ in place of $X + Y$. Following the notation of Equations (2.5), we associate a pair of functions $f\colon X \to Z$ and $g\colon Y \to Z$ to the function $[f, g]\colon X + Y \to Z$ given by

$$[f, g](w) := \begin{cases} f(w), & \text{if } w \in X, \\ g(w), & \text{if } w \in Y \end{cases}$$

for all $w \in X + Y$. Furthermore, to any function $h\colon X + Y \to Z$, we associate the pair of functions $h\kappa_1\colon X \to Z$ and $g\kappa_2\colon Y \to Z$, where κ_1 and κ_2 are the relevant coprojections. Then $[h\kappa_1, h\kappa_2] = h$ and $([f, g]\kappa_1, [f, g]\kappa_2) = (f, g)$. This establishes the bijective correspondence (2.6).

Continue fixing sets X, Y, Z. Following the notations of Equations (2.10), we associate a function $f\colon Z \times X \to Y$ to the function $\Lambda(f)\colon Z \to Y^X$ given by $\Lambda(f)(z) := f(z, -)$ for all $z \in Z$. Furthermore, to each function $g\colon Z \to Y^X$, we associate the function $U(g)\colon Z \times X \to Y$ given by $U(g)(z, x) := g(z)(x)$ for all $(z, x) \in Z \times X$. Then $\Lambda(U(g)) = g$ and $U(\Lambda(f)) = f$. So we have established the bijective correspondence (2.11).

Finally, fix sets X and Y. To each function $f\colon X \to \mathcal{P}(Y)$, we associate the relation

$$\mathsf{rel}(f) := \{\, (y, x) \in Y \times X \ : \ y \in f(x) \,\}.$$

Also, to each relation $R \subseteq Y \times X$, we associate the function $\mathsf{char}(R)\colon X \to \mathcal{P}(Y)$ given by

$$\mathsf{char}(R)(x) := \{\, y \in Y \ : \ R(y, x) \,\}$$

for all $y \in Y$. Then $\mathsf{rel}(\mathsf{char}(R)) = R$ and $\mathsf{char}(\mathsf{rel}(f)) = f$. We thus obtain the bijective correspondence (2.16). □

Exercise 2.1.2
Consider a poset $(D, \leq)$ as a category. Check that the product of two elements $d, e \in D$, if it exists, is the meet $d \wedge e$. And a coproduct of d, e, if it exists, is the join $d \vee e$.

Similarly, show that a final object is a top element $\top$ (with $d \leq \top$, for all $d \in D$) and that an initial object is a bottom element $\perp$ (with $\perp \leq d$, for all $d \in D$).

Solution. These follow immediately, as in a poset $(D, \leq)$, we have one (and only one) morphism $x \to y$ if and only if $x \leq y$, for $x, y \in D$, and that the only isomorphisms are identity morphisms. □

Exercise 2.1.3
Check that a product in a category $\mathbb{C}$ is the same as a coproduct in a category $\mathbb{C}^{\mathrm{op}}$.

Solution. Fix $X, Y, Z \in \mathsf{Obj}(\mathbb{C})$, and suppose the product $X \times Y$ exists in $\mathbb{C}$. For a pair of morphisms $f\colon Z \to X$ and $g\colon Z \to Y$, we have the following diagram

$$\begin{tikzcd}
X & & Y \\
& X \times Y \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] & \\
& Z \arrow[uul, bend left, "f"] \arrow[uur, bend right, "g"'] \arrow[u, dashed, "\exists! h"'] &
\end{tikzcd}$$

in $\mathbb{C}$ commuting. Thus we have the following diagram

$$\begin{tikzcd}
X \arrow[dr, "\pi_1"] \arrow[ddr, bend right, "f"'] & & Y \arrow[dl, "\pi_2"'] \arrow[ddl, bend left, "g"] \\
& X \times Y \arrow[d, dashed, "\exists! h"] & \\
& Z &
\end{tikzcd}$$

in $\mathbb{C}^{\mathrm{op}}$ commuting. This makes $X \times Y$ the coproduct of X and Y in $\mathbb{C}^{\mathrm{op}}$, with coprojections π_1 and π_2. Similarly, coproducts in $\mathbb{C}^{\mathrm{op}}$ correspond to products in $\mathbb{C}$. $\square$

Exercise 2.1.4
Fix a set A and prove that assignments $X \mapsto A \times X$, $X \mapsto A + X$ and $X \mapsto X^A$ are functorial and give rise to functors **Sets** $\to$ **Sets**.

Solution. Define $F, G, H\colon \mathbf{Sets} \to \mathbf{Sets}$ as follows. For a set X,

$$\begin{aligned}
FX &:= A \times X, \\
GX &:= A + X, \text{ and} \\
HX &:= X^A.
\end{aligned}$$

For a function $f\colon X \to Y$, define the functions $Ff\colon A \times X \to A \times Y$, $Gf\colon A + X \to A + Y$, and $Hf\colon X^A \to Y^A$ as follows:

$$\begin{aligned}
(Ff)(a, x) &:= (a, f(x)), && \text{for all } (a, x) \in A \times X, \\
(Gf)(w) &:= \begin{cases} w, & \text{if } w \in A, \\ f(w), & \text{if } w \in X, \end{cases} && \text{for all } w \in A + X, \\
(Hf)(h) &:= fh, && \text{for all functions } h\colon A \to X,
\end{aligned}$$

where we have assumed, without loss of generality, that A and X are disjoint so that $X + A$ is treated as $X \cup A$.

Then, for any set X,

$$\begin{aligned}
(F\mathrm{id}_X)(a, x) &= (a, \mathrm{id}_X(x)) \\
&= (a, x), && \text{for all } (a, x) \in A \times X,
\end{aligned}$$

$$
\begin{aligned}
(G\mathrm{id}_X)(w) &= \begin{cases} w, & \text{if } w \in A, \\ \mathrm{id}_X(w), & \text{if } w \in X, \end{cases} \\
&= w, && \text{for all } w \in A + X, \\
(H\mathrm{id}_X)(h) &= \mathrm{id}_X h \\
&= h, && \text{for all functions } h\colon A \to X,
\end{aligned}
$$

so $F\mathrm{id}_X = \mathrm{id}_{FX}$, $G\mathrm{id}_X = \mathrm{id}_{GX}$, and $H\mathrm{id}_X = \mathrm{id}_{HX}$. Now, for functions $X \xrightarrow{f} Y \xrightarrow{g} Z$,

$$
\begin{aligned}
(F(gf))(a, x) &= (a, g(f(x))) \\
&= (Fg)(a, f(x)) \\
&= (Fg \circ Ff)(a, x), && \text{for all } (a, x) \in A \times X, \\
(G(gf))(w) &= \begin{cases} w, & \text{if } w \in A, \\ g(f(w)), & \text{if } w \in X, \end{cases} \\
&= (Gg \circ Gf)(w), && \text{for all } w \in A + X, \\
(H(gf))(h) &= \lambda a \in A.(g(f(h(a)))), \\
&= (Hg)(fh) \\
&= (Hg \circ Hf)(h), && \text{for all functions } h\colon A \to X,
\end{aligned}
$$

so $F(gf) = (Fg)(Ff)$, $G(gf) = (Gg)(Gf)$, and $H(gf) = (Hg)(Hf)$. Thus F, G, and H are functors from **Sets** to **Sets**. $\square$

Exercise 2.1.5

Prove that the category **PoSets** *of partially ordered sets and monotone functions is a BiCCC. The definitions on the underlying sets X of a poset $(X, \leq)$ are like for ordinary sets but should be equipped with appropriate orders.*

Solution. The category **PoSets** has a terminal object, namely the singleton poset. Furthermore, given two posets $(X_1, \leq_1)$ and $(X_2, \leq_2)$, we can define a partial ordering $\leq_{1\times 2}$ on the product $X_1 \times X_2$ by

$$
(x_1, x_2) \leq_{1\times 2} (x_1', x_2') \quad \text{if and only if} \quad x_1 \leq x_1' \text{ and } x_2 \leq x_2'
$$

for all $(x_1, x_2), (x_1', x_2') \in X_1 \times X_2$. This poset $(X_1 \times X_2, \leq_{1\times 2})$ has the universal property of the product: given another poset $(X_3, \leq_3)$ and a pair of monotone functions $f\colon (X_3, \leq_3) \to (X_1, \leq_1)$ and $g\colon (X_3, \leq_3) \to (X_2, \leq_2)$, we have the diagram

$$
\begin{tikzcd}
(X_1, \leq_1) & & (X_2, \leq_2) \\
& (X_1 \times X_2, \leq_{1\times 2}) \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] & \\
& (X_3, \leq_3) \arrow[uul, bend left, "f"] \arrow[uur, bend right, "g"'] \arrow[u, dashed, "\exists! h"'] &
\end{tikzcd}
$$

in **PoSets** commuting, where π_1 and π_2 are the relevant projections (which are indeed monotone). The unique monotone function h is given by $h(x_3) \coloneqq (f(x_3), g(x_3))$ for all $x_3 \in X_3$. Therefore the category **PoSets** has finite products.

The category **PoSets** also has an initial object: the empty poset. Now, given two posets $(X_1, \leq_1)$ and $(X_2, \leq_2)$, we can define a partial ordering $\leq_{1+2}$ on the coproduct $X_1 + X_2$ by

$$w \leq_{1+2} w' \quad \text{if and only if} \quad (w, w' \in X_1 \text{ and } w \leq_1 w') \text{ or } (w, w' \in X_2 \text{ and } w \leq_2 w')$$

for all $w, w' \in X_1 + X_2$, where we have assumed without loss of generality that X_1 and X_2 are disjoint so that $X_1 + X_2$ may be identified with $X_1 \cup X_2$. Then, given any other poset $(X_3, \leq_3)$ and a pair of monotone functions $f\colon (X_1, \leq_1) \to (X_3, \leq_3)$ and $g\colon (X_2, \leq_2) \to (X_3, \leq_3)$, we have the diagram

$$\begin{tikzcd}
 & (X_3, \leq_3) & \\
 & (X_1 + X_2, \leq_{1+2}) \arrow[u, dashed, "\exists! h"'] & \\
(X_1, \leq_1) \arrow[ur, "\kappa_1"'] \arrow[uur, bend left, "f"] & & (X_2, \leq_2) \arrow[ul, "\kappa_2"] \arrow[uul, bend right, "g"']
\end{tikzcd}$$

in **PoSets** commuting, where κ_1 and κ_2 are the relevant coprojections (which are also monotone). The unique monotone function h is given by

$$h(w) := \begin{cases} f(w), & \text{if } w \in X_1, \\ g(w), & \text{if } w \in X_2, \end{cases}$$

for all $w \in X_1 + X_2$. Therefore **PoSets** also has finite coproducts.

Now we show that **PoSets** also has exponents. Fix any two posets $(X_1, \leq_1)$ and $(X_2, \leq_2)$. We define a partial ordering $\leq_{2^1}$ on the set $X_2^{X_1}$ as follows:

$$f \leq_{2^1} g \quad \text{if and only if} \quad f(x) \leq_2 g(x) \text{ for all } x \in X_1.$$

for all functions $f, g\colon X_1 \to X_2$. Then, for any poset $(X_3, \leq_3)$ and monotone function $f\colon (X_3, \leq_3) \to (X_2, \leq_2)$, we have the diagram

$$\begin{tikzcd}
(X_2^{X_1}, \leq_{2^1}) & & (X_1, \leq_1) & \\
(X_3, \leq_3) \arrow[u, dashed, "\exists! g"] & & (X_1, \leq_1) \arrow[u, "\mathrm{id}_{(X_1, \leq_1)}"'] & \\
 & (X_2^{X_1} \times X_1, \leq_{2^1 \times 1}) \arrow[uul, "p_1"'] \arrow[uur, "p_2"] \arrow[rr, "\mathrm{ev}"] & & (X_2, \leq_2) \\
 & (X_3 \times X_1, \leq_{3 \times 1}) \arrow[uul, bend left, "\pi_1"] \arrow[u, dashed, "g \times \mathrm{id}_{(X_1, \leq_1)}"] \arrow[uur, bend right, "\pi_2"'] \arrow[urr, "f"'] & &
\end{tikzcd}$$

in **PoSets** commuting, where $\mathrm{ev}(h, x_1) := h(x_1)$ for all $(h, x_1) \in X_2^{X_1} \times X_1$, and π_1, π_2, p_1, and p_2 are the relevant projections. The unique monotone function g is given by $g(x_3) := \lambda x_1 \in X_1 . f(x_3, x_1)$. Therefore **PoSets** also has exponents. $\square$

Exercise 2.1.6
Consider the category **Mon** *of monoids with monoid homomorphisms between them.*

1. *Check that the singleton monoid* 1 *is both an initial and a final object in* **Mon***; this is called a zero object.*

2. *Given two monoids* $(M_1, +_1, 0_1)$ *and* $(M_2, +_2, 0_2)$*, one defines a product monoid* $M_1 \times M_2$ *with componentwise addition* $(x, y) + (x', y') = (x{+_1}, x', y +_2 y')$ *and unit* $(0_1, 0_2)$*. Prove that* $M_1 \times M_2$ *is again a monoid, which forms a product in the category* **Mon** *with the standard projection maps* $M_1 \xleftarrow{\pi_1} M_1 \times M_2 \xrightarrow{\pi_2} M_2$*.*

3. *Note that there are also coprojections* $M_1 \xrightarrow{\kappa_1} M_1 \times M_2 \xleftarrow{\kappa_2} M_2$*, given by* $\kappa_1(x) = (x, 0_2)$ *and* $\kappa_2(y) = (0_1, y)$*, which are monoid homomorphisms and which makes* $M_1 \times M_2$ *at the same time the coproduct of* M_1 *and* M_2 *in* **Mon** *(and hence a biproduct). Hint: Define the cotuple* $[f, g]$ *as* $x \mapsto f(x) + g(x)$*.*

Solution.

1. Any monoid homomorphism $f\colon (M_1, +_1, 0_1) \to (M_2, +_2, 0_2)$ must satisfy $f(0_1) = 0_2$, so the singleton monoid is initial in **Mon**. It is also the final in **Mon** because the constant map to the unit is a monoid homomorphism.

2. Fix $(m_1, m_2), (m_1', m_2'), (m_1'', m_2'') \in M_1 \times M_2$. Then, using the associativity of $+_1$ and $+_2$,

$$\begin{aligned}(m_1, m_2) + ((m_1', m_2') + (m_1'', m_2'')) &= (m_1, m_2) + (m_1' +_1 m_1'',\ m_2' +_2 m_2'') \\ &= (m_1 +_1 m_1' +_1 m_1'',\ m_2 +_2 m_2' +_2 m_2'') \\ &= (m_1 +_1 m_1',\ m_2 +_2 m_2') + (m_1'', m_2'').\end{aligned}$$

Furthermore,

$$\begin{aligned}(m_1, m_2) + (0_1, 0_2) &= (m_1 +_1 0_1, m_2{+_2}, 0_2) \\ &= (m_1, m_2)\end{aligned}$$

and, similarly, $(0_1, 0_2) + (m_1, m_2) = (m_1, m_2)$. So $(M_1 \times M_2, +, (0_1, 0_2))$ is a monoid.

We now show that $M_1 \times M_2$ really is the categorical product of M_1 and M_2 in **Mon**. Fix any other monoid $(M_3, +_3, 0_3)$ and a pair of monoid homomorphisms $f\colon M_3 \to M_1$ and $g\colon M_3 \to M_2$. We need the diagram

$$\begin{tikzcd}
M_1 & & M_2 \\
& M_1 \times M_2 \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] & \\
& M_3 \arrow[uul, bend left, "f"] \arrow[uur, bend right, "g"'] \arrow[u, dashed, "\exists!\langle f,g\rangle"'] &
\end{tikzcd}$$

in **Mon** to commute. Indeed, we must have $\langle f, g\rangle(m_3) = (f(m_3), g(m_3))$ for all $m_3 \in M_3$. The fact that $\langle f, g\rangle\colon M_3 \to M_1 \times M_2$ is a monoid homomorphism follows from f and g being monoid homomorphisms.

3. Fix any monoid $(M_3, +_3, 0_3)$ and a pair of monoid homomorphisms $f\colon M_1 \to M_3$ and $g\colon M_2 \to M_3$. We need the diagram

$$\begin{tikzcd}
 & M_3 & \\
 & M_1 \times M_2 \arrow[u, dashed, "\exists ! [f,g]"'] & \\
M_1 \arrow[ur, "\kappa_1"'] \arrow[uur, bend left, "f"] & & M_2 \arrow[ul, "\kappa_2"] \arrow[uul, bend right, "g"']
\end{tikzcd}$$

in **Mon** to commute. This time we define $[f, g]\colon M_1 \times M_2 \to M_3$ by

$$[f, g](m_1, m_2) \coloneqq f(m_1) +_3 g(m_2)$$

for all $(m_1, m_2) \in M_1 \times M_2$. That $[f, g]$ is a monoid homomorphism follows from f and g being monoid homomorphisms. Then

$$\begin{aligned}
([f, g] \circ \kappa_1)(m_1) &= [f, g](m_1, 0_1) \\
&= f(m_1) +_3 g(0_1) \\
&= f(m_1)
\end{aligned}$$

for all $m_1 \in M_1$. Similarly, $([f, g] \circ \kappa_2) = g$.

Now suppose there is another monoid homomorphism $h\colon M_1 \times M_2 \to M_3$ satisfying

$$h\kappa_1 = f \quad \text{and} \quad h\kappa_2 = g.$$

Then, for any $(m_1, m_2) \in M_1 \times M_2$,

$$\begin{aligned}
h(m_1, m_2) &= h(m_1, 0_2) +_3 h(0_1, m_2)) \\
&= h(\kappa_1(m_1)) +_3 h(\kappa_2(m_2)) \\
&= f(m_1) +_3 g(m_2) \\
&= [f, g](m_1, m_2).
\end{aligned}$$

Therefore $[f, g]$ is the unique monoid homomorphism making the diagram above commute. □

Exercise 2.1.7

Show that in **Sets** *products distribute over coproducts, in the sense that the canonical maps*

$$(X \times Y) + (X \times Z) \xrightarrow{[\mathrm{id}_X \times \kappa_1, \mathrm{id}_X \times \kappa_2]} X \times (Y + Z)$$

$$0 \xrightarrow{\quad !\quad} X \times 0$$

are isomorphisms. Categories in which this is the case are called ***distributive****; see Cockett (1993) for more information on distributive categories in general and see Gumma, Hughes, and Schröder (2003) for an investigation of such distributivities in categories of coalgebras.*

Solution. In **Sets**, the initial object 0 is the empty set. Consequently, for any set X, the unique map $0 \xrightarrow{!} X \times 0$ is an isomorphism (in fact, $!$ is the identity morphism on 0) since $X \times 0 = 0$.

Now fix sets X, Y, and Z, and let $Y \xrightarrow{\kappa_1} Y+Z$ and $Z \xrightarrow{\kappa_2} Y+Z$ denote the appropriate coprojections. We may assume, without loss of generality, that Y and Z are disjoint, so that we may write $Y \cup Z$ in place of $Y+Z$, and have $\kappa_1 \colon Y \to Y \cup Z$ and $\kappa_2 \colon Z \to Y \cup Z$ be the appropriate inclusion functions.

The function $[\mathrm{id}_X \times \kappa_1, \mathrm{id}_X \times \kappa_2] \colon (X \times Y) + (X \times Z) \to X \times (Y+Z)$ is then given by

$$[\mathrm{id}_X \times \kappa_1, \mathrm{id}_X \times \kappa_2](x,w) = (x,w)$$

for all $(x,w) \in (X \times Y) + (X \times Z)$. This is clearly a bijection. $\square$

Exercise 2.1.8

1. *Consider a category with finite products* $(\times, 1)$*. Prove that there are isomorphisms:*

$$X \times Y \cong Y \times X, \qquad (X \times Y) \times Z \cong X \times (Y \times Z), \qquad 1 \times X \cong X.$$

2. *Similarly, show that in a category with finite coproducts* $(+, 0)$ *one has*

$$X + Y \cong Y + X, \qquad (X + Y) + Z \cong X + (Y + Z), \qquad 0 + X \cong X.$$

(This means that both the finite product and coproduct structure in a category yield so-called symmetric monoidal structure. See Mac Lane (1978) or Borceaux (1994) for more information.)

3. *Next, assume that our category also has exponents. Prove that*

$$X^0 \cong 1, \qquad X^1 \cong X, \qquad 1^X \cong 1.$$

And also that

$$Z^{X+Y} \cong Z^X \times Z^Y, \qquad Z^{X \times Y} \cong (Z^Y)^X, \qquad (X \times Y)^Z \cong X^Z \times Y^Z.$$

Solution.

1. Let $\mathbb{C}$ be a category with finite products. Fix $X, Y \in \mathsf{Obj}(\mathbb{C})$. Let $X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$ and $Y \xleftarrow{p_1} Y \times X \xrightarrow{p_2} X$ be the relevant projections. We have the diagram

$$\begin{tikzcd}
Y & & X \\
& Y \times X \arrow[ul, "p_1"'] \arrow[ur, "p_2"] & \\
X \arrow[uurr, "\mathrm{id}_X"] & & Y \arrow[uull, "\mathrm{id}_Y"'] \\
& X \times Y \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[uu, dashed, "\langle \pi_2, \pi_1 \rangle"] &
\end{tikzcd}$$

in $\mathbb{C}$ commuting. We claim that the unique induced morphism $X \times Y \xrightarrow{\langle \pi_2, \pi_1 \rangle} X \times Y$ is an isomorphism. Of course, its inverse would be the similarly obtained morphism $Y \times X \xrightarrow{\langle p_2, p_1 \rangle} Y \times X$. Indeed, looking at the diagram

$$\begin{tikzcd}
X & & Y \\
& X \times Y \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] & \\
Y \arrow[uurr, bend right, "\mathrm{id}_Y"] & & X \arrow[uull, bend left, "\mathrm{id}_X"'] \\
& Y \times X \arrow[ul, "p_1"'] \arrow[ur, "p_2"] \arrow[uu, dashed, "\langle p_2, p_1 \rangle"] & \\
X \arrow[uurr, bend right, "\mathrm{id}_X"] & & Y \arrow[uull, bend left, "\mathrm{id}_Y"'] \\
& X \times Y \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] \arrow[uu, dashed, "\langle \pi_2, \pi_1 \rangle"] &
\end{tikzcd}$$

in $\mathbb{C}$, we see that

$$\begin{aligned}
\pi_1 \circ \langle p_2, p_1 \rangle \circ \langle \pi_2, \pi_1 \rangle &= p_2 \circ \langle \pi_2, \pi_1 \rangle \\
&= \pi_1
\end{aligned}$$

and, similarly, $\pi_2 \circ \langle p_2, p_1 \rangle \circ \langle \pi_2, \pi_1 \rangle = \pi_2$. Consequently, $\langle p_2, p_1 \rangle \circ \langle \pi_2, \pi_1 \rangle = \mathrm{id}_{X \times Y}$. Similarly, we obtain $\langle \pi_2, \pi_1 \rangle \circ \langle p_2, p_1 \rangle = \mathrm{id}_{Y \times X}$. Therefore we have an isomorphism $X \times Y \xrightarrow[\cong]{\langle \pi_2, \pi_1 \rangle} Y \times X$.

Now fix $X, Y, Z \in \mathsf{Obj}(\mathbb{C})$. Consider the products $X \times Y$ and $(X \times Y) \times Z$ as in the diagram

$$\begin{tikzcd}
X & & Y & Z \\
X \times Y \arrow[u, "p_1"] \arrow[urr, "p_2"] & & & \\
& (X \times Y) \times Z \arrow[ul, "\pi_1"] \arrow[uurr, "\pi_2"'] & &
\end{tikzcd}$$

in $\mathbb{C}$. These come with associated projections $X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y$ and $X \times Y \xleftarrow{\pi_1} (X \times Y) \times Z \xrightarrow{\pi_2} Z$. We also have projections $X \xleftarrow{q_1} X \times (Y \times Z) \xrightarrow{q_2} Y \times Z$ and $Y \xleftarrow{r_1} Y \times Z \xrightarrow{r_2} Z$, as depicted in the diagram

$$
\begin{tikzcd}
X & Y & & Z \\
& & & Y \times Z \arrow[u, "r_2"'] \arrow[ull, "r_1"'] \\
& & X \times (Y \times Z) \arrow[uull, "q_1"] \arrow[ur, "q_2"']
\end{tikzcd}
$$

in $\mathbb{C}$. From these, we obtain the induced morphisms $(X \times Y) \times Z \xrightarrow{\langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle} X \times (Y \times Z)$ and $X \times (Y \times Z) \xrightarrow{\langle\langle q_1, r_1q_2\rangle, r_2q_2\rangle} (X \times Y) \times Z$.

$$
\begin{tikzcd}
X & & Y & & Z \\
X \times Y \arrow[u, "p_1"] \arrow[urr, "p_2"] & & & & Y \times Z \arrow[u, "r_2"'] \arrow[ull, "r_1"'] \\
& (X \times Y) \times Z \arrow[ul, "\pi_1"] \arrow[uurrr, "\pi_2"] \arrow[rr, "\langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle"'] & & X \times (Y \times Z) \arrow[uulll, "q_1"] \arrow[ur, "q_2"'] \arrow[ll, bend left, "\langle\langle q_1, r_1q_2\rangle, r_2q_2\rangle"]
\end{tikzcd}
$$

Then

$$
\begin{aligned}
p_1 \circ \pi_1 \circ \Big(\langle\langle q_1, r_1q_2\rangle, r_2q_2\rangle \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle\Big) &= p_1 \circ \Big(\pi_1 \circ \langle\langle q_1, r_1q_2\rangle, r_2q_2\rangle\Big) \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle \\
&= p_1 \circ \langle q_1, r_1q_2\rangle \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle \\
&= q_1 \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle \\
&= p_1\pi_1, \\
p_2 \circ \pi_1 \circ \Big(\langle\langle q_1, r_1q_2\rangle, r_2q_2\rangle \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle\Big) &= p_2 \circ \Big(\pi_1 \circ \langle\langle q_1, r_1q_2\rangle, r_2q_2\rangle\Big) \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle \\
&= p_2 \circ \langle q_1, r_1q_2\rangle \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle \\
&= r_1 \circ q_2 \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle \\
&= r_1 \circ \langle p_2\pi_1, \pi_2\rangle \\
&= p_2\pi_1, \text{ and} \\
\pi_2 \circ \Big(\langle\langle q_1, r_1q_2\rangle, r_2q_2\rangle \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle\Big) &= \Big(\pi_2 \circ \langle\langle q_1, r_1q_2\rangle, r_2q_2\rangle\Big) \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle \\
&= r_2 \circ q_2 \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle \\
&= r_2 \circ \langle p_2\pi_1, \pi_2\rangle \\
&= \pi_2.
\end{aligned}
$$

Thus $\langle\langle q_1, r_1q_2\rangle, r_2q_2\rangle \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle = \mathrm{id}_{(X\times Y)\times Z}$. Via a similar calculation, we also obtain $\langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle \circ \langle\langle q_1, r_1q_2\rangle, r_2q_2\rangle = \mathrm{id}_{X\times(Y\times Z)}$. Therefore, we have an isomorphism $(X \times Y) \times Z \xrightarrow[\cong]{\langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle} X \times (Y \times Z)$.

Now fix $X \in \mathsf{Obj}(\mathbb{C})$ and let 1 denote the terminal object in $\mathbb{C}$. We have the diagram

$$\begin{tikzcd}
1 & & X \\
& 1 \times X \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] & \\
& X \arrow[u, dashed, "{\langle f,\mathrm{id}_X\rangle}"] \arrow[uul, dashed, bend left, "\exists! f"] \arrow[uur, bend right, "\mathrm{id}_X"'] &
\end{tikzcd}$$

in $\mathbb{C}$, where $1 \xleftarrow{\pi_1} 1 \times X \xrightarrow{\pi_2} X$ are the relevant projections, and $X \xrightarrow{f} 1$ is the unique morphism from X to 1. As the diagram commutes, we have $\pi_2 \circ \langle f, \mathrm{id}_X \rangle = \mathrm{id}_X$. Furthermore, $\pi_1 \circ (\langle f, \mathrm{id}_X \rangle \circ \pi_2) = \pi_1$ because 1 is the terminal object and we already have the morphism $1 \times X \xrightarrow{\pi_1} 1$. Moreover, $\pi_2 \circ (\langle f, \mathrm{id}_X \rangle \circ \pi_2) = \pi_2$.

$$\begin{tikzcd}
1 & & X \\
& 1 \times X \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] & \\
& X \arrow[u, "{\langle f,\mathrm{id}_X\rangle}"] \arrow[uul, bend left, "f"] \arrow[uur, bend right, "\mathrm{id}_X"'] & \\
& 1 \times X \arrow[u, "\pi_2"] \arrow[uuul, bend left=60, "\pi_1"] \arrow[uuur, bend right=60, "\pi_2"'] &
\end{tikzcd}$$

Thus $\langle f, \mathrm{id}_X \rangle \circ \pi_2 = \mathrm{id}_{1 \times X}$. Therefore we have an isomorphism $1 \times X \xrightarrow[\cong]{\pi_2} X$.

2. This is dual to Exercise 2.1.8.1: coproducts in $\mathbb{C}$ coincide with products in $\mathbb{C}^{\mathrm{op}}$; the initial object in $\mathbb{C}$ is the terminal object in $\mathbb{C}^{\mathrm{op}}$; and isomorphisms in $\mathbb{C}$ are precisely isomorphisms in $\mathbb{C}^{\mathrm{op}}$.

3. Now suppose that the category $\mathbb{C}$ has all finite products, has all finite coproducts, and has exponents, i.e. $\mathbb{C}$ is a bicartesian closed category. Denote the initial and terminal objects of $\mathbb{C}$ by 0 and 1 respectively.

   Fix $X \in \mathsf{Obj}(\mathbb{C})$.

   #??

$\square$

Exercise 2.1.9
Show that the finite powerset also forms a functor $\mathcal{P}_{\mathrm{fin}} \colon \mathbf{Sets} \to \mathbf{Sets}$.

Solution. The proof that $\mathcal{P}_{\mathrm{fin}} \colon \mathbf{Sets} \to \mathbf{Sets}$ is a functor is identical to the proof that the usual power set operation $\mathcal{P} \colon \mathbf{Sets} \to \mathbf{Sets}$ is a functor. Given a function $f \colon X \to Y$, the function $\mathcal{P}_{\mathrm{fin}} f \colon \mathcal{P}_{\mathrm{fin}} X \to \mathcal{P}_{\mathrm{fin}} Y$ sends finite subsets $A \subseteq X$ to their image under f. That is, for finite subsets $A \subseteq X$, we define

$$(\mathcal{P}_{\mathrm{fin}} f)(A) \coloneqq \{\, f(x) \ : \ x \in A \,\},$$

which is indeed a finite set.

It is clear that $\mathcal{P}_{\text{fin}}\text{id}_X = \text{id}_{\mathcal{P}_{\text{fin}}X}$ for all sets X. Now given functions $f\colon X \to Y$ and $g\colon Y \to Z$,

$$\begin{aligned}
(\mathcal{P}_{\text{fin}}(gf))(A) &= \{\, g(f(x)) \;:\; x \in A \,\} \\
&= \{\, g(y) \;:\; y \in (\mathcal{P}_{\text{fin}}f)(A) \,\} \\
&= (\mathcal{P}_{\text{fin}}g)((\mathcal{P}_{\text{fin}}f)(A)) \\
&= (\mathcal{P}_{\text{fin}}g \circ \mathcal{P}_{\text{fin}}f)(A)
\end{aligned}$$

for all finite subsets $A \subseteq X$. Thus $\mathcal{P}_{\text{fin}}(gf) = (\mathcal{P}_{\text{fin}}g)(\mathcal{P}_{\text{fin}}f)$. □

Exercise 2.1.10
Check that

$$\mathcal{P}(0) \cong 1, \qquad \mathcal{P}(X+Y) \cong \mathcal{P}(X) \times \mathcal{P}(Y).$$

And similarly for the finite powerset $\mathcal{P}_{\text{fin}}$ instead of $\mathcal{P}$. This property says that $\mathcal{P}$ and $\mathcal{P}_{\text{fin}}$ are 'additive'; see Coumans and Jacobs (2013).

Solution. Let 0 and 1 respectively denote the initial and terminal objects in **Sets**. Then $\mathcal{P}(0) = \mathcal{P}_{\text{fin}}(0) = \mathcal{P}(\varnothing) = \{\varnothing\} \cong 1$.

Now fix sets X and Y and suppose, without loss of generality, that X and Y are disjoint so that we can write $X + Y = X \cup Y$. Then, we have a bijection $f\colon \mathcal{P}(X+Y) \to \mathcal{P}X \times \mathcal{P}Y$ defined by

$$f(A) \coloneqq \big(\{\, z \in A \;:\; z \in X \,\},\ \{\, z \in A \;:\; z \in Y \,\}\big)$$

for all $A \subseteq X + Y$. This is indeed a bijection as it has inverse $f^{-1}\colon \mathcal{P}X \times \mathcal{P}Y \to \mathcal{P}(X+Y)$ defined by

$$f^{-1}(A,B) \coloneqq A \cup B.$$

The proof that $\mathcal{P}_{\text{fin}}(X+Y) \cong \mathcal{P}_{\text{fin}}X \times \mathcal{P}_{\text{fin}}Y$ is similar. □

Exercise 2.1.11
Notice that a power set $\mathcal{P}(X)$ can also be understood as exponent 2^X, where $2 = \{0,1\}$. Check that the exponent functoriality gives rise to the contravariant powerset $\mathbf{Sets}^{\text{op}} \to \mathbf{Sets}$.

Solution. The identification of $\mathcal{P}(X)$ with 2^X is via the isomorphism $\alpha_X\colon \mathcal{P}(X) \to 2^X$ defined by

$$\alpha_X(A) \coloneqq \lambda x \in X. \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A, \end{cases}$$

for all $A \subseteq X$.

Fix a function $f\colon X \to Y$. The function $2^f\colon 2^Y \to 2^X$ is given by

$$(2^f)(k) \coloneqq \lambda x \in X. k(f(x)),$$

for all functions $k\colon Y \to 2$. We then see that $\alpha_X^{-1} \circ 2^f \circ \alpha_Y\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ satisfies

$$\begin{aligned}
\big(\alpha_X^{-1} \circ 2^f \circ \alpha_Y\big)(B) &= \big(\alpha_X^{-1} \circ 2^f\big)\left(\lambda y \in Y. \begin{cases} 1, & \text{if } y \in B, \\ 0, & \text{if } y \notin B \end{cases}\right) \\
&= \alpha_X^{-1}\left(\lambda x \in X. \begin{cases} 1, & \text{if } f(x) \in B, \\ 0, & \text{if } f(x) \notin B \end{cases}\right) \\
&= \{\, x \in X \;:\; f(x) \in B \,\}
\end{aligned}$$

for all $B \subseteq Y$. This is precisely how the contravariant power set functor is defined on morphisms. □

Exercise 2.1.12

Consider a function $f\colon X \to Y$. Prove that

1. *The direct image $\mathcal{P}(f) = \coprod_f \colon \mathcal{P}(X) \to \mathcal{P}(Y)$ preserves all joins and that the inverse image $f^{-1}(-)\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ preserves not only joins but also meets and negation (i.e. all the Boolean structure).*
2. *There is a Galois connection $\coprod_f(U) \subseteq V \iff U \subseteq f^{-1}(V)$, as claimed in (2.15).*
3. *There is a product function $\prod_f \colon \mathcal{P}(X) \to \mathcal{P}(Y)$ given by $\prod_f(U) = \{\, y \in Y \mid \forall x \in X. f(x) = y \Rightarrow x \in U \,\}$, with a Galois connection $f^{-1}(V) \subseteq U \iff V \subseteq \prod_f(U)$.*

Solution.

1. For a collection $\{A_i\}_{i\in I}$ of subsets of X, we see that

$$\begin{aligned}(\mathcal{P}f)\left(\bigcup_{i\in I} A_i\right) &= \left\{ f(x) \;:\; x \in \bigcup_{i\in I} A_i \right\} \\ &= \bigcup_{i\in I} \{\, f(x) \;:\; x \in A_i \,\} \\ &= \bigcup_{i\in I} (\mathcal{P}f)(A_i).\end{aligned}$$

So $\mathcal{P}f$ preserves all joins. Furthermore, for a collection $\{B_j\}_{j\in J}$ of subsets of X,

$$\begin{aligned}f^{-1}\left(\bigcup_{j\in J} B_j\right) &= \left\{ x \in X \;:\; f(x) \in \bigcup_{j\in J} B_j \right\} \\ &= \bigcup_{j\in J} \{\, x \in X \;:\; f(x) \in B_j \,\} \\ &= \bigcup_{j\in J} f^{-1}(B_j).\end{aligned}$$

So $f^{-1}(-)$ also preserves all joins. Moreover,

$$\begin{aligned}f^{-1}\left(\bigcap_{j\in J} B_j\right) &= \left\{ x \in X \;:\; f(x) \in \bigcap_{j\in J} B_j \right\} \\ &= \bigcap_{j\in J} \{\, x \in X \;:\; f(x) \in B_j \,\} \\ &= \bigcap_{j\in J} f^{-1}(B_j).\end{aligned}$$

So $f^{-1}(-)$ preserves all meets. Also, for any subset $B \subseteq Y$,

$$\begin{aligned}f^{-1}(Y \setminus B) &= \{\, x \in X \;:\; f(x) \in Y \setminus B \,\} \\ &= X \setminus \{\, x \in X \;:\; f(x) \in B \,\} \\ &= X \setminus f^{-1}(B).\end{aligned}$$

So $f^{-1}(-)$ preserves all negations.

2. Fix a pair of subsets $U \subseteq X$ and $V \subseteq Y$. Then

$$\begin{aligned}
(\mathcal{P}f)(U) \subseteq V \quad & \text{if and only if} \quad \{\, f(x) : x \in U \,\} \subseteq V \\
& \text{if and only if} \quad \text{for all } x \in U \text{ we have } f(x) \in V \\
& \text{if and only if} \quad U \subseteq \{\, x \in X : f(x) \in V \,\} \\
& \text{if and only if} \quad U \subseteq f^{-1}(V),
\end{aligned}$$

as claimed.

3. Fix a pair of subsets $U \subseteq X$ and $V \subseteq Y$. Then

$$\begin{aligned}
f^{-1}(V) \subseteq U \quad & \text{if and only if} \quad \{\, x \in X : f(x) \in V \,\} \subseteq U \\
& \text{if and only if} \quad \text{for all } x \in X \text{ with } f(x) \in V \text{ we have } x \in U \\
& \text{if and only if} \quad V \subseteq \{\, y \in Y : \text{for all } x \in X \text{ with } f(x) = y \text{ we have } x \in U \,\} \\
& \text{if and only if} \quad V \subseteq \prod_f (U),
\end{aligned}$$

as desired. ☐

Exercise 2.1.13

Assume a category $\mathbb{C}$ has arbitrary, set-indexed coproducts $\coprod_{i \in I} X_i$. Demonstrate, as in the proof of Proposition 2.1.5, that the category **CoAlg**(F) *of coalgebras of a functor $F \colon \mathbb{C} \to \mathbb{C}$ then also has such coproducts.*

Solution. Let I be a non-empty set and fix an I-indexed tuple $(X_i \xrightarrow{\alpha_i} FX_i)_{i \in I}$ of F-coalgebras. Let $C \coloneqq \coprod_{i \in I} X_i$ be the coproduct of $(X_i)_{i \in I}$ in $\mathbb{C}$ and, for $i \in I$, let $X_i \xrightarrow{\kappa_i} C$ denote the appropriate coprojection. We have the collection of morphisms $(X_i \xrightarrow{(F\kappa_i)\alpha_i} FC)_{i \in I}$. So there exists a unique morphism $\gamma \colon C \to FC$ such that $\gamma\kappa_i = (F\kappa_i)\alpha_i$ for all $i \in I$. That is, the diagram

$$\begin{tikzcd}
FC & & & & & \\
& FX_0 \arrow[ul, "F\kappa_0"'] & FX_1 \arrow[ull, "F\kappa_1"] & \cdots & FX_i \arrow[ullll, bend right=10, "F\kappa_i"'] & \cdots \\
C \arrow[uu, dashed, "\exists!\gamma"] & & & & & \\
& X_0 \arrow[uu, "\alpha_0"] \arrow[ul, "\kappa_0"] & X_1 \arrow[uu, "\alpha_1"] \arrow[ull, "\kappa_1"] & \cdots & X_i \arrow[uu, "\alpha_i"] \arrow[ullll, bend right=10, "\kappa_i"'] & \cdots
\end{tikzcd}$$

in $\mathbb{C}$ commutes. Consequently, we have a collection of homomorphisms of F-coalgebras $((X_i, \alpha_i) \xrightarrow{\kappa_i} (C, \gamma))_{i \in I}$.

Now, suppose we are given another F-coalgebra $Y \xrightarrow{\beta} FY$ and a collection of homomorphisms of F-coalgebras $((X_i, \alpha_i) \xrightarrow{f_i} (Y, \beta))_{i \in I}$. Then, as C is the coproduct of $(X_i)_{i \in I}$ in $\mathbb{C}$, there is a unique morphism $C \xrightarrow{g} Y$ in $\mathbb{C}$ such that $g\kappa_i = f_i$ for all $i \in I$.

$$
\begin{tikzcd}
 & FC \arrow[dl, dashed, "Fg"'] & & & & & \\
FY & & FX_0 \arrow[ul, "F\kappa_0"'] \arrow[ll, bend right, "Ff_0"'] & FX_1 \arrow[ull, "F\kappa_1"'] \arrow[lll, bend left, "Ff_1"] & \cdots & FX_i \arrow[ullll, bend right, "F\kappa_i"'] \arrow[lllll, bend left, "Ff_i"] & \cdots \\
 & C \arrow[uu, "\gamma"] \arrow[dl, dashed, "\exists ! g"'] & & & & & \\
Y \arrow[uu, "\beta"] & & X_0 \arrow[uu, "\alpha_0"] \arrow[ul, "\kappa_0"'] \arrow[ll, bend right, "f_0"'] & X_1 \arrow[uu, "\alpha_1"] \arrow[ull, "\kappa_1"'] \arrow[lll, bend left, "f_1"] & \cdots & X_i \arrow[uu, "\alpha_i"] \arrow[ullll, bend right, "\kappa_i"'] \arrow[lllll, bend left, "f_i"] & \cdots
\end{tikzcd}
$$

We now need to verify that g is actually a homomorphism of F-coalgebras from (C, γ) to (Y, β). We will use the universal property of C as the coproduct in $\mathbb{C}$: for all $i \in I$, we have

$$
\begin{aligned}
\beta g \kappa_i &= \beta f_i, && \text{since } g\kappa_i = f_i, \\
&= (Ff_i)\alpha_i, && \text{since } f_i \text{ is a homomorphism from } (X_i, \alpha_i) \text{ to } (Y, \beta), \\
&= (Fg)(F\kappa_i)\alpha_i, && \text{from } g\kappa_i = f_i \text{ and the functoriality of } F, \\
&= (Fg)\gamma\kappa_i, && \text{since } \kappa_i \text{ is a homomorphism from } (X_i, \alpha_i) \text{ to } (C, \gamma).
\end{aligned}
$$

Therefore $\beta g = (Fg)\gamma$, i.e. g is a homomorphism from (C, γ) to (Y, β). □

Exercise 2.1.14

*For two parallel maps $f, g \colon X \to Y$ between objects X, Y in an arbitrary category $\mathbb{C}$ a **coequaliser** $q \colon Y \to Q$ is a map in a diagram*

$$
\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] & Q
\end{tikzcd}
$$

with $q \circ f = q \circ g$ in a 'universal way': for an arbitrary map $h \colon Y \to Z$ with $h \circ f = h \circ g$ tehre is a unique map $k \colon Q \to Z$ with $k \circ q = h$.

1. *An **equalier** in a category $\mathbb{C}$ is a coequaliser in $\mathbb{C}^{\mathrm{op}}$. Formulate explicitly what an equaliser of two parallel maps is.*

2. *Check that in the category* **Sets** *the set Q can be defined as the quotient Y/R, where $R \subseteq Y \times Y$ is the least equivalence relation containing all pairs $(f(x), g(x))$ for $x \in X$.*

3. *Returning to the general case, assume a category $\mathbb{C}$ has coequalisers. Prove that for an arbitrary functor $F \colon \mathbb{C} \to \mathbb{C}$ the associated category of coalgebras* **CoAlg**(F) *also has coequalisers, as in $\mathbb{C}$: for two homomorphisms $f, g \colon X \to Y$ between coalgebras $c \colon X \to F(X)$ and $d \colon Y \to F(Y)$ there is by universality an induced coalgebra structure $Q \to F(Q)$ on the coequaliser Q of the underlying maps f, g, yielding a diagram of coalgebras*

$$
\begin{pmatrix} F(X) \\ \uparrow c \\ X \end{pmatrix} \overset{f}{\underset{g}{\rightrightarrows}} \begin{pmatrix} F(Y) \\ \uparrow d \\ Y \end{pmatrix} \xrightarrow{\quad q \quad} \begin{pmatrix} F(Q) \\ \uparrow \\ Q \end{pmatrix}
$$

with the appropriate universal property in $\mathbf{CoAlg}(F)$*: for each coalgebra* $e\colon Z \to F(Z)$ *with homomorphism* $h\colon Y \to Z$ *satisfying* $h \circ f = h \circ g$ *there is a unique homomorphism of coalgebras* $k\colon Q \to Z$ *with* $k \circ q = h$*.*

Solution.

1. An equaliser of a parallel pair $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is a morphism $E \xrightarrow{e} X$ such that both of the following hold:

   (a) we have $fe = ge$; and

   (b) for any morphism $Z \xrightarrow{h} X$ satisfying $fh = gh$ there exists a unique morphism $Z \xrightarrow{k} E$ in $\mathbb{C}$ such that $ek = h$.

2. Fix functions $f, g\colon X \to Y$. Let $R \subseteq Y \times Y$ be the smallest equivalence relation on Y such that $\{\, (f(x), g(x)) : x \in X \,\} \subseteq R$, and define $q\colon Y \to Y/R$ by $q(y) := [y]$ for all $y \in Y$, where $[y]$ denotes the R-equivalence class of $y \in Y$.

   Fix another function $h\colon Y \to Z$ such that $hf = hg$. We need to show that we have the diagram

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] \arrow[dr, "h"'] & Y/R \arrow[d, dashed, "\exists! k"] \\ & & Z \end{tikzcd}$$

   in **Sets** commuting. We define $k\colon Y/R \to Z$ by $k([y]) := h(y)$ for each R-equivalence class $[y] \in Y/R$. Note that this k is well-defined: if $y, y' \in Y$ are such that yRy' then we can prove by induction on the construction of R (as the reflexive symmetric transitive closure of $\{\, (f(x), g(x)) : x \in X \,\}$) that $h(y) = h(y')$. Then, by construction, $k\colon Y/R \to Z$ is the unique function satisfying $kq = h$.

3. Now suppose that $\mathbb{C}$ has coequalisers. Fix a parallel pair of morphisms $(X, c) \underset{g}{\overset{f}{\rightrightarrows}} (Y, d)$ in $\mathbf{CoAlg}(F)$.

   Let $Y \xrightarrow{q} Q$ be the coequaliser in $\mathbb{C}$ of the parallel pair $X \underset{g}{\overset{f}{\rightrightarrows}} Y$. Observe then that

$$\begin{aligned} (Fq)df &= (Fq)(Ff)c, && \text{since } f \text{ is a homomorphism from } (X,c) \text{ to } (Y,d), \\ &= F(qf)c, && \text{by functoriality of } F, \\ &= F(qg)c, && \text{since } qf = qg, \text{ because } q \text{ is the coequaliser of } f \text{ and } g, \\ &= F(q)F(g)c, && \text{by the functoriality of } F, \\ &= (Fq)dg, && \text{since } g \text{ is also a homomorphism from } (X,c) \text{ to } (Y,d). \end{aligned}$$

   So there must be a unique morphism $Q \xrightarrow{\alpha} FQ$ in $\mathbb{C}$ such that $\alpha q = (Fq)d$.

$$\begin{tikzcd} FX \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] & FY \arrow[r, "Fq"'] & FQ \\ X \arrow[u, "c"] \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \arrow[u, "d"'] \arrow[r, "q"] & Q \arrow[u, dashed, "\exists! \alpha"'] \end{tikzcd}$$

So we have an F-coalgebra structure on Q, namely $Q \xrightarrow{\alpha} FQ$, and the requirement $\alpha q = (Fq)d$ says that q is a homomorphism of F-coalgebras from (Y, d) to (Q, α).

Now suppose that there is another F-coalgebra $Z \xrightarrow{\beta} FZ$ and a homomorphism $(Y, d) \xrightarrow{h} (Z, \beta)$ such that $hf = hg$. Then there is a unique morphism $Q \xrightarrow{k} Z$ in $\mathbb{C}$ such that $kq = h$.

$$
\begin{tikzcd}
 & & & FZ \\
FX \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] & FY \arrow[r, "Fq"'] \arrow[urr, "Fh"] & FQ \arrow[ur, dashed, "Fk"] & \\
X \arrow[u, "c"] \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \arrow[u, "d"'] \arrow[r, "q"] \arrow[drr, "h"'] & Q \arrow[u, dashed, "\exists!\alpha"'] \arrow[dr, dashed, "\exists!k"] & \\
 & & & Z \arrow[uuu, "\beta"']
\end{tikzcd}
$$

We now just need to verify that k is a homomorphism from (Q, α) to (Z, β), i.e. $\beta k = (Fk)\alpha$. We will use the universal property of $Y \xrightarrow{q} Q$ as the coequaliser of $X \overset{f}{\underset{g}{\rightrightarrows}} Y$: we have

$$
\begin{aligned}
\beta kqf &= \beta hf, && \text{since } kq = h, \\
&= (Fh)df, && \text{since } h \text{ is a homomorphism from } (Y, d) \text{ to } (Z, \beta), \\
&= (Fk)(Fq)df, && \text{from } kq = h \text{ and the functoriality of } F, \\
&= (Fk)\alpha qf, && \text{since } q \text{ is a homomorphism from } (Y, d) \text{ to } (Q, \alpha), \\
&= (Fk)\alpha qg, && \text{since } qf = qg \text{ as } q \text{ coequalises } f \text{ and } g, \\
&= \beta hg, && \text{similarly as above}, \\
&= \beta kqg, && \text{since } kq = h.
\end{aligned}
$$

The equalities to take away from the calculation above are

$$\beta hf = \beta hg = (\beta k)qf = (\beta k)qg = ((Fk)\alpha)qf = ((Fk)\alpha)qg.$$

By the uniqueness clause in the universal property of coequalisers, we must have $\beta k = (Fk)\alpha$. □

2.2 Polynomial Functors and Their Coalgebras

Exercise 2.2.1
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Solution. #?? □

Exercise 2.2.2
#??

Solution. #?? □

Exercise 2.2.3
#??

Solution. #?? □

Exercise 2.2.4
#??

Solution. #?? □

Exercise 2.2.5
#??

Solution. #?? □

Exercise 2.2.6
#??

Solution. #?? □

Exercise 2.2.7
#??

Solution. #?? □

Exercise 2.2.8
#??

Solution. #?? □

Exercise 2.2.9
#??

Solution. #?? □

Exercise 2.2.10
#??

Solution. #?? □

Exercise 2.2.11
#??

Solution. #?? □

Exercise 2.2.12
#??

Solution. #?? □

2.3 Final Coalgebras

Exercise 2.3.1
#??

Solution. #?? □

Exercise 2.3.2
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Solution. #?? □

Exercise 2.3.3
#??

Solution. #?? □

Exercise 2.3.4
#??

Solution. #?? □

Exercise 2.3.5
#??

Solution. #?? □

Exercise 2.3.6
#??

Solution. #?? □

Exercise 2.3.7
#??

Solution. #?? □

Exercise 2.3.8
#??

Solution. #?? □

2.4 Algebras

Exercise 2.4.1
#??

Solution. #?? □

Exercise 2.4.2
#??

Solution. #?? □

Exercise 2.4.3
#??

Solution. #?? □

Exercise 2.4.4
#??

Solution. #?? □

Exercise 2.4.5
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Solution. #?? □

Exercise 2.4.6
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Solution. #?? □

Exercise 2.4.7
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Solution. #?? □

Exercise 2.4.8
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Solution. #?? □

Exercise 2.4.9
#??

Solution. #?? □

Exercise 2.4.10
#??

Solution. #?? □

2.5 Adjunctions, Cofree Coalgebras, Behaviour-Realisation

Exercise 2.5.1
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Solution. #?? □

Exercise 2.5.2
#??

Solution. #?? □

Exercise 2.5.3
#??

Solution. #?? □

Exercise 2.5.4
This exercise describes 'strength' for endofunctors on **Sets**. *In general, this is a useful notion in the theory of datatypes (Cockett and Spencer, 1992), (Cockett and Spencer, 1995) and of computations (Moggi, 1991); see Section 5.2 for a systemic description.*

Let $F\colon \mathbf{Sets} \to \mathbf{Sets}$ *be an arbitrary functor. Consider for sets* X, Y *the strength map*

$$F(X) \times Y \xrightarrow{\quad \mathrm{st}_{X,Y} \quad} F(X \times Y)$$

$$(u, y) \longmapsto F(\lambda x \in X.(x, y))(u)$$

1. *Prove that this yields a natural transformation* $F(-)\times(-) \stackrel{\mathrm{st}}{\Rightarrow} F((-)\times(-))$*, where both the domain and codomain are functors* $\mathbf{Sets} \times Sets \to \mathbf{Sets}$.

2. *Describe this strength map for the list functor* $(-)^{\star}$ *and for the powerset functor* $\mathcal{P}$.

Solution. #?? □

Exercise 2.5.5
#??

Solution. #?? □

Exercise 2.5.6
#??

Solution. #?? □

Exercise 2.5.7
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Solution. #?? □

Exercise 2.5.8
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Solution. #?? □

Exercise 2.5.9
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Solution. #?? □

Exercise 2.5.10
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Solution. #?? □

Exercise 2.5.11
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Solution. #?? □

Exercise 2.5.12
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Solution. #?? □

Exercise 2.5.13
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Solution. #?? □

Exercise 2.5.14
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Solution. #?? □

Exercise 2.5.15
#??

Solution. #?? □

Exercise 2.5.16
#??

Solution. #?? □

Exercise 2.5.17
#??

Solution. #?? □

3 Bisimulations

3.1 Relation Lifting, Bisimulations and Congruences

Exercise 3.1.1
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Solution. #?? □

Exercise 3.1.2
#??

Solution. #?? □

Exercise 3.1.3
#??

Solution. #?? □

Exercise 3.1.4
#??

Solution. #?? □

Exercise 3.1.5
#??

Solution. #?? □

Exercise 3.1.6
#??

Solution. #?? □

3.2 Properties of Bisimulations

Exercise 3.2.1
#??

Solution. #?? □

Exercise 3.2.2
#??

Solution. #?? □

Exercise 3.2.3
#??

Solution. #?? □

Exercise 3.2.4
#??

Solution. #?? □

Exercise 3.2.5
#??

Solution. #?? □

Exercise 3.2.6
#??

Solution. #?? □

Exercise 3.2.7
#??

Solution. #?? □

3.3 Bisimulations as Spans and Cospans

Exercise 3.3.1
#??

Solution. #?? □

Exercise 3.3.2
#??

Solution. #?? □

Exercise 3.3.3
#??

Solution. #?? □

Exercise 3.3.4
#??

Solution. #?? □

3.4 Bisimulations and the Coinduction Proof Principle

Exercise 3.4.1
#??

Solution. #?? □

Exercise 3.4.2
#??

Solution. #?? □

Exercise 3.4.3
#??

Solution. #?? □

Exercise 3.4.4
#??

Solution. #?? ☐

Exercise 3.4.5
#??

Solution. #?? ☐

Exercise 3.4.6
#??

Solution. #?? ☐

Exercise 3.4.7
#??

Solution. #?? ☐

3.5 Process Semantics

Exercise 3.5.1
#??

Solution. #?? ☐

Exercise 3.5.2
#??

Solution. #?? ☐

Exercise 3.5.3
#??

Solution. #?? ☐

Exercise 3.5.4
#??

Solution. #?? ☐

Bibliography and References

Francis Borceaux. *Handbook of Categorical Algebra*, volume 50–52 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 1994.
DOI:
Volume 1, https://doi.org/10.1017/CBO9780511525858;
Volume 2, https://doi.org/10.1017/CBO9780511525865;
Volume 3, https://doi.org/10.1017/CBO9780511525872.

J. Robin B. Cockett. Introduction to distributive categories. *Mathematical Structures in Computer Science*, 3:277–307, 1993.
DOI: https://doi.org/10.1017/S0960129500000232.

J. Robin B. Cockett and Dwight Spencer. Strong categorical datatypes I. In Robert A. G. Seely, editor, *International Meeting on Category Theory 1991*, volume 13, pages 141–169. Canadian Mathematical Society Proceedings, AMS, Montreal, 1992.

J. Robin B. Cockett and Dwight Spencer. Strong categorical datatypes II: A term logic for categorical programming. *Theoretical Computer Science*, 139:69–113, 1995.
DOI: https://doi.org/10.1016/0304-3975(94)00099-5.

Dion Coumans and Bart Jacobs. Scalars, monads, and categories. In Chris Heunen, Mehrnoosh Sadrzadeh, and Edward Grefenstette, editors, *Quantum Physics and Linguistics: A Compositional, Diagrammatic Discourse*, pages 182–216. Oxford University Press, 2013.
DOI: https://doi.org/10.1093/acprof:oso/9780199646296.003.0007.

Brian A. Davey and Hilary A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, 1990.
DOI: https://doi.org/10.1017/CBO9780511809088.

E. Allen Emerson. Temporal and modal logic. In Jan van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume B: Formal Models and Semantics, pages 995–1072. Elsevier B.V., 1990.
DOI: https://doi.org/10.1016/B978-0-444-88074-1.50021-4.

H. Peter Gumma, Jesse Hughes, and Tobias Schröder. Distributivity of categories of coalgebras. *Theoretical Computer Science*, 308:131–143, 2003.
DOI: https://doi.org/10.1016/S0304-3975(02)00582-0.

Bart Jacobs. *Introduction to Coalgebra: Towards Mathematics of States and Observation*. Cambridge University Press, 2017.
DOI: https://doi.org/10.1017/CBO9781316823187.

Saunders Mac Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer-Verlag New York, Inc., second edition, 1978.
DOI: https://doi.org/10.1007/978-1-4757-4721-8.

Eugenio Moggi. Notions of computation and monads. *Information and Computation*, 93(1):55–92, 1991.
DOI: https://doi.org/10.1016/0890-5401(91)90052-4.