Cichoń's Diagram and the Rearrangement Number

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25 April 2024

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Declaration

This essay is an exercise in self-learning and writing up a novel presentation of a certain mathematical topic for the *MA395 Essay* module at the University of Warwick. With the possible exception of a few relatively trivial examples, none of the results or proofs in this essay are original to me. The main sources for most of the results and proofs in this essay are *The Structure of the Real Line* by Lev Bukovský [6], *On the Structure of the Real Line* by Tomek Bartoszyński and Haim Judah [1], *Combinatorial Cardinal Characteristics of the Continuum* by Andreas Blass [2], *Classical Descriptive Set Theory* by Alexander Kechris [17], and *The Rearrangement Number* by Andreas Blass, Jörg Brendle, Will Brian, Joel Hamkins, Michael Hardy, and Paul Larson [3].

Generative artificial intelligence (AI) was not used in the writing of this essay.

1 Introduction

In light of the independence of the continuum hypothesis, a mystifying realm manifests in the crevice which divides $\aleph_1$ and $2^{\aleph_0}$. Explorers trained in the school of ZFC struggle to obtain a firm grasp on its inhabitants; they came here searching for answers, only to be teased away. Some strongly believe we should seal the gap shut to rid ourselves of any worries. Some suggest installing a "Danger! Keep clear!" sign to ward off passers-by. Some arm themselves with model theory and forcing for a second expedition.

We will take a guided tour through the landscape home to the *cardinal characteristics of the continuum*. We have a map, *Cichoń's diagram*, carefully crafted by the brave explorers of the past to aid us in our journey and to keep us away from dangerous waters. However, a new[1] cardinal has been brought to the world's attention — the *rearrangement number*. There are still unanswered questions about the nature of this cardinal; we are still unsure about its position in relation to Cichoń's diagram. This should add some spice to our journey...

1.1 ZFC Set Theory

We will subscribe to the ZFC school of thought under first-order logic. The list of axioms of set theory by Zermelo and Fraenkel, together with the axiom of choice, will be our foundational religious beliefs. Furthermore, the version of ZFC which we will adopt will insist that everything in the domain of the language of ZFC is a set. In the language of first-order logic, we have a primitive two-place identity predicate symbol $=$ which we will use to denote equality. So we write $x = y$ to mean "the sets x and y are equal". The language of set theory consists of another primitive two-place predicate symbol $\in$, which we use to denote *set membership*. That is, we write $x \in y$ to mean "the set x is an element of the set y". We will immediately start using all the usual two-place predicate symbols $\neq$, $\notin$, $\subseteq$, $\subset$ with their usual definitions in mathematics.

For brevity, though at the risk of losing rigour, we informally list axioms of ZFC [18, Chapter 1] [22, Section 2.10, Section 7.5, Section 8.1] [19, Section 9] below without completely rebuilding elementary ZFC set theory from the ground up.

Axiom 1.1.1 (Extensionality)**.** Let A and B be sets. If A is a subset of B, and B is a subset of A, then $A = B$.

Axiom 1.1.2 (Empty Set)**.** There exists a set $\varnothing$ with no elements.

Axiom Schema 1.1.3 (Separation)**.** Let $\varphi(x, A, a_1, \ldots, a_n)$ be a formula whose free variables are among $x, A, a_1, \ldots, a_n$. Fix any sets $A, a_1, \ldots, a_n$. Then there exists a set

$$\{\, x \in A : \varphi(x, A, a_1, \ldots, a_n) \,\}$$

consisting of all the elements $x \in A$ for which $\varphi(x, A, a_1, \ldots, a_n)$ is true, and nothing else.

Axiom 1.1.4 (Pairing)**.** Let A and B be sets. Then there exists a set $\{A, B\}$ whose elements are precisely A, B, and nothing else.

Axiom 1.1.5 (Union)**.** Let A be a set. Then there exists a set $\bigcup A$ consisting of all the elements of the sets which are elements of A, and nothing else.

Axiom 1.1.6 (Power Set)**.** Let A be a set. Then there exists a set $\mathscr{P}(A)$ consisting of all the subsets of A, and nothing else.

Axiom 1.1.7 (Infinity)**.** There exists a set $\mathbb{N}$ such that $\varnothing \in \mathbb{N}$, and for every $x \in \mathbb{N}$ we also have $x \cup \{x\} \in \mathbb{N}$.

[1] Relatively new [15] [3].

Axiom Schema 1.1.8 (Replacement)**.** Let $\varphi(x, y, A, a_1, \ldots, a_n)$ be a formula whose free variables are among $x, y, A, a_1, \ldots, a_n$. Fix any sets $A, a_1, \ldots, a_n$. Suppose the following holds:

$$\text{for all } x \in A \text{ there exists a unique } y \text{ such that } \varphi(x, y, A, a_1, \ldots, a_n) \text{ is true.}$$

Then there exists a set

$$\{\, y : \varphi(x, y, A, a_1, \ldots, a_n) \text{ is true for some } x \in A \,\}$$

consisting of all the sets y for which there exists some $x \in A$ such that $\varphi(x, y, A, a_1, \ldots, a_n)$ is true, and nothing else.

Axiom 1.1.9 (Foundation)**.** Let A be a non-empty set. Then there exists some $x \in A$ such that

$$\text{for every } y \in x \text{ we have } y \notin A.$$

Axiom 1.1.10 (Choice)**.** Let A be a set of non-empty sets. Then there exists a choice function $f\colon A \to \bigcup A$ such that

$$\text{for all } x \in A \text{ we have } f(x) \in x.$$

These axioms allow the development of most of "usual" mathematics. From here onwards, we will use tools from basic set theory without worry, including (but not limited to) intersections, functions, bijections, countability, and the real numbers. One piece notation which is important to note is that we will use $\mathbb{N}$ to denote the set of all natural numbers including 0.

1.2 Ordinals, Cardinals, and Cardinality

Perhaps the most important idea in all of set theory is using a bijection to say that two sets have the same "size". Given two sets A and B, we say that A and B "have the same number of elements" if there exists a bijection $f\colon A \to B$. We have the following classical result to establish the existence of bijections.

Theorem 1.2.1 (Cantor-Bernstein Theorem [22, Theorem 18 in Section 4.1], [16, Theorem 3.2])**.** *Let A and B be sets. Suppose there exists an injection $f\colon A \to B$, and suppose there exists another injection $g\colon B \to A$. Then there exists a bijection $h\colon A \to B$.*

We can also "flip" injections and surjections due to the following theorem.

Theorem 1.2.2 ([19, Theorem 2.7.25, Theorem 2.7.27])**.** *Let A and B be sets.*

(1) If A is non-empty and there exists an injection $f\colon A \to B$, then there exists a surjection $g\colon B \to A$.

(2) If there exists a surjection $f\colon A \to B$, then there exists an injection $g\colon B \to A$.

Remark. We need the axiom of choice to establish *(2)*.

Furthermore, the power set axiom allows us to keep obtaining larger and larger sets. This is demonstrated by Cantor's theorem below.

Theorem 1.2.3 (Cantor's Theorem, [16, Theorem 3.1])**.** *Let A be a set. Then there does not exist a bijection $f\colon A \to \mathscr{P}(A)$. More specifically, there does not exist a surjection $f\colon A \to \mathscr{P}(A)$.*

In particular, the set $\mathscr{P}(\mathbb{N})$ is uncountable. It can also be shown that $\mathbb{R}$ can be placed in bijection with $\mathscr{P}(\mathbb{N})$, and so $\mathbb{R}$ is also uncountable.

Bijections, while giving us a useful notion of two sets having the "same size", do not allow us to say what the size actually is. We can only compare it to other sets, rather than being able to *quantify* the size of the sets. To do this, we develop *ordinal* and *cardinal* numbers. Roughly speaking, ordinals are inspired from the numbers "first", "second", "third", etc. denoting positions in some ordering, and cardinals are inspired from the numbers "one", "two", "three", etc. denoting sizes of sets. They allow us to generalise these notions to talk about infinities.

Definition 1.2.4 (Ordinals and Cardinals, [16, Definition 2.10])**.** A set α is an *ordinal* if both of the following hold:

(1) for all $\beta \in \alpha$, we have $\beta \subseteq \alpha$,

(2) for any $A \subseteq \alpha$ with $A \neq \varnothing$, there exists $\beta \in A$ such that

$$\text{for all } \gamma \in A, \text{ either } \beta \in \gamma \text{ or } \beta = \gamma.$$

An ordinal α is said to be a *successor ordinal* if $\alpha = \beta \cup \{\beta\}$ for some ordinal β, and α is said to be a *limit ordinal* if $\alpha \neq \varnothing$ and α is not a successor ordinal.

An ordinal κ is a *cardinal* if for every $\alpha \in \kappa$, there does not exist a bijection $f\colon \alpha \to \kappa$.

Remark. A set satisfying (1) is said to be *transitive*, and a set satisfying (2) is said to be *well-ordered by* $\in$.

The sets $0 \coloneqq \varnothing$, $1 \coloneqq \{0\}$, $2 \coloneqq \{0, 1\}$, $3 \coloneqq \{0, 1, 2\}$, $\ldots$ are all cardinal numbers. Furthermore, $\omega \coloneqq \{0, 1, 2, \ldots\}$ and $\omega_1 \coloneqq \{\alpha : \alpha \text{ is a countable ordinal}\}$ are also cardinals, and ω_1 is uncountable.

For cardinals κ and λ, exactly one of the following holds [16, Lemma 2.11]:

- $\kappa = \lambda$,
- $\kappa \in \lambda$,
- $\lambda \in \kappa$.

We thus write $\kappa < \lambda$ to mean "$\kappa \in \lambda$", and $\kappa \leq \lambda$ to mean "$\kappa < \lambda$ or $\kappa = \lambda$". The following theorem, which is a consequence of the axiom of choice, justifies using cardinals to denote "size".

Theorem 1.2.5 ([16, Theorem 5.1])**.** *For any set X, there exists a unique cardinal κ such that there exists a bijection*

$$f\colon X \to \kappa.$$

Hence for any set X, we write $|X| = \kappa$ where κ is the unique cardinal such that there exists a bijection $f\colon X \to \kappa$. We may now define cardinal arithmetic.

Definition 1.2.6 ([16, Equation 3.3])**.** Let κ and λ be cardinals. Let X and Y be disjoint sets with $|X| = \kappa$ and $|Y| = \lambda$. We define

$$\begin{aligned} \kappa + \lambda &\coloneqq |X \cup Y| \\ \kappa \cdot \lambda &\coloneqq |X \times Y| \\ \kappa^\lambda &\coloneqq \left|X^Y\right| \end{aligned}$$

where $X \times Y$ denotes the Cartesian product of X and Y, and X^Y denotes the set of all functions $f\colon Y \to X$.

We already have notations for finite cardinals, namely the natural numbers 0, 1, 2, ... To talk about infinite cardinals, we appeal to transfinite recursion [16, Theorem 2.14 and Theorem 2.15] to introduce $\aleph$-numbers.

Definition 1.2.7 ($\aleph$ numbers, [16, Section 3])**.** Let α be an ordinal. We define

$$\aleph_\alpha := \begin{cases} \omega & \text{if } \alpha = 0, \\ \min\{\gamma : \gamma \text{ is an ordinal with } \aleph_\beta < \gamma\} & \text{if } \alpha \text{ is a sucessor ordinal with } \alpha = \beta \cup \{\beta\}, \\ \bigcup_{\gamma<\alpha} \aleph_\gamma & \text{if } \alpha > 0 \text{ and } \alpha \text{ is a limit ordinal,} \end{cases}$$

where the minimum is taken with respect to the well-ordering induced by $\in$ on the class of all ordinals, and ω is the first countably infinite ordinal.

Each $\aleph$-number is a cardinal. In particular, we have $\aleph_1 = \omega_1$, and $2^{\aleph_0} = |\mathscr{P}(\mathbb{N})| = |\mathbb{R}|$.

Theorem 1.2.8 ([16, Theorem 3.5])**.** *For all ordinals α, we have $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$.*

As a corollary of Theorem 1.2.8, we have $\aleph_0^{\aleph_0} = 2^{\aleph_0}$. This will be useful later when working with the set $\mathbb{N}^{\mathbb{N}}$ of functions $f : \mathbb{N} \to \mathbb{N}$.

1.3 The Continuum Hypothesis

Cantor showed, using his famous diagonal argument [7], that no bijection can be established between the set $\mathbb{N}$ of natural numbers and the set $\mathbb{R}$ of real numbers. In particular, he showed that $|\mathbb{N}| < |\mathbb{R}|$, that is, no injection from $\mathbb{N}$ to $\mathbb{R}$ can be a surjection. A natural question then arises: is there a set whose cardinality lies strictly between $|\mathbb{N}|$ and $|\mathbb{R}|$? Upon first attempts, it seems difficult to construct a set $X \subseteq \mathbb{R}$ with the property $|\mathbb{N}| < |X| < |\mathbb{R}|$.

- Any open interval in $\mathbb{R}$ can be placed in bijection with $\mathbb{R}$, so we should require X to not have any "region of continuity".
- The set $\mathbb{Q}$ of rational numbers is bijective to $\mathbb{N}$, and the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers is bijective to $\mathbb{R}$.
- The Cantor set, obtained by iteratively removing middle-thirds from the unit interval $[0, 1]$, has the same cardinality as $\mathbb{R}$.
- The Vitali set, which is an example of a non-Lebesgue-measurable set constructed using the axiom of choice, also has the same cardinality as $\mathbb{R}$.

Perhaps such a set X does not exist? Despite not being able to find a proof, Cantor believed that such a set X could not exist [9, pp. 134–137]. Cantor's claim is now known as the *continuum hypothesis* (CH).

Claim (The Continuum Hypothesis (CH), [6, Section 1.1])**.** *There does not exist a set X such that $\aleph_0 < |X| < 2^{\aleph_0}$. In particular, $\aleph_1 = 2^{\aleph_0}$.*

Two significant results in model theory followed, due to Kurt Gödel and Paul Cohen.

Theorem 1.3.1 ([14], [18, Corollary 4.9])**.** *If* ZFC *is consistent, then* ZFC+CH *is also consistent.*

Theorem 1.3.2 ([8], [18, Corollary 5.15])**.** *If* ZFC *is consistent, then* ZFC + ¬CH *is also consistent.*

We will continue the rest of this essay adopting the position of relaxing the continuum hypothesis, but not negating it. That is, we will explore cardinals κ for which we can prove that $\aleph_1 \leq \kappa \leq 2^{\aleph_0}$, but not prove equality. Such cardinals κ are called *cardinal characteristics of the continuum* [2].

Cichoń's diagram [1, Section 2.1], illustrated in Figure 1, consists of ten such cardinals with no further inequalities provable between them in ZFC, where an arrow $\kappa \longrightarrow \lambda$ denotes the inequality $\kappa \leq \lambda$ between cardinals κ and λ. At this point, none of the cardinals are supposed to mean anything to the reader. Figure 1 is simply a map of the journey we are about to embark on in the next twenty or so pages. If the continuum hypothesis is adopted, then the entire diagram collapses. Viewing it only from the lens of ZFC, however, allows the diagram to expand.

$$\begin{tikzcd}
 & 2^{\aleph_0} & \\
 & \mathbf{cof}(\mathcal{L}) \arrow[u] & \\
\mathbf{cof}(\mathcal{M}) \arrow[ur] & & \mathbf{non}(\mathcal{L}) \arrow[ul] \\
 & \mathfrak{d} \arrow[ul] & \\
\mathbf{non}(\mathcal{M}) \arrow[uu] & & \mathbf{cov}(\mathcal{M}) \arrow[uu] \arrow[ul] \\
 & \mathfrak{b} \arrow[ul] \arrow[uu] & \\
\mathbf{cov}(\mathcal{L}) \arrow[uu] & & \mathbf{add}(\mathcal{M}) \arrow[uu] \arrow[ul] \\
 & \mathbf{add}(\mathcal{L}) \arrow[ul] \arrow[ur] & \\
 & \aleph_1 \arrow[u] &
\end{tikzcd}$$

Figure 1: Cichoń's Diagram [2, End of Section 5].

In 2015, Michael Hardy asked the question "How many rearrangements must fail to alter the value of a sum before you conclude that none do?" on MathOverflow [15]. Answers to this question eventually turned into a research paper [3] which introduced a new cardinal characteristic of the continuum, the *rearrangement number* $\mathfrak{rr}$. After exploring the provable inequalities in Cichoń's diagram, we will explore where $\mathfrak{rr}$ lies in relation to Cichoń's diagram.

2 Small Subsets of the Real Line

In view of the independence of the continuum hypothesis from ZFC, we wish to search for cardinals α for which there exist models of ZFC such that $\aleph_0 < \alpha < 2^{\aleph_0}$ in that model. It is, however, fruitless to simply search for $\aleph$-numbers which satisfy this inequality, as it is consistent with ZFC that $2^{\aleph_0}$ equals any of the following cardinals:

$$\aleph_1, \qquad \aleph_2, \qquad \aleph_3, \qquad \aleph_{\omega+1}, \qquad \aleph_{\omega_1},$$

among many other $\aleph$ numbers [10, Theorem 1, Theorem 2] [21, Theorem 1].[2]

Following the naïve first attempts in Section 1.3 to construct a set whose cardinality lies strictly between $\aleph_0$ and $2^{\aleph_0}$, we are motivated to look for "small" subsets of $\mathbb{R}$ and use them to build sets with our desired cardinality property. As a trivial example, consider the cardinal

$$\alpha := \min\{\, |A| : A \subseteq \mathbb{R} \text{ and } A \text{ is uncountable} \,\}.$$

In this case, we start from the notion that countable sets are "small", and we then take α to be the smallest cardinality of any set which is "not small". Certainly, there are models of ZFC where $\aleph_0 < \alpha < 2^{\aleph_0}$. Indeed, by definition, we would have $\alpha = \aleph_1$. However this hardly an interesting example.

2.1 Ideals

We begin by generalising the notion of a "small" set. A "small" set could be a countable set, or a meagre set, or a set with measure zero [17, Section 8.A], among many other possible notions. We call a collection of "small" sets, whatever our definition of "small" is, an *ideal*, and we require several properties which agree with our intuitive ideas of a set being "small".

Definition 2.1.1 (Ideals, [16, Definition 7.1], [6, Section 1.1], [1, Definition 1.3.1])**.** Let X be a set and $\mathcal{I} \subseteq \mathscr{P}(X)$ be a collection of subsets of X. We say $\mathcal{I}$ is an *ideal* of X if all of the following four properties hold:

(1) $\varnothing \in \mathcal{I}$,

(2) if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$,

(3) if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$,

(4) for all $x \in X$, we have $\{x\} \in \mathcal{I}$.[3]

We say $\mathcal{I}$ is a *proper ideal* on X if $\mathcal{I}$ satisfies (1), (2), (3), (4) above, and also satisfies

(5) $X \notin \mathcal{I}$.[4]

We also say $\mathcal{I}$ is a *σ-ideal* on X if $\mathcal{I}$ satisfies (1), (2), (3), (4) above, and also satisfies

(6) if $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{I}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{I}$.

Finally, we say $\mathcal{I}$ is a *proper σ-ideal* on X if $\mathcal{I}$ satisfies (1), (2), (3), (4), (5), and (6) above.

[2] The equality $2^{\aleph_0} = \aleph_\omega$, however, is inconsistent with ZFC because $\aleph_\omega$ has countable cofinality [10].

[3] The requirement of ideals containing all singleton subsets may sometimes be omitted in some texts, as in *Set Theory* by Thomas Jech [16, Definition 7.1] and *Classical Descriptive Set Theory* by Alexander Kechris [17, Section 8.A]. However, the ideals we are concerned with will always have this property. For example, in Definition 2.1.3, the cardinal $\mathbf{cov}(\mathcal{I})$ would not be well-defined without this requirement, and the cardinal $\mathbf{non}(\mathcal{I})$ would trivially equal 1 if this requirement is not met.

[4] Some texts, for example *Set Theory* by Thomas Jech [16, Definition 7.1] and *The Structure of the Real Line* by Lev Bukovský [6, Section 1.1], bake this requirement $X \notin \mathcal{I}$ into the definition of an ideal, omitting the need for the term "proper ideal".

If one were to describe an ideal $\mathcal{I}$, there is a certain redundancy in simply listing out all the elements of $\mathcal{I}$. It is "redundant" in the sense that, if $A \in \mathcal{I}$ and $B \subseteq A$, then Definition 2.1.1 already requires that $B \in \mathcal{I}$, and so it would be redundant to further specify that $B \in \mathcal{I}$ after one has already said that $A \in \mathcal{I}$. A *basis* $\mathfrak{B}$ for an ideal allows one to generate the entire ideal by taking all the subsets of the elements of $\mathfrak{B}$.

Definition 2.1.2 (Basis for an Ideal, [2, Definition 2.7])**.** Let $\mathcal{I}$ be an ideal on a set X. A collection $\mathfrak{B} \subseteq \mathcal{I}$ is a *basis* for $\mathcal{I}$ if

$$\text{for all } A \in \mathcal{I} \text{ there exists } B \in \mathfrak{B} \text{ such that } A \subseteq B.$$

As an example, if X is an infinite set and $\mathcal{I}$ is the ideal of all the countable subsets of X, then the set $\mathfrak{B}$ of all countably *infinite* subsets of X will form a basis of $\mathcal{I}$. For a less contrived example, after introducing the Lebesgue measure in Section 2.2, we observe that every Lebesgue null set is contained in some Lebesgue null set which also happens to be Borel. Consequently, the set of all Borel null sets (sets which are both Borel and Lebesgue null) will form a basis for the ideal of all Lebesgue null subsets of $\mathbb{R}$. This observation leads to Theorem 2.3.1, but let us build up to that theorem slowly, as several definitions need to be introduced before Theorem 2.3.1 can be understood.

Given a proper σ-ideal $\mathcal{I}$ on a set X, we can now define several cardinals by asking very natural questions about "small" sets. First, note that we must require X to be uncountable in order for $\mathcal{I}$ to even be a proper σ-ideal. The cardinal $\mathbf{non}(\mathcal{I})$ is then defined to be the smallest cardinality of any subset of X which is not "small". The cardinal $\mathbf{cov}(\mathcal{I})$ is defined to be the smallest number of "small" sets needed to cover all of X. The cardinal $\mathbf{add}(\mathcal{I})$ is defined to be the smallest number of "small" sets needed to build a set which is not "small". Finally, the cardinal $\mathbf{cof}(\mathcal{I})$ is defined to be the smallest cardinality of any basis for $\mathcal{I}$. We write these definitions out formally in Definition 2.1.3.

Definition 2.1.3 (Uniformity, Covering Number, Additivity, and Cofinality of an Ideal, [2, Definition 2.7])**.** Let $\mathcal{I} \subseteq \mathscr{P}(X)$ be a proper σ-ideal on an uncountable set X.

(1) The *uniformity* of $\mathcal{I}$, denoted $\mathbf{non}(\mathcal{I})$, is defined by

$$\mathbf{non}(\mathcal{I}) \coloneqq \min\{\, |A| : A \subseteq X \text{ and } A \notin \mathcal{I}\,\}.$$

(2) The *covering number* of $\mathcal{I}$, denoted $\mathbf{cov}(\mathcal{I})$, is defined by

$$\mathbf{cov}(\mathcal{I}) \coloneqq \min\left\{\, |\mathcal{C}| : \mathcal{C} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{C} = X \,\right\}.$$

(3) The *additivity* of $\mathcal{I}$, denoted $\mathbf{add}(\mathcal{I})$, is defined by

$$\mathbf{add}(\mathcal{I}) \coloneqq \min\left\{\, |\mathcal{C}| : \mathcal{C} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{C} \notin \mathcal{I} \,\right\}.$$

(4) The *cofinality* of $\mathcal{I}$, denoted $\mathbf{cof}(\mathcal{I})$, is defined by

$$\mathbf{cof}(\mathcal{I}) \coloneqq \min\{\, |\mathfrak{B}| : \mathfrak{B} \subseteq \mathcal{I} \text{ and } \mathfrak{B} \text{ is a basis for } \mathcal{I}\,\}.$$

It is easy to check that these four cardinals $\mathbf{non}(\mathcal{I})$, $\mathbf{cov}(\mathcal{I})$, $\mathbf{add}(\mathcal{I})$, and $\mathbf{cof}(\mathcal{I})$ are uncountable. In fact, even without knowing the ideal $\mathcal{I}$ or the set X, we can already establish several inequalities between these cardinals.

Lemma 2.1.4 ([1, Lemma 1.3.2], [2, Section 2])**.** *Let $\mathcal{I}$ be a proper σ-ideal on an uncountable set X. Then all of the following hold:*

- $\mathbf{add}(\mathcal{I}) \geq \aleph_1$,
- $\mathbf{add}(\mathcal{I}) \leq \mathbf{cov}(\mathcal{I})$ *and* $\mathbf{add}(\mathcal{I}) \leq \mathbf{non}(\mathcal{I})$,
- $\mathbf{cov}(\mathcal{I}) \leq \mathbf{cof}(\mathcal{I})$ *and* $\mathbf{non}(\mathcal{I}) \leq \mathbf{cof}(\mathcal{I})$.

Proof. The inequality $\mathbf{add}(\mathcal{I}) \geq \aleph_1$ follows from $\mathcal{I}$ being a σ-ideal on X, as $\mathcal{I}$ must be closed under countable unions.

We clearly have $\mathbf{add}(\mathcal{I}) \leq \mathbf{cov}(\mathcal{I})$, since $X \notin \mathcal{I}$ as $\mathcal{I}$ is a proper ideal on X.

Next, let $A \in \mathscr{P}(X) \setminus \mathcal{I}$ be such that $|A| = \mathbf{non}(\mathcal{I})$. Then since $A = \bigcup\{\,\{x\} : x \in A\,\}$, we have

$$\mathbf{add}(\mathcal{I}) \leq |\{\,\{x\} : x \in A\,\}| = |A| = \mathbf{non}(\mathcal{I}),$$

since $\mathcal{I}$ contains all the singleton subsets of X.

Now let $\mathfrak{B} \subseteq \mathcal{I}$ be a basis for $\mathcal{I}$ with $|\mathfrak{B}| = \mathbf{cof}(\mathcal{I})$. Then for any $x \in X$ there exists $B \in \mathfrak{B}$ such that $\{x\} \subseteq B$. Hence $\bigcup \mathfrak{B} = X$, and so we obtain $\mathbf{cov}(\mathcal{I}) \leq |\mathfrak{B}| = \mathbf{cof}(\mathcal{I})$.

Finally suppose, for a contradiction, that $|\mathfrak{B}| < \mathbf{non}(\mathcal{I})$. For each $B \in \mathfrak{B}$, the set $X \setminus B$ is non-empty because $\mathcal{I}$ is a proper ideal on X. Now we form a set $C \subseteq X$ as follows: for each $B \in \mathfrak{B}$ choose exactly one $x_B \in X \setminus B$, and then let $C := \{\, x_B : B \in \mathfrak{B}\,\}$. Then we have $|C| \leq |\mathfrak{B}| < \mathbf{non}(\mathcal{I})$, and hence $C \in \mathcal{I}$. However, by construction, there does not exist any $B \in \mathfrak{B}$ for which $C \subseteq B$. This contradicts $\mathfrak{B}$ being a basis for $\mathcal{I}$. Therefore we must have $\mathbf{non}(\mathcal{I}) \leq |B| = \mathbf{cof}(\mathcal{I})$. □

$$\begin{tikzcd}
 & \mathbf{cof}(\mathcal{I}) & \\
\mathbf{cov}(\mathcal{I}) \arrow[ur] & & \mathbf{non}(\mathcal{I}) \arrow[ul] \\
 & \mathbf{add}(\mathcal{I}) \arrow[ul] \arrow[ur] & \\
 & \aleph_1 \arrow[u] &
\end{tikzcd}$$

Figure 2: A Hasse diagram representing the inequalities in Lemma 2.1.4.

The inequalities in Lemma 2.1.4 are represented in Figure 2. These are important inequalities, as they always hold regardless of the proper σ-ideal we are working with. Thus, even without knowing the definitions of $\mathcal{M}$ and $\mathcal{L}$, if we take on faith that $\mathcal{M}$ and $\mathcal{L}$ are proper σ-ideals on $\mathbb{R}$, then we can already see that four arrows in Cichoń's diagram (drawn in Figure 1) have already been established. In particular, we have the following inequalities:

- $\aleph_1 \leq \mathbf{add}(\mathcal{L}) \leq \mathbf{cov}(\mathcal{L})$,
- $\mathbf{add}(\mathcal{M}) \leq \mathbf{cov}(\mathcal{M})$, and
- $\mathbf{non}(\mathcal{L}) \leq \mathbf{cof}(\mathcal{L})$.

Without knowing more about the ideal $\mathcal{I}$ or the ambient space X, we cannot prove any other inequality between the five cardinals present in Figure 2. This is demonstrated by Example 2.1.5 and Example 2.1.6.

Example 2.1.5. Let X be an uncountable set, and let $\mathcal{I}$ be the collection of all subsets $A \subseteq X$ with $|A| \leq \kappa$.

- If $|X| = \aleph_1$ and $\kappa = \aleph_0$, then

$$\aleph_1 = \mathbf{add}(\mathcal{I}) = \mathbf{non}(\mathcal{I}) = \mathbf{cov}(\mathcal{I}) = \mathbf{cof}(\mathcal{I}).$$

  To find a basis $\mathfrak{B}$ for $\mathcal{I}$ with $|\mathfrak{B}| = \aleph_1$, choose a well-ordering of X of order-type ω_1 and take $\mathfrak{B}$ to be the collection of all the proper initial segments of that well-ordering. This is a basis because for any countable collection $\mathcal{C}$ of countable ordinals, we have that $\bigcup \mathcal{C}$ is also a countable ordinal.

- If $|X| = \aleph_3$ and $\kappa = \aleph_1$, then

$$\aleph_1 < \aleph_2 = \mathbf{add}(\mathcal{I}) = \mathbf{non}(\mathcal{I}) < \aleph_3 = \mathbf{cov}(\mathcal{I}) \leq \mathbf{cof}(\mathcal{I}).$$

These examples show that the arrow $\aleph_1 \longrightarrow \mathbf{add}(\mathcal{I})$, the arrow $\mathbf{add}(\mathcal{I}) \longrightarrow \mathbf{cov}(\mathcal{I})$, and the arrow $\mathbf{non}(\mathcal{I}) \longrightarrow \mathbf{cof}(\mathcal{I})$ in Figure 2 cannot be reversed, and that we cannot draw the arrow $\mathbf{cov}(\mathcal{I}) \longrightarrow \mathbf{non}(\mathcal{I})$.

Example 2.1.6. Let $\mathcal{C} := \{C_\alpha\}_{\alpha \in \omega_1}$ be a collection of $\aleph_1$ many pairwise disjoint sets such that $|C_\alpha| = \aleph_2$ for all $\alpha \in \omega_1$, and let $X = \bigcup \mathcal{C}$. Note that $|X| = \aleph_2$. Now, letting

$$\mathcal{I}_1 := \left\{ A \in \mathscr{P}(X) \;:\; A \subseteq \bigcup_{j \in \mathbb{N}} S_j \text{ for some } S_0, S_1, S_2, \dots \in \mathcal{C} \right\}$$

be the smallest σ-ideal containing each C_α, and letting

$$\mathcal{I}_2 := \{ A \in \mathscr{P}(X) \;:\; |A| \leq \aleph_1 \},$$

be the σ-ideal of all the subsets of X of cardinality at most $\aleph_1$, we define

$$\mathcal{I} := \{ A_1 \cup A_2 \;:\; A_1 \in \mathcal{I}_1 \text{ and } A_2 \in \mathcal{I}_2 \}.$$

Then we have

$$\aleph_1 = \mathbf{add}(\mathcal{I}) = \mathbf{cov}(\mathcal{I}) < \aleph_2 = \mathbf{non}(\mathcal{I}) \leq \mathbf{cof}(\mathcal{I}).$$

In particular, the arrows $\mathbf{add}(\mathcal{I}) \longrightarrow \mathbf{non}(\mathcal{I})$ and $\mathbf{cov}(\mathcal{I}) \longrightarrow \mathbf{cof}(\mathcal{I})$ in Figure 2 cannot be reversed, and the arrow $\mathbf{non}(\mathcal{I}) \longrightarrow \mathbf{cov}(\mathcal{I})$ cannot be drawn.

We can also generalise the definition of **add** and **cof** to two ideals $\mathcal{I}$ and $\mathcal{J}$ on the same underlying set X with $\mathcal{I} \subseteq \mathcal{J}$. The cardinal $\mathbf{add}(\mathcal{I}, \mathcal{J})$ is defined to be the smallest number of "$\mathcal{I}$-small" sets needed to build a set which is not "$\mathcal{J}$-small". The cardinal $\mathbf{cof}(\mathcal{I}, \mathcal{J})$ is defined to be the smallest cardinality of any subset of $\mathcal{J}$ which acts as a basis for $\mathcal{I}$. This is all written out formally in Definition 2.1.7.

Definition 2.1.7 (Additivity and Cofinality of Two Ideals, [1, Definition 2.1.3])**.** Let $\mathcal{I}$ and $\mathcal{J}$ be two proper σ-ideals on an uncountable set X, with $\mathcal{I} \subseteq \mathcal{J}$. We define

$$\mathbf{add}(\mathcal{I}, \mathcal{J}) := \min\left\{ |\mathcal{C}| : \mathcal{C} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{C} \notin \mathcal{J} \right\},$$

$$\mathbf{cof}(\mathcal{I}, \mathcal{J}) := \min\{ |\mathfrak{B}| : \mathfrak{B} \subseteq \mathcal{J}, \text{ and for all } A \in \mathcal{I} \text{ there exists } B \in \mathfrak{B} \text{ with } A \subseteq B \}.$$

The cardinals $\mathbf{add}(\mathcal{I}, \mathcal{J})$ and $\mathbf{cof}(\mathcal{I}, \mathcal{J})$ roughly quantify how much "smaller" the sets in $\mathcal{I}$ are compared to those in $\mathcal{J}$. Note that $\mathbf{add}(\mathcal{I}) = \mathbf{add}(\mathcal{I}, \mathcal{I})$ and $\mathbf{cof}(\mathcal{I}) = \mathbf{cof}(\mathcal{I}, \mathcal{I})$. We also have the following inequalities, some of which will be useful later in Section 3.3. These can be proven in a similar fashion to Lemma 2.1.4.

Lemma 2.1.8 ([1, Lemma 2.1.4])**.** *Let $\mathcal{I}$ and $\mathcal{J}$ be proper σ-ideals on an uncountable set X, and suppose that $\mathcal{I} \subseteq \mathcal{J}$. Then all of the following hold:*

- $\mathbf{add}(\mathcal{I}) \leq \mathbf{add}(\mathcal{I}, \mathcal{J})$ *and* $\mathbf{add}(\mathcal{J}) \leq \mathbf{add}(\mathcal{I}, \mathcal{J})$,
- $\mathbf{cof}(\mathcal{I}, \mathcal{J}) \leq \mathbf{cof}(\mathcal{J})$,
- $\mathbf{add}(\mathcal{I}, \mathcal{J}) \leq \mathbf{cov}(\mathcal{I})$ *and* $\mathbf{add}(\mathcal{I}, \mathcal{J}) \leq \mathbf{non}(\mathcal{J})$,
- $\mathbf{cov}(\mathcal{J}) \leq \mathbf{cof}(\mathcal{I}, \mathcal{J})$ *and* $\mathbf{non}(\mathcal{I}) \leq \mathbf{cof}(\mathcal{I}, \mathcal{J})$.

2.2 Alternative Representations of the Real Line

While we are mostly concerned with the small subsets of $\mathbb{R}$, we will often find ourselves needing to use alternative representations of $\mathbb{R}$ under which the cardinalities of the sets we are considering do not change. This will allow us to use whichever space is most convenient in our proofs. The alternatives to $\mathbb{R}$ which we will consider are the interval $[0, 1]$, the Cantor space $2^{\mathbb{N}}$, and the Baire space $\mathbb{N}^{\mathbb{N}}$. We present the definition of the Cantor space and the Baire space in Definition 2.2.1.

First, let us recall the definition of the *product topology*, which we will use regularly use. For a J-indexed collection of topological spaces $\{X_j\}_{j \in J}$, the basis of the *product topology* on the space $\prod_{j \in J} X_j$ consists of the sets of the form $\prod_{j \in J} U_j$ where U_j is open in X_j for all $j \in J$, and $U_j = X_j$ for all but finitely many $j \in J$ [17, Section 1.A].

Definition 2.2.1 (The Cantor Space and the Baire Space, [17, Section 3.A], [1, Section 1.1.B])**.** Let A be a non-empty set equipped with the discrete topology. The set $A^{\mathbb{N}}$ of functions $\mathbb{N} \to A$ is identified with the set $\prod_{n \in \mathbb{N}} A$ of infinite sequences of elements of A, equipped with the product topology. More explicitly, $f \in A^{\mathbb{N}}$ if and only if $(f(0), f(1), f(2), \dots) \in \prod_{n \in \mathbb{N}} A$.

The space $2^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}}$ is called the *Cantor space*, and the space $\mathbb{N}^{\mathbb{N}}$ is called the *Baire space*.

With the usual Euclidean topology on $[0, 1]$, the Cantor space $2^{\mathbb{N}}$ is homeomorphic to the Cantor (middle-third) set [6, Equation 2.32 in Section 2.4] equipped with the subspace topology [6, Theorem 3.1]. The homeomorphism is obtained by writing each number in the Cantor set as its 3-adic expansion [6, Equation 2.24 and Equation 2.25 in Section 2.4] using only the digits 0 and 2. One can also think of $2^{\mathbb{N}}$ as the set of all numbers in the interval $[0, 1]$ written in their binary expansion [3, Definition 10].[5]

With the usual Euclidean topology on $\mathbb{R}$, the Baire space $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers equipped with the subspace topology [6, Theorem 3.25]. This homeomorphism is obtained by representing each irrational number as a continued fraction of natural numbers [6, Equation 2.33 in Section 2.4].[6]

We now start with the first collection of "small" subsets of $\mathbb{R}$ which we will be focusing on — sets with Lebesgue measure zero. The definition of the Lebesgue measure is presented below.

Definition 2.2.2 (Lebesgue Measure and Lebesgue Null Sets, [6, Section 4.2], [3, Definition 10], [5, Definition 9], [2, Section 1])**.** The *Lebesgue measure* is defined on the space of real numbers $\mathbb{R}$, the real interval $[0, 1]$, the Cantor space $2^{\mathbb{N}}$ as follows:

- For each $X \in \{\mathbb{R}, [0, 1]\}$, the *Lebesgue outer measure on* X is $\lambda^*_X \colon \mathscr{P}(X) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ defined by

$$\lambda^*_X(A) := \inf \left\{ \sum_{j \in \mathbb{N}} (b_j - a_j) \ : \ A \subseteq \bigcup_{j \in \mathbb{N}} ((a_j, b_j) \cap X) \right\},$$

[5] Although this is not a bijection between $2^{\mathbb{N}}$ and $[0, 1]$, it is "close enough" to a bijection. For more details, see the proof of Theorem 2.2.5.

[6] A homeomorphism between $\mathbb{N}^{\mathbb{N}}$ and $[0, 1] \setminus \mathbb{Q}$ is presented in the proof of Theorem 2.2.5.

where (a_j, b_j) denotes an open interval in $\mathbb{R}$. Let

$$\mathfrak{M}_X := \{\, A \in \mathscr{P}(X) \;:\; \lambda_X^*(B) = \lambda_X^*(A \cap B) + \lambda_X^*(A \cap (X \setminus B)) \text{ for all } B \subseteq X \,\}$$

be the set of *Lebesgue measurable subsets* of X. The *Lebesgue measure* λ_X on X is the restriction of λ_X^* to $\mathfrak{M}_X$, that is, $\lambda_X := \lambda_X^*|_{\mathfrak{M}_X}$.

- The *Lebesgue measure on* $2^{\mathbb{N}}$ is the unique product measure $\lambda_{2^{\mathbb{N}}}$ defined on the σ-algebra generated by the collection of basic open sets

$$\mathfrak{B} := \left\{ \prod_{n\in\mathbb{N}} A_n \;:\; A_n \subseteq \{0,1\} \text{ and } A_n = \{0,1\} \text{ for all but finitely many } n \in \mathbb{N} \right\}$$

satisfying

$$\lambda_{2^{\mathbb{N}}}\left(\prod_{n\in\mathbb{N}} A_n\right) = \prod_{n\in\mathbb{N}} \mu(A_n) \qquad \text{for all } \prod_{n\in\mathbb{N}} A_n \in \mathfrak{B},$$

where $\mu\colon \mathscr{P}(\{0,1\}) \to \mathbb{R}$ is defined by

$$\mu(A) := \frac{|A|}{2}.$$

For $X \in \{\mathbb{R}, [0,1], 2^{\mathbb{N}}\}$, we say a set $N \subseteq X$ is a *Lebesgue null set* if $\lambda_X(N) = 0$. We write $\mathcal{L}(X)$ for the collection of all the Lebesgue null sets of X.

For $X \in \{\mathbb{R}, [0,1], 2^{\mathbb{N}}\}$, the set $\mathcal{L}(X)$ of Lebesgue null sets is a proper σ-ideal on X.

In topological spaces, nowhere dense sets and meagre sets also turn out to have properties which a "small" set should have. For a topological space X and a subset $A \subseteq X$, we write A° for the interior of A in X, and we write $\overline{A}$ for the closure of A in X.

Definition 2.2.3. Let X be a topological space. We say $A \subseteq X$ is *nowhere dense* in X if $(\overline{A})^\circ = \varnothing$. We say $B \subseteq X$ is *meagre* in X if $B = \bigcup_{j\in\mathbb{N}} A_j$ where $\{A_j\}_{j\in\mathbb{N}}$ is a countable[7] collection of nowhere dense sets of X. We write $\mathcal{M}(X)$ for the set of all meagre subsets of X.

For any non-meagre topological space X, the set $\mathcal{M}(X)$ of meagre subsets of X is a proper σ-ideal on X. Part of this is due to the Baire category theorem, from which we can show that the set $\mathbb{R}$ of real numbers endowed with its usual topology is not meagre.

With these two different notions of "small" subsets of $\mathbb{R}$, having Lebesgue measure zero and being meagre, we are now interested in the cardinals

$$\begin{matrix} \mathbf{add}(\mathcal{L}(\mathbb{R})), & \mathbf{cov}(\mathcal{L}(\mathbb{R})), & \mathbf{non}(\mathcal{L}(\mathbb{R})), & \mathbf{cof}(\mathcal{L}(\mathbb{R})), \\ \mathbf{add}(\mathcal{M}(\mathbb{R})), & \mathbf{cov}(\mathcal{M}(\mathbb{R})), & \mathbf{non}(\mathcal{M}(\mathbb{R})), & \mathbf{cof}(\mathcal{M}(\mathbb{R})). \end{matrix}$$

It will turn out that these cardinals have alternative representations using $[0,1]$, $2^{\mathbb{N}}$, or $\mathbb{N}^{\mathbb{N}}$. To establish this, we need the following lemma.

Lemma 2.2.4 ([6, Theorem 7.2])**.** *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be proper σ-ideals on uncountable sets X_1 and X_2 respectively. Fix $C_1 \in \mathcal{I}_1$ and $C_2 \in \mathcal{I}_2$. Suppose there exists a function $f\colon X_1 \to X_2$ such that $f|_{X_1\setminus C_1}\colon X_1 \setminus C_1 \to X_2 \setminus C_2$ is bijective and*

$$\text{for all } B \subseteq X_1 \setminus C_1, \text{ we have } B \in \mathcal{I}_1 \text{ if and only if } f(B) \in \mathcal{I}_2.$$

Then for each $\mathbf{ch} \in \{\mathbf{add}, \mathbf{cov}, \mathbf{non}, \mathbf{cof}\}$*, we have* $\mathbf{ch}(\mathcal{I}_1) = \mathbf{ch}(\mathcal{I}_2)$*.*

[7] Finite or countably infinite.

Proof. First, observe that

$$\text{for all } B \subseteq X_2 \setminus C_2, \text{ we have } B \in \mathcal{I}_2 \text{ if and only if } f^{-1}(B) \in \mathcal{I}_1. \tag{1}$$

Let $\mathcal{S}_1 \subseteq \mathcal{I}_1$ be such that $|\mathcal{S}_1| < \mathbf{add}(\mathcal{I}_2)$. We will show that $\bigcup \mathcal{S}_1 \in \mathcal{I}_1$, which shows that we must necessarily have $\mathbf{add}(\mathcal{I}_2) \leq \mathbf{add}(\mathcal{I}_1)$. Define $\mathcal{S}_2 \coloneqq \{ f(A) : A \in \mathcal{S}_1 \} \subseteq \mathcal{I}_2$ and observe that $\bigcup \mathcal{S}_2 \in \mathcal{I}_2$ since $|\mathcal{S}_2| \leq |\mathcal{S}_1| < \mathbf{add}(\mathcal{I}_2)$. Recalling that $f|_{X_1 \setminus C_1} \colon X_1 \setminus C_1 \to X_2 \setminus C_2$ is a bijection, we obtain

$$\bigcup \mathcal{S}_1 \subseteq f^{-1}\left(\bigcup \mathcal{S}_2\right) \cup C_1 \in \mathcal{I}_1 \qquad \text{due to (1).}$$

Now let $\mathcal{S}_1 \subseteq \mathcal{I}_1$ be such that $\bigcup \mathcal{S}_1 = X_1$ and $|\mathcal{S}_1| = \mathbf{cov}(\mathcal{I}_1)$. Define

$$\mathcal{S}_2 \coloneqq \{ f(A) : A \in \mathcal{S}_1 \} \cup \{C_2\} \subseteq \mathcal{I}_2$$

and observe that $\bigcup \mathcal{S}_2 = X_2$. Therefore $\mathbf{cov}(\mathcal{I}_2) \leq |\mathcal{S}_2| \leq |\mathcal{S}_1| = \mathbf{cov}(\mathcal{I}_1)$.

Next, let $A_1 \in \mathscr{P}(X_1) \setminus \mathcal{I}_1$ be such that $|A_1| = \mathbf{non}(\mathcal{I}_1)$. Then $A_1 \setminus C_1 \notin \mathcal{I}_1$, so we define $A_2 \coloneqq f(A_1 \setminus C_1) \notin \mathcal{I}_2$. Hence $\mathbf{non}(\mathcal{I}_2) \leq |A_2| \leq |A_1| = \mathbf{non}(\mathcal{I}_1)$.

Finally, let $\mathfrak{B}_1 \subseteq \mathcal{I}_1$ be a basis for $\mathcal{I}_1$ with $|\mathfrak{B}_1| = \mathbf{cof}(\mathcal{I}_1)$. Define

$$\mathfrak{B}_2 \coloneqq \{ f(A) \cup C_2 : A \in \mathfrak{B}_1 \},$$

and notice that $\mathfrak{B}_2$ is a basis for $\mathcal{I}_2$. Therefore, $\mathbf{cof}(\mathcal{I}_2) \leq |\mathfrak{B}_2| \leq |\mathfrak{B}_1| = \mathbf{cof}(\mathcal{I}_1)$.

The proofs of the reverse inequalities $\mathbf{ch}(\mathcal{I}_1) \leq \mathbf{ch}(\mathcal{I}_2)$ for each $\mathbf{ch} \in \{\mathbf{add}, \mathbf{cov}, \mathbf{non}, \mathbf{cof}\}$ are similar to the proofs above. ☐

Armed with Lemma 2.2.4, we can now show how the spaces $[0,1]$, $2^{\mathbb{N}}$, and $\mathbb{N}^{\mathbb{N}}$ really do act as alternative representations of $\mathbb{R}$ when talking about the cardinals we are interested in.

Theorem 2.2.5 ([6, Theorem 7.3])**.** *For each* $\mathbf{ch} \in \{\mathbf{add}, \mathbf{cov}, \mathbf{non}, \mathbf{cof}\}$*, we have*

$$\begin{aligned} &\mathbf{ch}(\mathcal{M}(\mathbb{R})) = \mathbf{ch}(\mathcal{M}([0,1])) = \mathbf{ch}\left(\mathcal{M}\left(2^{\mathbb{N}}\right)\right) = \mathbf{ch}\left(\mathcal{M}\left(\mathbb{N}^{\mathbb{N}}\right)\right), \quad \textit{and} \\ &\mathbf{ch}(\mathcal{L}(\mathbb{R})) \ \ = \mathbf{ch}(\mathcal{L}([0,1])) \ \ = \mathbf{ch}\left(\mathcal{L}\left(2^{\mathbb{N}}\right)\right). \end{aligned}$$

Proof (Sketch). Fix any $\mathbf{ch} \in \{\mathbf{add}, \mathbf{cov}, \mathbf{non}, \mathbf{cof}\}$. We will define functions that can be used with Lemma 2.2.4 to give the result.

To show that $\mathbf{ch}(\mathcal{M}([0,1])) = \mathbf{ch}(\mathcal{M}(\mathbb{R}))$ and $\mathbf{ch}(\mathcal{L}([0,1])) = \mathbf{ch}(\mathcal{L}(\mathbb{R}))$, define the function $f \colon [0,1] \to \mathbb{R}$ by

$$f(x) \coloneqq \begin{cases} 0, & \text{if } x = 0 \text{ or } x = 1, \\ -\frac{1}{x} + 2, & \text{if } 0 < x \leq \frac{1}{2}, \\ -\frac{1}{x-1} - 2, & \text{if } \frac{1}{2} < x < 1. \end{cases}$$

Define $C_1 \coloneqq \{0,1\}$, noting that C_1 is meagre and Lebesgue null in $[0,1]$. Then $f|_{[0,1]\setminus C_1} \colon (0,1) \to \mathbb{R}$ is a homeomorphism, and the images of Lebesgue null sets remain Lebesgue null under $f|_{(0,1)}$ and $f|_{(0,1)}^{-1}$.

Next, to show that $\mathbf{ch}\left(\mathcal{M}\left(2^{\mathbb{N}}\right)\right) = \mathbf{ch}\left(\mathcal{M}\left([0,1]\right)\right)$ and $\mathbf{ch}\left(\mathcal{L}\left(2^{\mathbb{N}}\right)\right) = \mathbf{ch}\left(\mathcal{L}\left([0,1]\right)\right)$, define the function $g \colon 2^{\mathbb{N}} \to [0,1]$ by

$$g(\alpha) \coloneqq \sum_{n \in \mathbb{N}} \frac{\alpha(n)}{2^{n+1}}.$$

Define $C_1 \coloneqq \left\{ \alpha \in 2^{\mathbb{N}} : \text{there exists } N \in \mathbb{N} \text{ such that for all } n > N \text{ we have } \alpha(n) = 1 \right\}$, noting that C_1 is meagre and Lebesgue null in $2^{\mathbb{N}}$. Then $g|_{2^{\mathbb{N}}\setminus C_1} \colon 2^{\mathbb{N}} \setminus C_1 \to [0,1]$ is a homeomorphism, and the images under $g|_{2^{\mathbb{N}}\setminus C_1}$ and $g|_{2^{\mathbb{N}}\setminus C_1}^{-1}$ of Lebesgue null sets remain Lebesgue null.

Finally, to show that $\mathbf{ch}\left(\mathcal{M}\left(\mathbb{N}^{\mathbb{N}}\right)\right) = \mathbf{ch}\left(\mathcal{M}\left([0,1]\right)\right)$, we first observe that $\mathbb{N}$ is homeomorphic to the discrete space $\mathbb{Z}_+ := \mathbb{N} \setminus \{0\}$ of strictly positive integers, so we may identify the Baire space $\mathbb{N}^{\mathbb{N}}$ with $(\mathbb{Z}_+)^{\mathbb{N}}$.[8] Define the function $h\colon (\mathbb{Z}_+)^{\mathbb{N}} \to [0,1]$ by

$$h(\alpha) := \cfrac{1}{\alpha(0) + \cfrac{1}{\alpha(1) + \cfrac{1}{\alpha(2) + \cfrac{1}{\alpha(3) + \cfrac{1}{\alpha(4) + \cdots}}}}}$$

where $h(\alpha)$ is computed as the limit of the continued fractions. Define $C_2 := [0,1] \cap \mathbb{Q}$, the set of all rational numbers between 0 and 1. Then C_2 is meagre in $[0,1]$, and $h\colon (\mathbb{Z}_+)^{\mathbb{N}} \to [0,1] \setminus C_2$ is a homeomorphism [6, Theorem 3.25]. □

Theorem 2.2.5 allows us to simply write the cardinals as $\mathbf{add}(\mathcal{L})$, $\mathbf{non}(\mathcal{M})$, etc. without ambiguity. It also allows us much more flexibility in proofs, as we may choose to work in $\mathbb{R}$, $[0,1]$, $2^{\mathbb{N}}$, or $\mathbb{N}^{\mathbb{N}}$ when proving results about the cardinals $\mathbf{add}(\mathcal{L})$, $\mathbf{non}(\mathcal{M})$, etc.

2.3 Lebesgue Null Sets and Meagre Subsets of the Real Line

Let us recall the fact that there are $2^{\aleph_0}$ many subsets of $\mathbb{R}$ [17, Section 11.B] which are Borel. Using this, we may establish an upper bound and lower bound on the cardinals $\mathbf{add}(\mathcal{L})$, $\mathbf{cov}(\mathcal{L})$, $\mathbf{non}(\mathcal{L})$, and $\mathbf{cof}(\mathcal{L})$. These bounds will show that these cardinals qualify for discussion as cardinal characteristics of the continuum.

Theorem 2.3.1 ([1, Section 2.1])**.** $\mathbf{add}(\mathcal{L}) \geq \aleph_1$ *and* $\mathbf{cof}(\mathcal{L}) \leq 2^{\aleph_0}$.

Proof. We work in $\mathbb{R}$.

Since $\mathcal{L}$ is a proper σ-ideal, the inequality $\mathbf{add}(\mathcal{L}) \geq \aleph_1$ follows from Lemma 2.1.4. Now recall that given any Lebesgue measurable set $M \subseteq \mathbb{R}$, we can write $M = B \setminus N$ for some Borel set $B \subseteq \mathbb{R}$ and some Lebesgue null set $N \subseteq \mathbb{R}$ [13, Proposition 3.6]. Since every Lebesgue null set is Lebesgue measurable [13, Lemma 3.2], every null set must be contained in some set which is both Borel and Lebesgue null. Therefore the collection of all subsets of $\mathbb{R}$ which are both Borel and Lebesgue null forms a basis for the ideal $\mathcal{L}$. Since there are only $2^{\aleph_0}$ many Borel subsets of $\mathbb{R}$, we conclude that $\mathbf{cof}(\mathcal{L}) \leq 2^{\aleph_0}$. □

In a similar spirit to Theorem 2.3.1, we can establish bounds for $\mathbf{add}(\mathcal{M})$, $\mathbf{cov}(\mathcal{M})$, $\mathbf{non}(\mathcal{M})$, and $\mathbf{cof}(\mathcal{M})$.

Theorem 2.3.2 ([1, Section 2.1])**.** $\mathbf{add}(\mathcal{M}) \geq \aleph_1$ *and* $\mathbf{cof}(\mathcal{M}) \leq 2^{\aleph_0}$.

Proof. We work in $\mathbb{R}$.

Since $\mathcal{M}$ is a proper σ-ideal on $\mathbb{R}$, the inequality $\mathbf{add}(\mathcal{M}) \geq \aleph_1$ follows from Lemma 2.1.4. Now given any meagre set $M \subseteq \mathbb{R}$, write $M = \bigcup_{n\in\mathbb{N}} A_n$, where $A_0, A_1, \cdots \subseteq \mathbb{R}$ are nowhere dense in $\mathbb{R}$. Then $\overline{A_n}$ is also nowhere dense for each $n \in \mathbb{N}$, and so $M \subseteq \bigcup_{n\in\mathbb{N}} \overline{A_n} \in \mathcal{M}(\mathbb{R})$. Noting that $\bigcup_{n\in\mathbb{N}} \overline{A_n}$ is Borel, we conclude that every meagre set is contained in some Borel set which is also a countable union of closed nowhere dense sets. Since there are only $2^{\aleph_0}$ many Borel subsets of $\mathbb{R}$, we obtain $\mathbf{cof}(\mathcal{M}) \leq 2^{\aleph_0}$. □

[8] There is no real reason to do this other than having a nicer-looking function h. Without doing this, the denominators in h will need to have consist of $(\alpha(n) + 1)$ instead of $\alpha(n)$ to avoid the case of division by zero.

While we think of Lebesgue null sets and meagre sets as being rather "small" subsets of $\mathbb{R}$, they capture the idea of "small" in very different ways [1, Section 2.1]. A Lebesgue null set may be dense (e.g. the set $\mathbb{Q}$ of rational numbers). Conversely, a nowhere dense set may have non-zero Lebesgue measure (e.g. the Smith-Volterra-Cantor set, also known as the *fat Cantor set*). The situation is even worse with meagre sets. We can actually decompose $\mathbb{R}$ into a disjoint union of a meagre set A and a null set B, so that the meagre set A has full measure, and the null set B is comeagre.

Lemma 2.3.3 ([1, Lemma 2.1.6])**.** *There exist $A \in \mathcal{L}(\mathbb{R})$ and $B \in \mathcal{M}(\mathbb{R})$ such that $A \cup B = \mathbb{R}$.*

Proof. Let $\{ q_n : n \in \mathbb{N} \}$ be an enumeration of all the rational numbers in $\mathbb{R}$. For each $m \in \mathbb{N}$, define

$$U_m := \bigcup_{\substack{n\in\mathbb{N},\\ n>m}} \left(q_n - \frac{1}{2^n},\ q_n + \frac{1}{2^n} \right).$$

Each U_m is open in $\mathbb{R}$ because it is a union of open intervals. Also, as the rational numbers are dense in $\mathbb{R}$, each U_m is also dense because they contain all but finitely many rational numbers. Let $A := \bigcap_{m\in\mathbb{N}} U_n$. Observe that

$$\lambda_{\mathbb{R}}(U_m) \leq \sum_{\substack{n\in\mathbb{N},\\ n\geq m}} \frac{2}{2^n} = \frac{1}{2^{m-1}}.$$

Therefore $\lambda_{\mathbb{R}}(A) \leq \frac{1}{2^{m-1}}$ for all $m \in \mathbb{N}$, and hence $A \in \mathcal{L}$. Now let

$$B := \mathbb{R} \setminus A = \bigcup_{m\in\mathbb{N}} (\mathbb{R} \setminus U_m).$$

Observe that for each $m \in \mathbb{N}$ the set U_m is open and dense in $\mathbb{R}$, so $\mathbb{R} \setminus U_m$ must be nowhere dense in $\mathbb{R}$. Therefore B is a countable union of nowhere dense sets, and so $B \in \mathcal{M}(\mathbb{R})$. □

We can use Lemma 2.3.3 to draw two more arrows in Cichoń's diagram. Working in the Cantor space $2^{\mathbb{N}}$, and recalling that there is a canonical mapping of $2^{\mathbb{N}}$ into $[0, 1]$, we exploit the fact that translation in $[0, 1]$ is a measure-preserving homeomorphism to establish the following theorem.

Theorem 2.3.4 (Rothberger's Theorem, [1, Theorem 2.1.7])**.**

$$\mathbf{cov}(\mathcal{L}) \leq \mathbf{non}(\mathcal{M}) \text{ and } \mathbf{cov}(\mathcal{M}) \leq \mathbf{non}(\mathcal{L}).$$

Proof. We work in $2^{\mathbb{N}}$.

By Lemma 2.3.3, we can write $[0, 1] = A' \cup B'$ for some Lebesgue null set $A' \subset [0, 1]$ and some meagre set $B' \subset [0, 1]$. Using the function g from the proof of Theorem 2.2.5, we can write $2^{\mathbb{N}} = A \cup B$ for some Lebesgue null set $A \subset 2^{\mathbb{N}}$ and some meagre set $B \subset 2^{\mathbb{N}}$.

Let $X \in \mathscr{P}\left(2^{\mathbb{N}}\right) \setminus \mathcal{M}\left(2^{\mathbb{N}}\right)$ be such that $|X| = \mathbf{non}(\mathcal{M})$. Observe that $x + A := \{ x + a : a \in A \}$ is meagre for all $x \in X$, where addition is understood to be addition of functions modulo 2.[9] We will show that $X + A := \bigcup_{x\in X}(x + A) = 2^{\mathbb{N}}$. Suppose, for a contradiction, that there exists some $z \in 2^{\mathbb{N}} \setminus (X + A)$. If $z + x \in A$ for some $x \in X$, then we obtain $z \in -x + A \subseteq X + A$,[10] contrary to the the definition of z. So $z + X$ must be disjoint from A, which yields $z + X \subseteq B$. However, this means $z + X \in \mathcal{M}\left(2^{\mathbb{N}}\right)$, which implies that $X \in \mathcal{M}\left(2^{\mathbb{N}}\right)$, contradicting the definition of X. Therefore,

$$\mathbf{cov}(\mathcal{L}) \leq |\{ x + A : x \in X \}| \leq |X| = \mathbf{non}(\mathcal{M}).$$

The proof for the inequality $\mathbf{cov}(\mathcal{M}) \leq \mathbf{non}(\mathcal{L})$ is similar. □

[9] For example, if $x(0) = 1$ and $a(0) = 1$ then $(x + a)(0) = 0$.

[10] Since we are working with addition modulo 2, we can guarantee that $-x \in X$ because $x = -x$ modulo 2.

The inequalities involving the cardinals $\mathbf{add}(\mathcal{L})$, $\mathbf{cov}(\mathcal{L})$, $\mathbf{non}(\mathcal{L})$, $\mathbf{cof}(\mathcal{L})$, $\mathbf{add}(\mathcal{M})$, $\mathbf{cov}(\mathcal{M})$, $\mathbf{non}(\mathcal{M})$, and $\mathbf{cof}(\mathcal{M})$ proven so far are presented in Figure 3. Specifically, the diagram represents the inequalities proven in Lemma 2.1.4, Theorem 2.3.1, Theorem 2.3.2, and Theorem 2.3.4, where an arrow $\kappa \longrightarrow \lambda$ denotes the inequality $\kappa \leq \lambda$.

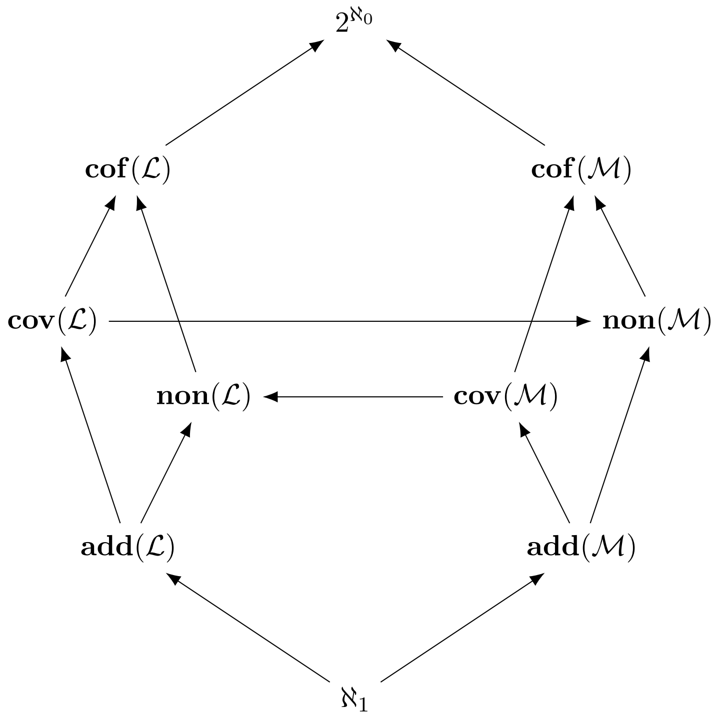

Figure 3: Our Hasse diagram of cardinal characteristics of the continuum.

3 Growth of Functions

We now turn our attention to the Baire space $\mathbb{N}^\mathbb{N}$. As this space consists of functions $f\colon \mathbb{N} \to \mathbb{N}$, we will define cardinals in terms of properties of functions. We will be exploring the long-term behaviour of functions in $\mathbb{N}^\mathbb{N}$, seeing if some function $f\colon \mathbb{N} \to \mathbb{N}$ will eventually dominate some other function $g\colon \mathbb{N} \to \mathbb{N}$.

3.1 The Dominating and The Unbounding Number

We begin by formalising a definition for $f \in \mathbb{N}^\mathbb{N}$ to "eventually dominate" $g \in \mathbb{N}^\mathbb{N}$. To do this, we define a new partial order $\leq^*$ on $\mathbb{N}^\mathbb{N}$.

Definition 3.1.1 (The $\leq^*$ relation, [2])**.** For functions $f, g \in \mathbb{N}^\mathbb{N}$, we write $f \leq^* g$ (and also write $g \geq^* f$) if there exists some $N \in \mathbb{N}$ such that

$$f(n) \leq g(n) \text{ for all } n \geq N.$$

Note that the relation $\leq^*$ is a partial order on $\mathbb{N}^\mathbb{N}$.

In other words, $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \mathbb{N}$. This is somewhat similar to the big-O notation in discrete mathematics. There, we write $f = O(g)$ if there exists some constant $k \in \mathbb{R}_{>0}$ and some $N \in \mathbb{N}$ such that $f(n) \leq kg(n)$ for all $n \geq N$. The $\leq^*$ relation simply requires that this constant k is actually *equal* to 1, so that the graph of f will eventually lie below the graph of g.

At this point, the reader may notice some similarities between the definition of $\leq^*$ and the product topology on $\mathbb{N}^\mathbb{N}$. We will establish this connection later in Section 3.2. For now, this relation $\leq^*$ serving as a new notion for function dominance allows us to define two new cardinals — the *dominating number* $\mathfrak{d}$ and the *unbounding number* $\mathfrak{b}$.

Definition 3.1.2 (Dominating Number, [2])**.** A subset $\mathcal{D} \subseteq \mathbb{N}^\mathbb{N}$ is a *dominating set* if for all $f \in \mathbb{N}^\mathbb{N}$ there exists $g \in \mathcal{D}$ such that $f \leq^* g$. The *dominating number* $\mathfrak{d}$ is the smallest cardinality of all dominating sets, i.e.

$$\mathfrak{d} := \min\{\, |\mathcal{D}| : \mathcal{D} \text{ is a dominating set}\,\}.$$

Definition 3.1.3 (Unbounding Number, [2])**.** A subset $\mathcal{B} \subseteq \mathbb{N}^\mathbb{N}$ is an *unbounded set* if there does not exist $g \in \mathbb{N}^\mathbb{N}$ such that for all $f \in \mathcal{B}$ we have $g \geq^* f$. The *unbounding number*[11] $\mathfrak{b}$ is the smallest cardinality of all unbounded sets, i.e.

$$\mathfrak{b} := \min\{\, |\mathcal{B}| : \mathcal{B} \text{ is an unbounded set}\,\}.$$

We will show that these cardinals $\mathfrak{b}$ and $\mathfrak{d}$ actually lie between $\aleph_1$ and $2^{\aleph_0}$. Furthermore, as Theorem 3.1.4 will show, all dominating sets are unbounded, giving $\mathfrak{b} \leq \mathfrak{d}$.

Theorem 3.1.4 ([2, Theorem 2.4])**.** $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq 2^{\aleph_0}$.

Proof. Let $\mathcal{B}$ be an unbounded set with $|\mathcal{B}| = \mathfrak{b}$, and let $\mathcal{D}$ be a dominating set with $|\mathcal{D}| = \mathfrak{d}$. First, we have

$$\mathfrak{d} = |\mathcal{D}| \leq \left|\mathbb{N}^\mathbb{N}\right| = {\aleph_0}^{\aleph_0} = 2^{\aleph_0}.$$

Next suppose, for a contradiction, that $\mathfrak{b} \leq \aleph_0$. Clearly $\mathfrak{b} > 0$, so we may enumerate $\mathcal{B}$ with $\mathcal{B} = \{\, g_n : n \in \mathbb{N}\,\}$.[12] Now define $f\colon \mathbb{N} \to \mathbb{N}$ by

$$f(x) := 1 + \max_{\substack{n\in\mathbb{N},\\ n\leq x}} g_n(x) \quad \text{for all } x \in \mathbb{N}.$$

[11] The unbounding number is also sometimes called the *bounding number*, which explains the choice of the letter $\mathfrak{b}$. The use of the term "bounding number" can be seen in, for example, Andreas Blass's *Combinatorial Cardinal Characteristics of the Continuum* [2] and relatively recently-published papers [3] [5].

[12] The g_n's do not need to be distinct, so this argument covers the case of $\mathfrak{b}$ being finite.

We observe that for each $n \in \mathbb{N}$, we have $f(x) > g_n(x)$ for all $x \geq n$, and so $f \geq^* g_n$. This contradicts $\mathcal{B}$ being an unbounded set. Therefore $\mathfrak{b} \geq \aleph_1$.

Finally, we will show that $\mathcal{D}$ must be an unbounded set. Suppose there exists $g \in \mathbb{N}^{\mathbb{N}}$ such that for all $f \in \mathcal{D}$ we have $g \geq^* f$. Define $h \colon \mathbb{N} \to \mathbb{N}$ by $h(x) := g(x) + 1$ for all $x \in \mathbb{N}$. Then for every $f \in \mathcal{D}$, there exists some $N \in \mathbb{N}$ such that $h(x) > f(x)$ for all $x \geq N$. Then it is not the case that $f \geq^* h$, contradicting $\mathcal{D}$ being a dominating set. Therefore $\mathcal{D}$ is an unbounded set, and we obtain $\mathfrak{b} \leq |\mathcal{D}| = \mathfrak{d}$. □

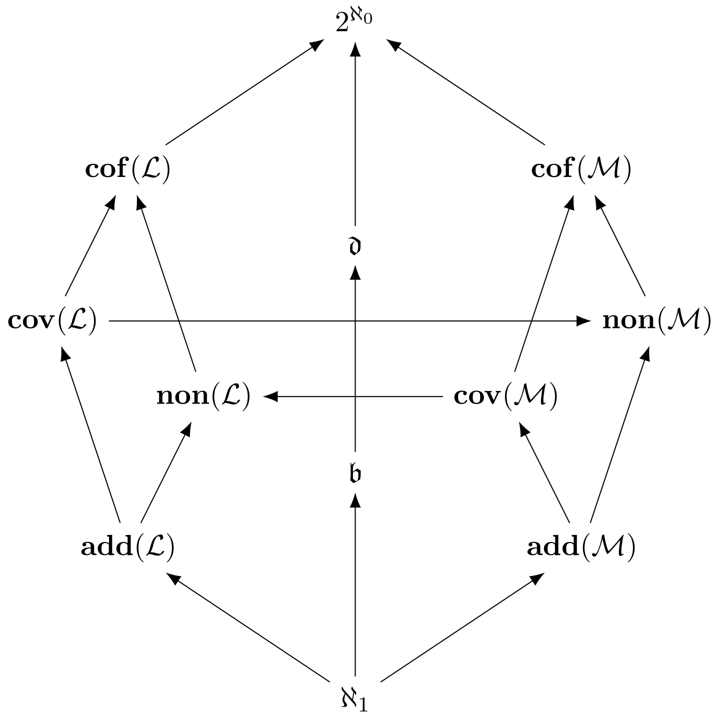

Figure 4: Our Hasse diagram of cardinal characteristics of the continuum.

Inserting $\mathfrak{b}$ and $\mathfrak{d}$ into our diagram of cardinal characteristics of the continuum, Theorem 3.1.4 allows us to obtain Figure 4. We wish to investigate how $\mathfrak{b}$ and $\mathfrak{d}$ interact with all the other cardinal characteristics defined on Lebesgue null and meagre subsets of $\mathbb{R}$.

3.2 Compact Subsets of the Baire Space

As $\mathfrak{b}$ and $\mathfrak{d}$ were defined on $\mathbb{N}^{\mathbb{N}}$, we are motivated to investigate the topological properties of $\mathbb{N}^{\mathbb{N}}$ to obtain more inequalities involving $\mathfrak{b}$ and $\mathfrak{d}$. For reasons that will become apparent soon, we define a new ideal $\mathcal{K}_\sigma$ to be the smallest ideal containing all the compact subsets of $\mathbb{N}^{\mathbb{N}}$.

Definition 3.2.1 ([2, Section 2], [1, Section 2.2], [6, Section 7.5])**.** We write $\mathcal{K}_\sigma$ for the collection of all $A \subseteq \mathbb{N}^{\mathbb{N}}$ such that $A \subseteq \bigcup_{j \in \mathbb{N}} K_j$ for some countable collection $\{K_j\}_{j \in \mathbb{N}}$ of compact subsets of $\mathbb{N}^{\mathbb{N}}$.

Compact subsets of $\mathbb{N}^{\mathbb{N}}$ can be completely classified, as Lemma 3.2.2 will show. We employ Tychonoff's theorem, which states that the product of compact sets remains compact in the product topology [17, Proposition 4.1].

Lemma 3.2.2 ([2, Theorem 2.8], [1, Lemma 1.2.3])**.** *Let $K \subseteq \mathbb{N}^{\mathbb{N}}$. Then K is compact if and only if K is closed and there exists a collection $\{B_n\}_{n \in \mathbb{N}}$ of finite sets such that $K \subseteq \prod_{n \in \mathbb{N}} B_n$.*

Proof. Suppose $K \subseteq \mathbb{N}^{\mathbb{N}}$ is compact. Then K must be closed since $\mathbb{N}^{\mathbb{N}}$ is Hausdorff. Now suppose, for a contradiction, that K is not contained in any product of finite subsets of $\mathbb{N}$. For each $j \in \mathbb{N}$, define the projection $\pi_j \colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ onto the j-th coordinate by $\pi_j(\alpha) \coloneqq \alpha(j)$. Note that this π_j is continuous. By our assumption, there must exist $j_0 \in \mathbb{N}$ such that $\pi_{j_0}(K)$ is an infinite set. But then $\pi_{j_0}(K)$ is not compact, contradicting π_{j_0} being continuous.

For the converse, suppose K is closed and $K \subseteq \prod_{n\in\mathbb{N}} B_n$ for some collection $\{B_n\}_{n\in\mathbb{N}}$ of finite subsets of $\mathbb{N}$. Let $\mathcal{U}$ be an open cover of K. Then $\mathcal{U} \cup \left\{\mathbb{N}^{\mathbb{N}} \setminus K\right\}$ is an open cover of $\mathbb{N}^{\mathbb{N}}$, and hence an open cover of $\prod_{n\in\mathbb{N}} B_n$. By Tychonoff's theorem, since each B_n is compact, the product $\prod_{n\in\mathbb{N}} B_n$ must also be compact, and so there must exist a finite subcover $\mathcal{U}' \subseteq \mathcal{U}$ such that $\prod_{n\in\mathbb{N}} B_n \subseteq \bigcup \mathcal{U}' \cup \left\{\mathbb{N}^{\mathbb{N}} \setminus K\right\}$. Therefore $K \subseteq \bigcup \mathcal{U}'$, and so K is compact. $\square$

With the characterisation of compact subsets of $\mathbb{N}^{\mathbb{N}}$ in Lemma 3.2.2, we can show how $\mathcal{K}_\sigma$ relates to $\mathcal{M}\left(\mathbb{N}^{\mathbb{N}}\right)$.

Lemma 3.2.3 ([1, Section 2.2], [1, Lemma 1.2.3], [2, Theorem 2.8])**.** *$\mathcal{K}_\sigma \subseteq \mathcal{M}\left(\mathbb{N}^{\mathbb{N}}\right)$, and $\mathcal{K}_\sigma$ is a proper σ-ideal on $\mathbb{N}^{\mathbb{N}}$.*

Proof. Recalling the definition of the product topology, every open set in $\mathbb{N}^{\mathbb{N}}$ is the union of sets of the form $\prod_{n\in\mathbb{N}} U_n$, where $U_n \subseteq \mathbb{N}$, and we must have $U_n = \mathbb{N}$ for all but finitely many $n \in \mathbb{N}$. So Lemma 3.2.2 implies that every compact subset of $\mathbb{N}^{\mathbb{N}}$ is nowhere dense, and therefore $\mathcal{K}_\sigma \subseteq \mathcal{M}\left(\mathbb{N}^{\mathbb{N}}\right)$.

By definition, $\mathcal{K}_\sigma$ is a σ-ideal on $\mathbb{N}^{\mathbb{N}}$. We need to show that it is a proper ideal. For each $f \in \mathbb{N}^{\mathbb{N}}$, define

$$\begin{aligned} C_f &\coloneqq \{\, g \in \mathbb{N}^{\mathbb{N}} : g(n) \le f(n) \text{ for all } n \in \mathbb{N} \,\}, \qquad \text{and} \\ C_f^* &\coloneqq \{\, g \in \mathbb{N}^{\mathbb{N}} : g \le^* f \,\}. \end{aligned}$$

We can rephrase Lemma 3.2.2 to say that a set $K \subseteq \mathbb{N}^{\mathbb{N}}$ is compact if and only if K is closed and there exists some $f \in \mathbb{N}^{\mathbb{N}}$ such that $K \subseteq C_f$. Arguing via a diagonal dominating function, similarly as in the proof for $\mathfrak{b} \ge \aleph_1$ in Theorem 3.1.4, we see that for any $A \in \mathcal{K}_\sigma$ there must exist some $f \in \mathbb{N}^{\mathbb{N}}$ such that $A \subseteq C_f^*$. Furthermore, for any $h \in \mathbb{N}^{\mathbb{N}}$, we must have $C_h^* \in \mathcal{K}_\sigma$ since

$$C_h^* = \bigcup_{n\in\mathbb{N}} \bigcup_{M\in\mathbb{N}} \left\{\, g \in \mathbb{N}^{\mathbb{N}} : g(k) \le h(k) \text{ for all } k \ge n, \text{ and } g(k) \le M \text{ for all } k < n \,\right\}.$$

Therefore $\mathcal{K}_\sigma = \left\{\, A \in \mathscr{P}\left(\mathbb{N}^{\mathbb{N}}\right) : A \subseteq C_f^* \text{ for some } f \in \mathbb{N}^{\mathbb{N}} \,\right\}$. Hence $\mathbb{N}^{\mathbb{N}} \notin \mathcal{K}_\sigma$. $\square$

Armed with Lemma 3.2.3, we can now establish inequalities between $\mathfrak{b}$, $\mathfrak{d}$, and several other cardinals introduced in Section 2.

Theorem 3.2.4 ([1, Lemma 2.2.1], [2, Theorem 2.8])**.**

$$\mathbf{add}(\mathcal{K}_\sigma) = \mathbf{non}(\mathcal{K}_\sigma) = \mathfrak{b} \text{ and } \mathbf{cov}(\mathcal{K}_\sigma) = \mathbf{cof}(\mathcal{K}_\sigma) = \mathfrak{d}.$$

Consequently, $\mathfrak{b} \le \mathbf{non}(\mathcal{M})$ and $\mathfrak{d} \ge \mathbf{cov}(\mathcal{M})$.

Proof. For $f \in \mathbb{N}^{\mathbb{N}}$, define $C_f^* \coloneqq \{\, g \in \mathbb{N}^{\mathbb{N}} : g \le^* f \,\}$.

From the proof of Lemma 3.2.3, a subset $A \subseteq \mathbb{N}^{\mathbb{N}}$ satisfies $A \in \mathcal{K}_\sigma$ if and only if there exists a function $f \in \mathbb{N}^{\mathbb{N}}$ with $A \subseteq C_f^*$. Furthermore, $f, g \in \mathbb{N}^{\mathbb{N}}$ satisfy $f \le^* g$ if and only if $C_f^* \subseteq C_g^*$. So the definitions of $\mathfrak{b}$ and $\mathfrak{d}$ immediately yield $\mathfrak{b} = \mathbf{non}(\mathcal{K}_\sigma)$ and $\mathfrak{d} = \mathbf{cov}(\mathcal{K}_\sigma)$.

Let $\mathcal{C} \subseteq \mathcal{K}_\sigma$ be such that $\bigcup \mathcal{C} \notin \mathcal{K}_\sigma$ and $|\mathcal{C}| = \mathbf{add}(\mathcal{K}_\sigma)$. For each $A \in \mathcal{C}$, choose a function $f_A \in \mathbb{N}^{\mathbb{N}}$ with $A \subseteq C_{f_A}^*$. Define $\mathcal{B} \coloneqq \{\, f_A : A \in \mathcal{C} \,\}$. If there exists a function $g \in \mathbb{N}^{\mathbb{N}}$ such that

$\mathcal{B} \subseteq C_g^*$, then $\bigcup \mathcal{C} \subseteq \bigcup_{f_A \in \mathcal{B}} C_{f_A}^* \subseteq C_g^* \in \mathcal{K}_\sigma$, contradicting the assumption of $\mathcal{C}$. Therefore $\mathcal{B}$ is an unbounded set, and so we obtain

$$\mathfrak{b} \leq |\mathcal{B}| \leq |\mathcal{C}| = \mathbf{add}(\mathcal{K}_\sigma).$$

Recalling Lemma 2.1.4, we obtain $\mathfrak{b} \leq \mathbf{add}(\mathcal{K}_\sigma) \leq \mathbf{non}(\mathcal{K}_\sigma) = \mathfrak{b}$ and so $\mathbf{add}(\mathcal{K}_\sigma) = \mathfrak{b}$.

Now let $\mathcal{D} \subseteq \mathbb{N}^\mathbb{N}$ be a dominating set with $|\mathcal{D}| = \mathfrak{d}$, and define $\mathfrak{B} := \{\, C_f^* : f \in \mathcal{D} \,\}$. For any $A \in K_\sigma$, there exists some $f \in \mathbb{N}^\mathbb{N}$ such that $A \subseteq C_f^*$. Then since $\mathcal{D}$ is a dominating set, there exists some $f' \in \mathcal{D}$ such that $C_f^* \subseteq C_{f'}^*$. Therefore $\mathfrak{B}$ is a basis for $\mathcal{K}_\sigma$, and hence

$$\mathbf{cof}(\mathcal{K}_\sigma) \leq |\mathfrak{B}| \leq |\mathcal{D}| = \mathfrak{d}.$$

Lemma 2.1.4 then yields $\mathfrak{d} = \mathbf{cov}(\mathcal{K}_\sigma) \leq \mathbf{cof}(\mathcal{K}_\sigma) \leq \mathfrak{d}$ and so $\mathbf{cof}(\mathcal{K}_\sigma) = \mathfrak{d}$.

Finally, the inequalities $\mathfrak{b} = \mathbf{non}(\mathcal{K}_\sigma) \leq \mathbf{non}(\mathcal{M})$ and $\mathbf{cov}(\mathcal{M}) \leq \mathbf{cov}(\mathcal{K}_\sigma) = \mathfrak{d}$ follow from the fact that $K_\sigma \subseteq \mathcal{M}\left(\mathbb{N}^\mathbb{N}\right)$, due to Lemma 3.2.3. □

3.3 Tukey Functions

At this point, we are four inequalities away from establishing all the arrows in Cichoń's diagram. We are still yet to establish the following inequalities to draw all the arrows in Figure 1:

- $\mathfrak{b} \geq \mathbf{add}(\mathcal{M})$,
- $\mathfrak{d} \leq \mathbf{cof}(\mathcal{M})$,
- $\mathbf{add}(\mathcal{M}) \geq \mathbf{add}(\mathcal{L})$,
- $\mathbf{cof}(\mathcal{M}) \leq \mathbf{cof}(\mathcal{L})$.

We will now turn to proving the first two inequalities. A *Tukey function* mapping elements of a partially ordered set to elements of another partially ordered set is a function such that the pre-image of any bounded set remains bounded. We concern ourselves with this discussion because $\subseteq$ is a partial order on any collection of sets.

Definition 3.3.1 (Tukey Functions, [1, Definition 2.1.1]). Let $(P, \preceq_P)$ and $(Q, \preceq_Q)$ be partially ordered sets. We say $\tau\colon P \to Q$ is a *Tukey function* if for every bounded $X \subseteq Q$, the set $\tau^{-1}(X)$ is bounded in P. More explicitly, τ is Tukey if: for every $X \subseteq Q$, if there exists some $q \in Q$ such that

$$x \preceq_Q q \quad \text{for all } x \in X,$$

then there exists some $p \in P$ such that

$$y \preceq_P p \quad \text{for all } y \in \tau^{-1}(X).$$

We now prove a significant result which will help establish the inequalities $\mathfrak{b} \geq \mathbf{add}(\mathcal{M})$ and $\mathfrak{d} \leq \mathbf{cof}(\mathcal{M})$.

First, let us set up some notation. Let A be a non-empty set. Given a natural number $j \geq 1$, we let A^j be the set of all functions $f\colon \{0, \ldots, j-1\} \to A$. Given two natural numbers $j_0, j_1 \geq 1$, and given two functions $f_0 \in A^{j_0}$ and $f_1 \in A^{j_1}$, we define the *concatenation* $f_0 \frown f_1$ to be the unique function in $A^{j_0+j_1}$ satisfying

$$(f_0 \frown f_1)(x) = \begin{cases} f_0(x) & \text{if } 0 \leq x < j_0, \\ f_1(x - j_0) & \text{if } j_0 \leq x < j_1. \end{cases}$$

Given a natural number $j \geq 1$ and a function $f \in A^j$, we define the *length* of f to be $\mathrm{len}(f) := j$, and we define the *open set generated by* f to be the countably infinite product of sets

$$[f] := \{f(0)\} \times \cdots \times \{f(j-1)\} \times A \times A \times A \times \cdots \in A^\mathbb{N},$$

where we view a function in $A^{\mathbb{N}}$ as an infinite ordered list of elements of A. Note that if A is given the discrete topology then $[f]$ is actually one of the basis elements for the product topology on $A^{\mathbb{N}}$.

Lemma 3.3.2 ([1, Theorem 2.2.2]). *Equip $\mathbb{N}^{\mathbb{N}}$ with the partial order $\leq^*$, and equip $\mathcal{M}\left(\mathbb{N}^{\mathbb{N}}\right)$ with the partial order $\subseteq$. Then there exists a Tukey function $\tau\colon \mathbb{N}^{\mathbb{N}} \to \mathcal{M}\left(\mathbb{N}^{\mathbb{N}}\right)$.*

Proof. For $f \in \mathbb{N}^{\mathbb{N}}$, define again $C_f^* := \left\{ g \in \mathbb{N}^{\mathbb{N}} : g \leq^* f \right\}$, and define $f^{\nearrow}\colon \mathbb{N} \to \mathbb{N}$ by

$$f^{\nearrow}(n) := 1 + \max\{ f(j) : j \leq n \}.$$

Define $\tau\colon \mathbb{N}^{\mathbb{N}} \to \mathcal{M}\left(\mathbb{N}^{\mathbb{N}}\right)$ by

$$\tau(f) := C^*_{f^{\nearrow}}.$$

From the proof of Lemma 3.2.3, we see that indeed the image of τ is contained in $\mathcal{K}_\sigma$, and hence contained in $\mathcal{M}\left(\mathbb{N}^{\mathbb{N}}\right)$. We will show that this function τ is a Tukey function.

Let $F \in \mathcal{M}\left(\mathbb{N}^{\mathbb{N}}\right)$. We need to show that there exists $h_F \in \mathbb{N}^{\mathbb{N}}$ such that for every $f \in \mathbb{N}^{\mathbb{N}}$, if $\tau(f) \subseteq F$ then $f \leq^* h_F$. Let $F_0, F_1, \dots \subseteq \mathbb{N}^{\mathbb{N}}$ be a sequence of closed nowhere dense sets such that $F \subseteq \bigcup_{j\in\mathbb{N}} F_j$. Define $h_F\colon \mathbb{N} \to \mathbb{N}$ as follows:

1. Choose a bijection $\beta\colon \mathbb{N} \to \bigcup_{j\in\mathbb{N}} \mathbb{N}^j$. So the sequence $(\beta(0),\ \beta(1),\ \beta(2),\ \dots)$ is an enumeration of all finite sequences of natural numbers.
2. Recursively define a sequence $k_0, k_1, \dots \in \mathbb{N}$ of natural numbers and a sequence $s_0, s_1, \dots \in \bigcup_{j\in\mathbb{N}} \mathbb{N}^j$ of finite sequences of natural numbers as follows:
   (a) Let $k_0 := 0$.
   (b) Assuming k_n is already defined for some $n \in \mathbb{N}$, define the formula $\varphi_n(s)$ to be true if and only if $s \in \bigcup_{j\in\mathbb{N}} \mathbb{N}^j$ satisfies the following property:
   $$\text{for every } t \in \bigcup_{\substack{m\in\mathbb{N},\\ m\leq k_n}} \{0,\ \dots,\ k_n\}^m, \text{ we have } [t \frown s] \cap \bigcup_{\substack{i\in\mathbb{N},\\ i\leq n}} F_i = \varnothing.$$
   For every $n \in \mathbb{N}$, we can always guarantee the existence of some $s \in \bigcup_{j\in\mathbb{N}} \mathbb{N}^j$ such that $\varphi_n(s)$ holds. This is because $F_0, \dots, F_n$ are closed and nowhere dense, and the complement of a closed nowhere dense set is open and dense.
   (c) For $n \in \mathbb{N}$, define
   $$s_n := \beta\left(\min\{ j \in \mathbb{N} \ :\ \varphi(n, \beta(j)) \}\right),$$
   and define
   $$k_{n+1} := k_n + \operatorname{len}(s_n) + \max\{ s_n(i) : i \in \mathbb{N} \text{ and } i < \operatorname{len}(s_n) \} + 1.$$
3. Finally define $h_F\colon \mathbb{N} \to \mathbb{N}$ by
$$h_F(n) := \max\{ s_n(i) : i \in \mathbb{N} \text{ and } i < \operatorname{len}(s_n) \}.$$

We wish to show that if $\tau(f) \subseteq F$ then $f \leq^* h_F$. We prove the contrapositive. Suppose $f \in \mathbb{N}^{\mathbb{N}}$ satisfies $f \not\leq^* h_F$. Define a strictly increasing infinite sequence of natural numbers $0 \leq x_0 < x_1 < x_2 < \cdots$ such that

$$\text{for all } j \in \mathbb{N}, \text{ we have } f(j) > h_F(j) \text{ if and only if } j \in \{x_0, x_1, x_2, \dots\}.$$

Let $g\colon \mathbb{N} \to \mathbb{N}$ have the property that for all $n \geq 1$, we have

$$g \in \left[g|_{\{0,\ \dots,\ k_{x_n}-1\}} \frown s_{x_n} \right].$$

An example of such a function g would be the following:

1. Let $\tilde{g}_1 \in \mathbb{N}^{k_{x_1}}$ be the function $\tilde{g}_1(j) \coloneqq 0$ for all $0 \le j < k_{x_1}$, and let $g_1 \in \mathbb{N}^{k_{x_1}+\mathrm{len}(s_{x_1})}$ be the function $g_1 \coloneqq \tilde{g}_1 \frown s_{x_1}$.
2. For each $n \ge 1$, let $\tilde{g}_{n+1} \in \mathbb{N}^{k_{x_{n+1}}}$ be the function defined by
$$\tilde{g}_{n+1}(j) \coloneqq \begin{cases} g_n(j) & \text{if } 0 \le j < k_{x_n} + \mathrm{len}(s_{x_n}), \\ 0 & \text{if } k_{x_n} + \mathrm{len}(s_{x_n}) \le j < k_{x_{n+1}}, \end{cases}$$
and let $g_n \in \mathbb{N}^{k_{x_{n+1}}+\mathrm{len}(s_{x_{n+1}})}$ be the function $g_{n+1} \coloneqq \tilde{g}_{n+1} \frown s_{x_{n+1}}$.
3. Define $g\colon \mathbb{N} \to \mathbb{N}$ to be the common extension of g_n for all $n \ge 1$.

Intuitively, we have
$$g = \tilde{g}_1 \frown s_{x_1} \frown \tilde{g}_2 \frown s_{x_2} \frown \tilde{g}_3 \frown s_{x_3} \frown \cdots$$
where each $\tilde{g}_n$ is a finite sequence of zeros such that $\tilde{g}_1 \frown s_{x_1} \frown \cdots \frown s_{x_{n-1}} \frown \tilde{g}_n$ has length k_{x_n}. Then, by the definition of s_n and φ_n above, we have $g \notin F_n$ for all $n \in \mathbb{N}$. Hence $g \notin F$. Now for all $j \in \mathbb{N}$, we have two cases:

- $g(j) = 0$, or
- there exist $n, t \in \mathbb{N}$ such that $k_{x_n} \le j < k_{x_{n+1}}$ and $g(j) = s_{x_n}(t)$.

In the first case, we have $g(j) = 0 \le f^{\nearrow}(j)$. In the second case, since $x_n \le k_{x_n} \le j$, we have
$$g(j) \le h_F(x_n) < f(x_n) \le f^{\nearrow}(j).$$
Therefore $g(j) \le f^{\nearrow}(j)$ for all $j \in \mathbb{N}$. In particular, we have $g \in \tau(f)$. Therefore $g \in \tau(f) \setminus F$, and so $\tau(f) \nsubseteq F$. $\square$

By virtue of Lemma 3.3.2, we can now establish two more inequalities in Cichoń's diagram. We will use the Tukey function τ in the proof of Lemma 3.3.2. Furthermore, we will make use of the generalised definitions of **add** and **cof** introduced in Definition 2.1.7.

Theorem 3.3.3 ([1, Theorem 2.2.3, Corollary 2.2.9, Theorem 2.2.11])**.**
$$\mathbf{add}(\mathcal{K}_\sigma) = \mathbf{add}(\mathcal{K}_\sigma, \mathcal{M}) \textit{ and } \mathbf{cof}(\mathcal{K}_\sigma) = \mathbf{cof}(\mathcal{K}_\sigma, \mathcal{M})$$
Consequently, $\mathfrak{b} \ge \mathbf{add}(\mathcal{M})$ *and* $\mathfrak{d} \le \mathbf{cof}(\mathcal{M})$*.*

Proof. For each $f \in \mathbb{N}^{\mathbb{N}}$, define again $C_f^* \coloneqq \{\, g \in \mathbb{N}^{\mathbb{N}} : g \le^* f \,\}$.

Since $\mathcal{K}_\sigma \subseteq \mathcal{M}\left(\mathbb{N}^{\mathbb{N}}\right)$, Lemma 2.1.8 gives us
$$\mathbf{add}(\mathcal{K}_\sigma) \le \mathbf{add}(\mathcal{K}_\sigma, \mathcal{M}) \text{ and } \mathbf{cof}(\mathcal{K}_\sigma) \ge \mathbf{cof}(\mathcal{K}_\sigma, \mathcal{M}).$$
So it only remains to prove the inequalities $\mathbf{add}(\mathcal{K}_\sigma) \ge \mathbf{add}(\mathcal{K}_\sigma, \mathcal{M})$ and $\mathbf{cof}(\mathcal{K}_\sigma) \le \mathbf{cof}(\mathcal{K}_\sigma, \mathcal{M})$.

Let $\mathcal{C} \subseteq \mathcal{K}_\sigma$ be such that $|\mathcal{C}| = \mathbf{add}(\mathcal{K}_\sigma)$ and $\bigcup \mathcal{C} \notin \mathcal{K}_\sigma$. For each $A \in \mathcal{C}$, choose a function $f_A \in \mathbb{N}^{\mathbb{N}}$ such that $A \subseteq C_{f_A}^*$. Take the Tukey function $\tau\colon \mathbb{N}^{\mathbb{N}} \to \mathcal{K}_\sigma$ from the proof of Lemma 3.3.2, and consider the set
$$\bigcup_{A \in \mathcal{C}} \tau(f_A).$$
We claim that this set is not meagre. Indeed, if $\bigcup_{A \in \mathcal{C}} \tau(f_A) = F$ for some meagre $F \subseteq \mathbb{N}^{\mathbb{N}}$, then taking the associated mapping h_F from the proof of Lemma 3.3.2 yields $f_A \le^* h_F$ for all $A \in \mathcal{C}$. However this means $\bigcup \mathcal{C} \subseteq C_{h_F}^* \in \mathcal{K}_\sigma$, which contradicts the definition of $\mathcal{C}$. We therefore obtain $\mathbf{add}(\mathcal{K}_\sigma, \mathcal{M}) \le |\{\, \tau(f_A) : A \in \mathcal{C} \,\}| \le |\mathcal{C}| = \mathbf{add}(\mathcal{K}_\sigma)$.

Now let $\mathfrak{B} \subseteq \mathcal{M}\left(\mathbb{N}^{\mathbb{N}}\right)$ be such that $|\mathfrak{B}| = \mathbf{cof}(\mathcal{K}_\sigma, \mathcal{M})$ and

$$\text{for all } A \in \mathcal{K}_\sigma \text{ there exists some } F \in \mathfrak{B} \text{ with } A \subseteq F.$$

Then using the mapping $F \mapsto h_F$ from the proof of Lemma 3.3.2, the set

$$\mathcal{D} := \{\, h_F : F \in \mathfrak{B} \,\}$$

forms a dominating set in $\mathbb{N}^{\mathbb{N}}$. This is because given any $f \in \mathbb{N}^{\mathbb{N}}$ we have $\tau(f) \in \mathcal{K}_\sigma$, and so there exists some $F \in \mathfrak{B}$ with $\tau(f) \subseteq F$, yielding $f \leq^* h_F$. Therefore $\{\, C_h^* : h \in \mathcal{D} \,\}$ is a basis for $\mathcal{K}_\sigma$, and so $\mathbf{cof}(\mathcal{K}_\sigma) \leq |\{\, C_h^* : h \in \mathcal{D} \,\}| \leq \mathbf{cof}(\mathcal{K}_\sigma, \mathcal{M})$.

Finally, Theorem 3.2.4 and Lemma 2.1.8 yield $\mathfrak{b} \geq \mathbf{add}(\mathcal{M})$ and $\mathfrak{d} \leq \mathbf{cof}(\mathcal{M})$. $\square$

Incorporating the inequalities proven in a result of Theorem 3.2.4 and Theorem 3.3.3 into Figure 4 and rearranging the diagram, we obtain Figure 5 representing all the inequalities proven between the cardinal characteristics of the continuum we have seen so far.

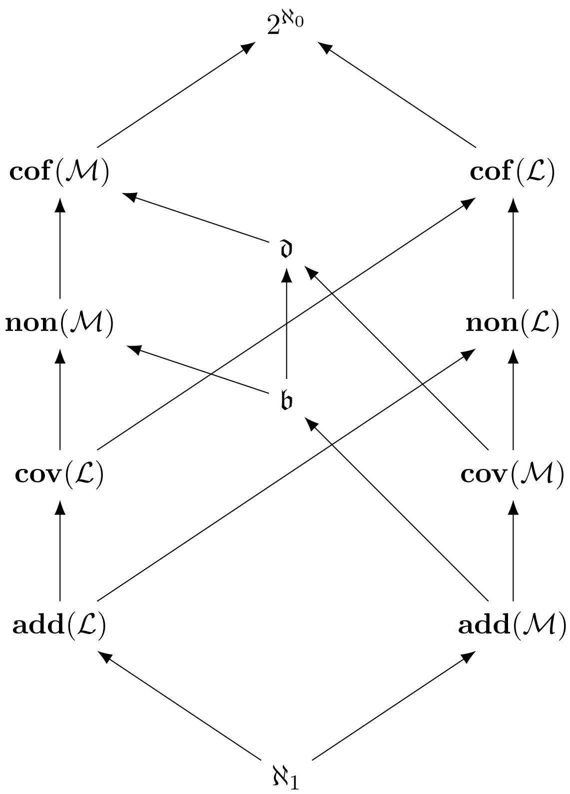

Figure 5: Our Hasse diagram of cardinal characteristics of the continuum.

Theorem 3.3.3 mainly relied on the specific Tukey function τ that was constructed in Lemma 3.3.2, rather than general properties of Tukey functions. Lemma 3.3.4 explains why we introduced the concept of Tukey functions, and how it may be useful.

Lemma 3.3.4 ([1, Lemma 2.1.2])**.** *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on a set X, equipped with the partial order $\subseteq$. Suppose there exists a Tukey function $\tau\colon \mathcal{I} \to \mathcal{J}$. Then*

$$\mathbf{add}(\mathcal{I}) \geq \mathbf{add}(\mathcal{J}) \text{ and } \mathbf{cof}(\mathcal{I}) \leq \mathbf{cof}(\mathcal{J}).$$

Proof. Let $\mathcal{C} \subseteq \mathcal{I}$ be such that $|\mathcal{C}| < \mathbf{add}(\mathcal{J})$. Then, because for every $A \in \mathcal{C}$ we have $\tau(A) \in \mathcal{J}$, it follows that $\tau(\bigcup \mathcal{C}) = \bigcup_{A \in \mathcal{C}} \tau(A) \in \mathcal{J}$. Then τ being Tukey yields $\bigcup \mathcal{C} \in \mathcal{I}$, and so we conclude that $\mathbf{add}(\mathcal{I}) \geq \mathbf{add}(\mathcal{J})$.

Now let $\mathfrak{B} \subseteq \mathcal{J}$ be a basis for $\mathcal{J}$. For each $B \in \mathfrak{B}$, choose $A_B \in \mathcal{I}$ such that

$$A \subseteq A_B \quad \text{for all } A \in \mathcal{I} \text{ such that } \tau(A) \subseteq B.$$

The existence of such A_B's is guaranteed due to τ being Tukey. Then the set $\{ A_B : B \in \mathfrak{B} \}$ forms a basis for $\mathcal{I}$. This is because for every $A \in \mathcal{I}$ there exists $B \in \mathfrak{B}$ such that $\tau(A) \subseteq B$, and consequently $A \subseteq A_B$. $\square$

With Lemma 3.3.4, the final two inequalities $\mathbf{add}(\mathcal{M}) \geq \mathbf{add}(\mathcal{L})$ and $\mathbf{cof}(\mathcal{M}) \leq \mathbf{cof}(\mathcal{L})$ in Cichoń's diagram can be proven. We will not include the proof, but will leave the result below for completeness.

Theorem 3.3.5 ([6, Theorem 7.51, Theorem 7.56], [1, Theorem 2.3.1, Theorem 2.3.7])**.** *Equip $\mathcal{M}\left(2^{\mathbb{N}}\right)$ and $\mathcal{L}\left(2^{\mathbb{N}}\right)$ with the partial order $\subseteq$. Then there exists a Tukey function $\tau \colon \mathcal{M}\left(2^{\mathbb{N}}\right) \to \mathcal{L}\left(2^{\mathbb{N}}\right)$. Consequently,*

$$\mathbf{add}(\mathcal{M}) \geq \mathbf{add}(\mathcal{L}) \text{ and } \mathbf{cof}(\mathcal{M}) \leq \mathbf{cof}(\mathcal{L}).$$

The inequalities in Theorem 3.3.5 are the final pieces of the puzzle needed to obtain all the arrows present in Cichoń's diagram, which we draw again in Figure 6.

$$\begin{tikzcd}
 & 2^{\aleph_0} & \\
 & \mathbf{cof}(\mathcal{L}) \arrow[u] & \\
\mathbf{cof}(\mathcal{M}) \arrow[ur] & & \mathbf{non}(\mathcal{L}) \arrow[ul] \\
 & \mathfrak{d} \arrow[ul] & \\
\mathbf{non}(\mathcal{M}) \arrow[uu] & & \mathbf{cov}(\mathcal{M}) \arrow[uu] \arrow[ul] \\
 & \mathfrak{b} \arrow[uu] \arrow[ul] & \\
\mathbf{cov}(\mathcal{L}) \arrow[uu] & & \mathbf{add}(\mathcal{M}) \arrow[uu] \arrow[ul] \\
 & \mathbf{add}(\mathcal{L}) \arrow[ul] \arrow[ur] & \\
 & \aleph_1 \arrow[u] &
\end{tikzcd}$$

Figure 6: Cichoń's Diagram [2, End of Section 5].

No other inequalities between any two distinct cardinals present in Cichoń's diagram are provable in ZFC [6, Metatheorem 9.1, Metatheorem 11.7]. More specifically:

- In the Cohen model [2, Section 11.2] [6, Equation 11.22], we have

$$\aleph_1 = \mathbf{non}(\mathcal{M}) < \mathbf{cov}(\mathcal{M}) = 2^{\aleph_0}.$$

- In the random reals model [2, Section 11.4] [6, Equation 11.23], we have

$$\aleph_1 = \mathfrak{d} = \mathbf{non}(\mathcal{L}) < \mathbf{cov}(\mathcal{L}) = 2^{\aleph_0}.$$

- In the Sacks model [2, Section 11.5] [6, Equation 11.24], we have

$$\aleph_1 = \mathbf{cof}(\mathcal{L}) < 2^{\aleph_0}$$

- In the Hechler model [2, Section 11.6] [6, Equation 11.27], we have

$$\aleph_1 = \mathbf{cov}(\mathcal{L}) < \mathbf{add}(\mathcal{M}) = 2^{\aleph_0}.$$

- In the Laver model [2, Section 11.7] [6, Equation 11.28], we have

$$\aleph_1 = \mathbf{cov}(\mathcal{L}) = \mathbf{non}(\mathcal{L}) < \mathfrak{b} = 2^{\aleph_0}.$$

- In the Mathias model [2, Section 11.8], we have

$$\aleph_1 = \mathbf{cov}(\mathcal{L}) = \mathbf{cov}(\mathcal{M}) < \mathfrak{b} = \mathbf{non}(\mathcal{L}) = 2^{\aleph_0}.$$

- In the Miller model [2, Section 11.9] [6, Equation 11.36], we have

$$\aleph_1 = \mathbf{non}(\mathcal{M}) = \mathbf{non}(\mathcal{L}) < \mathfrak{d} = 2^{\aleph_0}.$$

- Kunen and Tall showed that $\aleph_1 < \mathbf{add}(\mathcal{L})$ is consistent with ZFC [6, Equation 11.30].
- Kamburelis showed that $\mathbf{cof}(\mathcal{M}) < \mathbf{cof}(\mathcal{L})$ is consistent with ZFC [6, Equation 11.39].

We close this section, ending our tour of Cichoń's diagram, by stating the following two facts without proof:

Proposition 3.3.6 ([1, Corollary 2.2.9, Theorem 2.2.11])**.**

$$\mathbf{add}(\mathcal{M}) = \min\{\mathbf{cov}(\mathcal{M}), \mathfrak{b}\} \quad \textit{and} \quad \mathbf{cof}(\mathcal{M}) = \max\{\mathbf{non}(\mathcal{M}), \mathfrak{d}\}.$$

4 Rearrangements of Infinite Series

Throughout this section, we abbreviate a sequence $(a_n)_{n\in\mathbb{N}}$ to (a_n), and we abbreviate the series $\sum_{n\in\mathbb{N}} a_n$ to $\sum a_n$. Given a bijection $p\colon \mathbb{N}\to\mathbb{N}$, we abbreviate the rearranged sequence $(a_{p(n)})_{n\in\mathbb{N}}$ to $(a_{p(n)})$, and we abbreviate the rearranged series $\sum_{n\in\mathbb{N}} a_{p(n)}$ to $\sum a_{p(n)}$. We will also use the words "bijection" and "permutation" interchangeably.

Given a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers, the series $\sum_{n\in\mathbb{N}} a_n$ is said to be *absolutely convergent* if $\sum_{n\in\mathbb{N}} |a_n|$ converges. If instead $\sum a_n$ converges but $\sum |a_n|$ does not, we say the series $\sum a_n$ is *conditionally convergent*. It is well-known that $\sum a_n$ being conditionally convergent is equivalent to the existence of a permutation $p\colon \mathbb{N}\to\mathbb{N}$ such that

$$\sum a_{p(n)} \neq \sum a_n,$$

where we mean that either $\sum a_{p(n)}$ does not converge, or $\sum a_{p(n)}$ converges to a different real number [20, Theorem 3.55].

Given a convergent series $\sum a_n$, we may therefore say that it only takes *one* permutation $p\colon \mathbb{N}\to\mathbb{N}$ resulting in $\sum a_{p(n)} \neq \sum a_n$ for us to conclude that $\sum a_n$ is conditionally convergent. What happens when we are not able to find such a permutation p? As Michael Hardy asked on MathOverflow [15], "How many rearrangements must fail to alter the value of a sum before you conclude that none do?"

4.1 The Rearrangement Number

We see that the question does not come from nowhere. Despite Riemann's rearrangement theorem [20, Theorem 3.55], one need not check *all* permutations $p\colon \mathbb{N}\to\mathbb{N}$ to guarantee absolute convergence.

Example 4.1.1. Let $p\colon \mathbb{N}\to\mathbb{N}$ be a bijection, and suppose $\sum a_{p(n)}$ converges. Define $p_*\colon \mathbb{N}\to\mathbb{N}$ by

$$p_*(n) \coloneqq \begin{cases} p(1) & \text{if } n = 0, \\ p(0) & \text{if } n = 1, \\ p(n) & \text{if } n \geq 1. \end{cases}$$

Then $\sum a_{p_*(n)}$ converges, and $\sum a_{p_*(n)} = \sum a_{p(n)}$. Indeed, for any permutation $\tilde{p}$ which is a finite alteration of p, we will have that $\sum a_{\tilde{p}(n)}$ converges and $\sum a_{\tilde{p}(n)} = \sum a_{p(n)}$.

Effectively, checking just one permutation would check $\aleph_0$-many permutations simultaneously. Motivated by this, we define the *rearrangement number* $\mathfrak{rr}$ to be the minimum number of rearrangements we would need to check to guarantee absolute convergence.

Definition 4.1.2 (Rearrangement Number, [3, Definition 1])**.** A subset $\mathfrak{RR} \subseteq \mathbb{N}^{\mathbb{N}}$ is a *rearrangement set* if $\mathfrak{RR}$ only consists of bijections, and for any conditionally convergent series $\sum a_n$ there exists $p \in \mathfrak{RR}$ such that $\sum a_{p(n)} \neq \sum a_n$ (in the sense of $\sum a_{p(n)}$ diverging, or $\sum a_{p(n)}$ converging to a different limit). The *rearrangement number* $\mathfrak{rr}$ is the smallest cardinality of all rearrangement sets, i.e.

$$\mathfrak{rr} \coloneqq \min\{\, |\mathfrak{RR}| : \mathfrak{RR} \text{ is a rearrangement set} \,\}.$$

Arguing as in Theorem 3.1.4, we have that $\mathfrak{rr} \leq 2^{\aleph_0}$. Following the results in *The Rearrangement Number* [3], we will show that $\mathfrak{rr}$ is indeed a cardinal characteristic of the continuum by establishing several inequalities between $\mathfrak{rr}$ and cardinals which appear in Cichoń's diagram.

4.2 Connections with Cichoń's Diagram

A priori, we would first like to prove that $\mathfrak{rr} \geq \aleph_1$. We do this by proving the inequality $\mathfrak{rr} \geq \mathfrak{b}$, where $\mathfrak{b}$ is the unbounding number, because we already know that $\mathfrak{b} \geq \aleph_1$ by Theorem 3.1.4. The strategy of the proof, as named in *The Rearrangement Number* [3, Section 3], is to pad a conditionally convergent series with so many zeros that collections of permutations which are unbounded fail to affect the relative ordering of all but finitely many non-zero terms.

Theorem 4.2.1 ([3, Theorem 15, Theorem 16])**.** $\mathfrak{rr} \geq \mathfrak{b}$.

Proof. Let $\mathfrak{RR}$ be a rearrangement set with $|\mathfrak{RR}| = \mathfrak{rr}$, and suppose for a contradiction that $\mathfrak{rr} < \mathfrak{b}$. We will construct a conditionally convergent series $\sum a_n$ such that for all $p \in \mathfrak{RR}$, we have $\sum a_{p(n)} = \sum a_n$, contradicting $\mathfrak{RR}$ being a rearrangement set.

We will construct this $\sum a_n$ from the alternating harmonic series $\sum \frac{(-1)^n}{n+1}$, which we know is conditionally convergent. The strategy is to define (a_n) to have all the terms of $\left(\frac{(-1)^n}{n+1}\right)$ in the same order, but with a large number of zeroes in between each term so that any rearrangement in $\mathfrak{RR}$ only affect the relative ordering of finitely many non-zero terms. We do this by constructing a strictly increasing function $\zeta\colon \mathbb{N} \to \mathbb{N}$ so that $(a_n)_{n\in\mathbb{N}}$ is of the form

$$(a_n) = \left(0,\ \ldots,\ 0,\ 1,\ 0,\ \ldots\ ,\ 0,\ -\frac{1}{2},\ 0,\ \ldots\ ,\ 0,\ \frac{1}{3},\ 0,\ \ldots\ ,\ 0,\ -\frac{1}{4},\ \ldots\right), \tag{2}$$

where $a_{\zeta(0)} = 1$, $a_{\zeta(1)} = -\frac{1}{2}$, $a_{\zeta(2)} = \frac{1}{3}$, and so on. We would have that $\sum a_n$ is also conditionally convergent and that $\sum a_n = \sum \frac{(-1)^n}{n+1}$, because we only inserted zeroes into the series and did not rearrange any non-zero terms. For each $p \in \mathfrak{RR}$ we want to ensure that the series $\sum a_{p(n)}$ converges to the same value as $\sum a_n$. To achieve this, we will require that p only affects the relative positioning of the finitely many non-zero terms of (a_n). In particular, if we fixed any $p \in \mathfrak{RR}$, then we want there to exist some $N \in \mathbb{N}$ for which if $k \geq N$ then the relative positioning of the terms $a_{\zeta(k)}$ and $a_{\zeta(k+1)}$ in the series remain unchanged after applying the permutation p. Noting that the permutation p would send the term at position $\zeta(k)$ to position $p^{-1}(\zeta(k))$, we would want

$$p^{-1}(\zeta(k)) < p^{-1}(\zeta(k+1)) \quad \text{for all } k \geq N.$$

To achieve this, we want $\zeta\colon \mathbb{N} \to \mathbb{N}$ to have the following two properties:

(1) for all $k \in \mathbb{N}$, we have $\zeta(k+1) > \zeta(k)$,

(2) for all $p \in \mathfrak{RR}$, there exists some $N \in \mathbb{N}$ such that

$$\text{for all } k \geq N \text{ we have } \zeta(k+1) \notin \{\, p(j) : j \in \mathbb{N} \text{ and } j \leq p^{-1}(\zeta(k)) \,\}.$$

Since p is bijective, this is equivalent to the condition $p^{-1}(\zeta(k+1)) > p^{-1}(\zeta(k))$ for all $k \geq N$, which in turn ensures that $\sum a_{p(n)} = \sum a_n$.

We will define this ζ by exploiting the assumption that $\mathfrak{rr} < \mathfrak{b}$. For every $p \in \mathfrak{RR}$, define $f_p\colon \mathbb{N} \to \mathbb{N}$ by

$$f_p(x) \coloneqq \max_{\substack{j\in\mathbb{N},\\ j\leq p^{-1}(x)}} p(j) \quad \text{for all } x \in \mathbb{N}. \tag{3}$$

Intuitively, $f_p(x)$ is chosen so that, under p^{-1}, all the numbers strictly larger than $f_p(x)$ are mapped to numbers strictly larger than $p^{-1}(x)$. The family of functions $\mathcal{F} \coloneqq \{\, f_p : p \in \mathfrak{RR} \,\}$ then satisfies $\mathcal{F} \subseteq \mathbb{N}^{\mathbb{N}}$ and $|\mathcal{F}| \leq \mathfrak{rr} < \mathfrak{b}$. So there exists some $g \in \mathbb{N}^{\mathbb{N}}$ such that $g \geq^* f_p$ for all $f_p \in \mathcal{F}$. Now we recursively define $\zeta\colon \mathbb{N} \to \mathbb{N}$ by

$$\begin{aligned} \zeta(0) &\coloneqq g(0), \\ \zeta(k+1) &\coloneqq \zeta(k) + g(\zeta(k)) + 1 \quad \text{for all } k \in \mathbb{N}. \end{aligned}$$

By construction, the function ζ is strictly increasing. We can now define the sequence (a_n) in Equation (2) as follows: for every $n \in \mathbb{N}$,

$$a_n \coloneqq \begin{cases} \frac{(-1)^k}{k+1} & \text{if } n = \zeta(k) \text{ for some (unique) } k \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

We now only need to show that all permutations $p \in \mathfrak{RR}$ preserve the relative ordering of all but finitely many non-zero terms in the series $\sum a_n$ to complete the proof.

For any $p \in \mathfrak{RR}$, obtain the associated function f_p from Equation (3). Then there exists some $N \in \mathbb{N}$ such that if $k \geq N$ then $g(\zeta(k)) \geq f_p(\zeta(k))$, because ζ is strictly increasing and $g \geq^* f_p$. The definitions of f_p, g, and ζ yield

$$f_p(\zeta(k)) \leq g(\zeta(k)) < \zeta(k+1) \quad \text{for all } k \geq N.$$

So for any $j \in \mathbb{N}$ with $j \leq p^{-1}(\zeta(k))$, the definition of f_p in Equation (3) gives us $p(j) \leq f_p(\zeta(k))$, and so we have $p(j) < \zeta(k+1)$. This yields $\zeta(k+1) \notin \{\, p(j) : j \in \mathbb{N} \text{ and } j \leq p^{-1}(\zeta(k)) \,\}$ as desired. □

We can also establish that $\mathbf{cov}(\mathcal{L})$ is a lower bound for $\mathfrak{rr}$. To do this, we will work in $2^{\mathbb{N}}$ and make use of the following lemma by Hans Rademacher (as cited in *The Rearrangement Number* [3, Lemma 18]), which we will state but not prove.

Lemma 4.2.2 ([3, Lemma 18]). *Let (c_n) be a sequence of real numbers, and define*

$$A \coloneqq \left\{ s \in 2^{\mathbb{N}} : \sum_{n \in \mathbb{N}} (-1)^{s(n)} c_n \textit{ converges} \right\}.$$

If $\sum c_n^2$ converges, then the Lebesgue measure of A is 1. Otherwise if $\sum c_n^2$ diverges, then the Lebesgue measure of A is 0.

We will now prove that $\mathfrak{rr} \geq \mathbf{cov}(\mathcal{L})$. The strategy of the proof is to augment to a rearrangement set $\mathfrak{RR}$ a countable set of permutations so that the augmented set will make any conditionally convergent series diverge. Since $|\mathfrak{RR}| \geq \aleph_1$, by Theorem 4.2.1, the augmented set will still have the same cardinality as $\mathfrak{RR}$. We will then use Lemma 4.2.2 on this augmented set to extract a contradiction.

Theorem 4.2.3 ([3, Theorem 6, Lemma 7, Theorem 19]). $\mathfrak{rr} \geq \mathbf{cov}(\mathcal{L})$.

Proof. For ease of reading, let us use $\{\tau(0),\ \ldots,\ \tau(n)\}$ to denote $\{\, \tau(t) : t \in \mathbb{N} \text{ and } 0 \leq t \leq n \,\}$, where $\tau(x)$ is a term with free variable x.

For each bijection $q \colon \mathbb{N} \to \mathbb{N}$ we will define another bijection $g_q \colon \mathbb{N} \to \mathbb{N}$ with the following two properties:

- $\{g_q(0),\ \ldots,\ g_q(n)\} = \{q(0),\ \ldots,\ q(n)\}$ for infinitely many $n \in \mathbb{N}$,
- $\{g_q(0),\ \ldots,\ g_q(n)\} = \{0,\ \ldots,\ n\}$ for infinitely many $n \in \mathbb{N}$.

Intuitively, as we increase n, we want the range of $g_q|_{\{0,\ldots,n\}}$ to keep oscillating between the ranges of $q|_{\{0,\ldots,n\}}$ and $\mathbf{id}|_{\{0,\ldots,n\}}$, where $\mathbf{id} \colon \mathbb{N} \to \mathbb{N}$ denotes the identity function on $\mathbb{N}$. To do this, for each $k \in \mathbb{N}$, we define functions g_q^k recursively as follows:

(1) Define $g_q^0 \colon \{0\} \to \mathbb{N}$ by $g_q(0) \coloneqq q(0)$.

(2) For $k \in \mathbb{N}$, suppose $g_q^k \colon \{0, \ldots, n_k\} \to \mathbb{N}$ is already defined. Let

$$n_{k+1} := \begin{cases} 1 + \min\left\{ n \in \mathbb{N} : \operatorname{Range}\left(g_q^k\right) \subseteq \{0, \ldots, n\} \right\}, & \text{if } k \text{ is even,} \\ 1 + \min\left\{ n \in \mathbb{N} : \operatorname{Range}\left(g_q^k\right) \subseteq \{q(0), \ldots, q(n)\} \right\}, & \text{if } k \text{ is odd.} \end{cases}$$

Then define $g_q^{k+1} \colon \{0, \ldots, n_{k+1}\} \to \mathbb{N}$ by extending the domain of g_q^k and bijectively mapping the remaining elements of $\{n_k + 1, \ldots, n_{k+1}\}$ into $\{0, \ldots, n_{k+1}\} \setminus \operatorname{Range}\left(g_q^k\right)$ if k is even, or into $\{q(0), \ldots, q(n_{k+1})\} \setminus \operatorname{Range}\left(g_q^k\right)$ if k is odd.

If we considered functions as sets of ordered pairs, we would have $g_q^0 \subseteq g_q^1 \subseteq g_q^2 \subseteq \cdots$. We can thus let $g_q \colon \mathbb{N} \to \mathbb{N}$ be the common extension of all the g_q^k's, and this would be a bijection which satisfies our desired properties.

Now let $\mathfrak{RR}$ be a rearrangement set with $|\mathfrak{RR}| = \mathfrak{rr}$. Define $\mathfrak{RR}_{io} := \mathfrak{RR} \cup \{ g_q : q \in \mathfrak{RR} \}$. Then $\mathfrak{RR}_{io}$ is a rearrangement set such that for any conditionally convergent series $\sum a_n$, there exists some permutation $p \in \mathfrak{RR}_{io}$ such that $\sum a_{p(n)}$ diverges.[13] Indeed, if no rearrangements in $\mathfrak{RR}$ yields a divergence, then there exists some $q \in \mathfrak{RR}$ such that $\sum a_{q(n)}$ converges to a finite limit different to $\sum a_n$, and so the associated g_q will make $\sum a_{g_q(n)}$ diverge by oscillation. Also note that $|\mathfrak{RR}_{io}| = |\mathfrak{RR}| = \mathfrak{rr}$ since $\mathfrak{RR}$ has an infinite cardinality by Theorem 4.2.1.

Next, for each permutation $p \in \mathfrak{RR}_{io}$, define $f_p \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by

$$f_p(s) := s \circ p,$$

and define $A_p \subseteq 2^{\mathbb{N}}$ by

$$A_p := \left\{ s \in 2^{\mathbb{N}} : \sum_{n \in \mathbb{N}} \frac{(-1)^{s(n)}}{p(n) + 1} \text{ diverges} \right\}.$$

It is clear that f_p is bijective. Recall that the Lebesgue measure $\lambda_{2^{\mathbb{N}}}$ is defined on the σ-algebra generated by the π-system

$$\mathfrak{B} := \left\{ \prod_{n \in \mathbb{N}} B_n \ : \ B_n \subseteq \{0,1\}, \text{ and } B_n = \{0,1\} \text{ for all but finitely many } n \in \mathbb{N} \right\}$$

of open sets which form a basis for the topology of $2^{\mathbb{N}}$. For each $\prod_{n \in \mathbb{N}} B_n \in \mathfrak{B}$, observe that

$$\lambda_{2^{\mathbb{N}}} \left(\prod_{n \in \mathbb{N}} B_n \right) = \prod_{n \in \mathbb{N}} \frac{|B_n|}{2} = \prod_{n \in \mathbb{N}} \frac{\left| B_{f_p^{-1}(n)} \right|}{2} = \lambda_{2^{\mathbb{N}}} \left(f_p^{-1} \left(\prod_{n \in \mathbb{N}} B_n \right) \right)$$

because $\left| B_{f_p^{-1}(n)} \right| = 2$ for all but finitely many $n \in \mathbb{N}$. Thus, by Dynkin's π-λ theorem [13, Lemma 2.4, Theorem 3.2], the map $A \mapsto f_p^{-1}(A)$ is measure-preserving on the entire σ-algebra generated by $\mathfrak{B}$. Now since $\sum \frac{1}{(p(n)+1)^2}$ is a rearrangement of the absolutely convergent series $\sum \frac{1}{(n+1)^2}$, Lemma 4.2.2 implies that $A_p \in \mathcal{L}\left(2^{\mathbb{N}}\right)$. Hence $f_p^{-1}(A_p) \in \mathcal{L}\left(2^{\mathbb{N}}\right)$ for all $p \in \mathfrak{RR}_{io}$.

Finally, suppose for a contradiction that $\mathfrak{rr} < \mathbf{cov}(\mathcal{L})$. Then there exists some $\tilde{s} \in 2^{\mathbb{N}}$ such that

$$\tilde{s} \notin \bigcup_{p \in \mathfrak{RR}_{io}} f_p^{-1}(A_p) = \bigcup_{p \in \mathfrak{RR}_{io}} \left\{ s \in 2^{\mathbb{N}} : \sum_{n \in \mathbb{N}} \frac{(-1)^{s(p(n))}}{p(n) + 1} \text{ diverges} \right\},$$

as $|\{ f_p^{-1}(A_p) : p \in \mathfrak{RR}_{io} \}| \leq \mathfrak{rr}$. In particular, for all $p \in \mathfrak{RR}_{io}$, the series $\sum \frac{(-1)^{\tilde{s}(p(n))}}{p(n)+1}$ converges. This contradicts the construction of $\mathfrak{RR}_{io}$, as it fails to make that series diverge despite it being conditionally convergent. ☐

[13] In particular, $\sum a_{p(n)}$ will either diverge to *infinity* (∞ or $-\infty$) or will diverge by *oscillation*. This explains the choice of the name $\mathfrak{RR}_{io}$, which is inspired from the cardinal $\mathfrak{rr}_{io}$ in *The Rearrangement Number* [3, Definition 2].

We have established two lower bounds for $\mathfrak{rr}$, namely $\mathfrak{b} \leq \mathfrak{rr}$ and $\mathbf{cov}(\mathcal{L}) \leq \mathfrak{rr}$. It turns out that both the strict inequalities $\mathfrak{b} < \mathfrak{rr}$ and $\mathbf{cov}(\mathcal{L}) < \mathfrak{rr}$ are consistent with ZFC [3, Corollary 21].

We will now establish $\mathbf{non}(\mathcal{M})$ as an upper bound for $\mathfrak{rr}$. To do this, we will work with the topology of the subspace $\mathrm{Sym}(\mathbb{N})$ of all bijections in $\mathbb{N}^{\mathbb{N}}$. It turns out, that this subspace $\mathrm{Sym}(\mathbb{N})$ is homeomorphic to all of $\mathbb{N}^{\mathbb{N}}$. This is a fact we will state but not prove.

Lemma 4.2.4 ([3, Theorem 11], [17, Theorem 7.7])**.** *The set* $\mathrm{Sym}(\mathbb{N})$*, viewed as a subset of* $\mathbb{N}^{\mathbb{N}}$ *equipped with the subspace topology, is homeomorphic to the whole space* $\mathbb{N}^{\mathbb{N}}$*.*

As a corollary of Lemma 4.2.4 and Lemma 2.2.4, we have that

$$\mathbf{ch}\left(\mathcal{M}\left(\mathbb{N}^{\mathbb{N}}\right)\right) = \mathbf{ch}(\mathcal{M}(\mathrm{Sym}(\mathbb{N})))$$

for each $\mathbf{ch} \in \{\mathbf{add}, \mathbf{cov}, \mathbf{non}, \mathbf{cof}\}$. We are now ready to prove that $\mathfrak{rr} \leq \mathbf{non}(\mathcal{M})$. The strategy of the proof is to show that for any conditionally convergent series $\sum a_n$, the collection of all permutations $p \in \mathrm{Sym}(\mathbb{N})$ such that $\sum a_{p(n)} \neq \sum a_n$ forms a comeagre set. As such, any non-meagre set must intersect this collection.

Theorem 4.2.5 ([3, Theorem 11])**.** $\mathfrak{rr} \leq \mathbf{non}(\mathcal{M})$.

Proof. We work in $\mathrm{Sym}(\mathbb{N})$.

For any conditionally convergent series $\sum a_n$, let $D^{(a_n)}$ be the set of all $p \in \mathrm{Sym}(\mathbb{N})$ for which the sequence $\left(\sum_{n=0}^{m} a_{p(n)}\right)_{m\in\mathbb{N}}$ contains a subsequence which diverges to ∞. That is, letting

$$U_k^{(a_n)} := \bigcup_{\substack{m\in\mathbb{N},\\ m\geq k}} \left\{ p \in \mathrm{Sym}(\mathbb{N}) : \sum_{n=0}^{m} a_{p(n)} \geq k \right\} \qquad \text{for each } k \in \mathbb{N},$$

we define $D^{(a_n)} := \bigcap_{k\in\mathbb{N}} U_k^{(a_n)}$. Note that if $p \in D^{(a_n)}$ then $\sum a_{p(n)}$ cannot converge.

For each conditionally convergent series $\sum a_n$, and for each $k \in \mathbb{N}$, we will show that the set $U_k^{(a_n)}$ is open and dense in $\mathrm{Sym}(\mathbb{N})$. For every $p \in U_k^{(a_n)}$, by definition of $U_k^{(a_n)}$, there exists some $m \geq k$ such that $\sum_{n=0}^{m} a_{p(n)} \geq k$. Hence

$$p \in \Big(\{p(0)\} \times \cdots \times \{p(m)\} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \cdots\Big) \cap \mathrm{Sym}(\mathbb{N}) \subseteq U_k^{(a_n)}$$

where we view the function p as the infinite ordered list $(p(0), p(1), \dots)$. Recalling that the set

$$\Big(\{p(0)\} \times \cdots \times \{p(m)\} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \cdots\Big)$$

is open in $\mathbb{N}^{\mathbb{N}}$, we conclude that $U_k^{(a_n)}$ is open in $\mathrm{Sym}(\mathbb{N})$. So it remains to show that $U_k^{(a_n)}$ is dense in $\mathrm{Sym}(\mathbb{N})$. Fix any $p \in \mathrm{Sym}(\mathbb{N})$. We claim that for each $d \in \mathbb{N}$, there exists some $h \in U_k^{(a_n)}$ such that

$$h|_{\{0,\dots,d\}} = p|_{\{0,\dots,d\}}.$$

Indeed, we can construct such a bijection $h \colon \mathbb{N} \to \mathbb{N}$ as follows:

(1) For each $x \in \{0, \dots, d\}$, we set $h(x) := p(x)$.

(2) Choose a large enough, but finite, collection of distinct numbers $i_1, \dots, i_N$ from the set

$$\{d+1,\ d+2,\ d+3,\ \cdots\}$$

such that $\sum_{j=1}^{N} a_{p(i_j)} \geq k - \sum_{n=0}^{d} a_{p(n)}$.[14] Then we set

$$h(d+j) := p(i_j) \quad \text{for each } j \in \{1, \dots, N\}.$$

[14] This is always possible because $\sum a_n$ is conditionally convergent, and so the sum of all the positive terms must diverge to ∞.

(3) Choose any bijection $g\colon \mathbb{N} \setminus \{0, \ldots, d+N\} \to \mathbb{N} \setminus \{h(0), \ldots, h(d+N)\}$, and then set $h(x) \coloneqq g(x)$ for all $x \in \mathbb{N} \setminus \{0, \ldots, d+N\}$.

Step (1) in the construction above gives us $h|_{\{0,\ldots,d\}} = p|_{\{0,\ldots,d\}}$. Step (2) gives us the inequality $\sum_{n=0}^{d+N} a_{h(n)} \geq k$. Step (3) ensures h is surjective. Thus, for every open set $V \subseteq \mathrm{Sym}(\mathbb{N})$ with $p \in V$, there exists some $h \in U_k^{(a_n)}$ such that $h \in V$. Hence $U_k^{(a_n)}$ is dense in $\mathrm{Sym}(\mathbb{N})$.

Therefore $D^{(a_n)} = \bigcap_{k\in\mathbb{N}} U_k^{(a_n)}$ is a countable intersection of open and dense sets in $\mathrm{Sym}(\mathbb{N})$. In particular, the set $\mathrm{Sym}(\mathbb{N}) \setminus D^{(a_n)} = \bigcup_{k\in\mathbb{N}} \left(\mathrm{Sym}(\mathbb{N}) \setminus U_k^{(a_n)}\right)$ is meagre in $\mathrm{Sym}(\mathbb{N})$.

Finally, let $\mathcal{C} \subseteq \mathrm{Sym}(\mathbb{N})$ be non-meagre set with $|\mathcal{C}| = \mathbf{non}(\mathcal{M})$. Then for every conditionally convergent series $\sum a_n$, there exists some $p \in \mathcal{C}$ such that $p \in D^{(a_n)}$. This is because if $\mathcal{C} \subseteq \mathrm{Sym}(\mathbb{N}) \setminus D^{(a_n)}$ then $\mathcal{C}$ would be meagre. So, by the construction of $D^{(a_n)}$, the rearranged series $\sum a_{p(n)}$ cannot converge. Therefore $\mathcal{C}$ is a rearrangement set, and we thus conclude that $\mathfrak{rr} \leq |\mathcal{C}| = \mathbf{non}(\mathcal{M})$. □

We draw the rearrangement number with Cichoń's diagram in Figure 7.

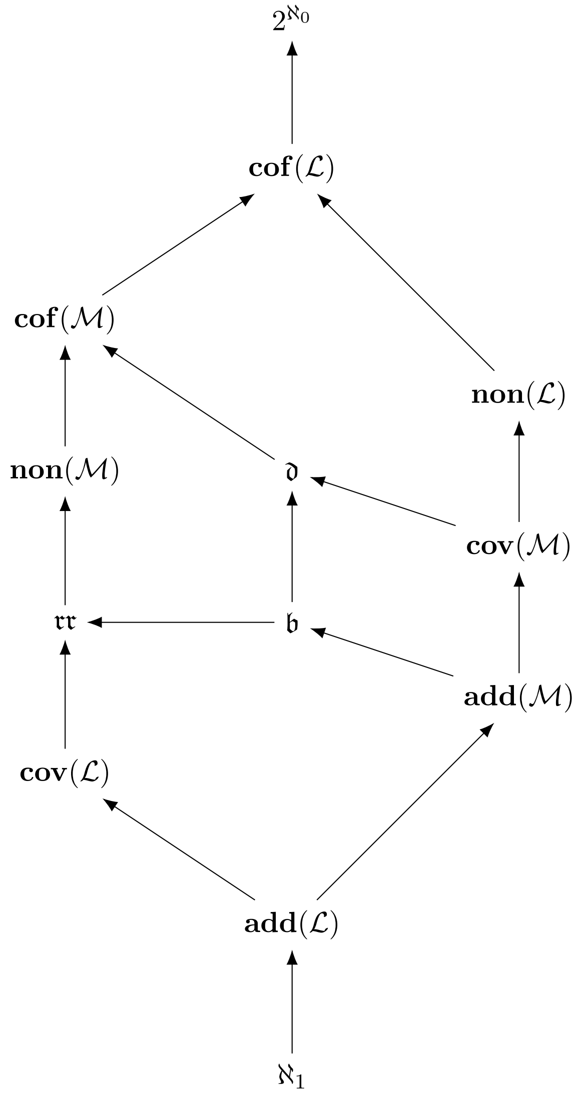

Figure 7: Cichoń's Diagram with the Rearrangement Number.

4.3 Open Questions

It is an open question whether $\mathfrak{rr} < \mathbf{non}(\mathcal{M})$ is consistent with ZFC [3, Question 54]. In a personal correspondence with me over email, Will Brian, one of the co-authors of *The Rearrangement Number* [3], wrote the following:

> We wanted to show the consistency of $\mathfrak{rr} < \mathbf{non}(\mathcal{M})$. The reason we wanted this, more or less, is that $\mathbf{non}(\mathcal{M})$ is the best candidate amongst the Cichoń diagram cardinals for being equal to $\mathfrak{rr}$, but we suspect they're not equal. . . . The main obstacle here is just that it's tricky to get models for $\mathbf{cov}(\mathcal{L}),\ \mathfrak{b} < \mathbf{non}(\mathcal{M})$ (and this would be necessary, since $\mathbf{cov}(\mathcal{L}),\ \mathfrak{b} \leq \mathfrak{rr}$), and the known ways of getting such models don't seem super flexible.
>
> — Will Brian, October 2023.

Furthermore, it is not known how $\mathfrak{rr}$ behaves in relation to other cardinals in Cichoń's diagram [3, Question 55], apart from the immediate inequalities obtained from Figure 7.

Another two cardinal characteristics of the continuum, which we have not discussed, are the *splitting number* $\mathfrak{s}$ and the *subseries number* ß.

Definition 4.3.1 (Splitting Number, [5, Definition 6])**.** The *splitting number* $\mathfrak{s}$ is the smallest cardinality of any collection $\mathcal{C} \subseteq \mathscr{P}(\mathbb{N})$ such that, for all infinite $B \subseteq \mathbb{N}$ there exists some $A \in \mathcal{C}$ such that both $A \cap B$ and $(\mathbb{N} \setminus A) \cap B$ are infinite.

Definition 4.3.2 (Subseries Number, [5, Definition 1])**.** The *subseries number* ß is the smallest cardinality of any collection $\mathcal{C} \subseteq \mathscr{P}(\mathbb{N})$ such that, for all conditionally convergent series $\sum_{n\in\mathbb{N}} a_n$ there exists some $A \in \mathcal{C}$ such that the subseries $\sum_{n\in A} a_n$ diverges.

The following are among the facts that were proven in the paper *The Subseries Number* [5]:

- $\mathfrak{s} \leq$ ß, [5, Theorem 7],
- $\mathbf{cov}(\mathcal{L}) \leq$ ß $\leq \mathbf{non}(\mathcal{M})$, [5, Theorem 11, Theorem 13],
- $\mathfrak{rr} \leq \max\{\mathfrak{b},$ ß$\}$, [5, Theorem 19],
- ß $< \mathfrak{b} = \mathfrak{rr}$ is consistent with ZFC, [5, Section 9].

It is an open question whether the inequalities $\mathfrak{rr} <$ ß and $\mathfrak{rr} < \max\{\mathfrak{b},$ ß$\}$ are consistent with ZFC [5, Question 20]. In the same email where the quote above appeared, Will Brian also wrote the following about the cardinals $\mathfrak{rr}$, $\mathfrak{s}$, and ß:

> We wanted to prove the consistency of $\mathfrak{rr} < \mathfrak{s}$. This problem arose after investigating the subseries number in a follow-up paper to the one about rearrangements. In this paper, one of our main theorems was proving that ß can be $< \mathfrak{rr}$. This happens in the Laver model. The hard part is showing that ß $= \aleph_1$ in the Laver model, but it's easy to see that $\mathfrak{rr} = 2^{\aleph_0}$, because we know that $\mathfrak{b} \leq \mathfrak{rr}$ and that $\mathfrak{b} = 2^{\aleph_0}$ in the Laver model. We were hoping for a "dual" sort of proof where we could show the consistency of $\mathfrak{rr} < \mathfrak{s}$ and then conclude (because $\mathfrak{s}$ is a lower bound for ß) that $\mathfrak{rr}$ can be less than ß. But we were never able to find a way of proving $\mathfrak{rr} < \mathfrak{s}$.
>
> — Will Brian, October 2023, adapted.

Acknowledgements

I would like to thank my lecturers and supervisors Dr Adam Epstein, Dr András Máthé, and Dr Josephine Evans at the University of Warwick for providing me with the foundational material and resources this essay is built on. This essay would not have been possible without their support.

This paragraph is dedicated to thanking my undergraduate peers at the University of Warwick. I thank Thomas King and Kyle Thompson for helping me understand the topologies of the spaces $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$. I thank Matthew Helm and Matthew Phillips for our many chats about the bizarre properties of measure-zero sets and meagre sets, solidifying my understanding of these objects. I thank Kit Liu for helping me with LaTeX and TikZ. I thank Alex Davies and Edward Taylor for assisting me in the preparation of the Bibliography and References section. Finally, I thanks both Kit Liu and Aris Mercier for spotting grammatical errors throughout this essay.

I also thank the co-authors of *The Rearrangement Number* [3] for corresponding with me over email about research updates on $\mathfrak{rr}$. I would especially like to thank Professor Will Brian for giving me permission quote excerpts from our email correspondence in Section 4.3.

Finally, I would like to thank all the anonymous people on the internet who engaged with me, providing me with more resources or giving me mathematical insights for this essay.

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