Semantics for Linear Logic

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a work in progress...

Contents

1 An introduction to linear logic

When one gives up the law of excluded middle, one creates the space for a number of strange phenomena to arise. Given any formula φ, we know that $\varphi \to \neg\neg\varphi$ holds intuitionistically. That is, it can be proven in a usual syntactic system for propositional (or first-order) logic without the use of the law of excluded middle. The converse $\neg\neg\varphi \to \varphi$, however, is not intuitionistically valid; one must appeal to the law of excluded middle or one of its equivalents to establish double negation elimination. As another example, given formulas φ and ψ, the following three de Morgan's laws

1. $(\neg\varphi \wedge \neg\psi) \to \neg(\varphi \vee \psi)$
2. $(\neg\varphi \vee \neg\psi) \to \neg(\varphi \wedge \psi)$
3. $\neg(\varphi \vee \psi) \to (\neg\varphi \wedge \neg\psi)$

can all be proven intuitionistically. However, the remaining de Morgan's law

4. $\neg(\varphi \wedge \psi) \to (\neg\varphi \vee \neg\psi)$

is not intuitionistically valid. Furthermore, the formula $(\neg\varphi \vee \psi) \to (\varphi \to \psi)$ is intuitionistically valid, while the formula $(\varphi \to \psi) \to (\neg\varphi \vee \psi)$ is not.

Must we settle for these asymmetries if we wish to have intuitionistic aspects in our logic?

1.1 Connectives

We will mainly concern ourselves with *propositional* linear logic, for there is already a stark enough difference in the propositional fragments of linear logic in contrast to intuitionistic or classical logic.

Definition 1.1.1
Let P be a countable set of propositional constants. The set $\mathcal{F}$ of formulas of propositional linear logic is generated by the grammar

$$\mathcal{F} ::= P \mid P^{\perp} \mid 0 \mid \top \mid \perp \mid 1 \mid \mathcal{F} \,\&\, \mathcal{F} \mid \mathcal{F} \oplus \mathcal{F} \mid \mathcal{F} \otimes \mathcal{F} \mid \mathcal{F} \parr \mathcal{F} \mid ?\mathcal{F} \mid !\mathcal{F}.$$

These constants and connectives are split into three different classifications.

The <u>additive</u> constants consist of 0 and $\top$, with 0 representing the additive falsity and $\top$ representing the additive truth. The additive conjunction is $\&$ and is called <u>with</u>. The additive disjunction is $\oplus$ and is called <u>plus</u>.

The <u>multiplicative</u> constants consist of $\perp$ and 1, with $\perp$ representing the multiplicative falsity and 1 representing the multiplicative truth. The symbol $\otimes$ is called <u>tensor</u>, and is to be interpreted as multiplicative conjunction. The symbol $\parr$ is called <u>par</u>, and is to be interpreted as multiplicative disjunction.

Finally, the symbols ? and ! make up the <u>exponential</u> class of connectives. We will later see that these allow us to have an interaction between the additive and the multiplicative connectives. Much like with real numbers, where we have the exponential law $a^{b+c} = a^b \cdot a^c$ (whenever $a \neq 0$), we will obtain, for instance, an equivalence between $?(A \,\&\, B)$ and $(?A) \otimes (?B)$ for any formulas A and B.

We remark that negation of formulas was only defined on the propositional constants. Our syntax for linear logic cannot make sense of $(p \,\&\, q)^{\perp}$ whenever p and q are propositional constants. Instead, we define negation as a meta-operation on formulas.

Definition 1.1.2
We define the meta-operation $(-)^{\perp}$ on formulas inductively as follows.

1. $(p)^\perp := p^\perp$ *for all propositional constants* p.
2. $(p^\perp)^\perp := p$ *for all propositional constants* p.
3. $\perp^\perp := 1$.
4. $1^\perp := \perp$.
5. $0^\perp := \top$.
6. $\top^\perp := 0$.
7. *If* A *and* B *are formulas of linear logic, then* $(A \mathbin{\&} B)^\perp := A^\perp \oplus B^\perp$.
8. *If* A *and* B *are formulas of linear logic, then* $(A \oplus B)^\perp := A^\perp \mathbin{\&} B^\perp$.
9. *If* A *and* B *are formulas of linear logic, then* $(A \otimes B)^\perp := A^\perp \parr B^\perp$.
10. *If* A *and* B *are formulas of linear logic, then* $(A \parr B)^\perp := A^\perp \otimes B^\perp$.
11. *If* A *is a formula of linear logic, then* $(?A)^\perp := \;!(A^\perp)$.
12. *If* A *is a formula of linear logic, then* $(!A)^\perp := ?(A^\perp)$.

We also define $A \multimap B := A^\perp \parr B$ *for any two formulas* A *and* B.

The symbol $\multimap$ is called <u>linear implication</u> or <u>lollipop</u>.

Exercise 1.1.3
Show that $(A^\perp)^\perp = A$ *for all formulas* A.

1.2 Inference rules

We will adopt a sequent calculus approach for proofs in linear logic. Capital Latin letters $A, B, C, \dots$ will be used to denote individual formulas, whereas capital Greek letters $\Gamma, \Delta, \Theta, \dots$ will be used to denote (possibly empty) sets of formulas. Those familiar with the sequent calculus would expect two rules for each connective, for example

$$\frac{\Gamma, A \vdash C, \Delta}{\Gamma, A \mathbin{\&} B \vdash C, \Delta}\,\&L \qquad\qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \mathbin{\&} B, \Delta}\,\&R$$

Notice, however, that there are *six* connectives and *four* propositional constants, making for an expected number of *twenty* inference rules if we had a left and right rule for each connective and constant. While manageable, this is far too many inference rules for pedagogical purposes. Instead, we shall identify the statement $A \vdash B$ with $\vdash A^\perp, B$. In general, $\Gamma \vdash \Delta$ is identified with $\vdash \Gamma^\perp, \Delta$, where $\Gamma^\perp := \{\, A^\perp : A \in \Gamma \,\}$, recalling that $(-)^\perp$ is a meta-operation. This makes all our contexts empty, and we only need a single rule for each connective and constant for the introduction of that symbol into the right-hand side of the turnstile. Henceforth, when we write

$$\frac{}{\Gamma}$$

we mean that we can prove Γ from the empty context, i.e.

$$\frac{}{\vdash \Gamma}$$

Let us start with the inference rules. The first is the identity rule, inspired from the rule $A \vdash A$ from the sequent calculus.

$$\frac{}{A, A^{\perp}}\ \text{id}$$

We will also adopt the exchange rule, so that the order of the formulas on either side of the turnstile does not matter.

$$\frac{\Gamma, A, B, \Delta}{\Gamma, B, A, \Delta}\ \text{ex}$$

We will often use this rule implicitly, for the sake of space.

The (infamous) cut rule will also be adopted.

$$\frac{\Gamma, A \qquad \Delta, A^{\perp}}{\Gamma, \Delta}\ \text{cut}$$

It can be shown that the calculus we are developing will allow for the omission of the cut rule.

We now move on to the additive rules. The additive conjunction rule is reminiscent of the $\wedge R$ rule from the LK sequent calculus.

$$\frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \mathbin{\&} B}\ \&$$

The additive disjunction rules are reminiscent of the $\vee R$ rules from the LK sequent calculus.

$$\frac{\Gamma, A}{\Gamma, A \oplus B}\ \oplus_0 \qquad\qquad \frac{\Gamma, A}{\Gamma, A \oplus B}\ \oplus_1$$

We will often not distinguish between the $\oplus_0$ and the $\oplus_1$ rule when writing down a proof involving the additive disjunction rules.

The additive truth rule is inspired from the $\top R$ rule from the LK sequent calculus.

$$\frac{}{\Gamma, \top}\ \top$$

In contrast, much like the LK sequent calculus having no $\perp R$ rule, we will have no inference rule for the additive falsity 0. This does *not* mean that the constant 0 can never appear in proofs.

We will have a rule for each multiplicative symbol. The multiplicative conjunction rule is

$$\frac{\Gamma, A \qquad \Delta, B}{\Gamma, \Delta, A \otimes B}\ \otimes$$

Note the difference between the multiplicative conjunction rule and the additive conjunction rule. Intuitively, the inference $\frac{\Gamma^{\perp}, A \qquad \Delta^{\perp}, B}{\Gamma^{\perp}, \Delta^{\perp}, A \otimes B}\ \otimes$ says that from $\Gamma \vdash A$ and $\Delta \vdash B$, we can conclude $\Gamma, \Delta \vdash A \otimes B$. It is perhaps most instructive to observe the case when $\Gamma = \Delta$. We would have $\frac{\Gamma^{\perp}, A \qquad \Gamma^{\perp}, B}{\Gamma^{\perp}, \Gamma^{\perp}, A \otimes B}\ \otimes$, whereas $\frac{\Gamma^{\perp}, A \qquad \Gamma^{\perp}, B}{\Gamma^{\perp}, A \mathbin{\&} B}\ \&$. That is, if $\Gamma \vdash A$ and $\Gamma \vdash B$, we would need two copies of Γ to prove $A \otimes B$, whereas we would only need one copy of Γ to prove $A \mathbin{\&} B$.

The multiplicative disjunction rule is

$$\frac{\Gamma, A, B}{\Gamma, A \parr B}\ \parr$$

Observe that, because $\vdash A, A^\perp$ for any formula A using the identity rule, the multiplicative disjunction rule gives a law of excluded middle $\vdash A \parr A^\perp$.

The multiplicative truth rule only allows us to instantiat 1 from no other premises.

$$\frac{}{1}\,1$$

Unlike the additive false 0, we *do* have a multiplicative falsity rule.

$$\frac{\Gamma}{\Gamma, \bot}\,\bot$$

We now move on to the exponential rules. Note that, so far, we have had no other structural rules other than the exchange rule; we do not have the weakening rule or the contraction rule for arbitrary formulas like in the LK sequent calculus. We will only have these structural rules for formulas with the question mark as their main connective.

$$\frac{\Gamma}{\Gamma, ?A}\,\text{wk} \qquad \frac{\Gamma, ?A, ?A}{\Gamma, ?A}\,\text{ctr}$$

We have two more rules associated with exponentials, which are rather unique to linear logic. The first is the dereliction rule, whose choice of name can be justified upon seeing the inference rule.

$$\frac{\Gamma, A}{\Gamma, ?A}\,\text{der}$$

Finally, we have the promotion rule for introducing the ! symbol.

$$\frac{?\Gamma, A}{?\Gamma, !A}\,\text{prom}$$

In the above inference rule, $?\Gamma \coloneqq \{\, ?B \ : \ B \in \Gamma \,\}$. We can only promote a formula A to $!A$ when every other formula on the same derivation line has ? as their main connective.

We collect all our rules below for ease of access.

$$\frac{}{A, A^\perp}\,\text{id} \qquad \frac{\Gamma, A, B, \Delta}{\Gamma, B, A, \Delta}\,\text{ex} \qquad \frac{\Gamma, A \qquad \Delta, A^\perp}{\Gamma, \Delta}\,\text{cut}$$

$$\frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \mathbin{\&} B}\,\& \qquad \frac{\Gamma, A}{\Gamma, A \oplus B}\,\oplus_0 \qquad \frac{\Gamma, A}{\Gamma, A \oplus B}\,\oplus_1 \qquad \frac{}{\Gamma, \top}\,\top$$

$$\frac{\Gamma, A \qquad \Delta, B}{\Gamma, \Delta, A \otimes B}\,\otimes \qquad \frac{\Gamma, A, B}{\Gamma, A \parr B}\,\parr \qquad \frac{}{1}\,1 \qquad \frac{\Gamma}{\Gamma, \bot}\,\bot$$

$$\frac{\Gamma}{\Gamma, ?A}\,\text{wk} \qquad \frac{\Gamma, ?A, ?A}{\Gamma, ?A}\,\text{ctr} \qquad \frac{\Gamma, A}{\Gamma, ?A}\,\text{der} \qquad \frac{?\Gamma, A}{?\Gamma, !A}\,\text{prom}$$

We should see a few examples of derivations using these rules.

Example 1.2.1
Let us prove the modus ponens rule: $A \otimes (A \multimap B) \vdash B$.

To show this, first recall that $A \multimap B$ means $A^\perp \parr B$. So we have to show that $A \otimes (A^\perp \parr B) \vdash B$. Recalling that we only have a one-sided sequent calculus and that we interpret $\Gamma \vdash C$ as $\vdash \Gamma^\perp, C$, we have to show that $\vdash \big(A \otimes (A^\perp \parr B)\big)^\perp, B$. Applying the $(-)^\perp$ operation repeatedly, this is asking us to show that $\vdash A^\perp \parr (A \otimes B^\perp), B$. Now that everything is in the syntactic language, we proceed with the proof. We will show the uses of the exchange rules for this example and this example only.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A, A^\perp}\,\text{id}}{A^\perp, A}\,\text{ex} \qquad \dfrac{}{B, B^\perp}\,\text{id}}{A^\perp, B, A \otimes B^\perp}\,\otimes}{B, A^\perp, A \otimes B^\perp}\,\text{ex}}{B, A^\perp \parr (A \otimes B^\perp)}\,\parr}{A^\perp \parr (A \otimes B^\perp), B}\,\text{ex}$$

This completes the proof. □

Exercise 1.2.2
Show that $\vdash A \multimap A$.

Example 1.2.3
Let us prove that $(A \mathbin{\&} B)^\perp \vdash A^\perp \oplus B^\perp$. This is one of de Morgan's laws.

We need to show that $\vdash A \mathbin{\&} B, A^\perp, B^\perp$.

$$\dfrac{\dfrac{\dfrac{}{A, A^\perp}\,\text{id}}{A, A^\perp \oplus B^\perp}\,\oplus_1 \qquad \dfrac{\dfrac{}{B, B^\perp}\,\text{id}}{B, A^\perp \oplus B^\perp}\,\oplus_0}{A \mathbin{\&} B, A^\perp \oplus B^\perp}\,\&$$

In the proof above, we distinguished between the $\oplus_0$ rule and the $\oplus_1$ rule. We will not do this anymore and simply write $\oplus$ for either rule from now onwards.

Let us now prove the converse direction to the de Morgan's law above: $A^\perp \oplus B^\perp \vdash (A \mathbin{\&} B)^\perp$. This time, we need to show that $\vdash (A^\perp \oplus B^\perp)^\perp, (A \mathbin{\&} B)^\perp$, or equivalently, $\vdash A \mathbin{\&} B, A^\perp \oplus B^\perp$. But this is precisely the de Morgan's law we have just established; we get this direction for free. □

If A and B are formulas $A \vdash B$ and $B \vdash A$, then we write $A \dashv\vdash B$ and say that A and B are linearly equivalent. The de Morgan's law established in Example 1.2.3 can thus be succinctly written as the linear equivalence $(A \mathbin{\&} B)^\perp \dashv\vdash A^\perp \oplus B^\perp$.

Exercise 1.2.4
Show the remaining de Morgan's laws:

1. $(A \oplus B)^\perp \dashv\vdash A^\perp \mathbin{\&} B^\perp$,
2. $(A \otimes B)^\perp \dashv\vdash A^\perp \parr B^\perp$, *and*
3. $(A \parr B)^\perp \dashv\vdash A^\perp \otimes B^\perp$.

Example 1.2.5
Let us prove that $!(A \& B) \vdash\ !A \otimes !B$, i.e. the exponential ! connective turns additive conjunction to multiplicative conjunction.

We need to show that $\vdash \big(!(A \& B)\big)^{\perp}, !A \otimes !B$. Fleshing this out, our goal is $\vdash ?(A^{\perp} \oplus B^{\perp}), !A \otimes !B$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A, A^{\perp}}\,\text{id}}{A, A^{\perp} \oplus B^{\perp}}\,\oplus}{A, ?(A^{\perp} \oplus B^{\perp})}\,\text{der}}{!A, ?(A^{\perp} \oplus B^{\perp})}\,\text{prom} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{B, B^{\perp}}\,\text{id}}{B, A^{\perp} \oplus B^{\perp}}\,\oplus}{B, ?(A^{\perp} \oplus B^{\perp})}\,\text{der}}{!B, ?(A^{\perp} \oplus B^{\perp})}\,\text{prom}}{\dfrac{?(A^{\perp} \oplus B^{\perp}), ?(A^{\perp} \oplus B^{\perp}), !A \otimes !B}{?(A^{\perp} \oplus B^{\perp}), !A \otimes !B}\,\text{ctr}}\,\otimes
$$

This completes the proof. $\square$

Exercise 1.2.6
Show that $!(A \& B) \dashv\vdash\ !A \otimes !B$. *Also show that* $?(A \oplus B) \dashv\vdash\ ?A \parr ?B$.

Exercise 1.2.7
Show that $1 \dashv\vdash\ !\top$ *and that* $\perp\ \dashv\vdash\ ?0$.

Exercise 1.2.8
Show that $A^{\perp} \dashv\vdash A \multimap \perp$.

2 Seely categories

Everyone loves a bit of category theory.

2.1 Category-theoretic preliminaries: symmetric monoidal closed categories

A symmetric monoidal category is precisely what its name suggests: we equip a category with a bifunctor $(-) \otimes (-)$ which gives it the structure of a symmetric monoid. This will be the natural categorical generalisation of equipping a set with a binary operation which turns it into a symmetric monoid. We will, in a categorical sense, require that $\otimes$ is associative, symmetric, and has a unit.

Definition 2.1.1
Equip a category $\mathcal{C}$ with

1. *a functor $(-) \otimes (-) \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the tensor product,*
2. *an object $\mathbf{1} \in \operatorname{ob} \mathcal{C}$, called the unit object,*
3. *a natural isomorphism $a \colon ((-) \otimes (-)) \otimes (-) \;\to\; (-) \otimes ((-) \otimes (-))$, called the associator,*
4. *a natural isomorphism $\lambda \colon \mathbf{1} \otimes (-) \;\to\; \mathrm{id}_{\mathcal{C}}$, called the left unitor,*
5. *a natural isomorphism $\rho \colon (-) \otimes \mathbf{1} \;\to\; \mathrm{id}_{\mathcal{C}}$, called the right unitor,*
6. *and a natural isomorphism $\sigma \colon (-)\otimes(-) \;\to\; (-)\otimes(-)$, called the braiding/symmetry of the tensor product.*

This category $\mathcal{C}$ is said to be a symmetric monoidal category if, for all $x, y, z, w \in \operatorname{ob} \mathcal{C}$, the following four diagrams in $\mathcal{C}$ commute.

1.
$$
\begin{tikzcd}
 & ((x \otimes y) \otimes z) \otimes w \arrow[dl, "a_{x,y,z} \otimes \mathrm{id}_w"', "\cong"] \arrow[dr, "a_{x \otimes y,z,w}", "\cong"'] & \\
(x \otimes (y \otimes z)) \otimes w \arrow[d, "a_{x,y \otimes z,w}"', "\cong"] & & (x \otimes y) \otimes (z \otimes w) \arrow[d, "\cong"', "a_{x,y,z \otimes w}"] \\
x \otimes ((y \otimes z) \otimes w) \arrow[rr, "\cong", "\mathrm{id}_x \otimes a_{y,z,w}"'] & & x \otimes (y \otimes (z \otimes w))
\end{tikzcd}
$$

2.
$$
\begin{tikzcd}
(x \otimes \mathbf{1}) \otimes y \arrow[rr, "a_{x,\mathbf{1},y}", "\cong"'] \arrow[dr, "\cong", "\rho_x \otimes \mathrm{id}_y"'] & & x \otimes (\mathbf{1} \otimes y) \arrow[dl, "\cong"', "\mathrm{id}_x \otimes \lambda_y"] \\
 & x \otimes y &
\end{tikzcd}
$$

3.

$$\begin{tikzcd}[column sep=small]
& (x \otimes y) \otimes z \arrow[dl, "\sigma_{x,y} \otimes \mathrm{id}_z"', "\cong"] \arrow[dr, "a_{x,y,z}", "\cong"'] & \\
(y \otimes x) \otimes z \arrow[d, "a_{y,x,z}"', "\cong"] & & x \otimes (y \otimes z) \arrow[d, "\sigma_{x,y\otimes z}", "\cong"'] \\
y \otimes (x \otimes z) \arrow[dr, "\mathrm{id}_y \otimes \sigma_{x,z}"', "\cong"] & & (y \otimes z) \otimes x \arrow[dl, "a_{y,z,x}", "\cong"'] \\
& y \otimes (z \otimes x) &
\end{tikzcd}$$

4.

$$\begin{tikzcd}
x \otimes y \arrow[r, "\sigma_{x,y}", "\cong"'] \arrow[dr, "\mathrm{id}_{x\otimes y}"', "\cong"] & y \otimes x \arrow[d, "\sigma_{y,x}", "\cong"'] \\
& x \otimes y
\end{tikzcd}$$

The assertions that the first three diagrams commute for all $x, y, z, w \in \operatorname{ob} \mathcal{C}$ are respectively called the pentagon identity, the triangle identity, and the hexagon identity.

In practice, one simply specifies the triple $(\mathcal{C}, \otimes, \mathbf{1})$ when equipping a category $\mathcal{C}$ with a symmetric monoidal structure.

We will soon become very familiar with the category **Rel** of relations. The objects of **Rel** are sets, and morphisms $A \xrightarrow{R} B$ in **Rel** are precisely subsets $R \subseteq A \times B$. It will turn out that this category **Rel** can be equipped with a lot more than just a symmetric monoidal structure, but let us establish this fact as a first stepping stone.

Example 2.1.2
Let **Rel** be the category of sets and relations between sets. Let us show that $(\mathbf{Rel}, \times, \{*\})$ is a symmetric monoidal category, where $\times$ denotes the usual cartesian product and $\{*\}$ denotes a singleton set.

We will only define the associator $a\colon ((-) \times (-)) \times (-) \to (-) \times ((-) \times (-))$ and the left unitor $\lambda\colon \{*\} \times (-) \to \mathrm{id}_{\mathbf{Rel}}$, as the rest are similar.

For $A, B, C \in \operatorname{ob} \mathbf{Rel}$, define the morphism $(A \times B) \times C \xrightarrow{a_{A,B,C}} A \times (B \times C)$ to be the relation

$$a_{A,B,C} \coloneqq \Big\{ \big(((x,y),z),\ (x,(y,z))\big) \ :\ x \in A, y \in B, z \in C \Big\}.$$

For $A \in \operatorname{ob} \mathbf{Rel}$, define the morphism $\{*\} \times A \xrightarrow{\lambda_A} A$ to be the relation

$$\lambda_A \coloneqq \Big\{ \big((*,x),\ x\big) \ :\ x \in A \Big\}.$$

The verification that these, along with the appropriate definitions for the right unitor and the symmetry, are natural isomorphisms satisfying the commuting diagrams in Definition 2.1.1 is straightforward. ☐

Exercise 2.1.3
Show that the category **Set** *with the usual cartesian product and any singleton set $\{*\}$ is a symmetric monoidal category.*

Exercise 2.1.4
Fix any field K and consider the category $\mathbf{Vect}_K$ of vector spaces over K. Let $(-) \otimes_K (-) \colon \mathbf{Vect}_K \times \mathbf{Vect}_K \to \mathbf{Vect}_K$ denote the usual tensor product over K of vector spaces over K. Also recall that we can view the field K as a vector space over itself. Show that $(\mathbf{Vect}_K, \otimes_K, K)$ is a symmetric monoidal category.

Just as we can define a homomorphism between monoids, there is a canonical notion of a morphism between symmetric monoidal categories: a symmetric monoidal functor. This is simply a functor between symmetric monoidal categories which preserves the symmetric monoidal structure of the domain category.

Definition 2.1.5
Let $(\mathcal{C}, \otimes, \mathbf{1})$ and $(\mathcal{D}, \oplus, \mathbf{0})$ be symmetric monoidal categories. A symmetric monoidal functor from $(\mathcal{C}, \otimes, \mathbf{1})$ to $(\mathcal{D}, \oplus, \mathbf{0})$ consists of a functor $F \colon \mathcal{C} \to \mathcal{D}$, a natural transformation $\mu \colon F(-) \oplus F(-) \to F((-) \otimes (-))$, and a morphism $\mathbf{0} \xrightarrow{\mu} F\mathbf{1}$ making the following four diagrams in $\mathcal{C}$ commute for all $x, y, z \in \operatorname{ob} \mathcal{C}$.

1.

$$\begin{tikzcd}
 & (Fx \oplus Fy) \oplus Fz \arrow[dl, "\mu_{x,y} \oplus \mathrm{id}_{Fz}"'] \arrow[dr, "a^{\mathcal{D}}_{Fx,Fy,Fz}", "\cong"'] & \\
F(x \otimes y) \oplus Fz \arrow[d, "\mu_{x \otimes y, z}"'] & & Fx \oplus (Fy \oplus Fz) \arrow[d, "\mathrm{id}_{Fx} \oplus \mu_{y,z}"] \\
F((x \otimes y) \otimes z) \arrow[dr, "\cong", "Fa^{\mathcal{C}}_{x,y,z}"'] & & Fx \oplus F(y \otimes z) \arrow[dl, "\mu_{x, y \otimes z}"] \\
 & F(x \otimes (y \otimes z)) &
\end{tikzcd}$$

where $a^{\mathcal{C}}$ and $a^{\mathcal{D}}$ are the associators for $(\mathcal{C}, \otimes, \mathbf{1})$ and $(\mathcal{D}, \oplus, \mathbf{0})$ respectively.

2.

$$\begin{tikzcd}
\mathbf{0} \oplus Fx \arrow[r, "\lambda^{\mathcal{D}}_{Fx}", "\cong"'] \arrow[d, "\mu \oplus \mathrm{id}_{Fx}"'] & Fx \\
F\mathbf{1} \oplus Fx \arrow[r, "\mu_{\mathbf{1},x}"'] & F(\mathbf{1} \otimes x) \arrow[u, "\cong", "F\lambda^{\mathcal{C}}_x"']
\end{tikzcd}$$

where $\lambda^{\mathcal{C}}$ and $\lambda^{\mathcal{D}}$ are the left unitors for $(\mathcal{C}, \otimes, \mathbf{1})$ and $(\mathcal{D}, \oplus, \mathbf{0})$ respectively.

3.

$$\begin{tikzcd}
Fx \oplus \mathbf{0} \arrow[r, "\rho^{\mathcal{D}}_{Fx}", "\cong"'] \arrow[d, "\mathrm{id}_{Fx} \oplus \mu"'] & Fx \\
Fx \oplus F\mathbf{1} \arrow[r, "\mu_{x,\mathbf{1}}"'] & F(x \otimes \mathbf{1}) \arrow[u, "\cong", "F\rho^{\mathcal{C}}_x"']
\end{tikzcd}$$

where $\rho^{\mathcal{C}}$ and $\rho^{\mathcal{D}}$ are the right unitors for $(\mathcal{C}, \otimes, \mathbf{1})$ and $(\mathcal{D}, \oplus, \mathbf{0})$ respectively.

4.

$$\begin{tikzcd} Fx \oplus Fy \arrow[r, "\sigma^{\mathcal{D}}_{Fx,Fy}", "\cong"'] \arrow[d, "\mu_{x,y}"'] & Fy \oplus Fx \arrow[d, "\mu_{y,x}"] \\ F(x \otimes y) \arrow[r, "\cong", "F\sigma^{\mathcal{C}}_{x,y}"'] & F(y \otimes x) \end{tikzcd}$$

where $\sigma^{\mathcal{C}}$ and $\sigma^{\mathcal{D}}$ are the symmetries for $(\mathcal{C}, \otimes, \mathbf{1})$ and $(\mathcal{D}, \oplus, \mathbf{0})$ respectively.

Note the abuse of notation by using the same letter μ for both the morphism $\mathbf{0} \xrightarrow{\mu} \mathbf{1}$ and the natural transformation $\mu\colon F(-) \oplus F(-) \;\to\; F((-) \otimes (-))$.

Recall that a <u>cartesian closed category</u> is a category $\mathcal{C}$ with all finite products such that the functor $(-) \times y\colon \mathcal{C} \to \mathcal{C}$ has a right adjoint $(-)^y\colon \mathcal{C} \to \mathcal{C}$. The category **Set** is cartesian closed, as we can perform currying to identify functions $f\colon A \times B \to C$ with functions $\bar{f}\colon A \to B^C$, where B^C denotes the set of all functions from C to B. We remark that this object B^C is precisely the set $\hom_{\mathbf{Set}}(C, B)$; for any two $B, C \in \operatorname{ob}\mathbf{Set}$, the category **Set** has another object B^C which acts as the collection $\hom_{\mathbf{Set}}(C, B)$ of morphisms from C to B.

The notion of cartesian closedness is now adapted for a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$, with $\otimes$ in place of $\times$.

Definition 2.1.6
A <u>symmetric monoidal closed category</u> is a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ equipped with a functor $(-) \multimap (-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$, called the <u>internal-hom functor</u>, such that, for each $y \in \operatorname{ob}\mathcal{C}$, the functor $(-) \otimes y\colon \mathcal{C} \to \mathcal{C}$ is left adjoint to the functor $y \multimap (-)\colon \mathcal{C} \to \mathcal{C}$.

Example 2.1.7
We know (from Example 2.1.2) that **Rel** can be given the structure of a symmetric monoidal category. Let us now equip it with an internal-hom functor to make it a symmetric monoidal closed category.

For $A, B, C \in \operatorname{ob}\mathbf{Rel}$, recall that a morphism $A \times B \xrightarrow{R} C$ in **Rel** is simply a subset $R \subseteq (A \times B) \times C$. Hence we have an isomorphism

$$\begin{array}{ccc} \hom_{\mathbf{Rel}}(A \times B, C) & \xrightarrow{\quad\cong\quad} & \hom_{\mathbf{Rel}}(A, B \times C) \\ \uplus & & \uplus \\ R & \longmapsto & \Big\{ \big(a, (b, c)\big) \;:\; \big((a, b), c\big) \in R \Big\} \end{array}$$

So we can take $B \multimap C$ to be $B \times C$. The verification that this isomorphism is natural in A and C is routine work. Following on from Example 2.1.2, this equips **Rel** with the structure of a symmetric monoidal closed category. □

Exercise 2.1.8
Let $\mathcal{C}$ be a cartesian closed category and let 1 denote the terminal object of $\mathcal{C}$. Show that $\Big(\mathcal{C}, \times, 1, (-)^{(-)}\Big)$ is a symmetric monoidal closed category.

Exercise 2.1.9
Fix any field K and consider the category $\mathbf{fdVect}_K$ of finite-dimensional vector spaces over K. Recall also that the field K itself can be viewed as a one-dimensional vector space over K. Let $\otimes_K$ denote the usual tensor product over K of vector spaces over K. Let $[-,-]\colon \mathbf{fdVect}_K^{\mathrm{op}} \times \mathbf{fdVect}_K \to \mathbf{fdVect}_K$ be

the functor such that $[V, W]$ *is the* K*-vector space of all linear maps from* V *to* W*, whenever* $V, W \in \operatorname{ob}\mathbf{fdVect}_K$.

Show that $\mathbf{fdVect}_K$*, together with* $\otimes_K$*,* K*, and* $[-,-]$*, is a symmetric monoidal closed category, eventhough* $\mathbf{fdVect}_K$ *is not cartesian closed.*

Definition 2.1.10
A *∗-autonomous category* *is a symmetric monoidal closed category* $(\mathcal{C}, \otimes, \mathbf{1}, \multimap)$ *equipped with a* *dualising object* $\perp \in \operatorname{ob}\mathcal{C}$ *such that, for all* $x \in \operatorname{ob}\mathcal{C}$*, the canonical morphism* $x \xrightarrow{\partial_x} (x \multimap \perp) \multimap \perp$ *below is an isomorphism,*

$$\begin{tikzcd}
\mathrm{id}_{x\multimap\perp} \arrow[drr, mapsto] & \in & \hom_{\mathcal{C}}(x \multimap \perp, x \multimap \perp) \arrow[r, "(-)\otimes x \,\dashv\, x\multimap(-)", "\cong"'] & \hom_{\mathcal{C}}((x \multimap \perp) \otimes x, \perp) \arrow[d, "(-)\circ\sigma_{x,x\multimap\perp}", "\cong"'] \\
& & & \hom_{\mathcal{C}}(x \otimes (x \multimap \perp), \perp) \arrow[d, "(-)\otimes x \,\dashv\, x\multimap(-)", "\cong"'] \\
& & \partial_x \quad \in & \hom_{\mathcal{C}}(x, (x \multimap \perp) \multimap \perp)
\end{tikzcd}$$

where $\sigma : (-) \otimes (-) \to (-) \otimes (-)$ *is the symmetry of* $(\mathcal{C}, \otimes, \mathbf{1})$*. In this case, we write* $x^* \coloneqq x \multimap \perp$*, giving rise to the* *dualisation functor* $(-)^*\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$*.*

Exercise 2.1.11
Let $(\mathcal{C}, \otimes, \mathbf{1}, \multimap, \perp)$ *be a ∗-autonomous category. For* $x \in \operatorname{ob}\mathcal{C}$*, let* $x \xrightarrow{\partial_x} x^{**}$ *denote the canonical morphism as defined in Definition 2.1.10. Show that the dualisation functor* $(-)^*\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ *is an equivalence of categories. Furthermore, show that* $\partial\colon \mathrm{id}_{\mathcal{C}} \to (-)^{**}$ *is a natural isomorphism satisfying* *#??*

Example 2.1.12
We know (from Example 2.1.7) that **Rel** can be given the structure of a symmetric monoidal closed category. Let us now show that it can be given the structure of a ∗-autonomous category.

Following on from Example 2.1.7, we take any singleton set $\{*\}$ as the dualising object. The diagram chase in Definition 2.1.10 then yields that the canonical morphism $A \xrightarrow{\partial_A} (A \times \{*\}) \times \{*\}$ is simply the relation

$$\partial_A = \Big\{ \big(x, ((x, *), *)\big) \ : \ x \in A \Big\},$$

which is clearly an isomorphism for any $A \in \operatorname{ob}\mathbf{Rel}$. □

Exercise 2.1.13
Fix a field K*. Following on Exercise 2.1.9, show that the field* K *viewed as a one-dimensional vector space over itself is a dualising object in* $\mathbf{fdVect}_K$*, making* $\mathbf{fdVect}_K$ *a ∗-autonomous category.*

2.2 Category-theoretic preliminaries: comonads and coKleisli categories

Definition 2.2.1
Fix a category $\mathcal{C}$*. A* *comonad* *on* $\mathcal{C}$ *is consists of a functor* $T\colon \mathcal{C} \to \mathcal{C}$ *and natural transformations*

$\varepsilon\colon T \to \mathrm{id}_{\mathcal{C}}$ and $\delta\colon T \to TT$ making the following diagrams

$$\begin{tikzcd}
T & TT \arrow[l, "\varepsilon_T"'] \arrow[r, "T\varepsilon"] & T \\
& T \arrow[ul, "\mathrm{id}_{\mathcal{C}}"] \arrow[u, "\delta"] \arrow[ur, "\mathrm{id}_{\mathcal{C}}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "\delta"] \arrow[d, "\delta"'] & TT \arrow[d, "\delta_T"] \\
TT \arrow[r, "T\delta"'] & TTT
\end{tikzcd}$$

in the functor category $[\mathcal{C}, \mathcal{C}]$ commute.

The following Exercise 2.2.2 shows that any adjoint pair of functors give rise to a comonad.

Exercise 2.2.2
Given an adjunction $\begin{tikzcd} \mathcal{C} \arrow[r, shift left=1.5, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left=1.5, "G"] \end{tikzcd}$ with unit $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$ and counit $\varepsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$, show that the triple $(FG, \varepsilon, F\eta_G)$ constitute a comonad on $\mathcal{D}$.

Definition 2.2.3
The <u>coKleisli category</u> for a comonad (T, ε, δ) on a category $\mathcal{C}$ is the category $\mathcal{C}_T$ consisting of the following.

1. *We declare $\mathrm{ob}\,\mathcal{C}_T \coloneqq \mathrm{ob}\,\mathcal{C}$.*
2. *For each $A, B \in \mathrm{ob}\,\mathcal{C}_T$, we declare $\hom_{\mathcal{C}_T}(A, B) \coloneqq \hom_{\mathcal{C}}(TA, B)$.*
3. *Given two morphisms $A \overset{f}{\rightsquigarrow} B \overset{g}{\rightsquigarrow} C$ in $\mathcal{C}_T$, the composite morphism $A \overset{gf}{\rightsquigarrow} C$ in $\mathcal{C}_T$ is defined to be the composite morphism*

$$TA \xrightarrow{\ \delta_A\ } TTA \xrightarrow{\ Tf\ } TB \xrightarrow{\ g\ } C$$

in $\mathcal{C}$.

4. *For each $A \in \mathrm{ob}\,\mathcal{C}_T$, the identity morphism $\begin{tikzcd} A \arrow[loop right, squiggly, "\mathrm{id}_A"] \end{tikzcd}$ in $\mathcal{C}_T$ is the morphism $TA \xrightarrow{\varepsilon_A} A$ in $\mathcal{C}$.*

The use of squiggly arrows $A \overset{f}{\rightsquigarrow} B$ for morphisms in the coKleisli category are for cosmetic reasons; they are there to remind us that when we draw an arrow $A \overset{f}{\rightsquigarrow} B$ in the coKleisli category, we really mean an arrow $A \xrightarrow{f} TB$ in the base category.

If we were really strict about notation, we should write $\mathcal{C}_{\mathbb{T}}$ for the coKleisli category for a comonad $\mathbb{T} = (T, \varepsilon, \delta)$ on a category $\mathcal{C}$; it could be the case that two different comonads have the same underlying endofunctor.

The coKleisi category is one of those rare instances where the fact that it is a category is not actually that obvious.

Exercise 2.2.4
Let (T, ε, δ) be a comonad on a category $\mathcal{C}$, and let $\mathcal{C}_T$ denote the coKleisli category for this comonad. Show that $\mathcal{C}_T$ is indeed a category, i.e. show that composition in $\mathcal{C}_T$ (as defined in Definition 2.2.3) is associative and that composing any morphism $A \overset{f}{\rightsquigarrow} B$ in $\mathcal{C}_T$ with either of the identity morphisms

$\begin{tikzcd} A \arrow[loop right, squiggly, "\mathrm{id}_A"] \end{tikzcd}$ or $\begin{tikzcd} B \arrow[loop right, squiggly, "\mathrm{id}_B"] \end{tikzcd}$ in $\mathcal{C}_T$ yield f.

Recall (from Exercise 2.2.2) that any adjunction gives rise to a comonad. The following Exercise 2.2.5 shows that any comonad stems from an adjunction in sense of Exercise 2.2.2.

Exercise 2.2.5
Let (T, ε, δ) be a comonad on a category $\mathcal{C}$, and let $\mathcal{C}_T$ denote the coKleisli category for this comonad. Find an adjoint pair of functors $\mathcal{C}_T \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}$ *with unit $\tilde{\eta}$ and counit $\tilde{\varepsilon}$ such that $(FG, \tilde{\varepsilon}, F\tilde{\eta}_G) = (T, \varepsilon, \delta)$.*

2.3 Modelling linear logic with Seely categories

Definition 2.3.1
A Seely category is a $*$-autonomous category $(\mathbb{L}, \otimes, \mathbf{1}, \multimap, \bot)$ which has all finite products with terminal object $\top \in \operatorname{ob}\mathbb{L}$, equipped with a comonad $(!, \varepsilon, \delta)$ on $\mathbb{L}$, an isomorphism $\mathbf{1} \underset{\cong}{\xrightarrow{\mu}} !\top$, and a natural isomorphism $\mu\colon !(-) \otimes !(-) \to !((-) \times (-))$ such that all of the following hold.

1. *The triple $(\mathbb{L}, \times, \top)$ is a symmetric monoidal category, where $\times$ denotes the categorical product in $\mathbb{L}$.*
2. *The pair $(!, \mu)$ is a symmetric monoidal functor from $(\mathbb{L}, \times, \top)$ to $(\mathbb{L}, \otimes, \mathbf{1})$.*
3. *For each $x, y \in \operatorname{ob}\mathbb{L}$, the following diagram*

$$\begin{tikzcd}
!x \otimes !y \arrow[r, "\mu_{x,y}", "\cong"'] \arrow[dd, "\delta_x \otimes \delta_y"'] & !(x \times y) \arrow[d, "\delta_{x \times y}"] \\
 & !!(x \times y) \arrow[d, "!(!\pi_1, !\pi_2)"] \\
!!x \otimes !!y \arrow[r, "\cong", "\mu_{!x,!y}"'] & !(!x \times !y)
\end{tikzcd}$$

in $\mathbb{L}$ commutes, where $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$ are the projections associated with the product $A \times B$.

We note the abuse of notation by using the same letter μ to denote the isomorphism $\mathbf{1} \underset{\cong}{\xrightarrow{\mu}} !\top$ as well as the natural isomorphism $\mu\colon !(-) \otimes !(-) \to !((-) \times (-))$. These isomorphisms denoted by μ are known as the Seely isomorphisms.

We should see an example of a Seely category. We have been working with the category **Rel** of sets and relations so far, making it into a $*$-autonomous category (cf. Example 2.1.12). We will now upgrade **Rel** to a Seely category. The endofunctor of the comonad will be the finite multiset functor, which we define below.

A finite multiset on a set S is a function $m\colon S \to \mathbb{N}$ such that $m(m^{-1}(\mathbb{N} \setminus \{0\}))$ is a finite subset of $\mathbb{N}$. We write

$$m = \Big[\underbrace{x_1, \ldots, x_1}_{n_1 \text{ copies of } x_1}, \ldots, \underbrace{x_k, \ldots, x_k}_{n_k \text{ copies of } x_k}\Big]$$

for the finite multiset m on a set $S \supseteq \{x_1, \ldots, x_k\}$ with $m(x_i) = n_i$ for all $1 \leq i \leq k$. In this case, we abuse notation and say that $x_1, \ldots, x_k \in m$ and that $|m| = n_1 + \cdots + n_k$.

For instance, if $S = \{x, y, z\}$, we write $m = [x, x, x, y]$ to denote the finite multiset $m\colon S \to Y$ where $m(x) = 3$, $m(y) = 1$, and $m(z) = 0$, and we say that $x, y \in m$, $z \notin m$, and $|m| = 4$. We do not distinguish between permutations of elements in a finite multiset m, so $m = [x, x, x, y] = [x, x, y, x] = [x, y, x, x] = [y, x, x, x]$. The notation $[\,]$ refers to the empty finite mutliset, i.e. the multiset which sends all elements of S to 0.

Given finite multisets m and m' on a set S, we define $m + m'$ to be the finite multiset on S given by $(m + m')(x) \coloneqq m(x) + m'(x)$ for all $x \in S$. For example, if $m = [x, x, x, y]$ and $m' = [x, y, z]$, then $m + m' = [x, x, x, x, y, y, z]$.

We use $\mathcal{M}_{\text{fin}}(S)$ to denote the set of all finite multisets on a set S. Defining

$$\begin{aligned}
&\mathcal{M}_{\text{fin}}(A \xrightarrow{R} B) \\
&\coloneqq \Big\{ (m, m') \in \mathcal{M}_{\text{fin}}(A) \times \mathcal{M}_{\text{fin}}(B) \ : |m| = |m'| \text{ and there exist permutations } m = [x_1, \dots x_k] \\
&\qquad\qquad \text{and } m' = [x'_1, \dots, x'_k] \text{ such that } x_i R x'_i \text{ for all } 1 \leq i \leq k \Big\}
\end{aligned}$$

for relations $R \subseteq A \times B$ makes $\mathcal{M}_{\text{fin}}$ into an endofunctor on **Rel**.

Example 2.3.2

We know (from Example 2.1.12) that **Rel** can be given the structure of a $*$-autonomous category. Let us now show that **Rel** can be given the structure of a Seely category.

We will take $! \coloneqq \mathcal{M}_{\text{fin}}$ for the endofunctor of our comonad. For a set A define the morphism $\mathcal{M}_{\text{fin}}(A) \xrightarrow{\varepsilon_A} A$ in **Rel** to be the relation

$$\varepsilon_A \coloneqq \big\{ ([x], x) \ : \ x \in A \big\}.$$

Furthermore, we define the morphism $\mathcal{M}_{\text{fin}}(A) \xrightarrow{\delta_A} \mathcal{M}_{\text{fin}}(\mathcal{M}_{\text{fin}}(A))$ in **Rel** to be the relation

$$\delta_A \coloneqq \Big\{ (m, [m_1, \dots, m_k]) \in \mathcal{M}_{\text{fin}}(A) \times \mathcal{M}_{\text{fin}}\big(\mathcal{M}_{\text{fin}}(A)\big) \ : \ m = m_1 + \dots + m_k \Big\}.$$

These make $\varepsilon\colon \mathcal{M}_{\text{fin}} \to \mathrm{id}_{\mathbf{Rel}}$ and $\delta\colon \mathcal{M}_{\text{fin}} \to \mathcal{M}_{\text{fin}}\mathcal{M}_{\text{fin}}$ into natural transformations, and it is elementary to check that the diagrams

$$\begin{tikzcd}
{[x_1, \dots, x_k]} & & & {[[x_1, \dots, x_k]]} \arrow[lll] \\
& \mathcal{M}_{\text{fin}}(A) \arrow[ul, phantom, "\in"] & \mathcal{M}_{\text{fin}}\mathcal{M}_{\text{fin}}(A) \arrow[l, "\varepsilon_{\mathcal{M}_{\text{fin}}(A)}"'] \arrow[ur, phantom, "\ni"] & \\
& & \mathcal{M}_{\text{fin}}(A) \arrow[ul, "\mathrm{id}_{\mathcal{M}_{\text{fin}}(A)}"] \arrow[u, "\delta_A"] & \\
& & & {[x_1, \dots, x_k]} \arrow[uuu, mapsto] \arrow[ul, phantom, "\ni"]
\end{tikzcd}$$

$$
\begin{tikzcd}
{[[x_1],\dots,[x_k]]} \arrow[rrr] & & & {[x_1,\dots,x_k]} \\
& \mathcal{M}_{\mathrm{fin}}\mathcal{M}_{\mathrm{fin}}(A) \arrow[r, "\mathcal{M}_{\mathrm{fin}}\varepsilon_A"] \arrow[ul, phantom, "\in"] & \mathcal{M}_{\mathrm{fin}}(A) \arrow[ur, phantom, "\ni"] & \\
& \mathcal{M}_{\mathrm{fin}}(A) \arrow[u, "\delta_A"] \arrow[ur, "\mathrm{id}_{\mathcal{M}_{\mathrm{fin}}(A)}"'] & & \\
{[x_1,\dots,x_k]} \arrow[uuu] \arrow[ur, phantom, "\in"] & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\sum_{i=1}^{k}(\sum_{j=1}^{n_i} m_{i,j}) \arrow[rrr] \arrow[ddd] & & & {[m_{1,1},\dots,m_{1,n_1},\dots,m_{k,1},\dots,m_{k,n_k}]} \arrow[ddd] \\
& \mathcal{M}_{\mathrm{fin}}(A) \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] \arrow[ul, phantom, "\in"] & \mathcal{M}_{\mathrm{fin}}\mathcal{M}_{\mathrm{fin}}(A) \arrow[d, "\delta_{\mathcal{M}_{\mathrm{fin}}(A)}"] \arrow[ur, phantom, "\ni"] & \\
& \mathcal{M}_{\mathrm{fin}}\mathcal{M}_{\mathrm{fin}}(A) \arrow[r, "\mathcal{M}_{\mathrm{fin}}\delta_A"'] \arrow[dl, phantom, "\in"] & \mathcal{M}_{\mathrm{fin}}\mathcal{M}_{\mathrm{fin}}\mathcal{M}_{\mathrm{fin}}(A) \arrow[dr, phantom, "\ni"] & \\
\left[\sum_{j=1}^{n_1} m_{1,j},\dots,\sum_{j=1}^{n_k} m_{k,j}\right] \arrow[rrr] & & & {[[m_{1,1},\dots,m_{1,n_1}],\dots,[m_{k,1},\dots,m_{k,n_k}]]}
\end{tikzcd}
$$

in **Rel** commute for all sets A, making $(\mathcal{M}_{\mathrm{fin}}, \varepsilon, \delta)$ a comonad on **Rel**.

Recall that the categorical product in **Rel** is the set-theoretic disjoint union $\sqcup$, so the terminal object in **Rel** is the empty set $\varnothing$. This makes $(\mathbf{Rel}, \sqcup, \varnothing)$ a symmetric monoidal category. Then there is an obvious isomorphism $\{*\} \xrightarrow[\cong]{\mu} \mathcal{M}_{\mathrm{fin}}(\varnothing)$ in **Rel**, namely $\mu \coloneqq \Big\{ \big(*, [\,]\big) \Big\}$. Also, for sets A and B which we may assume without loss of generality to be disjoint, we define the morphism $\mathcal{M}_{\mathrm{fin}}(A) \times \mathcal{M}_{\mathrm{fin}}(B) \xrightarrow{\mu_{A,B}} \mathcal{M}_{\mathrm{fin}}(A \sqcup B)$ in **Rel** to be the relation

$$\mu_{A,B} \coloneqq \Big\{ \big((m', m''), m\big) \in \big(\mathcal{M}_{\mathrm{fin}}(A) \times \mathcal{M}_{\mathrm{fin}}(B)\big) \times \mathcal{M}_{\mathrm{fin}}(A \sqcup B) \ : \ m' + m'' = m \Big\}.$$

Observe that this $\mu \colon \mathcal{M}_{\mathrm{fin}}(-) \times \mathcal{M}_{\mathrm{fin}}(-) \ \to \ \mathcal{M}_{\mathrm{fin}}((-) \sqcup (-))$ is a natural isomorphism since A and B are disjoint: any finite multiset on $A \sqcup B$ can be viewed as a finite multiset on A together with a finite multiset on B.

It is then easy to check that $(\mathcal{M}_{\mathrm{fin}}, \mu)$ is a symmetric monoidal functor from $(\mathbf{Rel}, \times, \{*\})$ to $(\mathbf{Rel}, \sqcup, \varnothing)$. Finally, for all sets A and B, which we may again assume without loss of generality to be disjoint, the diagram

$$\begin{tikzcd}
\mathcal{M}_{\text{fin}}(A) \times \mathcal{M}_{\text{fin}}(B) \arrow[rr, "\mu_{A,B}"] \arrow[dd, "\delta_A \times \delta_B"'] & & \mathcal{M}_{\text{fin}}(A \sqcup B) \arrow[d, "\delta_{A\sqcup B}"] \\
& & \mathcal{M}_{\text{fin}}\mathcal{M}_{\text{fin}}(A \sqcup B) \arrow[d, "\mathcal{M}_{\text{fin}}\big((\mathcal{M}_{\text{fin}}\pi_A)\cup(\mathcal{M}_{\text{fin}}\pi_B)\big)"] \\
\mathcal{M}_{\text{fin}}\big(\mathcal{M}_{\text{fin}}(A)\big) \times \mathcal{M}_{\text{fin}}\big(\mathcal{M}_{\text{fin}}(B)\big) \arrow[rr, "\mu_{\mathcal{M}_{\text{fin}}(A),\mathcal{M}_{\text{fin}}(B)}"'] & & \mathcal{M}_{\text{fin}}\big(\mathcal{M}_{\text{fin}}(A) \sqcup \mathcal{M}_{\text{fin}}(B)\big)
\end{tikzcd}$$

in **Rel** commutes, where $A \xleftarrow{\pi_A} A \sqcup B \xrightarrow{\pi_B} B$ are the projections associated with the categorical product $A \sqcup B$ in **Rel**, i.e. $\pi_A := \{\, (x, x) \ : \ x \in A \,\} \subseteq (A \sqcup B) \times A$ and $\pi_B := \{\, (y, y) \ : \ y \in B \,\} \subseteq (A \sqcup B) \times B$. The diagonal of this commuting square relates a pair $(m', m'') \in \mathcal{M}_{\text{fin}}(A) \times \mathcal{M}_{\text{fin}}(B)$ to (some permutation of) a multiset $[m_1, \ldots, m_k] \in \mathcal{M}_{\text{fin}}\big(\mathcal{M}_{\text{fin}}(A) \sqcup \mathcal{M}_{\text{fin}}(B)\big)$ if and only if there is some $i \in \{1, \ldots, k\}$ such that $m' = m_1 + \ldots m_i$ and $m'' = m_{i+1} + \cdots + m_k$.

All of the above makes **Rel** into a Seely category. $\square$

We now start to model linear logic with Seely categories. We observe that the structure of a Seely category already gives us canonical ways to define the interpretations of many connectives. The remaining connectives will be defined via their duality properties. For instance, we will interpret $A \oplus B$ as $(A^\perp \,\&\, B^\perp)^\perp$.

Definition 2.3.3

Fix a Seely category $\mathbb{L}$. We define the interpretation $[\![A]\!]$ of a formula A inductively as follows.

1. *For propositional constants p, we are free to choose an assignment $[\![p]\!] \in \operatorname{ob} \mathbb{L}$.*
2. *If $A \equiv 1$, then $[\![1]\!]$ is the unit of the tensor product in $\mathbb{L}$ from the $*$-autonomous structure of $\mathbb{L}$.*
3. *If $A \equiv \perp$, then $[\![\perp]\!]$ is the dualising object in $\mathbb{L}$ from the $*$-autonomous structure of $\mathbb{L}$.*
4. *If $A \equiv \top$, then $[\![\top]\!]$ is the terminal object in $\mathbb{L}$.*
5. *If $A \equiv B \multimap C$, then $[\![B \multimap C]\!] := [\![B]\!] \multimap [\![C]\!]$, where the $\multimap$ on the right-hand side is the internal-hom functor from the $*$-autonomous structure of $\mathbb{L}$.*
6. *If $A \equiv 0$, then $[\![0]\!] := [\![\top \multimap \perp]\!]$.*
7. *If $A \equiv p^\perp$ for some propositional constant p, then $[\![p^\perp]\!] := [\![p \multimap \perp]\!]$.*
8. *If $A \equiv B \,\&\, C$, then $[\![B \,\&\, C]\!] := [\![B]\!] \times [\![C]\!]$, where $\times$ is the categorical product in $\mathbb{L}$.*
9. *If $A \equiv B \oplus C$, then $[\![B \oplus C]\!] := [\![((B \multimap \perp) \,\&\, (C \multimap \perp)) \multimap \perp]\!]$.*
10. *If $A \equiv B \otimes C$, then $[\![B \otimes C]\!] := [\![B]\!] \otimes [\![C]\!]$, where the $\otimes$ on the right-hand side is the tensor product from the $*$-autonomous structure of $\mathbb{L}$.*
11. *If $A \equiv B \parr C$, then $[\![B \parr C]\!] := [\![((B \multimap \perp) \otimes (C \multimap \perp)) \multimap \perp]\!]$.*
12. *If $A \equiv {!B}$, then $[\![!B]\!] := ![\![B]\!]$, where the $!$ on the right-hand side is the endofunctor in the comonad on $\mathbb{L}$.*

13. If $A \equiv ?B$, then $[\![?A]\!] \coloneqq [\![(!(B \multimap \bot)) \multimap \bot]\!]$.

We chose to define dual connectives via the $(-) \multimap \bot$ operator. We should check that this agrees with the $(-)^\bot$ operator.

Lemma 2.3.4
Fix a Seely category $\mathbb{L}$ and a formula A. Then $[\![A^\bot]\!] \cong [\![A \multimap \bot]\!]$ in $\mathbb{L}$.

Proof. We induct on the complexity of A.

If $A \equiv p$ for some propositional constant p, then we have $[\![p^\bot]\!] = [\![p \multimap \bot]\!]$ by definition.

If $A \equiv 1$, then for any $x \in \operatorname{ob}\mathbb{L}$ we have

$$\hom_{\mathbb{L}}(x, [\![1^\bot]\!]) = \hom_{\mathbb{L}}(x, [\![\bot]\!]) \cong \hom_{\mathbb{L}}(x \otimes 1, [\![\bot]\!]) \cong \hom_{\mathbb{L}}(x, [\![1 \multimap \bot]\!]),$$

and so $[\![1^\bot]\!] = [\![\bot]\!] \cong [\![1 \multimap \bot]\!]$.

If $A \equiv \bot$, then for any $x \in \operatorname{ob}\mathbb{L}$ we have

$$\hom_{\mathbb{L}}(x, [\![\bot^\bot]\!]) = \hom_{\mathbb{L}}(x, [\![1]\!]) \cong \hom_{\mathbb{L}}(x, [\![(1 \multimap \bot) \multimap \bot]\!]) \cong \hom_{\mathbb{L}}(x, [\![\bot \multimap \bot]\!])$$

since $[\![1]\!] \cong [\![(1 \multimap \bot) \multimap \bot]\!]$ and $[\![(1 \multimap \bot)]\!] \cong [\![\bot]\!]$. Thus $[\![1^\bot]\!] = [\![\bot]\!] \cong [\![1 \multimap \bot]\!]$.

If $A \equiv \top$, then $[\![\top^\bot]\!] = [\![0]\!] = [\![\top \multimap \bot]\!]$ by definition.

If $A \equiv 0$, then $[\![0^\bot]\!] = [\![\top]\!] \cong [\![(\top \multimap \bot) \multimap \bot]\!] = [\![0 \multimap \bot]\!]$ since $\top \cong (\top \multimap \bot) \multimap \bot$.

If $A \equiv B \mathbin{\&} C$, then

$$\begin{aligned}
[\![(B \mathbin{\&} C)^\bot]\!] &= [\![B^\bot \oplus C^\bot]\!] \\
&= [\![((B^\bot) \multimap \bot) \mathbin{\&} (C^\bot \multimap \bot)) \multimap \bot]\!] \\
&\cong [\![(B \mathbin{\&} C) \multimap \bot]\!]
\end{aligned}$$

where, in the last line, we used the inductive hypothesis which asserts that $B^\bot \cong B \multimap \bot$ and $C^\bot \cong C \multimap \bot$.

If $A \equiv B \oplus C$, then

$$\begin{aligned}
[\![(B \oplus C)^\bot]\!] &= [\![B^\bot \mathbin{\&} C^\bot]\!] \\
&\cong [\![((B^\bot \mathbin{\&} C^\bot) \multimap \bot) \multimap \bot]\!], && \text{since } x \cong (x \multimap \bot) \multimap \bot \text{ for all } x \in \operatorname{ob}\mathbb{L}, \\
&\cong [\![(((B \multimap \bot) \mathbin{\&} (C \multimap \bot)) \multimap \bot) \multimap \bot]\!], && \text{by the inductive hypothesis,} \\
&= [\![(B \oplus C) \multimap \bot]\!].
\end{aligned}$$

If $A \equiv B \otimes C$, then

$$\begin{aligned}
[\![(B \otimes C)^\bot]\!] &= [\![B^\bot \parr C^\bot]\!] \\
&= [\![((B^\bot \multimap \bot) \otimes (C^\bot \multimap \bot)) \multimap \bot]\!] \\
&\cong [\![(B \otimes C) \multimap \bot]\!],
\end{aligned}$$

where we used the inductive hypothesis in the last line.

If $A \equiv B \parr C$, then

$$\begin{aligned}
[\![(B \parr C)^\bot]\!] &= [\![B^\bot \otimes C^\bot]\!] \\
&\cong [\![((B^\bot \otimes C^\bot) \multimap \bot) \multimap \bot]\!], && \text{since } x \cong (x \multimap \bot) \multimap \bot \text{ for all } x \in \operatorname{ob}\mathbb{L}, \\
&\cong [\![(((B \multimap \bot) \otimes (C \multimap \bot)) \multimap \bot) \multimap \bot]\!], && \text{by the inductive hypothesis,} \\
&= [\![(B \parr C) \multimap \bot]\!].
\end{aligned}$$

If $A \equiv\, !B$, then

$$\begin{aligned}
[\![(!B)^\perp]\!] &= [\![?B^\perp]\!] \\
&= [\![!(B^\perp \multimap \bot) \multimap \bot]\!] \\
&\cong [\![!B \multimap \bot]\!]
\end{aligned}$$

where we used the inductive hypothesis in the last line.

If $A \equiv\, ?B$, then

$$\begin{aligned}
[\![(?B)^\perp]\!] &= [\![!B^\perp]\!] \\
&\cong [\![(!B^\perp \multimap \bot) \multimap \bot]\!] \\
&\cong [\![(!(B \multimap \bot) \multimap \bot) \multimap \bot]\!] \\
&= [\![?B \multimap \bot]\!],
\end{aligned}$$

again using that $x \cong (x \multimap \bot) \multimap \bot$ for all $x \in \operatorname{ob}\mathbb{L}$ as well as the inductive hypothesis. □

For brevity and the ease of reading, we will introduce an abuse of notation. For a Seely category $\mathbb{L}$ and a formula A, we shall simply write A in place of $[\![A]\!]$ for the interpretation of A in $\mathbb{L}$. So the statement of the previous Lemma 2.3.4 will be expressed as follows: "Fix a Seely category $\mathbb{L}$ and a formula A. Then $A^\perp \cong A \multimap \bot$ in $\mathbb{L}$."

If Seely categories are to be any good as a model of linear logic, we should at least have a soundness theorem for them.

Exercise 2.3.5
For formulas $A_1, \ldots, A_k, B_1, \ldots, B_n$, show that

$$A_1, \ldots, A_k \vdash B_1, \ldots, B_n \quad \textit{if and only if} \quad 1 \vdash (A_1 \otimes \cdots \otimes A_k)^\perp \parr B_1 \parr \cdots \parr B_n.$$

Theorem 2.3.6 (Soundness)
Fix a Seely category $\mathbb{L}$. For a finite set of formulas $\{A_1, \ldots, A_n\}$, if $\vdash A_1, \ldots, A_n$ then there exists a morphism $1 \to A_1 \parr \cdots \parr A_n$ in $\mathbb{L}$.

Proof. We induct on the complexity of proofs.

Our first base case is the identity rule $\dfrac{}{A, A^\perp}\,\mathrm{id}$.

#?? □

3 Phase semantics

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