CHAPTER 7

Noise Performance of Various Modulation Schemes

7.1 Introduction

The process of (electronic) communication becomes quite challenging because of the unwanted electrical signals in a communications system. These undesirable signals, usually termed as *noise*, are random in nature and interfere with the message signals. The receiver input, in general, consists of (message) signal plus noise, possibly with comparable power levels. The purpose of the receiver is to produce the desired signal with a *signal-to-noise ratio* that is above a specified value.

In this chapter, we will analyze the noise performance of the modulation schemes discussed in chapters 4 to 6. The results of our analysis will show that, under certain conditions, FM is superior to the linear modulation schemes in combating noise and PCM can provide better signal-to-noise ratio at the receiver output than FM. The trade-offs involved in achieving the superior performance from FM and PCM will be discussed.

We shall begin our study with the noise performance of various CW modulations schemes. In this context, it is the performance of the detector (demodulator) that would be emphasized. We shall first develop a suitable receiver model in which the role of the demodulator is the most important one.

7.2 Receiver Model and Figure of Merit: Linear Modulation

7.2.1 Receiver model

Consider the superheterodyne receiver shown in Fig. 4.75. To study the noise performance we shall make use of simplified model shown in Fig. 7.1. Here, $H_{eq}(f)$ is the equivalent IF filter which actually represents the cascade filtering characteristic of the RF, mixer and IF sections of Fig. 4.75. $s(t)$ is the desired modulated carrier and $w(t)$ represents a sample function of the white Gaussian noise process with the two sided spectral density of $\frac{N_0}{2}$. We treat $H_{eq}(f)$ to be an ideal narrowband, bandpass filter, with a passband between $f_c - W$ to $f_c + W$ for the double sideband modulation schemes. For the case of SSB, we take the filter passband either between $f_c - W$ and f_c (LSB) or f_c and $f_c + W$ (USB). (The transmission bandwidth B_T is $2W$ for the double sideband modulation schemes whereas it is W for the case of SSB). Also, in the present context, f_c represents the carrier frequency measured at the mixer output; that is $f_c = f_{IF}$.

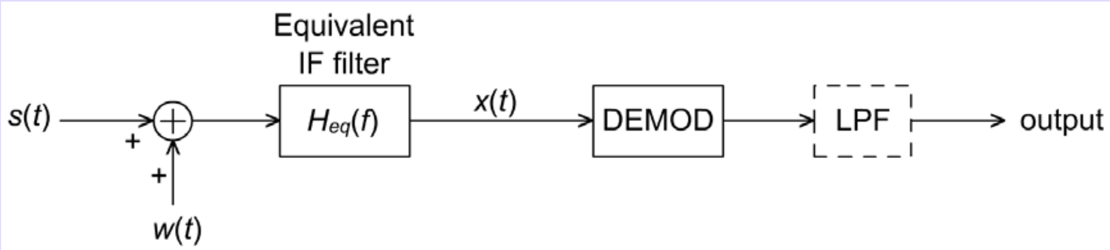


Fig. 7.1: Receiver model (linear modulation)

The input to the detector is $x(t) = s(t) + n(t)$, where $n(t)$ is the sample function of a bandlimited (narrowband) white noise process $N(t)$ with the power spectral density $S_N(f) = \frac{N_0}{2}$ over the passband of $H_{eq}(f)$. (As $H_{eq}(f)$ is treated as a narrowband filter, $n(t)$ represents the sample function of a narrowband noise process.)

7.2.2 Figure-of-merit

The performance of analog communication systems are measured in terms of Signal-to-Noise Ratio (SNR). The SNR measure is meaningful and unambiguous provided the signal and noise are additive at the point of measurement. We shall define two (SNR) quantities, namely, (i) $(SNR)_0$ and (ii) $(SNR)_r$.

The output signal-to-noise ratio is defined as,

$$(SNR)_0 = \frac{\text{Average power of the message at receiver output}}{\text{Average noise power at the receiver output}} \tag{7.1}$$

The reference signal-to-noise ratio is defined as,

$$(SNR)_r = \frac{\begin{pmatrix}\text{Average power of the modulated} \\ \text{message signal at receiver input}\end{pmatrix}}{\begin{pmatrix}\text{Average noise power in the message} \\ \text{bandwidth at receiver input}\end{pmatrix}} \tag{7.2}$$

The quantity, $(SNR)_r$ can be viewed as the output signal-to-noise ratio which results from baseband or direct transmission of the message without any modulation as shown in Fig. 7.2. Here, $m(t)$ is the baseband message signal with the same power as the modulated wave. For the purpose of comparing different modulation systems, we use the Figure-of-Merit (FOM) defined as,

$$FOM = \frac{(SNR)_0}{(SNR)_r} \tag{7.3}$$

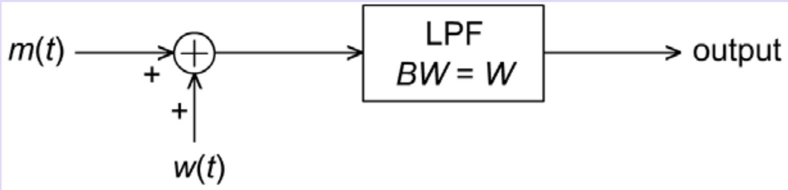


Fig. 7.2: Ideal Baseband Receiver

FOM as defined above provides a normalized $(SNR)_0$ performance of the various modulation-demodulation schemes; larger the value of FOM, better is the noise performance of the given communication system.

Before analyzing the SNR performance of various detectors, let us quantify the outputs expected of the (idealized) detectors when the input is a narrowband signal. Let $x(t)$ be a real narrowband bandpass signal. From Eq. 1.55, $x(t)$ can be expressed as

$$x(t) = \begin{cases} x_c(t)\cos(\omega_c t) - x_s(t)\sin(\omega_c t) & (7.4a) \\ A(t)\cos\left[\omega_c t + \varphi(t)\right] & (7.4b) \end{cases}$$

$x_c(t)$ and $x_s(t)$ are the in-phase and quadrature components of $x(t)$. The envelope $A(t)$ and the phase $\varphi(t)$ are given by Eq. 1.56. In this chapter, we will analyze the performance of a coherent detector, envelope detector, phase detector and a frequency detector when signals such as $x(t)$ are given as input. The outputs of the (idealized) detectors can be expressed mathematically in terms of the quantities involved in Eq. 7.4. These are listed below. (Table 7.1)

Table 7.1: Outputs from various detectors

	$x(t)$ is input to an ideal	Detector output proportional to
i)	Coherent detector	$x_c(t)$
ii)	Envelope detector	$A(t)$
iii)	Phase detector	$\varphi(t)$
iv)	Frequency detector	$\frac{1}{2\pi}\frac{d\varphi(t)}{dt}$

$x(t)$ could be used to represent any of the four types of linear modulated signals or any one of the two types of angle modulated signals. In fact, $x(t)$ could even represent (signal + noise) quantity, as will be seen in the sequel.

Table 7.2 gives the quantities $x_c(t)$, $x_s(t)$, $A(t)$ and $\varphi(t)$ for the linear and angle modulated signals of Chapter 4 and 5.

Table 7.2: Components of linear and angle modulated signals

	Signal	$x_c(t)$	$x_s(t)$	$A(t)$	$\varphi(t)$
1	DSB-SC $A_c\, m(t) \cos(\omega_c t)$	$A_c\, m(t)$	zero	$A_c \lvert m(t) \rvert$	$0,\ m(t) > 0$ $\pi,\ m(t) < 0$
2	DSB-LC (AM) $A_c [1 + g_m m(t)] \cos(\omega_c t)$, $A_c [1 + g_m m(t)] \geq 0$	$A_c [1 + g_m m(t)]$	zero	$A_c [1 + g_m m(t)]$	zero
3	SSB $\frac{A_c}{2} m(t) \cos(\omega_c t)$ $\pm \frac{A_c}{2} \widehat{m}(t) \sin(\omega_c t)$	$\frac{A_c\, m(t)}{2}$	$\pm \frac{A_c\, \hat{m}(t)}{2}$	$\frac{A_c}{2} \sqrt{m^2(t) + \widehat{m}^2(t)}$	$\tan^{-1}\left[-\frac{\hat{m}(t)}{m(t)}\right]$
4	Phase modulation $A_c \cos[\omega_c t + \varphi(t)]$, $\varphi(t) = k_p\, m(t)$	$A_c \cos\varphi(t)$	$A_c \sin\varphi(t)$	A_c	$k_p\, m(t)$
5	Frequency modulation $A_c \cos[\omega_c t + \varphi(t)]$, $\varphi(t) = 2\pi k_f \int_{-\infty}^{t} m(\tau)\, d\tau$	$A_c \cos\varphi(t)$	$A_c \sin\varphi(t)$	A_c	$2\pi k_f \int_{-\infty}^{t} m(\tau)\, d\tau$

Example 7.1

Let $s(t) = A_c \cos(\omega_m t) \cos(\omega_c t)$ where $f_m = 10^3$ Hz and $f_c = 10^6$ Hz. Let us compute and sketch the output $v(t)$ of an ideal frequency detector when $s(t)$ is its input.

From Table 7.1, we find that an ideal frequency detector output will be proportional to $\frac{1}{2\pi} \frac{d\varphi(t)}{dt}$. For the DSB-SC signal,

$$\varphi(t) = \begin{cases} 0\,, & m(t) > 0 \\ \pi\,, & m(t) < 0 \end{cases}$$

For the example, $m(t) = \cos\left[\left(2\pi \times 10^3\right)t\right]$. Hence $\varphi(t)$ is shown in Fig. 7.3(b).

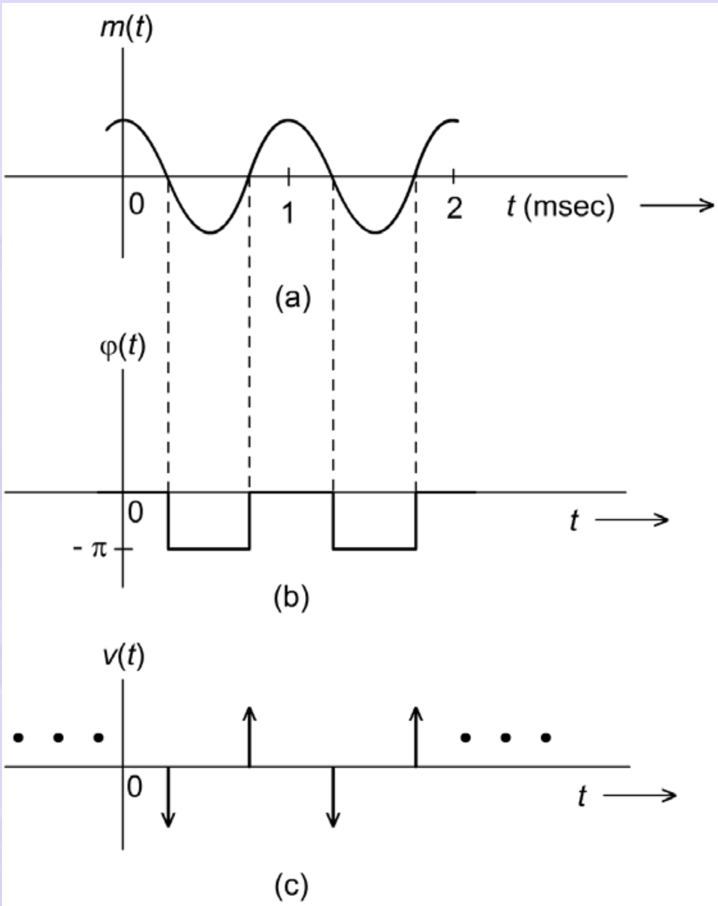


Fig. 7.3: (Ideal) frequency detector output of example 7.1

Differentiating the waveform in (b), we obtain $v(t)$, which consists of a sequence of impulses which alternate in polarity, as shown in (c). ♦

Example 7.2

Let $m(t) = \frac{1}{1 + t^2}$. Let $s(t)$ be an SSB signal with $m(t)$ as the message signal. Assuming that $s(t)$ is the input to an ideal ED, let us find the expression for its output $v(t)$.

From Example 1.24, we have

$$\widehat{m}(t) = \frac{1}{1 + t^2}$$

As the envelope of $s(t)$ is $\left\{\left[m(t)\right]^2 + \left[\widehat{m}(t)\right]^2\right\}^{\frac{1}{2}}$, we have

$$v(t) = \frac{1}{\sqrt{1 + t^2}}.$$ ♦

7.3 Coherent Demodulation

7.3.1 DSB-SC

The receiver model for coherent detection of DSB-SC signals is shown in Fig. 7.4. The DSB-SC signal is, $s(t) = A_c\, m(t) \cos(\omega_c t)$. We assume $m(t)$ to be sample function of a WSS process $M(t)$ with the power spectral density, $S_M(f)$, limited to $\pm W$ Hz.

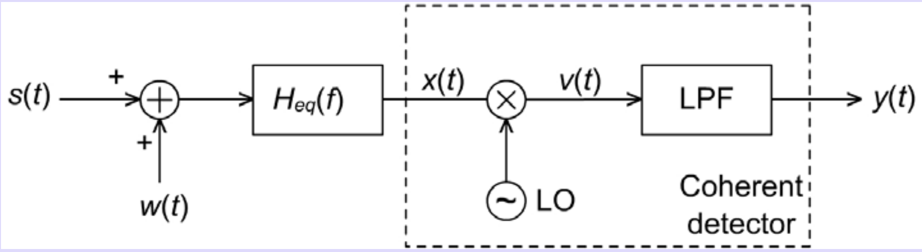


Fig. 7.4: Coherent Detection of DSB-SC.

The carrier, $A_c \cos(\omega_c t)$, which is independent of the message $m(t)$ is actually a sample function of the process $A_c \cos(\omega_c t + \Theta)$ where Θ is a random variable, uniformly distributed in the interval 0 to 2π. With the random phase added to the carrier term, $R_s(\tau)$, the autocorrelation function of the process $S(t)$ (of which $s(t)$ is a sample function), is given by,

$$R_s(\tau) = \frac{A_c^2}{2} R_M(\tau) \cos(\omega_c \tau) \tag{7.5a}$$

where $R_M(\tau)$ is the autocorrelation function of the message process. Fourier transform of $R_s(\tau)$ yields $S_s(f)$ given by,

$$S_s(f) = \frac{A_c^2}{4}\left[S_M(f - f_c) + S_M(f + f_c)\right] \tag{7.5b}$$

Let P_M denote the message power, where

$$P_M = \int_{-\infty}^{\infty} S_M(f)\, df = \int_{-W}^{W} S_M(f)\, df$$

Then, $\displaystyle\int_{-\infty}^{\infty} S_s(f)\, df = 2\,\frac{A_c^2}{4}\int_{f_c - W}^{f_c + W} S_M(f - f_c)\, df = \frac{A_c^2\, P_M}{2}$.

That is, the average power of the modulated signal $s(t)$ is $\frac{A_c^2\, P_M}{2}$. With the (two sided) noise power spectral density of $\frac{N_0}{2}$, the average noise power in the message bandwidth $2W$ is $2W \times \frac{N_0}{2} = W N_0$. Hence,

$$\left[(SNR)_r\right]_{DSB-SC} = \frac{A_c^2 P_M}{2 W N_0} \tag{7.6}$$

To arrive at the FOM, we require $(SNR)_0$. The input to the detector is $x(t) = s(t) + n(t)$, where $n(t)$ is a narrowband noise quantity. Expressing $n(t)$ in terms of its in-phase and quadrature components, we have

$$x(t) = A_c\, m(t) \cos(\omega_c t) + n_c(t) \cos(\omega_c t) - n_s \sin(\omega_c t)$$

Assuming that the local oscillator output is $\cos(\omega_c t)$, the output $v(t)$ of the multiplier in the detector (Fig. 7.4) is given by

$$v(t) = \frac{1}{2} A_c\, m(t) + \frac{1}{2} n_c(t) + \frac{1}{2}\left[A_c\, m(t) + n_c(t)\right] \cos(2\omega_c t) - \frac{1}{2} A_c\, n_s(t) \sin(2\omega_c t)$$

As the LPF rejects the spectral components centered around $2 f_c$, we have

$$y(t) = \frac{1}{2} A_c\, m(t) + \frac{1}{2} n_c(t) \qquad (7.7)$$

From Eq. 7.7, we observe that,

i) Signal and noise which are additive at the input to the detector are additive even at the output of the detector

ii) Coherent detector completely rejects the quadrature component $n_s(t)$.

iii) If the noise spectral density is flat at the detector input over the passband $(f_c - W, f_c + W)$, then it is flat over the baseband $(-W, W)$, at the detector output. (Note that $n_c(t)$ has a flat spectrum in the range $-W$ to W.)

As the message component at the output is $\left(\frac{1}{2}\right) A_c\, m(t)$, the average message power at the output is $\left(\frac{A_c^2}{4}\right) P_M$. As the spectral density of the in-phase noise component is N_0 for $|f| \leq W$, the average noise power at the receiver output is $\frac{1}{4}(2W \cdot N_0) = \frac{W N_0}{2}$. Therefore,

$$\left[(SNR)_0\right]_{DSB-SC} = \frac{\left(A_c^2 / 4\right) P_M}{(W N_0)/2}$$

$$= \frac{A_c^2 P_M}{2W N_0} \tag{7.8}$$

From Eq. 7.6 and 7.8, we obtain

$$[FOM]_{DSB-SC} = \frac{(SNR)_0}{(SNR)_r} = 1 \tag{7.9}$$

7.3.2 SSB

Assuming that LSB has been transmitted, we can write $s(t)$ as follows:

$$s(t) = \frac{A_c}{2} m(t) \cos(\omega_c t) + \frac{A_c}{2} \widehat{m}(t) \sin(\omega_c t)$$

where $\widehat{m}(t)$ is the Hilbert transform of $m(t)$. Generalizing,

$$S(t) = \frac{A_c}{2} M(t) \cos(\omega_c t) + \frac{A_c}{2} \widehat{M}(t) \sin(\omega_c t).$$

We can show that the autocorrelation function of $S(t)$, $R_s(\tau)$ is given by

$$R_s(\tau) = \frac{A_c^2}{4}\left[R_M(\tau)\cos(\omega_c \tau) + \widehat{R}_M(\tau)\sin(\omega_c \tau)\right]$$

where $\widehat{R}_M(t)$ is the Hilbert transform of $R_M(t)$. Hence the average signal power, $R_s(0) = \frac{A_c^2}{4} P_M$

and $$(SNR)_r = \frac{A_c^2 P_M}{4W N_0} \tag{7.10}$$

Let $n(t) = n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t)$

(Note that with respect to f_c , $n(t)$ does not have a locally symmetric spectrum).

$$y(t) = \frac{1}{4} A_c\, m(t) + \frac{1}{2} n_c(t)$$

Hence, the output signal power is $\frac{A_c^2 P_M}{16}$ and the output noise power as $\left(\frac{1}{4}\right) W N_0$. Thus, we obtain,

$$(SNR)_{0, SSB} = \frac{A_c^2 P_M}{16} \times \frac{4}{W N_0}$$

$$= \frac{A_c^2 P_M}{4 W N_0} \quad (7.11)$$

From Eq. 7.10 and 7.11,

$$(FOM)_{SSB} = 1 \quad (7.12)$$

From Eq. 7.9 and 7.12, we find that under synchronous detection, *SNR* performance of DSB-SC and SSB are identical, when both the systems operate with the same signal-to-noise ratio at the input of their detectors.

In arriving at the RHS of Eq. 7.11, we have used the narrowband noise description with respect to f_c. We can arrive at the same result, if the noise quantity is written with respect to the centre frequency $\left(f_c - \frac{W}{2}\right)$.

7.4 Envelope Detection

DSB-LC or AM signals are normally envelope detected, though coherent detection can also be used for message recovery. This is mainly because envelope detection is simpler to implement as compared to coherent detection. We shall now compute the $(FOM)_{AM}$.

The transmitted signal $s(t)$ is given by

$$s(t) = A_c \left[1 + g_m m(t)\right] \cos(\omega_c t)$$

Then the average signal power in $s(t) = \frac{A_c^2 \left[1 + g_m^2 P_M\right]}{2}$. Hence

$$(SNR)_{r,\, DSB-LC} = \frac{A_c^2 \left(1 + g_m^2 P_M\right)}{2 W N_0} \quad (7.13)$$

Using the in-phase and quadrature component description of the narrowband noise, the quantity at the envelope detector input, $x(t)$, can be written as

$$x(t) = s(t) + n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t)$$

$$= \left[A_c + A_c\, g_m\, m(t) + n_c(t)\right]\cos(\omega_c t) - n_s(t)\sin(\omega_c t) \qquad (7.14)$$

The various components of Eq. 7.14 are shown as phasors in Fig. 7.5. The receiver output $y(t)$ is the envelope of the input quantity $x(t)$. That is,

$$y(t) = \left\{\left[A_c + A_c\, g_m\, m(t) + n_c(t)\right]^2 + n_s^2(t)\right\}^{\frac{1}{2}}$$

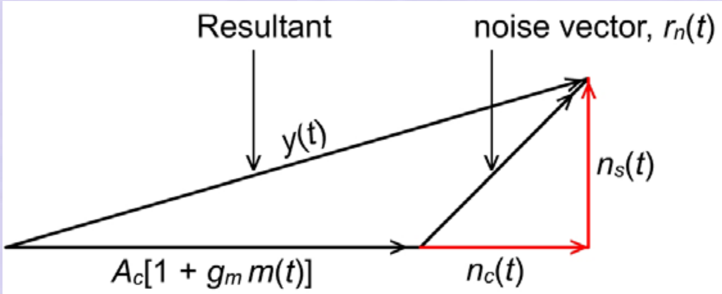


Fig. 7.5: Phasor diagram to analyze the envelope detector

We shall analyze the noise performance of envelope detector for two different cases, namely, (i) large *SNR* at the detector input and (ii) weak *SNR* at the detector input.

7.4.1 Large predetection *SNR*

Case (i): If the signal-to-noise ratio at the input to the detector is sufficiently large, we can approximate $y(t)$ as (see Fig. 7.5)

$$y(t) \approx A_c + A_c\, g_m\, m(t) + n_c(t) \qquad (7.15)$$

On the RHS of Eq. 7.15, there are three quantities: A DC term due to the transmitted carrier, a term proportional to the message and the in-phase noise component. In the final output, the DC is blocked. Hence the average signal power at the output is given by $A_c^2\, g_m^2\, P_M$. The output noise power being equal to $2W N_0$ we have,

$$\left[(SNR)_0\right]_{AM} \approx \frac{A_c^2\, g_m^2\, P_M}{2W N_0} \tag{7.16}$$

It is to be noted that the signal and noise are additive at the detector output and power spectral density of the output noise is flat over the message bandwidth. From Eq. 7.13 and 7.16 we obtain,

$$(FOM)_{AM} = \frac{g_m^2\, P_M}{1 + g_m^2\, P_m} \tag{7.17}$$

As can be seen from Eq. 7.17, the FOM with envelope detection is less than unity. That is, the noise performance of DSB-LC with envelope detection is inferior to that of DSB-SC with coherent detection. Assuming $m(t)$ to be a tone signal, $A_m \cos(\omega_m t)$ and $\mu = g_m A_m$, simple calculation shows that $(FOM)_{AM}$ is $\frac{\mu^2}{(2 + \mu^2)}$. With the maximum permitted value of $\mu = 1$, we find that the $(FOM)_{AM}$ is $\frac{1}{3}$. That is, other factors being equal, DSB-LC has to transmit three times as much power as DSB-SC, to achieve the same quality of noise performance. Of course, this is the price one has to pay for trying to achieve simplicity in demodulation.

7.4.2 Weak predetection *SNR*

In this case, noise term dominates. Let $n(t) = r_n(t)\cos\left[\omega_c t + \psi(t)\right]$. We now construct the phasor diagram using $r_n(t)$ as the reference phasor (Fig. 7.6). Envelope detector output can be approximated as

$$y(t) \approx r_n(t) + A_c \cos\left[\psi(t)\right] + A_c\, g_m\, m(t) \cos\left[\psi(t)\right] \tag{7.18}$$

From Eq. 7.18, we find that detector output has no term strictly proportional to $m(t)$. The last term on the RHS of Eq. 7.18 contains the message signal $m(t)$

multiplied by the noise quantity, $\cos \psi(t)$, which is random; that is, the message signal is hopelessly mutilated beyond any hope of signal recovery. Also, it is to be noted that signal and noise are no longer additive at the detector output. As such, $(SNR)_0$ is not meaningful.

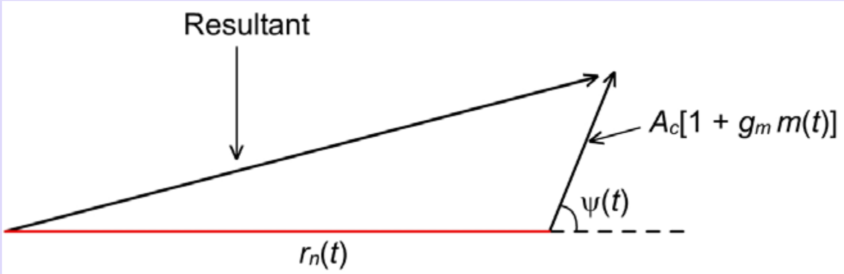


Fig. 7.6: Phase diagram to analyze the envelope detector for case (ii)

The mutilation or loss of message at low input SNR is called the **threshold effect**. That is, there is some value of input SNR, above which the envelope detector operates satisfactorily whereas if the input SNR falls below this value, performance of the detector deteriorates quite rapidly. Actually, threshold is not a unique point and we may have to use some reasonable criterion in arriving it. Let R denote the random variable obtained by observing the process $R(t)$ (of which $r(t)$ is a sample function) at some fixed point in time. It is quite reasonable to assume that the detector is operating well into the threshold region if $P[R \geq A_c] \geq 0.5$; where as, if the above probability is 0.01 or less, the detector performance is quite satisfactory. Let us define the quantity, *carrier-to-noise ratio*, ρ as

$$\rho = \frac{\text{average carrier power}}{\text{average noise power in the transmission bandwidth}}$$

$$= \frac{A_c^2 / 2}{2W N_0} = \frac{A_c^2}{4W N_0}$$

We shall now compute the threshold SNR in terms of ρ defined above. As R is Rayleigh variable, we have

$$f_R(r) = \frac{r}{\sigma_N^2} e^{-\frac{r^2}{2\sigma_N^2}}$$

where $\sigma_N^2 = 2W N_0$

$$P[R \geq A_c] = \int_{A_c}^{\infty} f_R(r)\, dr$$

$$= e^{-\frac{A_c^2}{4W N_0}}$$

$$= e^{-\rho}$$

Solving for ρ from $e^{-\rho} = 0.5$, we get $\rho = \ln 2 = 0.69$ or - 1.6 dB. Similarly, from the condition $P[R \geq A_c] = 0.01$, we obtain $\rho = \ln 100 = 4.6$ or 6.6 dB.

Based on the above calculations, we state that if $\rho \leq -1.6$ dB, the receiver performance is controlled by the noise and hence its output is not acceptable whereas for $\rho \geq 6.6$ dB, the effect of noise is not deleterious. However, reasonable intelligibility and naturalness in voice reception requires a post detection SNR of about 25 dB. That is, for satisfactory reception, we require a value of ρ much greater than what is indicated by the threshold considerations. In other words, additive noise makes the signal quality unacceptable long before multiplicative noise mutilates it. Hence threshold effect is usually not a serious limitation in AM transmission.

We now present two oscilloscope displays of the ED output of an AM signal with tone modulation. They are in flash animation.

ED - Display 1: SNR at the input to the detector is about 0 dB. ($m(t)$ is a tone signal at 3 kHz.) Output resembles the sample function of the noise process. Threshold effect is about to be set in.

ED - Display 2: SNR at the detector input is about 10 dB. Output of the detector, though resembling fairly closely a tone at 3 kHz, is still not a pure tone signal. Some amount of noise is seen riding on the output sine wave and the peaks of the sinewave are not perfectly aligned.

Example 7.3

In a receiver meant for the demodulation of SSB signals, $H_{eq}(f)$ has the characteristic shown in Fig. 7.7. Assuming that USB has been transmitted, let us find the FOM of the system.

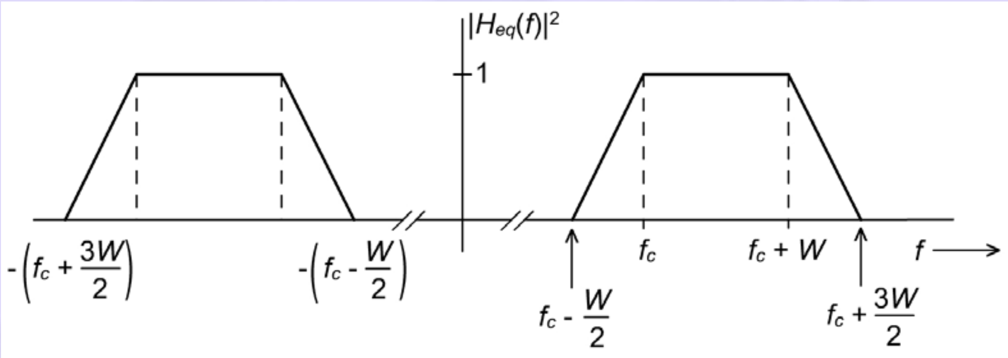


Fig. 7.7: $H_{eq}(f)$ for the Example 7.3

Because of the non-ideal $H_{eq}(f)$, $S_{N_c}(f)$ will be as shown in Fig. 7.8.

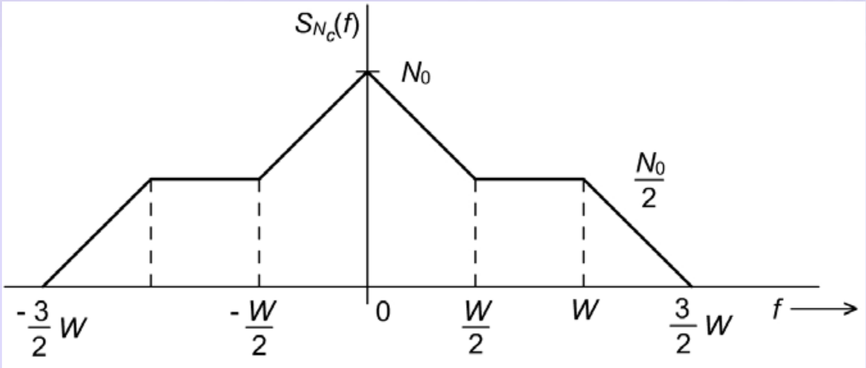


Fig. 7.8: $S_{N_c}(f)$ of Example 7.3

For SSB with coherent demodulation, we have

Signal quantity at the output $= \frac{A_c}{4} m(t)$

Noise quantity at the output $= \frac{n_c(t)}{2}$

Output noise power $= \frac{1}{4} \int_{-W}^{W} S_{N_c} \, df$

$$= \frac{5}{16} N_0 W$$

$$(SNR)_0 = \frac{\left(\frac{A_c^2 P_M}{16}\right)}{\frac{5}{16} N_0 W}$$

$$= \frac{A_c^2 P_M}{5 N_0 W}$$

$$(SNR)_r = \frac{A_c^2 P_M}{4 W N_0}$$

Hence $FOM = \frac{(SNR)_0}{(SNR)_r} = \frac{4}{5} = 0.8$. ♦

Exercise 7.1

In a receiver using coherent demodulation, the equivalent IF filter has the characteristics shown in Fig. 7.9. Compute the output noise power in the range $|f| \leq 100$ Hz assuming $N_0 = 10^{-3}$ Watts/Hz.

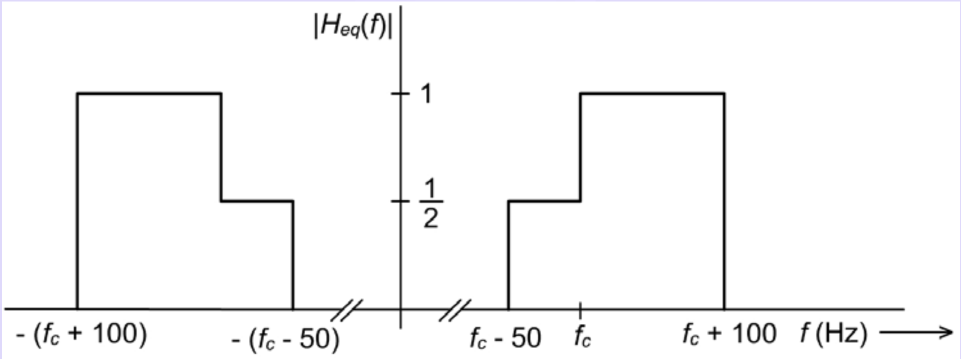


Fig. 7.9: $\left|H_{eq}(f)\right|$ for the Exercise 7.1

Ans: 0.225 Watts

Example 7.4

In a laboratory experiment involving envelope detection, AM signal at the input to ED, has the modulation index 0.5 with the carrier amplitude of 2 V. $m(t)$ is a tone signal of frequency 5 kHz and $f_c >> 5$ kHz. If the (two-sided) noise PSD at the detector input is 10^{-8} Watts/Hz, what is the expected $(SNR)_0$ of this scheme? By how many dB, this scheme is inferior to DSB-SC?

Spectrum of the AM signal is as shown in Fig. 7.10.

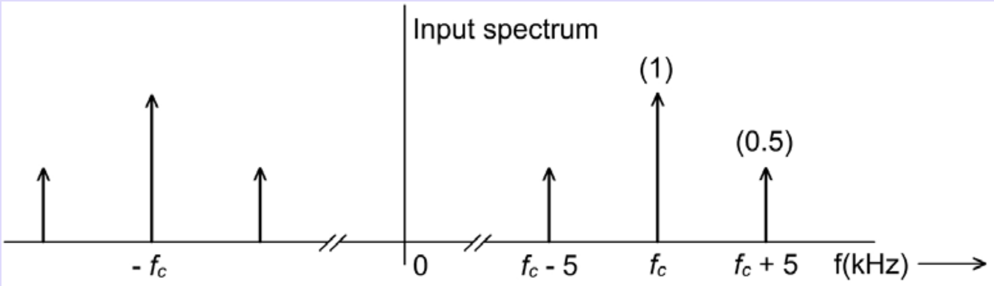


Fig. 7.10: Spectrum of the AM signal of Example 7.4

$$(SNR)_{0,AM} = \frac{A_c^2 g_m^2 P_M}{2W N_0}$$

As $P_M = \frac{A_m^2}{2}$,

$$(SNR)_{0,\,AM} = \frac{A_c^2 (g_m A_m)^2}{4W N_0}$$

But $g_m A_m = \mu = 0.5$. Hence,

$$(SNR)_{0,\,AM} = \frac{4 \cdot \frac{1}{4}}{4 \times 5 \times 10^3 \times 2 \times 10^{-8}}$$

$$= \frac{1}{40 \times 10^{-5}}$$

$$= \frac{10^4}{4}$$

$$= 36 \text{ dB}$$

$$(FOM)_{AM} = \frac{\mu^2}{2 + \mu^2} = \frac{\frac{1}{4}}{2 + \frac{1}{4}} = \frac{1}{9}$$

$$(FOM)_{DSB-SC} = 1$$

DSB-SC results in an increase in the $(SNR)_0$ by a factor of 9; that is by 9.54 dB.

♦

7.5 Receiver Model: Angle Modulation

The receiver model to be used in the noise analysis of angle modulated signals is shown in Fig. 7.11. (The block de-emphasis filter is shown with broken lines; the effect of pre-emphasis, de-emphasis will be accounted for subsequently).

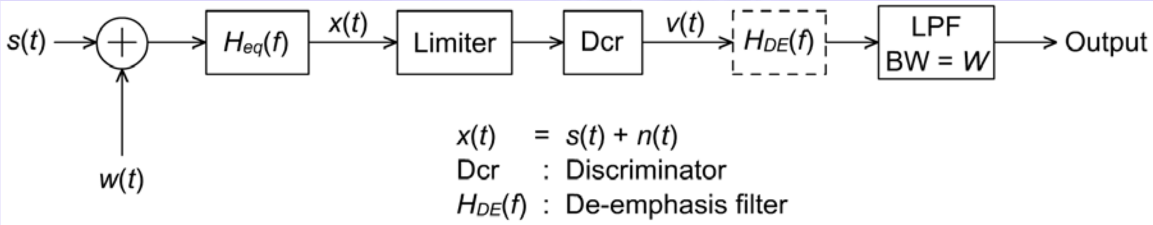


Fig. 7.11: Receiver model for the evaluation of noise performance

The role of $H_{eq}(f)$ is similar to what has been mentioned in the context of Fig. 7.1, with suitable changes in the centre frequency and transmission bandwidth. The centre frequency of the filter is $f_c = f_{IF}$, which for the commercial FM is 10.7 MHz. The bandwidth of the filter is the transmission bandwidth of the angle modulated signals, which is about 200 kHz for the commercial FM. Nevertheless, we treat $H_{eq}(f)$ to be a narrowband bandpass filter which passes the signal component $s(t)$ without any distortion whereas $n(t)$, the noise component at its output is the sample function of a narrowband noise process with a flat spectrum within the passband. The limiter removes any amplitude variations in the output of the equivalent IF filter. We assume the discriminator to be ideal, responding to either phase variations (phase discriminator) or derivative of the phase variations (frequency discriminator) of the quantity present at its input. The figure of merit (FOM) for judging the noise performance is the same as defined in section 7.2.2, namely, $\dfrac{(SNR)_0}{(SNR)_r}$.

7.6 Calculation of *FOM*

Let,

$$s(t) = A_c \cos\left[\omega_c t + \varphi(t)\right] \qquad (7.19)$$

where

$$\varphi(t) = \begin{cases} k_p\, m(t) & , \text{ for PM} \quad \cdots \quad (7.20a) \\ 2\pi k_f \displaystyle\int_{-\infty}^{t} m(\tau)\, d\tau , & \text{for FM} \quad \cdots \quad (7.20b) \end{cases}$$

The output of $H_{eq}(f)$ is,

$$x(t) = s(t) + n(t) \qquad (7.21a)$$

$$= A_c \cos\left[\omega_c t + \varphi(t)\right] + r_n(t)\cos\left[\omega_c t + \psi(t)\right] \qquad (7.21b)$$

where, on the RHS of Eq. 7.21(b) we have used the envelope $\left(r_n(t)\right)$ and phase $\left(\psi(t)\right)$ representation of the narrowband noise. As in the case of envelope detection of AM, we shall consider two cases:

i) Strong predetection *SNR* , $\left(A_c >> r_n(t) \textit{ most of the time}\right)$ and

ii) Weak predetection *SNR* , $\left(A_c << r_n(t) \textit{ most of the time}\right)$.

7.6.1 Strong Predetection *SNR*

Consider the phasor diagram shown in Fig. 7.12, where we have used the unmodulated carrier as the reference. $r(t)$ represents the envelope of the resultant (signal + noise) phasor and $\theta(t)$, the phase angle of the resultant. As far as this analysis is concerned, $r(t)$ is of no consequence (any variations in $r(t)$ are taken care of by the limiter). We express $\theta(t)$ as

$$\theta(t) = \varphi(t) + \tan^{-1}\left\{\frac{r_n(t)\sin\left[\psi(t) - \varphi(t)\right]}{A_c + r_n(t)\cos\left[\psi(t) - \varphi(t)\right]}\right\} \qquad (7.22a)$$

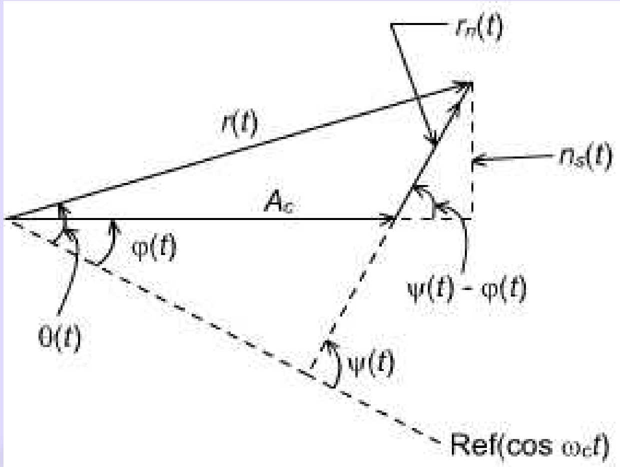


Fig. 7.12: Phasor diagram for the case of strong predetection SNR.

If we make the assumption that $A_c >> r_n(t)$ most of the time, we can write,

$$\theta(t) \approx \varphi(t) + \frac{r_n(t)}{A_c} \sin\left[\psi(t) - \varphi(t)\right] \tag{7.22b}$$

Notice that the second term on the RHS of Eq. 7.22(b) has the factor $\frac{r_n(t)}{A_c}$. Thus when the FM signal is much stronger than the noise, it will suppress the small random phase variations caused by noise; then the FM signal is said to *capture* the detector. $v(t)$, the output of the discriminator is given by,

$$v(t) = k_d\, \theta(t) \qquad \text{(phase detector)}$$

$$= \frac{k_d}{2\pi} \frac{d\,\theta(t)}{d\,t} \qquad \text{(frequency detector)}$$

where k_d is the gain constant of the detector under consideration.

a) Phase Modulation

For PM, $\varphi(t) = k_p\, m(t)$. For convenience, let $k_p\, k_d = 1$. Then,

$$v(t) \approx m(t) + \frac{k_d\, r_n(t)}{A_c} \sin\left[\psi(t) - \varphi(t)\right] \tag{7.23}$$

Again, we treat $m(t)$ to be a sample function of a WSS process $M(t)$. Then, output signal power = $P_M = \overline{M^2(t)} = R_M(0)$ (7.24)

Let $n_P(t) = \dfrac{k_d\, r_n(t)}{A_c} \sin\left[\psi(t) - \varphi(t)\right]$ (7.25)

To calculate the output noise power, we require the power spectral density of $n_P(t)$. This is made somewhat difficult because of $\varphi(t)$ in $n_P(t)$. The analysis becomes fairly easy if we assume $\varphi(t) = 0$. Of course, it is possible to derive the PSD of $n_P(t)$ without making the assumption that $\varphi(t) = 0$. This has been done in Appendix A7.1. In this appendix, it has been shown that the effect of $\varphi(t)$ is to produce spectral components beyond W, which are anyway removed by the final, LPF. Hence, we proceed with our analysis by setting $\varphi(t) = 0$ on the RHS of Eq. 7.25. Then $n_P(t)$ reduces to,

$$n_P(t) = \frac{k_d\, r_n(t)}{A_c} \sin\left[\psi(t)\right]$$

$$= \frac{k_d}{A_c} n_s(t)$$

Hence,

$$S_{N_P}(f) = \left(\frac{k_d}{A_c}\right)^2 S_{N_s}(f)$$

But, $S_{N_s}(f) = \begin{cases} N_0 \ , & |f| \le \dfrac{B_T}{2} \\ 0 \ , & otherwise \end{cases}$

B_T is the transmission bandwidth, which for the PM case can be taken as the value given by Eq. 5.26.

Post detection LPF passes only those spectral components that are within $(-W, W)$. Hence the output noise power $= \left(\dfrac{k_d}{A_c}\right)^2 2W N_0$, resulting in,

$$(SNR)_{0,PM} = \frac{P_M}{2W N_0 \left(\frac{k_d}{A_c}\right)^2}$$

$$= \frac{A_c^2}{2W N_0} k_p^2 P_M \qquad (7.26)$$

As, $(SNR)_{r,PM} = \dfrac{\left(A_c^2\right)/2}{N_0 W}$

we have,

$$(FOM)_{PM} = \frac{(SNR)_0}{(SNR)_r} = k_p^2 P_M \qquad (7.27a)$$

We can express $(FOM)_{PM}$ in terms of the RMS bandwidth. From Eq. A5.4.7, (Appendix A5.4), we have

$$(B_{rms})_{PM} = 2 k_p \sqrt{R_M(0)} \, (B_{rms})_M$$

$$= 2 k_p \sqrt{P_M} \, (B_{rms})_M$$

Hence $k_p^2 P_M = \dfrac{(B_{rms})_{PM}^2}{4 (B_{rms})_M^2}$

Using this value in Eq. 7.27(a), we obtain

$$(FOM)_{PM} = \frac{\left[(B_{rms})_{PM}\right]^2}{4 \left[(B_{rms})_M\right]^2} \qquad (7.27b)$$

b) Frequency Modulation

$$v(t) = \frac{k_d}{2\pi} \frac{d\,\theta(t)}{d t} \qquad (7.28a)$$

$$= k_f k_d m(t) + \frac{k_d}{2\pi A_c} \frac{d n_s(t)}{d t} \qquad (7.28b)$$

Again, letting $k_f k_d = 1$, we have

$$v(t) = m(t) + \frac{k_d}{2\pi A_c} \frac{d\,n_s(t)}{d\,t}$$

output signal power $= P_M$

Let $\quad n_F(t) = \frac{k_d}{2\pi A_c} \frac{d\,n_s(t)}{d\,t}$

Then, $S_{N_F}(f) = \left(\frac{k_d}{2\pi A_c}\right)^2 |j\,2\pi f|^2 \, S_{N_S}(f)$

The above step follows from the fact that $\frac{d\,n_s(t)}{d\,t}$ can be obtained by passing $n_s(t)$ through a differentiator with the transfer function $j\,2\pi f$. Thus,

$$S_{N_F}(f) = \frac{k_d^2\, f^2}{A_c^2} S_{N_S}(f)$$

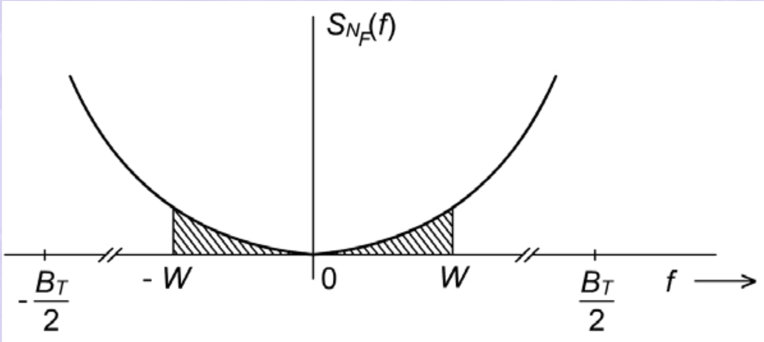


Fig. 7.13: Noise spectra at the FM discriminator output

As $S_{N_S}(f)$ is flat for $|f| \le \frac{B_T}{2}$, we find that $S_{N_F}(f)$ is parabolic as shown in Fig. 7.13.

The post detection filter eliminates the spectral components beyond $|f| > W$. Hence, The output noise power $= \int_{-W}^{W} \frac{k_d^2 \, f^2 N_0}{A_c^2} \, d f$

$$= \frac{k_d^2 \, N_0}{A_c^2} \left(\frac{2}{3}\right) W^3 \qquad (7.29)$$

This is equal to the hatched area in Fig 7.13.

Again, as in the case of PM, we find that increasing the carrier power has a noise quietening effect. But, of course, there is one major difference between $S_{N_P}(f)$ and $S_{N_F}(f)$; namely, the latter is parabolic whereas the former is a fiat spectrum.

The parabolic nature of the output FM noise spectrum implies, that high frequency end of the message spectrum is subject to stronger degradation because of noise. Completing our analysis, we find that

$$(SNR)_{0,\, FM} = \frac{3 A_c^2 P_M}{2 k_d^2 N_0 W^3} \qquad (7.30a)$$

$$= \frac{3}{2} k_f^2 \frac{A_c^2 P_M}{N_0 W^3} \qquad (7.30b)$$

$$= \left(\frac{A_c^2}{2 N_0 W}\right) \frac{3 k_f^2 P_M}{W^2} \qquad (7.30c)$$

Let us express $(FOM)_{FM}$ in terms of $(B_{rms})_{FM}$. From Appendix A5.4, Eq. A5.4.5,

$$(B_{rms})_{FM} = 2 \, k_f \sqrt{R_M(0)}$$

$$= 2 \, k_f \sqrt{P_M}$$

That is, $k_f^2 \, P_M = \frac{\left[(B_{rms})_{FM}\right]^2}{4}$

Hence, $(SNR)_{0,\,FM} = \dfrac{A_c^2}{2 N_0 W} \dfrac{3}{4}\left[\dfrac{(B_{rms})_{FM}}{W}\right]^2$

As $(SNR)_r$ is the same as in the case of PM, we have

$$(FOM)_{FM} = 3\,\frac{k_f^2 P_M}{W^2} = \frac{3}{4}\left[\frac{(B_{rms})_{FM}}{W}\right]^2 \qquad (7.31)$$

For a given peak value of the input signal, we find that the deviation ratio D is proportional to $\dfrac{k_f}{W}$; hence $(FOM)_{FM}$ is a quadratic function of D. The price paid to achieve a significant value for the FOM is the need for increased transmission bandwidth, $B_T = 2(D + k)W$. Of course, we should not forget the fact that the result of Eq. 7.31 is based on the assumption that SNR at the detector input is sufficiently large.

How do we justify that increasing D, (that is, the transmission bandwidth), will result in the improvement of the output SNR? Let us look at Eq. 7.28(b). On the RHS, we have two quantities, namely $k_f\,k_d\,m(t)$ and $\dfrac{k_d}{2\pi A_c}\dfrac{d\,n_s(t)}{d\,t}$. *The latter quantity is dependent only on noise and is independent of the message signal*. $\dfrac{k_d}{2\pi A_c}$ being a constant, $\dfrac{d\,n_s(t)}{d\,t}$ is the quantity that causes the perturbation of the instantaneous frequency due to the noise. Let us that assume that it is less than or equal to $(\Delta f)_n$, most of the time. For a given detector, k_d is fixed. Hence, as k_f increases, frequency deviation increases, thereby increasing the value of D. Let $k_{f,2} > k_{f,1}$. Then, to transmit the same $m(t)$, we require more bandwidth if we use a modulator with the frequency sensitivity $k_{f,2}$ instead of $k_{f,1}$. In other words, $(\Delta f)_2 = k_{f,2}\,m_p > (\Delta f)_1 = k_{f,1}\,m_p$. Hence,

$$\frac{(\Delta f)_n}{(\Delta f)_2} < \frac{(\Delta f)_n}{(\Delta f)_1}$$

In other words, as the frequency derivation due to the modulating signal keeps increasing, the effect of noise becomes less and less significant , thereby increasing the output SNR.

Example 7.5

A tone of unit amplitude and frequency 600 Hz is sent via FM. The FM receiver has been designed for message signals with a bandwidth upto 1 kHz. The maximum phase deviation produced by the tone is 5 rad. We will show that the $(SNR)_0 = 31.3$ dB, given that $\frac{A_c^2}{2 N_0} = 10^5$.

From Eq. 7.29, output noise power for a message of bandwidth W is $\frac{k_d^2 N_0}{A_c^2}\left(\frac{2}{3}\right) W^3$. For the problem on hand, $W = 1$ kHz. Hence output noise power $= k_d^2 \cdot \frac{1}{10^5} \cdot \frac{(1000)^3}{3}$. We shall assume $k_f k_d = 1$ so that $k_d = \frac{1}{k_f}$.

$$[s(t)]_{FM} = A_c [\cos(\omega_c t + \beta \sin \omega_m t)]$$

This maximum phase deviation produced is β.

But $\quad \beta = \frac{\Delta f}{f_m} = \frac{k_f A_m}{f_m}$.

As $A_m = 1$, we have

$$5 = \frac{k_f}{600}. \text{ That is, } k_f = 3000. \text{ Then,}$$

$$k_d = \frac{1}{3000}.$$

$$\text{Output noise power} = \frac{1}{(3000)^2} \cdot \frac{1}{10^5} \cdot \frac{10^9}{3}$$

$$= \frac{1}{2700}$$

Output signal power $= \frac{1}{2}$

Hence $(SNR)_0 = 1350$.

$$= 31.3 \text{ dB}$$

♦

Example 7.6

Compare the FOM of PM and FM when $m(t) = \cos\left(2\pi \times 5 \times 10^3\right)t$. The frequency deviation produced in both cases is 50 kHz.

For the case of PM, we have, $\Delta f = \left(\frac{k_p}{2\pi}\right) m_p'$

As $m_p' = 2\pi \times 5 \times 10^3$ and $\Delta f = 50 \times 10^3$,

$$k_p = \frac{2\pi \times 50 \times 10^3}{2\pi \times 5 \times 10^3} = 10$$

Therefore,

$$(FOM)_{PM} = k_p^2 P_M$$

$$= \frac{1}{2} \times 100 = 50$$

For the case of FM,

$$\Delta f = k_f m_p$$

As $m_p = 1$, we have $k_f = \Delta f = 50 \times 10^3$

Therefore,

$$(FOM)_{FM} = 3 \frac{k_f^2 P_M}{W^2}$$

$$= 3\,\frac{\left(50\times 10^3\right)^2 \frac{1}{2}}{\left(5\times 10^3\right)^2}$$

$$= \frac{3}{2}\times 100 = 150 \qquad \blacklozenge$$

The above result shows, that for tone modulation and for a given frequency deviation, FM is superior to PM by a factor of 3. In fact, FM results in superior performance as long as $\left(2\pi W\, m_p\right)^2 < 3\left(m_p'\right)^2$. Evidently, the Example 7.6 falls under this category. $\blacklozenge$

Example 7.7

Let $m(t) = 3\cos\left(2\pi\times 10^3\right)t + \cos\left(2\pi\times 5\times 10^3\right)t$. Assuming that the frequency deviation produced is 50 kHz, find $\dfrac{(FOM)_{PM}}{(FOM)_{FM}}$.

$$m_p' = 6\pi\times 10^3 + 10\pi\times 10^3 = 16\pi\times 10^3$$

$$m_p = 3 + 1 = 4$$

We have $\dfrac{k_p}{2\pi}\, m_p' = k_f\, m_p = 50\times 10^3$. That is,

$$\frac{k_p}{k_f} = \frac{2\pi m_p}{m_p'} = \frac{1}{2\times 10^3}$$

Hence,

$$\frac{(FOM)_{PM}}{(FOM)_{FM}} = \frac{1}{3}\left(\frac{k_p}{k_f}\right)^2 W^2 = \frac{1}{3}\times\frac{1}{4\times 10^6}\times 25\times 10^6$$

$$= \frac{25}{12} \approx 2.1 \qquad \blacklozenge$$

This example indicates that PM is superior to FM. It is the PSD of the input signal that decides the superiority or otherwise of the FM over PM. We can gain further insight into this issue by looking at the expressions for the FOM in terms of the RMS bandwidth.

From Eq. 7.27(b) and 7.31, we have

$$\frac{(FOM)_{PM}}{(FOM)_{FM}} = \frac{1}{3} \frac{\left[\left(B_{rms}\right)_{PM}\right]^2 W^2}{\left[\left(B_{rms}\right)_M\right]^2 \left[\left(B_{rms}\right)_{FM}\right]^2}$$

Assuming the same RMS bandwidth for both PM and FM, we find that PM is superior to FM, if

$$W^2 > 3\left[\left(B_{rms}\right)_M\right]^2$$

If $W^2 = 3\left[\left(B_{rms}\right)_M\right]^2$, then both PM and FM result in the same performance. This case corresponds to the PSD of the message signal, $S_M(f)$ being uniformly distributed in the range $(-W, W)$. If $S_M(f)$ decreases with frequency, as it does in most cases of practical interest, then $W^2 > 3\left[\left(B_{rms}\right)_M\right]^2$ and PM is superior to FM. This was the situation for the Example 7.7. If, on the other hand, the spectrum is more heavily weighted at the higher frequencies, then $W^2 < 3\left[\left(B_{rms}\right)_M\right]^2$, and FM gives rise to better performance. This was the situation for the Example 7.6, where the entire spectrum was concentrated at the tail end (at 5 kHz) with nothing in between.

In most of the real world information bearing signals, such as voice, music etc. have spectral behavior that tapers off with increase in frequency. Then, why not have PM broadcast than FM transmission? As will be seen in the context of pre-emphasis and de-emphasis in FM, the so called FM transmission is really a combination of PM and FM, resulting in a performance which is better than either PM or FM alone.

We have developed two different criteria for comparing the *SNR* performance of PM and FM, namely, PM is superior to FM, if either

C1) $W^2 > \dfrac{3 m_p'}{4 \pi^2 m_p^2}$, or

C2) $W^2 > 3\left[\left(B_{rms}\right)_M\right]^2$

is satisfied.

Then which criterion is to be used in practice? C1 is based on the transmission bandwidth where as C2 is based on the RMS bandwidth. Though C1 is generally preferred, in most cases of practical interest, it may be difficult to arrive at the parameters required for C1. Then the only way to make comparison is through C2.

7.6.2 Weak predetection SNR: Threshold effect

Consider the phasor diagram shown in Fig. 7.14, where the noise phasor is of a much larger magnitude, compared to the carrier phasor. Then, $\theta(t)$ can be approximated as

$$\theta(t) \approx \psi(t) + \frac{A_c}{r_n(t)} \sin\left[\varphi(t) - \psi(t)\right] \tag{7.32}$$

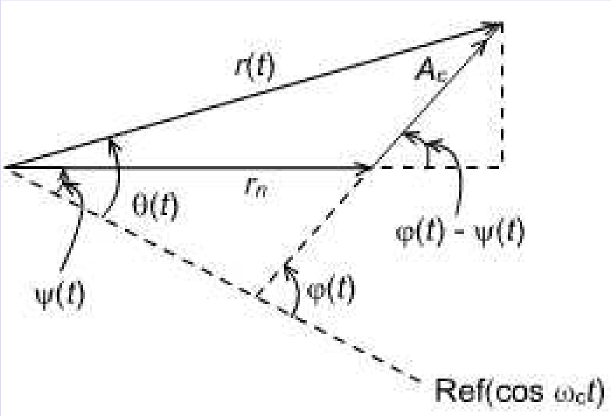


Fig. 7.14: Phasor diagram for the case of weak predetection SNR

As can be seen from Eq. 7.32, there is no term in $\theta(t)$ that represents only the signal quantity; the term that contains the signal quantity in $\theta(t)$ is actually multiplied by $\dfrac{A_c}{r_n(t)}$, which is random. This situation is somewhat analogous to the envelope detection of AM with low predetection SNR. Thus, we can expect a threshold effect in the case of FM demodulation as well. As $\dfrac{A_c}{r_n(t)}$, is small most of the time, phase of $r(t)$ is essentially decided by $\psi(t)$. As $\psi(t)$ is uniformly distributed, it is quite likely that in short time intervals such as (t_1, t_2), (t_3, t_4) etc., $\theta(t)$ changes by 2π (i.e., $r(t)$ rotates around the origin) as shown in Fig. 7.15(a).

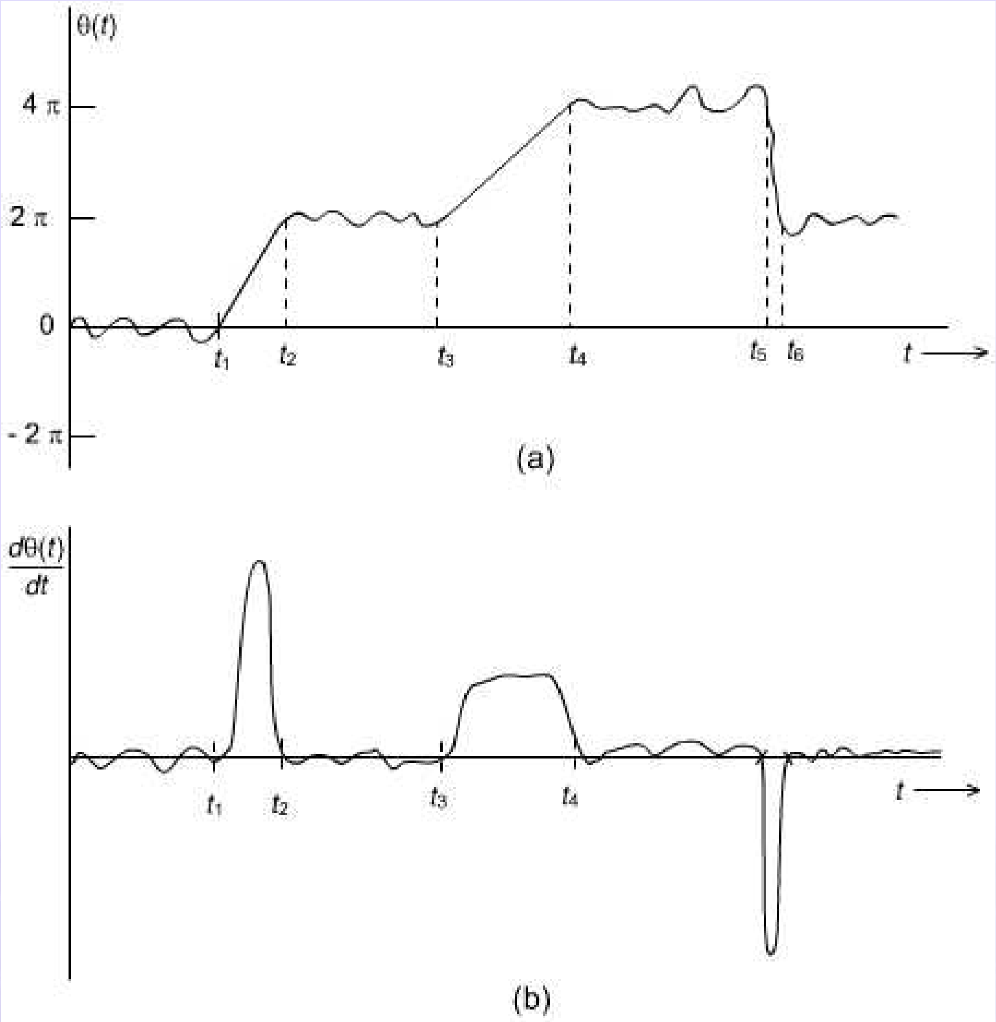


Fig. 7.15: Occurrence of short pulses at the frequency discriminator output for low predetection SNR.

When such phase variations go through a circuit responding to $\frac{d\theta}{dt}$, a series of short pulses appear at the output (Fig. 7.15(b)). The duration and frequency (average number of pulses per unit time) of such pulses will depend on the predetection SNR. If SNR is quite low, the frequency of the pulses at the discriminator output increases. As these short pulses have enough energy at the low frequencies, they give rise to crackling or sputtering sound at the receiver (speaker) output. The $(SNR)_0$ formula derived earlier, for the large input SNR case is no longer valid. As the input SNR keeps decreasing, it is even meaningless to talk of $(SNR)_0$. In such a situation, the receiver is captured by noise and is said to be working in the threshold region. (To gain some insight into the occurrence of the threshold phenomenon, let us perform the following experiment. An unmodulated sinewave + bandlimited white noise is applied as input to an FM discriminator. The frequency of the sinusoid can be set to the centre frequency of the discriminator and the PSD of the noise is symmetrical with respect to the frequency of the sinusoid. To start with, the input SNR is made very high. If the discriminator output is observed on an oscilloscope, it may resemble the sample function of a bandlimited white noise. As the noise power is increased, impulses start appearing in the output. The input SNR value at which these spikes or impulses start appearing is indicative of the setting in of the threshold behavior).

We now present a few oscilloscopic displays of the experiment suggested above. Display-1 and Display-2 are in flash animation.

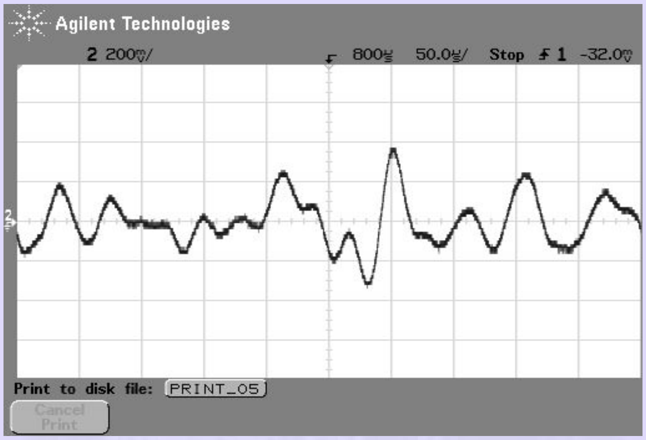
FM: Display - 1: (Carrier + noise) at the input to PLL with a Carrier-to-Noise Ratio (CNR) of about 15 dB.

FM: Display - 2: Output of the PLL for the above input. Note that the response of the PLL to a signal at the carrier frequency is zero. Hence, display-2 is the response of the PLL for the noise input which again looks like a noise waveform.

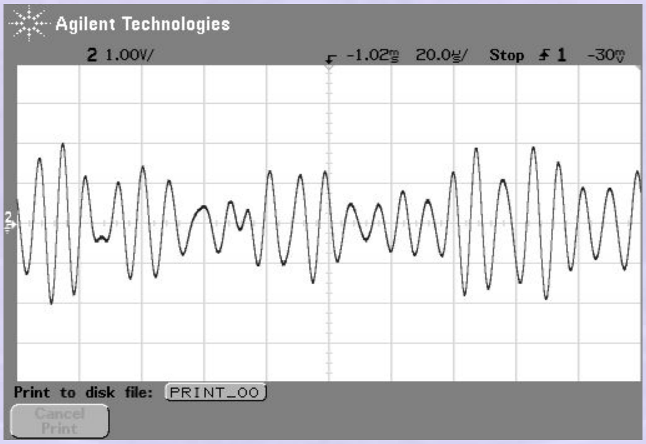
FM: Display - 3: Expanded version of a small part of display - 2. This could be treated as a part of a sample function of the output noise process.

FM: Display - 4: (Carrier + noise) at the input to the PLL. CNR is 0 dB. The effect of noise is more prominent in this display when compared to the 15 dB case.

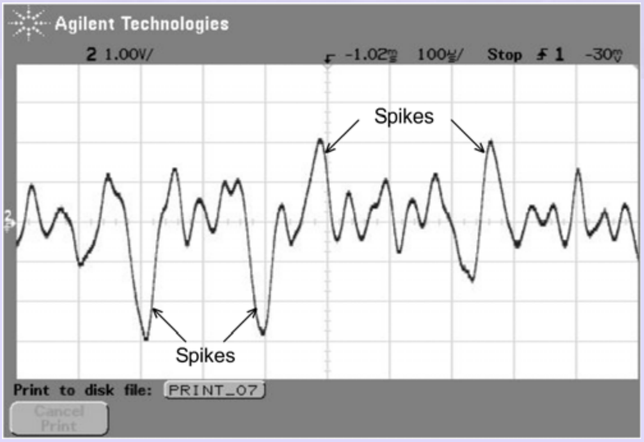
FM: Display - 5: Output of the PLL with the input corresponding to 0 dB CNR. Appearance of spikes is clearly evident.



FM: Display - 3



FM: Display - 4



FM: Display - 5

As in the case of AM noise analysis, if we set the limit that for the FM detector to operate above the threshold as, $P\left[R_n > A_c\right] \leq 0.01$, then we find that the minimum carrier-to-noise ratio $\rho = \dfrac{A_c^2}{2 B_T N_0}$ required is about 5. But, experimental results indicate that to obtain the predicted *SNR* improvement of the WBFM, ρ is of the order of 20, or 13 dB. That is, if $\dfrac{A_c^2}{2} > 20 B_T N_0$, then the FM detector will be free from the threshold effect.

Fig. 7.16 gives the plots of $(SNR)_0$ vs. $(SNR)_r = \dfrac{A_c^2}{2 N_0 W}$ for the case of FM with tone modulation. If we take $B_T = 2(\beta + 1)W$, threshold value of $(SNR)_r$ will approximately be $\left[13 + 10 \log(\beta + 1)\right]$ dB. More details on the threshold effect in FM can be found in [1, 2].

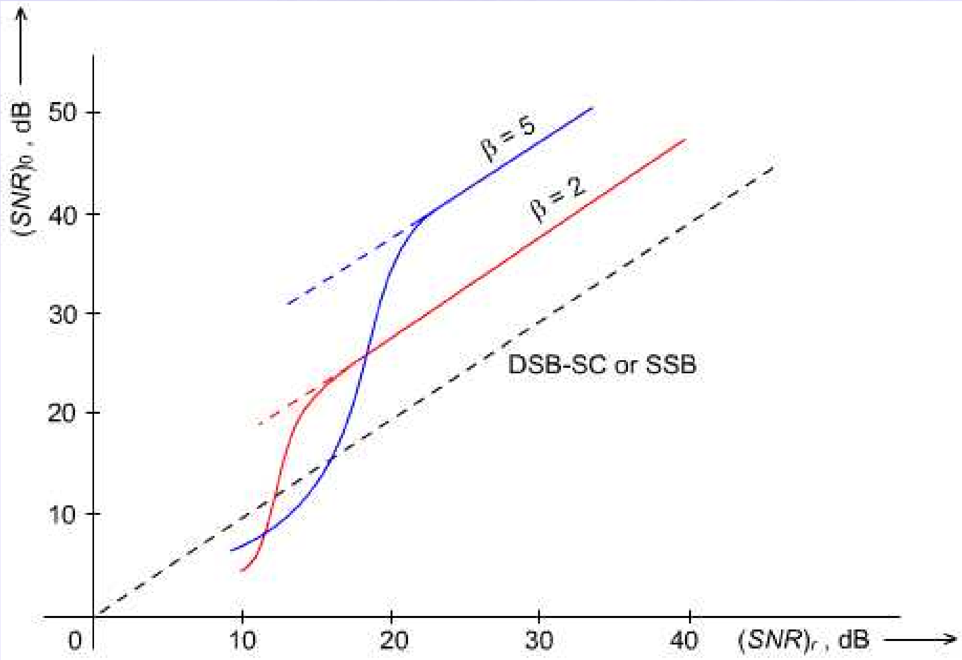


Fig. 7.16: $(SNR)_0$ performance of WBFM

For the FM demodulator operating above the threshold, we have (Eq. 7.31),

$$\frac{(SNR)_0}{(SNR)_r} = \frac{3 k_f^2 P_m}{W^2}$$

For a tone signal, $A_m \cos(\omega_m t)$, $P_m = \frac{A_m^2}{2}$, $W = f_m$ and $k_f A_m = \Delta f$. As $\beta = \frac{\Delta f}{f_m}$, we have

$$\frac{(SNR)_0}{(SNR)_r} = \frac{3}{2}\beta^2 .$$

That is,

$$10\log_{10}\left[(SNR)_0\right]_{FM} = 10\log_{10}\left[(SNR)_r\right]_{FM} + 10\log_{10}\left(\frac{3\beta^2}{2}\right) \qquad (7.33)$$

That is, WBFM operating above threshold provides an improvement of $10\log_{10}\left(\frac{3\beta^2}{2}\right)$ dB, with respect to $(SNR)_r$. For $\beta = 2$, this amounts to an improvement of about 7.7 dB and $\beta = 5$, the improvement is about 15.7 dB. This is evident from the plots in Fig. 7.16.

We make a few observations with respect to the plots in Fig. 7.16.

(i) Above threshold [i.e. $(SNR)_r$ above the knee for each curve), WBFM gives rise to impressive $(SNR)_0$ performance when compared to DSB-SC or SSB with coherent detection. For the latter, $(SNR)_0$ is best equal to $(SNR)_r$. (Using pre-emphasis and de-emphasis, the performance of FM can be improved further).

(ii) Simply increasing the bandwidth without a corresponding increase in the transmitted power does not improve $(SNR)_0$, because of the threshold effect. For example with $(SNR)_r$ about 18dB, $\beta = 2$ and $\beta = 5$ give rise to the same kind of performance. If $(SNR)_r$ is reduced a little, say to about 16dB, the $(SNR)_0$ performance with $\beta = 5$ is much inferior to that of $\beta = 2$.

7.7 Pre-Emphasis and De-Emphasis in FM

For many signals of common interest, such as speech, music etc., most of the energy concentration is in the low frequencies and the frequency components near about W have very little energy in them. When these low energy, high-frequency components frequency modulate a carrier, they will not give rise to full frequency deviation and hence the message will not be utilizing fully the allocated bandwidth. Unfortunately, as was established in the previous section, the noise PSD at the discriminator output increases as f^2. The net result is an unacceptably low SNR at the high frequency end of the message spectrum. Nevertheless, proper reproduction of the high frequency (but low energy) spectral components of the input spectrum becomes essential from the point of view of final tonal quality or aesthetic appeal. To offset this undesirable occurrence, a clever but easy-to-implement signal processing scheme has been proposed which is popularly known as *pre-emphasis* and *de-emphasis technique*.

Pre-emphasis consists in artificially boosting the spectral components in the latter part of the message spectrum. This is accomplished by passing the message signal $m(t)$, through a filter called the pre-emphasis filter, denoted $H_{PE}(f)$. The pre-emphasized signal is used to frequency modulate the carrier at the transmitting end. In the receiver, the inverse operation, de-emphasis, is performed. This is accomplished by passing the discriminator output through a filter, called the de-emphasis filter, denoted $H_{DE}(f)$. (See Fig. 7.11.) The de-emphasis operation will restore all the spectral components of $m(t)$ to their original level; this implies the attenuation of the high frequency end of the demodulated spectrum. In this process, the high frequency noise components are also attenuated, thereby improving the overall SNR at the receiver output.

Let $S_{N_F}(f)$ denote the PSD of the noise at the discriminator output. Then the noise power spectral density at the output of the de-emphasis filter is $|H_{DE}(f)|^2 S_{N_F}(f)$. Hence,

$$\text{Output noise power with de-emphasis} = \int_{-W}^{W} |H_{DE}(f)|^2 S_{N_F}(f)\, df \qquad (7.34)$$

As the message power is unaffected because of PE-DE operations. $\left(\text{Note that } H_{DE}(f) = \frac{1}{H_{PE}(f)}\right)$, it follows that the improvement in the output SNR is due to the reduced noise power after de-emphasis. We quantify the improvement in output SNR, produced by PE-DE operation by the *improvement factor* I, where

$$I = \frac{\text{average output noise power without PE-DE}}{\text{average output noise power with PE-DE}} \qquad (7.35)$$

The numerator of Eq. 7.35 is $\frac{2 k_d^2 N_0 W^3}{3 A_c^2}$. As the frequency range of interest is only $|f| \le W$, let us take

$$S_{N_F}(f) = \begin{cases} \frac{2 k_d^2 N_0 f^2}{A_c^2}, & |f| \le W \\ 0, & \textit{otherwise} \end{cases}$$

We can now compute the denominator of Eq. 7.35 and thereby the improvement factor, which is given by

$$I = \frac{2 W^3}{3 \int_{-W}^{W} f^2 \left|H_{DE}(f)\right|^2 d f} \tag{7.36}$$

We shall now describe the commonly used PE-DE networks in the commercial FM broadcast and calculate the corresponding improvement in output SNR.

Fig 7.17(a) gives the PE network and 7.17(b), the corresponding DE network used in commercial FM broadcast. In terms of the Laplace transform,

$$H_{PE}(s) = \sqrt{K_1} \frac{s + \frac{1}{r C}}{s + \frac{R + r}{r R C}} \tag{7.37}$$

where K_1 is a constant to be chosen appropriately. Usually $R \ll r$. Hence,

$$H_{PE}(s) \approx \sqrt{K_1} \frac{s + \frac{1}{r C}}{s + \frac{1}{R C}} \tag{7.38}$$

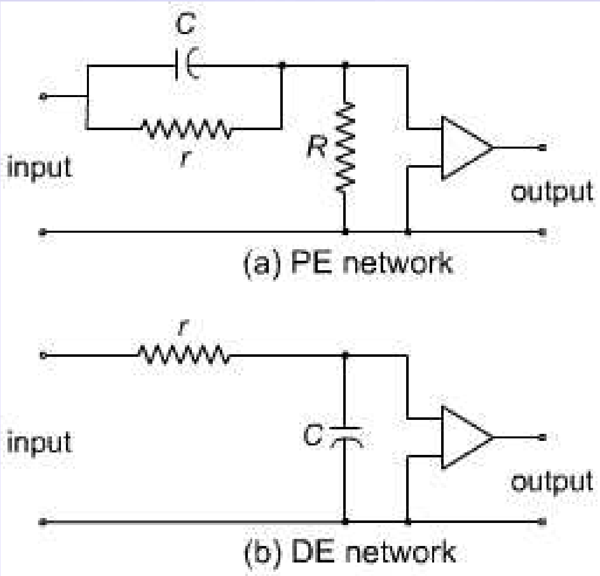


Fig. 7.17: Circuit schematic of a PE-DE network

The time constant $T_{C1} = rC$ normally is 75 μsec. If $\omega_1 = 2\pi f_1 = \frac{1}{T_{C1}}$, then $f_1 = 2.1$ kHz[1]. The value of $T_{C2} = RC$ is not very critical, provided $f_2 = \frac{1}{2\pi T_{C2}}$ is not less than the highest audio frequency for which pre-emphasis is desired (15 kHz).

Bode plots for the PE and DE networks are given in Fig. 7.18. Eq. 7.38 can be written as

$$H_{PE}(s) \approx \frac{RC}{rC}\sqrt{K_1}\,\frac{1 + srC}{1 + sRC} \tag{7.39a}$$

with $s = j2\pi f$, $H_{PE}(f) \approx \sqrt{K}\,\dfrac{1 + j\left(\frac{f}{f_1}\right)}{1 + j\left(\frac{f}{f_2}\right)}$ (7.39b)

where $\sqrt{K} = \frac{R}{r}\sqrt{K_1}$.

For $|f| \leq f_2$, we can take

$$H_{PE}(f) = \sqrt{K}\left[1 + j\left(\frac{f}{f_1}\right)\right]$$

Hence $H_{DE}(f) = \dfrac{1}{\sqrt{K}}\left[1 + j\left(\dfrac{f}{f_1}\right)\right]^{-1}$

[1] The choice of f_1 was made on an experimental basis. It is found that this choice of f_1 maintained the same peak amplitude m_p with or without PE-DE. This satisfies the constraint of a fixed B_T.

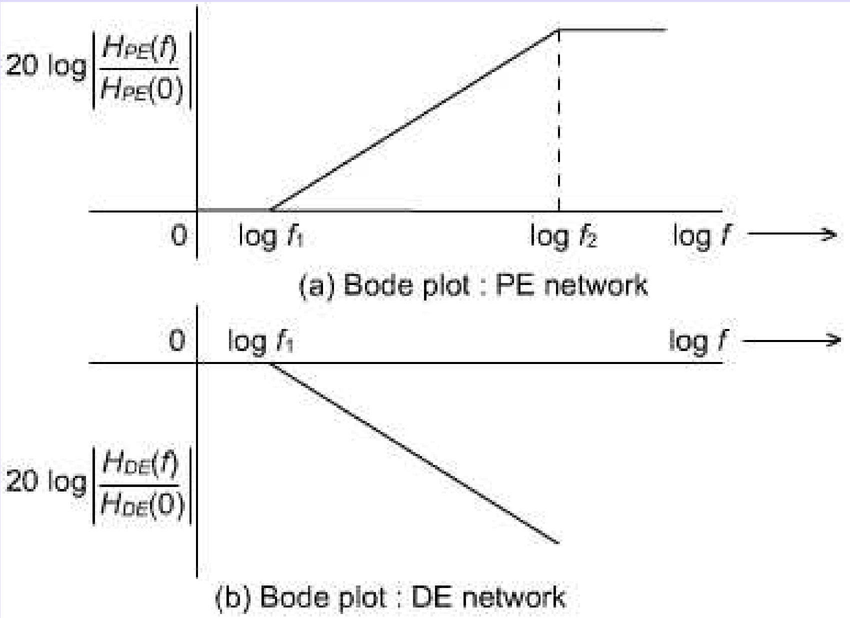


Fig. 7.18: Bode plots of the response of PE-DE networks

The factor K is chosen such that the average power of the emphasized message signal is the same as that of the original message signal $m(t)$. That is, K is such that

$$\int_{-W}^{W} S_M(f)\, df = \int_{-W}^{W} \left|H_{PE}(f)\right|^2 S_M(f)\, df = P_M \tag{7.40}$$

This will ensure the same RMS bandwidth for the FM signal with or without PE. Note that $\left(B_{rms}\right)_{FM} = 2 k_f \sqrt{P_M}$.

Example 7.8

Let $S_M(f) = \begin{cases} \dfrac{1}{1 + \left(\dfrac{f}{f_1}\right)^2}, & |f| \leq W \\ 0, & outside \end{cases}$

and $H_{PE}(f) = \sqrt{K}\left[1 + j\left(\dfrac{f}{f_1}\right)\right]$

Let us find (i) the value K and (ii) the improvement factor I, assuming $f_1 = 2.1$ kHz and $W = 15$ kHz.

From Eq. 7.40, we have

$$\int_{-W}^{W} \frac{1}{1+\left(\frac{f}{f_1}\right)^2} d f = \int_{-W}^{W} K \, d f$$

or $\quad K = \frac{f_1}{W} \tan^{-1}\left(\frac{W}{f_1}\right)$

and $$I = \frac{K \int_{-W}^{W} C f^2 d f}{\int_{-W}^{W} C f^2 \left[1+\left(\frac{f}{f_1}\right)^2\right]^{-1} d f} \qquad (7.41a)$$

where $S_N(f)$ is taken as $C f^2$, C being a constant.

Carrying out the integration, we find that

$$I = \left(\frac{W}{f_1}\right)^2 \frac{\tan^{-1}\left(\frac{W}{f_1}\right)}{3\left[\frac{W}{f_1} - \tan^{-1}\left(\frac{W}{f_1}\right)\right]} \qquad (7.41b)$$

With $W = 15$ kHz and $f_1 = 2.1$ kHz, $I \approx 4 = 6$ dB. ♦

Pre-emphasis and de-emphasis also finds application in phonographic and tape recording systems. Another application is the SSB/FM transmission of telephone signals. In this, a number of voice channels are frequency division multiplexed using SSB signals; this composite signal frequency modulates the final carrier. (See Exercise 7.3.) PE-DE is used to ensure that each voice channel gives rise to almost the same signal-to-noise ratio at the destination.

Example 7.9

Pre-emphasis - de-emphasis is used in a DSB-SC system. PSD of the message process is,

$$S_M(f) = \frac{1}{1 + \left(\frac{f}{f_1}\right)^2}, \quad |f| \le W$$

Let $\quad H_{PE}(f) = \sqrt{K}\left[1 + j\left(\frac{f}{f_1}\right)\right]$

where f_1 is a known constant.

Transmitted power with pre-emphasis remains the same as without pre-emphasis. Let us calculate the improvement factor I.

As the signal power with pre-emphasis remains unchanged, we have

$$\int_{-W}^{W} S_M(f)\, df = \int_{-W}^{W} S_M(f)\, K\left[1 + \left(\frac{f}{f_1}\right)^2\right] df$$

$$= K \int_{-\infty}^{\infty}\left[\frac{1}{1 + \left(\frac{f}{f_1}\right)^2}\right]\left[1 + \left(\frac{f}{f_1}\right)^2\right] df$$

That is,

$$K = \int_{-W}^{W} S_M(f)\, df = \frac{f_1}{W}\tan^{-1}\left(\frac{W}{f_1}\right)$$

Noise power after de-emphasis,

$$N_{out} = \int_{-W}^{W} \frac{N_0}{4}\left|H_{DE}(f)\right|^2 df$$

$$= \frac{N_0}{4}\int_{-W}^{W} \frac{\left[1 + \left(\frac{f}{f_1}\right)^2\right]^{-1}}{K}\, df$$

$$= W \frac{N_0}{2}$$

Note that the noise quantity at the output of the coherent demodulator is $\frac{1}{2} n_c(t)$.

Noise power without de-emphasis $= 2W \cdot \frac{N_0}{4} = \frac{W N_0}{2}$

$$\text{Hence, } I = \frac{\frac{W N_0}{2}}{\frac{W N_0}{2}} = 1$$

This example indicates that PE-DE is of no use in the case of DSB-SC. ♦

Exercise 7.2

A signal $m(t) = 2\cos\left[(1000\pi)\,t\right]$ is used to frequency modulate a very high frequency carrier. The frequency derivation produced is 2.5 kHz. At the output of the discriminator, there is bandpass filter with the passband in the frequency range $100 < |f| < 900$ Hz. It is given that $\frac{A_c^2}{2N_0} = 2 \times 10^5$ and $k_d = 1$.

a) Is the system operating above threshold?

b) If so, find the $(SNR)_0$, dB.

Ans: (b) 34 dB

Exercise 7.3

Consider the scheme shown in Fig. 7.19.

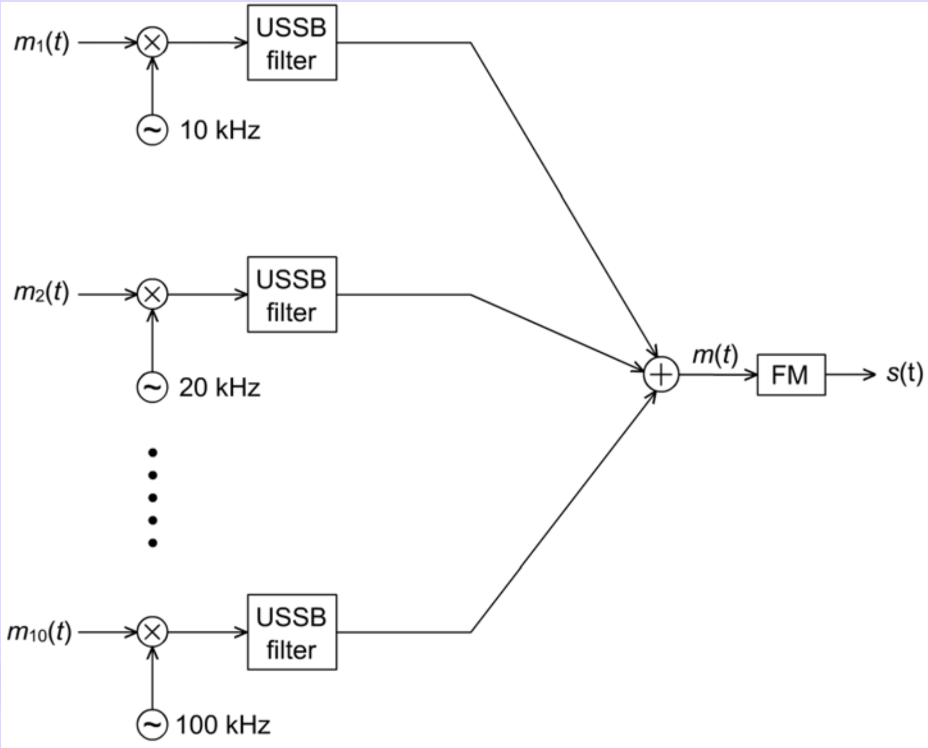


Fig. 7.19: Transmission scheme of Exercise 7.3

Each one of the USSB signals occupies a bandwidth of 4 kHz with respect to its carrier. All the message signals, $m_i(t)$, $i = 1, 2, \cdots, 10$, have the same power. $m(t)$ frequency modulates a high frequency carrier. Let $s(t)$ represent the FM signal.

a) Sketch the spectrum of $s(t)$. You can assume suitable shapes for $M_1(f), M_2(f), \cdots, M_{10}(f)$

b) At the receiver $s(t)$ is demodulated to recover $m(t)$. (Note that from $m(t)$ we can retrieve $\left[m_j(t)\right]$, $j = 1, 2, \cdots, 10$.) If $m_1(t)$ can give rise to signal-to-noise ratio of 50 dB, what is the expected signal-to-noise ratio from $m_{10}(t)$?

7.8 Noise Performance of a PCM system

There are two sources of error in a PCM system: errors due to quantization and the errors caused by channel noise, often referred to as detection errors. We shall treat these two sources of error as independent noise sources and derive an expression for the signal-to-noise ratio expected at the output of a PCM system. As we have already studied the quantization noise, let us now look into the effects of channel noise on the output of a PCM system.

We will assume that the given PCM system uses polar signaling. Even if the transmitted pulse is rectangular, the received pulse $p_r(t)$ will be distorted due to a band-limited and imperfect channel; hence $p_r(t)$ may look like as shown in Fig. 7.20 (a).

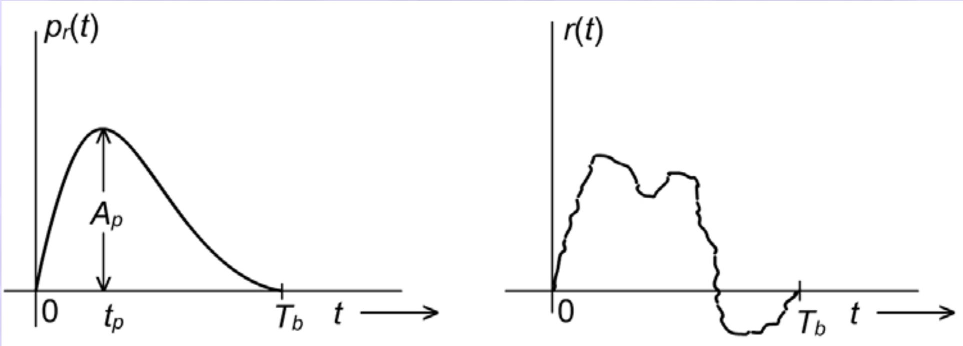


Fig. 7.20: (a) Typical received pulse (without noise)
(b) Received pulse with noise

The input to the PCM receiver will be $r(t) = \pm\, p_r(t) + n(t)$, where $n(t)$ is a sample function of a zero-mean Gaussian process with variance σ_N^2. A possible pulse shape of $r(t)$ is shown in Fig. 7.20(b). The detection process of the PCM scheme consists of sampling $r(t)$ once every T_b seconds and comparing it to a threshold. For the best performance, it would be necessary to sample $p_r(t)$ at its peak amplitude $\left(k\,t_s = k\,T_b + t_p\right)$ resulting in a signal component of $\pm A_p$. Hence $r(k\,t_s) = \pm A_p + N$ whose N is a zero mean Gaussian variable, representing the noise sample of a band-limited white Gaussian process with a bandwidth greater than or equal to the bandwidth of the PCM signal. If binary '1' corresponds to $+A_p$ and '0' to $-A_p$, then,

$$f_R(r/\text{'1' transmitted}) \text{ is } N\left(A_p, \sigma_N^2\right) \text{ and}$$

$$f_R(r/\text{'0' transmitted}) \text{ is } N\left(-A_p, \sigma_N^2\right)$$

where R (as a subscript) represents the received random variable. We assume that 1's and 0's are equally likely to be transmitted. The above conditional densities are shown in Fig. 7.21. As can be seen from the figure, the optimum decision threshold is zero. Let $P_{e,0}$ denote the probability of wrong decision, given that '0' is transmitted (area hatched in red); similarly $P_{e,1}$ (area hatched in blue). Then P_e, the probability of error is given by $P_e = \frac{1}{2} P_{e,0} + \frac{1}{2} P_{e,1}$.

From Fig. 7.21, we have

$$P_{e,0} = \int_0^{\infty} f_R(r/\text{'0'})\, d\,r = Q\left(\frac{A_p}{\sigma_N}\right)$$

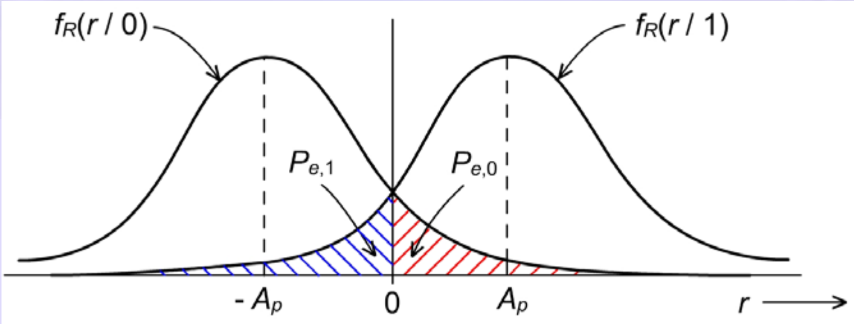


Fig. 7.21: Conditional PDFs at the detector input

Similarly $P_{e,1} = Q\left(\frac{A_p}{\sigma_N}\right)$, which implies

$$P_e = Q\left(\frac{A_p}{\sigma_N}\right) \tag{7.42}$$

For optimum results, the receiver uses a matched filter whose output is sampled once every T_b seconds, at the appropriate time instants so as to obtain the best possible signal-to-noise ratio at the filter output. In such a situation, it can be shown that we can replace $\left(\frac{A_p}{\sigma_N}\right)$ by $\sqrt{\frac{2E_b}{N_0}}$ where E_b is the energy of the received binary pulse and $\frac{N_0}{2}$ represents the spectral height of the, band-limited white Gaussian noise process. Using the above value for $\left(\frac{A_p}{\sigma_N}\right)$ yields,

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{7.43}$$

Assuming that there are R binary pulses per sample and $2W$ samples/second, we have $T_b = \frac{1}{(2RW)}$ where T_b represents the duration of each pulse. Hence the received signal power S_r is given by, $S_r = \frac{E_b}{T_b} = 2RW\,E_b$ or $E_b = \frac{S_r}{2RW}$.

Therefore Eq. 7.43 can also be written as

$$P_e = Q\left(\sqrt{\frac{S_r}{RW\,N_0}}\right)$$

$$= Q\left(\sqrt{\frac{\gamma}{R}}\right) \tag{7.44}$$

where $\gamma = \frac{S_r}{W\,N_0}$. Eq. 7.42 to 7.44 specify the probability of any received bit being in error. In a PCM system, with R bits per sample, error in the reconstructed sample will depend on which of these R bits are in error. We would like to have an expression for the variance of the reconstruction error. Assume the following:

i) The quantizer used is a uniform quantizer

ii) The quantizer output is coded according to natural binary code

iii) P_e is small enough so that the probability of two or more errors in a block of R bits is negligible.

Then, it can be shown that σ_c^2, the variance of the reconstruction error due to channel noise is,

$$\sigma_c^2 = \frac{4 m_p^2 P_e \left(2^{2R} - 1\right)}{3\left(2^{2R}\right)} \tag{7.45}$$

Details can be found in [3]. In addition to the reconstruction error due to channel noise, PCM has the inevitable quantization noise with variance $\sigma_Q^2 = \frac{\Delta^2}{12}$, where $\Delta = \frac{2 m_p}{L}$ and $L = 2^R$. Treating these two error sources as independent noise sources, total reconstruction noise variance σ_e^2, can be written as

$$\sigma_e^2 = \sigma_Q^2 + \sigma_c^2$$

$$= \frac{m_p^2}{3L^2} + \frac{4 m_p^2 P_e \left(L^2 - 1\right)}{3L^2} \tag{7.46}$$

Let $\overline{M^2(t)} = P_M$

Then, $(SNR)_0 = \frac{P_M}{\sigma_e^2}$

$$= \frac{3L^2}{1 + 4P_e\left(L^2 - 1\right)}\left(\frac{P_M}{m_p^2}\right) \tag{7.47}$$

Using Eq. 7.44 in Eq. 7.47, we have

$$(SNR)_0 = \frac{3L^2}{1 + 4\left(L^2 - 1\right) Q\left(\sqrt{\frac{\gamma}{R}}\right)}\left(\frac{P_M}{m_p^2}\right) \tag{7.48}$$

Figure 7.22 shows the plot of $(SNR)_0$ as a function of γ for tone modulation $\left(\frac{P_M}{m_p^2} = \frac{1}{2}\right)$. However, with suitable modifications, these curves are applicable even in a more general case.

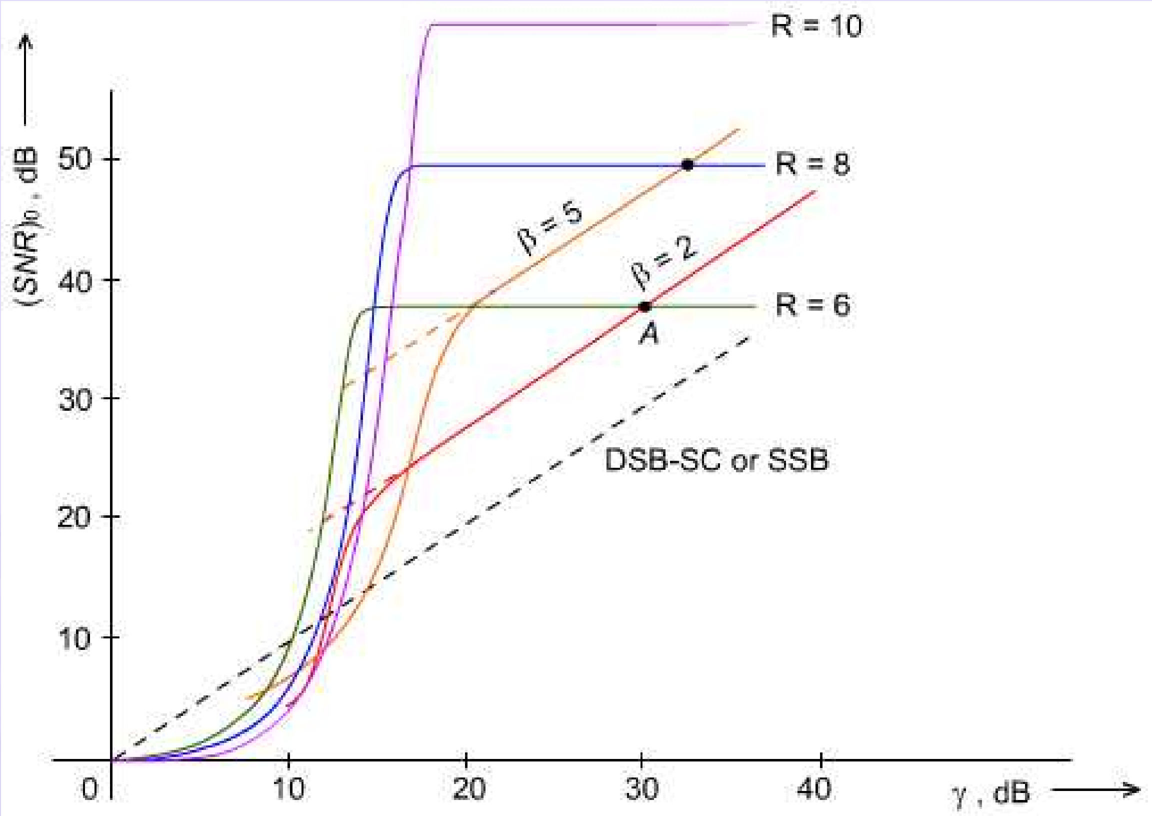


Fig. 7.22: $(SNR)_0$ performance of PCM

Referring to the above figure, we find that when γ is too small, the channel noise introduces too many detection errors and as such reconstructed waveform has little resemblance to the transmitted waveform and we encounter the threshold effect. When γ is sufficiently large, then $P_e \rightarrow 0$ and $\sigma_e^2 \approx \sigma_Q^2$ which is a constant for a given n. Hence $(SNR)_0$ is essentially independent of γ resulting in saturation.

For the saturation region, $(SNR)_0$ can be taken as

$$(SNR)_0 = 3\left(2^{2R}\right)\left(\frac{P_M}{m_p^2}\right) \qquad (7.49)$$

The transmission bandwidth B_{PCM} of the PCM system is $k(2WR)$ where k is a constant that is dependent on the signal format used. A few values of k are given below:

S. No.	Signal Format	*k*
1	NRZ polar	1
2	RZ polar	2
3	Bipolar (RZ or NRZ)	1
4	Duobinary (NRZ)	$\frac{1}{2}$

(For a discussion on duobinary signaling, refer Lathi [3]). As $B_{PCM} = k(2WR)$, we have $R = \frac{B_{PCM}}{k(2W)}$. Using this in the expression for $(SNR)_0$ in the saturation region, we obtain

$$(SNR)_0 = 3\left(\frac{P_M}{m_p^2}\right)2^{\frac{B_{PCM}}{kW}} \qquad (7.50)$$

It is clear from Eq. 7.50 that in PCM, $(SNR)_0$ increases exponentially with the transmission bandwidth. Fig. 7.22 also depicts the $(SNR)_0$ performance of DSB-SC and FM $(\beta = 2, 5)$. A comparison of the performance of PCM with that of FM is appropriate because both the schemes exchange the bandwidth for the signal -to -noise ratio and they both suffer from threshold phenomenon. In FM, $(SNR)_0$ increases as the square of the transmission bandwidth. Hence, doubling the transmission bandwidth quadruples the output SNR. In the case of

PCM, as can be seen from Eq. 7.49, increasing R by 1 quadruples the output SNR, where as the bandwidth requirement increases only by $\frac{1}{R}$. As an example, if R is increased from 8 to 9, the additional bandwidth is 12.5% of that required for R = 8. Therefore, in PCM, the exchange of SNR for bandwidth is much more efficient than that in FM, especially for large values of R[1]. In addition, as mentioned in the introduction, PCM has other beneficial features such as use of regenerative repeaters, ease of mixing or multiplexing various types of signals etc. All these factors put together have made PCM a very important scheme for modern-day communications.

The PCM performance curves of Fig. 7.22 are based on Eq. 7.48 which is applicable to polar signaling. By evaluating P_e for other signaling techniques (such as bipolar, duobinary etc.) and using it in Eq. 7.47, we obtain the corresponding expressions for the $(SNR)_0$. It can be shown that the PCM performance curves of Fig.7.19 are applicable to bipolar signaling if 3 dB is added to each value of γ.

Example 7.10

A PCM encoder produces ON-OFF rectangular pulses to represent '1' and '0' respectively at the rate of 1000 pulses/sec. These pulses amplitude modulate a carrier, $A_c \cos(\omega_c t)$, where $f_c >> 1000$ Hz. Assume that '1's and '0's are equally likely. Consider the receiver scheme shown in Fig. 7.23.

---

[1] Note that the FM curve for $\beta = 2$ and PCM curve for $R = 6$ intersect at point A. The corresponding γ is about 30 dB. If we increase γ beyond this value, there is no further improvement in $(SNR)_0$ of the PCM system where as no saturation occurs in FM. If we take $B_T = 2(\beta + 1) f_m$, then the transmission bandwidth requirements of PCM with $R = 6$ and FM with $\beta = 2$ are the same. This argument can be extended to other values of R and β.

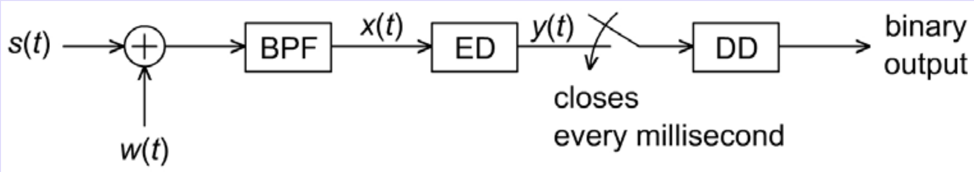


Fig. 7.23: Receiver for the Example 7.11

$$s(t) = \begin{cases} 2\cos(\omega_c t), & \text{if the input is '1'} \\ 0 & \text{, if the input is '0'} \end{cases}$$

$w(t)$: sample function of a white Guassian noise process, with a two sided spectral density of 0.25×10^{-4} Watts/Hz.

BPF : Bandpass filter, centered at f_c with a bandwidth of 2 kHz so that the signal is passed with negligible distortion.

DD : Decision device (comparator)

a) Let Y denote the random variable at the sampler output. Find $f_{Y/0}(y/0)$ and $f_{Y/1}(y/1)$, and sketch them.

b) Assume that the DD implements the rule: if $Y \geq 1$, then binary '1' is transmitted, otherwise it is binary '0'.

If $f_{Y/1}(y/1)$ can be well approximated by $N(2, 0.1)$, let us find the overall probability of error.

$$x(t) = s(t) + n(t), \text{ where } n(t) \text{ is the noise output of the BPF.}$$

$$= A_k \cos(\omega_c t) + n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t)$$

where $A_k = \begin{cases} 2, & \text{if the k}^{\text{th}} \text{ transmission bit is '1'} \\ 0, & \text{if the k}^{\text{th}} \text{ transmission bit is '0'} \end{cases}$

The random process $Y(t)$ at the output of the ED, is

$$Y(t) = R(t)\cos[\omega_c t + \Theta]$$

$$R(t) = \left\{\left[A_k + n_c(t)\right]^2 + \left[n_s(t)\right]^2\right\}^{\frac{1}{2}}$$

If '0' is transmitted, $Y(t)$ represents the envelope of narrowband noise. Hence, the random variable Y obtained by sampling $Y(t)$ is Rayleigh distributed.

$$f_{Y/0}(y/0) = \frac{y}{N_0} e^{-\frac{y^2}{2N_0}}, \quad y \geq 0$$

where $N_0 = \left(0.25 \times 10^{-4}\right) 4 \times 10^3 = 0.1$

when '1' is transmitted, the random variable Y is Rician, given by

$$f_{Y/1}(y/1) = \frac{y}{N_0} I_0\left(\frac{2y}{N_0}\right) e^{-\left(\frac{y^2+4}{2N_0}\right)}, \quad y \geq 0$$

These are sketched in Fig. 7.24.

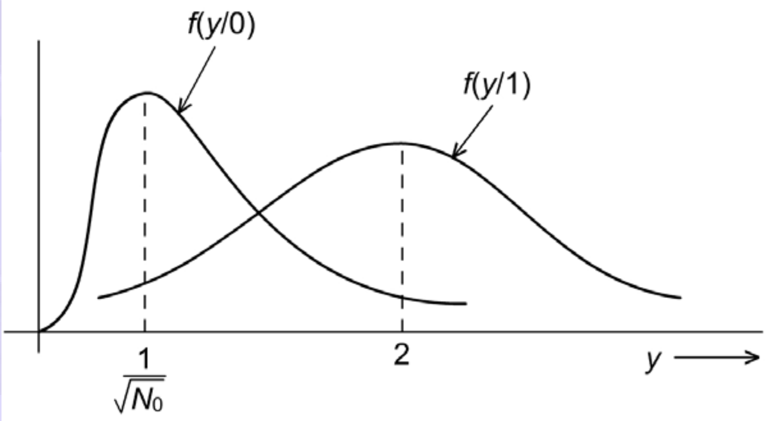


Fig. 7.24: The conditional PDFs of Example 7.10

b) As the decision threshold is taken as '1', we have

$$P_{e,0} = \int_1^{\infty} 10y\, e^{-5y^2}\, dy = e^{-5}$$

$$P_{e,1} = \int_{-\infty}^{1} \frac{1}{\sqrt{2\pi}\,\sqrt{0.1}}\, e^{-\frac{(y-2)^2}{0.2}}\, dy$$

$$= Q\left(\sqrt{10}\right)$$

$$P\left(error\right) = P_e = \frac{1}{2}\left[e^{-5} + Q\left[\sqrt{10}\right]\right]$$ ♦

Appendix A7.1

PSD of Noise for Angle Modulated Signals

For the case of strong predetection SNR, we have (Eq. 7.22b),

$$\theta(t) \simeq \varphi(t) + \frac{r_n(t)}{A_c} \sin\left[\psi(t) - \varphi(t)\right]$$

Let $$\lambda(t) = \frac{r_n(t)}{A_c} \sin\left[\psi(t) - \varphi(t)\right]$$

$$= \frac{1}{A_c} I_m\left[e^{-j\varphi(t)} r_n(t)\, e^{j\psi(t)}\right]$$

Note that $e^{j\varphi(t)}$ is the complex envelope of the FM signal and $r_n(t)\, e^{j\psi(t)}$ is $n_{ce}(t)$, the complex envelope of the narrow band noise.

Let $x_{ce}(t) = e^{-j\varphi(t)} n_{ce}(t) = x_c(t) + j\, x_s(t)$.

Then, $\lambda(t) = \dfrac{1}{A_c} x_s(t)$

We treat $x_{ce}(t)$ to be a sample function of a WSS random process $X_{ce}(t)$ where,

$$X_{ce}(t) = e^{-j\Phi(t)} N_{ce}(t) = X_c(t) + j\, X_s(t) \qquad \text{(A7.1.1)}$$

Similarly,

$$\Lambda(t) = \frac{1}{A_c} X_s(t) \qquad \text{(A7.1.2)}$$

Eq. A7.1.2 implies, the ACF of $\Lambda(t)$, $R_\Lambda(\tau)$ is

$$R_\Lambda(\tau) = \frac{1}{A_c^2} R_{X_s}(\tau) \qquad \text{(A1.7.3a)}$$

and the PSD

$$S_\Lambda(f) = \frac{1}{A_c^2} S_{X_s}(f) \qquad \text{(A7.1.3b)}$$

We will first show that

$$E\left[X_{ce}(t+\tau)\, X_{ce}(t)\right] = 0$$

$$E\left[X_{ce}(t+\tau)\, X_{ce}(t)\right] = E\left[N_{ce}(t+\tau)\, N_{ce}(t)\, e^{-j\left[\Phi(t+\tau)+\Phi(t)\right]}\right]$$

(A7.4.1a)

$$= E\left[N_{ce}(t+\tau)\, N_{ce}(t)\right] E\left[e^{-j\Phi(t+\tau)+\Phi(t)}\right]$$

(A7.4.1b)

Eq. A7.4.1(b) is due to the condition that the signal and noise are statistically independent.

$$E\left[N_{ce}(t+\tau)\, N_{ce}(t)\right] = \left[R_{N_c}(\tau) - R_{N_s}(\tau)\right] + j\left[R_{N_c N_s}(\tau) + R_{N_s N_c}(\tau)\right]$$

As $R_{N_c}(\tau) = R_{N_s}(\tau)$ and $R_{N_c N_s}(\tau) = -R_{N_s N_c}(\tau)$,

we have $E\left[N_{ce}(t+\tau)\, N_{ce}(t)\right] = 0$

Therefore

$$E\left[X_{ce}(t+\tau)\, X_{ce}(t)\right] = \left[R_{X_c}(\tau) - R_{X_s}(\tau)\right] + j\left[R_{X_c X_s}(\tau) + R_{X_s X_c}(\tau)\right] = 0$$

This implies $R_{X_c}(\tau) = R_{X_s}(\tau)$.

Now consider the autocorrelation of $X_{ce}(t)$,

$$E\left[X_{ce}(t+\tau)\, X_{ce}^*(t)\right] = E\left\{\left[e^{-j\Phi(t+\tau)}\, N_{ce}(t+\tau)\right]\left[e^{j\Phi(t)}\, N_{ce}^*(t)\right]\right\}$$

(A7.1.5)

$$E\left[X_{ce}(\;)\, X_{ce}^*(\;)\right] = \underbrace{E\left[N_{ce}(t+\tau)\, N_{ce}^*(t)\right]}_{a_1 + j b_1}\; \underbrace{E\left[e^{j\left[\Phi(t)-\Phi(t+\tau)\right]}\right]}_{a_2 + j b_2} \qquad \text{(A7.1.6)}$$

$$= \left[R_{X_c}(\tau) + R_{X_s}(\tau)\right] + j\left[R_{X_s X_c}(\tau) - R_{X_c X_s}(\tau)\right]$$

$$= 2\, R_{X_s}(\tau) + j\left[R_{X_s X_c}(\tau) - R_{X_c X_s}(\tau)\right]$$

(Note that $R_{X_c}(\tau) = R_{X_s}(\tau)$ as proved earlier.)

$$R_{X_s}(\tau) = \frac{1}{2}\,\mathrm{Re}\left[(a_1 + j b_1)(a_2 + j b_2)\right]$$

$$= \frac{1}{2}\left(a_1 a_2 - b_1 b_2\right)$$

Now $b_1 = 2 R_{N_s N_c}(\tau) = 0$ for symmetric noise PSD, and $a_1 = 2 R_{n_s}(\tau)$. Hence,

$$R_{X_s}(\tau) = \frac{1}{2}\left[2 R_{N_s}(\tau)\right] \underbrace{E\left\{\cos\left[\Phi(t) - \Phi(t+\tau)\right]\right\}}_{g(\tau)}$$

That is,

$$S_{X_s}(f) = S_{N_s}(f) * G(f)$$

where $g(\tau) \longleftrightarrow G(f)$.

As $S_\Lambda(f) = \frac{1}{A_c^2} S_{X_s}(f)$, we have,

$$S_\Lambda(f) = \frac{1}{A_c^2}\left[S_{N_s}(f) * G(f)\right]$$

By definition, $g(\tau)$ is the real part of the ACF of $e^{-j\Phi(t)}$. We know that $e^{j\Phi(t)}$ is the complex envelope of the FM process. For a wideband PM or FM, the bandwidth of $G(f) >> W$, the signal bandwidth. The bandwidth of $G(f)$ is approximately $\frac{B_T}{2}$, as shown in Fig. A7.1.

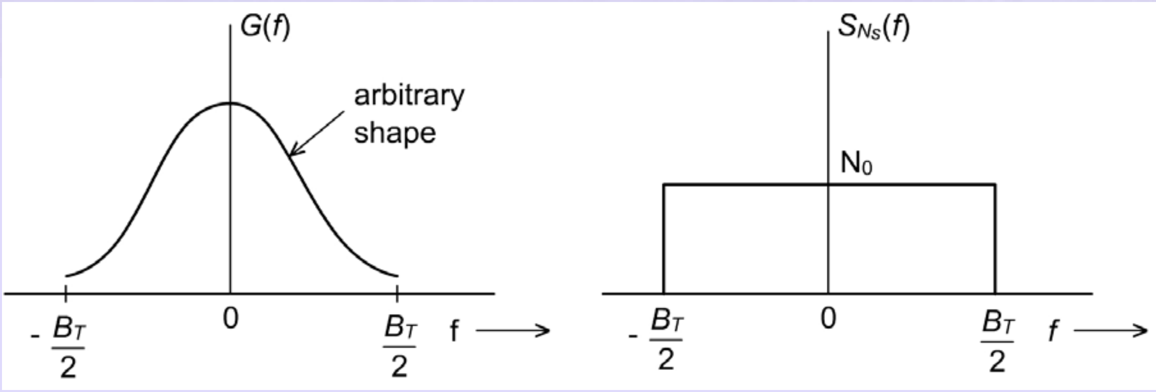


Fig. A7.1: Components of $S_\Lambda(f)$: (a) typical $G(f)$ (b) $S_{N_s}(f)$

Note that $g(0) = \int_{-\infty}^{\infty} G(f)\, df = E[\cos(0)] = 1$.

For $|f| \le W$, $G(f) * S_{N_s}(f) \simeq N_0 \int_{-\infty}^{\infty} G(f)\, df = N_0$. That is, $\Phi(t)$ is immaterial as far as PSD of $N_s(t)$ in the range $|f| \le W$ is concerned. Hence, we might as well set $\varphi(t) = 0$ in the Eq. 7.22 (b), for the purpose of calculating the noise PSD at the output of the discriminator.

References

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