CHAPTER 3

Random Signals and Noise

3.1 Introduction

The concept of 'random variable' is adequate to deal with unpredictable voltages; that is, it enables us to come up with the probabilistic description of the numerical value of a random quantity, which we treat for convenience to be a voltage quantity. In the real world, the voltages vary not only in amplitude but also exhibit variations with respect to the time parameter. In other words, we have to develop mathematical tools for the probabilistic characterization of random signals. The resulting theory, which extends the mathematical model of probability so as to incorporate the time parameter, is generally called the theory of Random or Stochastic Processes.

Before we get started with the mathematical development of a random process, let us consider a few practical examples of random processes and try to justify the assertion that the concept of random process is an extension of the concept of a random variable.

Let the variable X denote the temperature of a certain city, say, at 9 A.M. In general, the value of X would be different on different days. In fact, the temperature readings at 9 A.M. on two different days could be significantly different, depending on the geographical location of the city and the time separation between the days of observation (In a place like Delhi, on a cold winter morning, temperature could be as low as $40^\circ\ F$ whereas at the height of the summer, it could have crossed 100° F even by 9 A.M.!).

To get the complete statistics of X, we need to record values of X over many days, which might enable us to estimate $f_X(x)$.

But the temperature is also a function of time. At 12 P.M., for example, the temperature may have an entirely different distribution. Thus the random variable X is a function of time and it would be more appropriate to denote it by $X(t)$. At least, theoretically, the PDF of $X(t_1)$ could be very much different from that of $X(t_2)$ for $t_1 \neq t_2$, though in practice, they may be very much similar if t_1 and t_2 are fairly closely spaced.

As a second example, think of a situation where we have a very large number of speakers, each one of them uttering the same text into their individual microphones of identical construction. The waveforms recorded from different microphones would be different and the output of any given microphone would vary with time. Here again, the random variables obtained from sampling this collection of waveforms would depend on the sampling instants.

As a third example, imagine a large collection of resistors, each having the same value of resistance and of identical composition and construction. Assume that all these resistors are at room temperature. It is well known that thermal voltage (usually referred to as thermal noise) develops across the terminals of such a resistor. If we make a simultaneous display of these noise voltages on a set of oscilloscopes, we find amplitude as well as time variations in these signals. In the communications context, these thermal voltages are a source of interference. Precisely how they limit our capacity to enjoy, say, listening to music in an AM receiver, is our concern. The theory of random processes enables one to come up with a quantitative answer to this kind of problem.

3.2 Definition of a Random Process

Consider a sample space $\mathcal{S}$ (pertaining to some random experiment) with sample points $s_1, s_2, ..., s_n, ...$. To every $s_j \in \mathcal{S}$, let us assign a real valued function of time, $x(s_j, t)$ which we denote by $x_j(t)$. This situation is illustrated in Fig. 3.1, which shows a sample space $\mathcal{S}$ with four points and four waveforms, labeled $x_j(t)$, $j = 1, 2, 3, 4$.

Now, let us think of observing this set of waveforms at some time instant $t = t_1$ as shown in the figure.

Since each point s_j of $\mathcal{S}$ has associated with it, a number $x_j(t_1)$ and a probability P_j, the collection of numbers, $\{x_j(t_1), j = 1, 2, 3, 4\}$ forms a random variable. Observing the waveforms at a second time instant, say t_2, yields a different collection of numbers, and hence a different random variable. Indeed this set of four waveforms defines a random variable for each choice of the observation instant. The above situation can easily be extended to the case where there are infinite numbers of sample points and hence, the number of waveforms associated with them are correspondingly rich.

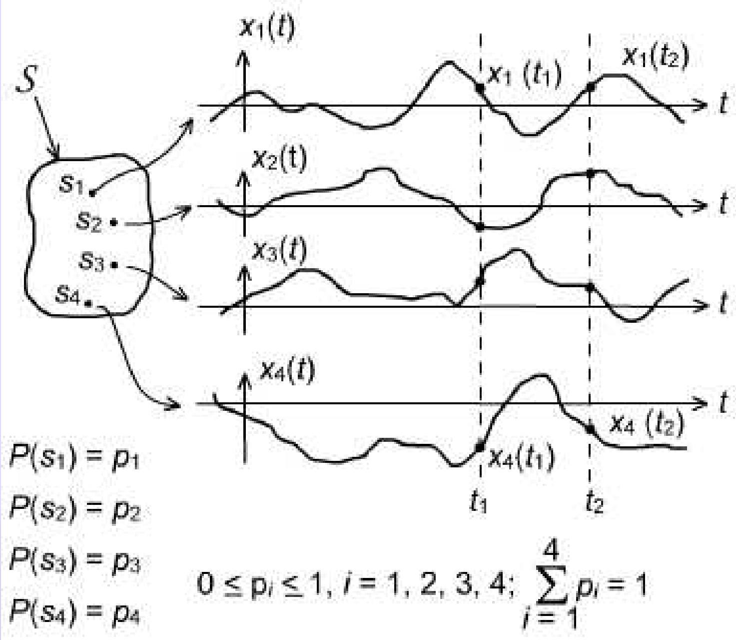

Fig 3.1: A simple random process

The probability system composed of a sample space, an ensemble (collection) of time functions and a probability measure is called a *Random Process* (RP) and is denoted $X(t)$. Strictly speaking, a random process is a function of two variables, namely $s \in \mathcal{S}$ and $t \in (-\infty, \infty)$. As such, a better notation would be $X(s, t)$. For convenience, we use the simplified notation $X(t)$ to denote a random process. The individual waveforms of $X(t)$ are called sample functions and the probability measure is such that it assigns a probability to any meaningful event associated with these sample functions.

Given a random process $X(t)$, we can identify the following quantities:

$X(t)$: The random process

$x_j(t)$: The sample function associated with the sample point s_j

$X(t_i)$: The random variable obtained by observing the process at $t = t_i$

$x_j(t_i)$: A real number, giving the value of $x_j(t)$ at $t = t_i$.

We shall now present a few examples of random processes.

Example 3.1

Consider the experiment of tossing a fair coin. The random process $X(t)$ is defined as follows: $X(t) = \sin(\pi t)$, if head shows up and $X(t) = 2t$, if the toss results in a tail. Sketch the sample functions. We wish to find the expression for the PDF of the random variables obtained from sampling the process at (a) $t = 0$ and (b) $t = 1$.

There are only two sample functions for the process. Let us denote them by $x_1(t)$ and $x_2(t)$ where $x_1(t) = \sin(\pi t)$ and $x_2(t) = 2t$ which are shown in Fig. 3.2.

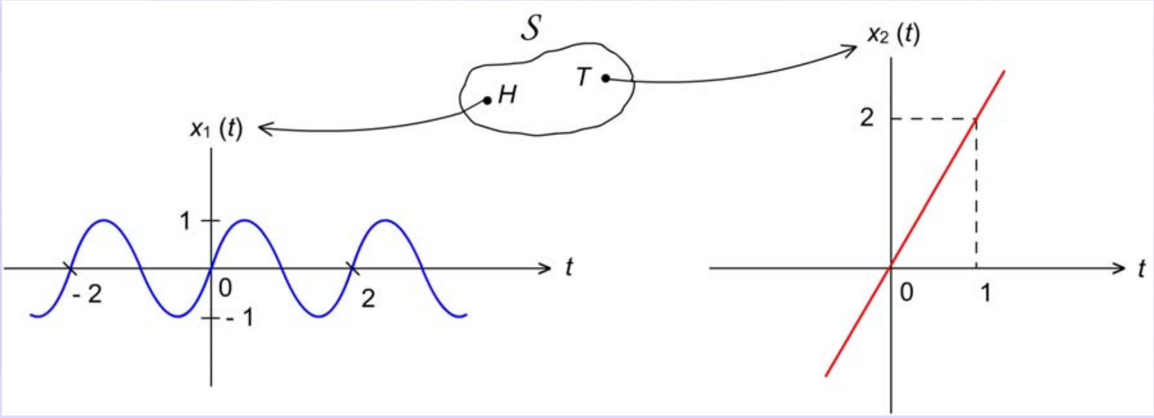

Fig. 3.2: The ensemble for the coin tossing experiment

As heads and tails are equally likely, we have $P[x_1(t)] = P[x_2(t)] = \frac{1}{2}$. Let X_0 denote the random variable $X(t)|_{t=0}$ and X_1 correspond to $X(t)|_{t=1}$. Then, we have $f_{X_0}(x) = \delta(x)$ and $f_{X_1}(x) = \frac{1}{2}[\delta(x) + \delta(x-2)]$. ♦

Note that this is one among the simplest examples of RPs that can be used to construe the concept.

Example 3.2

Consider the experiment of throwing a fair die. The sample space consists of six sample points, $s_1, ..., s_6$ corresponding to the six faces of the die. Let the sample functions be given by $x_i(t) = \frac{1}{2} t + (i - 1)$ for $s = s_i$, $i = 1, ..., 6$. Let us find the mean value of the random variable $X = X(t)|_{t = 1}$.

A few of the sample functions of this random process are shown below (Fig 3.3).

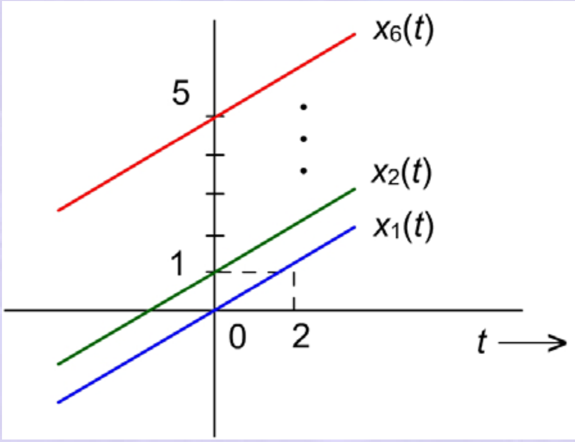

Fig. 3.3: A few sample functions of the RP of Example 3.2

The PDF of X is

$$f_X(x) = \frac{1}{6} \sum_{i = 1}^{6} \delta\left[x - \left(\frac{1}{2} + (i - 1)\right)\right]$$

$$E[X] = \frac{1}{12}(1 + 3 + 5 + 7 + 9 + 11) = 3.0$$ ♦

The examples cited above have two features in common, namely (i) the number of sample functions are finite (in fact, we could even say, quite small) and (ii) the sample functions could be mathematically described. In quite a few situations involving a random process, the above features may not be present.

Consider the situation where we have at our disposal N identical resistors, N being a very large number (of the order of a million!).

Let the experiment be 'picking a resistor at random' and the sample functions be the thermal noise voltage waveforms across these resistors. Then, typical sample functions might look like the ones shown in Fig. 3.4.

Assuming that the probability of any resistor being picked up is $\frac{1}{N}$, we find that this probability becomes smaller and smaller as N becomes larger and larger. Also, it would be an extremely difficult task to write a mathematical expression to describe the time variation of any given voltage waveform. However, as we shall see later on in this chapter, statistical characterization of such noise processes is still possible which is adequate for our purposes.

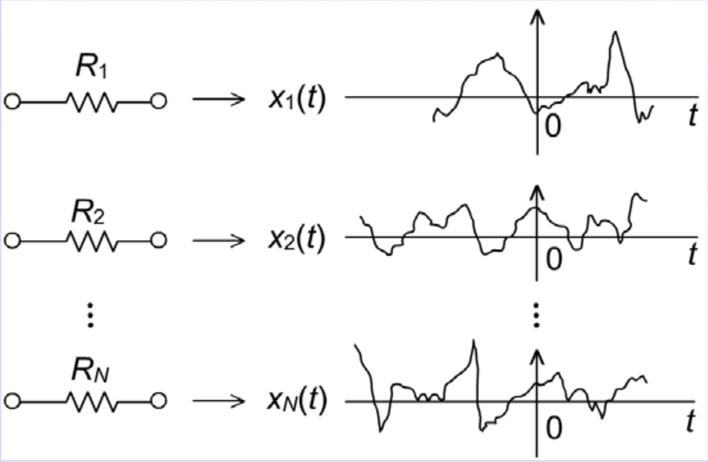

Fig. 3.4: The ensemble for the experiment 'picking a resistor at random'

One fine point deserves a special mention; that is, the waveforms (sample functions) in the ensemble are not random. They are deterministic. Randomness in this situation is associated not with the waveforms but with the uncertainty as to which waveform will occur on a given trial.

3.3 Stationarity

By definition, a random process $X(t)$ implies the existence of an infinite number of random variables, one for every time instant in the range, $-\infty < t < \infty$. Let the process $X(t)$ be observed at n time instants, $t_1, t_2, ..., t_n$. We then have the corresponding random variables $X(t_1), X(t_2), ..., X(t_n)$. We define their *Joint distribution function* by $F_{X(t_1), ..., X(t_n)}(x_1, x_2, ..., x_n) = P\{X(t_1) \le x_1, ... X(t_n) \le x_n\}$. Using a vector notation is quite often convenient and we denote the joint distribution by $F_{\boldsymbol{X}(\boldsymbol{t})}(x)$ where the n-component random vector $\boldsymbol{X}(\boldsymbol{t}) = (X(t_1), ..., X(t_n))$ and the dummy vector $\boldsymbol{x} = (x_1, x_2, ..., x_n)$. The joint PDF of $X(t)$, $f_{\boldsymbol{X}(\boldsymbol{t})}(\boldsymbol{x})$, is given by

$$f_{\boldsymbol{X}(\boldsymbol{t})}(\boldsymbol{x}) = \frac{\partial^n}{\partial x_1 \, \partial x_2 \, ... \, \partial x_n} F_{\boldsymbol{X}(\boldsymbol{t})}(\boldsymbol{x})$$

We say a random process $X(t)$ is specified if and only if a rule is given or implied for determining $F_{X(t)}(x)$ or $f_{X(t)}(x)$ for any finite set of observation instants $(t_1, t_2, ..., t_n)$.

In application, we encounter three methods of specification. The first (and simplest) is to state the rule directly. For this to be possible, the joint density function must depend in a known way on the time instants. For the second method, a time function involving one or more parameters is given. For example, $X(t) = A\cos(\omega_c t + \Theta)$ where A and ω_c are constants and Θ is a random variable with a known PDF. The third method of specifying a random process is to generate its ensemble by applying a stated operation to the sample functions of a known process. For example, a random process $Y(t)$ may be the result of linear filtering on some known process $X(t)$. We shall see later on in this lesson, examples of all these methods of specification.

Once $f_{X(t)}(x)$ is known, it is possible for us to compute the probability of various events. For example, we might be interested in the probability of the random process $X(t)$ passing through a set of windows as shown in Fig. 3.5.

Let A be the event:

$$A = \{s : a_1 < X(t_1) \le b_1, a_2 < X(t_2) \le b_2, a_3 < X(t_3) \le b_3\}$$

That is, the event A consists of all those sample points $\{s_j\}$ such that the corresponding sample functions $\{x_j(t)\}$ satisfy the requirement, $a_i \le x_j(t_i) \le b_i$, $i = 1, 2, 3$. Then the required quantity is $P(A)$. A typical sample function which would contribute to $P(A)$ is shown in the same figure. $P(A)$ can be calculated as

$$P(A) = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f_{\mathbf{X}(\mathbf{t})}(\mathbf{x})\, \mathbf{d\,x}$$

where $\mathbf{x}(\mathbf{t}) = (X(t_1), X(t_2), X(t_3))$ and $\mathbf{d\,x} = d\,x_1\, d\,x_2\, d\,x_3$

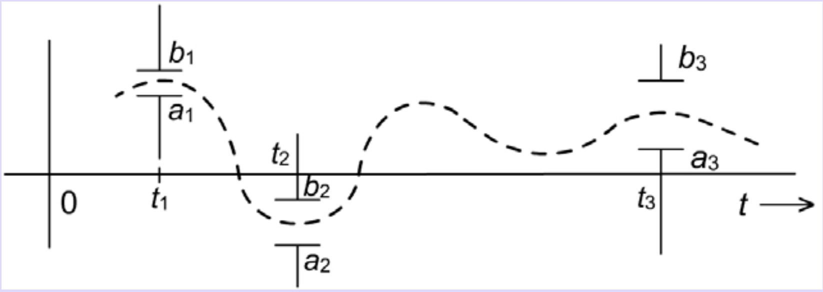

Fig. 3.5: Set of windows and a waveform that passes through them

The above step can easily be generalized to the case of a random vector with n-components.

Stationary random processes constitute an important subset of the general class of random processes. We shall define stationarity here. Let the process $X(t)$ be observed at time instants $t_1, t_2, ..., t_n$ and $\boldsymbol{X}(\boldsymbol{t})$ be the corresponding random vector.

Def. 3.1: A random process $X(t)$ is said to be *strictly stationary or Stationary in a Strict Sense* (SSS) if the joint PDF $f_{\boldsymbol{X}(\boldsymbol{t})}(\boldsymbol{x})$ is invariant to a translation of the time origin; that is, $X(t)$ is SSS, only if

$$f_{X(t+T)}(x) = f_{X(t)}(x) \tag{3.1}$$

where $(t + T) = (t_1 + T, t_2 + T, ..., t_n + T)$. ♦

For $X(t)$ to be SSS, Eq. 3.1 should be valid for every finite set of time instants $\{t_j\}$, $j = 1, 2, ..., n$, and for every time shift T and dummy vector $\boldsymbol{x}$. If $X(t)$ is not stationary, then it is called a nonstationary process.

One implication of stationarity is that the probability of the set of sample functions of this process which passes through the windows of Fig. 3.6(a) is equal to the probability of the set of sample functions which passes through the corresponding time shifted windows of Fig. 3.6(b). Note, however, that it is not necessary that these two sets consist of the same sample functions.

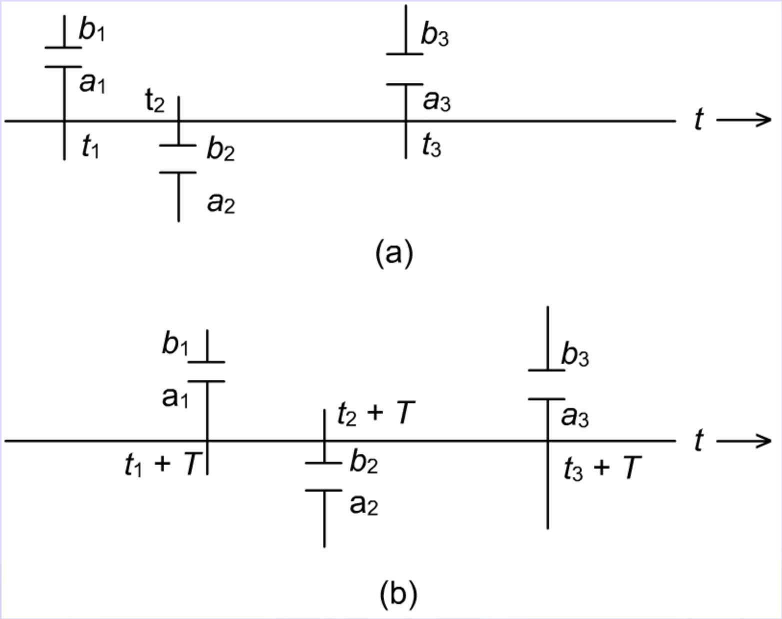

Fig. 3.6: (a) Original set of Windows (b) The translated set

Example 3.3

We cite below an example of a nonstationary process (It is easier to do this than give a nontrivial example of a stationary process). Let

$$X(t) = \sin(2\pi F t)$$

where F is a random variable with the PDF

$$f_F(f) = \begin{cases} \dfrac{1}{100}, & 100 \leq f \leq 200\ Hz \\ 0, & otherwise \end{cases}$$

(Note that this specification of $X(t)$ corresponds to the second method mentioned earlier on in this section). We now show that $X(t)$ is nonstationary.

$X(t)$ consists of an infinite number of sample functions. Each sample function is a sine wave of unit amplitude and a particular frequency f. Over the ensemble, the random variable F takes all possible values in the range $(100, 200\ Hz)$. Three members of this ensemble, (with $f = 100, 150$ and 200 Hz) are plotted in Fig. 3.7.

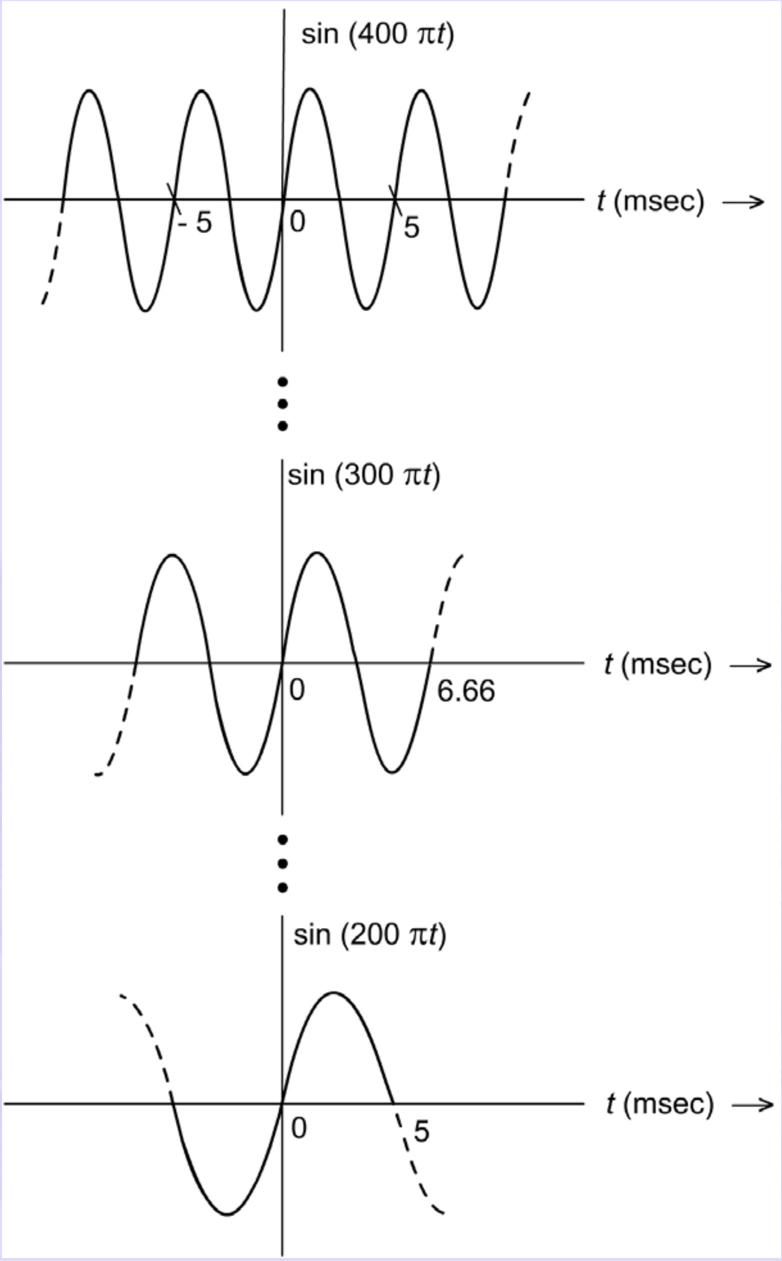

Fig. 3.7: A few sample functions of the process of example 3.3

To show that $X(t)$ is nonstationary, we need only observe that *every* waveform in the ensemble is,

zero at $t = 0$,

positive for $0 < t < 2.5\, m\sec$

negative for $(-2.5\, m\sec) < t < 0$.

Thus the density function of the random variable $X(t_1)$ for $t_1 = 1\, m\sec$ is identically zero for negative arguments whereas the density function of the RV $X(t_2)$ for $t_2 = -1\, m\sec$ is non-zero only for negative arguments (Of course, the PDF of $X(0)$ is an impulse). For a process that is SSS, the one-dimensional PDF is independent of the observation instant, which is evidently not the case for this example. Hence $X(t)$ is nonstationary. ♦

3. 4 Ensemble Averages

We had mentioned earlier that a random process is completely specified, if $f_{\boldsymbol{X}(t)}(\boldsymbol{x})$ is known. Seldom is it possible to have this information and we may have to be content with a partial description of the process based on certain averages. When these averages are derived from the ensemble, they are called *ensemble averages*. Usually, the mean function and the autocorrelation function, (or the auto-covariance function) of the process provide a useful description of the processes. At times, we require the cross-correlation between two different processes. (This situation is analogous to the random variable case, wherein we had mentioned that even if the PDF of the variable is not available, certain averages such as mean value, variance etc., do provide adequate or useful information).

Def. 3.2: The Mean Function

The mean function of a process $X(t)$ is

$$m_X(t) = E\left[X(t)\right] = \overline{X(t)} = \int_{-\infty}^{\infty} x\, f_{X(t)}(x)\, dx \qquad (3.2)$$ ♦

For example, if X_i and X_j are the random variables obtained by sampling the process at $t = t_i$ and $t = t_j$ respectively, then

$$\overline{X_i} = \int_{-\infty}^{\infty} x\, f_{X_i}(x)\, dx = m_X(t_i)$$

$$\overline{X_j} = \int_{-\infty}^{\infty} x\, f_{X_j}(x)\, dx = m_X(t_j)$$

In general, $m_X(t)$ is a function of time.

Def. 3.3: The Autocorrelation Function

The autocorrelation function of a random process $X(t)$ is a function of two variables t_k and t_i, and is given by

$$R_X(t_k, t_i) = E\left[X(t_k)\, X(t_i)\right] \qquad (3.3)$$

♦

Denoting the joint PDF of the random variables $X(t_k)$ and $X(t_i)$ by $f_{X_k, X_i}(x, y)$ we may rewrite Eq. 3.3 as

$$R_X(t_k, t_i) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, y\, f_{X_k, X_i}(x, y)\, dx\, dy$$

We also use ACF to denote the Auto Correlation Function.

Def. 3.4: The Auto-covariance Function

Let $C_X(t_k, t_i)$ denote the auto covariance function of $X(t)$. It is given by

$$C_X(t_k, t_i) = E\left[\left(X(t_k) - m_X(t_k)\right)\left(X(t_i) - m_X(t_i)\right)\right] \qquad (3.4a)$$

It is not too difficult to show that

$$C_X(t_k, t_i) = R_X(t_k, t_i) - m_X(t_k)\, m_X(t_i) \qquad (3.4b)$$

♦

In general, the autocorrelation and the auto-covariance would be a function of both the arguments t_k and t_i. If the process has a zero mean value (that is, $m_X(t) = 0$ for all t), then $C_X(t_k, t_i) = R_X(t_k, t_i)$.

For a stationary process, the ensemble averages defined above take a simpler form. In particular, we find that mean function of the process is a constant. That is,

$$m_X(t) = m_X \qquad (3.5a)$$

m_X being a constant. In such a case, we can simply mention the mean value of the process. Also, for a stationary process, we find that the autocorrelation and auto-covariance functions depend only on the time difference $(t_k - t_i)$, rather than on the actual values of t_k and t_i. This can be shown as follows.

$$R_X(t_k, t_i) = E\left[X(t_k)\, X(t_i)\right]$$

$$= E\left[X(t_k + T)\, X(t_i + T)\right], \text{ as } X(t) \text{ is stationary.}$$

In particular if $T = -t_i$, then

$$R_X(t_k, t_i) = E\left[X(t_k - t_i)\, X(0)\right] \tag{3.5b}$$

$$R_X(t_k, t_i) = R_X(t_k - t_i, 0)$$

In order to simplify the notation, it is conventional to drop the second argument on the RHS of Eq. 3.5(b) and write as

$$R_X(t_k, t_i) = R_X(t_k - t_i)$$

In view of Eq. 3.4(b), it is not difficult to see that for a stationary process

$$C_X(t_k, t_i) = C_X(t_k - t_i)$$

It is important to realize that for a process that is SSS, Eq. 3.5(a) and Eq. 3.5(b) hold. However, we should not infer that any process for which Eq. 3.5 is valid, is a stationary process. In any case, the processes satisfying Eq. 3.5 are sufficiently useful and are termed Wide Sense Stationary (WSS) processes.

Def. 3.5: Wide Sense Stationarity

A process $X(t)$ is WSS or *stationary in a wide sense*[1], provided

$$m_X(t) = m_X \tag{3.6a}$$

and $$R_X(t_k, t_i) = R_X(t_k - t_i) \tag{3.6b}$$

♦

Wide sense stationarity represents a weak kind of stationarity in that all processes that are SSS are also WSS; but the converse is not necessarily true. When we simply use the word stationary, we imply stationarity in the strict sense.

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[1] For definitions of other forms of stationarity (such as Nth order stationary) see [1, P302]

3.4.1 Properties of ACF

The autocorrelation function of a WSS process satisfies certain properties, which we shall presently establish. ACF is a very useful tool and a thorough understanding of its behavior is quite essential for a further study of the random processes of interest to us. For convenience of notation, we define the ACF of a wide-sense stationary process $X(t)$ as

$$R_X(\tau) = E\left[X(t+\tau)\, X(t)\right] \tag{3.7}$$

Note that τ is the difference between the two time instants at which the process is being sampled.

P1) The mean-square value of the process is $R_X(\tau)\big|_{\tau = 0}$.

This follows from Eq. 3.7 because,

$$R_X(0) = E\left[X^2(t)\right]$$

As $R_X(0)$ is a constant, we infer that for a WSS process, mean and mean-square values are independent of time.

P2) The ACF is an even function of τ; that is,

$$R_X(-\tau) = R_X(\tau)$$

This is because $E\left[X(t+\tau)\, X(t)\right] = E\left[X(t)\, X(t+\tau)\right]$. That is,

$$R_X\left[(t+\tau) - t\right] = R_X\left[t - (t+\tau)\right]$$

which is the desired result.

P3) ACF is maximum at the origin.

Consider the quantity, $E\left\{\left[X(t) \pm X(t+\tau)\right]^2\right\}$

Being the expectation of a squared quantity, it is nonnegative. Expanding the squared quantity and making use of the linearity property of the expectation, we have

$$R_X(0) \pm R_X(\tau) \geq 0$$

which implies $R_X(0) \geq \left|R_X(\tau)\right|$.

P4) If the sample functions of the process $X(t)$ are periodic with period T_0, then the ACF is also periodic with the same period.

This property can be established as follows.

Consider $E\left[X(t+\tau)\,X(t)\right]$ for $\tau \geq T_0$. As each sample function repeats with period T_0, the product repeats and so does the expectation of this product.

The physical significance of $R_X(\tau)$ is that it provides a means of describing the inter-dependence of two random variables obtained by observing the process $X(t)$ at two time instants τ seconds apart. In fact if $X(t)$ is a zero mean process, then for any $\tau = \tau_1$, $\dfrac{R_X(\tau_1)}{R_X(0)}$ is the correlation coefficient of the two random variables spaced τ_1 seconds apart. It is therefore apparent that the more rapidly $X(t)$ changes with time, the more rapidly $R_X(\tau)$ decreases from its maximum value $R_X(0)$ as τ increases (Fig. 3.8). This decrease may be characterized by a *de-correlation time* τ_0, such that for $|\tau| \geq \tau_0, |R_X(\tau)|$ remains below some prescribed value, say $\dfrac{R_X(0)}{100}$. We shall now take up a few examples to compute some of the ensemble averages of interest to us.

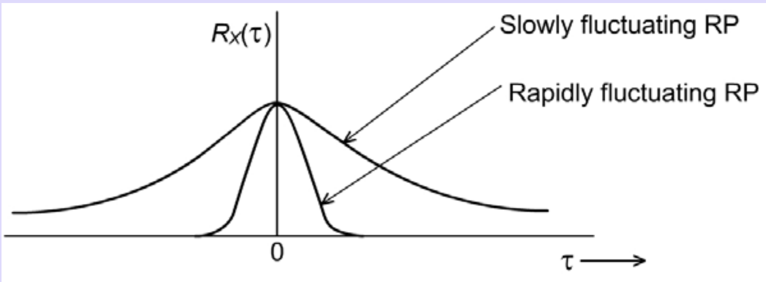


Fig. 3.8: ACF of a slowly and rapidly fluctuating random process

Example 3.4

For the random process of Example 3.2, let us compute $R_X(2, 4)$.

$$R_X(2, 4) = E\left[X(2)\, X(4)\right]$$

$$= \sum_{j=1}^{6} P\left[x_j\right] x_j(2)\, x_j(4)$$

A few of the sample functions of the process are shown below.

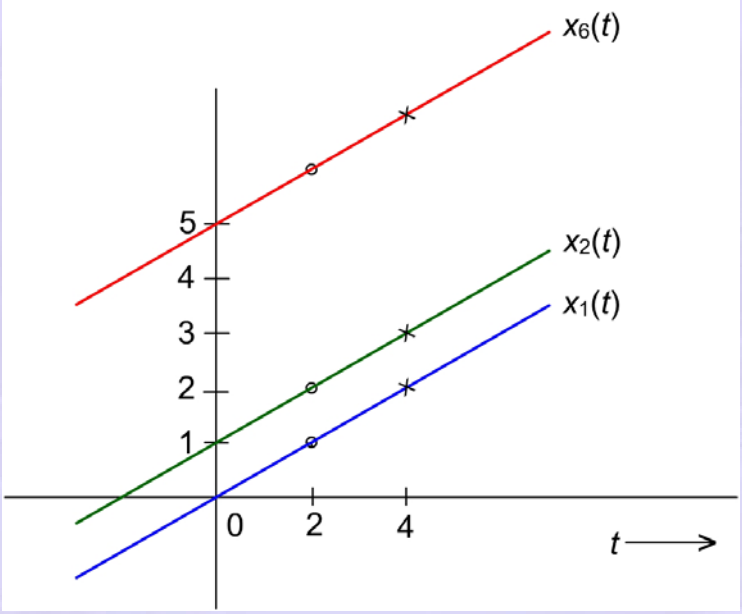


Fig. 3.9: Sampling the process of example 3.2 at $t = 2$ and 4

As $P\left[x_j\right] = \frac{1}{6}$ for all j, we have

$$R_X(2, 4) = \frac{1}{6} \sum_{j=1}^{6} x_j(2)\, x_j(4)$$

for $j = 1$, the two samples that contribute to the ACF have the values 1 (indicated by the $\circ$ on $x_1(t)$) and 2 (indicated by $\times$ on $x_1(t)$). The other sample pairs are $(2, 3)$, $(3, 4)$, $(4, 5)$, $(5, 6)$, $(6, 7)$.

$$\text{Hence } R_X(2, 4) = \frac{1}{6}[2 + 6 + 12 + 20 + 30 + 42]$$

$$= \frac{112}{6} = 18.66 \qquad \blacklozenge$$

Exercise 3.1

Let a random process $X(t)$ consist of 6 equally likely sample functions, given by $x_i(t) = i\,t$, $i = 1, 2, ..., 6$. Let X and Y be the random variables obtained by sampling process at $t = 1$ and $t = 2$ respectively.

Find

a) $E[X]$ and $E[Y]$

b) $f_{X,Y}(x, y)$

c) $R_X(1, 2)$

Exercise 3.2

A random process $X(t)$ consists of 5 sample functions, each occurring with probability $\frac{1}{5}$. Four of these sample functions are given below.

$$x_1(t) = \cos(2\pi t) - \sin(2\pi t)$$

$$x_2(t) = \sin(2\pi t) - \cos(2\pi t)$$

$$x_3(t) = -\sqrt{2}\cos t$$

$$x_4(t) = -\sqrt{2}\sin t$$

a) Find the fifth sample function $x_5(t)$ of the process $X(t)$ such that the process $X(t)$ is

i) zero mean

ii) $R_X(t_1, t_2) = R_X(|t_1 - t_2|)$

b) Let V be the random variable $X(t)\big|_{t=0}$ and W be the random variable $X(t)\big|_{t=\pi/4}$. Show that, though the process is WSS, $f_V(v) \neq f_W(v)$.

Example 3.5

Let $X(t) = A\cos(\omega_c t + \Theta)$,

where A and ω_c are constants and Θ is a random variable with the PDF,

$$f_\Theta(\theta) = \begin{cases} \dfrac{1}{2\pi}, & 0 \leq \theta < 2\pi \\ 0 \quad, & otherwise \end{cases}$$

Let us compute a) $m_X(t)$ and b) $R_X(t_1, t_2)$.

(Note that the random processes is specified in terms of a random parameter, namely Θ , another example of the second method of specification)

The ensemble of $X(t)$ is composed of sinusoids of amplitude A and frequency f_c but with a random initial phase. Of course, a given sample function has a fixed value for $\theta = \theta_1$, where $0 \leq \theta_1 < 2\pi$.

Three sample functions of the process are shown in Fig. 3.10 for $f_c = 10^6\ Hz$ and $A = 1$.

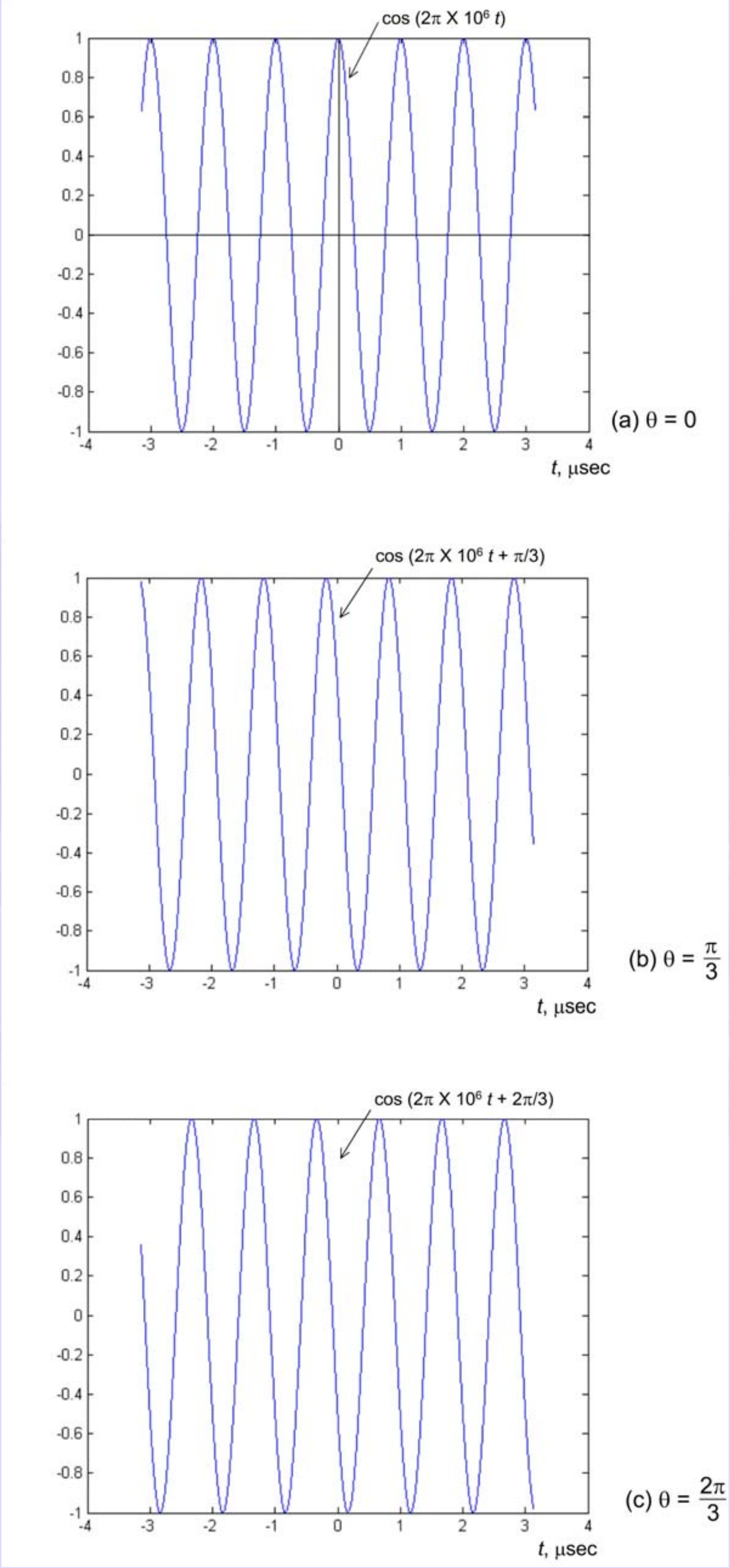


Fig 3.10: Three sample functions of the process of example 3.5

a) $m_X(t) = E\left[A\cos(\omega_c\, t + \Theta)\right]$. As Θ is the only random variable, we have,

$$m_X(t) = \frac{A}{2\pi}\int_0^{2\pi}\cos(\omega_c\, t + \theta)\, d\,\theta$$

$$= 0$$

Note that Θ is uniform in the range 0 to 2π.

This is a reasonable answer because, for any given $\theta = \theta_1$, the ensemble has both the waveforms, namely, $A\cos(\omega_c t + \theta_1)$ and $A\cos(\omega_c t + \theta_1 + \pi) = -\left[A\cos(\omega_c t + \theta_1)\right]$. Both the waveforms are equally likely and sum to zero for all values of t.

b) $R_X(t_1, t_2) = E\left[A\cos(\omega_c\, t_1 + \Theta)\, A\cos(\omega_c\, t_2 + \Theta)\right]$

$$= \frac{A^2}{2}\left\{\overline{\cos\left[\omega_c\,(t_1 + t_2) + 2\,\Theta\right]} + \overline{\cos\left[\omega_c\,(t_1 - t_2)\right]}\right\}$$

As the first term on the RHS evaluates to zero, we have

$$R_X(t_1, t_2) = \frac{A^2}{2}\cos\left[\omega_c\,(t_1 - t_2)\right]$$

As the ACF is only a function of the time difference, we can write

$$R_X(t_1, t_2) = R_X(t_1 - t_2) = R_X(\tau) = \frac{A^2}{2}\cos(\omega_c\,\tau)$$

Note that $X(t)$ is composed of sample functions that are periodic with period $\frac{1}{f_c}$. In accordance with property P4, we find that the ACF is also periodic with period $\frac{1}{f_c}$. Of course, it is also an even function of τ (property P3). ♦

Exercise 3.3

For the process $X(t)$ of example 3.5, let $\omega_c = 2\pi$. let Y be the random variable obtained from sampling $X(t)$ and $t = \frac{1}{4}$. Find $f_Y(y)$.

Exercise 3.4

Let $X(t) = A\cos(\omega_c t + \Theta)$ where A and ω_c are constants, and θ is a random variable, uniformly distributed in the range $0 \leq \theta \leq \pi$. Show the process is not WSS.

Exercise 3.5

Let $Z(t) = X\cos(\omega_c t) + Y\sin(\omega_c t)$ where X and Y are independent Gaussian variables, each with zero mean and unit variance. Show that $Z(t)$ is WSS and $R_Z(\tau) = \cos(\omega_c \tau)$. Let $Z(t_1) = V$. Show that V is $N(0, 1)$.

Example 3.6

In this example, we shall find the ACF of the random impulse train process specified by

$$X(t) = \sum_n A_n \, \delta(t - nT - T_d)$$

where the amplitude A_n is a random variable with $P(A_n = 1) = P(A_n = -1) = \frac{1}{2}$. Successive amplitudes are statistically independent. The time interval t_d of the first impulse from the origin has uniform PDF in the range $(0, T)$. That is, $f_{T_D}(t_d) = \frac{1}{T}$, $0 \leq t_d \leq T$ and zero elsewhere. Impulses are spaced T seconds apart and A_n is independent of T_d. (The symbol $\sum_n$ indicates summation with respect to n where n, an integer, ranges from $(-\infty, \infty)$.)

A typical sample function of the process $X(t)$ is shown in Fig. 3.11.

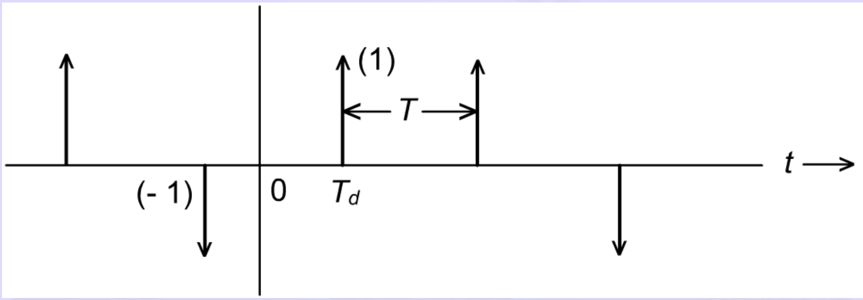


Fig. 3.11: Sample function of the random impulse train process

$$R_X(t+\tau, t) = E\left\{\left[\sum_m A_m \delta(t - mT + \tau - T_d)\right]\left[\sum_n A_n \delta(t - nT - T_d)\right]\right\}$$

$$= E\left[\sum_m \sum_n A_m A_n \delta(t - mT + \tau - T_d)\,\delta(t - nT - T_d)\right]$$

As A_n is independent of T_d, we can write (after interchanging the order of expectation and summation),

$$R_X(t+\tau, t) = \sum_m \sum_n \overline{A_m A_n}\;\overline{\delta(t - nT + \tau - T_d)\,\delta(t - nT - T_d)}$$

But, $\overline{A_m A_n} = \begin{cases} 1, & m = n \\ 0, & otherwise \end{cases}$

This is because when $m = n$, $\overline{A_m A_n} = \overline{A_m^2} = \frac{1}{2}(1)^2 + \frac{1}{2}(-1)^2 = 1$. If $m \neq n$, then $\overline{A_m A_n} = \overline{A_m}\;\overline{A_n}$, as successive amplitudes are statistically independent. But $\overline{A_m} = \overline{A_n} = \frac{1}{2}(1) + \frac{1}{2}(-1) = 0$.

Using this result, we have

$$R_X(t+\tau, t) = \sum_n \overline{\delta(t - nT + \tau - T_d)\,\delta(t - nT - T_d)}$$

$$= \sum_n \frac{1}{T}\int_0^T \delta(t - nT + \tau - t_d)\,\delta(t - nT - t_d)\,d t_d$$

(Note that T_d is uniformly distributed in the range 0 to T)

Let $t - nT - t_d = x$. Then,

$$R_X(t+\tau, t) = \sum_n \frac{1}{T}\int_{t-(n+1)T}^{t-nT} \delta(x)\,\delta(x+\tau)\,dx$$

$$= \frac{1}{T}\int_{+\infty}^{-\infty} \delta(x)\,\delta(x+\tau)\,dx$$

Letting $y = -x$, we have

$$R_X(t+\tau, t) = \frac{1}{T}\int_{-\infty}^{\infty} \delta(-y)\,\delta(\tau - y)\,dy = \frac{1}{T}\int_{-\infty}^{\infty} \delta(y)\,\delta(\tau - y)\,dy$$

$$= \frac{1}{T}\left[\delta(\tau) * \delta(\tau)\right] = \frac{1}{T}\,\delta(\tau)$$

That is, the ACF is a function of τ alone and it is an impulse! ♦

It is to be pointed out that in the case of a random process, we can also define time averages such as *time-averaged mean value* or *time-averaged ACF etc.*, whose calculation is based on the individual sample functions. There are certain processes, called *ergodic processes* where it is possible to interchange the corresponding ensemble and time averages. More details on ergodic processes can be found in [2].

3.4.2 Cross-correlation

Consider two random processes $X(t)$ and $Y(t)$. We define the two *cross-correlation functions* of $X(t)$ and $Y(t)$ as follows:

$$R_{X,Y}(t_1, t_2) = E\left[X(t_1)\,Y(t_2)\right]$$

$$R_{Y,X}(t_1, t_2) = E\left[Y(t_1) X(t_2)\right]$$

where t_1 and t_2 are the two time instants at which the processes are observed.

Def. 3.6:

Two processes are said to be *jointly wide-sense stationary* if,

i) $X(t)$ and $Y(t)$ are WSS and

ii) $R_{X,Y}(t_1, t_2) = R_{X,Y}(t_1 - t_2) = R_{XY}(\tau)$

Cross-correlation is not generally an even function of τ as is the case with ACF, nor does it have a maximum at the origin. However, it does obey a symmetrical relationship as follows:

$$R_{XY}(\tau) = R_{YX}(-\tau) \tag{3.8}$$

Def. 3.7:

Two random process $X(t)$ and $Y(t)$ are called (mutually) orthogonal if

$R_{XY}(t_1, t_2) = 0$ for every t_1 and t_2. ♦

Def. 3.8:

Two random process $X(t)$ and $Y(t)$ are uncorrelated if

$C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - m_X(t_1)\, m_Y(t_2) = 0$ for every t_1 and t_2. ♦

Exercise 3.6

Show that the cross-correlation function satisfies the following inequalities.

a) $\left|R_{XY}(\tau)\right| \leq \sqrt{R_X(0)\, R_Y(0)}$

b) $R_{XY}(\tau) \leq \dfrac{1}{2}\left[R_X(0) + R_Y(0)\right]$

3.5 Systems with Random Signal Excitation

In Chapter 1, we discussed the transmission of deterministic signals through linear systems. We had developed the relations for the input-output spectral densities. We shall now develop the analogous relationships for the case when a linear time-invariant system is excited by random signals.

Consider the scheme shown in Fig. 3.12. $h(t)$ represents the (known) impulse response of a linear time-invariant system that is excited by a random process $X(t)$, resulting in the output process $Y(t)$.

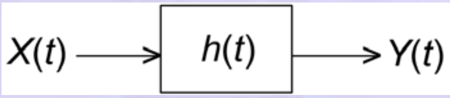


Fig 3.12: Transmission of a random process through a linear filter

We shall now try to characterize the output process $Y(t)$ in terms of the input process $X(t)$ and the impulse response $h(t)$ [third method of specification]. Specifically, we would like to develop the relations for $m_Y(t)$ and $R_Y(t_1, t_2)$ when $X(t)$ is WSS.

Let $x_j(t)$ be a sample function of $X(t)$ which is applied as input to the linear time-invariant system. Let $y_j(t)$ be the corresponding output where $y_j(t)$ belongs to $Y(t)$. Then,

$$y_j(t) = \int_{-\infty}^{\infty} h(\tau)\, x_j(t - \tau)\, d\tau$$

As the above relation is true for every sample function of $X(t)$, we can write

$$Y(t) = \int_{-\infty}^{\infty} h(\tau)\, X(t - \tau)\, d\tau$$

Consider first the mean of the output process

$$m_Y(t) = E\left[Y(t)\right] = E\left[\int_{-\infty}^{\infty} h(\tau)\, X(t-\tau)\, d\tau\right] \quad (3.9a)$$

Provided that $E\left[X(t)\right]$ is finite for all t, and the system is stable, we may interchange the order of the expectation and integration with respect to τ in Eq. 3.9(a) and write

$$m_Y(t) = \int_{-\infty}^{\infty} h(\tau)\, E\left[X(t-\tau)\right] d\tau \quad (3.9b)$$

where we have used the fact that $h(\tau)$ is deterministic and can be brought outside the expectation. If $X(t)$ is WSS, then $E\left[X(t)\right]$ is a constant m_X, so that Eq. 3.9(b) can be simplified as

$$m_Y(t) = m_X \int_{-\infty}^{\infty} h(\tau)\, d\tau$$

$$= m_X\, H(0) \quad (3.10)$$

where $H(0) = H(f)\big|_{f\,=\,0}$ and $H(f)$ is the transfer function of the given system. We note that $m_Y(t)$ is a constant.

Let us compute $R_Y(t, u)$, where t and u denote the time instants at which the output process is observed. We have,

$$R_Y(t, u) = E\left[Y(t)\, Y(u)\right] = E\left[\int_{-\infty}^{\infty} h(\tau_1)\, X(t-\tau_1)\, d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)\, X(u-\tau_2)\, d\tau_2\right]$$

Again, interchanging the order of integration and expectation, we obtain

$$R_Y(t, u) = \int_{-\infty}^{\infty} d\tau_1\, h(\tau_1) \int_{-\infty}^{\infty} d\tau_2\, h(\tau_2)\, E\left[X(t-\tau_1)\, X(u-\tau_2)\right]$$

$$= \int_{-\infty}^{\infty} d\tau_1\, h(\tau_1) \int_{-\infty}^{\infty} d\tau_2\, h(\tau_2)\, R_X(t-\tau_1,\, u-\tau_2)$$

If $X(t)$ is WSS, then

$$R_Y(t, u) = \int_{-\infty}^{\infty} d\tau_1\, h(\tau_1) \int_{-\infty}^{\infty} d\tau_2\, h(\tau_2)\, R_X(\tau - \tau_1 + \tau_2) \qquad (3.11)$$

where $\tau = t - u$. Eq. 3.11 implies that $R_Y(t, u)$ is only a function of $t - u$. Hence, the LHS of Eq. 3.11 can be written as $R_Y(\tau)$. Eq. 3.10 and 3.11 together imply that if $X(t)$ is WSS, then so is $Y(t)$.

3.6 Power Spectral Density

The notion of Power Spectral Density (PSD) is an important and useful one. It provides the frequency domain description of a stationary (at least WSS) random process. From Eq. 3.11, we have

$$E\left[Y^2(t)\right] = R_Y(0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\, h(\tau_2)\; R_X(\tau_2 - \tau_1)\, d\tau_1\, d\tau_2$$

But, $h(\tau_1) = \int_{-\infty}^{\infty} H(f)\, e^{j 2\pi f \tau_1}\, df$

Hence, $E\left[Y^2(t)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[H(f)\, e^{j 2\pi f \tau_1}\, df\right] h(\tau_2)\, R_X(\tau_2 - \tau_1)\, d\tau_1\, d\tau_2$

$$= \int_{-\infty}^{\infty} H(f)\, df \int_{-\infty}^{\infty} h(\tau_2)\, d\tau_2 \int_{-\infty}^{\infty} R_X(\tau_2 - \tau_1)\, e^{j 2\pi f \tau_1}\, d\tau_1$$

Let $\tau_2 - \tau_1 = \lambda$; that is, $\tau_1 = \tau_2 - \lambda$.

$$E\left[Y^2(t)\right] = \int_{-\infty}^{\infty} H(f)\, df \int_{-\infty}^{\infty} h(\tau_2)\, e^{j 2\pi f \tau_2}\, d\tau_2 \int_{-\infty}^{\infty} R_X(\lambda)\, e^{-j 2\pi f \lambda}\, d\lambda$$

The second integral above is $H^*(f)$, the complex conjugate of $H(f)$. The third integral will be a function f, which we shall denote by $S_X(f)$. Then

$$E\left[Y^2(t)\right] = \int_{-\infty}^{\infty} S_X(f)\left|H(f)\right|^2 df \qquad (3.12)$$

where $S_X(f) = \int_{-\infty}^{\infty} R_X(\lambda)\, e^{-j2\pi f \lambda}\, d\lambda$.

We will now justify that $S_X(f)$ can be interpreted as the power spectral density of the WSS process $X(t)$. Suppose that the process $X(t)$ is passed through an ideal narrowband, band-pass filter with the passband centered at f_c and having the amplitude response

$$|H(f)| = \begin{cases} 1, & |f \pm f_c| < \frac{1}{2}\Delta f \\ 0, & |f \pm f_c| > \frac{1}{2}\Delta f \end{cases}$$

Then, from Eq. 3.12, we find that if the filter band-width is sufficiently small and $S_X(f)$ is a continuous function, then the mean square value of the filter output is approximately,

$$E\left[Y^2(t)\right] \simeq (2\Delta f)\, S_X(f_c)$$

The filter however passes only those frequency components of the input random process $X(t)$ that lie inside the narrow frequency band of width Δf centered about $\pm f_c$. Thus, $S_X(f_c)$ represents the frequency density of the average power in the process $X(t)$, evaluated at $f = f_c$. The dimensions of $S_X(f)$ are watts/Hz.

3.6.1 Properties of power spectral density

The PSD $S_X(f)$ and the ACF $R_X(\tau)$ of a WSS process $X(t)$ form a Fourier transform pair and are given by

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j2\pi f \tau}\, d\tau \qquad (3.13)$$

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)\, e^{j 2 \pi f \tau}\, df \tag{3.14}$$

Eq. 3.13 and 3.14 are popularly known as Weiner-Khinchine Relations. Using this pair of equations, we shall derive some general properties of PSD of a WSS random process.

P1) The zero frequency value of the PSD of a WSS process equals the total area under the graph of ACF; that is

$$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau)\, d\tau$$

This property follows directly from Eq. 3.13 by putting $f = 0$.

P2) The mean square value of a WSS process equals the total area under the graph of the PSD; that is,

$$E\left[X^2(t)\right] = \int_{-\infty}^{\infty} S_X(f)\, df$$

This property follows from Eq. 3.14 by putting $\tau = 0$ and noting $R_X(0) = E\left[X^2(t)\right]$.

P3) The PSD is real and is an even function of frequency; that is, $S_X(f)$ is real and $S_X(-f) = S_X(f)$.

This result is due to the property of the ACF, namely, $R_X(\tau)$ is real and even.

P4) The PSD of a WSS process is always non-negative; that is,

$S_X(f) \geq 0$, for all f.

To establish this, assume that $S_X(f)$ is negative, for a certain frequency interval, say $(f_1, f_1 + \Delta f)$. Let $X(t)$ be the input to a narrowband filter with the transfer function characteristic,

$$|H(f)| = \begin{cases} 1, & f_1 \leq |f| \leq f_1 + \Delta f \\ 0, & otherwise \end{cases}$$

Then from Eq. 3.12, we have $E\left[Y^2(t)\right] = \int_{-\infty}^{\infty} S_X(f)\left|H(f)^2\right| df$, as a negative quantity which is a contradiction.

We shall now derive an expression for the power spectral density, $S_Y(f)$, of the output of Fig. 3.12.

Using the relation $R_X(\tau) = F^{-1}\left[S_X(f)\right]$, Eq. 3.11 can be written as

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\, h(\tau_2)\left[\int_{-\infty}^{\infty} S_X(f)\, e^{j2\pi f(\tau - \tau_1 + \tau_2)}\, df\right] d\tau_1\, d\tau_2$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\tau_1)\, e^{-j2\pi f\tau_1}\, d\tau_1\right]\left[\int_{-\infty}^{\infty} h(\tau_2)\, e^{j2\pi f\tau_2}\, d\tau_2\right] S_X(f)\, e^{j2\pi f\tau}\, df$$

$$= \int_{-\infty}^{\infty} H(f)\cdot H^*(f)\, S_X(f)\, e^{j2\pi f\tau}\, df$$

$$= \int_{-\infty}^{\infty}\left[\left|H(f)\right|^2 S_X(f)\right] e^{j2\pi f\tau}\, df \qquad (3.15)$$

As $R_Y(\tau) \longleftrightarrow S_Y(f)$, Eq. 3.15 implies

$$S_Y(f) = S_X(f)\left|H(f)\right|^2 \qquad (3.16)$$

Note that when the input to an LTI system is deterministic, we have the input-output FT relationship, $Y(f) = X(f)\, H(f)$. The corresponding time domain relationship is $y(t) = x(t) * h(t)$. Let $R_h(\tau)$ denote the ACF of the $h(t)$. Then $R_h(\tau) \longleftrightarrow \left|H(f)\right|^2$ (see P4, sec 1.6.2). Hence

$$R_Y(\tau) = R_x(\tau) * R_h(\tau)$$

$$= R_x(\tau) * h(\tau) * h^*(-\tau) \qquad (3.17a)$$

If the impulse response is real, then

$$R_Y(\tau) = R_x(\tau) * h(\tau) * h(-\tau) \qquad (3.17b)$$

We shall now take up a few examples involving the computation of PSD.

Example 3.7

For the random process $X(t)$ of example 3.5, let us find the PSD.

Since $R_X(\tau) = \frac{A^2}{2}\cos(\omega_c \tau)$, we have

$$S_X(f) = \frac{A^2}{4}\left[\delta(f - f_c) + \delta(f + f_c)\right] \qquad \blacklozenge$$

Example 3.8: (Modulated Random Process)

Let $Y(t) = X(t)\cos(\omega_c t + \Theta)$ where $X(t)$ is a WSS process with known $R_X(\tau)$ and $S_X(f)$. Θ is a uniformly distributed random variable in the range $(0 - 2\pi)$. $X(t)$ and Θ are independent. Let us find the ACF and PSD of $Y(t)$.

$$R_Y(\tau) = E\left[Y(t+\tau)\,Y(t)\right]$$

$$= E\left\{X(t+\tau)\cos\left[\omega_c(t+\tau)+\Theta\right]X(t)\cos(\omega_c t + \Theta)\right\}$$

As $X(t)$ and Θ are independent, we have

$$R_Y(\tau) = E\left[X(t+\tau)\,X(t)\right]E\left[\cos(\omega_c(t+\tau)+\Theta)\cos(\omega_c t + \Theta)\right]$$

$$= \frac{1}{2}R_X(\tau)\cos(\omega_c \tau)$$

$$S_Y(f) = F\left[R_Y(\tau)\right] = \frac{1}{4}\left[S_X(f - f_c) + S_X(f + f_c)\right] \qquad \blacklozenge$$

Example 3.9: (Random Binary Wave)

Fig. 3.13 shows the sample function $x_j(t)$ of a process $X(t)$ consisting of a random sequence of binary symbols 1 and 0. It is assumed that:

1. The symbols 1 and 0 are represented by pulses of amplitude $+A$ and $-A$ volts, respectively and duration T seconds.
2. The pulses are not synchronized, so that the starting time of the first pulse, t_d is equally likely to lie anywhere between zero and T seconds. That is, t_d is the sample value of a uniformly distributed random variable T_d with its probability density function defined by

$$f_{T_d}(t_d) = \begin{cases} \dfrac{1}{T}, & 0 \leq t_d \leq T \\ 0, & elsewhere \end{cases}$$

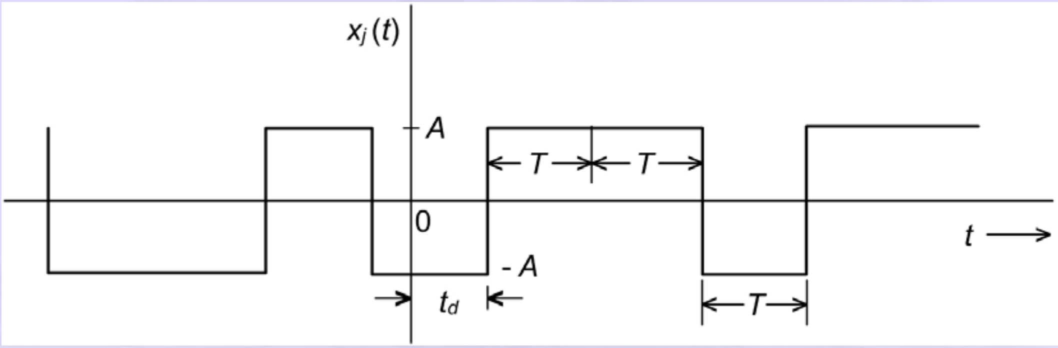


Fig. 3.13: Random binary wave

3. During any time interval $(n-1)T \leq t - t_d \leq nT$, where n is an integer, the presence of a 1 or 0 is determined by tossing a fair coin; specifically, if the outcome is ‘heads’, we have a 1 and if the outcome is 'tails', we have a 0. These two symbols are thus equally likely, and the presence of a 1 or 0 in anyone interval is independent of all other intervals. We shall compute $S_X(f)$ and $R_X(\tau)$.

The above process can be generated by passing the random impulse train (Example 3.6) through a filter with the impulse response,

$$h(t) = \begin{cases} A, & 0 \leq t \leq T \\ 0, & otherwise \end{cases}$$

$$\left|H(f)\right| = AT\left|\text{sinc}(fT)\right|$$

Therefore, $S_X(f) = \frac{1}{T} A^2 T^2 \,\text{sinc}^2(fT)$

$$= A^2 T \,\text{sinc}^2(fT)$$

Taking the inverse Fourier transform, we have

$$R_X(\tau) = \begin{cases} A^2\left(1 - \frac{|\tau|}{T}\right), & 0 \le \tau \le T \\ 0, & otherwise \end{cases}$$

The ACF of the random binary wave process can also be computed by direct time domain arguments. The interested reader is referred to [3].

We note that the energy spectral density of a rectangular pulse $x(t)$ of amplitude A and duration T, is $E_x(f) = A^2 T^2 \,\text{sinc}^2(fT)$

Hence, $S_x(f) = E_x(f)/T$. ♦

Exercise 3.7

In the random binary wave process of example 3.9, let 1 be represented by a pulse of amplitude A and duration T sec. The binary zero is indicated by the absence of any pulse. The rest of the description of the process is the same as in the example 3.9. Show that

$$S_X(f) = \frac{A^2}{4}\left[\delta(f) + \frac{\sin^2(\pi f T)}{\pi^2 f^2 T}\right]$$

Exercise 3.8

The input voltage to an RLC series circuit is a WSS process $X(t)$ with $\overline{X(t)} = 2$ and $R_X(\tau) = 4 + e^{-2|\tau|}$. Let $Y(t)$ be the voltage across the capacitor. Find

a) $\overline{Y(t)}$

b) $S_Y(f)$

3.7 Cross-Spectral Density

Just as the PSD of a process provides a measure of the frequency distribution of the given process, the cross spectral density provides a measure of frequency interrelationship between two processes. (This will become clear from the examples that follow). Let $X(t)$ and $Y(t)$ be two jointly WSS random processes with their cross correlation functions $R_{XY}(\tau)$ and $R_{YX}(\tau)$. Then, we define, $S_{XY}(f)$ and $S_{YX}(f)$ as follows:

$$S_{XY}(f) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-j2\pi f\tau}\, d\tau \tag{3.18}$$

$$S_{YX}(f) = \int_{-\infty}^{\infty} R_{YX}(\tau)\, e^{-j2\pi f\tau}\, d\tau \tag{3.19}$$

That is,

$$R_{XY}(\tau) \longleftrightarrow S_{XY}(f)$$

$$R_{YX}(\tau) \longleftrightarrow S_{YX}(f)$$

The cross-spectral density is in general complex. Even if it is real, it need not be positive. However, as $R_{XY}(\tau) = R_{YX}(-\tau)$, we find

$$S_{XY}(f) = S_{YX}(-f) = S^{*}_{YX}(f)$$

We shall now give a few examples that involve the cross-spectrum.

Example 3.10

Let $Z(t) = X(t) + Y(t)$

where the random processes $X(t)$ and $Y(t)$ are jointly WSS and $\overline{X(t)} = \overline{Y(t)} = 0$. We will show that to find $S_Z(f)$, we require the cross-spectrum.

$$R_Z(t,u) = E\left[Z(t)\, Z(u)\right]$$

$$= E\left[\left(X(t) + Y(t)\right)\left(X(u) + Y(u)\right)\right]$$

$$= R_X(t, u) + R_{XY}(t, u) + R_{YX}(t, u) + R_Y(t, u)$$

Letting $\tau = t - u$, we have

$$R_Z(\tau) = R_X(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_Y(\tau) \quad (3.20a)$$

Taking the Fourier transform, we have

$$S_Z(f) = S_X(f) + S_{XY}(f) + S_{YX}(f) + S_Y(f) \quad (3.20b)$$

♦

If $X(t)$ and $Y(t)$ are uncorrelated, that is,

$$R_{XY}(\tau) = \overline{X(t + \tau)\, Y(t)} = 0$$

then,

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau) \text{ and}$$

$$S_Z(f) = S_X(f) + S_Y(f)$$

Hence, we have the superposition of the autocorrelation functions as well as the superposition of power spectral densities.

Example 3.11

Let $X(t)$ be the input to an LTI system with the impulse response $h(t)$. If the resulting output is $Y(t)$, let us find an expression for $S_{YX}(f)$.

$$R_{YX}(t, u) = E\left[Y(t)\, X(u)\right]$$

$$= E\left[\int_{-\infty}^{\infty} h(\tau_1)\, X(t - \tau_1)\, d\tau_1\, X(u)\right]$$

Interchanging the order of expectation and integration, we have

$$R_{YX}(t, u) = \int_{-\infty}^{\infty} h(\tau_1)\, E\left[X(t - \tau_1)\, X(u)\right] d\tau_1$$

If $X(t)$ is WSS, then

$$R_{YX}(t, u) = \int_{-\infty}^{\infty} h(\tau_1)\, R_X(t - u - \tau_1)\, d\tau_1$$

Letting $t - u = \tau$ gives the result

$$R_{YX}(t, u) = \int_{-\infty}^{\infty} h(\tau_1)\, R_X(\tau - \tau_1)\, d\tau_1$$

$$= h(\tau) * R_X(\tau) \qquad (3.21a)$$

That is, the *cross-correlation between the output and input is the convolution of the input ACF with the filter impulse response*. Taking the Fourier transform,

$$S_{YX}(f) = S_X(f)\, H(f) \qquad (3.21b)$$

Eq. 3.21(b) tells us that $X(t)$ and $Y(t)$ will have strong cross-correlation in those frequency components where $\left|S_X(f)\, H(f)\right|$ is large.

Example 3.12

In the scheme shown (Fig. 3.14), $X(t)$ and $Y(t)$ are jointly WSS. Let us compute $S_{VZ}(f)$.

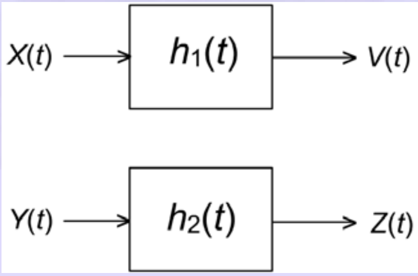


Fig. 3.14: Figure for the example 3.12

$$R_{VZ}(t, u) = E\left[V(t)\, Z(u)\right]$$

$$= E\left\{\int_{-\infty}^{\infty} h_1(\tau_1)\, X(t - \tau_1)\, d\tau_1 \int_{-\infty}^{\infty} h_2(\tau_2)\, Y(u - \tau_2)\, d\tau_2\right\}$$

Interchanging the order of expectation and integration,

$$R_{VZ}(t, u) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\tau_1)\, h_2(\tau_2)\, E\left[X(t - \tau_1)\, Y(u - \tau_2)\right] d\tau_1\, d\tau_2$$

As $X(t)$ and $Y(t)$ are jointly WSS, we have

$$R_{VZ}(t, u) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\tau_1)\, h_2(\tau_2)\, R_{XY}(\tau - \tau_1 + \tau_2)\, d\tau_1\, d\tau_2$$

where $\tau = t - u$. That is,

$$R_{VZ}(t, u) = R_{VZ}(t - u) = R_{VZ}(\tau)$$

It not too difficult to show that

$$S_{VZ}(f) = H_1(f)\, H_2^*(f)\, S_{XY}(f) \qquad (3.22)$$

♦

We shall now consider some special cases of Example 3.12.

i) Let $H_1(f)$ and $H_2(f)$ of Fig. 3.14 have non-overlapping passbands. Then $H_1(f)\, H_2^*(f) = 0$ for all f. That is, $S_{VZ}(f) \equiv 0$; this implies $R_{VZ}(\tau) \equiv 0$ and we have $V(t)$ and $Z(t)$ being orthogonal. In addition, either $\overline{V(t)}$ or $\overline{Z(t)}$ (or both) will be zero (note that $\overline{V(t)} = H_1(0)\, m_X$ and $\overline{Z(t)} = H_2(0)\, m_Y$), because atleast one of the quantities, $H_1(0)$ or $H_2(0)$ has to be zero. That is, the two random variables, $V(t_i)$ and $Z(t_j)$ obtained from sampling the processes $V(t)$ at t_i, and $Z(t)$ at t_j respectively, are uncorrelated. That is, the processes $V(t)$ and $Z(t)$ are uncorrelated.

ii) Let $X(t) = Y(t)$ and $X(t)$ is WSS. Then we have the situation shown in Fig. 3.15.

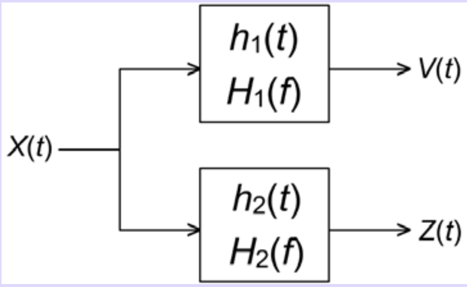


Fig. 3.15: Two LTI systems with a common input

Then from Eq. 3.22, we obtain,

$$S_{VZ}(f) = H_1(f)\, H_2^*(f)\, S_X(f) \tag{3.23}$$

iii) In the scheme of Fig. 3.15, if $h_1(t) = h_2(t) = h(t)$, then we have the familiar result

$$S_V(f) = |H(f)|^2\, S_X(f)$$

3.8 Gaussian Process

Gaussian processes are of great practical and mathematical significance in the theory of communication. They are of great practical significance because a large number of noise processes encountered in the real world (and that interfere with the communication process) can be characterized as Gaussian processes. They are of great mathematical significance because Gaussian processes possess a neat mathematical structure which makes their analytical treatment quite feasible.

Let a random process $X(t)$ be sampled at time instants $t_1, t_2, ..., t_n$. Let $X(t_1) = X_1, X(t_2) = X_2, ..., X(t_n) = X_n$ and $\boldsymbol{X}$ denote the row vector $(X_1, X_2, ..., X_n)$. The process $X(t)$ is called Gaussian, if $f_{\boldsymbol{X}}(\boldsymbol{x})$ is an n-dimensional joint Gaussian density for every $n \geq 1$ and $(t_1, t_2, ..., t_n) \in (-\infty, \infty)$. The n-dimensional Gaussian PDF is given by

$$f_X(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2} |C_X|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{m}_x) C_X^{-1} (\boldsymbol{x} - \boldsymbol{m}_x)^T\right] \quad (3.24)$$

where C_x is the covariance matrix, $|\ |$ denotes its determinant and C_X^{-1} is its inverse. $\boldsymbol{m}_X$ is the mean vector, namely, $\left(\overline{X_1}, \overline{X_2}, ..., \overline{X_n}\right)$ and the superscript T denotes the matrix transpose. The covariance matrix C_x is given by

$$C_x = \begin{pmatrix} \lambda_{11} & \lambda_{12} \cdots \lambda_{1n} \\ \lambda_{21} & \lambda_{22} \cdots \lambda_{2n} \\ \vdots & \\ \lambda_{n1} & \lambda_{n2} \cdots \lambda_{nn} \end{pmatrix}$$

where $\lambda_{ij} = \text{cov}\left[X_i, X_j\right]$

$$= E\left[\left(X_i - \overline{X_i}\right)\left(X_j - \overline{X_j}\right)\right]$$

Similarly, $(\boldsymbol{x} - \boldsymbol{m}_X) = \left(x_1 - \overline{X_1}, x_2 - \overline{X_2}, ..., x_n - \overline{X_n}\right)$

Specification of a Gaussian process as above corresponds to the first method mentioned earlier. Note that an n-dimensional Gaussian PDF depends only on the means and the covariance quantities of the random variables under consideration. If the mean value of the process is constant and $\lambda_{ij} = \text{cov}\left[X(t_i), X(t_j)\right]$ depends only on $t_i - t_j$, then the joint PDF is independent of the time origin. In other words, a WSS Gaussian process is also stationary in a strict sense. To illustrate the use of the matrix notation, let us take an example of a joint Gaussian PDF with $n = 2$.

Example 3.13

X_1 and X_2 are two jointly Gaussian variables with $\overline{X_1} = \overline{X_2} = 0$ and $\sigma_1 = \sigma_2 = \sigma$. The correlation coefficient between X_1 and X_2 is ρ. Write the joint PDF of X_1 and X_2 in i) the matrix form and ii) the expanded form.

i) As $\lambda_{11} = \lambda_{22} = \sigma^2$ and $\lambda_{12} = \lambda_{21} = \rho\,\sigma^2$, we have

$$C_x = \begin{bmatrix} \sigma^2 & \rho\,\sigma^2 \\ \rho\,\sigma^2 & \sigma^2 \end{bmatrix}, \quad |C_X| = \sigma^4 \left(1 - \rho^2\right)$$

$$C_X^{-1} = \frac{1}{\sigma^4 \left(1 - \rho^2\right)} \begin{bmatrix} \sigma^2 & -\rho\,\sigma^2 \\ -\rho\,\sigma^2 & \sigma^2 \end{bmatrix}$$

$$= \frac{1}{\sigma^2 \left(1 - \rho^2\right)} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$$

Therefore,

$$f_{X_1, X_2}(x_1, x_2) = \frac{1}{2\pi\sigma^2 \sqrt{1 - \rho^2}} \exp\left\{ \frac{-1}{2\sigma^2\left(1 - \rho^2\right)} \left(x_1 \; x_2\right) \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\}$$

ii) Taking the matrix products in the exponent above, we have the expanded result, namely,

$$f_{X_1, X_2}(x_1, x_2) = \frac{1}{2\pi\sigma^2 \sqrt{1 - \rho^2}} \exp\left\{ -\frac{x_1^2 - 2\rho\, x_1 x_2 + x_2^2}{2\sigma^2\left(1 - \rho^2\right)} \right\}, \quad -\infty < x_1, x_2 < \infty$$

This is the same as the bivariate Gaussian PDF of Chapter 2, section 6 with $x_1 = x$ and $x_2 = y$, and $m_X = m_Y = 0$, and $\sigma_X = \sigma_Y = \sigma$. ♦

Example 3.14

$X(t)$ is a Gaussian process with $m_X(t) = 3$ and

$C_X(t_1, t_2) = 4\, e^{-0.2\,|t_1 - t_2|} = 4\, e^{-0.2\,|\tau|}$. Let us find, in terms of $Q(\;)$,

a) $P\left[X(5) \leq 2\right]$

b) $P\left[\left|X(8) - X(5)\right| \leq 1\right]$

a) Let Y be the random variable obtained by sampling $X(t)$ at $t = 5$. Y is Gaussian with the mean value 3, and the variance $= C_X(0) = 4$. That is, Y is $N(3, 4)$.

$$P[Y \leq 2] = P\left[\frac{Y - 3}{2} \leq -\frac{1}{2}\right] = Q(0.5)$$

b) Let $Z = X(8) - X(5)$ and $Y = X(8)$, and $W = X(5)$. Note that $Z = Y - W$ is Gaussian. We have $\overline{Z} = 0$ and

$$\begin{aligned}
\sigma_Z^2 &= \sigma_Y^2 + \sigma_W^2 - 2\rho_{Y,W}\,\sigma_Y\,\sigma_W \\
&= \sigma_Y^2 + \sigma_W^2 - 2\,\text{cov}[Y, W] \\
&= 4 + 4 - 2 \times 4\, e^{-0.2 \times 3} \\
&= 8\left(1 - e^{-0.6}\right) = 3.608
\end{aligned}$$

$$P[|Z| \leq 1] = 2\,P[0 \leq Z \leq 1] = 2\left\{\frac{1}{2} - P[Z > 1]\right\}$$

$$= 1 - 2P\left[\frac{Z}{\sqrt{3.6}} > \frac{1}{\sqrt{3.6}}\right] = 1 - 2Q(0.52)$$

♦

Exercise 3.9

Let $\boldsymbol{X}$ be a zero-mean Gaussian vector with four components, X_1, X_2, X_3 and X_4. We can show that

$$\overline{X_1 X_2 X_3 X_4} = \overline{X_1 X_2} \cdot \overline{X_3 X_4} + \overline{X_1 X_3} \cdot \overline{X_2 X_4} + \overline{X_1 X_4} \cdot \overline{X_2 X_3}$$

The above formula can also be used to compute the moments such as $\overline{X_1^4}$, $\overline{X_1^2 X_2^2}$, etc. $\overline{X_1^4}$ can be computed as

$$\overline{X_1^4} = \overline{X_1 X_1 X_1 X_1} = 3 \sigma_1^4$$

Similarly, $\overline{X_1^2 X_2^2} = \overline{X_1 X_1 X_2 X_2} = \sigma_1^2 \sigma_2^2 + 2 \left(\overline{X_1 X_2}\right)^2$

A zero mean stationary Gaussian process is sampled at $t = t_1$ and t_2. Let X_1 and X_2 denote the corresponding random variables. The covariance matrix of X_1 and X_2 is

$$C_X = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Show that $E\left[X_1^2 X_2^2\right] = 6$

We shall now state (without proof) some of the important properties of n jointly Gaussian random variables, $\boldsymbol{X} = \left(X_1, X_2, ..., X_n\right)$. (These properties are generalizations of the properties mentioned for the bivariate case in Chapter 2).

P1) If $\boldsymbol{X} = \left(X_1, X_2, ..., X_n\right)$ are jointly Gaussian, then any subset of them are jointly Gaussian.

P2) The X_i's are statistically independent if their covariance matrix is diagonal; that is, $\lambda_{i j} = \sigma_i^2 \, \delta_{i j}$ where

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

P3) Let $\boldsymbol{Y} = (Y_1, Y_2, ..., Y_n)$ be a set of vectors obtained from $\boldsymbol{X} = (X_1, X_2, ..., X_n)$ by means of the linear transformation

$$\boldsymbol{Y}^T = A\,\boldsymbol{X}^T + \boldsymbol{a}^T$$

where A is any $n \times n$ matrix and $\boldsymbol{a} = (a_1, a_2, ..., a_n)$ is a vector of constants $\{a_i\}$. If $\boldsymbol{X}$ is jointly Gaussian, then so is $\boldsymbol{Y}$.

As a consequence of **P3)** above, we find that if a Gaussian process $X(t)$ is input to a linear system, then the output $Y(t)$ is also Gaussian process. We shall make use of this result in developing the properties of narrowband noise.

Exercise 3.10

Let $\boldsymbol{Y}^T = A\,\boldsymbol{X}^T + \boldsymbol{a}^T$

where the notation is from **P3)** above. Show that

$$C_Y = A C_X A^T$$

3.9 Electrical Noise

Electronic communication systems are made up of circuit elements such as R, L and C, and devices like diodes, transistors, etc. All these components give rise to what is known as internal circuit noise. It is this noise that sets the fundamental limits on communication of acceptable quality.

Two of the important internal circuit noise varieties are: i) Thermal noise ii) Shot noise

Historically, Johnson and Nyquist first studied **thermal noise** in metallic resistors; hence it is also known as Johnson noise or resistance noise. The random motion of the *free electrons* in a conductor caused by thermal agitation gives rise to a voltage $V(t)$ at the open ends of the conductor and is present whether or not an electrical field is applied. Consistent with the central limit theorem, $V(t)$ is a Gaussian process with zero mean and the variance is a function of R and T, where R is the value of the resistance and T is the temperature of R. It has also been found that the spectral density of $V(t)$ $\left(\text{in } (\text{Volts})^2 / \text{Hz}\right)$, denoted $S_V(f)$ is essentially constant for $|f| \leq 10^{12}$ Hz, if T is 290° K or 63° F (290° K is taken as the standard room temperature). 10^{12} Hz is already in the infrared region of EM spectrum. This constant value of $S_V(f)$ is given by

$$S_V(f) = 2RkT \quad V^2/Hz \tag{3.25}$$

where T : the temperature of R, in degrees Kelvin $\left({}^\circ K = {}^\circ C + 273\right)$

k : Boltzman's constant $= 1.37 \times 10^{-23}$ Joules$/{}^\circ$K.

It is to be remembered that Eq. 3.25 is valid only upto a certain frequency limit. However this limit is much, much higher than the frequency range of interest to us.

If this open circuit voltage is measured with the help of a true RMS voltmeter of bandwidth B (frequency range: $-B$ to B), then the reading on the instrument would be $\sqrt{2RkT \cdot 2B} = \sqrt{4RkTB}$ V.

Thermal noise sources are also characterized in terms of *available noise PSD.*

Def. 3.9:

Available noise PSD is the maximum PSD that can be delivered by a source. ♦

Let us treat the resistor to be noise free. But it is in series with a noise source with $S_V(f) = 2RkT$. In other words, we have a noise source with the source resistance R (Fig. 3.16(a)).

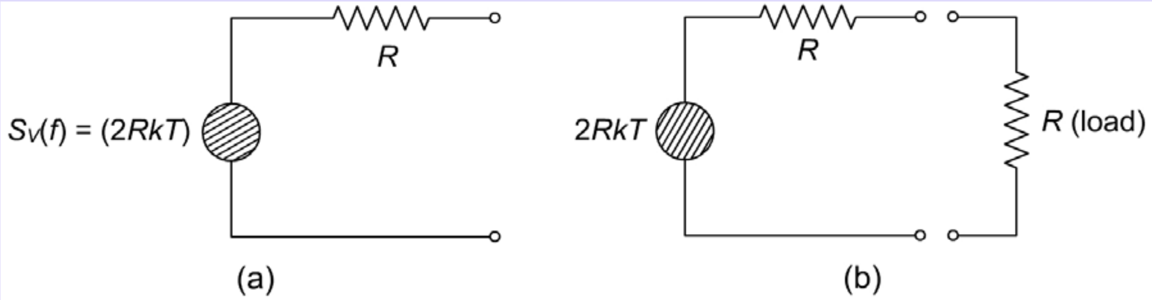


Fig. 3.16: (a) Resistor as noise source (b) The noise source driving a load

We know that the maximum power is transferred to the load, when the load is matched to the source impedance. That is, the required load resistance is R (Fig. 3.16(b)). The transfer function of this voltage divider network is $\frac{1}{2}$ $\left(\text{i.e. } \frac{R}{R+R}\right)$. Hence,

$$\text{available } PSD = \frac{S_V(f)}{R}\,|H(f)|^2$$

$$= \frac{2RkT}{4R} = \frac{kT}{2} \text{ Watts / Hz} \tag{3.26}$$

It is to be noted that available PSD does not depend on R, though the noise voltage is produced by R.

Because of Eq. 3.26, the available power in a bandwidth of B Hz is,

$$P_a = \frac{kT}{2}\cdot 2B = kTB \tag{3.27}$$

In many solid state devices (and of course, in vacuum tubes!) there exists a noise current mechanism called **shot noise.** In 1918, Schottky carried out the first theoretical study of the fluctuations in the anode current of a temperature limited vacuum tube diode (This has been termed as shot noise). He showed that at frequencies well below the reciprocal of the electron transit time (which extends upto a few Giga Hz), the spectral density of the mean square noise current due to the *randomness of emission of electrons from the cathode* is given by,

$$S_I(f) = 2qI \ (\text{Amp})^2 / \text{Hz}$$

where q is the electronic charge and I is the average anode current. (Note that units of spectral density could be any one of the three, namely, (a) Watts/Hz, (b) $(\text{Volts})^2/\text{Hz}$ and (c) $(\text{Amp})^2/\text{Hz}$. The circuit of Fig. 3.16(a) could be drawn with $S_I(f)$ for the source quantity. Then, we will have the Norton equivalent circuit with $S_I(f) = \dfrac{2RkT}{R^2} = \dfrac{2kT}{R}$ in parallel with the resistance R.)

Semiconductor diodes, BJTs and JFETs have sources of shot noise in them. Shot noise (which is non-thermal in nature), occurs whenever charged particles cross a potential barrier. Here, we have an applied potential and there is an average flow in some direction. However, there are going to be fluctuations about this average flow and it is these fluctuations that contribute a noise with a very wide spectrum.

In p-n junction devices, fluctuations of current occurs because of (i) randomness of the transit time across space charge region separating p and n regions, (ii) randomness in the generation and recombination of electron-hole pairs, and (iii) randomness in the number of charged particles that diffuse etc.

Schottky's result also holds for the semiconductor diode. The current spectral density of shot noise in a p-n junction diode is given by

$$S_I(f) = 2q(I + 2I_S) \qquad (3.28)$$

where I is the net DC current and I_S is the reverse saturation current.

A BJT has two semiconductor junctions and hence two sources of shot noise. In addition, it contributes to thermal noise because of internal ohmic resistance (such as base resistance).

A JFET has a reverse biased junction between the gate terminal and the semiconductor channel from the source to the drain. Hence, we have the gate shot noise and the channel thermal noise in a JFET. (Note that gate shot noise could get amplified when the device is used in a circuit.)

3.9.1 White noise

Eq. 3.25 and Eq. 3.28 indicate that we have noise sources with a flat spectral density with frequencies extending upto the infrared region of the EM spectrum.

The concept of *white noise* is an idealization of the above. Any noise quantity (thermal or non-thermal) which has a fiat power spectrum (that is, it contains all frequency components in equal proportion) for $-\infty < f < \infty$ is called *white noise*, in analogy with white light. We denote the PSD of a white noise process $W(t)$ as

$$S_W(f) = \frac{N_0}{2} \text{ watts / Hz} \tag{3.29}$$

where the factor $\frac{1}{2}$ has been included to indicate that half the power is associated with the positive frequencies and half with negative frequencies. In addition to a fiat spectrum, if the process happens to be Gaussian, we describe it as *white Gaussian noise*. Note that *white* and *Gaussian* are two different attributes. White noise need not be Gaussian noise, nor Gaussian noise need be white. Only when "whiteness" together with "Gaussianity" simultaneously exists, the process is qualified as White Gaussian Noise (WGN) process.

White noise, whether Gaussian or not, must be fictitious (as is the case with everything that is "ideal") because its total mean power is infinity. The utility of the concept of white noise stems from the fact that such a noise, when passed through a linear filter for which

$$\int_{-\infty}^{\infty} |H(f)|^2 < \infty \qquad (3.30)$$

the filter output is a stationary zero mean noise process $N(t)$ that is meaningful (Note that white noise process, by definition, is a zero mean process).

The condition implied by Eq.3.30 is not too stringent as we invariably deal with systems which are essentially band limited and have finite value for $|H(f)|$ within this band. In so far as the power spectrum at the output is concerned, it makes little difference how the input power spectrum behaves outside of the pass band. Hence, if the input noise spectrum is flat within the pass band of the system, we might as well treat it to be white as this does not affect the calculation of output noise power. However, assuming the input as 'white' will simplify the calculations.

As $S_W(f) = \frac{N_0}{2}$, we have, for the ACF of white noise

$$R_W(\tau) = \frac{N_0}{2}\,\delta(\tau), \qquad (3.31)$$

as shown in Fig. 3.17. This is again a nonphysical but useful result.

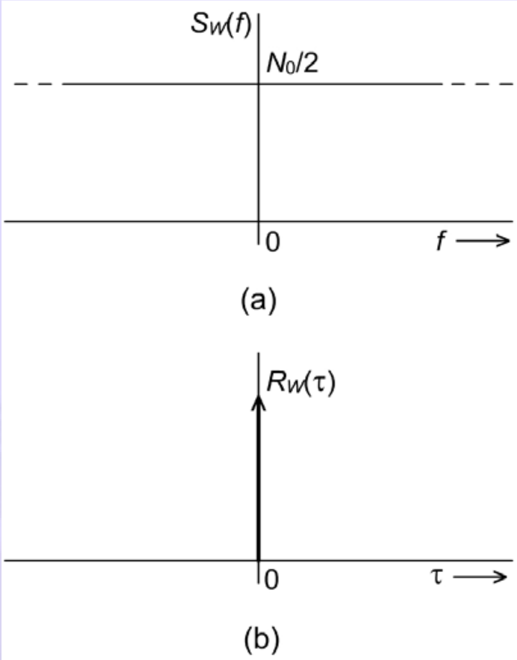


Fig.3.17: Characterization of white noise: a) Power spectral density

b) Auto correlation function

Eq. 3.31 implies that any two samples of white noise, no matter how closely together in time they are taken, are uncorrelated. (Note that $R_W(\tau) = 0$ for $\tau \neq 0$). In addition, if, the noise process is Gaussian, we find that any two samples of WGN are statistically independent. In a sense, WGN represents the ultimate in randomness!

Imagine that white noise is being displayed on an oscilloscope. Though the waveforms on successive sweeps are different, the display on the oscilloscope always *appears* to be the same, no matter what sweep speed is used. In the case of display of a deterministic waveform (such as a sinusoid) changing the time-base makes the waveform to 'expand' or 'contract'. In the case of white noise, however, the waveform changes randomly from one instant to the next, no matter what time scale is used and as such the display on the scope appears to be the same for all time instants. If white noise drives a speaker, it should sound monotonous because the waveform driving the speaker appears to be the same for all time instants.

As mentioned earlier, white noise is fictitious and cannot be generated in the laboratory. However, it is possible to generate what is known as Band Limited White Noise (BLWN) which has a flat power spectral density for $|f| \leq W$, where W is finite. (BLWN sources are commercially available. The cost of the instrument depends on the required bandwidth, power level, etc.) We can give partial demonstration (audio-visual) of the properties of white noise with the help of these sources.

We will begin with the audio. By clicking on the speaker symbol, you will listen to the speaker output when it is driven by a BLWN source with the frequency range upto 110 kHz. (As the speaker response is limited only upto 15 kHz, the input to the speaker, for all practical purposes, is white.)

We now show some flash animation pictures of the spectral density and time waveforms of BLWN.

1. Picture 1 is the display put out by a spectrum analyzer when it is fed with a BLWN signal, band limited to 6 MHz. As can be seen from the display, the spectrum is essentially constant upto 6 MHz and falls by 40 to 45 dB at about 7MHz and beyond.

2. Picture 2 is time domain waveform (as seen on an oscilloscope) when the 6 MHz, BLWN signal is the input to the oscilloscope. Four sweep speeds have been provided for you to observe the waveform. These speeds are: 100 μsec/div, 200 μsec/div, 500 μsec/div and 50 μsec/div. As you switch from 100 μsec/div to 200 μsec/div to 500 μsec/div, you will find that the

display is just about the same, whereas when you switch from 500 µsec/div to 50 µsec/div, there is a change, especially in the display level.

Consider a 500 kHz sine wave being displayed on an oscilloscope whose sweep is set to 50 µsec/div. This will result in 25 cycles per division which implies every cycle will be seen like a vertical line of appropriate height. As the frequency increases, the display essentially becomes a band of fluorescence enclosed between two lines. (The spikes in the display are due to large amplitudes in noise voltage which occur with some non-zero probability.)

Now consider a 20 kHz sine wave being displayed with the same sweep speed, namely, 50 µsec/div. This will result in 1 cycle/div which implies we can see the fine structure of the waveform. If the sweep speed were to be 500 µsec/div, then a 20 kHz time will result in 10 cycles/div, which implies that fine structure will not be seen clearly. Hence when we observe BLWN, band limited to 6 MHz, on an oscilloscope with a sweep speed of 50 µsec/div, fine structure of the low frequency components could interfere with the constant envelope display of the higher frequency components, thereby causing some reduction in the envelope level.

3. Picture 3 is the display from the spectrum analyzer when it is fed with a BLWN signal, band limited to 50 MHz. As can be seen from the display, PSD is essentially constant upto 50 MHz and then starts falling, going 40 dB down by about 150 MHz.

4. Picture 4 is the time domain signal, as seen on an oscilloscope, when the input is the 50 MHz wide, BLWN signal.

You have again the choice of the same four sweep rates. This time you will observe that when you switch from 500 µsec/div to 50 µsec/div, the change is much less. This is because, this signal has much wider spectrum and the power in the frequency range of $|f| \leq 100$ kHz is $0.002P$ where P is the total power of the BLWN signal.

From the above, it is clear that as the BLWN tends towards white noise, variations in the time waveform keep reducing, resulting in a steady picture on the oscilloscope no matter what sweep speed is used for the display.

Example 3.15: (White noise through an ILPF)

White Gaussian noise with the PSD of $\frac{N_0}{2}$ is applied as input to an ideal LPF of bandwidth B. Find the ACF of the output process. Let Y be the random variable obtained from sampling the output process at $t = 1$ sec. Let us find $f_Y(y)$.

Let $N(t)$ denote the output noise process when the WGN process $W(t)$ is the input. Then,

$$S_N(f) = \begin{cases} \frac{N_0}{2}, & -B \leq f \leq B \\ 0, & otherwise \end{cases}$$

Taking the inverse Fourier transform of $S_N(f)$, we have

$$R_N(\tau) = N_0 \, B \, \text{sinc}(2B\tau)$$

A plot of $R_N(\tau)$ is given in Fig. 3.18.

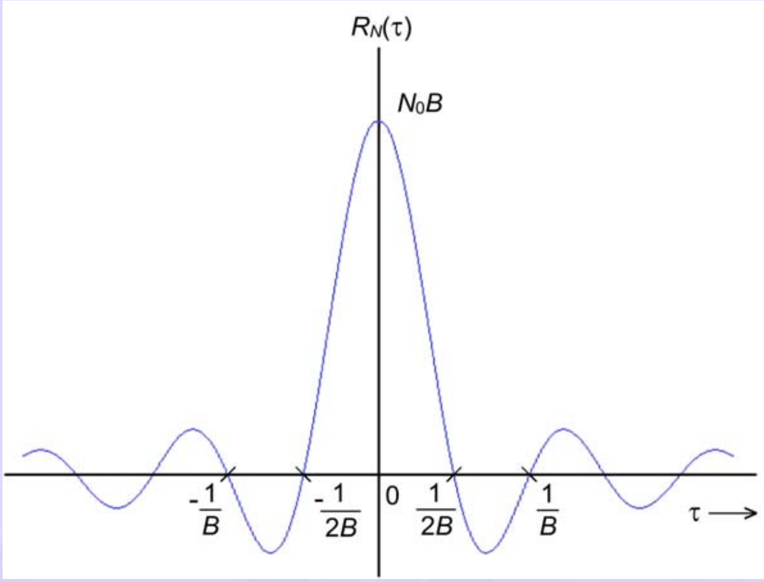


Fig 3.18: $R_N(\tau)$ of example 3.15

The filter output $N(t)$ is a stationary Gaussian process. Hence the PDF is independent of the sampling time. As the variable Y is Gaussian, what we need is the mean value and the variance of Y. Evidently $E[Y] = 0$ and $\sigma_Y^2 = \overline{Y^2} = R_Y(0) = N_0 B$. Hence Y is $N(0, N_0 B)$. ♦

Note that ACF passes through zeros at $\tau = \dfrac{n}{2B}$ where $n = \pm 1, \pm 2, ...$. Hence any two random variables obtained by sampling the output process at times t_1 and t_2 such that $|t_1 - t_2|$ is multiple of $\dfrac{1}{2B}$, are going to be statistically independent.

Example 3.16: (White Noise through an RC-LPF)

Let $W(t)$ be input to an RC-LPF. Let us find the ACF of the output $N(t)$. If X and Y are two random variables obtained from sampling $N(t)$ with a separation of 0.1 sec, let us find ρ_{XY}.

The transfer function of the RC LPF is given by $H(f) = \frac{1}{1 + j\,2\pi f RC}$.

Therefore $S_N(f) = \frac{N_0/2}{1 + (2\pi f RC)^2}$, and

$$R_N(\tau) = \frac{N_0}{4RC} \exp\left(\frac{-|\tau|}{RC}\right)$$

$$\rho_{XY} = \frac{\lambda_{XY}}{\sigma_X\,\sigma_Y} = \frac{E[XY]}{E[X^2]}$$

(Note that $\overline{N(t)} = 0$ and $\sigma_X = \sigma_Y$).

$$E[XY] = R_N(\tau)\Big|_{\tau = 0.1\,\text{sec}}$$

$$E[X^2] = R_N(\tau)\Big|_{\tau = 0} = \frac{N_0}{4RC}$$

Hence, $\rho_{XY} = e^{-(0.1/RC)}$ ♦

Exercise 3.11

$X(t)$ is a zero mean Gaussian process with $R_X(\tau) = \frac{1}{1 + \tau^2}$. Let $Y(t) = \widehat{X}(t)$ where $\widehat{X}(t)$ is the Hilbert transform of $X(t)$. The process $X(t)$ and $Y(t)$ are sampled at $t = 1$ and $t = 2$ sec. Let the corresponding random variables be denoted by (X_1, X_2) and (Y_1, Y_2) respectively.

a) Write the covariance matrix of the four variables X_1, X_2 Y_1 and Y_2.

b) Find the joint PDF of X_2, Y_1 and Y_2.

Exercise 3.12

The impulse response of a filter (LTI system) is given by

$$h(t) = \delta(t) - \alpha e^{-\alpha t} u(t)$$

where α is a positive constant. If the input to the filter is white Gaussian noise with the PSD of $\frac{N_0}{2}$ watts/Hz, find

a) the output PSD

b) Show that the ACF of the output is

$$\frac{N_0}{2}\left[\delta(\tau) - \frac{\alpha}{2} e^{-\alpha|\tau|}\right]$$

Exercise 3.13

White Gaussian noise process is applied as input to a zero-order-hold circuit with a delay of T sec. The output of the ZOH circuit is sampled at $t = T$. Let Y be the corresponding random variable. Find $f_Y(y)$.

Exercise 3.14

Noise from a 10 $k\Omega$ resistor at room temperature is passed through an ILPF of bandwidth 2.5 MHz. The filtered noise is sampled every microsecond. Denoting the random variables corresponding to the two adjacent samples as Y_1 and Y_2, obtain the expression for the joint PDF of Y_1 and Y_2.

Let τ_0 denote the de-correlation time where τ_0 is defined such that if $|\tau| > \tau_0$, then $|R_N(\tau)| \leq 0.01 R_N(0)$. Then for the RC-LPF, $\tau_0 = 4.61 RC$. That is, if the output process $N(t)$ is sampled at t_1 and t_2 such that $|t_1 - t_2| \geq 4.61 RC$, then the random variables $N(t_1)$ and $N(t_2)$ will be essentially uncorrelated. If $N(t)$ happens to be a Gaussian process, then $N(t_1)$ and $N(t_2)$ will be, for all practical purposes, independent.

In example 3.15, we found that the average output noise power is equal to $N_0 B$ where B is the bandwidth of the ILPF. Similarly, in the case of example 3.16, we find the average output noise power being equal to $\left(\frac{N_0}{4RC}\right)$ where $\frac{1}{2\pi RC}$ is the 3-dB bandwidth of the RC-LPF. That is, in both the cases, we find that the average output noise power is proportional to some measure of the bandwidth. We may generalize this statement to include all types of low pass filters by defining the *noise equivalent bandwidth* as follows.

Suppose we have a source of white noise with the PSD $S_W(f) = \frac{N_0}{2}$ connected to an arbitrary low pass filter of transfer function $H(f)$. The resulting average output noise power N is given by,

$$N = \frac{N_0}{2} \int_{-\infty}^{\infty} \left| H(f)^2 \right| df$$

$$= N_0 \int_{0}^{\infty} \left| H(f) \right|^2 df$$

Consider next the same source of white noise connected to the input of an ILPF of zero frequency response $H(0)$ and bandwidth B. In this case, the average output noise power is,

$$N' = N_0 B H^2(0).$$

Def. 3.10:

The *noise equivalent bandwidth* of an arbitrary filter is defined as the bandwidth of an ideal rectangular filter that would pass as much white noise power as the filter in question and has the same maximum gain as the filter under consideration. ♦

Let B_N be the value of B such that $N = N'$. Then B_N, the noise equivalent bandwidth of $H(f)$, is given by

$$B_N = \frac{\int_0^{\infty} |H(f)|^2 \, df}{H^2(0)} \tag{3.32}$$

The notion of equivalent noise bandwidth is illustrated in Fig. 3.19.

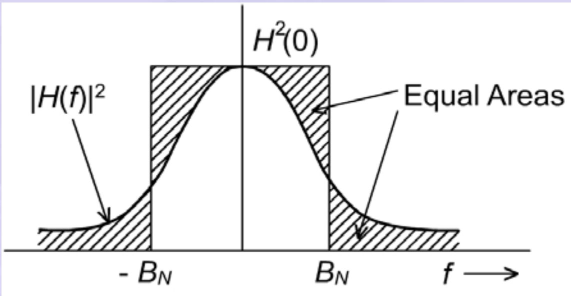


Fig. 3.19: Pictorial representation of noise equivalent bandwidth

The advantage of the noise equivalent bandwidth is that if B_N is known, we can simply calculate the output noise power without worrying about the actual shape of $|H(f)|$. The definition of noise equivalent bandwidth can be extended to a band pass filter.

Example 3.17

Compute the noise equivalent bandwidth of the RC-LPF.

We have $|H(0)| = 1$. Hence,

$$B_N = \int_0^{\infty} \frac{df}{1 + (2 \pi f R C)^2} = \frac{1}{4 R C}$$ ♦

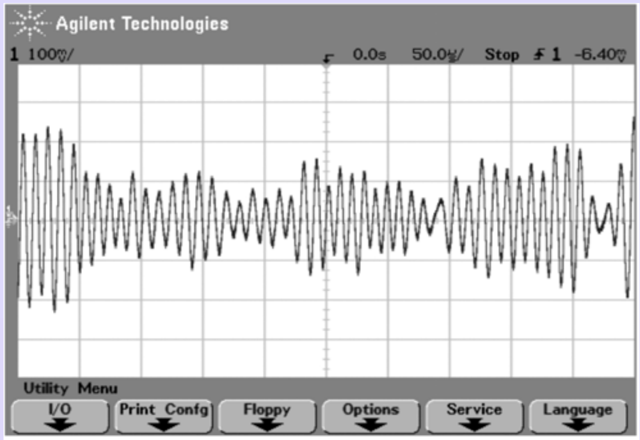
3.10 Narrowband Noise

The communication signals that are of interest to us are generally NarrowBand BandPass (NBBP) signals. That is, their spectrum is concentrated around some (nominal) centre frequency, say f_c, with the signal bandwidth usually being much smaller compared to f_c. Hence the receiver meant for such signals usually consists of a cascade of narrowband filters; this means that even if the noise process at the input to the receiver is broad-band (may even considered to be white) the noise that may be present at the various stages of a receiver is essentially narrowband in nature. We shall now develop the statistical characterization of such NarrowBand Noise (NBN) processes.

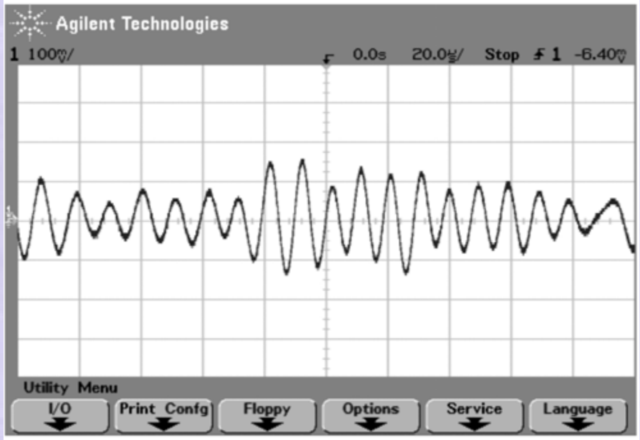
Let $N(t)$ denote the noise process at the output of a narrowband filter produced in response to a white noise process, $W(t)$, at the input. $S_W(f)$ is taken as $\frac{N_0}{2}$. If $H(f)$ denotes the filter transfer function, we have

$$S_N(f) = \frac{N_0}{2}|H(f)|^2 \qquad (3.33)$$

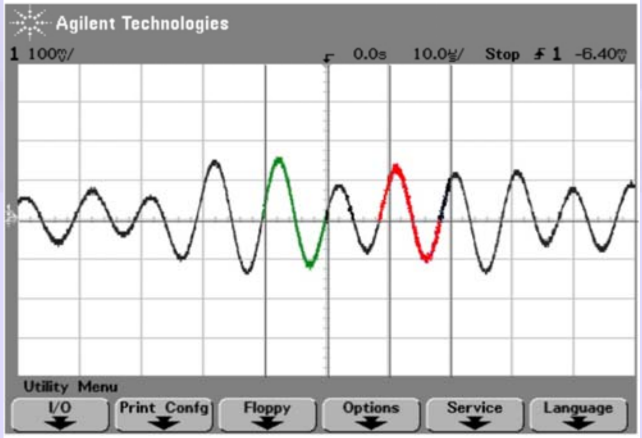
In fact, any narrowband noise encountered in practice could be modeled as the output of a suitable filter. In Fig. 3.20, we show the waveforms of experimentally generated NBBP noise. This noise process is generated by passing the BLWN (with a flat spectrum upto 110 kHz) through a NBBP filter, whose magnitude characteristic is shown in Fig. 3.21. This filter has a centre frequency of 101 kHz and a 3-dB bandwidth of less than 3 kHz.



(a)



(b)



(c)

Fig. 3.20: Some oscilloscope displays of narrowband noise

X-axis: time

Y-axis: voltage

The magnitude response of the filter is obtained by sweeping the filter input from 87 kHz to 112 kHz.

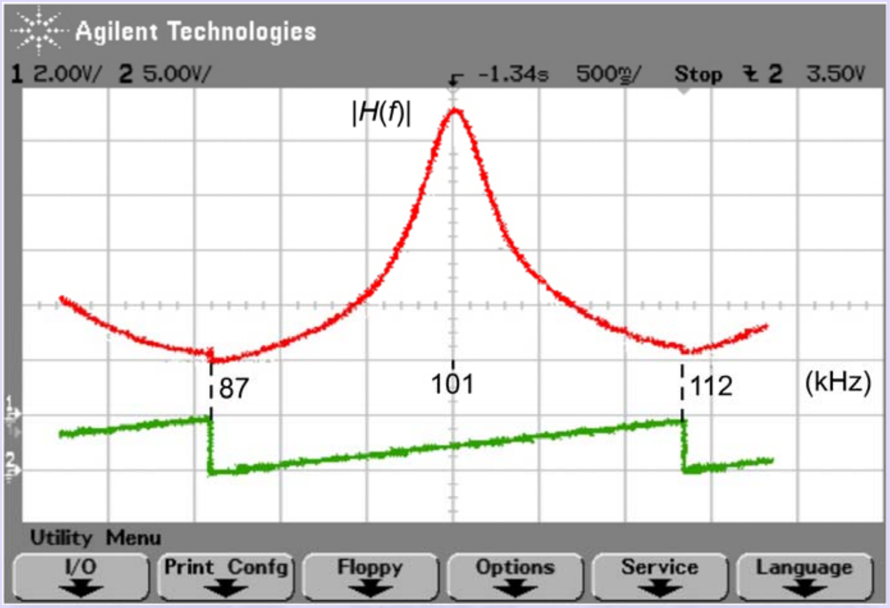


Fig. 3.21: Frequency response of the filter used to generate NBN

Plot in Fig. 3.20(a) gives us the impression that it is a 101 kHz sinusoid with slowly changing envelope. This is only partially correct. Waveforms at (b) and (c) are expanded versions of a part of the waveform at (a). From (b) and (c), it is clear that zero crossings are not uniform. In Fig. 3.20(c), the cycle of the waveform shown in green fits almost into the space between two adjacent vertical lines of the graticule, whereas for the cycle shown in red, there is a clear offset. Hence the proper time-domain description of a NBBP signal would be: *it is a sinusoid undergoing slow amplitude and phase variations*. (This will be justified later on by expressing the NBBP noise signal in terms of its envelope and phase.)

We shall now develop two representations for the NBBP noise signal. These are (a) canonical (also called in-phase and quadrature component) representation and (b) Envelope and Phase representation.

3.10.1 Representation of narrowband noise

a) Canonical representation

Let $n(t)$ represent a sample function of a NBBP noise process $N(t)$, and $n_{pe}(t)$ and $n_{ce}(t)$, its pre-envelope and complex envelope respectively. We will assume f_c to be the (nominal) centre frequency of the noise process. Then, we can write:

$$n_{pe}(t) = n(t) + j\hat{n}(t) \text{ and} \tag{3.34}$$

$$n_{ce}(t) = n_{pe}(t)\exp(-j2\pi f_c t) \tag{3.35}$$

where $\hat{n}(t)$ is the Hilbert transform of $n(t)$.

The complex envelope $n_{ce}(t)$ itself can be expressed as

$$n_{ce}(t) = n_c(t) + j n_s(t) \tag{3.36}$$

With the help of Eq. 3.34 to 3.36, it is easy to show,

$$n_c(t) = n(t)\cos(2\pi f_c t) + \hat{n}(t)\sin(2\pi f_c t) \tag{3.37}$$

$$n_s(t) = \hat{n}(t)\cos(2\pi f_c t) - n(t)\sin(2\pi f_c t) \tag{3.38}$$

As $\quad n(t) = \mathrm{Re}\left[n_{ce}(t)e^{j2\pi f_c t}\right]$

$$= \mathrm{Re}\left\{\left[n_c(t) + j n_s(t)\right]e^{j2\pi f_c t}\right\}$$

We have,

$$n(t) = n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t) \tag{3.39}$$

As in the deterministic case, we call $n_c(t)$ the in-phase component and $n_s(t)$ the quadrature component of $n(t)$.

As Eq. 3.39 is valid for any sample function $n(t)$ of $N(t)$, we shall write

$$N(t) = N_c(t)\cos(2\pi f_c t) - N_s(t)\sin(2\pi f_c t) \tag{3.40a}$$

Eq. 3.40(a) is referred to as the canonical representation of $N(t)$. $N_c(t)$ and $N_s(t)$ are low pass random processes; $n_c(t)$ and $n_s(t)$ are sample functions of $N_c(t)$ and $N_s(t)$ respectively.

Exercise 3.15

Show that $\hat{N}(t) = N_c(t)\sin(2\pi f_c t) + N_s(t)\cos(2\pi f_c t)$ (3.40b)

b) Envelope and phase representation

Let us write $n(t)$ as

$$n(t) = r(t)\cos\left[2\pi f_c t + \psi(t)\right] \quad (3.41a)$$

$$= r(t)\left[\cos\psi(t)\cos(2\pi f_c t) - \sin\psi(t)\sin(2\pi f_c t)\right] \quad (3.41b)$$

Comparing Eq. 3.41(b) with Eq. 3.39, we find that

$$n_c(t) = r(t)\cos\psi(t)$$

$$n_s(t) = r(t)\sin\psi(t)$$

or $$r(t) = \left[n_c^2(t) + n_s^2(t)\right]^{\frac{1}{2}} \quad (3.42a)$$

and $$\psi(t) = arc\tan\left[\frac{n_s(t)}{n_c(t)}\right] \quad (3.42b)$$

$r(t)$ is the envelope of $n(t)$ and $\psi(t)$ its phase. Generalizing, we have

$$N(t) = R(t)\left[\cos\left(2\pi f_c t + \Psi(t)\right)\right] \quad (3.43)$$

where $R(t)$ is the envelope process and $\Psi(t)$ is the phase process. Eq. 3.43 justifies the statement that the NBN waveform exhibits both amplitude and phase variations.

3.10.2 Properties of narrowband noise

We shall now state some of the important properties of NBN. (For proofs, refer to Appendix A3.1.)

P1) *If $N(t)$ is zero mean, then so are $N_c(t)$ and $N_s(t)$.*

P2) *If $N(t)$ is a Gaussian process, then $N_c(t)$ and $N_s(t)$ are jointly Gaussian.*

P3) *If $N(t)$ is WSS, then $N_c(t)$ and $N_s(t)$ are WSS.*

P4) *Both $N_c(t)$ and $N_s(t)$ have the same PSD which is related to $S_N(f)$ of the original narrowband noise as follows:*

$$S_{N_c}(f) = S_{N_s}(f) = \begin{cases} S_N(f - f_c) + S_N(f + f_c), & -B \le f \le B \\ 0, & \text{elsewhere} \end{cases}$$

where it is assumed that $S_N(f)$ occupies the frequency interval $f_c - B \le |f| \le f_c + B$ and $f_c > B$.

P5) I*f the narrowband noise $N(t)$ is zero mean, then $N_c(t)$ and $N_s(t)$ have the same variance as the variance of $N(t)$.*

P6) *The cross-spectral densities of the quadrature components of narrowband noise are purely imaginary, as shown by*

$$S_{N_c N_s}(f) = -S_{N_s N_c}(f) = \begin{cases} j\left[S_N(f + f_c) - S_N(f - f_c)\right], & -B \le f \le B \\ 0, & \text{elsewhere} \end{cases}$$

P7) *If $N(t)$ is zero-mean Gaussian and its PSD, $S_N(f)$ is locally symmetric about $\pm f_c$, then $N_c(t)$ and $N_s(t)$ are statistically independent.*

Property **P7)** implies that if $S_N(f)$ is locally symmetric about $\pm f_c$, then $R_{N_c N_s}(\tau)$ is zero for all τ. Since $N_c(t)$ and $N_s(t)$ are jointly Gaussian, they become independent. However, it is easy to see that $R_{N_c N_s}(0) = 0$ whether or not there is local symmetry. In other words the variables $N_c(t_1)$ and $N_s(t_1)$, for any sampling instant t_1, are always independent.

Note: Given the spectrum of an arbitrary band-pass signal, we are free to choose any frequency f_c as the (nominal) centre frequency. The spectral shape $S_{N_c}(f)$ and $S_{N_s}(f)$ will depend on the f_c chosen. As such, the canonical representation of a narrowband signal is not unique. For example, for the narrowband spectra shown in Fig. 3.22, if f_1 is chosen as the representative carrier frequency, then the noise spectrum is $2\,B_1$ wide, whereas if f_2 (which is actually the mid-frequency of the given band) is chosen as the representative carrier frequency, then the width of the spectrum is $2\,B_2$. Note for the $S_N(f)$ of Fig. 3.22, it is not possible for us to choose an f_c such that $S_N(f)$ exhibits local symmetry with respect to it.

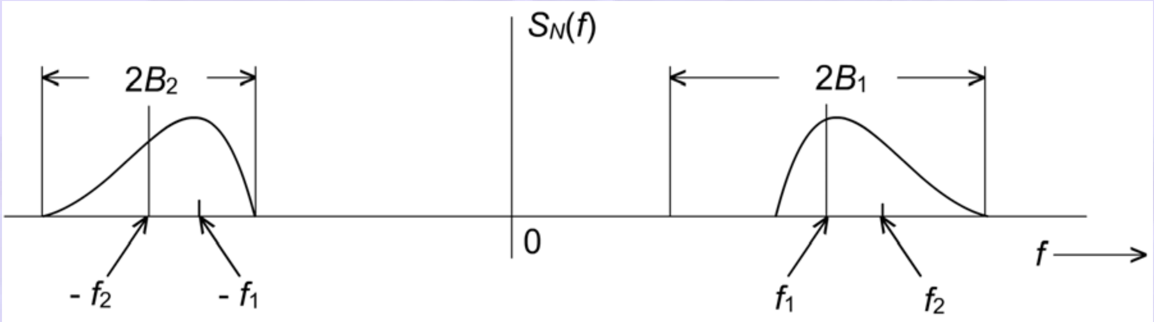


Fig. 3.22: Narrowband noise spectrum with two different centre frequencies

Example 3.18

For the narrowband noise spectrum $S_N(f)$ shown in Fig. 3.23, sketch $S_{N_c}(f)$ for the two cases, namely a) $f_c = 10\,kHz$ and b) $f_c = 11\,kHz$. c) What is the variance $N_c(t)$?

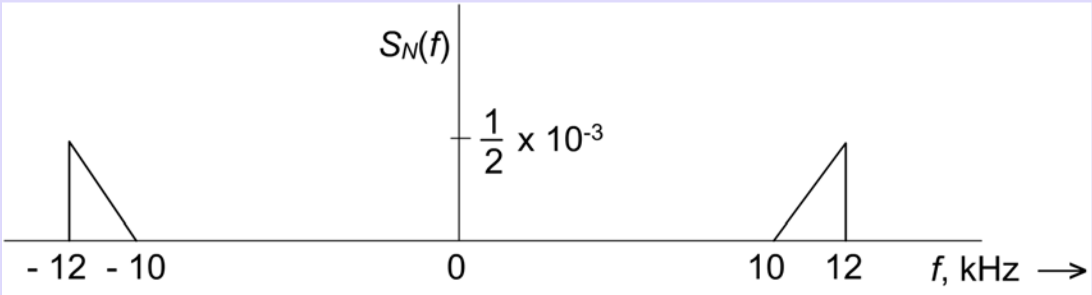


Fig. 3.23: $S_N(f)$ for the example 3.18

a) From **P4)**, we have,

$$S_{N_c}(f) = \begin{cases} S_N(f - f_c) + S_N(f + f_c), & -B \leq f \leq B \\ 0 & , \textit{elsewhere} \end{cases}$$

Using $f_c = 10\, kHz$, and plotting $S_N(f - f_c)$ and $S_N(f + f_c)$, we obtain Fig. 3.24.

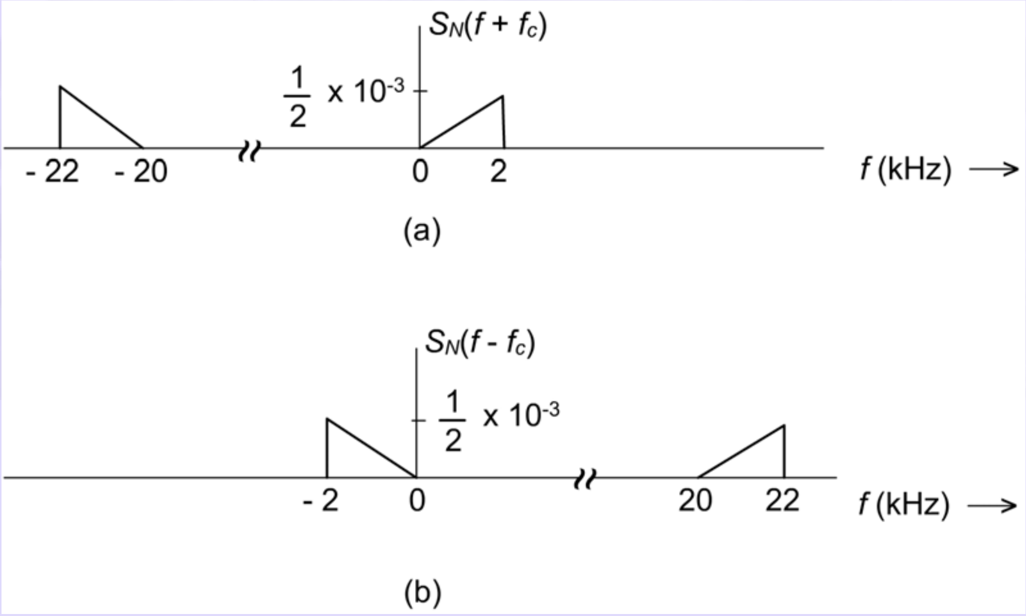


Fig. 3.24: Shifted spectra: (a) $S_N(f + f_c)$ and (b) $S_N(f - f_c)$

Taking $B = 2\, kHz$ and extracting the relevant part of $S_N(f + f_c)$ and $S_N(f - f_c)$, we have $S_{N_c}(f)$ as shown in Fig. 3.25.

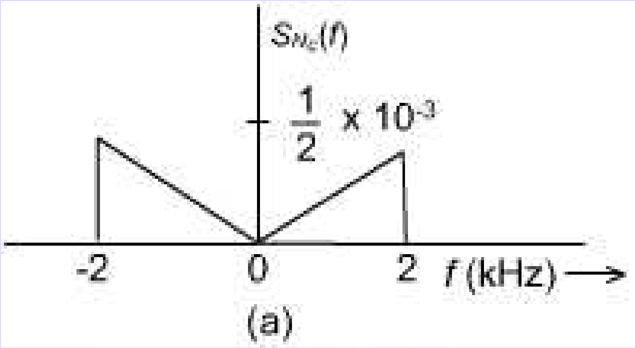


Fig. 3.25: $S_{N_c}(f)$ with $f_c = 10\,kHz$

b) By repeating the above procedure, we obtain $S_{N_c}(f)$ (the solid line) shown in Fig. 3.26.

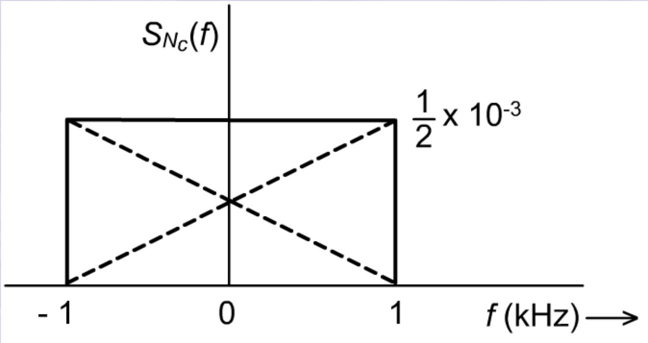


Fig. 3.26: $S_{N_c}(f)$ with $f_c = 11\,kHz$

c) $\sigma^2_{N_c} = \sigma^2_N = \int_{-\infty}^{\infty} S_N(f)\, d f = 1.0\, Watt$ ♦

Exercise 3.16

Assuming $N(t)$ to be WSS and using Eq. 3.40(a), establish

$$R_N(\tau) = R_{N_c}(\tau)\cos(2\pi f_c \tau) - R_{N_s N_c}(\tau)\sin(2\pi f_c \tau) \qquad (3.44)$$

Exercise 3.17

Let $N(t)$ represent a narrowband, zero-mean Gaussian process with

$$S_N(f) = \begin{cases} \dfrac{N_0}{2}, & f_c - B \le |f| \le f_c + B \\ 0, & otherwise \end{cases}$$

Let X and Y be two random variables obtained by sampling $N_c(t)$ and $\dfrac{d N_c(t)}{d t}$ at $t = t_1$, where $N_c(t)$ is the in phase component of $N(t)$.

a) Show that X and Y are independent.

b) Develop the expression for $f_{X,Y}(x, y)$.

Exercise 3.18

Let $N(t)$ represent a NBN process with the PSD shown in Fig. 3.27 below.

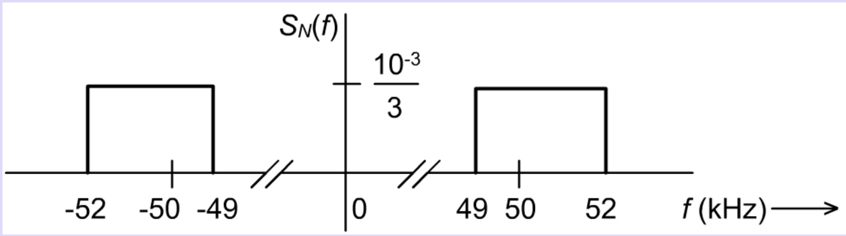


Fig. 3.27: Figure for exercise 3.18

Let $N(t) = N_c(t)\cos(\omega_c t) - N_s(t)\sin(\omega_c t)$ with $f_c = 50$ kHz.

Show that $R_{N_s N_c}(\tau) = \dfrac{2}{3}\,\text{sinc}(10^3\tau)\sin(3\pi 10^3\tau)$.

We shall now establish a relation between $R_N(\tau)$ and $R_{N_{ce}}(\tau)$, where $R_{N_{ce}}(\tau)$ evidently stands for the ACF of the complex envelope of the narrowband noise. For complex signals, such as $N_{ce}(t)$, we define its ACF as

$$R_{N_{ce}}(t+\tau, t) = E\left[N_{ce}(t+\tau)\, N^*_{ce}(t)\right]$$

$$= E\left[\left(N_c(t+\tau) + j N_s(t+\tau)\right)\left(N_c(t) - j N_s(t)\right)\right]$$

$$= R_{N_c}(\tau) + R_{N_s}(\tau) + j R_{N_s N_c}(\tau) - j R_{N_c N_s}(\tau)$$

But $R_{N_c}(\tau) = R_{N_s}(\tau)$ (Eq. A3.1.7)

and $R_{N_s N_c}(\tau) = R_{N_c N_s}(-\tau)$ (Property of cross correlation)

$$= -R_{N_c N_s}(\tau)$$

The last equality follows from Eq. A3.1.8. Hence, $R_{N_{ce}}(t+\tau, t) = R_{N_{ce}}(\tau) = 2\left(R_{N_c}(\tau) + j R_{N_s N_c}(\tau)\right)$. From Eq. 3.44, we have

$$R_N(\tau) = \frac{1}{2}\operatorname{Re}\left[R_{N_{ce}}(\tau)\, e^{+j 2\pi f_c \tau}\right]$$

$$= \frac{1}{4}\left[R_{N_{ce}}(\tau)\, e^{j 2\pi f_c \tau} + R^*_{N_{ce}}(\tau)\, e^{-j 2\pi f_c \tau}\right] \qquad (3.45)$$

Taking the Fourier transform of Eq. 3.45, we obtain

$$S_N(f) = \frac{1}{4}\left[S_{N_{ce}}(f - f_c) + S^*_{N_{ce}}(-f - f_c)\right] \qquad (3.46)$$

Note: For complex signals, the ACF is conjugate symmetric; that is, if $X(t)$ represents a complex random process that is WSS, then $R_X(-\tau) = R^*_X(\tau)$. The PSD $S_X(f) = F\left[R_X(\tau)\right]$ is real, nonnegative but not an even function of frequency.

3.10.3 PDF of the envelope of narrowband noise

From Eq. 3.43, we have,

$$N(t) = R(t)\cos\left[\left(2\pi f_c t + \Psi(t)\right)\right]$$

with $$N_c(t) = R(t)\cos\left(\Psi(t)\right) \quad (3.47a)$$

and $$N_s(t) = R(t)\sin\left(\Psi(t)\right) \quad (3.47b)$$

That is,

$$R(t) = \left[N_c^2(t) + N_s^2(t)\right]^{\frac{1}{2}} \quad (3.48a)$$

$$\Psi(t) = \tan^{-1}\left[\frac{N_s(t)}{N_c(t)}\right] \quad (3.48b)$$

Our interest is the PDF of the random variable $R(t_1)$, for any arbitrary sampling instant $t = t_1$.

Let N_c and N_s represent the random variables obtained from sampling $N_c(t)$ and $N_s(t)$ at any time $t = t_1$. Assuming N(t) to be a Gaussian process, we have N_c and N_s as zero mean, independent Gaussian variables with variance σ^2 where $\sigma^2 = \int_{-\infty}^{\infty} S_N(f)\, df$.

Hence,

$$f_{N_c N_s}(n_c, n_s) = \frac{1}{2\pi\sigma^2}\exp\left[-\frac{n_c^2 + n_s^2}{2\sigma^2}\right]$$

For convenience, let $R(t_1) = R$ and $\Psi(t_1) = \Psi$. Then,

$$N_c = R\cos\Psi \text{ and } N_s = R\sin\Psi$$

and the Jacobian of the transformation is,

$$J = \begin{bmatrix} \cos\psi & \sin\psi \\ \frac{-\sin\psi}{r} & \frac{\cos\psi}{r} \end{bmatrix} = r^{-1}$$

Therefore,

$$f_{R,\Psi}(r, \psi) = \begin{cases} \dfrac{r}{2\pi\sigma^2} \exp\left(-\dfrac{r^2}{2\sigma^2}\right), & r \geq 0, 0 \leq \psi < 2\pi \\ 0 & , \textit{otherwise} \end{cases}$$

$$f_R(r) = \int_0^{2\pi} f_{R,\Psi}(r, \psi)\, d\psi$$

It is easy to verify that

$$f_R(r) = \begin{cases} \dfrac{r}{\sigma^2} \exp\left(-\dfrac{r^2}{2\sigma^2}\right), & r \geq 0 \\ 0 & , \textit{otherwise} \end{cases} \tag{3.49}$$

Similarly, it can be shown that

$$f_\Psi(\psi) = \begin{cases} \dfrac{1}{2\pi}, & 0 \leq \psi < 2\pi \\ 0 & , \textit{otherwise} \end{cases} \tag{3.50}$$

As $f_{R,\psi}(r, \psi) = f_R(r)\, f_\Psi(\psi)$, we have R and Ψ as independent variables.

The PDF given by Eq. 3.49 is the Rayleigh density which was introduced in Chapter 2. From the above discussion, we have the useful result, namely, *the envelope of narrowband Gaussian noise is Rayleigh distributed.*

Let us make a normalized plot of the Rayleigh PDF by defining a new random variable V as $V = \dfrac{R}{\sigma}$ (transformation by a multiplicative constant). Then,

$$f_V(v) = \begin{cases} v \exp\left(-\dfrac{v^2}{2}\right), & v \geq 0 \\ 0 & , \textit{otherwise} \end{cases}$$

$f_V(v)$ is plotted in Fig. 3.28.

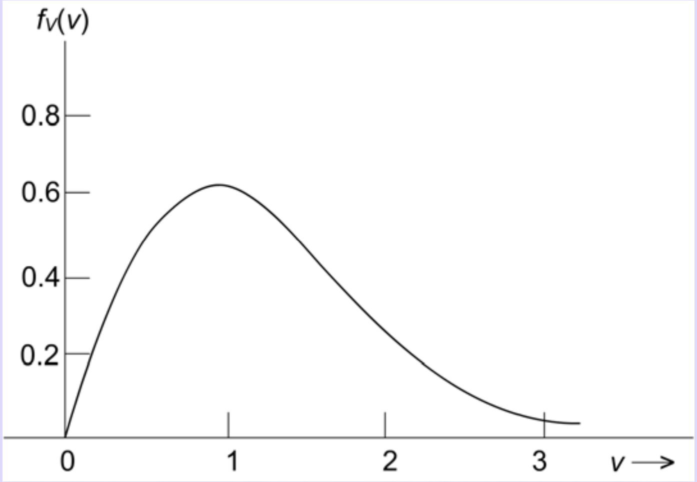


Fig. 3.28: Normalized Rayleigh PDF

The peak value of the PDF occurs at $v = 1$ and $f_V(1) = 0.607$.

3.10.4 Sine wave plus narrowband noise

Let a random process $X(t)$ be given by

$$X(t) = A\cos(2\pi f_c t) + N(t) \tag{3.51}$$

where A and f_c are constants, $N(t)$ represents the narrowband noise process whose centre frequency is taken as f_c. Our interest is to develop an expression for the PDF of the envelope of $X(t)$.

Using the canonical form for $N(t)$, Eq. 3.51 can be written as

$$X(t) = A\cos(2\pi f_c t) + N_c(t)\cos(2\pi f_c t) - N_s(t)\sin(2\pi f_c t)$$

Let $N_c'(t) = A + N_c(t)$. Assume $N(t)$ to be a Gaussian process with zero mean and variance σ^2. Then, for any sampling instant $t = t_1$, let N_c' denote the random variable $N_c'(t_1)$ and let N_s denote the random variable $N_s(t_1)$. From our earlier discussion, we find that N_c' is $N(A, \sigma^2)$, N_s is $N(0, \sigma^2)$ and N_c' is independent of N_s. Hence,

$$f_{N_c',N_s}\left(n_c', n_s\right) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{\left(n_c' - A\right)^2 + n_s^2}{2\sigma^2}\right] \quad (3.52)$$

Let $\quad R(t) = \left\{\left[N_c'(t)\right]^2 + \left[N_s(t)\right]^2\right\}^{\frac{1}{2}}$

$$\Psi(t) = \tan^{-1}\left[\frac{N_s(t)}{N_c'(t)}\right]$$

where $R(t)$ and $\Psi(t)$ are the envelope and phase, respectively of $X(t)$. By a procedure similar to that used in the computation of the PDF of the envelope of narrowband noise, we find that

$$f_{R,\Psi}(r, \psi) = \frac{r}{2\pi\sigma^2} \exp\left[-\frac{r^2 + A^2 - 2Ar\cos\psi}{2\sigma^2}\right], \quad r \geq 0, 0 \leq \psi \leq 2\pi$$

where $R = R(t_1)$ and $\Psi = \Psi(t_1)$

The quantity of interest is $f_R(r)$, where

$$f_R(r) = \int_0^{2\pi} f_{R,\Psi}(r, \psi)\, d\psi$$

That is,

$$f_R(r) = \frac{r}{2\pi\sigma^2} \exp\left(-\frac{r^2 + A^2}{2\sigma^2}\right) \int_0^{2\pi} \exp\left(\frac{Ar}{\sigma^2}\cos\psi\right) d\psi \quad (3.53)$$

The integral on the RHS of Eq. 3.53 is identified in terms of the defining integral for the *modified Bessel function of the first kind and zero order.*

$$\text{Let } I_0(y) = \frac{1}{2\pi}\int_0^{2\pi} \exp(y\cos\psi)\, d\psi \quad (3.54)$$

A plot of $I_0(y)$ is shown in Fig. 3.29. In Eq. 3.54, if we let $y = \frac{Ar}{\sigma^2}$, Eq. 3.53 can be rewritten as

$$f_R(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2 + A^2}{2\sigma^2}\right) I_0\left(\frac{A r}{\sigma^2}\right) \quad (3.55)$$

The PDF given by Eq. 3.55 is referred to as *Rician distribution.*

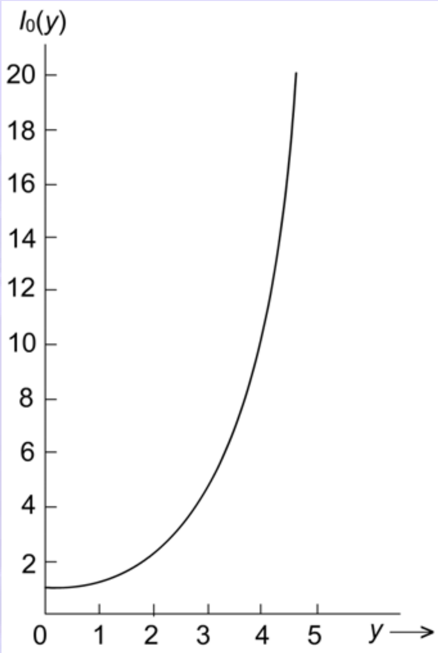


Fig. 3.29: Plot of $I_0(y)$

The graphical presentation of the Rician PDF can be simplified by introducing two new variables, namely, $V = \frac{R}{\sigma}$ and $\alpha = \frac{A}{\sigma}$. Then, the Rician density of Eq. 3.55 can be written in a *normalized form*,

$$f_V(v) = v \exp\left(-\frac{v^2 + \alpha^2}{2}\right) I_0(\alpha v) \quad (3.56)$$

which is plotted in Fig. 3.30 for various values of α.

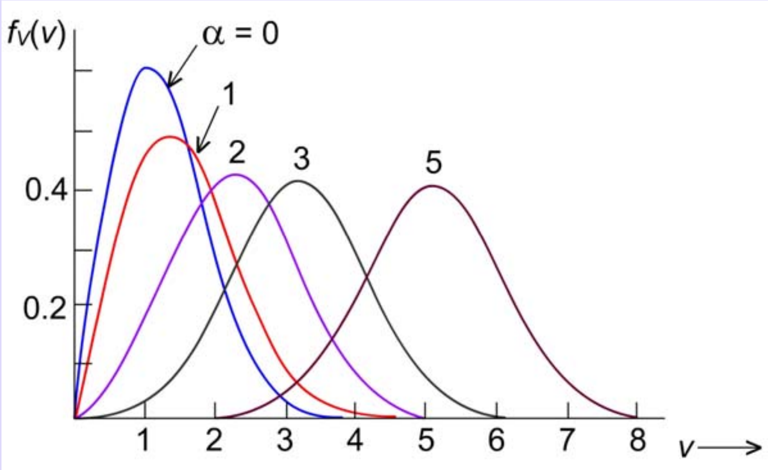


Fig. 3.30: Normalized Rician PDF for various values of α

Based on these curves, we make the following observations:

i) From Fig. 3.28, we find $I_0(0) = 1$. If $A = 0$, then $\alpha = 0$ and

$$f_V(v) = \begin{cases} v \exp\left(-\dfrac{v^2}{2}\right), & v \geq 0 \\ 0 & , \; otherwise \end{cases}$$

which is the normalized Rayleigh PDF shown earlier in Fig. 3.27. This is justified because if $A = 0$, then $R(t)$ is the envelope of only the narrowband noise.

ii) For $y >> 1$, $I_0(y) \simeq \dfrac{e^y}{\sqrt{2\pi y}}$. Using this approximation, it can be shown that $f_V(v)$ is approximately Gaussian in the vicinity of $v = \alpha$, when α is sufficiently large. That is, when the sine-wave amplitude A is large compared with σ (which is the square root of the average power in $N(t)$), $f_V(v)$ can be approximated by a Gaussian PDF over a certain range. This can be seen from Fig. 3.29 for the cases of $\alpha = 3$ and 5.

Appendix A3.1

Properties of Narrowband Noise: Some Proofs.

We shall now give proofs to some of the properties of the NBN, mentioned in sec.3.10.2.

P1) *If $N(t)$ is zero mean Gaussian, then so are $N_c(t)$ and $N_s(t)$.*

Proof: Generalizing Eq. 3.37 and 3.38, we have

$$N_c(t) = N(t)\cos(2\pi f_c t) + \widehat{N}(t)\sin(2\pi f_c t) \quad \text{(A3.1.1)}$$

$$N_s(t) = \widehat{N}(t)\cos(2\pi f_c t) - N(t)\sin(2\pi f_c t) \quad \text{(A3.1.2)}$$

$\widehat{N}(t)$ is obtained as the output of an LTI system with $h(t) = \frac{1}{\pi t}$, with the input $N(t)$. This implies that if $N(t)$ is zero mean, then so is $\widehat{N}(t)$. Taking the expectation of A3.1.1, we have

$$\overline{N_c(t)} = \overline{N(t)}\cos(2\pi f_c t) + \overline{\widehat{N}(t)}\sin(2\pi f_c t)$$

As $\overline{N(t)} = \overline{\widehat{N}(t)} = 0$, we obtain $\overline{N_c(t)} = 0$. Taking the expectation of A3.1.2, we obtain $\overline{N_s(t)} = 0$. ♦

P2) *If $N(t)$ is a Gaussian process, then $N_c(t)$ and $N_s(t)$ are jointly Gaussian.*

Proof: This property follows from Eq. 3.40(a) because, $N(t)$ is guaranteed to be Gaussian only if $N_c(t)$ and $N_s(t)$ are jointly Gaussian. ♦

P3) *If $N(t)$ is WSS, then $N_c(t)$ and $N_s(t)$ are WSS.*

Proof: We will first establish that

$$R_{N_c}(t+\tau, t) = R_{N_c}(\tau) = R_N(\tau)\cos(2\pi f_c \tau) + \widehat{R}_N(\tau)\sin(2\pi f_c \tau) \quad \text{(A3.1.3)}$$

(In Eq. A3.1.3, $\widehat{R}_N(\tau)$ is the Hilbert transform of $R_N(\tau)$.)

Consider the scheme shown in Fig A3.1.1.

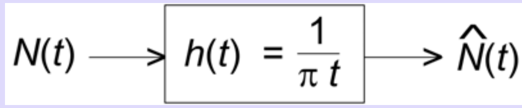


Fig. A3.1.1: Scheme to obtain $\hat{N}(t)$ from $N(t)$

Eq. 3.21(a) gives us

$$R_{\hat{N} N}(\tau) = \frac{1}{\pi \tau} * R_N(\tau) \quad \text{(A3.1.4a)}$$

$$= \hat{R}_N(\tau) \quad \text{(A3.1.4b)}$$

$$R_{N \hat{N}}(\tau) = R_{\hat{N} N}(-\tau),$$

$$= \frac{1}{\pi(-\tau)} * R_N(-\tau)$$

$$= -\frac{1}{\pi \tau} * R_N(\tau)$$

$$= -\hat{R}_N(\tau)$$

That is, $R_{\hat{N} N}(\tau) = -R_{N \hat{N}}(\tau)$ (A3.1.5)

$$R_{N_c}(t + \tau, t) = E\left[N_c(t + \tau) N_c(t)\right]$$

Expressing $N_c(t + \tau)$ and $N_c(t)$ in terms of the RHS quantities of Eq. A3.1.1 and after some routine manipulations, we will have

$$\begin{aligned} R_{N_c}(t + \tau, t) = \ & \frac{1}{2}\left[R_N(\tau) + R_{\hat{N}}(\tau)\right] \cos(2\pi f_c \tau) \\ & + \frac{1}{2}\left[R_N(\tau) - R_{\hat{N}}(\tau)\right] \cos\big(2\pi f_c (2t + \tau)\big) \\ & + \frac{1}{2}\left[R_{\hat{N} N}(\tau) - R_{N \hat{N}}(\tau)\right] \sin(2\pi f_c \tau) \\ & + \frac{1}{2}\left[R_{\hat{N} N}(\tau) + R_{N \hat{N}}(\tau)\right] \sin\big(2\pi f_c (2t + \tau)\big) \end{aligned} \quad \text{(A3.1.6)}$$

$R_{\widehat{N}}(\tau)$ is the autocorrelation of the Hilbert transform of $N(t)$. As $N(t)$ and $\widehat{N}(t)$ have the same PSD, we have $R_N(\tau) = R_{\widehat{N}}(\tau)$ and from Eq. A3.1.5, we have $R_{\widehat{N}N}(\tau) = -R_{N\widehat{N}}(T)$. Using these results in Eq. A3.1.6, we obtain Eq. A3.1.3.

♦

Exercise A3.1.1

Show that $R_{N_s}(t) = R_{N_c}(\tau)$ (A3.1.7)

and $R_{N_c N_s}(\tau) = R_N(\tau)\sin(2\pi f_c \tau) - \widehat{R}_N(\tau)\cos(2\pi f_c \tau)$ (A3.1.8)

P4) *Both $N_c(t)$ and $N_s(t)$ have the same PSD which is related to $S_N(f)$ of the original narrowband noise as follows:*

$$S_{N_c}(f) = S_{N_s}(f) = \begin{cases} S_N(f - f_c) + S_N(f + f_c), & -B \le f \le B \\ 0, & elsewhere \end{cases}$$

where it is assumed that $S_N(f)$ occupies the frequency interval $f_c - B \le |f| \le f_c + B$ and $f_c > B$.

Proof: Taking the Fourier transform of Eq. A3.1.3, we have

$$S_{N_c}(f) = \frac{1}{2}\left[S_N(f - f_c) + S_N(f + f_c)\right]$$
$$-\frac{1}{2}\left[S_N(f - f_c)\,\mathrm{sgn}(f - f_c) - S_N(f + f_c)\,\mathrm{sgn}(f + f_c)\right]$$
$$= \frac{1}{2} S_N(f - f_c)\left[1 - \mathrm{sgn}(f + f_c)\right] + \frac{1}{2} S_N(f + f_c)\left[1 + \mathrm{sgn}(f + f_c)\right]$$

(A3.1.9)

But $1 - \mathrm{sgn}(f - f_c) = \begin{cases} 2, & f < f_c \\ 0, & outside \end{cases}$ (A3.1.10a)

$1 + \mathrm{sgn}(f + f_c) = \begin{cases} 2, & f > -f_c \\ 0, & outside \end{cases}$ (A3.1.10b)

Using Eq. A3.1.10 in Eq. A3.1.9 completes the proof. ♦

Exercise A3.1.2

Establish properties P5 to P7.

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3) K. Sam Shanmugam and A. M. Breiphol, Random Signals: Detection, Estimation and data analysis, John Wiley, 1988