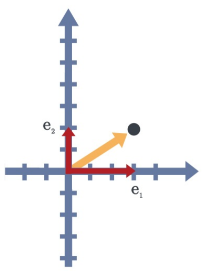
Linear Algebra: Low Dimension

What you'll do

- Write a vector and perform basic operations
- Use the dot product to test whether two vectors are orthogonal
- Define a line in a plane
- Compute distance to a line in a plane



Course Description

To perform basic computations in the eCornell Machine Learning certificate, you need the ability to solve elementary linear algebra problems in two dimensions. This course will provide you with the theory and activities that you can apply to build those skills. In this course, we are focusing on solving problems in low dimension; that is, two dimensions. You will practice a limited number of topics; this course is not a complete linear algebra course.

If you've taken an eCornell course before, this one will seem different. There is no required discussion, although we will provide a discussion board where you can ask questions. That said, our faculty expert Dr. Steve Bennoun has provided videos, examples, and exercises to guide you in solving linear algebra problems in two-dimensional space.

If you are new to linear algebra or forgot what you previously learned, we suggest that you take the whole course. Otherwise, feel free to choose from among those videos and activities that support the problems you want to solve. This is self-paced, just-in-time training for the eCornell Machine Learning certificate.

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Steve Bennoun is an Active Learning Lecturer in the Department of Mathematics. Dr. Bennoun completed his undergraduate degree at the École Polytechnique Fédérale de Lausanne in Switzerland and his PhD at the University of British Columbia in Canada. His research is focused on algebra and category theory, and he has presented his work at several national and international conferences. Dr. Bennoun has also published articles in journals such as the *Journal of Algebra*. At Cornell, Dr. Bennoun's research is focused on how to enhance student learning in mathematics courses by introducing active learning methods, a topic about which he is writing a book.

Welcome Transcript

In this course, we'll learn the fundamental concepts of linear algebra. We'll talk about the vectors, about how to compute what we call the dot product, computing distances between a point and a line, or to know on which side of the line a point is lying, all these kind of things that we use in linear algebra all the time. In this course, we'll do everything in two dimensions. Because on the one hand we can easily draw and visualize what we're doing very easily, on the other hand, we can still define all the concepts that we need and that would then be used in higher dimensions.

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Module 1: Write a Vector and Perform Basic Operations

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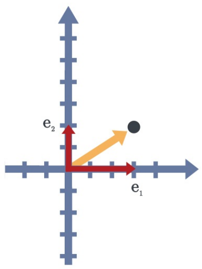
Module Introduction: Write a Vector and Perform Basic Operations



Understanding vectors and performing basic operations such as addition, subtraction, and multiplication are key linear algebra skills. In this module, you will perform basic operations and calculate the length of a vector.

In addition to viewing the videos, you can practice a few exercises and then check your work with the solution set. Finally, there's a discussion board if you have a question.

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Read: Review a Few Algebra Concepts

Download this summarized
refresher of these topics.

You've taken algebra and trigonometry in the past. We think it will be helpful for you to review a few concepts from those subject areas. Take a moment to scan the following topics that Dr. Bennoun will use to solve linear algebra exercises.

X-Y Plane
The Slope of a Line
Right Triangles and Cosines

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Watch: Explore Vectors

If you've given someone directions at a store, pointing that person down one aisle, then to the right past the next two aisles, for example, then you've used vectors. In this video, Dr. Bennoun will describe vectors and how to identify their components.

Video Transcript

Imagine you are at home with a friend and you want to explain to them how to go to the bookstore. Well, in that case, that's going to be quite easy. You want to tell them to follow Willow Street for two blocks, and then Main Street for three blocks. What you could also do is use some arrows. So we will have a Willow Street arrow and a Main Street arrow. Then that would be followed, two Willow Street arrows and then three Main Street arrows. The advantage with that is that you don't have to tell them, "Oh, go on the left, on the right," because the arrows are going to give the direction. So there's no way you get lost. So we see with this example that, in general, arrows are then what we're going to call vectors. Are extremely efficient to describe a space. Can be something in two-dimension, as on this map, or in three-dimension or more dimensions.

So that's why we want to use vectors in general. Let's think about the two-dimensional plane. So to describe it, just as in the map, we need two basic directions. In that case, these two basic directions, we're going to call them e1 and e2. These are two arrows of length one. They are on the two axis; on the x and y-axis. Now, imagine you want to go from the origin to the 0.32. Well, in that case, you just need to follow e1 for three blocks, and then e2 two times or two blocks. Therefore, the vector or the arrow that goes from the origin to (3, 2) is going to be (3, 2). We're going to write that as a v and a column with 3 for 3 times e1 and 2 for 2 times e2. You see that in terms of notation, vectors like this one, we write it with bold letters. Or if it's written by hand or in textbooks, you have a small arrow above the letter, to indicate very clearly, this is a vector and not just a number. Let's take another example. Here, we have the vector u that goes from the origin to the point (1, -2). So this time, I'm following e1 for one block, and then I have to go down, not up, for two blocks. So because of that, the second component of u is going to be -2, and not 2 because I'm going downward in the other direction. For this vector here, u, the components are going to be 1 and then -2.

Now, let's look at examples that don't start from the origin. Vectors that don't start at the origin. So let's take x, that goes from (1, 1) to (3, 2). Well, in that case, I'm following e1 for two blocks, and then e2 for one block. So this vector x has components, (2, 1). If I want to go let's say from (-2, -2), to (-4, 2), what do I have to do? Well, I have to go to the left for two lengths of e2, so this is -2, and then I'm going up for four blocks along e2, so this is four. So this vector here is going to be (-2, 4). So finally, let's look at the vector z. The vector z starts at the point (3, -3), and goes to (5, -2). So this time here, while I'm going, I'm following e1 for two blocks. So first component is 2, and I am going upward for one block, so this is 1. So what do we see? That the components of z, so the numbers that compose z are (2, 1), which is just the same as x that we had before. So that's something very important. When we talk about vectors, vectors are arrows like this, but they don't recur the starting point. So this means that if you move your vector or your arrow all around, this is the same vector. Two vectors are equal, the components are the same, and here, it's the case. So that's why this is the same vector. Z and x are just the same vector.

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Watch: Perform Basic Operations with Vectors

As with numbers, you can add and subtract vectors. You can also multiply a vector by a number. In linear algebra, that number is called a scalar. When you multiply a vector by a positive scalar, the new vector points in the same direction. When you multiply a vector by a negative scalar, the new vector points in the opposite direction.

Dr. Bennoun will demonstrate these basic operations, and also how to calculate the components of a vector that does not start at the origin.

Video Transcript

Let's take the vector u, which is (3, 1), and v, (2, 4). Lets see what are the basic operations that we can do with these vectors. Well, the first thing we can do is add them. In terms of components, adding two vectors is very straight forward. You simply add component by component. So to have the vector u plus v, we're just going to get (5, 5), that we get by adding 3 plus 2 and 1 plus 4. As with numbers, if I have u plus v or v plus u, this is just exactly the same thing. Geometrically, that means that I have a new arrow that starts at the beginning of u and goes up to the end of v. I can also subtract them, the same rule applies.

So in that case if want to have u minus v, I will get (1, -3). If I want to compute v minus u, well this is going to be -1, which is 2 minus 3, and 3, that I get by 4 minus 1. Now another thing that we can do is called multiplication by a scalar. So scalar it's just a term we use in linear algebra to say a number or coefficient. We want to be really clear to make a clear difference between a scalar or a number and a vector. So here if I take the vector 2, 1, and multiply by 3, then I'm just going to get the vector, 3 times 2, 6 and 3 times 1, 3. This is the same thing as adding the vector v three times. I can also multiply by a negative number in that case, -2v which was -4 and -2. In terms of notation, notice that the scalar is not important. It's on the vectors that are important. So that's to make clear the difference between scalars and vectors. Let's finish with one more application of subtraction, which is how to compute the components of a vector.

Let's say you want to compute the components of the vector that starts at (1, 2), and goes all the way to (4, 6). The way we have done that so far is, to use e1 and e2, and you see how many times I need to use e1 and e2 to go from the starting point to the endpoint. In that case, v is (3, 4). But there's another rule. You can take the components of the endpoint and subtract the components of the starting point. So in that case, that would give us 4 minus 1, 3 and 6 minus 2, 4. So we do get the same thing. So that's another way of computing the components of a vector. Let's see why it works. I can define the two final vectors, the vector (1, 2). So I'm looking at the point as a vector from the origin, and the vector (4, 6). So now if I subtract two vectors, I do get the same vector as the vector v I'm interested in. That's just shifted and starts at the origin, but that's really the same vector. So now we have a new way of computing components of a vector by subtracting from the components of the endpoint, the components of the starting point.

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Watch: Determine the Length of a Vector

One aspect of a vector you might want to know is its length or, as it is called in linear algebra, its norm. The length of a vector is also called its magnitude and can be any positive number. In the following video, Dr. Bennoun will show you how to draw a right triangle with the vector as its hypotenuse, and use the Pythagorean theorem to solve for the norm of a vector.

Please note that at 1:41 and 1:54, when Dr. Bennoun says, "It can really be any number," he means to say, "It can really be any positive number." Distance is always a positive number; in other words, a negative distance is impossible.

Please note that at 1:52, when Dr. Bennoun says "four squared," he means to say "two squared." The calculation is correct on the screen.

Video Transcript

Let's take a vector v. So we have seen that a vector it's an arrow on the plane. We don't record the starting point. So this arrow can be anywhere, and it has certain length. So how do we compute this length? In linear algebra, the length is called the norm of v, and we write that with two small bars around the letter v. Let's take an example to see how we compute the length of a vector. Consider the vector that's (3, 4). To compute the length, we're going to build a small right triangle. The horizontal side has length 3, and the vertical side has length 4. Now, if we use the Pythagorean theorem, so what does it say? What the Pythagorean theorem says is that when you have a right triangle as this one, if you sum the square of the two small sides, you get the square of the bigger side. So in that case, that means that the norm of v squared is going to be equal to 3 squared plus 4 squared which in total is 25.

So now, if I want just the length of v, I just need to take the square root of this 25 which is 5. So the length or the norm of v is 5. That's the same thing in general. If I have a vector v, which has components v1 and v2, its norm is just going to be the square root of v1 squared plus v2 squared. In general, the length does not have to be a whole number. It can be really any number. So if I take the vector x, (1, 2), then its norm is going to be the square root of 1 squared plus 4 squared, square root of 5. So it can really be any number. If I multiply a vector by a scalar as we saw before, then the length it's just multiplied by the scalar which is exactly what we would expect. So for example, if I have the vector v 3, 4 and then I compute the vector 3v, which is going to be (9, 12). Then when I compute the norm or the length of 3v, this is going to be the square root of 9 squared plus 12 squared, which is the square root of 225, which is 15 and 15 is 3 times 5, and 5 was the length of v. So this is exactly what we would expect.

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Read: Examine Three Additional Examples

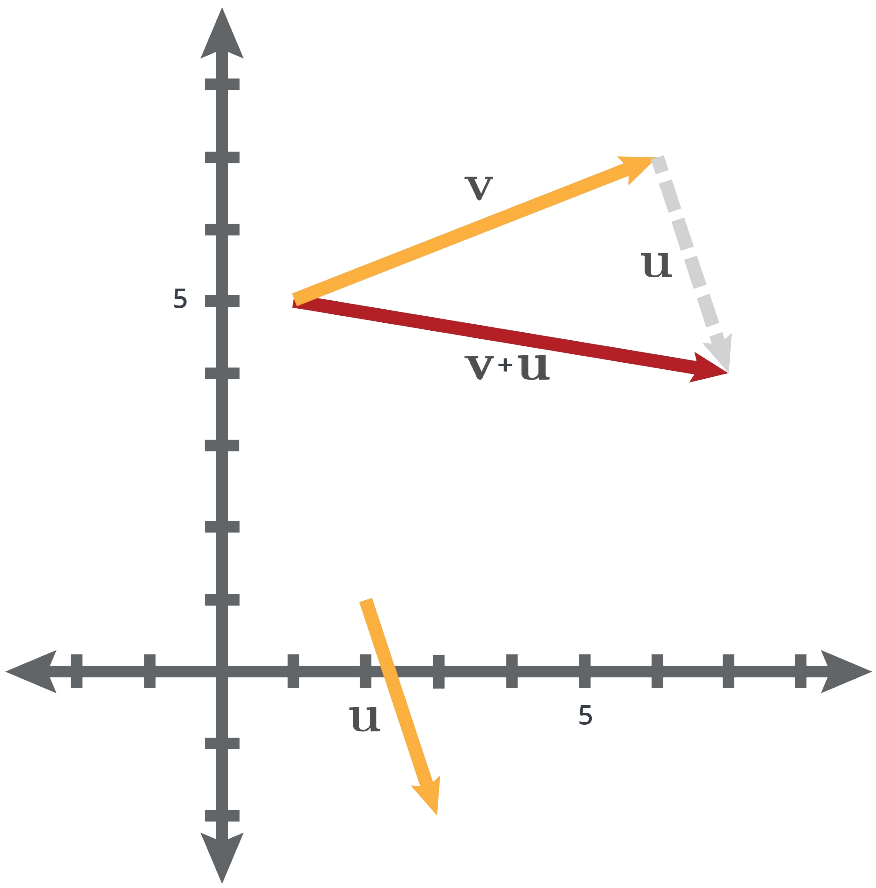
Here are three additional examples of basic operations with vectors. Click on each tab to open and view the problem.

reveal2Example 1Example 1
reveal4Example 2Example 2
reveal6Example 3Example 3

Compute $\mathbf{v} + \mathbf{u}$.

Take the following vectors: $\mathbf{u} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

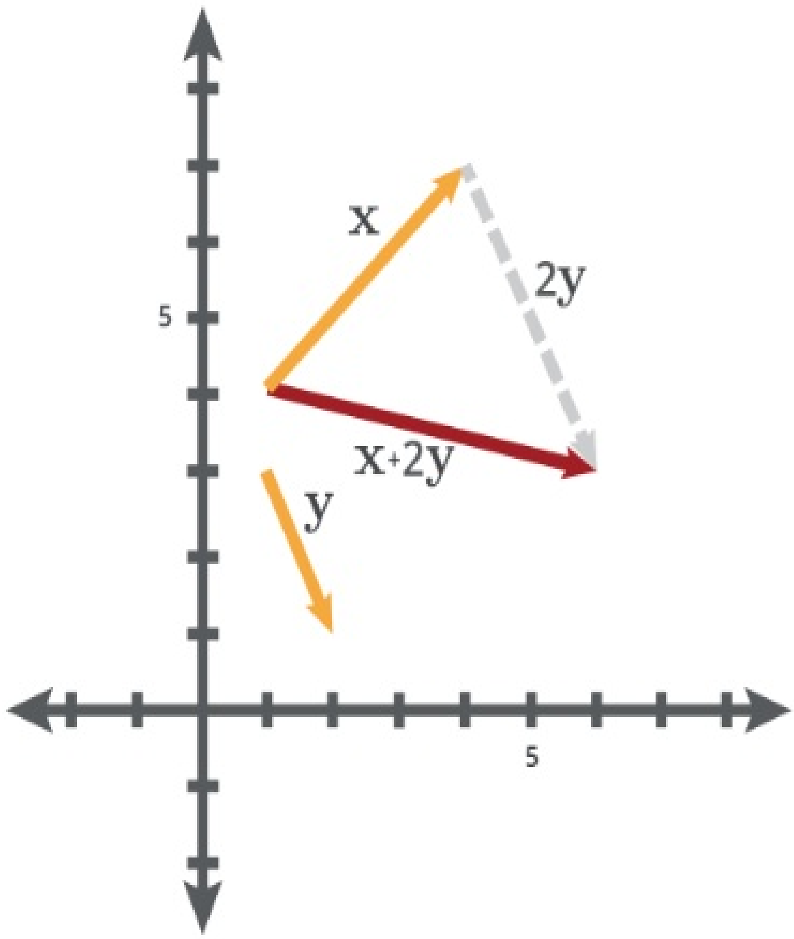
$$\mathbf{v} + \mathbf{u} = \begin{pmatrix} 5+1 \\ 2+(-3) \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$$



Compute $\mathbf{x} + 2\mathbf{y}$.

Take the following vectors: $\mathbf{x} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

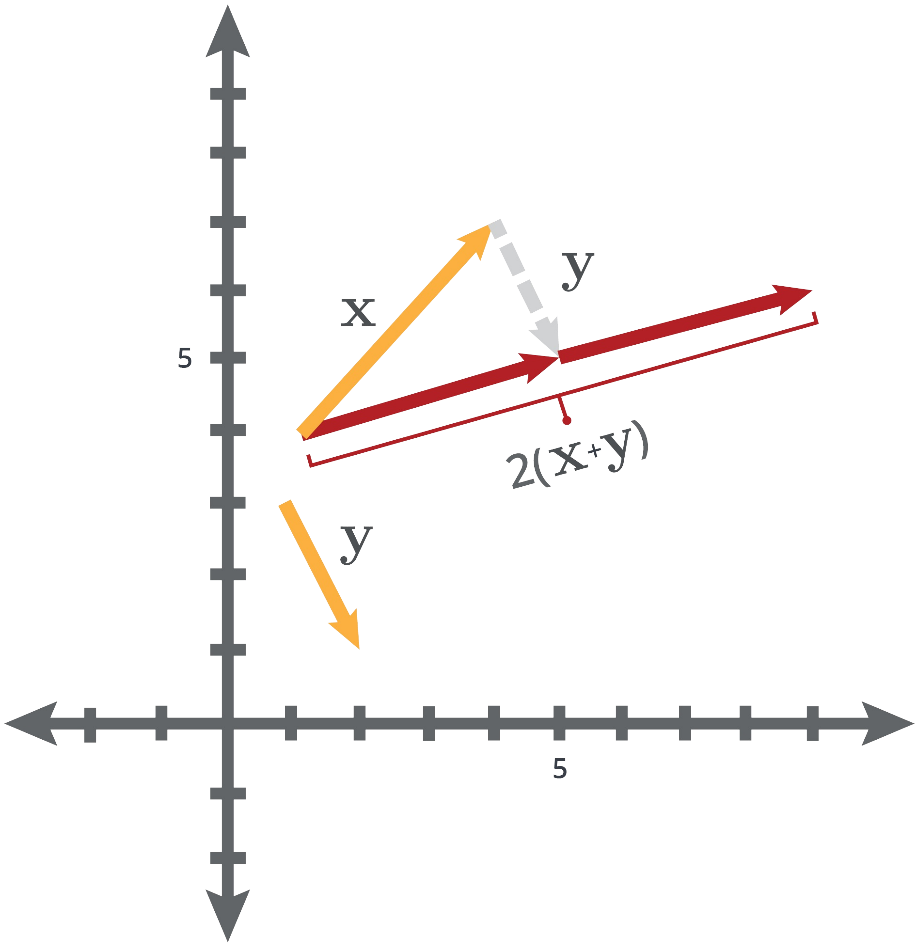
$$\mathbf{x} + 2\mathbf{y} = \begin{pmatrix} 3 + 2 \cdot 1 \\ 3 + 2 \cdot (-2) \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$$



Compute $2(\mathbf{x}+\mathbf{y})$.

Take the following vectors: $\mathbf{x} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$and $\mathbf{y} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

$$2(\mathbf{x}+\mathbf{y}) = 2\begin{pmatrix} 3+1 \\ 3-2 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$$



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Activity: Module 1 Exercises

Print this Module One Exercise Worksheet to practice basic operations with vectors.

You have explored basic operations with vectors and computed the norm of a vector. Now you should practice a few problems before proceeding to the next topic. Dr. Bennoun recommends that you print this exercise set and solve them by hand. You will find the solutions on the next page.

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Tool: Module 1 Exercises Solution Set

Use the Solution Set for Module 1
Exercises to review your work.

If you haven't completed the exercises for Module 1, we suggest you do so before you check the answers. As in sports, you have to perform math to learn it. Use this solution set to assess where you succeeded in solving problems correctly. For any problem you performed incorrectly, we recommend you review the video that covers that topic and try it again.

If you need additional assistance, we have provided a discussion board on the next page where you can ask an instructor for clarification. Be sure to search the discussions first to see if your question has already been answered.

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Module 1 Q&A

This Q&A discussion board is a place for you to post questions about the exercises. Common questions and important topics may be addressed by the course assistant.

Instructions:

Enter a keyword in the search box — for example, "norm" or "Exercise 2" — to find an existing discussion.
Upvote posts that contain a question that you also have by "liking" it with the thumbs-up button at the bottom of the post.
If you don't see your question, write a discussion post. Be specific: include the exercise number and the topic, along with the tactics you already used to try and solve it.
If you have a potential answer or suggestion that might resolve a question thread, feel free to share your understanding in a reply. This will help guide the instructor's feedback.

The questions that are the most upvoted or replied to that haven't been resolved will likely be addressed by an instructor.

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Module Wrap-up: Write a Vector and Perform Basic Operations

You practiced identifying the components of vectors that begin at the origin, and those that do not begin at the origin. You added and subtracted vectors, and multiplied vectors by scalars. Finally, you calculated the norm of a vector.

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Module 2: Use the Dot Product to Test for Orthogonality

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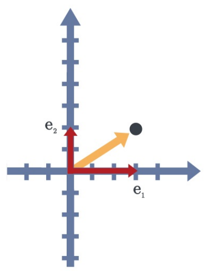
Module Introduction: Use the Dot Product to Test for Orthogonality



You might recall from earlier math studies that if two lines are perpendicular to one another, they intersect at right angles. Well, in linear algebra, we say that if two vectors (or planes) are perpendicular, they are orthogonal. This condition is called orthogonality. It is specialized terminology, but it means the same thing as a concept you likely already know.

In this module, you will learn to use a calculation called the dot product to determine if two vectors are orthogonal. The dot product is simply a method of multiplication. You will solve for the dot product geometrically as well. Dr. Bennoun has provided several examples of both computations for you to explore, and you can practice solving problems afterward.

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Watch: Consider the Dot Product

The dot product is the sum of the products of the components of two vectors. Note that the dot product is sometimes called the scalar product or the inner product. Dr. Bennoun explains further in this video and provides examples.

Video Transcript

In linear algebra, there is an operation that we called the dot product. Let's consider two vectors x and y. In that case, let's take x is 1, 2 and y is 3, 4. Then the dot product of x and y is going to be a number that we compute as follows. It's going to be 1 times 3 plus 2 times 4, or in total 11. So we said that the dot product of x and y is 11. Really notice that we take two vectors and we get a number, we don't get another vector. That's really important. We get a scalar. We write it as x small dot y, hence the name, and the general rule is really for two vectors, if I have a general vector x which has components x1, x2 and y, y1, y2, the dot product of x and y is going to be x1 times y1 plus x2 times y2. Let's see another concrete example. If now x is -2, 3, and y 4,1, then the dot product of x and y is going to be -2 times 4 plus 3 times 1 which is -5. So you see that the dot product can be negative or positive. Again it can be any sort of number. Let's also note that the dot product of x with y is the same thing as the dot product of y with x. That's because multiplication and addition are commutative. The order in which you do them doesn't matter. So that's the same for the dot product. One last thing, the dot product is also often called the scalar product because it gives us a scalar or sometimes the inner product. The notation x dot y is the most common, but there's also another notation that you'll see sometimes which is x, superscript t, y.

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Watch: Examine Geometrical Representations of the Dot Product

You can also compute the dot product geometrically. In this video, Dr. Bennoun will illustrate how to do this. Here's a reminder about terminology: In the first example, x is the hypotenuse of the right triangle. The adjacent is the side that shares the angle θ with the hypotenuse, in this case, d_1.

Video Transcript

Take the vectors x and y and the dot product. The dot product is going to be x1 times y1 plus x2 times y2. There is another way of computing the dot product. Namely it's the norm of x times the norm of y times the cosine of the angle between these two vectors. Let's see how it looks like geometrically. To see the cosine geometrically, we need to form a right triangle. So let's draw the line that is perpendicular to y and passes through the endpoint of x. We're going to call the distance between this perpendicular line and the origin of the two vectors d1. You can think of this length d1 as the shadow of x onto y if we had the spotlight high above the two vectors. This length is called the length of the orthogonal projection and orthogonal really means just perpendicular. That's a word that we use in linear algebra. So here the cosine of theta is going to be the adjacent over the hypotenuse of d1 over the norm of x.

So now if I look at my dot product, we said the dot product it's going to be the normal x times the norm of y times the cosine. Now the cosine is d1 over the norm of x. So in the end I really get d1 times the norm of y. If we want to know how long this distance d1 is, I just need to take the dot product and then divide by the norm of y. Of course we could also think about the dot product the other way round, namely I could think of it as the projection of y onto x, and that would be completely similar. Now the dot product would be the norm of x times the normal y times another distance d2 which would be projecting y onto x divided by the norm of y. So the end we would have d2 times the norm of x. This formula is very useful to compute our projection and we're going, that's something that we do a lot. The angle between two vectors can also be computed this way.

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Watch: Test for Orthogonality

If the dot product of two vectors is zero, then the vectors are orthogonal (perpendicular to one another). Likewise, if two vectors are orthogonal, then the dot product is zero. Watch as Dr. Bennoun expands on these ideas.

Video Transcript

What does that mean for two vectors to be orthogonal? Well, orthogonal is a word we use in linear algebra to say perpendicular, what you have a 90 degree angle between two vectors. Let's take some examples. Here we have u and v. They're orthogonal whereas w, x and y, z they're not, because the angle that is formed between those two aspects of vectors are not 90 degrees. Now, if I give you two vectors, how can we determine for sure to have a test whether they are orthogonal or not? Because when we look at the vectors it's not always completely clear. So let's look at an example. If I take the vectors u and v, u is (1, 2), and v is (-4, 2), and x and y, x is (-1, 5), and y is (4, 1). How can I see whether these two pairs are orthogonal or not? Well, let's think about the dot product. We saw that the dot product is the norm of y and the norm of x times the norm y times the cosine of the angle. So now remember, what is cosine of 90 degrees? Well, it is zero. So if the cosine is zero it means the dot product is going to be zero. So that gives us a test for orthogonality. If the dot product is is zero, my two vectors are orthogonal, that's for sure. So now if you go back to our example, let's compute the dot products. So dot product of u and v is going to be 1 times -4 plus 2 times 2. Well, that's zero. So these two vectors are orthogonal. If I look at x and y and the dot product is -1 times 4 plus 5 times 1, this is 1. So these two vectors are not orthogonal.

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Read: Understand the Zero Vector

The zero vector is essentially a
point.

A special kind of vector is the zero vector. It behaves similarly to zero in arithmetic. Consider how you use zero with numbers. When you add zero to or subtract zero from another number, the result is that number. This is called the "identity property of zero."

Similarly, when working with vectors, we have the zero vector:

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

As with numbers, if we add the zero vector to or subtract the zero vector from any other vector, nothing changes; the result is that vector.

Geometrically, the zero vector is essentially a point. We can also see this by computing its length, which is zero. Indeed, we have

$$||\mathbf{v}|| = \sqrt{0^2 + 0^2} = \sqrt{0 + 0} = \sqrt{0} = 0.$$

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Activity: Module 2 Exercises

Print this Module 2 Exercise Set
to practice solving the dot
product.

You have explored the dot product to determine if two vectors are orthogonal. Now you should practice a few problems before proceeding to the next topic. Dr. Bennoun recommends that you print this exercise set and solve the exercises by hand. You will find the solutions on the next page.

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Tool: Module 2 Exercises Solution Set

Use the Module 2 Solutions to review your work.

If you haven't completed the Module 2 exercises, we suggest that you do so before you check the answers. Use the solution set to assess where you succeeded. For any problem you performed incorrectly, we recommend you review the video that covers that topic and try it again.

If you still have questions, use the discussion board on the next page. Be sure to review the instructions before posting.

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Module 2 Q&A

This Q&A discussion board is a place for you to post questions about the exercises. Common questions and important topics may be addressed by the course assistant.

Instructions:

Enter a keyword in the search box — for example, "dot product" or "Exercise 2" — to find an existing discussion.
Upvote posts that contain a question that you also have by "liking" it with the thumbs-up button at the bottom of the post.
If you don't see your question, write a discussion post. Be specific: include the exercise number and the topic, along with the tactics you already used to try and solve it.
If you have a potential answer or suggestion that might resolve a question thread, feel free to share your understanding in a reply. This will help guide the instructor's feedback.

The questions that are the most upvoted or replied to that haven't been resolved will likely be addressed by an instructor.

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Module Wrap-up: Use the Dot Product to Test for Orthogonality

In this module, you used the dot product to test if two vectors are orthogonal. You also examined what the dot product represents geometrically. You then practiced solving exercises around this topic.

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Module 3: Define a Line in the Plane

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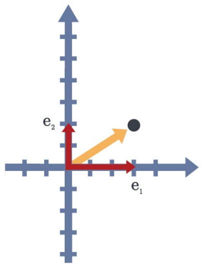
Module Introduction: Define a Line in a Plane



Now you can perform basic operations with vectors and calculate the dot product for two vectors. You can also test for orthogonality. In this module, you will combine these skills to define a line in a plane using vectors.

As before, we have provided exercises where you can practice the skills you will develop in this module.

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Watch: Determine a Vector from a Point

Given a point, you can determine the vector that goes from the origin to that point. There are many times when you want to ask a question about the point itself, but the best way to answer the question is to use the vector. For that reason, you will often find yourself switching back and forth between point and vector, as if they were the same thing. Dr. Bennoun explains this issue and gives an illustration in the following video.

Video Transcript

Take a point in the plane, for example the point (2, 1). Now, imagine you want to compute the distance between this point and a line on the plane or you want to know on which side of a line of this point is the line. How can we do that? Well, to do that, we need to use the dot product. Now, the problem with dot product is that we need a vector. We can't compute the dot product with a point. So what can we do?

Well, there is an easy way around there. What we're going to do is, instead of taking the point (2, 1), we're going to take the vector that starts at the origin and goes to the point (2, 1). The components of this vector is just (2, 1). So on the one hand I have a point and on the other hand I have the vector that goes from the origin to that point, and they have the same components. Now that I have a vector I can do all these operations, the dot product and all these things for which I need a vector. So we see that between this point and this vector, we can just use whichever we need for the operations we are doing. So that's why when we compute distances or all these things, such as the dot product, we're going to use the vector and not the point. So don't be surprised when we talk about a point and so we'll be using a vector instead. That's really this vector that is uniquely associated to that point. Let's just know that we can do that in the plane as we do here but it works in any dimension.

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Watch: Explore the Relationship between the Dot Product and the Norm

In this video, Dr. Bennoun will demonstrate the relationship between the dot product and the norm.

Video Transcript

What is the relationship between the dot product, and the norm of a vector? So it's length. Well, think the vector v, v1, v2. Its norm is going to be the square root of v1 squared plus v2 squared. Now let's look at the dot product. If I take the dot product of v with itself, it's going to be v1 times v1 plus v2 times v2. In other words, the dot product, this is the same thing as the norm but squared. Because, for the norm I have the square root, whereas for the dot product I don't have it. So the dot product of v with v is the norm squared. The other way of saying that, is using the other formula we have for the dot product, the geometrical one, where the dot product of v times v would be the norm of v times the norm of v times the cosine of the angle between these two vectors. Now since I'm taking the same vector of v and v, the angle is actually zero. So it's the same vector. And remember that cosine of 0 is 1. So again, I get that, the dot product of v with v is just the norm of v squared.

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Watch: Use a Vector to Determine the Direction of a Line

You can use a vector and the equation of a line to determine the line's direction. Watch as Dr. Bennoun describes this method.

Video Transcript

Take a line in the plane. Consider the line y equals 2x plus 3. Here the slope is 2 and the y-intercept is 3. So it means we cut the y-axis at 3. How can we define this sign using vectors? Well first we need a direction for the line. So this is going to be given by a vector v. Now isn't enough to just give a vector. Well remember that when we define a vector we don't register the origin of the vector it can stop anywhere. So if I just give you a direction, It could be a line, can be another parallel line also another one. So we need something more which is one point on the line to specify which of these parallel lines we are talking about.

So here for this example how do we find the direction? Well we want a vector that has a slope 2. The slope is just the rise over the run. So for a vector in two-dimension it's going to be the second component divided by the first component. So here are the simplest is just to take the vector (1, 2), when they divide the second component by the first I do get a slope of 2. We could take another one. We could take (2, 4), (3, 6), that would work as well. Let's just take the simplest one. So we have our direction the vector v. Now we need a point. what point are we in on this line? Well we know that the line is cutting the y-axis at 3. So that means it is passing through the (0, 3).

Now I have all I need to define my line and my line in vectors is going to be written as r equals the point p plus c times the vector v and c is a Scalar. So it's just any coefficient and the point in the plane that can write in this form p plus c times v is going to be on my line. For example, if I take c equals two, then I get r is (0, 3) plus 2 times (1, 2). So this is (2, 7). This means that the point (2, 7) is on the line. Conversely, if I'm given a point and I have to check whether it is on the line or not. How am I supposed to do that, let's imagine we want to check whether the point

(-1, 1) is on the line. What do we need to do is determine whether we can find a coefficient c as a scalar c so that in the end we got (-1, 1). So here, if I take c equals -1, what do I get? Well I am going to get (0, 3), plus -1 times 1, 2. So I really get 0 minus 1, and 3 minus 2, which is the point we want, (-1, 1). So this way we can see that this point is really on the line.

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Watch: Define a Line That Passes through the Origin

Can you define a line that goes through the origin using the dot product? Yes, you can, as Dr. Bennoun will describe here.

Note: at 1:49, the point (8, 5) is shown, while Dr. Bennoun says (8,4). The computations should be done using the point (8, 4).

Video Transcript

Consider a line that goes through the origin. I can really find this line using the dot product. Let's consider a vector w that is perpendicular to this line. Then let's take a point on this sign. Let's say x which has coordinates x1, x2, and we consider that as a vector. So now, if I take the dot product of w with x, what is it going to be? By definition of w, these two vectors are orthogonal to each other, and therefore, their dot product is 0. So this is a way of defining a line that goes through the origin. So I have very simple equation. W dot product with x is 0, and what is grid is that, it works in any dimension. Let's take a concrete example. If I take w is (-1, 2), then my equation is just w dot product with x is 0 or a -1 times x1 plus 2 times x2 is 0. So if I want to check whether a point is on this line, well, I just need to compute this dot product. So let's see with two points (6, 5) and (8, 4). The dot product with (6, 5), what does that give us? Well, that's going to be negative 1 times 6 plus 2 times 5, which is 4. It's not 0. So this point is not on the line. Now, if I take the dot product of w with (8, 4), what do I get? Negative 1 times 8 plus 2 times 4 this is 0. So this point is on the line. So this is how we check whether a point is on this line or not using this form, this definition of the line with the dot product.

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Watch: Define a Line That Does Not Pass through the Origin

In the previous video, you defined a line that passes through the origin. In this video, Dr. Bennoun will describe how to use the dot product to define a line that does NOT pass through the origin.

Video Transcript

Let's take a line that does not go through the origin. I have my line and I have my orthogonal vector w. Let's pick a point x, (x1, x2), that is on the line, and they compute the dot product of w with x. Well, it is not going to be equal to zero, because these two vectors are not orthogonal. So we'll need the different criteria to define this line. But let's see what happens geometrically. Let's look at the geometry of the situation. Well, this dot product w with x is the norm of w times the norm of x times the distance d divided by the number of x. So in the end is just d times w, the norm of w. Where d is just the distance between the origin and line following the direction given by w.

So now if I pick another point on that line, point y, (y1, y2). What do I get? When I take the dot product, at now it's going to be the norm of w times the norm of y times d over the norm of y. So it's just d times w, the norm of w again. What was really important here is that it is the same, d is exactly the same thing as before. This distance is always going to be the same. So it means that any point I pick on the line when I do the dot product with w, always get the same number. Let's call that number b. So it means that a general equation for a line is given by dot product of w with x equals b. When b is zero it means my line goes through the origin, and when it's different from zero, it does not go through the origin. Let's also note that b could be negative depending on the geometry of the situation and the orientation of the vectors. Let's take a concrete example. If w is (2, 2), and b is six, so my line is defined by dot product of w with x equals six. Let's pick some point and that's determined whether they're on the line them up.

So I'm going to take p1 is (4, 3), p2 is (0, 3), and p3 is (-2, 3). To check whether they are on the line or not, what I need to do is compute the dot product. So the dot product of w with p1 is 2 times 4 plus 2 times 3, well, this is 14 it's different from six. So this point is not on the line. For the second point p2, the dot product this time is 2 times 0 plus 2 times 3. Well, this is 6. So this point is on the line. Finally, the point 3, the dot product this time is 2 times negative 2 plus 2 times 3, well, this is 2. So again, this is not on the line. We can see something a bit more about p1 and p3. Note that, the dot product of w with p1 is 14, which is greater than six. Whereas, the dot product of w with p3 is 2 which is smaller than 6. So we can see thanks to that, that they're on different sides of the line, and we can even see that because the dot product of w with p1 is greater than 6, p1 lies on the side of w, of the side to which w is pointing, where as p3 is pointing on the other side where w is not pointing, because the dot product of w with p3 is smaller than 6. If we draw the situation this is exactly what we see.

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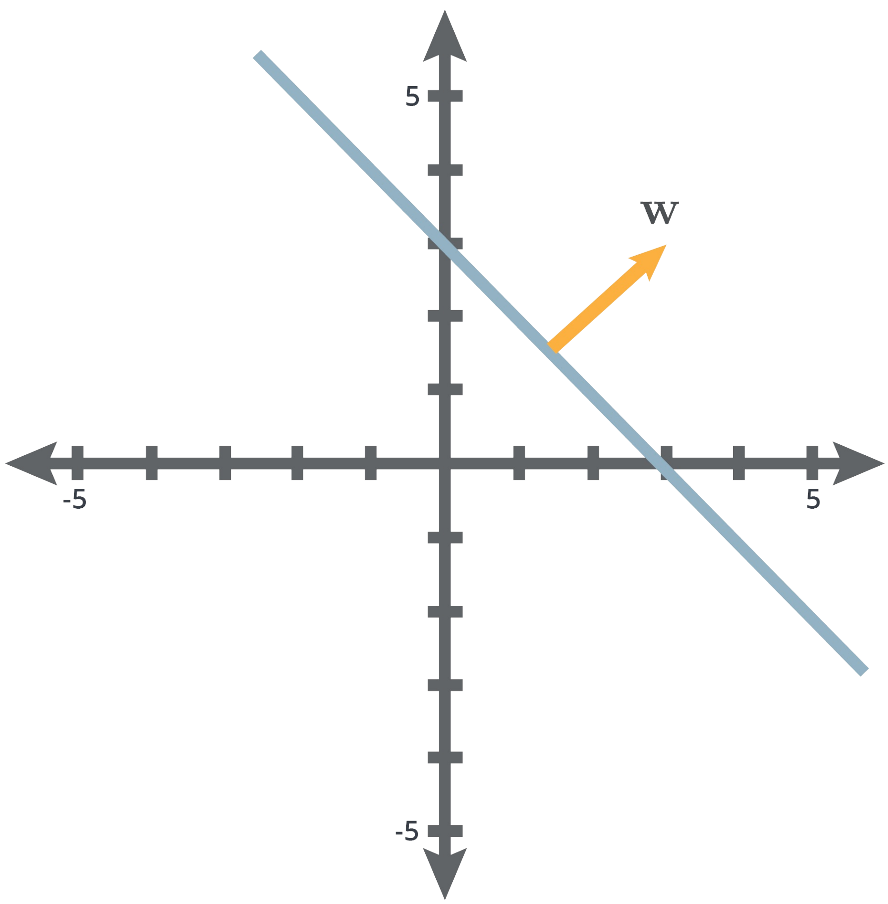
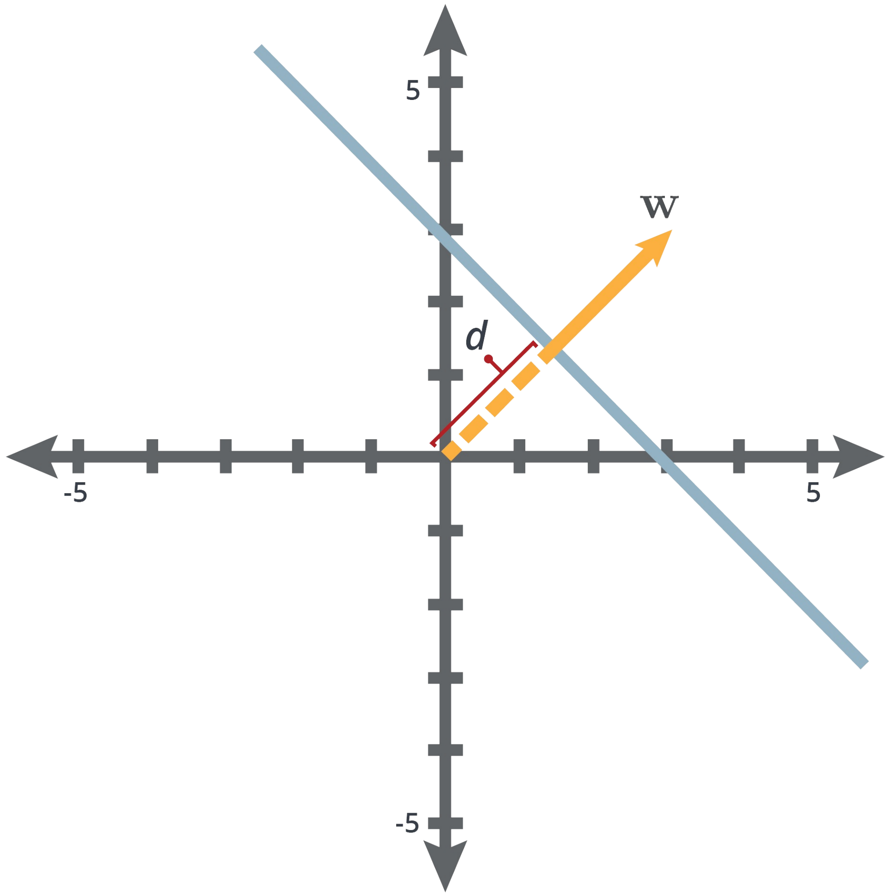
Read: Examine an Additional Example

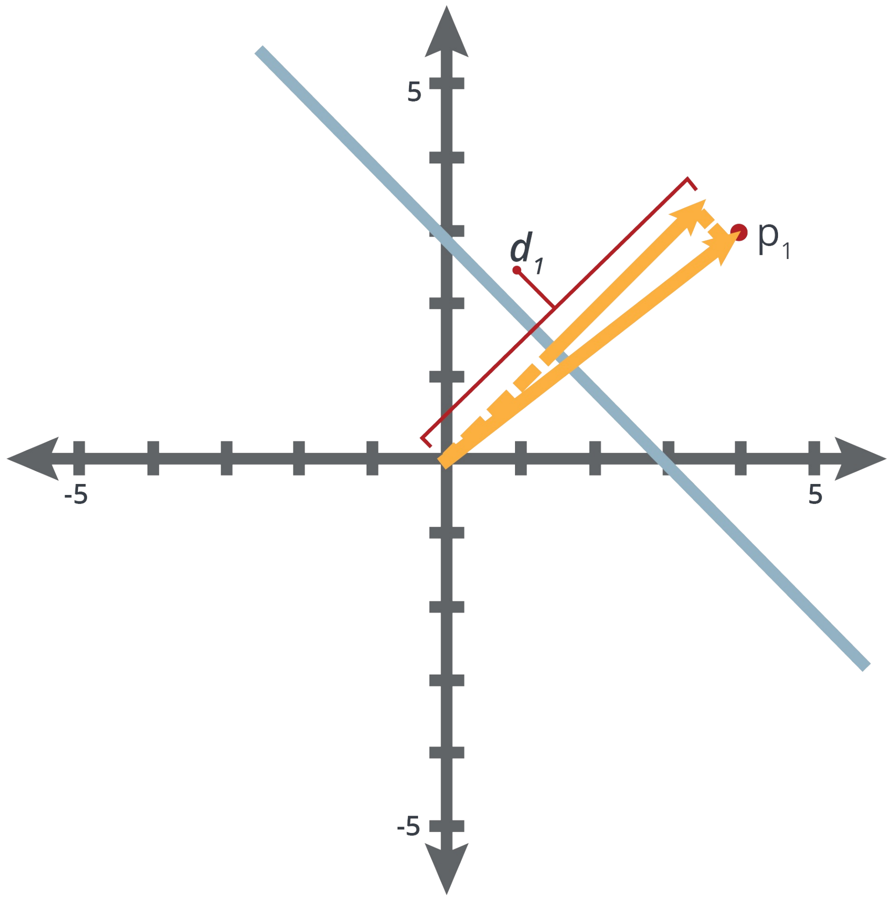
You can compute on which side of the line that a point P lies. Consider the line $\mathbf{w} \cdot \mathbf{x} = b$ with $\mathbf{w} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $b = 6$.



Remember that $b = d \cdot ||\mathbf{w}||$, where d is the distance between the origin and the line along the direction given by $\mathbf{w}$.

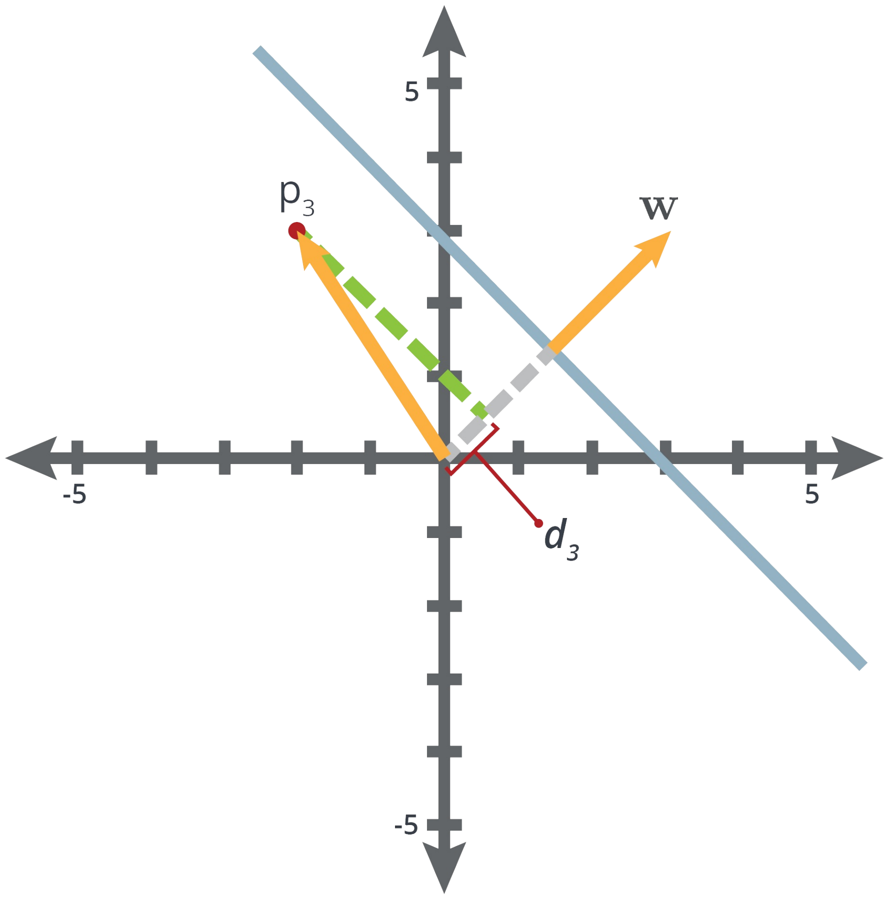


Consider the following points: $P_1 = (4, 3)$ and $P_2 = (-2, 3)$. Graphically, the dot product $\mathbf{w} \cdot P_1 = d_1 \cdot ||\mathbf{w}||$ where d_1 is shown below.



Clearly, d_1 is greater than d. Thus, $d_1 \cdot ||\mathbf{w}|| > d \cdot ||\mathbf{w}|| = b$. This is true for any point on the "same side" as $\mathbf{w}$,in this example, the "right-hand" side of the line.

Now do the same for P_2. Here, $\mathbf{w} \cdot P_2 = d_2 \cdot ||\mathbf{w}||$.



This time, d_2 is less than d. Therefore, $d_2 \cdot ||\mathbf{w}|| < d \cdot ||\mathbf{w}|| = b$. The same reasoning holds for any point on the "opposite" side of $\mathbf{w}$.

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Activity: Module 3 Exercises

Print this Module 3 Exercises document to practice defining a plane and a line.

You have explored how to define a line in a plane, whether it passes through the origin or not. Now you should practice a few problems before proceeding to the next topic. Dr. Bennoun recommends that you print this exercise set and solve them by hand. You will find the solutions on the next page.

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Tool: Module 3 Exercises Solution Set

Use the Solution Set for Module 3
Exercises to review your work.

If you haven't completed the Module 3 exercises, we suggest that you do so before you check the answers. Use the solution set to assess where you succeeded. For any problem you performed incorrectly, we recommend you review the video that covers that topic and try it again.

If you still have questions, use the discussion board on the next page. Be sure to review the instructions before posting.

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Module 3 Q&A

This Q&A discussion board is a place for you to post questions about the exercises. Common questions and important topics may be addressed by the course assistant.

Instructions:

Enter a keyword in the search box — for example, "norm" or "Exercise 3" — to find an existing discussion.
Upvote posts that contain a question that you also have by "liking" it with the thumbs-up button at the bottom of the post.
If you don't see your question, write a discussion post. Be specific: include the exercise number and the topic, along with the tactics you already used to try and solve it.
If you have a potential answer or suggestion that might resolve a question thread, feel free to share your understanding in a reply. This will help guide the instructor's feedback.

The questions that are the most upvoted or replied to that haven't been resolved will likely be addressed by an instructor.

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Module Wrap-up: Define a Line in a Plane

In this module, you practiced your skills in basic operations with vectors to define lines in a plane. You explored how the dot product and the norm are related to one another. Finally, for a given line, you determined whether a point lies on it or not. You practiced additional exercises where you defined a plane.

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Module 4: Compute Distance to a Line in a Plane

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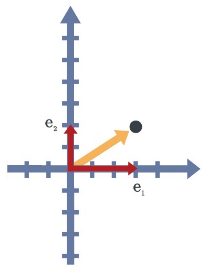
Module Introduction: Compute Distance to a Line in a Plane



You can use the orthogonal projection of a point onto a line to solve problems in linear algebra. In this module, Dr. Bennoun will provide instruction in this method of computing a distance, whether the line passes through the origin or not. Again, you will have the opportunity to practice in a few exercises.

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Watch: Project a Vector onto Another Vector

Computing orthogonal projections is useful for computing distances. In this video, Dr. Bennoun will demonstrate how to compute orthogonal projections.

Video Transcript

Take two vectors, x and y. We want to compute the orthogonal projection of x onto y. This means that we want to find the distance d1 as well as the vector whose length is d1 and has the same direction as y. How do we do that? Well, remember the geometry of the dot product. If I take the dot product of (x, y), this is going to be the norm of x times the norm of y times d1 over the norm of x. So in the end, I just have d1 times the norm of y. So if I want to find this distance d1 what they need to do is compute the dot product and divide by the norm of y. Let's take a concrete example. If x is (3, 4) and y is (7,0), then their dot product is 3 times 7 plus 4 times 0 which is 21. The norm of y is going to be the square root of 7 squared plus 0 squared which is the square root of 49 which is 7. Therefore, the distance d1 is 21 divided by 7 which is 3.

Now, how do we find a vector that has the correct length and direction? The direction is given by y. We want a vector that is going along the same direction, and that has length d1 which in that case 3. So to find a vector that has the same direction and length 1, and then we multiply by distance d1. That's the simplest way of doing that. What we do simply is, we take y and we multiply it by 1 over its norm. So this way we will have the same vector or the vector going in the same direction but with length 1. So if I take here y is (7, 0). Its length is 7. So this new vector y divided by its norm is going to be 1 over 7 times (7, 0) which is (1, 0). This procedure of dividing a vector by its norm works just for any vector. So now, I have my distance which is 3 and I have my new vector which is (1, 0). So the projection of x onto y is just 3 times the vector (1, 0). In other words, it is (3, 0).

Let's just finish by pointing out that when we talk about the orthogonal projection of x onto y what do we really mean is the projection of x onto the line given by y. So if we look at this situation now where the orientation of the two vectors are different, we can do exactly the same. It doesn't matter that the projection is not going to be on y per se, but it's just going to be on the line. We can still compute the distance d1 and we will still have our vector projection of x onto y.

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Watch: Compute the Distance between a Point and a Line

You can compute the distance between a line and a point by using a perpendicular line, in other words, orthogonally.

Explore the Computation for a Line That Goes through the Origin

In this video, Dr. Bennoun will describe how to compute the distance between a line and a point on a plane orthogonally.

Video Transcript

Let's take a line and a point anywhere on the plane. How can we compute the distance between the line and the point, knowing that the distance is going to be computed orthogonally? So taking a perpendicular line. Let's start by taking a line that goes through the origin. So I have that w dot product with x is 0. Well in this situation, notice that this is just the same as the orthogonal projection of p onto w, where p is the vector representing my point. So here, the product of p with w, or w with p, that's the same thing, is going to be equal to the distance d I am looking for, times the norm of w.

So take a concrete example. Let's take w as 5, 5, and my point is 4, -2. Then, the dot product of p with w is 4 times 5 plus -2 times 5, which is 10. The norm of w is the square root of 5 squared plus 5 squared or the square root of 50. Therefore in this situation, the distance d between the line and the point is going to be 10 divided by square root of 50, which is approximately 1.41. So now what happens if the line does not pass through the origin? Let's take a line w dot product with x equals b, and a point p. So the first thing that we would like to do, that first thing that comes to mind is to take the same dot-product as we did before. Now, as you can see in this situation, and it's not going to work because the line is not going through the origin anymore. So we need to take another vector.

We need to do something else. So how should we define this other vector that we're going to project onto w? Well, let's pick a point on the line, just any point. Then, if I define my vector v, that is the vector that goes from q to the point p, this vector is going to work. When I project that onto w, I am going to get the correct distance.

Explore the Computation for a Line That Does Not Go through the Origin

In this video, Dr. Bennoun will provide two examples to illustrate how to use orthogonality to solve this kind of problem.

Please note that in the second video at 3:30, when Dr. Bennoun says, "If the point on the other side of the plane...," he meant to say "...on the other side of the line."

Video Transcript

Let's see that with a concrete example. If w is 2,2 and b is 6, so this is my line. Then I take the 4, 2 well, first we need to pick a point Q on the line and we can pick really any point. We're going to see that Q is just Q_1, Q_2, and it is on the line. Then the vector is just going to be p minus q endpoint minus the starting point. Now let's compute the dot product of v with w, which is going to give us the distance, d we are looking for times the norm of w. Well, this dot product here, if I replace v by p minus q that is going to give me the dot product of p with w minus the dot product of q with w. Now, what is this last term, dot-product of q with w?

Well, keep in mind that q is on the line. By definition, the dot product of q with w, which is the same thing as the dot product of w with q is equal to b because it is on the line. This dot product, v times w is really the dot-product of p with w minus b. In our example, p is 5,2 and w is 2,2 and b is 6. Therefore, the dot product of p with w is going to be 5 times 2 plus 2 times 2, which is 14. Therefore, the dot product of v with w is going to be 14 minus b, which is 14 minus 6, which is 8. Now, the last thing I need is under the length of w. Here the norm of w is the square root of two squared plus two squared or square root of 8. Finally, the distance we are looking for is 8. The dot product divided by square root of 8, which in this case is approximately 2.83. Note that the way we compute the distance here, the distance could be negative if the point was on the other side of the plane. In general, if we're not interested on what side we are on, but just the distance, we can just take the absolute value of the number we get so we always get a positive number.

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Examine an Additional Example

Consider the following vectors: $\mathbf{x} = \begin{pmatrix} -5 \\ 1 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

To compute $\mathbf{proj}_{\mathbf{y}}(\mathbf{x})$, the projection of $\mathbf{x}$ onto $\mathbf{y}$, first compute the dot product.

$$\mathbf{x} \cdot \mathbf{y} = (-5) \cdot 3 + 1 \cdot 4 = -11$$

This is $d_1 \cdot ||\mathbf{y}||$.

Then, compute $||\mathbf{y}||$.

$$||\mathbf{y}|| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

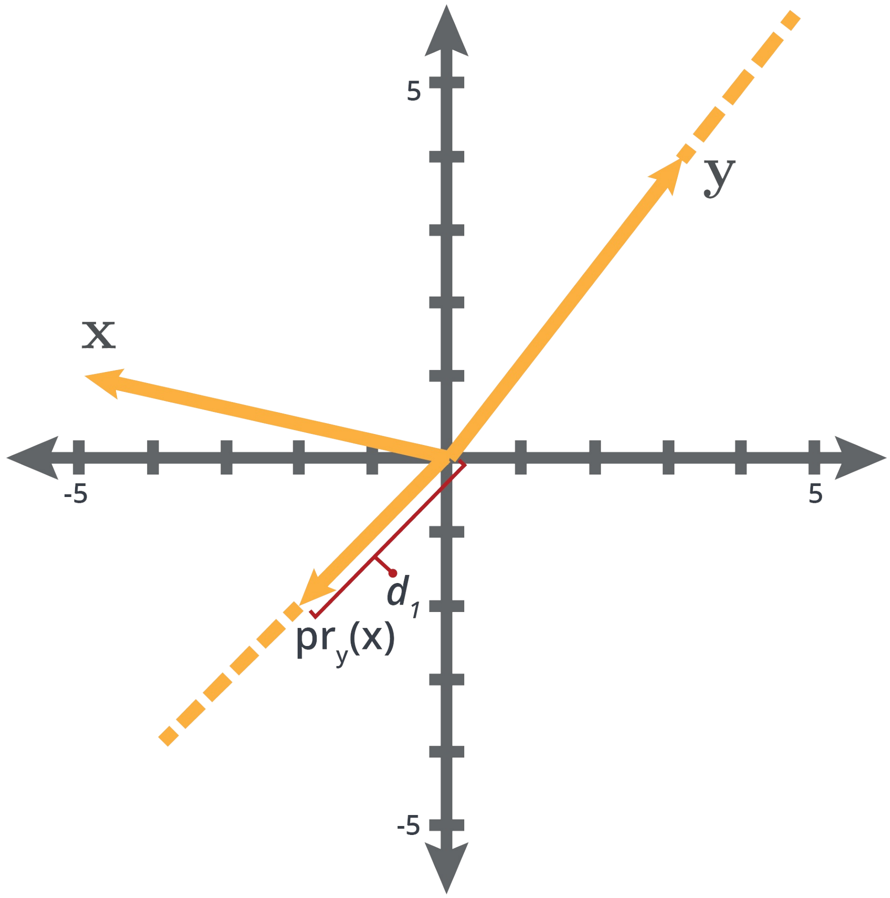
Thus, $d_1 = -\frac{11}{5}$.

Next, compute the vector $\mathbf{v}$ that has length 1 with the same direction as $\mathbf{y}$:

$$\mathbf{v} = \frac{1}{||\mathbf{y}||}\mathbf{y} = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}.$$

Finally, the projection of $\mathbf{x}$ onto $\mathbf{y}$ is

$$\mathbf{proj}_{\mathbf{y}}(\mathbf{x}) = d_1 \cdot \mathbf{v} = -\frac{11}{5} \cdot \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} = \begin{pmatrix} -\frac{33}{25} \\ -\frac{44}{25} \end{pmatrix}.$$



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Activity: Module 4 Exercises

Print this Module 4 Exercises document to practice using the orthogonal projection to compute distance.

You have explored how to use the orthogonal projection of a point onto a line to compute distances. Now you should practice a few problems before proceeding to the final assessment. Dr. Bennoun recommends that you print this exercise set and solve them by hand. You will find the solutions on the next page.

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Tool: Module 4 Exercises Solution Set

Use the Solution Set for Module 4 Exercises to review your work.

If you haven't completed the Module 4 exercises, we suggest that you do so before you check the answers. Use the solution set to assess where you succeeded. For any problem you performed incorrectly, we recommend you review the video that covers that topic and try it again.

If you still have questions, use the discussion board on the next page. Be sure to review the instructions before posting.

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Module 4 Q&A

This Q&A discussion board is a place for you to post questions about the exercises. Common questions and important topics may be addressed by the course assistant.

Instructions:

Enter a keyword in the search box — for example, "orthogonal projection" or "Exercise 1" — to find an existing discussion.
Upvote posts that contain a question that you also have by "liking" it with the thumbs-up button at the bottom of the post.
If you don't see your question, write a discussion post. Be specific: include the exercise number and the topic, along with the tactics you already used to try and solve it.
If you have a potential answer or suggestion that might resolve a question thread, feel free to share your understanding in a reply. This will help guide the instructor's feedback.

The questions that are the most upvoted or replied to that haven't been resolved will likely be addressed by an instructor.

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Tool: Linear Algebra Cheat Sheet

Use the Linear Algebra Cheat Sheet to solve problems on the final assessment.

The Linear Algebra Cheat Sheet is a list of the formulas you used to solve the problems in this course. Download it and save it to your desktop to use when you complete the final assessment, and later as a reference tool for the eCornell Machine Learning certificate. Dr. Bennoun recommends that you use the cheat sheet as a starting point to creating your own summary of the course.

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Final Assessment

Now that you have practiced solving linear algebra problems in low dimension, you can test your knowledge with this self-graded assessment. We strongly recommend that you take this quiz to assess your preparation for the computations in the eCornell Machine Learning certificate.

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Module Wrap-up: Compute Distance to a Line in a Plane

In this module, you used the orthogonal projection to find the distance from a point to a line. Then, you practiced solving problems to build your skills. Finally, you took the final assessment. You can proceed to the second course to apply these methods to problems in more than two dimensions.

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Read: Thank You and Farewell

Steve Bennoun
Active Learning Lecturer
College of Arts and Sciences
Cornell University

Congratulations on completing "Linear Algebra: Low Dimension." I hope that you practiced the skills you need to be able to solve problems in the eCornell Machine Learning certificate.

The next course will cover similar topics in higher dimension, or three or more dimensions. You will also explore matrix operations. I suggest that you practice these skills as well.

From all of us at Cornell University and eCornell, thank you for participating in this course.

Sincerely,

Steve Bennoun

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Glossary

Dimension

The number of coordinates used to define a point or vector. You can think of the dimension as the size of space you are working in.

Dot product

The dot product is also known as the inner product or scalar product.

Line

A line is straight and extends in both directions infinitely.

Norm

The norm of a vector is its length.

Origin

The origin is the point at which the horizontal and vertical axes intersect each other.

Orthogonal

Orthogonal means perpendicular. Two vectors, $\mathbf{u}$ and $\mathbf{v}$, are orthogonal to each other if and only if their dot product is zero.

Perpendicular

A line or vector is perpendicular to another line or vector if they meet or intersect at a right angle.

Projection

A projection is the transformation of a vector into another vector. If you imagine a light shining from above onto the vector, the projection is the shadow it casts.

Scalar

A scalar is a number.

Vector

A vector is a directed line segment. Its notation is a bold letter, such as $\mathbf{v}$, or a letter with an arrow above it, such as $\vec{v}$.

Zero vector

The zero vector is essentially a point. If you add or subtract the zero vector to any other vector, the value of the other vector does not change.