Digital Signal Analysis

* Dirichlet conditions:

→ signal should have finite number of maxima & minima over the range of time period

→ signal should have finite number of discontinuities over the range of time period

→ signal should be absolutely integrable over the time period

→ signal is periodic, single-valued, finite & piece-wise continuous

* Trigonometric Fourier Series

$T = 2\pi/\omega_0$

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right]$$

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,dt$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos(n\omega_0 t)\,dt$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin(n\omega_0 t)\,dt$$

* Exponential Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} F_n \cdot e^{j(n\omega_0 t)}$$

$$F_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\cdot e^{-j(n\omega_0 t)}\,dt$$

* $X(jw) = \int_{-\infty}^{\infty} x(t) \cdot e^{-jwt} dt$ → Fourier Transform

$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(jw) \cdot e^{jwt} dw$ → Inverse fourier transform

* DTFT

$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-jwn}$

$x(n) = 1/2\pi \int_{-\pi}^{\pi} X(e^{jw}) \cdot e^{jwn} dw$

* DFT

$X(K) = \sum_{n=0}^{N-1} x(n) \cdot e^{-\frac{2\pi nK}{N}(j)}$

$X(jw) = X_R(jw) + i \cdot X_I(jw)$

$|X(w)| = \sqrt{X_R^2 + X_I^2}$

$X(w) = |X(w)| \cdot e^{-j\phi}$

phase, $\phi = Tan^{-1}\left(\frac{X_i(w)}{X_j(w)}\right)$

→ $X(e^{jw})$ is periodic version of $X(w)$ with period dependent on sampling rate

---

⇒ $x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau) \cdot h(t-\tau) d\tau$

$\big\downarrow$ FT

$X(jw) \cdot H(jw)$

⇒ **Odd Fn:** $f(-x) = f(x)$ holds $\forall x \in D_f$

* Periodic — Fourier series

Non-periodic — Fourier Transform $\leftrightarrow X(\omega)$

$\downarrow$

DTFT $\longrightarrow X(e^{j\omega})$

$\downarrow$

DFT $\longrightarrow X(k)$

* Duality:

$$f(t) \rightarrow F(\omega)$$
$$F(t) \rightarrow 2\pi f(-\omega)$$

* Linearity: $a x_1(t) + b x_2(t) \rightarrow a X_1(\omega) + b X_2(\omega)$

Homogeneity

superposition

* Time-shifting:

$$x(t) \rightarrow X(\omega)$$
$$x(t-t') \rightarrow X(\omega) \cdot e^{-j\omega t'}$$

* Scaling:

$$x(t) \rightarrow X(\omega)$$
$$x(at) \rightarrow \frac{1}{a} X(\omega/a)$$

(compression in 1 domain leads to expansion in other)

* $x(t) \rightarrow X(\omega)$

$$x(-t) \rightarrow X(-\omega)$$

* $\frac{d}{dt} x(t) \rightarrow (j\omega) X(\omega)$ when $x(t) \rightarrow X(\omega)$

* **Parseval's Theorem:**

$$\int_{-\infty}^{\infty} x(t)\cdot\overline{x(t)}\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\cdot\overline{X(\omega)}\,d\omega$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega}$$

* $N \geq L$ → Length of sequence

N-point DFT

* $\boxed{W_N = e^{-j(2\pi/N)}}$

* Decimation in Frequency

$$X(2m) = \sum_{n=0}^{N/2-1} \left(x(n) + x(n+N/2)\right) W_{N/2}^{mn}$$

$$X(2m+1) = \sum_{n=0}^{N/2-1} \left(x(n) - x(n+N/2)\right)\cdot W_{N/2}^{mn}\cdot W_N^{n}$$

* Nyquist Theorem: periodic signal must be sampled at more than twice the highest frequency component of the signal.

$$f_s \geq 2f_m$$

$$X_d(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(e^{j(\omega-k\omega_0)})$$

* **Uniform - Quantizer:**

$$\text{step-size} = \frac{V_{max} - V_{min}}{2^N}$$

$$\text{Max-quantization error} = \frac{V_{max} - V_{min}}{2^{N+1}}$$

* Energy of signal: $\sum_{n=-\infty}^{\infty} |x(n)|^2$

* Power of signal: $\lim_{N \to \infty} \frac{1}{2N+1} \left(\sum_{n=-N}^{N} |x(n)|^2 \right)$

* Energy signal: Energy = finite, power = 0
* Power signal: power = finite, energy = ∞

$\boxed{\text{Periodic signals are power signals}}$

* Types of Discrete systems:

$x(n) \rightarrow \square \rightarrow y(n)$

→ delay
→ adder
→ multiplier

→ Linear:
$a x_1(t) + b x_2(t) \longrightarrow a y_1(t) + b y_2(t)$

→ Time-invariance:
$x(n) \rightarrow y(n)$
$x(n-d) \rightarrow y(n-d)$

→ Stable-systems
$x(n) \leq |M_x|, \; y(n) \leq |M_y|$

→ Invariability
$x(n) \longrightarrow \boxed{T} \xrightarrow{y(n)} \boxed{\bar{T}} \longrightarrow x(n)$

→ Causal
y(n)/output depends only on present, past inputs and past outputs

* Bit-rate: $f_s \times N$

* Memory-less: depends only on $x(n)$

* $$\boxed{x_1(n) * x_2(n) = \sum_{n=-\infty}^{\infty} x_1(m) \cdot x_2(n-m)}$$

* Linear-convolution:

$\{1,2,3,4\} * \{4,5,6\} =$

	1	2	3	4
4	4	8	12	16
5	-	-	-	-
6	-	-	-	-

$\|y(n)\|$
$= \|x(n)\| + \|h(n)\| - 1$

* Circular-convolution:

$\{1,2,3,4\} * \{4,5,6$

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \\ 0 \end{bmatrix} =$$

* Circular convolution $\longrightarrow$ Linear convolution

$\{1,2,3,4,0,0\}$
$*$
$\{4,5,6,0,0,0\}$