DSA

Assignment - 3

2021101133
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1. Z-transform :

① ⓐ Given,

$$x(n) = \{2, 4, \underset{\uparrow}{5}, 7, 0, 1\}$$

$$z\text{-tranform for } x(n) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= x[-2]z^{2} + x[-1]z + x[0]z^{0} + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3}$$

$$= 2z^{2} + 4z + 5 + 7z^{-1} + 0 + z^{-3}$$

$$\boxed{\therefore X[z] = 2z^{2} + 4z + 5 + 7z^{-1} + z^{-3}}$$

ROC → Entire z plane except $z = 0$.

ⓑ Given, $x(n) = a^{n} \cdot u(n) + b \cdot u(-n-1)$

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & o/w \end{cases}$$

$$= \sum_{n=-\infty}^{\infty} \left(a^{n} u(n) + b^{n} u(-n-1)\right) z^{-n}$$

$$u(-n-1) = \begin{cases} 1, & -n-1 \geq 0 \to n \leq -1 \\ 0, & o/w \end{cases}$$

$$= \sum_{n-\infty}^{\infty} [a^{n} \cdot u(n)] z^{-n} + \sum_{n=-\infty}^{\infty} [b^{n} \cdot u(-n-1)] z^{-n}$$

$$= \sum_{n=0}^{\infty} a^{n} (1) z^{-n} + \sum_{n=-\infty}^{-1} b^{n} (1) z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^{n} + \sum_{n=-\infty}^{-1} (bz^{-1})^{n}$$

$$\left[1 + az^{-1} + (az^{-1})^{2} + \cdots\right] + \left[(bz^{-1})^{-1} + (bz^{-1})^{-2} + \cdots\right]$$

$$= \frac{1}{1 - az^{-1}} + \frac{(bz^{-1})^{-1}}{1 - (bz^{-1})^{-1}}$$

→ ROC $|(bz^{-1})^{-1}| < 1$

$$\left|\frac{z}{b}\right| < 1$$

$$|z| < |b| \to R_2$$

ROC : $|az^{-1}| < 1$

$$\left|\frac{a}{z}\right| < 1$$

$$|a| < |z| \to R_1$$

ROC of $X(z)$ is the intersection of R_1 and R_2

~~$R_1 \Rightarrow |\frac{a}{z}| < 1$~~

$R_1 \Rightarrow |a| < |z|$ and $R_2 \Rightarrow |z| < |b|$

$\boxed{ROC \Rightarrow |a| < |z| < |b|}$

$\boxed{z\text{-transform} = \frac{1}{1-az^{-1}} + \frac{(bz^{-1})^{-1}}{1-(bz^{-1})^{-1}}}$

(2) Given 2 sequences

$x_1(n) = 3\delta(n) + 2\delta(n-1)$

$x_2(n) = 2\delta(n) - \delta(n-1)$

(1) z-transform of the convolution,

$x_1(n) * x_2(n)$

Let $x(n) = x_1(n) * x_2(n)$

$\Rightarrow \boxed{X(z) = X_1(z) \cdot X_2(z)} \rightarrow (1)$

Given,

$x_1(n) = 3\delta(n) + 2\delta(n-1)$

$X_1(z) = z(3\delta(n) + 2\delta(n-1))$

$= 3(1) + 2(z^{-1})$

$\boxed{X_1(z) = 3 + 2z^{-1}} \rightarrow (2)$

We have,

$\delta(n) \xrightarrow{z} 1$

$\delta(n-t) \xrightarrow{z} z^{-t}$

From time shifting Property.

$x_2(n) = 2\delta(n) - \delta(n-1)$

$X_2(z) = z(2\delta(n) - \delta(n-1))$

$\boxed{X_2(z) = 2 - z^{-1}} \rightarrow (3)$

Substituting (2) and (3) in (1).

$X(z) = X_1(z) \cdot X_2(z)$

$= (3 + 2z^{-1})(2 - z^{-1})$

$= 6 + 4z^{-1} - 3z^{-1} - 2z^{-2}$

$= 6 + z^{-1} - 2z^{-2}$

$\therefore \boxed{z(x_1(n) * x_2(n)) = 6 + z^{-1} - 2z^{-2}}$

(2) convolution sum using Z-transform:

$$x(n) = x_1(n) * x_2(n) = \sum_{k=0}^{\infty} x_1(k) x_2(n-k)$$

The z-transform for $x_1(n) * x_2(n) = 6 + z^{-1} - 2z^{-2}$, from (a1)

$$Z(x_1(n) * x_2(n)) = 6 + z^{-1} - 2z^{-2}$$

$$Z(x(n)) = 6 + z^{-1} - 2z^{-2}$$

$$x(n) = \text{InvZ}(6 + z^{-1} - 2z^{-2})$$

$$x(n) = \text{InvZ}(6) + \text{InvZ}(z^{-1}) - 2\,\text{InvZ}(z^{-2})$$

$$x(n) = 6\delta(n) + \delta(n-1) - 2\delta(n-2)$$

Since,
$\delta(n) \xrightarrow{Z} 1$
$\delta(n-t) \xrightarrow{Z} z^{-t}$
Inverse:
$1 \xrightarrow{\text{InvZ}} \delta(n)$
$z^{-t} \xrightarrow{\text{InvZ}} \delta(n-t)$.

$$\boxed{\therefore \text{The convolution sum } x(n) = 6\delta(n) + \delta(n-1) - 2\delta(n-2)}$$

(2) LTI Analysis: Z-transform:

(1) Given,
relaxed LTI System with impulse response

$$h[n] = a^n u[n], \quad |a| < 1$$

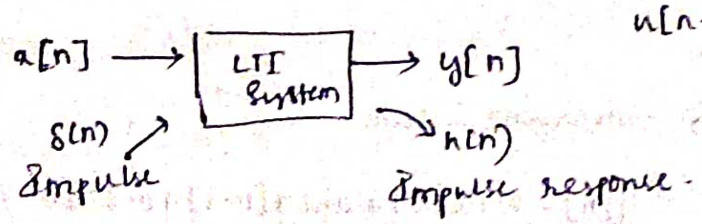
and input is a unit step fn.

$$x[n] = u[n]$$

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0. \end{cases}$$

output $y[n] = ?$

$$u[n-k] = \begin{cases} 1, & n \geq k \\ 0, & n < k \end{cases}$$



We know that,

$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$$

$$= \sum_{k=\infty}^{\infty} u(n-k)\, a^k u(k)$$

$$= \sum_{k=-\infty}^{\infty} u(n-k) u(k) a^k.$$

$$= \sum_{k=0}^{\infty} u(n-k)(1)a^k.$$

$$= \sum_{k=0}^{\infty} u(n-k)\, a^k$$

[if $n<0$, $k>0 \Rightarrow n-k<0 \Rightarrow u(n-k)=0$
if $n\geq 0$, $k\leq n \Rightarrow n-k>0 \Rightarrow u(n-k)=1$]

Hence, for $n<0 \Rightarrow y[n]=0$.

for $n\geq 0$.

$$y[n] = \sum_{k=0}^{\infty} u(n-k)\, a^k.$$

$$\Rightarrow \sum_{k=0}^{n} \underbrace{u(n-k)}_{1} a^k + \sum_{k=n+1}^{\infty} (0)\, a^k.$$

$$= \sum_{k=0}^{n} a^k.$$

$$= a^0 + a^1 + a^2 + \dots + a^n$$

$$= 1 + a + a^2 + \dots a^n.$$

$$= 1\left(\frac{a^{n+1}-1}{a-1}\right).$$

$$\Rightarrow y[n] = \frac{1-a^{n+1}}{1-a} \quad \because |a|<1.$$

$$\therefore y[n] = \begin{cases} 0, & \text{if } n<0 \\ \frac{1-a^{n+1}}{1-a}, & n\geq 0 \end{cases}$$

$$\Rightarrow \boxed{y[n] = \frac{1-a^{n+1}}{1-a}\, u[n]}$$

(2) Given,

Shift-invariant system,

$$y[n] = 0.1x[n] + 0.2x[n-1] + 0.3x[n-2] + 0.4x[n-4]$$

Applying z-transform,

$$Y[z] = 0.1X[z] + 0.2X[z]z^{-1} + 0.3X[z]z^{-2} + 0.4X[z]z^{-4}$$

$$y[n] = x[n] * h[n]$$
$$Y[z] = X[z]H[z]$$
$$H[z] = \frac{Y[z]}{X[z]}$$

$$\Rightarrow Y[z] = (0.1 + 0.2z^{-1} + 0.3z^{-2} + 0.4z^{-4})X[z]$$

$$\Rightarrow \frac{Y[z]}{X[z]} = 0.1 + 0.2z^{-1} + 0.3z^{-2} + 0.4z^{-4}$$

$$\Rightarrow H[z] = 0.1 + 0.2z^{-1} + 0.3z^{-2} + 0.4z^{-4}.$$

$$h[n] = \text{Invz}\{0.1 + 0.2z^{-1} + 0.3z^{-2} + 0.4z^{-4}\}$$

$$h[n] = 0.1\delta(n) + 0.2\delta(n-1) + 0.3\delta(n-2) + 0.4\,\delta(n-4).$$

↳ Impulse response of the given Shift-Invariant System.

wkt $\delta(n) = \begin{cases} 1, & n=0 \\ 0, & o/w. \end{cases}$

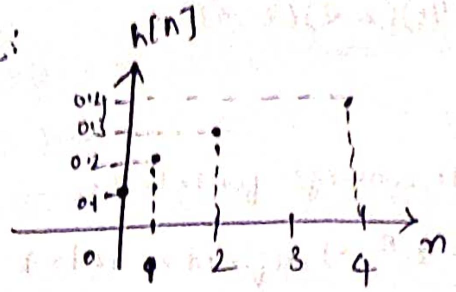
$\delta(n-1) = \begin{cases} 1, & n=1 \\ 0, & o/w \end{cases}$

$\delta(n-2) = \begin{cases} 1, & n=2 \\ 0, & o/w \end{cases}$

$\delta(n-4) = \begin{cases} 1, & n=4 \\ 0, & o/w \end{cases}$

$$\Rightarrow h[n] = \begin{cases} 0.1, & n=0 \\ 0.2, & n=1 \\ 0.3, & n=2 \\ 0.4, & n=4 \\ 0, & o/w \end{cases}$$

Sketch:



zero, otherwise

(3) Given,

Digital System,

$$y[n] = x[n] - 0.5x[n-1] + 0.36x[n-2]$$

Applying z-transform,

$$Y[z] = X[z] - 0.5X[z]z^{-1} + 0.36\,X[z]\,z^{-2}$$

$$Y[z] = [1 - 0.5z^{-1} + 0.36z^{-2}]X[z]$$

$$\frac{Y[z]}{X[z]} = 1 - 0.5z^{-1} + 0.36z^{-2}$$

$\therefore$ $\boxed{\text{Transfer function}, \; H[z] = 1 - 0.5z^{-1} + 0.36z^{-2}}$

$$H[z] = \frac{1 - 0.5z^{-1} + 0.36z^{-2}}{1} = \frac{B[z]}{A[z]}$$

$\therefore$ Numerator polynomial, $B[z] = 1 - 0.5z^{-1} + 0.36z^{-2}$

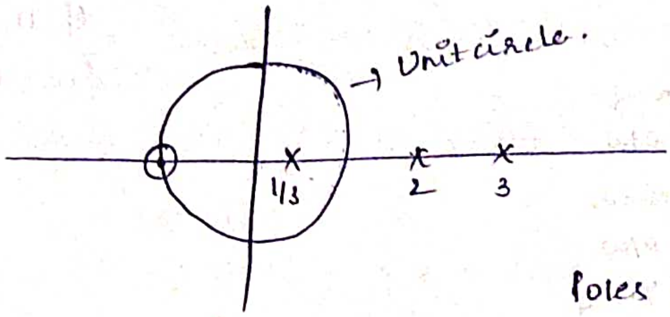
Denominator polynomial, $A[z] = 1$

① z-Transform :

③

Given,

the pole-zero plot of a System :



Poles : $z = 1/3, 2, 3$

Zeroes : $z = -1$.

Then, H(z) can be written as,

$$H(z) = \frac{(z+1)}{(z-1/3)(z-2)(z-3)}$$

(i) Given,

H(z) is known to converge for $|z| = 1$.

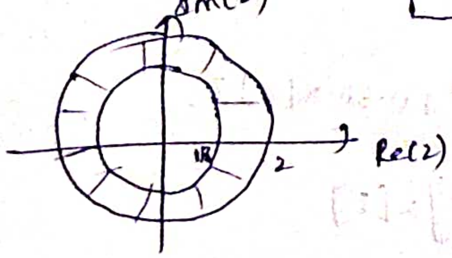
⇒ finite sum over z^{-n} → defined at $|z| = 1$

↳ ROC contains $|z| = 1$.

As, we define ROC at consecutive poles,

ROC which converge at $|z| = 1$ is :

$$\boxed{1/3 < |z| < 2}$$



→ Annular Shape ⇒ h[n] is both sided.

(ii) Given,

It is unknown that H(z) converge for $|z| = 1$.

As, ROC is defined only b/w the poles [and not zeroes],

we can define different ROCs for different poles given.

Poles : $z = 1/3, 2, 3$.

The possible ROC's are : 1 → $0 \le |z| < 1/3$ Not Stable, Not causal

2 → $1/3 \le |z| < 2$ Stable, Not causal

3 → $2 \le |z| < 3$ Not Stable, Not causal

4 → $|z| \ge 1/3$ Not Stable, causal

System is Stable if ROC contains unit circle

System is causal, upper limit of ROC ie infinity.

Hence,

(i) A stable and causal system :

(ii) Stable but not causal : $1/3 \le |z| < 2$

(iii) Causal but unstable : $|z| \ge 3$.