DSA-Assignment-2

Gowlapalli Rohit

20211011113

1. SIGNALS

(1)

(a) $x[n] = \sin^2(\pi + 3n)$ $\left(\begin{aligned} \cos 2x &= 1 - 2\sin^2 x \\ &= \cos^2 x - \sin^2 x \end{aligned} \right)$

$= \dfrac{1 - \cos(2\pi + 6n)}{2}$

Let N be the fundamental period of $x[n]$

$\Rightarrow x(n+N) = x(n)$

$\dfrac{\cancel{1} - \cos(2\pi + 6(n+N))}{\cancel{2}} = \dfrac{\cancel{1} - \cos(2\pi + 6n)}{\cancel{2}}$

$\rightarrow \cos(2\pi + 6n + 6N) = \cos(2\pi + 6n)$

(If $\cos x = \cos\theta$, then $\Rightarrow x = \theta \pm 2k\pi$, $k \in W$)

$\rightarrow \cancel{2\pi} + \cancel{6n} + 6N = \cancel{2\pi} + \cancel{6n} + 2k\pi$ where $k \in I$

$N = k\pi/3$ where $k \in I$

But N is an irrational value, which is not possible as it can take only +ve integral values

$\boxed{x(n) \text{ is not a periodic function}}$

(b) $e^{j\pi n/8} = x[n]$

$= \cos \pi n/8 + j \cdot \sin(\pi n/8)$

Let N be the fundamental period of $x[n]$

$\Rightarrow x(n+N) = x(n)$

real part of LHS = real part of RHS

and vice-versa for imaginary

$\Rightarrow \cos\left(\frac{\pi(n+N)}{8}\right) + j\sin\left(\frac{\pi(n+N)}{8}\right) = \cos\frac{\pi n}{8} + j\sin \pi n/_8$

$\Rightarrow \cos\left(\frac{\pi(n+N)}{8}\right) = \cos\left(\frac{\pi n}{8}\right)$

If $\cos x = \cos\theta$, then $x = \theta \pm 2k\pi, k \in N$

$\frac{\pi(n+N)}{8} = \frac{\pi n}{8} \pm 2k\pi$

$\pi N/_8 = 2k\pi$

$N = 16k \rightarrow N_1$ say

$\Rightarrow \sin\left(\frac{\pi(n+N)}{8}\right) = \sin\left(\frac{\pi n}{8}\right)$

If $\sin\theta = \sin x$, $\theta = (-1)^n x + n\pi$, $n \in I$

$\frac{\pi(n+N)}{8} = (-1)^k\left(\frac{\pi n}{8}\right) + k\pi$

k is odd

$\pi N/_8 = k\pi - 2\pi n/8$

$N = 8k - 2n$

$k = 2m+1$

$N = 16m - 2n + 8$

$\rightarrow N_3 \rightarrow$ doesn't contribute as when k is odd, $\cos\theta = \cos x$ is not satisfied

k is even

$\frac{\pi(\cancel{n}+N)}{8} = \cancel{\frac{\pi n}{8}} + k\pi$

$k = 2m$

$N = 8k$

$N = 16m$

$\rightarrow N_2$

$N = \text{LCM of } (N_1, N_2)$

$= 16k'$

$\boxed{\text{fundamental period} = 16}$

$k' = 1$

16

$\boxed{N = 16}$

x[n] is a periodic function

ⓒ $x[n] = \cos\left(\frac{\pi n}{10}\right)\cos\left(\pi n/30\right)$

$$= \frac{1}{2}\cdot\left[2\cos(\pi n/10)\cos(\pi n/30)\right]$$

$$= \frac{1}{2}\left[\underbrace{\cos(2\pi n/15)}_{x_1} + \underbrace{\cos(\pi n/15)}_{x_2}\right]$$

This is of the form $x_1(n) + x_2(n) \rightarrow$ Hence, period of $x(n)$ is L.C.M of period of $x_1(n)$ and $x_2(n)$

Let N_1 and N_2 be the periods of $x_1(n)$ & $x_2(n)$ respectively

$x_1(n+N_1) = x_1(n)$ and $x_2(n+N_2) = x_2(n)$

$$\tfrac{1}{2}\cos\left(\frac{2\pi}{15}(n+N_1)\right) = \tfrac{1}{2}\cos\left(\frac{2\pi n}{15}\right)$$

$$\text{and } \tfrac{1}{2}\cos\left(\frac{\pi}{15}(n+N_2)\right) = \tfrac{1}{2}\cos(\pi n/15)$$

$\rightarrow$ If $\cos x = \cos\theta$, then $x = \theta \pm 2k\pi$, then $\xrightarrow{} k \in I$

$$\cos\left(\frac{2\pi}{15}(n+N_1)\right) = \cos\left(\frac{2\pi n}{15}\right) \;\&\; \cos\left(\frac{\pi}{15}(n+N_2)\right) = \cos\left(\frac{\pi n}{15}\right)$$

$$\frac{2\pi}{15}(\cancel{n}+N_1) = \frac{\cancel{2\pi n}}{\cancel{15}} \pm 2k_1\pi \;\&\; \frac{\pi}{15}(\cancel{n}+N_2) = \frac{\cancel{\pi n}}{\cancel{15}} + 2k_2\pi$$

$N_1 = 15k_1$ $\qquad$ $N_2 = 30k_2$

where $k_1, k_2 \in I$

$N_1 = 15$ and $N_2 = 30$ $\quad$ [N_1, N_2 are fundamental frequencies, they are the smallest numbers]

Fundamental period of $x(n) = \text{LCM}(N_1, N_2) = \text{LCM}(15, 30)$
$= 30$

$x[n]$ is periodic function
with fundamental period 30

(d) $x[n] = \sin(4\pi n + 3)$

Let N be the fundamental period of $x(n)$

$\Rightarrow x(n+N) = x(n)$

$\sin(4\pi(n+N)+3) = \sin(4\pi n+3)$

$\rightarrow$ If $\sin\theta = \sin x \Rightarrow \theta = (-1)^n x + n\pi$, $n \in I$

$4\pi(n+N)+3 = (-1)^k(4\pi n+3) + k\pi$

K is odd

$4\pi(n+N)+3$
$= -(4\pi n+3) + k\pi$

$4\pi N = k\pi - (8\pi n+6)$

$N = k/4 - (2n + \frac{3}{2\pi})$

$\downarrow$

N is irrational which is a contradiction

K is even

$k = 2m$

$m \in I$

$4\pi(\cancel{n}+N)+\cancel{3}$
$= \cancel{4\pi n}+\cancel{3} + 2m\pi$

$4\pi N = 2m\pi$

$N = m/2$

$\downarrow$

smallest possible such integer is $\underline{1}$

$N = 1$

$\downarrow$

N can't take fractional values as $x(n)$ is discrete

$\rightarrow$ $x(n)$ is periodic function with a fundamental period of $\underline{1}$

Scanned with CamScanner

(e) $x[n] = \cos(\pi n^2/3)$

Let N be the fundamental period, then

$$x(n+N) = x(n)$$

$$\cos\left(\pi/3\,(n+N)^2\right) = \cos(\pi n^2/3)$$

→ If $\cos\theta = \cos x \Rightarrow \theta = x \pm 2k\pi$ where $k \in N$

$$\pi/3\,(n+N)^2 = \pi n^2/3 + 2k\pi \quad (k \in I)$$

$$\pi/3\,(\cancel{n^2} + N^2 + 2nN) = \cancel{\pi n^2/3} + 2k\pi$$

$$N^2 + 2nN - 6k = 0$$

$$N(N+2n) = 6k \Rightarrow N(N/2 + n) = 3k$$

↳ i.e for any n ; $N(N+2n)$ should be a multiple of 6, Hence

N should be a multiple of 6

↓

Fundamental period → smallest number satisfying

→ So, $\boxed{N = 6}$

Fundamental period of $\cos(\pi n^2/3)$ is given by 6

$x(n)$ is periodic

(f) $x[n] = \sum_{k=-\infty}^{\infty} (-1)^k \cdot \delta(n-k)$

To determine if the signal $x[n]$ is periodic, we need to check if there exists a positive integer N such that $x[n] = x[n+N]$ $\forall n$

$$x[n+N] = \sum_{k=-\infty}^{\infty} (-1)^k \cdot \delta(n+N-k))$$

$$= \sum_{k=-\infty}^{\infty} (-1)^k \cdot \delta((n-(k-N))$$

Since $\delta(n-(k-N)) = 0$ $\forall$ $k \neq n+N$, we can simplify the above expression

$$x[n+N] = (-1)^{n+N}$$

So, for $x[n]$ to be periodic, we need to find an integer N such that $x[n] = x[n+N]$ $\forall n$

$$\sum_{k=-\infty}^{\infty} (-1)^k \cdot \delta(n-k) = \sum_{k=-\infty}^{\infty} (-1)^k \cdot \delta(n+N-k)$$

$$\cancel{(-1)^n} = (-1)^{n+N}$$

$$= \cancel{(-1)^n} \cdot (-1)^N$$

$(-1)^N = 1 \Rightarrow$ N is an even integer

Fundamental period, smallest such number

$N = 2$ $\rightarrow$ $x[n]$ is periodic function

Fundamental period of $x(n)$ is 2

(2)

(a) $x[n] = \sqrt{2}\cos(\pi(an+1/4))$

oddpart of $x[n] = \dfrac{x(n) - x(-n)}{2}$

$= \frac{1}{2}\left[\sqrt{2}\cos(\pi(an+1/4)) - \sqrt{2}\cos(\pi(-an+1/4))\right]$

$= \frac{1}{2} \times \sqrt{2} \times \left[\cos(\pi an + \pi/4) - \cos(\pi an - \pi/4)\right]$

$= \frac{1}{\cancel{2}} \times \cancel{\sqrt{2}} \times (-\cancel{2}) \times \sin(\pi an) \times \cancel{\sin(\pi/4)}$

$= -\sin(\pi an)$

even-part of $x[n] = \dfrac{x(n) + x(-n)}{2}$

$= \frac{1}{2}\left[\sqrt{2}\cos(\pi(an+1/4)) + \sqrt{2}\cos(\pi(-an+1/4))\right]$

$= \frac{1}{2} \times \sqrt{2} \times \left[\cos(\pi an + \pi/4) + \cos(-\pi an + 1/4)\right]$

$= 1/\cancel{2} \times \cancel{\sqrt{2}} \times \cancel{2}\cos(\pi an)\cancel{\cos(\pi/4)}$

$= \cos(\pi an)$

(b) $x[n] = e^{jan\pi} + e^{jn\pi/b}$

odd-part of $x[n] = \dfrac{x[n] - x[-n]}{2}$

$$= \frac{(e^{jan\pi} + e^{jn\pi/b}) - (e^{-jan\pi} + e^{-jn\pi/b})}{2}$$

$$= (\sin(\pi a n) + \sin(\pi n/b)) \cdot j$$

even-part of $x[n] = \dfrac{x(n) + x(-n)}{2}$

$$= \frac{(e^{jan\pi} + e^{jn\pi/b}) + (e^{-jan\pi} + e^{-jn\pi/b})}{2}$$

$$= \cos(\pi a n) + \cos(\pi n/b)$$

(3)

(a) Energy of the signal is given by

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

Given that the signal $x[n] = \begin{cases} 0, & n<0 \\ n, & n\geq 0 \end{cases}$

$$E = \sum_{n=-\infty}^{0} |x(n)|^2 + \sum_{n=1}^{\infty} |x(n)|^2$$

$$= 0 + \sum_{n=1}^{\infty} (n^2) \rightarrow \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6}$$

$$\downarrow$$

$$\infty$$

$E = \infty$ (tends to infinity for $x(n)$)

Power of the signal is given by,

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \underset{N\to\infty}{Lt} \left(\frac{1}{2N+1}\right) \cdot \left[\sum_{n=-N}^{0} 0^2 + \sum_{n=1}^{N} n^2 \right]$$

$$= \underset{N\to\infty}{Lt} \left(\frac{1}{\cancel{2N+1}}\right) \left(\frac{N(N+1)\cancel{(2N+1)}}{6}\right)$$

$$= \underset{N\to\infty}{Lt} \frac{N(N+1)}{6} \longrightarrow \infty$$

$P = \infty$, $E = \infty$ $\longrightarrow$ $x(n)$ is neither a power signal, nor an Energy signal.

(b) $x(n) = \cos(n\pi/2)$

Energy of the signal is given by

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} \cos^2 n\pi/2$$

$$= \sum_{n=2,4,\ldots}^{\infty} \cos^2(2k\pi) + \sum_{-\infty}^{n=-2,-4} \cos^2(2k\pi)$$

where k is an integer

$+ \, 0 \rightarrow$ when n is odd

$$= \underbrace{(1+1+ \text{------} 1)}_{\infty \text{ times}} = \rightarrow \infty$$

Energy is infinite $\rightarrow$ for $x(n)$.

Power of the signal, $P = \lim_{N\to\infty} \left(\frac{1}{2N+1}\right) \sum_{n=-N}^{N} |x(n)|^2$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \cos^2 n\pi/2 = \lim_{N\to\infty} \left(\frac{1}{2N+1}\right) \begin{cases} N+1, & \text{if } N \text{ is even} \\ N, & \text{if } N \text{ is odd} \end{cases}$$

$$\Rightarrow \quad 0 < P < \lim_{N\to\infty} \frac{N}{2N+1} = 1/2 \Rightarrow 0 < P < 1/2$$

Scanned with CamScanner

$\Rightarrow$ power is finite and Energy is finite for $x(n)$

$\boxed{x(n) \text{ is a power-signal}}$

ⓒ $x[n] = \begin{cases} 3^n, & n<0 \\ 1/2^n, & n \geq 0 \end{cases}$

Energy of the signal is given by

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{n=-1} 3^{2n} + \sum_{n=0}^{\infty} 1/2^{2n}$$

$$= \left(1/3^2 + 1/3^4 + \cdots\right) + \left(1 + 1/2^2 + 1/2^4 + \cdots\right)$$

$$= \left(\frac{1/3^2}{1-1/3^2}\right) + \left(\frac{1}{1-1/2^2}\right)$$

$$= \frac{(1/9)}{(8/9)} + \left(\frac{1}{3/4}\right) = \frac{4}{3} + \frac{1}{8}$$

$$= 35/24$$

Power of the signal is given by,

$$0 \leq P = \lim_{N\to\infty} \left(\frac{1}{2N+1}\right) \sum_{n=-N}^{N} |(x(n))|^2$$

$$\leq \lim_{N\to\infty} \left(\frac{1}{2N+1}\right) \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{35}{24} \cdot \left(\frac{1}{2N+1}\right) = 0$$

$P = 0$

→ Energy of the signal is finite and power of the signal is zero

$\boxed{\text{The given signal is an energy signal}}$

(d) $x[n] = a^n u(n)$, $a \in \mathbb{R}$

Energy of the signal is given by

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} |a^n . u(n)|^2$$

$$= \sum_{n=-\infty}^{-1} |a^n . 0|^2 + \sum_{n=0}^{\infty} |a^n . 1|^2$$

$$= \sum_{n=0}^{\infty} a^{2n} = 1 + a^2 + a^4 + \cdots + \cdots$$

$$E = \begin{cases} \frac{1}{1-a^2} & , \text{if } |a| < 1 \\ \infty & , \text{if } |a| \geq 1 \end{cases}$$

Power of the signal is given by,

$$P = \underset{N \to \infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \underset{N \to \infty}{Lt} \frac{1}{2N+1} \left(\sum_{n=-\infty}^{-1} |a^n . 0|^2 + \sum_{n=0}^{\infty} |a^n . 1|^2 \right)$$

$$= \underset{N \to \infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{\infty} a^{2n}$$

$$= \underset{N \to \infty}{Lt} \frac{1}{2N+1} \cdot \left[\frac{1 - a^{(2N+2)}}{1-a^2} \right]$$

$$= \frac{1}{1-a^2} \cdot \underset{N \to \infty}{Lt} \frac{1}{2N+1} \left(1 - a^{2(N+1)}\right)$$

$$P = \begin{cases} 0, & \text{if } |a| < 1 \\ 1/2, & \text{if } |a| = 1 \\ \infty, & \text{if } |a| > 1 \end{cases}$$

When $|a| < 1$, Energy is finite and Power is zero
$= 1/(1-a^2)$ → given signal is energy signal

When $|a| = 1$, Energy is infinite, Power $= 1/2$ → finite
↳ given signal is power signal

When $|a| > 1$, Energy and Power are infinite
↳ Given signal is neither an Energy signal nor an Power signal

(e) $x[n] = e^{n} \cdot \delta(n-4)$

We know that

Energy of the signal, $E = \sum_{n=-\infty}^{\infty} |x(n)|^2$

$$= \sum_{n=-\infty}^{\infty} |e^{n} \cdot \delta(n-4)|^2$$

$$= \sum_{n=-\infty}^{n=3} |e^{n} \cdot 0|^2 + (e^4 \cdot 1)^2 + \sum_{n=5}^{\infty} |e^{n} \cdot 0|^2$$

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & \text{else} \end{cases}$$

$= e^8$ → finite

Power of the Signal,

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \left(\sum_{n=-\infty}^{n=3} |e^n \cdot 0|^2 + (e^4 \cdot 1)^2 + \sum_{n=5}^{\infty} |e^n \cdot 0|^2 \right)$$

$$= \underset{N\to\infty}{Lt} \frac{e^8}{2N+1} = 0$$

$\boxed{P=0}$

→ Energy of the signal is finite and power of the signal is zero

↳ The given signal is an Energy signal

Scanned with CamScanner

2. SYSTEMS

② Systems

① If $x(n)$ is input & corresponding output is $y(n)$, then if $y(n-k)$ is same as the output produced when $x(n-k)$ is given as input, the system is time-invariant

ⓐ $y(t) = t^2 x(t-1)$

Consider an arbitrary input $x_1(t)$.

Let $y_1(t) = t^2 x_1(t-1)$ be the corresponding output

→ Consider $x_2(t)$ by shifting $x_1(t)$ in time

$$x_2(t) = x_1(t-t_0)$$

The output corresponding to this input is

$$y_2(t) = t^2 x_2(t-1)$$
$$= t^2 x_1(t-t_0-1)$$

$$y_1(t-t_0) = (t-t_0)^2 x_1(t-t_0-1) \neq y_2(t)$$

Therefore, the system is not time-invariant

(b) $y[n] = x[n-1] + x[n+1]$

Now, if input is $x(n-k)$, let the output be y'

$$y'(n) = 3\cdot x(n-k-1) + x(n-k+1)$$

$$= x((n-k)-1) + x((n-k)+1)$$

$$= y(n-k)$$

→ Same-delay is produced in output

↳ Hence the above system is time-invariant

(c) $y[n] = 1/x[n]$

$$y(n-k) = 1/x(n-k)$$

Now, if input is $x(n-k)$, let the output be y'

$$y'(n) = \frac{1}{x(n-k)} = y(n-k)$$

→ Same delay is produced in output

↳ Hence the system is time-invariant

(d) Given the system S with input $x[n]$ and output $y[n]$ related by

$$y[n] = x[n] \cdot (g(n) + g(n-1))$$

(i) Given that $g(n) = 1 \ \forall n$

Then,

$$y[n] = x[n] \cdot (1+1)$$
$$= 2x[n]$$

→ Consider a shift in $y[n]$ by n_0

$$y[n-n_0] = 2 \cdot x(n-n_0)$$

⇒ If the input is shifted by n_0, then the output corresponding to this input is given by,

$x[n-n_0]$

$$y'(n) = 2 \cdot x(n-n_0) = y(n-n_0)$$ (same delay is produced in output)

→ Hence, the above system is $\boxed{\text{time-invariant}}$

(ii) Given that $g(n) = n$

Then,

$$y(n) = x[n] \cdot (n+n-1)$$
$$= (2n-1) \cdot x[n]$$

Now if input is shifted by n_0, let the output be y'

$x(n-n_0)$

$$y'(n) = (2n-1) \cdot x(n-n_0) \neq y(n-n_0)$$

$$(2n-2n_0-1) \cdot x(n-n_0)$$

Scanned with CamScanner

→ Consider a shift in $y[n]$ by n_0

$$y[n-n_0] = (2(n-n_0)-1)\, x(n-n_0)$$
$$= (2n+1-2n_0)\, x(n-n_0)$$

shift by n_0 in the input doesn't have a corresponding shift in the output

→ Hence the above system is time-variant

(c) Given that

$$g(n) = 1 + (-1)^n$$

Then,

$$y(n) = x(n)\left[1+(-1)^n + 1 + (-1)^{n-1}\right]$$
$$= x(n)\left[2 + (-1)^{n-1}\left[1+(-1)\right]\right]$$
$$= 2x(n)$$

Let the input be shifted by n_0, let the output be y'

↳ $x[n-n_0]$

$$y'(n) = 2x(n-n_0) = y(n-n_0)$$

a shift of n_0 in the output

→ A shift of n_0 in the input have a corresponding shift in the output with same delay

Hence the system S is time-invariant

(2.) Let $x_1(n) \rightarrow y_1(n)$, $x_2 \rightarrow y_2(n)$

The system is linear when

$$a_1 x_1(n) + a_2 x_2(n) \rightarrow a_1 y_1(n) + a_2 y_2(n)$$

(a) $y(t) = x(\sin t)$.

Consider 2 arbitrary inputs $x_1(t)$ and $x_2(t)$

$x_1(t) \rightarrow y_1(t) = x_1(\sin t)$

input

$x_2(t) \rightarrow y_2(t) = x_2(\sin t)$

(a & b are arbitrary)

Let $x_3(t) = a x_1(t) + b x_2(t)$ ⇒ linear combination of $x_1(t)$ & $x_2(t)$

If $x_3(t)$ is the input to given system, then corresponding output $y_3(t)$ is

$$y_3(t) = x_3(\sin t)$$

$$= a x_1(\sin t) + b x_2(\sin t)$$

$$= a y_1(t) + b y_2(t)$$

⇒ Therefore, the system is linear

Scanned with CamScanner

(b) $y(t) = \begin{cases} 0 & , t<0 \\ x(t)+x(t-2) & , t\geq 0 \end{cases}$

consider 2 arbitrary inputs $x_1(t)$ & $x_2(t)$

$x_1(t) \rightarrow y_1(t) = \begin{cases} 0 & , t<0 \\ x_1(t)+x_1(t-2) & ; t\geq 0 \end{cases}$

$x_2(t) \rightarrow y_2(t) = \begin{cases} 0 & , t<0 \\ x_2(t)+x_2(t-2) & ; t\geq 0 \end{cases}$

Let $x_3(t) = ax_1(t) + bx_2(t)$ → a & b are arbitrary

→ Linear combination of $x_1(t)$ & $x_2(t)$

→ If $x_3(t)$ is input to given system, then corresponding

$y_3(t) = \begin{cases} 0 & , t<0 \\ x_3(t)+x_3(t-2) & , t\geq 0 \end{cases}$

$= \begin{cases} 0 & , t<0 \\ ax_1(t)+ax_1(t-2)+ bx_2(t)+bx_2(t-2) & , t\geq 0 \end{cases}$

$= \begin{cases} 0 & , t<0 \\ ay_1(t)+by_2(t) & , t\geq 0 \end{cases}$

→ Therefore, the system is linear

satisfies both additivity & Homogeneity

(i) $y(t) = \frac{d(x(t))}{dt}$

$\Rightarrow$ Consider 2 arbitrary inputs $x_1(t)$ & $x_2(t)$

$x_1(t) \rightarrow y_1(t) = \frac{dx_1(t)}{dt}$

$x_2(t) \rightarrow y_2(t) = \frac{dx_2(t)}{dt}$

Let $x_3(t) = ax_1(t) + bx_2(t)$ $\rightarrow$ a & b are arbitrary scalars

$\hookrightarrow$ linear combination of $x_1(t)$ & $x_2(t)$

$\rightarrow$ If $x_3(t)$ is input to given system, then

$y_3(t) = \frac{dx_3(t)}{dt}$

$= \frac{d}{dt}(ax_1(t) + bx_2(t))$

$= a \cdot \frac{d}{dt}(x_1(t)) + b \cdot \frac{d}{dt}(x_2(t))$

$$\boxed{y_3(t) = ay_1(t) + by_2(t)}$$

$\downarrow$ satisfies both additivity & homogeneity properties

$\rightarrow$ Therefore, the system is linear

(d) $y[n] = \sum_{m=0}^{M} a \cdot x[n-m] + \sum_{m=1}^{N} b \cdot x[n-m]$

$$y_1(n) = \sum_{m=0}^{M} a \cdot x_1(n-m) + \sum_{m=1}^{N} b \cdot x_1(n-m)$$

$$y_2(n) = \sum_{m=0}^{M} a \cdot x_2(n-m) + \sum_{m=1}^{N} b \cdot x_2(n-m)$$

Let $y_3(n)$ be the output when $a_1 x_1(n) + a_2 x_2(n)$ is given as input

$$y_3(n) = \sum_{m=0}^{M} a \cdot [a_1 x_1(n-m) + a_2 x_2(n-m)] + \sum_{m=1}^{N} b \cdot [a_1 x_1(n-m) + a_2 x_2(n-m)]$$

$$= a_1 \cdot \sum_{m=0}^{M} a x_1(n-m) + a_2 \sum_{m=0}^{M} a x_2(n-m) + a_1 \sum_{m=1}^{N} b \cdot x_1(n-m) + a_2 \cdot \sum_{m=1}^{N} b x_2(n-m)$$

$$= a_1 \left[\sum_{m=0}^{M} a \cdot x(n-m) + \sum_{m=1}^{N} b \cdot x(n-m) \right] + a_2 \left[\sum_{m=0}^{M} a \cdot x(n-m) + \sum_{m=1}^{N} b \cdot x(n-m) \right]$$

$$= a_1 y_1(n) + a_2 y_2(n)$$

→ Therefore, the system is Linear

(e) $y[n] = a \cdot x[n] + \frac{b}{x[n-1]}$

Let $y_1(n) = a x_1(n) + \frac{b}{x_1(n-1)}$.

$y_2(n) = a x_2(n) + \frac{b}{x_2(n-1)}$

Let $y_3(n)$ be the output when $a_1 x_1(n) + a_2 x_2(n)$ is input

$$y_3(n) = a\left(a_1 x_1(n) + a_2 x_2(n)\right) + \frac{b}{a_1 x_1(n-1) + a_2 x_2(n-1)}$$

Now,

$$a_1 y_1(n) + a_2 y_2(n) = a_1\left(a x_1(n) + \frac{b}{x_1(n-1)}\right) + a_2\left(a x_2(n) + \frac{b}{x_2(n-1)}\right)$$

$$\neq y_3(n)$$

$$\frac{a_1 b}{x_1(n-1)} + \frac{a_2 b}{x_2(n-1)} \neq \frac{b}{a_1 x_1(n-1) + a_2 x_2(n-1)}$$

$\Rightarrow$ Therefore, the system is non-linear

(3) A system is causal if it doesn't depend on future-inputs

(a) $y(t) = x(t-2) + x(2-t)$

Consider the output at $t=0$; i.e,

$$y(0) = x(0-2) + x(2-0)$$
$$= x(-2) + x(2)$$

Output $y(0)$, at $t=0$, depends upon the past value $x(-2)$ and future value $x(2)$

→ Therefore, the system is not causal

(b) $y(t) = x(t) \cdot \cos 3t$

Consider the output at $t=t'$, i.e

$$y(t') = x(t') \cdot \cos(3t')$$

The output at $t=t'$, depends on the present value $x(t')$

→ Therefore, the system is causal

(c) $y(t) = \int_{-\infty}^{2t} x(k)\,dk$

Consider the output at $t=t'$, i.e

$$y(t') = \int_{-\infty}^{2t'} x(k)\,dk$$

The output $y(t')$ at $t=t'$ depends on the past-inputs and the future-inputs i.e, i.e, $-\infty < t \leq t'-1$ and the future-inputs i.e,

$t'+1 \leq t < 2t'$

→ Therefore, the system is not causal

(d) $y[n] = \sum_{k=0}^{\infty} x[n+k]$

as $k \geq 0$

$n+k \geq n$

→ $y(n)$ depends on future inputs

Therefore, the system is not causal system

(e) $y[n] = \sum_{k=0}^{\infty} x[n-k]$

as $k \geq 0 \Rightarrow -k \leq 0$

$n-k \leq n$

→ $y(n)$ depends only on present & past inputs

Therefore, the system is causal

3. SAMPLING FREQUENCY

③ Sampling Frequency

① Aliasing refers to the disortion or artifacts that arise when a continuous analog signal is sampled at a lower-rate than the Nyquist-rate. [The Nyquist-rate is defined as twice the highest frequency component in the signal, and sampling at a rate below this leads to loss of information & distortion]

For Example:

sine wave - 500HZ

↓

sample atleast at 1000HZ

If sample at 700HZ

⇓

Aliasing

→ To reducing aliasing, there are several techniques that can be used

i) Increase the sampling rate:
This is the most effective way to reduce aliasing. By sampling the signal at a higher rate, the Nyquist criterion can be met and the resulting digital signal will accurately represent the original analog signal

ii) Use an anti-aliasing filter: It's an low-pass filter that is used to remove high-frequency components from the analog signal before it sampled. This ensures that no high-frequency components are present in the signal that could lead to aliasing

iii) Oversampling: Here the signal is sampled at a much higher rate than the Nyquist rate This allows for more accurate reconstruction of the original signal and reduces the effect of aliasing.

iv) Bandlimiting the signal: If the frequency content of the signal is known to be within a certain frequency range, the signal can be bandlimited to that range before sampling. This ensures that only the desired frequency components are present in the signal and reduces the chances of aliasing

v) Interpolation: This technique is used to estimate the value of a signal b/w 2 sampled points. By using cubic spline, the signal can be reconstructed with higher accuracy and aliasing can be reduced

vi) Using High-resolution ADC's: By this technique, number of quantization levels increases, reducing the quantization noise & accuracy is improved

vii) Windowing: By applying a window function to the signal before sampling, the spectral leakage can be reduced, resulting in clearer signal & less aliasing

Given that

$$x(t) = \frac{1}{2\pi}\cos(4000\pi t)\cos(1000\pi t)$$

$$= \frac{1}{4\pi}\left(2\cos(4000\pi t)\cos(1000\pi t)\right)$$

$$= \frac{1}{4\pi}\left[\cos(3000\pi t) + \cos(5000\pi t)\right]$$

$$= \frac{1}{4\pi}\left[\cos(2\pi[1500]t) + \cos(2\pi[2500]t)\right]$$

→ Frequencies in the signal are 1500Hz, 2500Hz

$f_{max} = max(f_1, f_2)$

$= max(1500, 2500) = 2500$

Hence, Nyquist-rate $= 2 \times f_{max}$

$= 2 \times 2500$

$\boxed{\text{Nyquist-rate} = 5000Hz}$

→ Nyquist interval $= \dfrac{1}{\text{Nyquist rate}}$

$= \dfrac{1}{5000} = 2 \times 10^{-4}$

$= 200$ milliseconds

② Given $x(t) = 10\cos((1000t + \pi/3)) + 20\cos(2000t + \pi/6)$

ω_1 , ϕ_1 , ω_2 , ϕ_2

$\omega_1 = 1000$, $f_1 = \omega_1/2\pi = \frac{500}{\pi}$

$\omega_2 = 2000$, $f_2 = \omega_2/2\pi = \frac{1000}{\pi}$

Therefore, the sampling rate should be,

$$f_s \geq 2f_{max} = 2 \times \frac{1000}{\pi} = \frac{2000}{\pi}$$

$$= 636.6 \text{ samples/sec}$$

The sampling interval should satisfy,

$$T_m = 1/f_s \leq \frac{\pi}{2000} = 0.00157 \text{ sec}$$

maximum allowable time-interval between sample values that will ensure perfect signal reproduction

→ If we want to reproduce 1 hour of this waveform,

Number of sample values that are needed to be stored $= \frac{2000}{\pi}$ samples/sec $\times$ 3600 sec

$= 2.291831 \times 10^6$ samples

For Q3)

It was mentioned in the class by Anil Kumar sir that Ω_s can be assumed to be any of angular frequency / frequency of oscillation

For parts a, b → I assumed Ω_s to be angular frequency

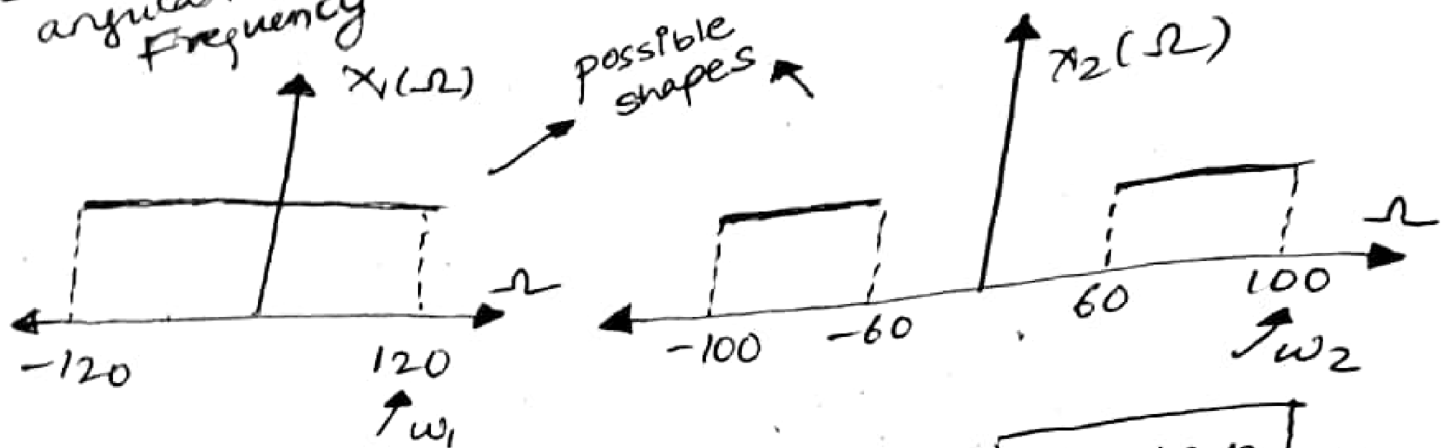
For part c → I have written answers in both methods

(3) Consider 2 signals $x_1(t)$ and $x_2(t)$ with Fourier transforms satisfying:

$X_1(\Omega) = 0, \quad 120 \leq |\Omega|$

$X_2(\Omega) = 0, \quad |\Omega| \leq 60, \ |\Omega| \geq 100$

Hence, Ω is angular Frequency



$\boxed{\begin{aligned} \omega_1 &= 120 \\ \omega_2 &= 100 \end{aligned}}$ → rad/s

(a) $x(t) = x_1(t) + x_2(t)$

F-transform gives

$X(\Omega) = X_1(\Omega) + X_2(\Omega)$

$\omega_s = 2 * [\text{maximum}(\omega_1, \omega_2)]$

$= 2 * \max(120, 100)$

$= 2 * 120$

$\boxed{\omega_s = 240 \text{ rad/s}}$ → $\Omega_s = 240$

$f_s = \dfrac{240}{2\pi}$ Hz

$\boxed{f_s = \dfrac{120}{\pi} \text{ Hz}}$

(b) $x(t) = x_1(t) \cdot x_2(t)$

$\downarrow$ F-transform $\qquad$ convolution

$X(\Omega) = X_1(\Omega) * X_2(\Omega)$

$$X_1(\Omega) * X_2(\Omega) = \int_{k=-\infty}^{\infty} x_1(k) \cdot x_2(\Omega - k)\, dk$$

$$= \int_{k=-\infty}^{-120} x_1(k) \cdot x_2(\Omega - k)\, dk + \int_{-120}^{120} x_1(k) \cdot x_2(\Omega - k)\, dk + \int_{120}^{\infty} x_1(k) \cdot x_2(\Omega - k)\, dk$$

$$= \int_{-120}^{120} x_1(k) \cdot x_2(\Omega - k)\, dk$$

Now for both $x_1(k)$ and $x_2(\Omega - k)$ to be non-zero functions

$$k < 120 \quad \text{— (1)}$$

$$60 < \Omega - k < 100 \quad \text{— (2)}$$

from (1) & (2),

$$60 + k < \Omega < 100 + k < 100 + 120 \quad \text{(From Eqn - (1))}$$

$$\boxed{\Omega_s/2 = 220} \rightarrow \text{max}$$

$\omega_s = 220 \text{ rad/s} \times 2 = 440 \text{ rad/s}$

$f_s = \frac{220}{2\pi} \times 2 = \frac{220}{\pi} \text{ Hz}$ $\qquad \boxed{\Omega_s = 440}$

(c) $x(t) = \cos(3.6\pi t + 9.23) = \cos(\omega' t + \phi)$

$\omega_s = 2 * \omega' = 2 * 3.6\pi$

$\boxed{\omega_s = 7.2\pi \text{ rad/sec}}$

$\boxed{\Omega_s = 7.2\pi}$

if Ω_s is angular frequency

$f_s = \omega_s / 2\pi = \frac{7.2\pi}{2\pi} = 3.6 \text{ Hz}$

$\Omega_s = 3.6$ $\boxed{f_s = 3.6 \text{ Hz}}$

if Ω_s is frequency of oscillation

4. QUANTIZATION

④ Quantization

① Given the waveform $x(t) = \begin{cases} -2\sin\left(\frac{\pi t}{4}\right), & 0 \le t < 4 \\ t-4, & 4 \le t < 5 \\ 1, & 5 \le t < 7 \\ 8-t, & 7 \le t \le 10 \end{cases}$

Also given that it is sampled at 1000Hz and quantised with a 2-bit quantizer with input range −2V to 2V

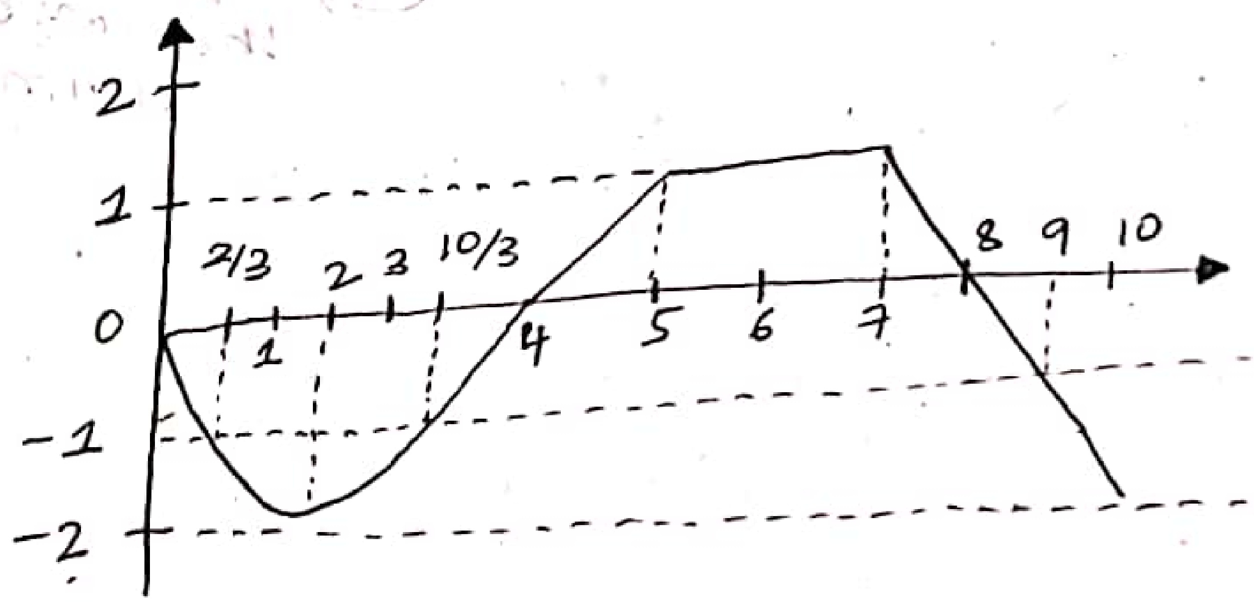
2-bit quantizer

ⓐ Given the sampling frequency

$f_s = 1000Hz$

Sampling points are given by $1/f_s$, $2/f_s$, ------

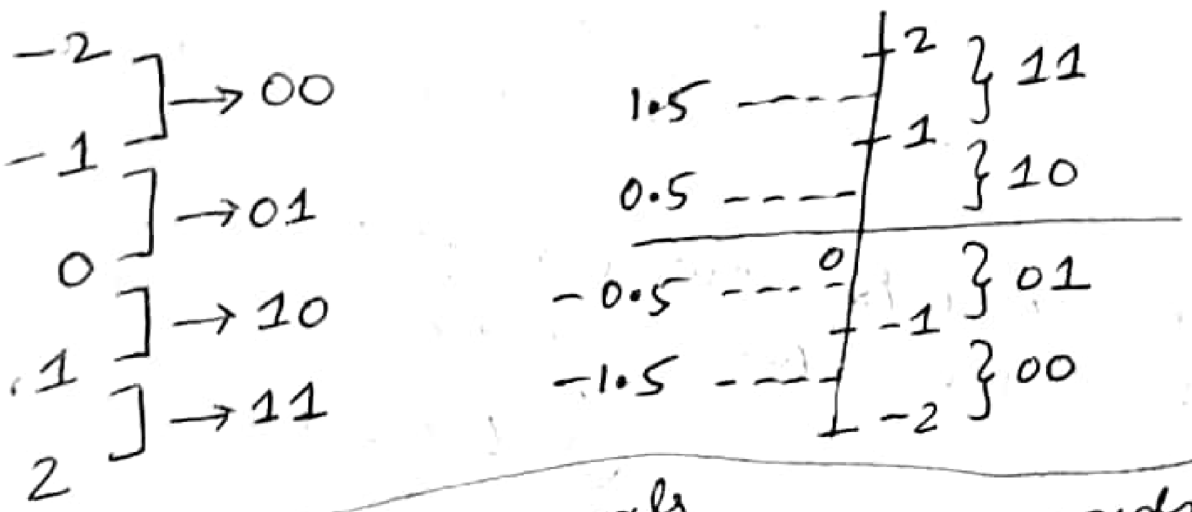
$= \frac{1}{1000}, \frac{2}{1000}, \frac{3}{1000}, ------ 5$



ⓑ Given input range $-2V$ to $2V$ with a 2-bit quantizer

$$\text{Quantization step-size} = \frac{V_{max} - V_{min}}{2^N}$$

$$= \frac{2-(-2)}{2^2} = 1$$

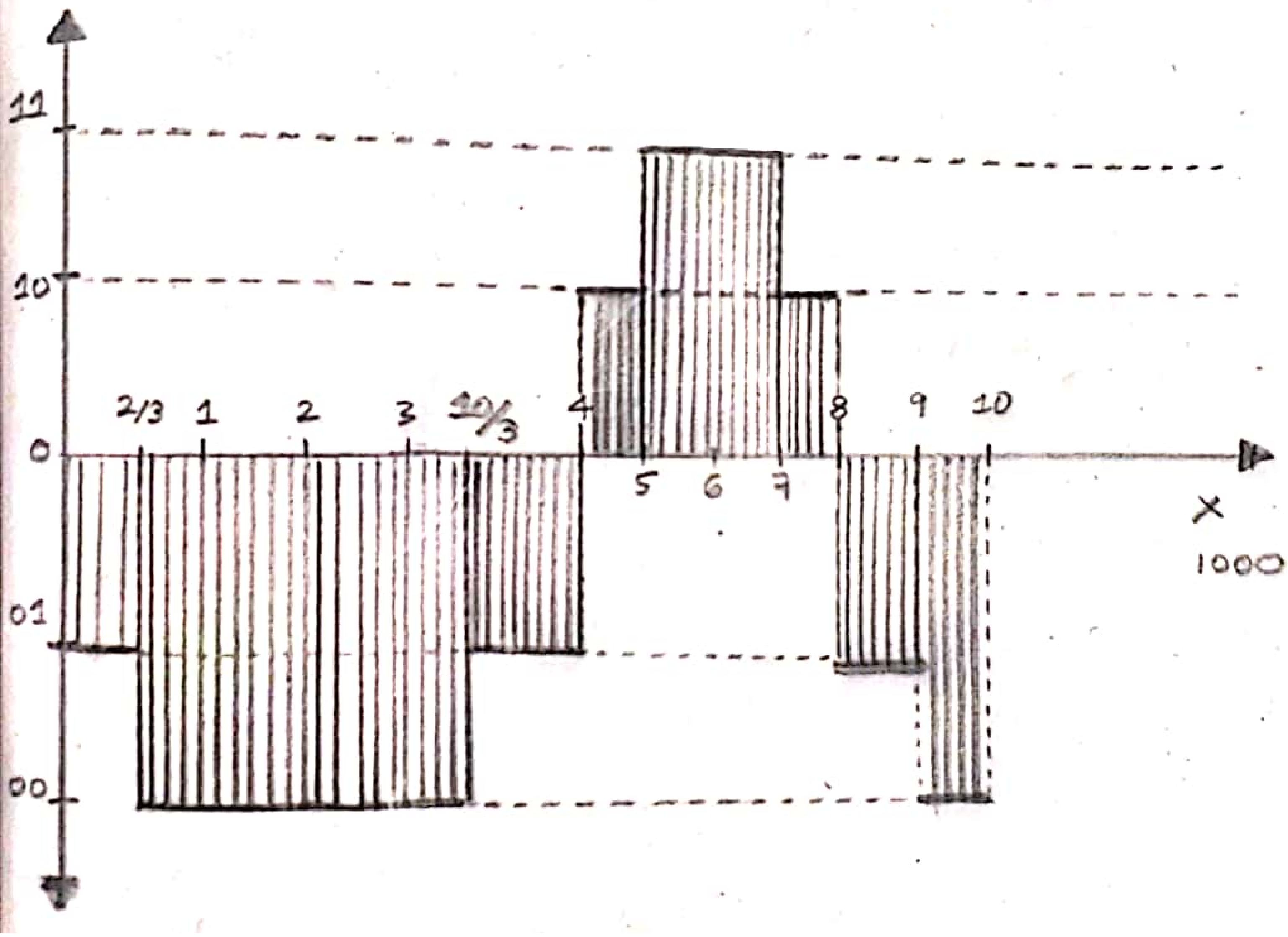


Quantization Intervals & the corresponding digital words

$\Rightarrow$ $\{-2,-1\}$, $\{-1,0\}$, $\{0,1\}$ and $\{1,2\}$ are quantised to the digital words 00, 01, 10 and 11 respectively

(Here $\{m,n\}$ → depicts the range (m,n))

(c) Ranges $[b_n, b_{n+1})$ are considered for mapping to digital ranges



(d) $[0, 2/3) \rightarrow 01$

$[2/3, \frac{10}{3}) \rightarrow 00$

$[\frac{10}{3}, 4) \rightarrow 01$

$[4,5] \rightarrow 10$

$[5,7] \rightarrow 11$

$[7,8] \rightarrow 10$

$[8,9] \rightarrow 01$

$[9,10] \rightarrow 00$

$\underbrace{\boxed{01}\,0101---}_{\frac{2000}{3} \approx 666 \text{ times}} --\underbrace{\boxed{00}\,0000--}_{\frac{8000}{3} = 2666+1 \text{ times}} \underbrace{\boxed{01}\,01----}_{\frac{2000}{3} \approx 667 \text{ times}}$

$---\underbrace{\boxed{10}\,1010---}_{1000 \text{ times}} \underbrace{\boxed{11}\,1111---}_{2000 \text{ times}}$

$----\underbrace{\boxed{10}\,1010---}_{1000 \text{ times}} \underbrace{\boxed{01}\,0101--}_{1000 \text{ times}} \underbrace{\boxed{00}\,0000---}_{1000 \text{ times}}$

Scanned with CamScanner

(e)

Resulting bit-rate = Sampling Frequency × No. of bits in Quantizer

= 1000×2

= 2000 bits/sec

(f)

Maximum Quantization error = $\frac{V_{max} - V_{min}}{2^{N+1}}$

$= \frac{2-(-2)}{2^{2+1}}$

$= 4/2^3 = 4/8 = 0.5$

3-bit Quantizer

(a) Given the sampling frequency

$$f_s = 1000\,\text{Hz}$$

sampling points are given by

$$1/f_s,\ 2/f_s,\ \text{------}\quad \text{sec}$$

$$= \frac{1}{1000},\ \frac{2}{1000},\ \frac{3}{1000},\ \text{---------}\ s$$

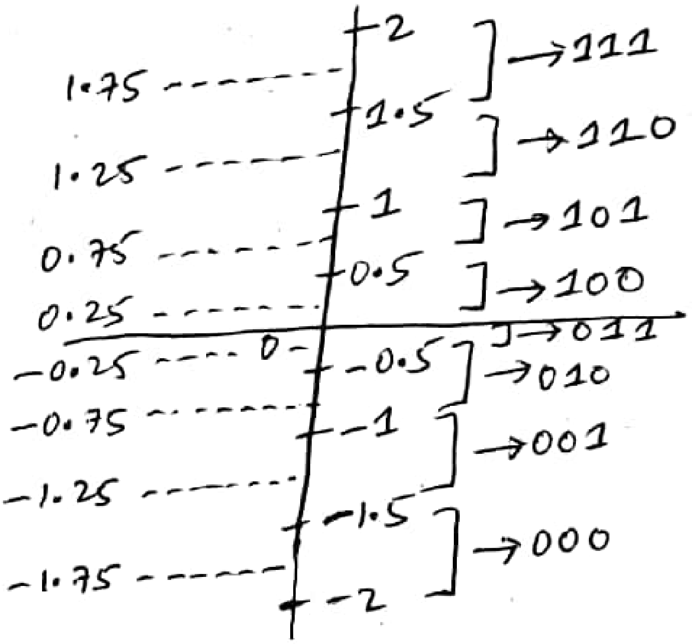
(b) Given input-range $-2V$ to $2V$ with a 3-bit quantizer

$$\text{Quantization step-size} = \frac{V_{max} - V_{min}}{2^N}$$

$$= \frac{2-(-2)}{2^3} = 4/8$$

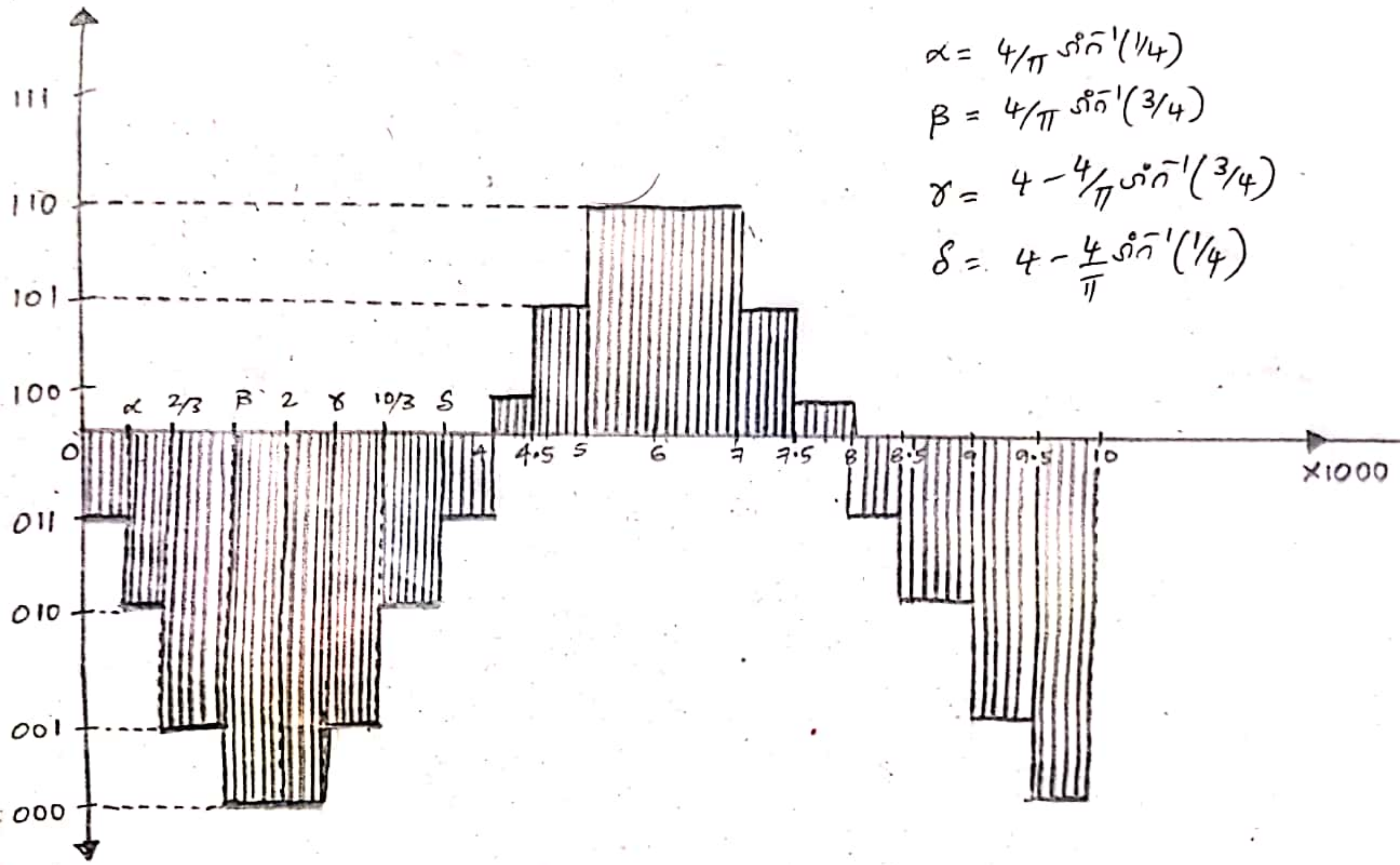
$$= 0.5$$

-2 to -1.5	→ 000
-1.5 to -1	→ 001
-1 to -0.5	→ 010
-0.5 to 0	→ 011
0 to 0.5	→ 100
0.5 to 1	→ 101
1 to 1.5	→ 110
1.5 to 2	→ 111



⇒ $\{-2,-1.5\}$, $\{-1.5,-1\}$, $\{-1,-0.5\}$, $\{-0.5,0\}$, $\{0,0.5\}$, $\{0.5,1\}$, $\{1,1.5\}$ and $\{1.5,2\}$ are quantised to the digital words 000, 001, 010, 011, 100, 101, 110 and 111

(c) Ranges $[b_n, b_{n+1})$ are considered for mapping to digital ranges



$$\alpha = \frac{4}{\pi}\sin^{-1}(1/4)$$

$$\beta = \frac{4}{\pi}\sin^{-1}(3/4)$$

$$\gamma = 4 - \frac{4}{\pi}\sin^{-1}(3/4)$$

$$\delta = 4 - \frac{4}{\pi}\sin^{-1}(1/4)$$

(d)

$\left(0, \frac{4}{\pi}\sin^{-1}\left(\frac{1}{4}\right)\right) \rightarrow 011$

$\left(\frac{4}{\pi}\sin^{-1}\left(\frac{1}{4}\right), \frac{2}{3}\right) \rightarrow 010$

$\left(\frac{2}{3}, \frac{4}{\pi}\sin^{-1}(3/4)\right) \rightarrow 001$

$\left(\frac{4}{\pi}\sin^{-1}\left(\frac{3}{4}\right), 4-\frac{4}{\pi}\sin^{-1}\left(\frac{3}{4}\right)\right) \rightarrow 000$

$\left(4-\frac{4}{\pi}\sin^{-1}\left(\frac{3}{4}\right), \frac{10}{3}\right) \rightarrow 001$

$\left(\frac{10}{3}, 4-\frac{4}{\pi}\sin^{-1}\left(\frac{1}{4}\right)\right) \rightarrow 010$

$\left(4-\frac{4}{\pi}\sin^{-1}(1/4), 4\right) \rightarrow 011$

$(4, 4.5) \rightarrow 100$

$(4.5, 5) \rightarrow 101$

$(5, 7) \rightarrow 110$

$(7, 7.5) \rightarrow 101$

$(7.5, 8) \rightarrow 100$

$(8, 8.5) \rightarrow 011$

$(8.5, 9) \rightarrow 010$

$(9, 9.5) \rightarrow 001$

$(9.5, 10) \rightarrow 000$

$\underbrace{\boxed{011}\,011\cdots}_{321\text{ times}}\ \underbrace{\boxed{010}\,010\cdots}_{345\text{ times}}\ \underbrace{\boxed{001}\,001\cdots}_{413\text{ times}}$

$\cdots\underbrace{\boxed{000}\,000\cdots}_{1841\text{ times}}\ \underbrace{\boxed{001}\,001\cdots}_{413\text{ times}}\ \underbrace{\boxed{010}\,010\cdots}_{345\text{ times}}$

$\cdots\underbrace{\boxed{011}\,011\cdots}_{321\text{ times}}\ \underbrace{\boxed{100}\,100\cdots}_{500\text{ times}}\ \underbrace{\boxed{101}\,101\cdots}_{500\text{ times}}$

$\cdots\underbrace{\boxed{110}\,110\cdots}_{2000\text{ times}}\ \underbrace{\boxed{101}\,101\cdots}_{500\text{ times}}\ \underbrace{\boxed{100}\,100\cdots}_{500\text{ times}}$

$\cdots\underbrace{\boxed{011}\,011\cdots}_{500\text{ times}}\ \underbrace{\boxed{010}\,010\cdots}_{500\text{ times}}\ \underbrace{\boxed{001}\,001\cdots}_{500\text{ times}}$

$\cdots\underbrace{\boxed{000}\,000\cdots}_{500\text{ times}}$

(e)

Resulting bit-rate = Sampling Frequency × No. of bits in Quantizer

= 1000 Hz × 3

= 3000 bits/sec

(f) Maximum Quantization Error = $\frac{V_{max} - V_{min}}{2^{N+1}}$

$= \frac{2-(-2)}{2^{3+1}} = 4/2^4$

$= 0.25$

(2) Advantages:

(1) Improved precision: Increasing the number of quantization bits can lead to greater precision in representing values, which may be important

(2) Reduced quantization error: With more quantization bits, the quantization error can be reduced, which can lead to more accurate measurements or calculations

(3) Improved dynamic range: Increasing the number of quantization bits can increase the dynamic range of the quantized values which is important in applications where a wide range of values must be represented

(4) Increased accuracy: Increase the number of quantization bits can lead to higher accuracy, which is particularly amount for applications such as medical imaging where precise measurements are required

(5) Improved signal-to-noise ratio: By increasing the number of quantization bits, the signal to noise ratio can be improved, resulting in a cleaner / more accurate signal

Disadvantages :

1. Increased memory requirements : Increasing the number of quantization bits increases the amount of memory required to store the digital signal which can be disadvantage in applications where memory is limited

2. Increased computational requirements : Higher quantization bit depths require more complex mathematical operations, which can be computationally intensive and may require more powerful hardware

3. Reduced compatibility : Increasing the number of quantization bits may lead to compatibility issues with hardware or software that cannot handle signals with a high-bit depth

4. Increased power consumption : Higher-bit depths require more power to process, which may be a concern in applications where power consumption in a limiting factor