Digital Signal Analysis

* Signal: varies with an independent parameter
  (parameter ... another)
  ↓
  carries information

* Digital because we want to process signals using digital devices
  ↓ compact, flexible
  ↳ but loss of information from Analog to Digital

analog – generally continuous

Analog → discrete ⇒ sampling
discrete → digital ⇒ Quantization

bitrates in audio

Sampling Theorem

Sampling Frequency → 2 times maximum
↓
8kHz → 8000 samples/sec

* If periodic signal is a vector-space, we can represent the space using harmonics of sines & cosines

* For all periodic signals, Fourier Series may not exist, it has to follow Dirichlet conditions
  i) f(t) is a singular valued function in any time interval
  ii) Any time interval should have finite number of discontinuities
  iii) within a time interval, signal should have finite number of maximas & minimas
  iv) Signal should be Integrable

① Trigonometric Fourier Series $T = \frac{2\pi}{\omega_0}$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,dt \rightarrow \text{DC component}$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos(n\omega_0 t)\,dt$$

synthesis
↓
original get back

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin(n\omega_0 t)\,dt$$

② Exponential Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \longrightarrow \text{synthesis}$$

$$F_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,e^{-jn\omega_0 t}\,dt$$

↓
analysis eqn

⇒ Assume $T \to \infty$ → Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\,e^{-j\omega t}\,dt$$

↳ Not possible on digital devices

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,e^{j\omega t}\,dt$$

DTFT → Discrete Time Fourier Transform :

$$F(e^{jw}) = \sum_{n=-\infty}^{\infty} f(n)\, e^{-jwn}$$

$$f(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{jw})\, dw$$

DFT → Discrete Fourier Transform

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-2\pi j/N\,(nk)}$$

Fourier Transform

(b)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad \omega = 2\pi f$$

$\hookrightarrow$ $\cos\omega t - j\sin\omega t$

CTFT or FT

$\rightarrow X_R(\omega) + j X_I(\omega)$

$|X(\omega)|$ $\qquad \angle X(\omega) = \tan^{-1}\left(\frac{\text{img}}{\text{Real}}\right)$

$X(\omega) = |X(\omega)|\, e^{-j\phi}$

$X(e^{j\omega}) = X(\omega) * H(\omega)$

DTFT

$x(t) \xrightarrow{\text{sampling}} x(n) = x(t) \times h(n)$

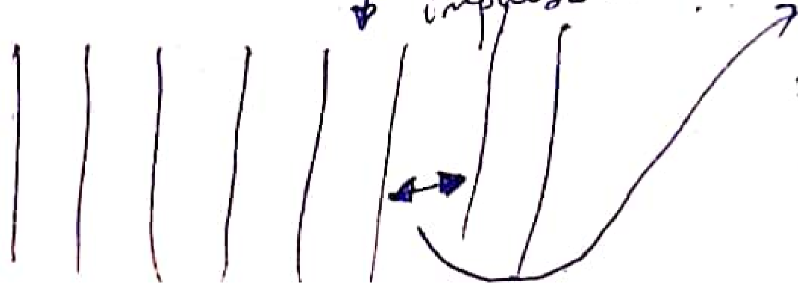
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$X(e^{j\omega})$ is periodic version of $X(\omega)$ with period dependent on sampling rate

$x(t)$ → FT, DFT

multiply by impulse strain



Sampling

What should be the seperation so that I can construct back

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convolution

$$x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$x(t) \xrightarrow{FT} X(\omega)$$

$$h(t) \xrightarrow{FT} H(\omega)$$

$$x(t) * h(t) \xrightarrow{FT} X(\omega) \cdot H(\omega)$$

$$x(t) \cdot h(t) \xrightarrow{FT} X(\omega) * H(\omega)$$

$$\int_{-\infty}^{\infty} x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \, e^{-j\omega t} dt$$

$$t-\tau = t' \Rightarrow t = t'+\tau$$

$$= \int_{t=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} x(\tau) h(t') e^{-j\omega(t'+\tau)} d\tau dt'$$

$$= \int_{t=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} x(\tau) h(t') e^{-j\omega t'} \cdot e^{-j\omega \tau} d\tau dt'$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau) \left[\int_{t'=-\infty}^{\infty} h(t') e^{-j\omega t'} dt' \right] e^{-j\omega \tau} d\tau$$

$$X(e^{j\omega}) \xrightarrow{\text{N-DFT}} X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\left(\frac{2\pi k n}{N}\right)}$$

$$x_1(n) \cdot x_2(n) \xrightarrow{} X_1(k) * X_2(k)$$

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periodic signal $\longrightarrow$ fourier series

Non-periodic $\longrightarrow$ Fourier Transform

$\swarrow$ $X(\omega)$

$\downarrow$ DTFT $\longrightarrow$ DFT

DTFT $\downarrow$ $X(e^{j\omega})$ DFT $\downarrow$ $X(k)$

Duality $f(t) \longleftrightarrow F(\omega)$

$F(t) \longleftrightarrow 2\pi f(-\omega)$

when hard to integrate

① Linearity:

$a x_1(t) + b x_2(t) \longrightarrow a X_1(\omega) + b X_2(\omega)$

Homogeneity

$X_1(t) \oplus X_2(t) \longrightarrow X_1(\omega) + X_2(\omega)$

& Superposition

② Time Shifting:

$x(t) \longrightarrow X(\omega)$

$x(t-t') \longrightarrow X(\omega) e^{-j\omega t'}$

* $x(t) \rightarrow X(\omega)$

$x(at) \rightarrow \frac{1}{a} X\left(\frac{w}{a}\right)$

* $\frac{d\,x(t)}{dt} \rightarrow j\omega X(\omega)$, $x(-t) \rightarrow X(-\omega)$

Parvel's Theorem

Energy is neither created nor destroyed

→ conjugate

$$\int_{-\infty}^{\infty} x(t)\, x^T(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, X^T(\omega) \cdot d\omega$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

⇒ Magnitude spectrum → Frequency vs Amplitude

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

↓ sampling

$x(n)$

DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

DFT

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\left(\frac{2\pi}{N}\right) kn}$$

$N \geq L$, length of sequence

N-point DFT

spectral resolution

$$X(k) = \sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi k}{N}(n)}$$

$x(n) = \{1, 2, 3, 4\}$

$n=0$, $n=1$, $n=2$, $n=3$

when $N=4$

$X(0) = 1+2+3+4 = 10$

$$X(1) = \sum_{n=0}^{3} x(n)\, e^{-j\left(\frac{\pi}{2}\right)(1)(n)}$$

$= 1(e^{-j\pi/2}) + 2e^{-j\pi} + 3e^{-3j\pi/2} + 4e^{-2j\pi}$

$= -2+2j$

$X(2) = -2$

$X(3) = -2-2j$

when $N=8$

$N=8$ $\{1,2,3,4,5,6,7,8\}$

$X(0) = 10$

$X(1) = x(0)e^{-\frac{j\pi(2)(1)(0)}{8}} + \dots$

$\vdots$

$X(7)$

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$$W_N = e^{-j\left(\frac{2\pi}{N}\right)}$$

Twiddle factor

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\left(\frac{2\pi}{N}\right)kn} = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

assumed to be periodic

$$W_N^{k+N/2} = -W_N^{k}$$

$$W_N^{k+N} = W_N^{k}$$

FFT

N = power of 2

Dessimation In Frequency

$$x(k) = \sum_{n=0}^{N-1} x(n) \left(W_N^{kn}\right)$$

$$= \sum_{n=0}^{N/2-1} x(n)\, W_N^{kn} + \sum_{n=N/2}^{N-1} x(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) \cdot W_N^{kn} + \sum_{m=0}^{N/2-1} x\left(m+\frac{N}{2}\right) W_N^{k\left(m+\frac{N}{2}\right)}$$

$$= \sum_{n=0}^{N/2-1} x(n)\, W_N^{kn} + (-1)^k \sum_{m=0}^{N/2-1} x\left(m+\frac{N}{2}\right) W_N^{kn}$$

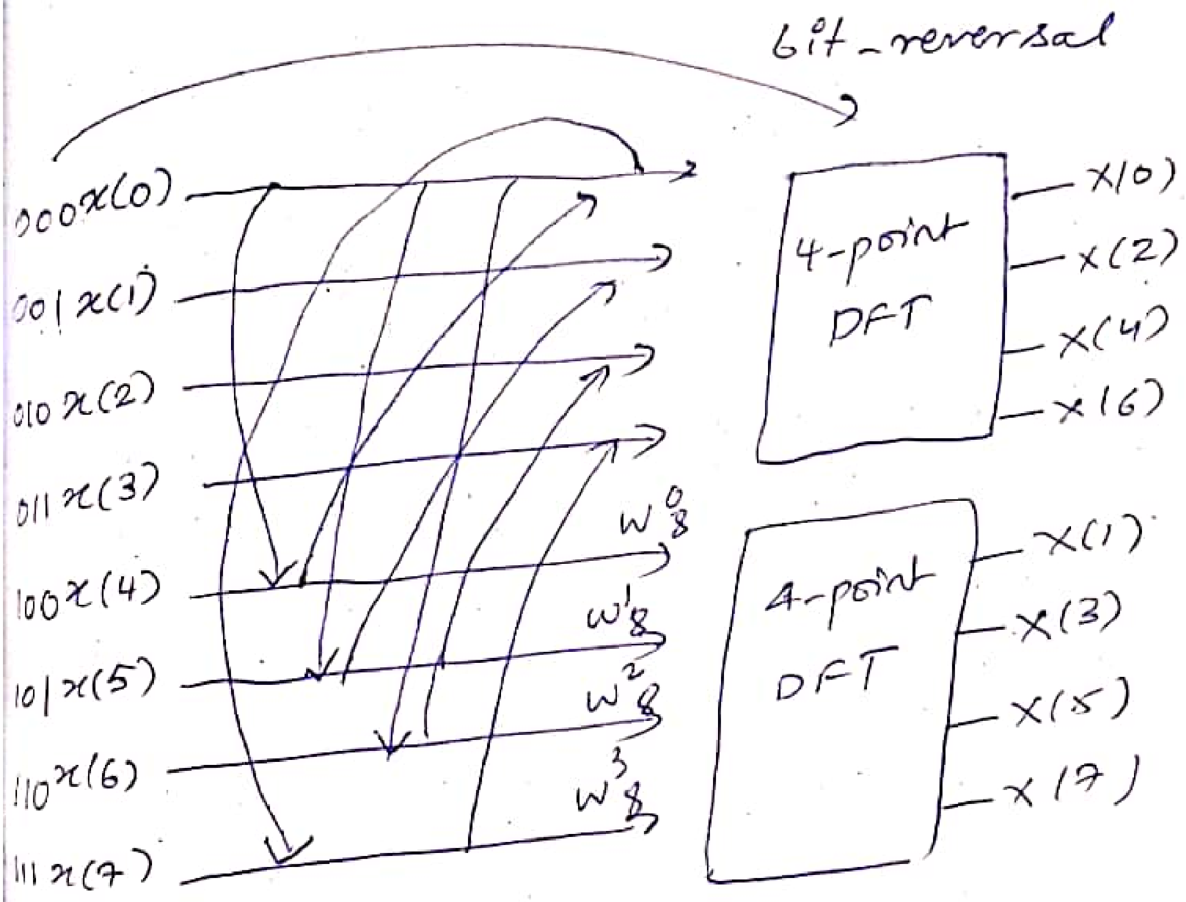
$$= \sum_{n=0}^{N/2-1} \left[x(n) + (-1)^k \left(x\left(n+\frac{N}{2}\right)\right)\right] W_N^{kn}$$

$k = 2m$

$$X(2m) = \sum_{n=0}^{N-1} \left(x(n) + x\left(n+\frac{N}{2}\right)\right) W_N^{2mn}$$

$$= \sum_{n=0}^{N-1} a_n W_{N/2}^{mn}$$

$$X(2m+1) = \sum_{n=0}^{N/2-1} \left(x(n) - x\left(n+\frac{N}{2}\right)\right) W_{N/2}^{mn} \cdot W_N^{n}$$



But write in
0, 1, 2, 3, 4, 5, 6, 7
order
at last

X(0) → sum of all
conjugate pairs
Verify using Parsevals Theorem

$x(0)$	$X(0)$
$x(4)$	$X(1)$
$x(2)$	$X(2)$
$x(6)$	$X(3)$
$x(1)$	$X(4)$
$x(5)$	$X(5)$
$x(3)$	$X(6)$
$x(7)$	$X(7)$

If dissipation in energy asked take conjugate in Last 4