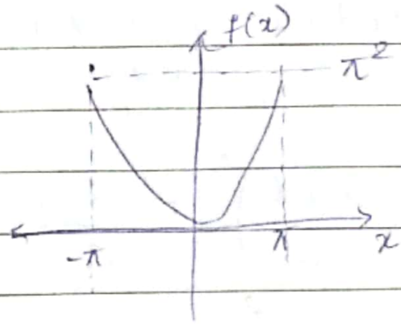
Assignment 1 Solutions

1. i)



$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\left[x^3/3\right]_{-\pi}^{\pi}$$

$$= 2\pi^2/3.$$

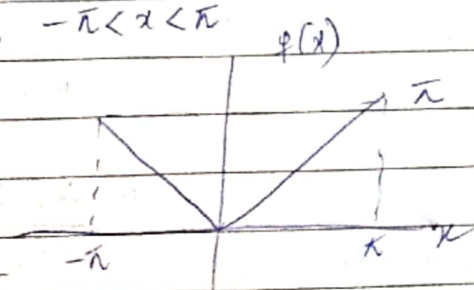
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx$$

$$= 2\cdot\frac{1}{\pi}\int_{0}^{\pi} x^2 \cos nx\,dx = \frac{2}{\cancel{\pi}}\left(\frac{2\cancel{\pi}(-1)^n}{n^2}\right) = \frac{4(-1)^n}{n^2}.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin(nx)\,dx = 0.$$

$$f(x) = \pi^2/3 + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx).$$

ii) $f(x) = |x|,\ -\pi < x < \pi$



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx \, dx = \frac{2}{\pi}\int_{0}^{\pi} x \cos nx \, dx$$

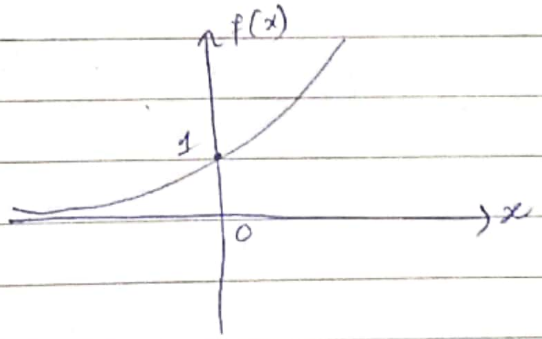
$$= \frac{2}{\pi}\left(\frac{\cos(nx)}{n^2}\right)_0^{\pi} = \frac{2}{\pi}\left(\frac{\cos n\pi - 1}{n^2}\right)$$

$$= \frac{2}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right).$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \sin(nx)\, dx = 0.$$

$$f(x) = \pi/2 + 2/\pi \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n^2}\right)\cos nx.$$

(iii) $f(x) = e^{2x}, \quad -\pi < x < \pi.$



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{2x}\, dx = \frac{e^{2\pi} - e^{-2\pi}}{2\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{2x}\cos(nx)\, dx = \frac{2\cos n\pi}{(n^2+4)\pi}\left(e^{2\pi} - e^{-2\pi}\right)$$

$$= \frac{2(-1)^n}{(n^2+4)\pi}\left(e^{2\pi} - e^{-2\pi}\right).$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{2x}\sin(nx)\, dx$$

$$= \frac{n(-1)^{n+1}}{(n^2+4)\pi}\left(e^{2\pi} - e^{-2\pi}\right).$$

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$$f(x) = \left(\frac{e^{2\pi} - e^{-2\pi}}{4\pi}\right) + \sum_{n=1}^{\infty} \frac{2\cos(nx)(-1)^n}{}$$

$$f(x) = \frac{e^{2\pi} - e^{-2\pi}}{4\pi} + \sum_{n=1}^{\infty} \left(\left(\frac{e^{2\pi} - e^{-2\pi}}{\pi(n^2+4)}\right)(-1)^n(2\cos nx - n\sin nx)\right)$$

2. i) odd ii) neither even nor odd iii) neither even nor odd

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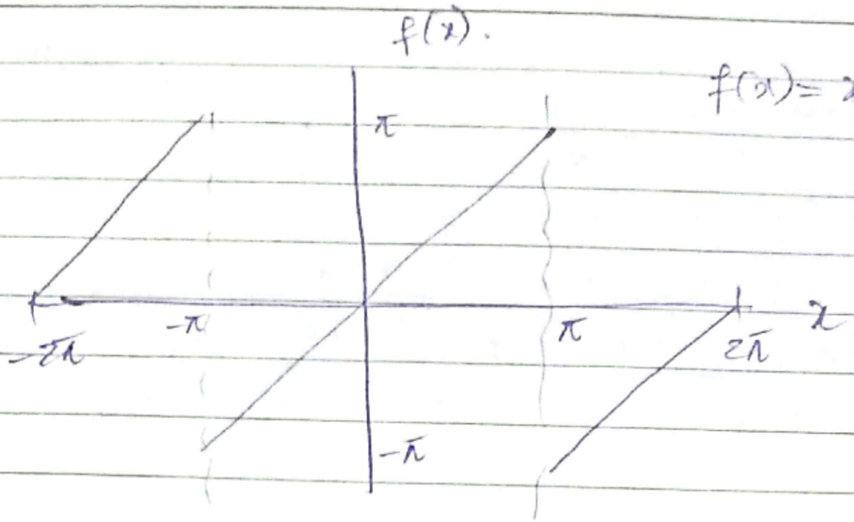
3. i)

$f(x) = x, \quad -\pi < x < \pi$

Period $= 2\pi$.



ii) $a_0 = 0, \; a_n = 0, \; b_n = -2/n\,(-1)^n.$

$$f(x) = \sum_{n=1}^{\infty} -2/n\,(-1)^n \sin(nx).$$

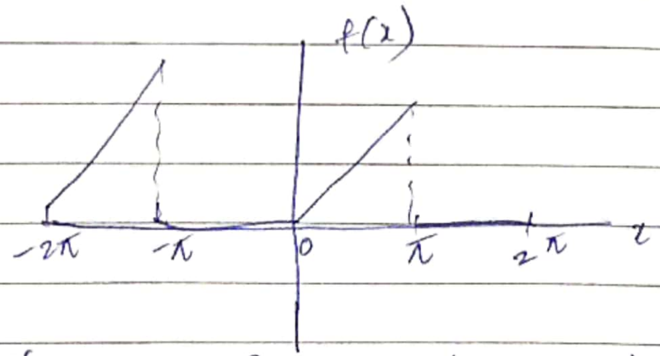
iii) $$x = -2 \sum_{n=1}^{\infty} \sin(nx) \cdot \frac{(-1)^n}{n}$$

$x = \pi/2$

$\pi/2 = -2\left(-1 + 1/3 - 1/5 + 1/7\right)$

$\Rightarrow \pi/4 = \left(1 - 1/3 + 1/5 - -\right).$

4. i)



ii) $a_0 = \pi/2, \; a_n = \dfrac{-2}{\pi(2n-1)^2}, \; b_n = \dfrac{-1}{n}(-1)^n$

$$f(x) = \pi/4 - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos nx - \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}(-1)^n.$$

iii) $x = 0$,

$$0 = \pi/4 - 2/\pi \sum_{n=1}^{\infty} 1/(2n-1)^2 \quad -① $$

$$= (\pi/4)(\pi/2) = 1 + 1/3^2 + 1/5^2 + \cdots$$

$$= \pi^2/8 = 1 + 1/3^2 + 1/5^2 + \cdots$$

5. $x(t) = \begin{cases} 1 & 1 \le |t| \le 3 \\ -1 & |t| < 1 \\ 0 & \text{otherwise} \end{cases}$

$$x(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(\omega) = \int_{-3}^{-1} e^{-j\omega t} dt + \int_{-1}^{1} e^{-j\omega t}(-1)\, dt + \int_{1}^{3} e^{-j\omega t} dt$$

$$x(\omega) = \frac{1}{j\omega}\left[e^{j3\omega} - e^{-3j\omega} + 2\left(e^{-j\omega} - e^{j\omega}\right)\right]$$

$$e^{j\omega} = \cos\omega + j\sin\omega$$

$$x(\omega) = \frac{2}{\omega}(\sin 3\omega - 2\sin\omega) = \frac{2\sin 3\omega}{\omega} - \frac{4\sin\omega}{\omega}$$

$$x(\omega) = 6\,\mathrm{sinc}(3\omega) - 4\,\mathrm{sinc}(\omega)$$

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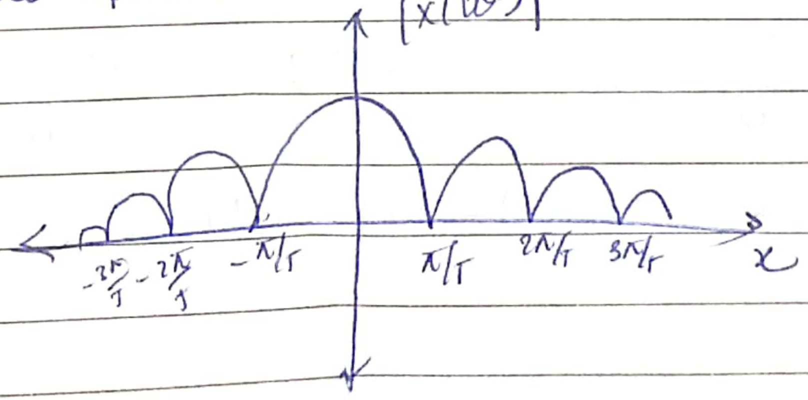
6. a. $x(t) = \begin{cases} a, & -T \le t \le T \\ 0, & \text{Otherwise} \end{cases}$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$X(\omega) = \left. \frac{a e^{-j\omega t}}{-j\omega} \right|_{-T}^{T} + 0 = \frac{A(e^{j\omega T} - e^{-j\omega T})}{j\omega}$$

$$\Rightarrow \frac{2A \sin(\omega T)}{\omega T} = 2A \operatorname{sinc}(\omega T)$$

Magnitude spectrum



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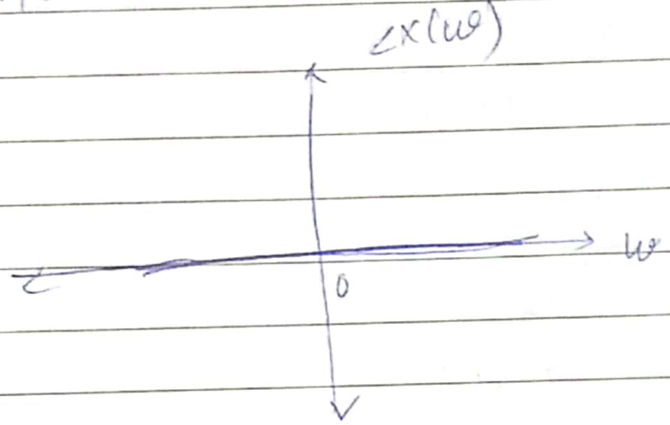
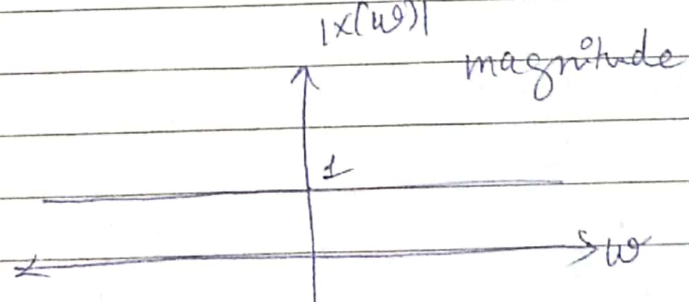
phase spectrum

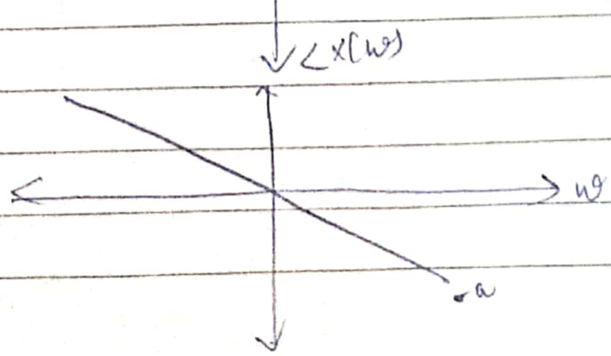


b. $x(t) = \delta(t+a)$, a is real

$$X(\omega) = \int_{-\infty}^{\infty} x(\omega) e^{-j\omega t} dt = \cos(a\omega) - j\sin(a\omega).$$

Magnitude spectrum = 1
phase spectrum = $-a\omega$





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7. i) $x(t) = e^{-|a|t} \cdot u(t)$.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(\omega) = \frac{1}{|a| + j\omega} = \frac{1}{|a| + j\omega} \frac{(|a| - j\omega)}{(|a| - j\omega)}$$

$$X(\omega) = \frac{|a| - j\omega}{(|a|^2 + \omega^2)}$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}$$

$$\angle X(\omega) = \tan^{-1}\left(\frac{-\omega}{|a|}\right).$$

$$Re(X(\omega)) = \frac{|a|}{|a|^2 + \omega^2}$$

$$Im(X(\omega)) = \frac{-\omega}{a^2 + \omega^2}.$$

ii) $x(t) = e^{(-1+2j)t}$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{(2j - j\omega - 1)} \left[e^{(2j - \omega j - 1)t} \right]_0^{\infty}$$

$$= \frac{1}{(1 + j\omega - 2j)}.$$

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a) $|X(\omega)| = \frac{1}{\sqrt{1+(\omega-2)^2}}$

b) $\angle X(\omega) = \tan^{-1}(-(\omega-2)) = \tan^{-1}(2-\omega)$.

c) $Re(X(\omega)) = \frac{1}{1+(\omega-2)^2}$

d) $Im(X(\omega)) = \frac{(2-\omega)}{1+(\omega-2)^2}$.

8. i) $x[n] = (1/5)^n u(n+1)$.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}.$$

$$= \sum_{n=-\infty}^{\infty} \left(1/5\right)^n u(n+1) e^{-j\omega n}$$

$$= \sum_{n=-1}^{\infty} \left(1/5\right)^n e^{-j\omega n}$$

$$= \left(\frac{e^{-j\omega}}{5}\right)^{-1} + \left(\frac{e^{-j\omega}}{5}\right)^{0} + \left(\frac{e^{-j\omega}}{5}\right)^{1} + \dots$$

$$\Rightarrow X(e^{j\omega}) = \frac{\left(e^{-j\omega}/5\right)^{-1}}{1-\left(e^{-j\omega}/5\right)} = \frac{5e^{j\omega}}{\frac{5-e^{-j\omega}}{5}}$$

$$= \frac{25e^{2j\omega}}{5e^{j\omega}-1}.$$

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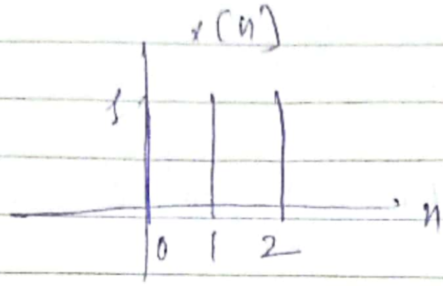
9. ii)



$$X(e^{j\omega}) = 1 \cdot e^{0} + 1 \cdot e^{-j\omega} + 1 \cdot e^{-2j\omega} + 0.$$

$$= 1 + e^{-j\omega} + e^{-2j\omega}$$

iii) $x(n) = \left(\frac{1}{2}\right)^{n+2} u(n).$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{n+1} u(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{n+2} e^{-j\omega n}$$

$$= \frac{1}{4} \sum_{n=0}^{\infty} \frac{e^{-j\omega n}}{2^{n}}$$

$$= \frac{1}{4}\left(\frac{1}{1 - \frac{e^{-j\omega}}{2}}\right) = \frac{e^{j\omega}}{4e^{j\omega} - 2}.$$

iv) $x(n) = \left(\frac{1}{2}\right)^{n} u(n-4)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{n} u(n-4)\, e^{-j\omega n}$$

$$= \sum_{n=4}^{\infty} \left(\frac{1}{2}\right)^{n} e^{-j\omega n}$$

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$$x(e^{jw}) = \frac{\left(\frac{1}{2}\right)^4 \cdot e^{-jw4}}{1 - \left(\frac{1}{2}\right)e^{-jw}}$$

$$= \frac{\left(\frac{1}{2e^{+jw}}\right)^4}{1 - \frac{1}{2e^{jw}}}$$

$$\frac{\frac{1}{(2e^{+jw})^4}}{\frac{2e^{jw}-1}{2e^{jw}}} = \frac{1}{(2e^{jw})^3(2e^{jw}-1)}.$$

9. i) $x(n) = \frac{1}{4}\delta(n)$.

$$x[K] = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn}.$$

$$x[K] = \frac{1}{4}\delta(0)\,e^{-2\pi k/N(0)} = \frac{1}{4}(1) = 1/4.$$

$$x[K] = \{1/4, 1/4, 1/4, 1/4, 1/4, 1/4, 1/4, 1/4\}$$

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ii) $x[n] = \{1, -1, 1, -1, 1\}$.

$$X[K] = 1 \cdot e^{0} + (-1)\, e^{-\pi j K/4} + e^{-\pi j K/2} + (-1)\, e^{-3\pi j K/4}$$
$$+ (-1)\, e^{-\pi j K}$$

$$x[K] = \left(1 + e^{-\pi j K} + e^{-\pi j K/2}\right) - \left(e^{-\pi j K/4} + e^{-3\pi j K/4}\right)$$
$$= \{1, (\sqrt{2}-1)j, 1, (\sqrt{2}+1)j, 5, -(\sqrt{2}+1)j, (1-\sqrt{2})j\}.$$

iii) (a) $x(n) = \cos\left(\frac{\pi n}{4}\right)$.

$$x[K] = \sum_{n=0}^{7} \cos\left(\frac{n\pi}{4}\right) e^{-j\pi n k/4}$$
$$= \sum_{n=0}^{7} \left(\frac{e^{j\pi n} + e^{-j\pi n}}{2}\right) e^{-j\pi n k/4}$$
$$= \frac{1}{2}\sum_{n=0}^{7} e^{j\frac{\pi n(1-k)}{4}} + \frac{1}{2}\sum_{n=0}^{7} e^{-j\frac{\pi n(1+k)}{4}}$$
$$= \frac{1}{2}\left[\frac{1 - e^{2\pi j(1-k)}}{1 - e^{\pi j/4(1-k)}}\right] + \left(\frac{1 - e^{-2\pi j(1+k)}}{1 - e^{-\pi j/4(1+k)}}\right)$$
$$= \{0, 4, 0, 0, 0, 0, 0, 4\}.$$

(iv) $x[n] = \{j, 0, -j, 1\}$

$$X[K] = \sum_{n=0}^{7} x[n]\, e^{-j\frac{\pi n K}{4}} = e^{-3\pi j K/4} + j\left(1 - e^{-\pi j K/2}\right)$$

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10. i) $X(e^{j\omega}) = \dfrac{e^{-j\omega} - (1/4)}{1-\left(\frac{1}{4}\right)e^{-j\omega}} = e^{-j\omega}\left(\dfrac{1}{1-\frac{1}{4}e^{-j\omega}}\right) - \dfrac{1}{4}\left(\dfrac{1}{1-1/4\, e^{-j\omega}}\right)$

$x[n-n_0] \leftrightarrow e^{-j\omega n_0} X(e^{j\omega})$

$a^n u[n] \leftrightarrow \dfrac{1}{1-ae^{-j\omega}}$

$x[n] = \left(\frac{1}{4}\right)^{n-1} u[n-1] - \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)^n u[n]$

$\therefore x[n] = \left(\frac{1}{4}\right)^{n-1} u[n-1] - \left(\frac{1}{4}\right)^{n+1} u[n].$

ii) $X(e^{j\omega}) = \dfrac{1-(1/2)^4 \cdot e^{-4j\omega}}{1-(1/2)e^{-j\omega}}$

$\dfrac{1-\left(\frac{1}{2e^{j\omega}}\right)^4}{1-\left(\frac{1}{2e^{j\omega}}\right)} = \sum_{i=0}^{3}\left(\dfrac{1}{2e^{j\omega}}\right)^i$

So, $= 1 + \dfrac{1}{2e^{j\omega}} + \dfrac{1}{4e^{2j\omega}} + \dfrac{1}{8e^{3j\omega}}$

$x(t) = \delta(t) + \frac{1}{2}\delta(t-1) + \frac{1}{4}\delta(t-2) + \frac{1}{8}\delta(t-3)$

iii) $X(e^{j\omega}) = \cos^2 3\omega + \sin^2\omega$

$= \dfrac{\cos 6\omega - \cos 2\omega + 2}{2} =$

$\dfrac{e^{6j\omega} + e^{-6j\omega}}{4} - \dfrac{e^{2j\omega} + e^{-2j\omega}}{4} + 1$

$x(t) = \delta(t) - \dfrac{\delta(t+2) + \delta(t-2)}{4} + \dfrac{\delta(t+6) + \delta(t-6)}{4}$

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11) $x[n] = (n-1)\left(\frac{1}{a}\right)^{|n|}$

from the properties,

If $x[n] \longleftrightarrow X(e^{j\omega})$

then $n x[n] \longleftrightarrow j\frac{d}{d\omega}(Xe^{j\omega})$

$$x[n] = (n-1)\left(\frac{1}{a}\right)^{|n|}$$
$$= n\left(\frac{1}{a}\right)^{|n|} - \left(\frac{1}{a}\right)^{|n|}$$

If $y[n] = \left(\frac{1}{a}\right)^{|n|} \rightarrow ①$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n} \quad \text{[DTFT]}$$
$$= \sum_{n=-\infty}^{0} \left(\frac{1}{a}\right)^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} \left(\frac{1}{a}\right)^{n} e^{-j\omega n}$$
$$= \sum_{n=0}^{\infty} \left(\frac{a}{e^{j\omega}}\right)^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{ae^{j\omega}}\right)^{n}$$
$$= \frac{1}{1-\frac{e^{j\omega}}{a}} + \frac{1}{1-\frac{1}{ae^{j\omega}}} \quad \text{[Infinite GP]}$$

$$\therefore Y(e^{j\omega}) = \frac{a}{a-e^{j\omega}} + \frac{ae^{j\omega}}{ae^{j\omega}-1} \rightarrow ②$$

$$x[n] = n\left(\frac{1}{a}\right)^{|n|} - \left(\frac{1}{a}\right)^{|n|}$$

finding DTFT

$$X(e^{j\omega}) = j\frac{d}{d\omega}\left(Y(e^{j\omega})\right) - Y(e^{j\omega})$$

[from eq① & property]

$$= j\frac{d}{d\omega}\left[\frac{a}{a-e^{j\omega}} + \frac{ae^{j\omega}}{ae^{j\omega}-1}\right] - \left[\frac{a}{a-e^{j\omega}} + \frac{ae^{j\omega}}{ae^{j\omega}-1}\right]$$

$$= \frac{ae^{j\omega}}{(ae^{j\omega}-1)^2} - \frac{ae^{j\omega}}{(a-e^{j\omega})^2} - \frac{a}{a-e^{j\omega}} - \frac{ae^{j\omega}}{ae^{j\omega}-1}$$

$$= \frac{2ae^{j\omega} - a^2e^{2j\omega}}{(ae^{j\omega}-1)^2} - \frac{a^2}{(a-e^{j\omega})^2}$$

$$\therefore X(e^{j\omega}) = \frac{-a^2e^{2j\omega} + 2ae^{j\omega}}{(ae^{j\omega}-1)^2} - \frac{a^2}{(a-e^{j\omega})^2}$$

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12. a) $$\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

Parseval's theorem,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

$$\Rightarrow \int_{-2\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= 2\pi\,[1+4+49+0+64+100] = 436\pi.$$

b) $$\int_{-\pi}^{\pi} \left|\frac{dX(e^{j\omega})}{d\omega}\right|^2 d\omega$$

$$nx[n] \longleftrightarrow j\,\frac{dX(e^{j\omega})}{d\omega}$$

$$= \int_{-\pi}^{\pi} \left|\frac{dX(e^{j\omega})}{d\omega}\right|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} \left|\frac{n}{j}\,x[n]\right|^2$$

$$= +2\pi\,[4+4+0+256+900)$$

$$= 2328\pi$$

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