Digital Signal Analysis - Quiz Solutions

February 4, 2025

Q1

State whether the following system is time-invariant, linear, or causal, with explanation.

$$y(n) = x(n-1) + x(n+1) + y(n+1) + 2$$

Solution:

First lets reorder and simplify the equation:

$$\begin{aligned}
&\implies y(n) = x(n-1) + x(n+1) + y(n+1) + 2 \\
&\implies y(n+1) = y(n) - x(n-1) - x(n+1) - 2 \\
&\implies y((n-1)+1) = y(n-1) - x((n-1)-1) - x((n-1)+1) - 2 \qquad \text{(replacing n with n-1)} \\
&\implies y(n) = y(n-1) - x(n-2) - x(n) - 2
\end{aligned}$$

1. **Time Invariance**

   Given response input $x[n]$ and its corresponding output $y[n]$, the system is time invariant is $T\{x[n-n_0]\} = y[n-n_0] \ \forall n_0$.

   Simplify the response for the shifted signal, $y'[n] = T\{x'[n]\}$ where $x'[n] = x[n-n_0]$:

   $$y'[n] = y'[n-1] - x'[n-2] - x'[n] - 2$$
   $$y'[n] = y'[n-1] - x[n-n_0-2] - x'[n-n_0] - 2$$

   Simplifying $y[n-n_0]$:

   $$y[n-n_0] = y[n-n_0-1] - x[n-n_0-2] - x'[n-n_0] - 2$$

   Comparing the 2 equations, $y'[n]$ aligns with $y[n-n_0]$ and $y'[n-1]$ aligns with $y[n-n_0-1]$. So, the system is time-invariant.

   [**TIME-INVARIANT**]

2. **Linearity**

   Given 2 response inputs $x_1[n]$ and $x_2[n]$ and their corresponding response outputs $y_1[n]$ and $y_2[n]$. The system is Linear if:

$$T\{ax_1[n] + bx_2[n]\} = ay_1[n] + by_2[n]$$

let $y'[n]$ be $T\{ax_1[n] + bx_2[n]\}$, then:

$$y'[n] = y'[n-1] - (ax_1[n-2] + bx_2[n-2]) - (ax_1[n] + bx_2[n]) - 2$$

$$y'[n] = y'[n-1] - a(x_1[n-2] + x_1[n]) - b(x_2[n-2] + x_2[n]) - 2$$

Also, $ay_1[n] + by_2[n]$ is:

$$ay_1[n] + by_2[n]$$

$$a(y_1[n-1] - x_1[n-2] - x_1[n] - 2) + b(y_2[n-1] - x_2[n-2] - x_2[n] - 2)$$

$$(ay_1[n-1] + by_2[n-1]) - a(x_1[n-2] + x_1[n]) - b(x_2[n-2] + x_2[n]) - 2(a+b)$$

Clearly, $y'[n]$ and $ay_1[n]+by_2[n]$ are not the same due to the presence of different constant terms: -2 and $-2(a+b)$, respectively. Therefore, the system is not linear.

[**NOT LINEAR**]

3. **Causality**

   The system is causal if $y[n]$ only depends in present and past values of $x[n]$. In the given system, after simplification, we see that $y[n]$ only depends on $x[n-2]$, $x[n]$ and $y[n-1]$. So, the system is causal.

   Note: You might initially think that in the original equation, $y[n]$ depends on $x[n+1]$ and $y[n+1]$. However, remember that any term of the form $y[n+k]$ in the equation should be rewritten by substituting $n \to n-k$. After simplifying the equation, you can then determine whether the system is causal or non-causal based on the interpretation.

   [**CAUSAL**]

Q2

Calculate the linear convolution of $x(n) = \{1, 2, 3, 2\}$ and $h(n) = \{2, 3, 2\}$ using circular convolution.
Solution:

$$\text{Length of linear convolution} = l_1 + l_2 - 1 = 4 + 3 - 1$$

$$\text{Length of circular convolution} = \max(l_1, l_2) = 4$$

In order to calculate linear convolution from circular convolution, we pad $x(n)$ with 2 zeroes and $h(n)$ with 3 zeroes:

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & 2 \\ 2 & 1 & 0 & 0 & 2 & 3 \\ 3 & 2 & 1 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 0 & 0 \\ 0 & 2 & 3 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 14 \\ 17 \\ 12 \\ 4 \end{bmatrix}$$

Q3

Calculate the Nyquist sampling rate for the signal $x(t) = \cos(200\pi t)\cos(400\pi t)$.

Solution: Using the trigonometric identity:

$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

we rewrite the given signal as:

$$x(t) = \frac{1}{2}\left[\cos(200\pi t - 400\pi t) + \cos(200\pi t + 400\pi t)\right]$$

$$= \frac{1}{2}\left[\cos(-200\pi t) + \cos(600\pi t)\right]$$

Since $\cos(-\theta) = \cos(\theta)$, we get:

$$x(t) = \frac{1}{2}\left[\cos(200\pi t) + \cos(600\pi t)\right]$$

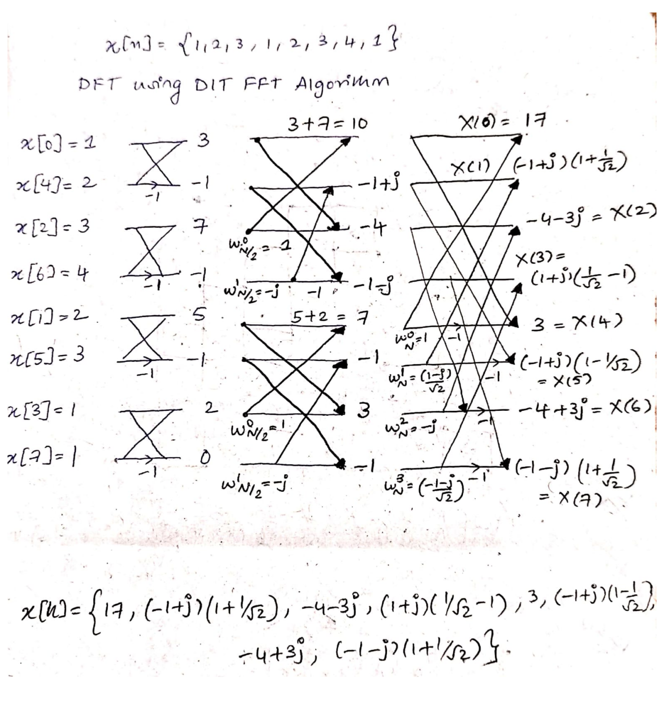
The highest frequency component is 600π rad/s, which corresponds to a frequency $f_{max} = 300$ Hz. The Nyquist sampling rate is given by:

$$f_s \geq 2 f_{max} = 600 \text{ Hz}$$

Q4

Calculate the DFT using Decimation-in-Time FFT for $x(n) = \{1, 2, 3, 1, 2, 3, 4, 1\}$.

Solution: The 8-point DFT is computed using the Decimation-in-Time FFT algorithm.



Q5

Calculate the DTFT for $x(n) = \left(\frac{1}{3}\right)^n u(n+3)$.

Solution: The DTFT is defined as:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

Given $x(n) = \left(\frac{1}{3}\right)^n u(n+3)$, we rewrite the summation as:

$$X(\omega) = \sum_{n=-3}^{\infty} \left(\frac{1}{3}\right)^n e^{-j\omega n}$$

This forms a geometric series with first term $a = \left(\frac{1}{3}\right)^{-3} e^{j3\omega}$ and common ratio $r = \frac{1}{3}e^{-j\omega}$, which converges for $|r| < 1$:

$$X(\omega) = \frac{\left(\frac{1}{3}\right)^{-3} e^{j3\omega}}{1 - \frac{1}{3}e^{-j\omega}}$$

Simplifying:

$$X(\omega) = \frac{81 e^{j3\omega}}{3 - e^{-j\omega}}$$