

Digital Signal Analysis - Midsem Solutions

April 10, 2024

SET-1

Q1

(a) State whether below signals are periodic or not. Justify:

(i) $x(n) = \cos^2(2n + \frac{\pi}{2})$

(ii) $x(n) = u(n) + u(n-1)$

(iii) $x(n) = \delta(n)$

Solution:

a) $x(n) = \cos^2(2n + \frac{\pi}{2})$

Let N be the fundamental period of $x[n]$. Then,

$$\begin{aligned}x[N+n] &= x[n] \\ \cos^2\left(2n + \frac{\pi}{2}\right) &= \cos^2\left(2(N+n) + \frac{\pi}{2}\right) \\ \cos(4n + \pi) &= \cos(4(n+N) + \pi) \\ 4n + \pi &= 4(N+n) \pm 2k\pi \\ N &= (2k-1)\frac{\pi}{4}\end{aligned}$$

But N is an irrational value, and hence cannot be a period as it is supposed to be taking only integral values for a discrete signal.

b) To check if this signal is periodic, we need to verify if there exists a positive integer N such that:

$$x(n) = x(n+N)$$

for all values of n .

Let's analyze the signal:

$$x(n) = u(n) + u(n-1)$$

$$\text{Given signal: } x(n) = \begin{cases} 2 & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

To check if this signal is periodic, we need to verify if there exists a positive integer N such that:

$$x(n) = x(n + N)$$

for all values of n .

Let's specifically check the condition for $n = 0$:

$$x(0) = x(0 + N)$$

$$1 \neq x(N) = 2$$

Since $1 \neq 2$ for any positive integer N , the signal $x(n)$ is not periodic.

- c) To check if $x[n] = \delta(n)$ is periodic, we need to verify if there exists a positive integer N such that $x[n] = x[n + N]$ for all n . However, since $\delta(n)$ is only nonzero at $n = 0$, we have $x[n] = 1$ at $n = 0$ and $x[n + N] = 0$ for $N > 0$. Thus, $x[n]$ does not satisfy the periodicity condition for any $N > 0$. Hence, $x[n] = \delta(n)$ is not periodic.
- (b) State whether below signal is energy signal or power signal or neither : $x[n] = -a^n u(-n - 1)$
Solution: Energy of the signal is given by

$$\begin{aligned} E &= \sum_{n=-\infty}^{\infty} |x(n)|^2 \\ &= \sum_{n=-\infty}^{\infty} |(-a^n u(-n - 1))^2| \\ &= \sum_{n=-\infty}^{\infty} |(a^n u(-n - 1))^2| \\ &= \sum_{n=-\infty}^{-1} |(-a^n \cdot 1)^2| + \sum_{n=0}^{\infty} |(-a^n \cdot 0)^2| \\ &= \sum_{n=-\infty}^{-1} |(-a^{2n})| \\ &= \begin{cases} \frac{1}{a^2 - 1} & \text{if } |a| > 1 \\ \infty & \text{otherwise} \end{cases} \end{aligned}$$

Power of the signal is given by

$$P = \lim_{N \rightarrow \infty} \frac{1}{2N + 1} \sum_{n=-N}^N |x(n)|^2 = \lim_{N \rightarrow \infty} \frac{1}{2N + 1} \sum_{n=-N}^{-1} |(a^n)^2| = \frac{1}{a^{-2} - 1} \lim_{N \rightarrow \infty} \frac{a^{-2N-2} - a^{-2}}{2N + 1} = \begin{cases} \infty & \text{if } |a| < 1 \\ \frac{1}{2} & \text{if } |a| = 1 \\ 0 & \text{if } |a| > 1 \end{cases}$$

When $|a| < 1$ - Energy and power are infinite, so it's neither power nor energy signal.

When $|a| = 1$ - Energy is infinite, power is finite. Hence it's a power signal.

When $|a| > 1$ - Energy is finite, power is 0. Hence it's an energy signal.

Q2

State and Prove Convolution property of Fourier Transform

Solution:

Let $f(t)$ and $g(t)$ be two functions with Fourier transforms $F(\omega)$ and $G(\omega)$ respectively. Then, the convolution of $f(t)$ and $g(t)$ is given by:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau) d\tau$$

The Fourier transform of the convolution $f * g$ is given by:

$$\mathcal{F}[f * g] = \int_{-\infty}^{\infty} (f * g)(t)e^{-i\omega t} dt$$

Using the definition of convolution, we have:

$$\begin{aligned}\mathcal{F}[f * g] &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\tau)g(t - \tau) d\tau \right) e^{-i\omega t} dt \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau)g(t - \tau)e^{-i\omega t} d\tau dt\end{aligned}$$

Now, we change the order of integration:

$$\begin{aligned}&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau)g(t - \tau)e^{-i\omega t} dt d\tau \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau)g(t - \tau)e^{-i\omega(t - \tau)}e^{-i\omega\tau} dt d\tau \\ &= \int_{-\infty}^{\infty} f(\tau) \left(\int_{-\infty}^{\infty} g(t - \tau)e^{-i\omega(t - \tau)} dt \right) e^{-i\omega\tau} d\tau\end{aligned}$$

Now, let $v = t - \tau$, then $dt = dv$ and when $t = -\infty$, $v = -\infty$, and when $t = \infty$, $v = \infty$.

$$\begin{aligned}&= \int_{-\infty}^{\infty} f(\tau) \left(\int_{-\infty}^{\infty} g(v)e^{-i\omega v} dv \right) e^{-i\omega\tau} d\tau \\ &= \left(\int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau} d\tau \right) \left(\int_{-\infty}^{\infty} g(v)e^{-i\omega v} dv \right) \\ &= F(\omega) \cdot G(\omega)\end{aligned}$$

Hence, the Fourier transform of the convolution of $f(t)$ and $g(t)$ is equal to the pointwise product of their individual Fourier transforms.

Q3

A signal has amplitude of -5V to 5V. If maximum quantisation error should be less than 0.1 , how many bits are required for Quantisation ?

Solution: To find the number of bits required for quantization such that the maximum quantization error is less than 0.1, we can use the formula:

$$\text{Maximum Quantisation Error} = \frac{V_{\max} - V_{\min}}{2^{N+1}}$$

Given that $V_{\max} = 5$ V, $V_{\min} = -5$ V, and the maximum quantization error should be less than 0.1 V, we can rearrange the formula to solve for N , the number of bits:

$$N = \log_2 \left(\frac{V_{\max} - V_{\min}}{\text{Maximum Quantisation Error}} \right) - 1$$

Substituting the given values:

$$N = \log_2 \left(\frac{5 - (-5)}{0.1} \right) - 1$$

$$N \geq \log_2(100) - 1$$

$$N = \log_2(2^7) - 1$$

$$N = 7 - 1$$

$$N = 6$$

So, 6 bits are required for quantization.

Q4

Let $x[n] = \{1, 0, 1, 0, 1, 0, 1, 0\}$. Find the Discrete Fourier Transform (DFT).

Solution: To find the Discrete Fourier Transform (DFT) of the given sequence $x[n] = \{1, 0, 1, 0, 1, 0, 1, 0\}$, we'll use the formula for the DFT:

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-i2\pi \frac{kn}{N}}, \quad k = 0, 1, \dots, N-1$$

Where N is the length of the sequence, $x[n]$ is the sequence, and $X[k]$ is the DFT coefficient at frequency bin k .

Given $x[n] = \{1, 0, 1, 0, 1, 0, 1, 0\}$, and $N = 8$ (since the sequence has 8 elements), let's compute $X[k]$ for $k = 0, 1, \dots, 7$.

For $k = 0$:

$$\begin{aligned}
 X[0] &= \sum_{n=0}^7 x[n] \cdot e^{-i2\pi \frac{0 \cdot n}{8}} \\
 &= \sum_{n=0}^7 x[n] \cdot e^0 \\
 &= \sum_{n=0}^7 x[n] \\
 &= 1 + 0 + 1 + 0 + 1 + 0 + 1 + 0 \\
 &= 4
 \end{aligned}$$

Similarly , $X[4] = 4$

For $k = 1$:

$$\begin{aligned}
 X[1] &= \sum_{n=0}^7 x[n] \cdot e^{-i2\pi \frac{1 \cdot n}{8}} \\
 &= \sum_{n=0}^7 x[n] \cdot e^{-i \frac{\pi n}{4}} \\
 &= 1 \cdot e^{-i \frac{\pi \cdot 0}{4}} + 0 \cdot e^{-i \frac{\pi \cdot 1}{4}} + 1 \cdot e^{-i \frac{\pi \cdot 2}{4}} + 0 \cdot e^{-i \frac{\pi \cdot 3}{4}} + 1 \cdot e^{-i \frac{\pi \cdot 4}{4}} + 0 \cdot e^{-i \frac{\pi \cdot 5}{4}} + 1 \cdot e^{-i \frac{\pi \cdot 6}{4}} + 0 \cdot e^{-i \frac{\pi \cdot 7}{4}} \\
 &= 0
 \end{aligned}$$

For $k = 2, 3, \dots, 7$:

$$X[k] = 0$$

So, the DFT of the sequence $x[n] = \{1, 0, 1, 0, 1, 0, 1, 0\}$ is $X[k] = \{4, 0, 0, 0, 4, 0, 0, 0\}$.

Q5

Let $x_1[n] = \{4, 5, 6\}$ and $x_2[n] = \{1, 2, 3, 4\}$. Calculate Linear convolution using Circular convolution.

Solution:

$$\text{Length of linear convolution} = l_1 + l_2 - 1 = 4 + 3 - 1$$

$$\text{Length of circular convolution} = \max(l_1, l_2) = 4$$

In order to calculate linear convolution from circular convolution, we pad $x_2(n)$ with 2 zeroes and $x_1(n)$ with 3 zeroes:

$$\begin{bmatrix} 1 & 0 & 0 & 4 & 3 & 2 \\ 2 & 1 & 0 & 0 & 4 & 3 \\ 3 & 2 & 1 & 0 & 0 & 4 \\ 4 & 3 & 2 & 1 & 0 & 0 \\ 0 & 4 & 3 & 2 & 1 & 0 \\ 0 & 0 & 4 & 3 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 4 \\ 5 \\ 6 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \\ 28 \\ 43 \\ 38 \\ 24 \end{bmatrix}$$

Q6

- (a) Check whether the below system is causal or not. Justify.

$$y(n+4) = x(n+3) + x(n+2) + y(n+3) + x(n-4)$$

Solution: Causal system is a system in which the output depends only on present-past inputs and past outputs.

Re-adjusting the above equation by replacing n with $n-4$, the equation becomes:

$$y((n-4)+4) = x((n-4)+3) + x((n-4)+2) + y((n-4)+3) + x((n-4)-4)$$

$$\Rightarrow y(n) = x(n-1) + x(n-2) + y(n-1) + x(n-8)$$

Since the equation $y(n) = x(n-1) + x(n-2) + y(n-1) + x(n-8)$ satisfies the condition where the output depends only on present and past inputs and past outputs, it is a causal system.

- (b) Check whether the below system is LTI or not. Justify.

(i)