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DSA - Assignment - 1

i and j are used in sync in the below assignment to represent $\sqrt{-1}$. It is evident that $i^2 = j^2 = ij = -1$

1. FOURIER SERIES

(1)

(a)

Square-wave function:

$$f(x) = \begin{cases} 1, & 0 \le x < \pi \\ -1, & \pi \le x < 2\pi \end{cases}$$

$$T = \frac{2\pi}{\omega_0} = 2\pi$$

$$\boxed{\omega_0 = 1}$$

$$f(x+2\pi) = f(x)$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt = \frac{1}{2\pi}\int_0^{\pi} 1 \cdot dt + \frac{1}{2\pi}\int_{\pi}^{2\pi} -1 \cdot dt$$

$$= \frac{1}{2\pi}\left[t\right]_0^{\pi} + \frac{1}{2\pi}\left[-t\right]_{\pi}^{2\pi}$$

$$= \frac{1}{2\pi}(\pi - 0) + \frac{1}{2\pi}(-2\pi - (+\pi)(-1))$$

$$\boxed{a_n = 0}$$

$$a_n = \frac{2}{2\pi}\int_0^{2\pi} f(t)\cos(n\omega_0 t)\,dt = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)\,dt$$

$$= \frac{1}{\pi}\int_0^{\pi}\cos(nt)(1)\cdot dt + \frac{1}{\pi}\int_{\pi}^{2\pi}(-1)\cos nt \cdot dt$$

$$= \frac{1}{\pi}\left[\frac{\sin nt}{n}\right]_0^{\pi} + \frac{-1}{\pi}\left[\frac{\sin nt}{n}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}(0-0) - \frac{1}{\pi}(0-0) = 0$$

$$b_n = \frac{2}{2\pi}\int_0^{2\pi} f(t)\sin(n\omega_0 t)\,dt$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)\,dt$$

$$= \frac{1}{\pi}\int_0^{\pi} 1\cdot\sin(nt)\,dt + \frac{1}{\pi}\int_{\pi}^{2\pi} -1\cdot\sin(nt)\,dt$$

$$= \frac{1}{\pi}\left[\frac{-\cos nt}{n}\right]_0^{\pi} + \frac{1}{\pi}\left[\frac{\cos nt}{n}\right]_{\pi}^{2\pi}$$

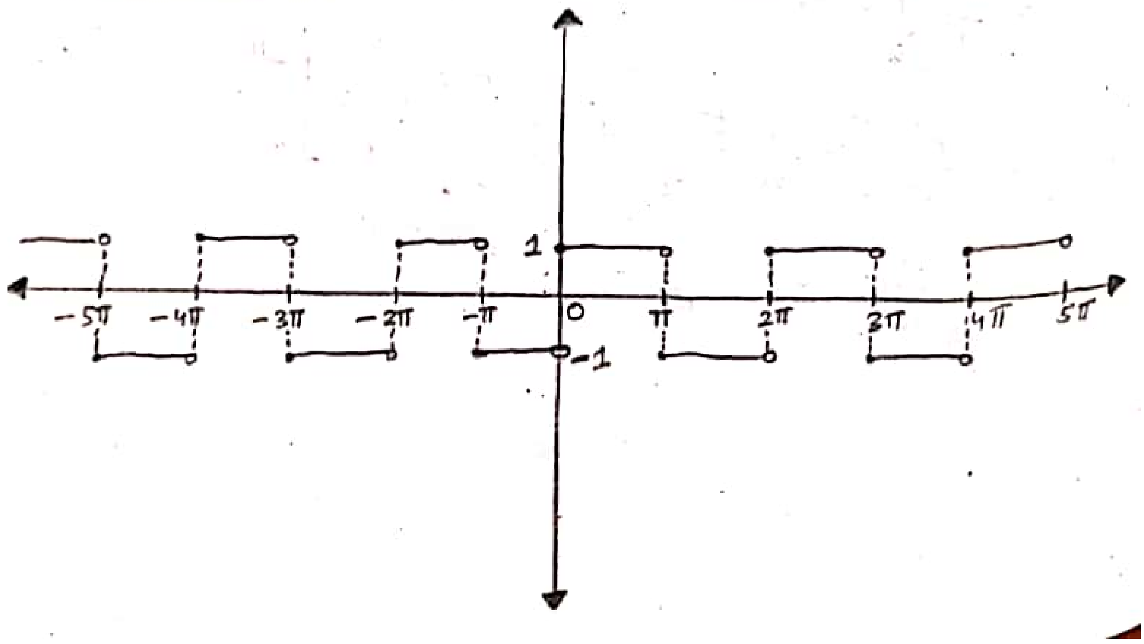
$$= \begin{cases} 0, & \text{if } n = 2,4,6,8,---- \\ \frac{4}{n\pi}, & \text{if } n = 1,3,5,7,---- \end{cases}$$

$$= \frac{4}{n\pi}\begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$$

$$f(t) = \sum_{n=1,3,5,7,--}^{\infty} \frac{4}{n\pi}\sin(nt)$$

$$= \frac{4}{n\pi}\sum_{n=1,3,5,--}^{\infty} \sin(nt)$$

$$\boxed{f(t) = \frac{4}{\pi}\sum_{n=1,3,5,7,--}^{\infty} \sin(nt)/n}$$



(b) Saw tooth wave function:

$$T = \frac{2\pi}{\omega_0} = 2\pi$$

$$\omega_0 = 1$$

$$f(x) = x, \quad -\pi \le x < \pi$$

$$f(x+2\pi) = f(x)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} t\,dt = \frac{1}{2\pi}\left[\frac{t^2}{2}\right]_{-\pi}^{\pi}$$

$$= 0$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\cos(n\omega_0 t)$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos(nt)\,dt = \frac{t}{\pi}\int_{-\pi}^{\pi}\cos nt\,dt$$

$$- \int_{-\pi}^{\pi}\frac{\sin(nt)}{n}\,dt$$

$$= \frac{t}{\pi}\left[\frac{\sin(nt)}{n}\right]_{-\pi}^{\pi} - \frac{1}{n}\left[\frac{-\cos nt}{n}\right]_{-\pi}^{\pi}$$

$$= 0 + \frac{1}{n}\left[\frac{\cos nt}{n}\right]_{-\pi}^{\pi} = 0$$

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin(nt)\,dt$$

$$= \frac{1}{\pi}\left(t\int_{-\pi}^{\pi}\sin(nt)\,dt - \int_{-\pi}^{\pi}\frac{-\cos(nt)}{n}\,dt\right.$$

$$= \frac{1}{\pi}\left[t\left[-\frac{\cos(nt)}{n}\right]_{-\pi}^{\pi} + \left[\frac{\sin(nt)}{n^2}\right]_{-\pi}^{\pi}\right)$$

$$= -\frac{2}{n}(-1)^n$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} -\frac{2*(-1)^n}{n}\sin(nx)}$$

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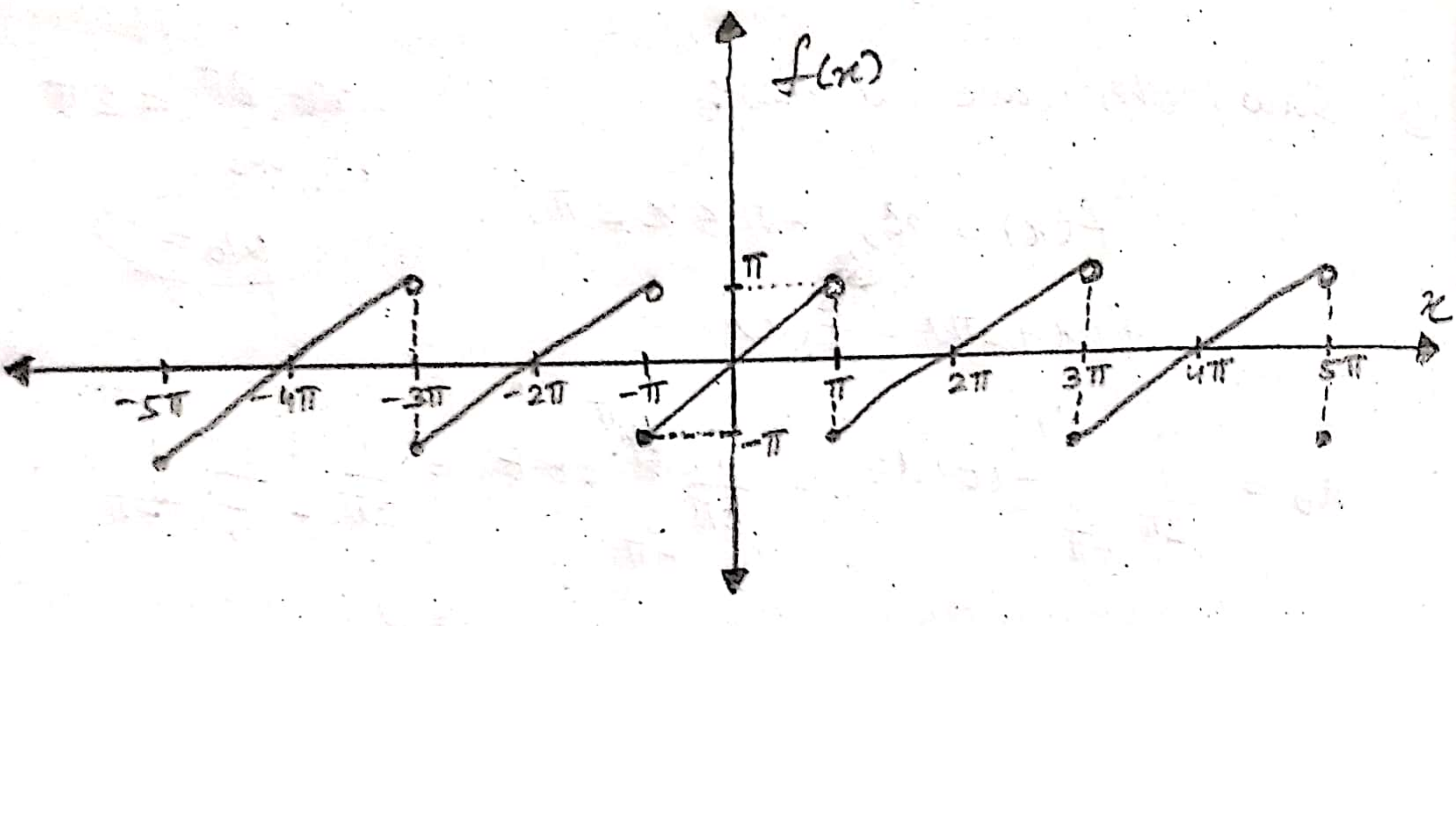

f(x)
π
-π
x
-5π
-4π
-3π
-2π
-π
π
2π
3π
4π
5π

(i) $f(x) = e^{x}, \quad -\pi \le x < \pi$

$f(x+2\pi) = f(x)$ $\qquad T = \frac{2\pi}{\omega_0} = 2\pi$

$\omega_0 = 1$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{t}\,dt = \frac{1}{2\pi}\left[e^{t}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left[e^{\pi} - e^{-\pi}\right]$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\cos(n\omega_0 t)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{t}\cos(nt)\,dt$$

$$= \frac{1}{\pi}\left[\frac{e^{t}}{n^2+1}\left[\cos nt + n\sin nt\right]\right]_{-\pi}^{\pi}$$

$$= \frac{1}{(n^2+1)\pi}\left[e^{\pi}\cos n\pi - e^{-\pi}\cos n\pi\right]$$

$$= \frac{(-1)^n}{\pi.(n^2+1)}(e^{\pi} - e^{-\pi})$$

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\sin(n\omega_0 t)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{t}\sin(nt)\,dt$$

$$= \frac{1}{\pi}\left[\frac{e^{t}}{n^2+1}\left[\sin(nt) - n\cos nt\right]\right]_{-\pi}^{\pi}$$

$$= \frac{1}{(n^2+1)\pi}\left[e^{\pi}(-n\cos n\pi) - e^{-\pi}(-n\cos n\pi)\right]$$

$$= \frac{(-1)^{n+1}}{\pi(n^2+1)}(e^{\pi} - e^{-\pi}).n$$

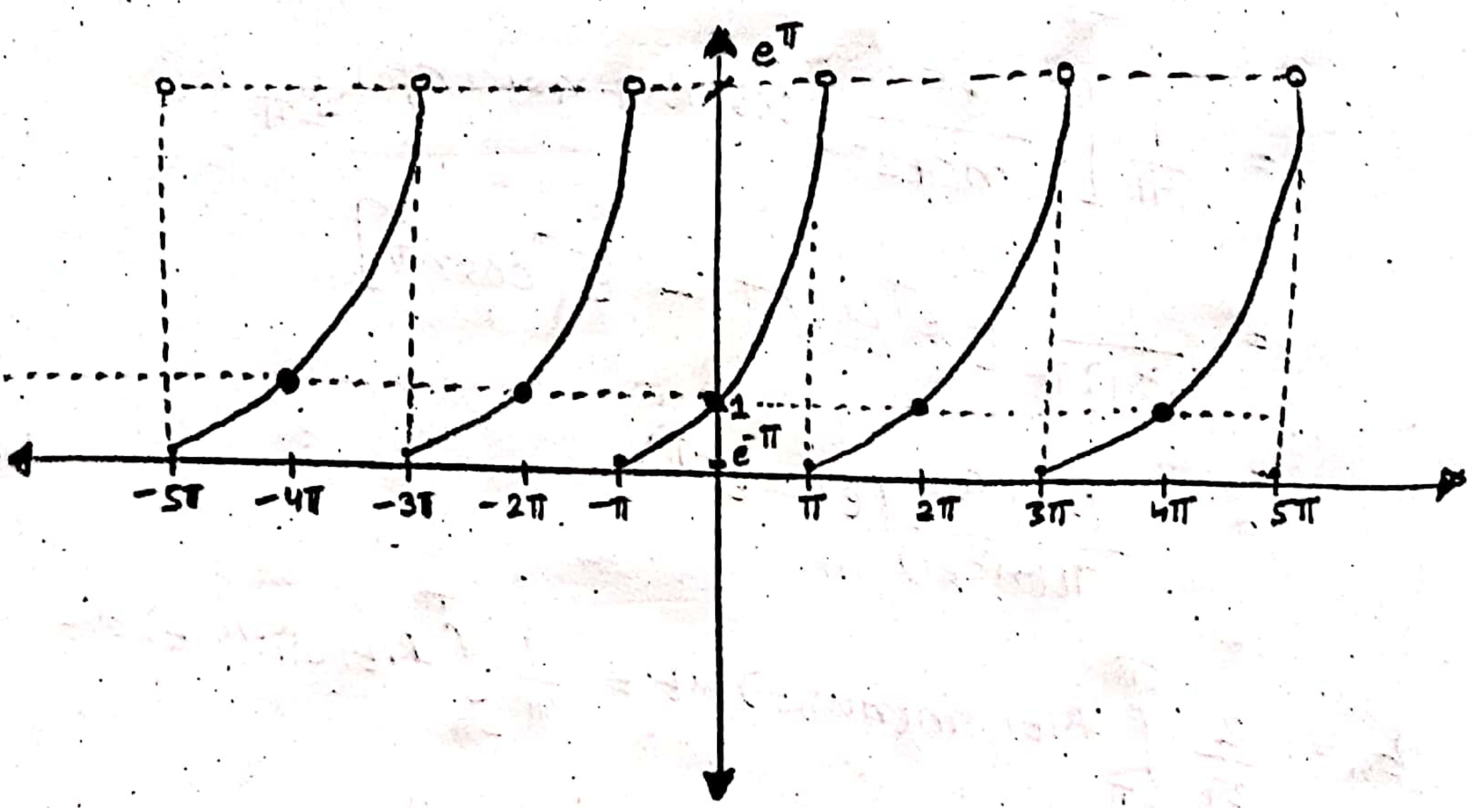
$$f(t) = \frac{1}{2\pi}\left[e^{\pi}-e^{-\pi}\right] + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)$$

$$= \frac{1}{2\pi}\left[e^{\pi}-e^{-\pi}\right] + \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi(n^2+1)}(e^{\pi}-e^{-\pi})\cos(nt)$$

$$+ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi(n^2+1)}(e^{\pi}-e^{-\pi})\cdot n\sin(nt)$$

$$= \frac{(e^{\pi}-e^{-\pi})}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2+1}(-1)^n + \frac{(-1)^{n+1}}{n^2+1}(n\sin(nt))\right]$$

$$= \left(\frac{e^{\pi}-e^{-\pi}}{\pi}\right)\left(\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n\cdot\left[\cos(nt) - n\sin(nt)\right]}{n^2+1}\right)$$



(d) $f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ x, & 0 \le x < \pi \end{cases}$ $\qquad T = \frac{2\pi}{\omega_0} = 2\pi$

$f(x+2\pi) = f(x)$ $\qquad \omega_0 = 1$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(t)\,dt + \int_{0}^{\pi} f(t)\,dt\right]$$

$$= \frac{1}{2\pi}\left(0 + \int_{0}^{\pi} t\,dt\right) = \frac{1}{2\pi}\left(\frac{\pi^2}{2}\right) = \frac{\pi}{4}$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\cos(n\omega_0 t)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt$$

$$= \underbrace{\frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\cos(nt)\,dt}_{0} + \frac{1}{\pi}\int_{0}^{\pi} f(t)\cos(nt)\,dt$$

$$= \frac{1}{\pi}\int_{0}^{\pi} t\cos(nt)\,dt = \frac{1}{\pi}\left[\frac{nt\sin(nt) + \cos(nt)}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{n^2\pi}\left[\cos(n\pi)-1\right] = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{-2}{n^2\pi}, & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\sin(n\omega_0 t)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

$$= \frac{1}{\pi}\underbrace{\int_{-\pi}^{0} 0\cdot\sin(nt)\,dt}_{0} + \frac{1}{\pi}\int_{0}^{\pi} t\sin(nt)\,dt$$

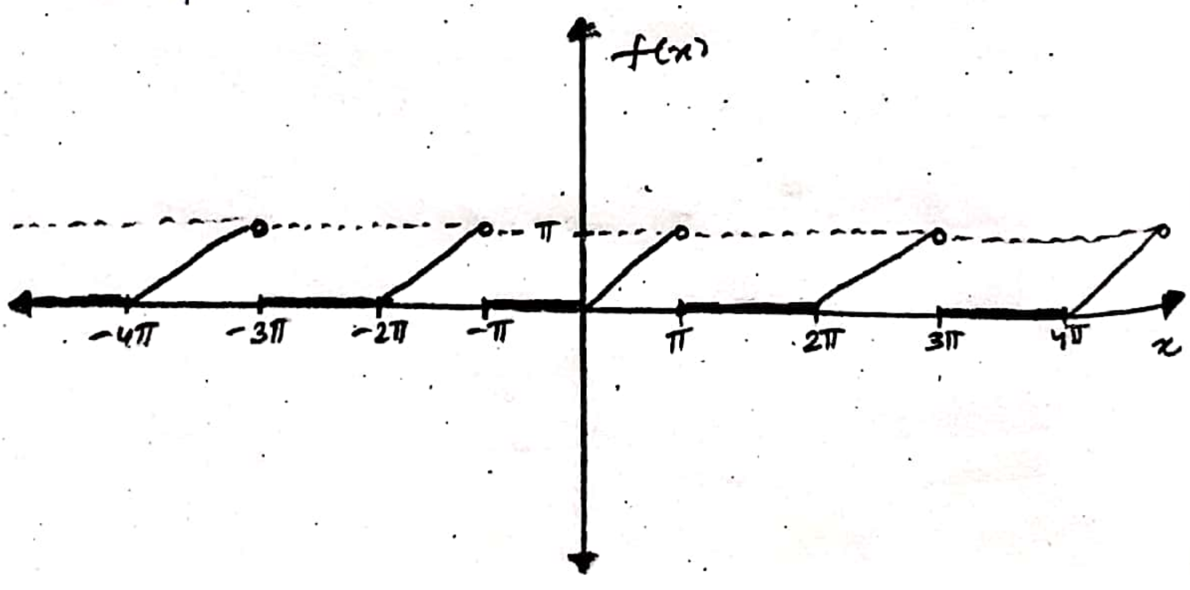
$$= \frac{1}{\pi}\left[\frac{\sin(nt) - nt\cos(nt)}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-n\pi\cos(n\pi)}{n^2}\right] = \frac{-1}{n}\cos(n\pi) = \frac{(-1)^{n+1}}{n}$$

$$f(x) = \pi/4 + \sum_{n=1,3,5,\dots}^{\infty} \frac{-2}{n^2\pi}\cos(nt) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nt)$$

$$= \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{-2}{(2n-1)^2\pi}\cos nt - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nt)$$

$$= \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{-2}{\pi(2n-1)^2}\cos nt - \left[\frac{(-1)^n}{n}\sin(nt)\right]\right)$$



(2)

Odd Functions: A function f is odd if the following equation holds $\forall x$ & $-x$ in the domain of f.

$$f(-x) = -f(x)$$

$\Rightarrow$ Geometrically, the graph of an odd function has rotational symmetry w.r.t origin of $180°$

Even functions: A function f is even if the following equation holds $\forall x$ & $-x$ in the domain of f:

$$f(x) = f(-x)$$

$\Rightarrow$ Geometrically, the graph of an even function is symmetric w.r.t y-axis

(a) $$f(x) = \begin{cases} x-1, & 0 \le x \le 2 \\ -1-x, & -2 < x < 0 \end{cases}$$

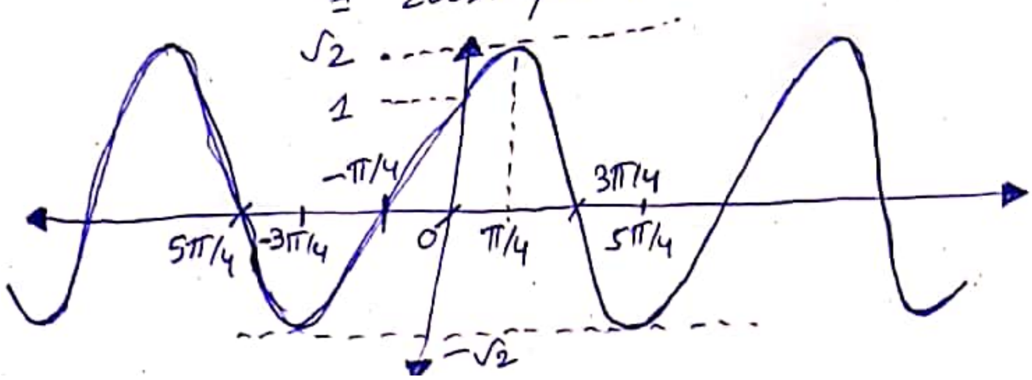
$\Rightarrow$ $f(x)$ is an even function

(b) $f(x) = \sin x + \cos x$

$\Rightarrow$ $f(x)$ is neither an even function nor an odd function

$$f(x) - f(-x) = \sin x + \cos x - (\sin(-x) + \cos(-x))$$
$$= 2\sin x \neq 0 \quad \forall x \in \mathbb{R}$$

$$f(x) + f(-x) = \sin x + \cos x + [\sin[x(-1)] + \cos(-x)]$$
$$= \sin x + \cos x - \sin x + \cos x$$
$$= 2\cos x \neq 0 \quad \forall x \in \mathbb{R}$$



For example:

Let $x = \pi/4$

$f(x) = \sin \pi/4 + \cos \pi/4 = 1/\sqrt{2} + 1/\sqrt{2} = \sqrt{2}$

$f(-x) = \sin(-\pi/4) + \cos(-\pi/4) = -1/\sqrt{2} + 1/\sqrt{2} = 0$

$|f(x)| \neq |f(-x)|$ → Hence $f(x)$ is neither even nor odd function

↳ for $x = \pi/4 \in D_f\,(f(x))$

(c) $f(x) = |x| - 2 \quad , -2 \leq x < 2$

$f(x+4) = f(x)$

→ $f(x)$ is an even function

3

(a) Conditions for Existence of Fourier Series (Dirichlet conditions)

Condition-1: Signal should have finite number of maxima and minima over the range of time-period

Condition-2: Signals should have finite number of discontinuities over the range of time period

Condition-3: Signal should be absolutely integrable over the range of time period

Condition-4: Signal is periodic, single-valued and finite and piece-wise continuous

(b) From part-b of Qn-1,

For a saw-tooth wave function defined by:

$$\boxed{\begin{array}{l} f(x) = x, \quad -\pi \le x < \pi \\ f(x+2\pi) = f(x) \end{array}}$$

Fourier-Series representation of $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin(nx)$$

Let $x = \pi/2$ (substitute it in expression of $f(x)$)

$$f(\pi/2) = \pi/2 = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin(n\pi/2)$$

$$\pi/2 = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi/2)$$

$$\pi/2 = 2\left[1-0-1/3-0+\frac{1}{5} ----\right]$$

$$\pi/2 = 2\left[1-1/3+1/5-1/7+-----+\frac{(-1)^{n+1}}{2n-1}\right]$$

$n\to\infty$

Dividing by 2 on both sides,

$$\boxed{\pi/4 = 1-1/3+1/5-1/7+----+\frac{(-1)^{n+1}}{2n-1}}$$

(c) Consider part-d of Qn-1 given by,

$$\boxed{f(x)=\begin{cases}0, & -\pi\le x<0\\ x, & 0\le x<\pi\end{cases} \qquad f(x+2\pi)=f(x)}$$

Fourier-series representation of f(x) is given by,

$$\therefore f(x)=\pi/4+\sum_{n=1}^{\infty}\left[\frac{-2}{\pi(2n-1)^2}\cos nt+\frac{(-1)^{n+1}}{n}\sin(nt)\right]$$

Let $x=0$ [substituting $x=0$ in f(x)'s expression]

$$f(0)=0=\pi/4+\sum_{n=1}^{\infty}\frac{-2}{\pi(2n-1)^2}\cos(0^\circ)+0$$

$$=\pi/4+\sum_{n=1}^{\infty}\frac{-2}{\pi(2n-1)^2}$$

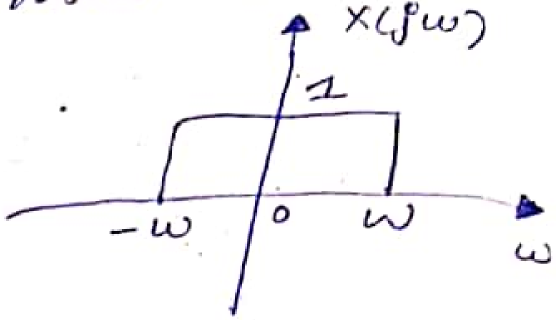
$$\pi/4=\sum_{n=1}^{\infty}\frac{2}{\pi(2n-1)^2}=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\pi^2/8=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=1+1/3^2+1/5^2+---$$

$$\boxed{\pi^2/8 = 1+1/3^2+1/5^2+-----+\frac{1}{(2n-1)^2}}$$

2. FOURIER TRANSFORM

(1) Consider the signal $x(t)$ whose Fourier-Transform is



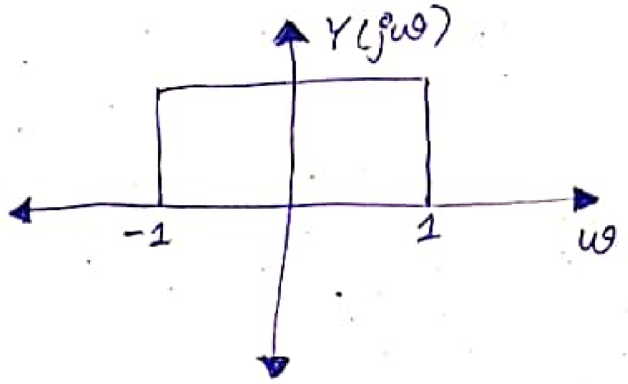
(a) $X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$

$$x(t) = \frac{1}{2\pi} \int_{-W}^{W} e^{j\omega t} \, d\omega$$

$$= \frac{\sin(Wt)}{\pi t}$$

Let $y(t) = \frac{\sin(Wt)}{\pi t}$ (i.e, $W = 1$) $= \frac{\sin t}{\pi t}$

$$Y(j\omega) = \begin{cases} 1, & |\omega| < 1 \\ 0, & |\omega| > 1 \end{cases} = u(\omega+1) - u(\omega-1)$$



where u is unit step fn

$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$

⇒ Fourier-Transform of

$$a(t) \cdot b(t) \xleftrightarrow{FT} \frac{1}{2\pi} A(j\omega) * B(j\omega)$$

Let $x(t) = \frac{\sin^2 t}{\pi^2 t}$ and $z(t) = \left(\frac{\sin t}{\pi t}\right)^2$

$$= \left(\frac{\sin t}{\pi t}\right) \cdot \left(\frac{\sin t}{\pi t}\right)$$

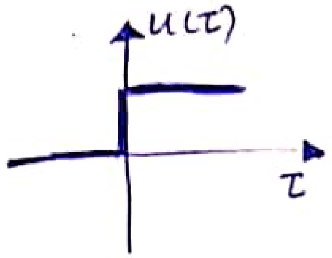
$$= y(t) \cdot y(t)$$

$$Z(j\omega) = \frac{1}{2\pi} \left[Y(j\omega) * Y(j\omega) \right]$$

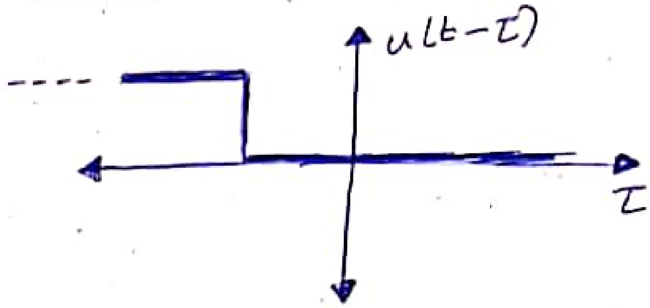
Theorem: Let $u(t)$ be a unit-step function, convolution of two unit-step functions is given by $r(t) = u(t) \cdot u(t) = t\,u(t)$

proof:-

Let $y(t) = u(t) * u(t) = \int_{-\infty}^{\infty} u(\tau) \cdot u(t-\tau)\, d\tau$



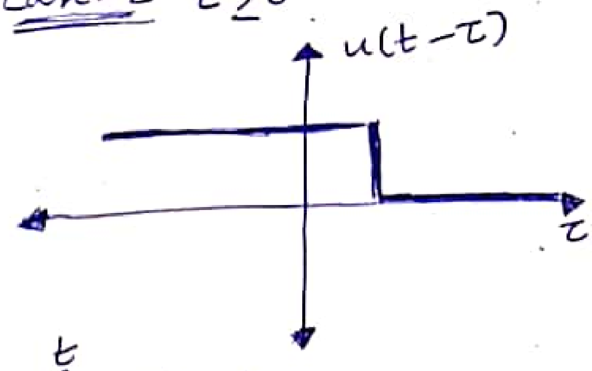
Case-1:- $t<0$



$u(\tau) \cdot u(t-\tau) = 0$

$\int_{-\infty}^{\infty} 0 \cdot dt = 0$

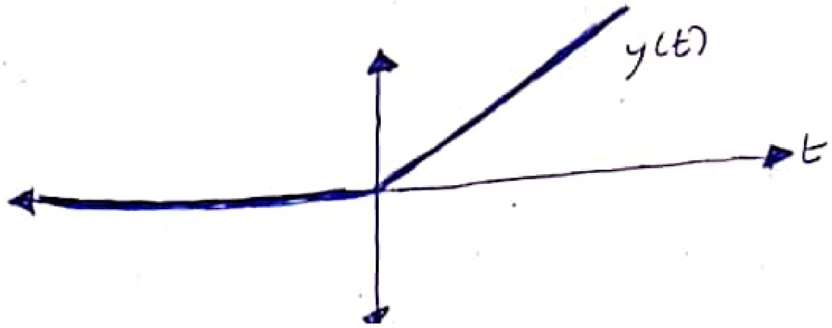
Case-2 $t \geq 0$



$\int_{-\infty}^{\infty} u(\tau) \cdot u(t-\tau)\, d\tau = \int_{0}^{t} 1 \cdot d\tau = t$

$$y(t) = u(t) * u(t) = \begin{cases} 0, & t<0 \\ t, & t \geq 0 \end{cases}$$

called as ramp function



$$Z(j\omega) = \frac{1}{2\pi}\left(Y(j\omega) * Y(j\omega)\right)$$

$$= \frac{1}{2\pi}\left((u(\omega+1) - u(\omega-1)) * (u(\omega+1) - u(\omega-1)\right\}$$

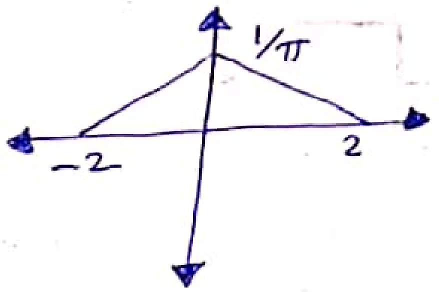
$$= \frac{1}{2\pi}\left[r(\omega+2) + r(\omega-2) - 2r(\omega)\right]$$

where

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$= \frac{1}{2\pi}\left[(\omega+2)u(\omega+2) + (\omega-2)u(\omega-2) - 2\omega u(\omega)\right]$$

$$= \begin{cases} \dfrac{-|\omega|+2}{2\pi}, & |\omega| < 2 \\ 0, & |\omega| \geq 2 \end{cases}$$



<u>Theorem</u>:- $t \cdot x(t) \xleftrightarrow{FT} j \cdot \dfrac{d(X(\omega))}{d\omega}$

By definition, $X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$

Differentiating w.r.t ω

$$\frac{d}{d\omega}(X(\omega)) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot (-jt)\, dt$$

$$= -j \int_{-\infty}^{\infty} [t \cdot (x(t))]\, e^{-j\omega t} \cdot dt$$

$$\frac{d}{d\omega} X(\omega) = -j\, FT[t \cdot x(t)]$$

(or)

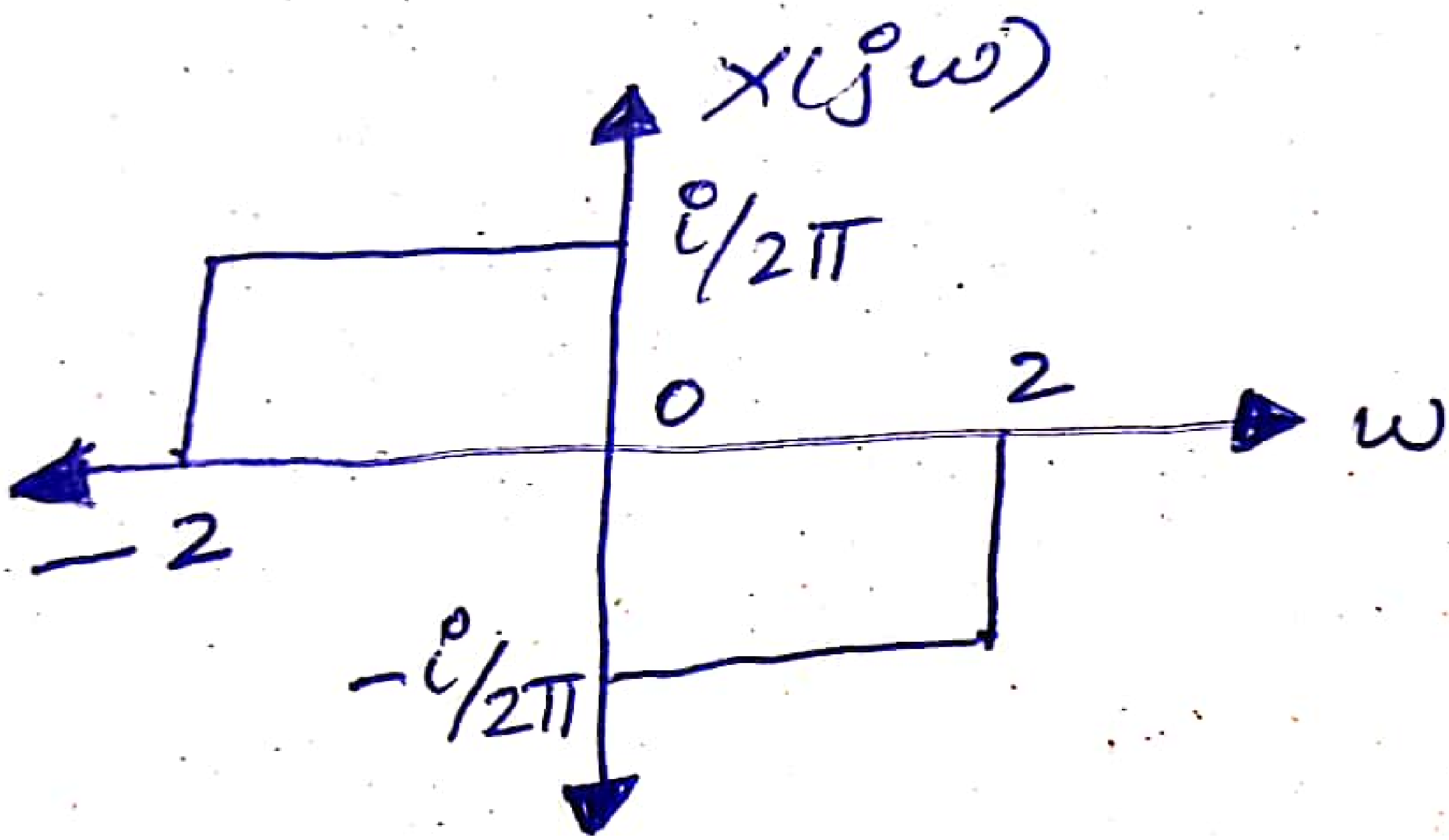
$$\boxed{FT[t \cdot x(t)] = j \cdot \frac{d}{d\omega} X(\omega)}$$

If the convolution of $x(t)$ & $y(t)$ is $h(t)$, then the convolution of $x(t-\alpha)$ & $y(t-\beta)$ is given by $h(t-\alpha-\beta)$

$$x(t) = t \cdot z(t) \overset{F.T}{\Longleftrightarrow} j \cdot \frac{d}{d\omega} Z(j\omega)$$

$$X(j\omega) = \begin{cases} j \cdot \frac{d}{d\omega}\left(\frac{-|\omega|+2}{2\pi}\right), & |\omega| \leq 2 \\ 0, & |\omega| > 2 \end{cases}$$

$$= \begin{cases} j/2\pi, & \text{if } -2 \leq \omega < 0 \\ -j/2\pi, & \text{if } 0 < \omega \leq 2 \\ 0, & \text{else} \end{cases}$$



X(jω)
i/2π
0
2
ω
−2
−i/2π

(b) $\int_{-\infty}^{+\infty} |x(t)|^2 dt = 1/2\pi \cdot \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$

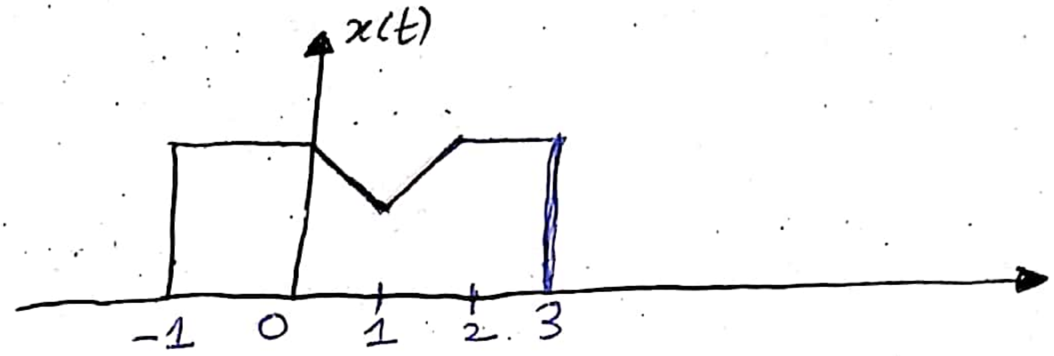
$$A = \int_{-\infty}^{\infty} t^2 \cdot \left(\frac{\sin t}{\pi t}\right)^4 \cdot dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-2}^{0} \left| j/2\pi \right|^2 d\omega + \int_{0}^{2} \left| \frac{-j}{2\pi} \right|^2 d\omega\right]$$

$$= \frac{1}{2\pi} \times 2 \times \left[\frac{1}{4\pi^2} [\omega]_{-2}^{0}\right]$$

$$\boxed{A = 1/2\pi^3}$$

(2)



Let $X(\omega)$ denote the Fourier Transform of the signal $x(t)$ depicted in the figure above

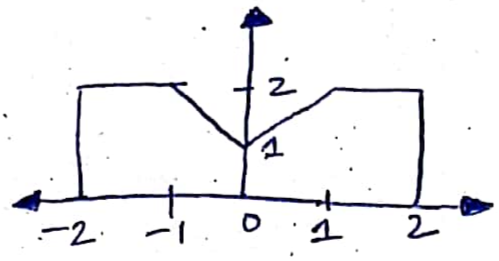
(a)

$$x(t) = \begin{cases} 2, & t \in [-1,0] \cup [2,3] \\ 2-t, & t \in [0,1] \\ t, & t \in [1,2] \\ 0, & \text{otherwise} \end{cases}$$

$$y(t) = x(t+1) = \begin{cases} 2, & t \in [-2,-1] \cup [1,2] \\ 1-t, & t \in [-1,0] \\ 1+t, & t \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

both real and even function

Hence $Y(j\omega)$ is also even and real



$\angle Y(j\omega) = 0$

$Y(j\omega) = X(j\omega) \cdot e^{j\omega} \Rightarrow X(j\omega) = Y(j\omega) \cdot e^{-j\omega}$

Hence phase of $X(j\omega) = \angle X(j\omega) = 0 - \omega = -\omega$

$$\boxed{\text{phase of } X(j\omega) = -\omega}$$

(b)

$$X(j0) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt \Big|_{\omega=0}$$

$$= \int_{-\infty}^{\infty} x(t) \cdot e^{-j0} dt$$

$$= \int_{-\infty}^{\infty} (x(t) \cdot 1) dt = \int_{-1}^{0} 2 \cdot dt + \int_{0}^{1} (2-t) dt + \int_{1}^{2} t \cdot dt + \int_{2}^{3} 2 \cdot dt$$

$$= [2t]_{-1}^{0} + [2t - t^2/2]_{0}^{1} + [t^2/2]_{1}^{2} + [2t]_{2}^{3}$$

$$= 2(0-(-1)) + \left(2(1) - \left(\tfrac{1}{2}\right)\right) + \left(2^2/2 - 1^2/2\right) + 2(3-2)$$

$$= 2 + 3/2 + 3/2 + 2 = 7$$

$$\boxed{X(j0) = 7}$$

(c)

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \cdot e^{j\omega t} dt$$

$$\int_{-\infty}^{\infty} X(j\omega) \cdot d\omega$$

$$= \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) \cdot e^{j\omega t} d\omega\right]_{t=0} \cdot 2\pi$$

$$= x[0] \cdot 2\pi = (2-0) \cdot 2\pi = 4\pi$$

(d) $\int_{-\infty}^{\infty} X(j\omega) \cdot \frac{2\sin\omega}{\omega} \cdot e^{j2\omega} d\omega$

$\Rightarrow$ Let $Y(j\omega) = \frac{2 \cdot \sin\omega}{\omega} \cdot e^{j2\omega}$

square signal shifted by 2

$$y(t) = \begin{cases} 1, & \text{if } t \in (-3,-1) \\ 0, & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} X(j\omega) \cdot Y(j\omega)\, d\omega = 2\pi \{x(t) * y(t)\}_{t=0}$$

$$= 2\pi \cdot (x(t) * y(t))\Big|_{t=0}$$

$$= 2\pi \cdot \left[\int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau\right]_{t=0}$$

$$= \left(\int_{-\infty}^{\infty} x(\tau) \cdot y(-\tau)\, d\tau\right) \cdot 2\pi$$

$$= \left(\int_{-\infty}^{1} (x(\tau) \cdot 0)\, d\tau + \int_{1}^{3} (x(\tau) \cdot 1)\, d\tau + \int_{3}^{\infty} (x(\tau) \cdot 0)\, d\tau\right) \cdot 2\pi$$

$$= \left(\int_{1}^{3} x(\tau) \cdot d\tau\right) \cdot 2\pi = \left(\int_{1}^{2} \tau \cdot d\tau + \int_{2}^{3} 2 \cdot d\tau\right) \cdot 2\pi$$

$$= \left(\left[\tau^2/2\right]_1^2 + \left[2\tau\right]_2^3\right) \cdot 2\pi$$

$$= \left(3/2 + 2\right) \cdot 2\pi = \frac{7}{2} \cdot 2\pi = 7\pi$$

(e)

Parseval's Relation:

$$\boxed{\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega}$$

$$\Rightarrow \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega = 2\pi \cdot \left(\int_{-\infty}^{+\infty} |x(t)|^2 dt \right)$$

Given that

$$x(t) = \begin{cases} 2, & t \in [-1,0] \cup [2,3] \\ 2-t, & t \in [0,1] \\ t, & t \in [1,2] \\ 0, & \text{otherwise}. \end{cases}$$

$$|x(t)|^2 = \begin{cases} 4, & t \in [-1,0] \cup [2,3] \\ (t-2)^2, & t \in [0,1] \\ t^2, & t \in [1,2] \\ 0, & \text{otherwise} \end{cases}$$

$$2\pi \cdot \left(\int_{-\infty}^{+\infty} |x(t)|^2 dt \right)$$

$$= 2\pi \cdot \left[\int_{-1}^{0} 4 \cdot dt + \int_{0}^{1} (t-2)^2 \cdot dt + \int_{1}^{2} t^2 \cdot dt + \int_{2}^{3} 4 \cdot dt \right]$$

$$= 2\pi \cdot \left([4t]_{-1}^{0} + \left[\frac{(t-2)^3}{3} \right]_0^1 + \left[t^3/3 \right]_1^2 + [4t]_2^3 \right)$$

$$= 2\pi \cdot \left(4 + (-1/3) - \left(-\frac{8}{3}\right) + 7/3 + 4 \right)$$

$$= 2\pi \left(8 + 14/3 \right) = 76\pi/3$$

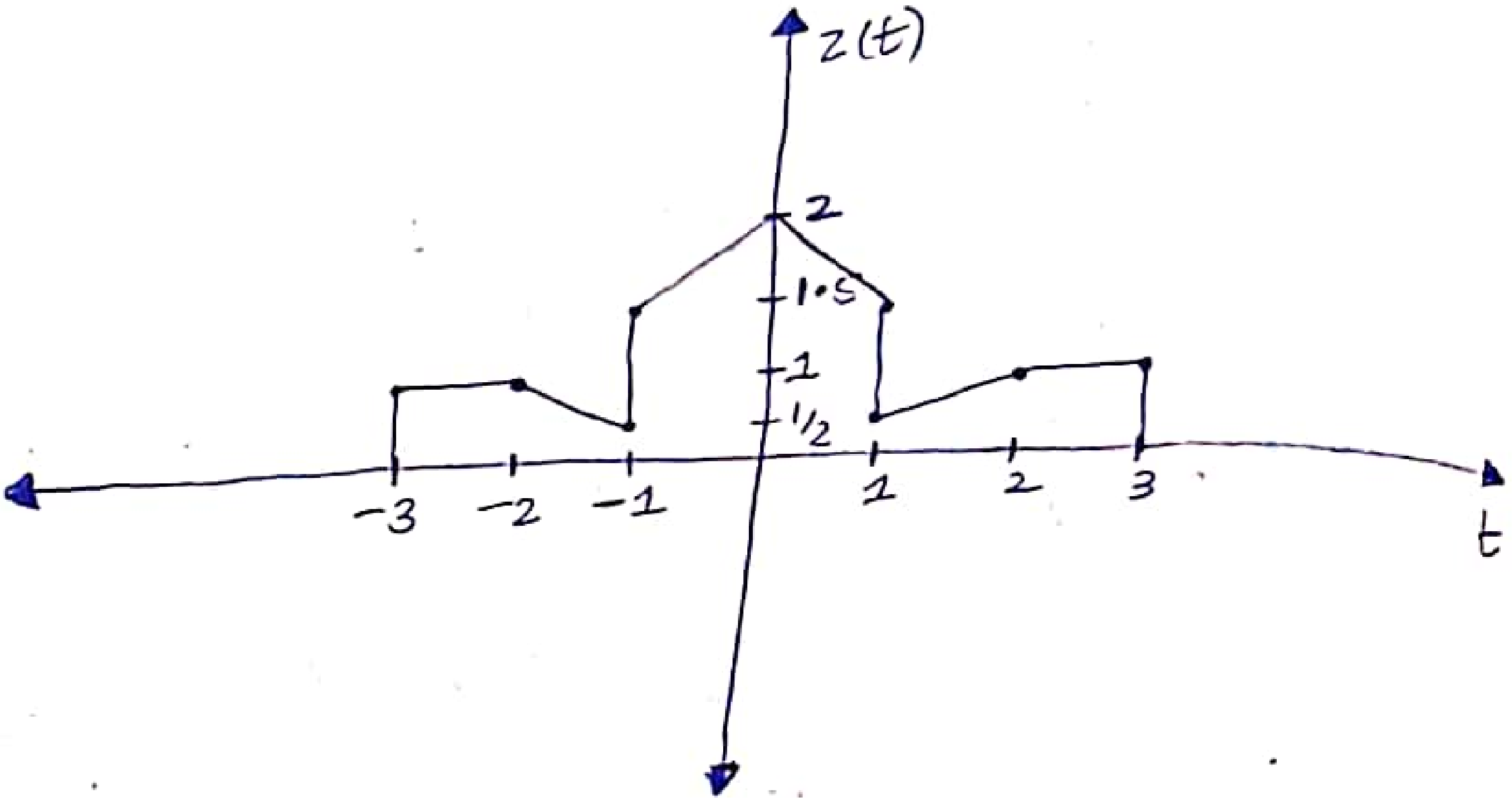
(f) Let $y(t) = x(-t) = \begin{cases} 2, & t \in [0,1] \cup [-3,-2] \\ 2+t, & t \in [-1,0] \\ -t, & t \in [-2,-1] \\ 0, & \text{otherwise} \end{cases}$

$$x(t) = \begin{cases} 2, & t \in [-1,0] \cup [2,3] \\ 2-t, & t \in [0,1] \\ t, & t \in [1,2] \\ 0, & \text{otherwise} \end{cases}$$

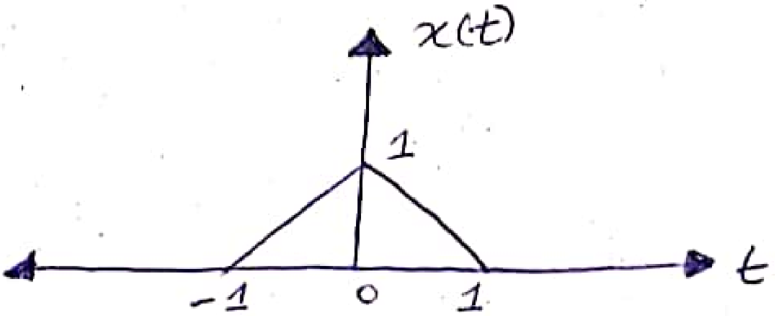
let $z(t) = \frac{1}{2}\left(x(t) + y(t)\right)$

$$z(t) = \begin{cases} 1, & t \in [-3,-2] \cup [2,3] \\ -t/2, & t \in [-2,-1] \\ 2+t/2, & t \in [-1,0] \\ 2-t/2, & t \in [0,1] \\ t/2, & t \in [1,2] \end{cases}$$

The inverse fourier transform of $Re(X(jw))$ is given by $\frac{(x(t)+x(-t))}{2} = z(t)$



③



$$x(t) = \begin{cases} 0 \ , & |t| \geq 1 \\ 1-|t| \ , & |t| < 1 \end{cases}$$

ⓐ $$X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt$$

$$= \int_{-1}^{1} (1-|t|) \cdot e^{-j\omega t} \, dt + \underbrace{\int_{-\infty}^{-1} 0 \cdot e^{-j\omega t} \, dt}_{=0}$$

$$+ \underbrace{\int_{1}^{\infty} 0 \cdot e^{-j\omega t} \, dt}_{=0}$$

$$= \int_{0}^{1} (1-t) \cdot e^{-j\omega t} \, dt + \int_{-1}^{0} (1+t) \cdot e^{-j\omega t} \, dt$$

$$= \int_0^1 (1-t)\cdot e^{-j\omega t}\,dt + \int_{-1}^{0} (1+t)\cdot e^{-j\omega t}\,dt$$

$$= \left[(1-t)\cdot e^{-j\omega t}/-j\omega + \frac{e^{-j\omega t}}{j^2\omega^2} \right]_0^1$$

$$+ \left[(1+t)\cdot \frac{e^{-j\omega t}}{-j\omega} + (-1)\int \frac{e^{-j\omega t}}{-j\omega} \right]_{-1}^{0}$$

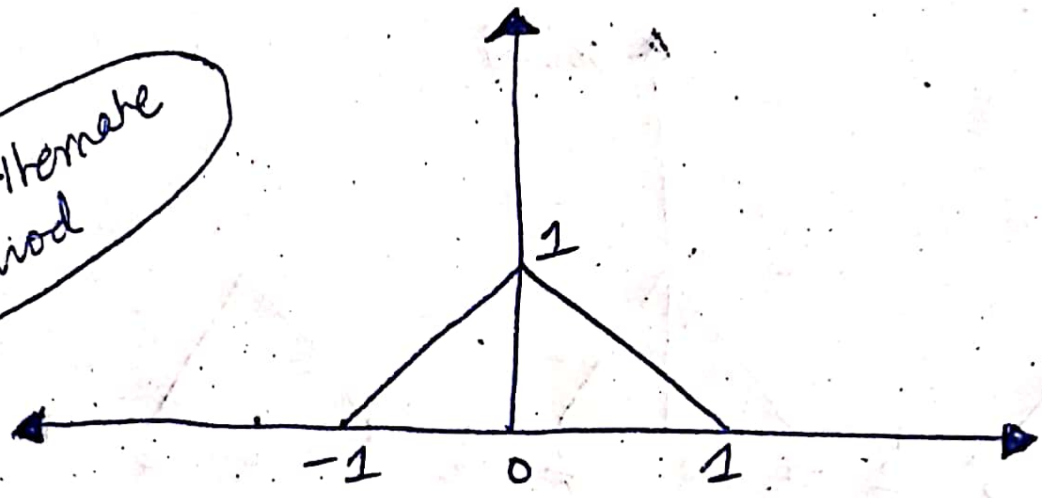
$$= \left[1/j\omega + 1/\omega^2 - \frac{e^{-j\omega}}{\omega^2} \right] + \left[\frac{(1+t)\cdot e^{-j\omega t}}{-j\omega} + \frac{e^{-j\omega t}\cdot(-1)}{j^2\omega^2} \right]_{-1}^{0}$$

$$= \left[\left(\cancel{1/j\omega} + 1/\omega^2 - \frac{e^{-j\omega}}{\omega^2} \right) + \left(\cancel{\frac{1}{-j\omega}} - \frac{1}{j^2\omega^2} \right) - \left(\frac{e^{j\omega}(-1)}{j^2\omega^2} \right) \right]$$

$$= 2/\omega^2 + \frac{-1}{\omega^2}\left[e^{-j\omega} + e^{j\omega} \right]$$

$$= 2/\omega^2 - \frac{2\cos\omega}{\omega^2} = \frac{4\sin^2\omega/2}{\omega^2}$$

Alternate Method



ⓐ $x(t)$ can be expressed as

$$x(t) = x_1(t) * x_1(t) \quad \hookrightarrow \quad x_1(t) = \begin{cases} 1, & |w| < 1/2 \\ 0, & \text{otherwise} \end{cases}$$

Fourier Transform $X_1(jw)$ of $x_1(t)$ is

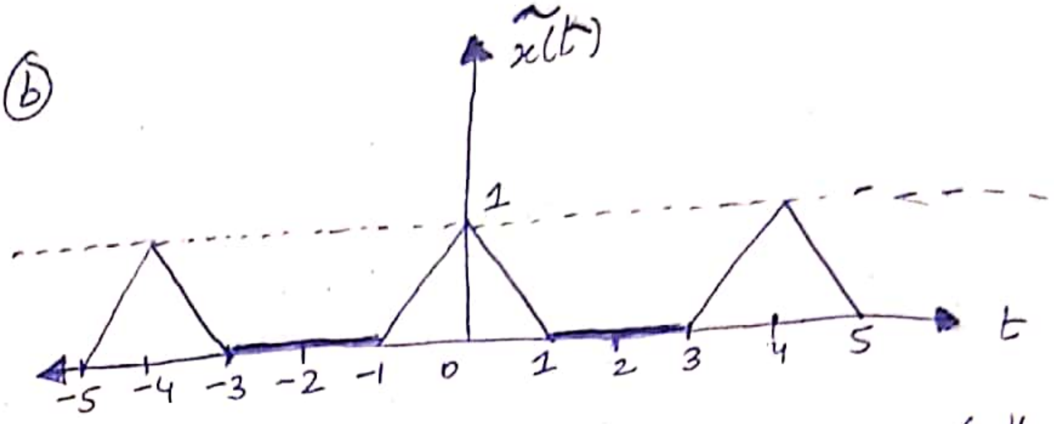
$$X_1(jw) = \frac{2 \cdot \sin(w/2)}{w}$$

From property of convolution we have,

$$X(jw) = X_1(jw) \cdot X_1(jw)$$

$$= \left(\frac{2 \cdot \sin(w/2)}{w}\right)^2 = \frac{4\sin^2(w/2)}{w^2}$$

(b)



$$\tilde{x}(t) = x(t) * \sum_{k=-\infty}^{\infty} \delta(t-4k)$$

(because t & k are independent)

$$= \sum_{k=-\infty}^{\infty} x(t) * \delta(t-4k)$$

$$= \sum_{k=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau) \cdot \delta(t-\tau-4k) \cdot d\tau \right)$$

$$= \sum_{k=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau) \cdot \delta((t-4k)-\tau)\, d\tau \right)$$

consider this as α

$$= \sum_{k=-\infty}^{\infty} x(t-4k)$$

From the property of convolution over impulse function

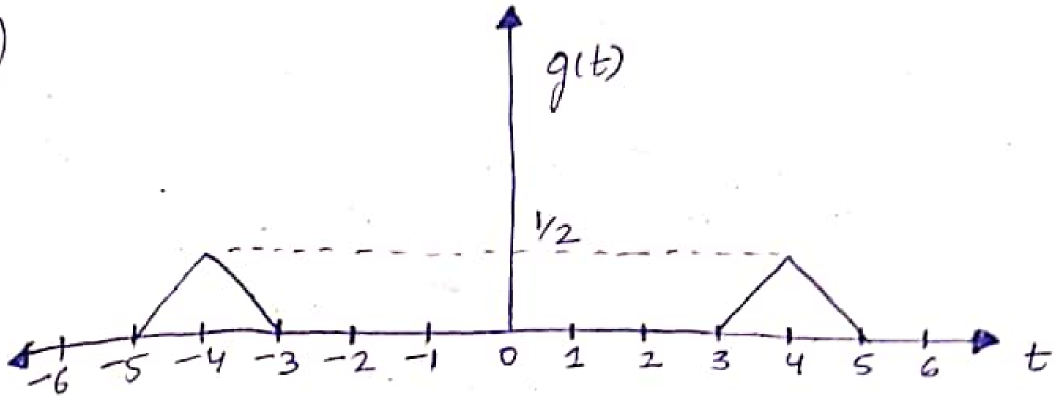
$$\int_{-\infty}^{\infty} x(\tau) \cdot \delta(\alpha-\tau)\, d\tau = x(\alpha)$$

$\tilde{x}(t)$ is a shifted version of $x(t)$ by $4k$ units

along horizontal-axis

Repeats with a period of 4

(c)



Let $g(t) = \dfrac{x(t+4) + x(t-4)}{2}$

$$\tilde{g} = \tilde{x}' = g(t) * \sum_{k=-\infty}^{\infty} \delta(t-4k)$$

$$= \frac{1}{2} \cdot \sum_{k=-\infty}^{\infty} \left(\left(\int_{-\infty}^{\infty} x(\tau+4) \cdot \delta(t-\tau-4k) \cdot d\tau \right) + \int_{-\infty}^{\infty} x(\tau-4) \cdot \delta(t-\tau-4k) \cdot d\tau \right)$$

Let

$\tau_1 = \tau + 4$ and $\tau_2 = \tau - 4$

$d\tau_1 = d\tau$ & $d\tau_2 = d\tau$

$$= \frac{1}{2} \cdot \sum_{k=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau_1) \cdot \delta(t-\tau_1+4-4k) \cdot d\tau_1 \right)$$

$= \delta((t-4(k-1))-\tau_1))$

$$+ \frac{1}{2} \cdot \sum_{k=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau_2) \cdot \delta(t-\tau_2-4-4k) \cdot d\tau_1 \right)$$

$= \delta((t-4(k+1))-\tau_2')$

$$= \frac{1}{2} \cdot \sum_{k=-\infty}^{\infty} \left(x(t-4(k-1)) + x(t-4(k+1)) \right)$$

$$= \frac{1}{2} \cdot \sum_{k'=-\infty}^{\infty} \left(x(t-4k') + \frac{1}{2} \sum_{k''=-\infty}^{\infty} x(t-4k'') \right)$$

$k' = k-1$ $\quad$ $k'' = k+1$

$$= \frac{1}{2} \cdot \sum_{k=-\infty}^{\infty} x(t-4k) + \frac{1}{2} \sum_{k=-\infty}^{\infty} x(t-4k)$$

$$= \sum_{k=-\infty}^{\infty} x(t-4k)$$

(changing both variables k', k'' to k as in summation doesn't matter)

same as $\tilde{x}$ from part (b)

shifted version of $x(t)$ by $4k$ units along horizontal axis

Repeats with a period of 4

(d) Given that $\tilde{x}(t) = x(t) \cdot \sum_{k=-\infty}^{\infty} \delta(t-4k)$

Hence,

($\delta(t-4k)$ → creates 4 portions of 2)

$$\tilde{x}(j\omega) = X(j\omega) \cdot \left(\frac{2\pi}{4}\right) \cdot \sum_{k=-\infty}^{\infty} \delta(j(\omega - k\pi/2))$$

$$= X(j\omega) \cdot \pi/2 \cdot \sum_{k=-\infty}^{\infty} \delta(j(\omega - k\pi/2))$$

From part c

$$= G(j\omega) \cdot \pi/2 \cdot \sum_{k=-\infty}^{\infty} \delta(j(\omega - k\pi/2))$$

$$= \pi/2 \cdot \sum_{k=-\infty}^{\infty} X(j\pi k/2) \cdot \delta(j(\omega - k\pi/2))$$

$$= \pi/2 \cdot \sum_{k=-\infty}^{\infty} G(j\pi k/2) \cdot \delta(j(\omega - k\pi/2))$$

Hence, equal at all integers from $-\infty$ to ∞

exactly equal bcz $X(j\pi k/2) \cdot \delta(j(\omega - k\pi/2))$ ↓ not zero only when input is zero

Hence all terms have to be exactly equal

General solution

Example :-

Let $g(t) = \dfrac{x(t+4) + x(t-4)}{2}$

F.T$(x(t-4b))$ = F.T$(x(t)) \cdot e^{-j\omega(4b)}$

From property of Time-shifting

$$G(j\omega) = \frac{1}{2} X(j\omega) \cdot \left[e^{4j\omega} + e^{-4j\omega}\right]$$

It is evident that $2\cos 4\omega \neq 2$ at all times

$e^{4j\omega} + e^{-4j\omega}$

$$\Rightarrow G(j\pi k/2) = \frac{X(j\pi k/2)}{2} \cdot \left[2\cos\left(4\left(\frac{\pi k}{2}\right)\right)\right]$$

$$= X(j\pi k/2) \cdot \cos(2\pi k)$$

1 for all integers k

$k \in \mathbb{Z}$

Hence,

$G(j\pi k/2) = X(j\pi k/2)$ for all integers k

3. DTFT

③ DTFT

① To show that DTFT of a discrete time-signal $x[n]$ is periodic with period 2π, we need to demonstrate that

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

for all frequencies ω and all signals $x[n]$ of length N.

Using the definition of DTFT, we have:

$$X(e^{j(\omega+2\pi)}) = \sum_{n=0}^{N-1} x[n] \cdot e^{-jn(\omega+2\pi)}$$

$$= \sum_{n=0}^{N-1} x[n] \cdot e^{-jn\omega} \cdot e^{-2n\pi j}$$

$$= \sum_{n=0}^{N-1} x[n] \cdot e^{-jn\omega} \left[\cos(2\pi n) - \sin(2\pi n) \cdot j\right]$$

Since $\cos(2\pi n) = 1$ and $\sin(2\pi n) = 0$ for all integers n, we have:

$$X(e^{j(\omega+2\pi)}) = \sum_{n=0}^{N-1} x[n] \cdot e^{-jn\omega} = X(e^{j\omega})$$

where we have used the fact that $x[n]$ is periodic with period N, so that $x[n+N] = x[n]\ \forall n$.

$\Rightarrow$ Thus, we have shown that the DTFT of a discrete time-signal is periodic with period 2π

(2) We can find the DTFT of y[n] in terms of $X(e^{jw})$ using the definition of DTFT:

$$Y(e^{jw}) = \sum_{n=-\infty}^{\infty} y[n] \cdot e^{-jwn}$$

Substituting $y[n] = x[n-m]$, we get:

$$Y(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n-m] \cdot e^{-jwn}$$

Now, we can write the summation variable by replacing n with n+m

$$Y(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j(n+m)w}$$

$$= e^{-jmw} \cdot \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-jnw}$$

$$\boxed{Y(e^{jw}) = e^{-jmw} \cdot X(e^{jw})}$$

where we have used the fact that $X[e^{jw}]$ is the DTFT of $x[n]$. Therefore, the DTFT of $y[n]$ in terms of $x[e^{jw}]$ is given by:

$$\boxed{Y(e^{jw}) = e^{-jmw} \cdot X(e^{jw})}$$

(3)

(a) $x[n] = (1/4)^n u[n+2]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^n u[n+2]\, e^{-j\omega n}$$

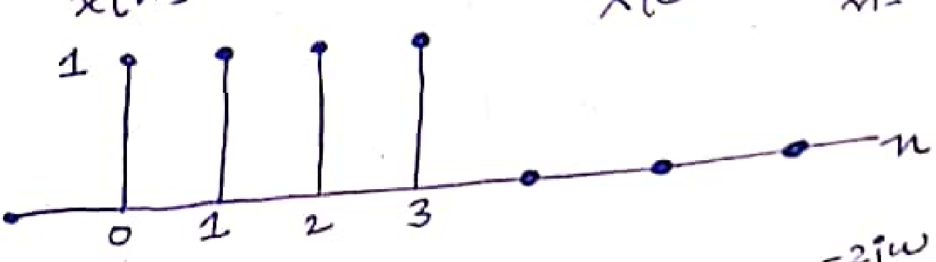
$$= \sum_{n=-2}^{\infty} (1/4)^n e^{-j\omega n}$$

$$= \left(\frac{e^{+2j\omega}}{4^{-2}}\right) + \left(\frac{e^{j\omega}}{4^{-1}}\right) + \frac{(1)}{(1)} + \text{------}$$
$$+ \left(\frac{e^{-j\omega}}{4}\right) + \left(\frac{e^{-2j\omega}}{4^2}\right) + \text{----}$$

$$= \frac{e^{2j\omega} \times 4^2}{1 - \left[\frac{e^{-j\omega}}{4}\right]} = \frac{16 e^{2j\omega}}{1 - \frac{1}{4e^{j\omega}}}$$

$$\boxed{X(e^{j\omega}) = \frac{64 e^{3j\omega}}{4e^{j\omega} - 1}}$$

(b)



$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$$

$$X(e^{j\omega}) = (1+0) \cdot e^{-j\omega(0)} + 1 \cdot e^{-j\omega} + 1 \cdot e^{-2j\omega} + 1 \cdot e^{-3j\omega}$$

$$= 1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega}$$

4. DFT

④ DFT

① ⓐ $x[n] = \frac{1}{4} u(4-n)$ $N=8$

$$x[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j\left(\frac{2\pi}{N}\right)kn}$$

$$= \sum_{n=0}^{7} u(4-n) \cdot e^{\frac{-\pi jkn}{4}}$$

$$= \underbrace{\sum_{n=0}^{4} u(4-n)}_{1} \cdot e^{\frac{-j\pi kn}{4}} + \underbrace{\sum_{n=5}^{7} u(4-n)}_{0} \cdot e^{\frac{-j\pi kn}{4}}$$

$$= \sum_{n=0}^{4} e^{-j\pi kn/4} + 0$$

$$= \sum_{n=0}^{4} e^{\frac{-j\pi kn}{4}}$$

$$x[0] = \frac{1}{4} \cdot (\underbrace{1 + 1 + --- 1}_{5 \text{ times}}) = 5/4$$

$$x[1] = \left(e^{0} + e^{-j\pi/4} + e^{-2j\pi/4} + e^{-3j\pi/4} + e^{-j\pi}\right) \cdot 1/4$$

$$= \left(\cancel{1} + \cancel{\frac{1}{\sqrt{2}}} - \frac{i}{\sqrt{2}} - i + \cancel{\left(-\frac{1}{\sqrt{2}}\right)} - \frac{i}{\sqrt{2}} + \cancel{-1}\right) \cdot 1/4$$

$$= -i(\sqrt{2}+1)/4$$

$$x[2] = \left(1 + e^{-j\pi/2} + e^{-j\pi} + e^{-3j\pi/2} + e^{-2j\pi}\right) \cdot 1/4$$

$$= \left(\cancel{1} \cancel{-i} + \cancel{(-1)} + \cancel{i} + 1\right)/4 = 1/4$$

$$x[3] = \left(\sum_{n=0}^{4} e^{-\frac{j\pi(3)(n)}{4}}\right) \cdot 1/4$$

$$= \left(1 + e^{-3j\pi/4} + e^{-3j\pi/2} + e^{-9j\pi/4} + e^{-3j\pi}\right) \cdot 1/4$$

$$= \left(\cancel{1} + \cancel{\left(\frac{-1}{\sqrt{2}}\right)} - \frac{i}{\sqrt{2}} + i + \cancel{\left(1/\sqrt{2}\right)} - \frac{i}{\sqrt{2}} + \cancel{(-1)}\right) \cdot 1/4$$

$$= i(1-\sqrt{2})/4$$

$$x[4] = \sum_{n=0}^{4} e^{-j\pi n}/4 = \left(e^{0} + \cancel{e^{-j\pi}} + \cancel{e^{-2j\pi}} + \cancel{e^{-3j\pi}} + \cancel{e^{-4j\pi}}\right)/4$$

$$= 1/4$$

$$x[5] = \sum_{n=0}^{4} \frac{e^{-5j\pi n/4}}{4}$$

$$= \left(\cancel{1} + e^{-5j\pi/4} + e^{-5j\pi/2} + e^{-15j\pi/4} + \cancel{e^{-5j\pi}}\right) \cdot \frac{1}{4}$$

$$= \left(\cancel{-1} + \cancel{\left(\frac{-1}{\sqrt{2}}\right)} + \frac{i}{\sqrt{2}} - i + \cancel{\frac{1}{\sqrt{2}}} + \frac{i}{\sqrt{2}} + \cancel{1}\right) \cdot 1/4$$

$$= i(\sqrt{2}-1)/4$$

$$x[6] = \sum_{n=0}^{4} e^{-3j\pi n/2} = \left(1 + e^{-3j\pi/2} + e^{-3j\pi} + e^{-9j\pi/2} + e^{-6j\pi}\right) \cdot 1/4$$

$$= \frac{1 + \cancel{i} - \cancel{1} - \cancel{i} + \cancel{1}}{4} = 1/4$$

$$x[7] = \sum_{n=0}^{4} e^{-7j\pi n/4} = \left(1 + e^{-7j\pi/4} + e^{-7j\pi/2} + e^{-21j\pi/4} + e^{-7j\pi}\right) \cdot \frac{1}{4}$$

$$= \left(\cancel{1} + \cancel{\frac{1}{\sqrt{2}}} + \frac{i}{\sqrt{2}} + i - \cancel{1/\sqrt{2}} + i/\sqrt{2} - \cancel{1}\right) \cdot \frac{1}{4}$$

$$= i(\sqrt{2}+1)/4$$

$$x[k] = \left\{5/4,\ \frac{-i(\sqrt{2}+1)}{4},\ 1/4,\ \frac{i(1-\sqrt{2})}{4},\ 1/4,\ \frac{i(\sqrt{2}-1)}{4},\ \frac{1}{4},\ \frac{i(\sqrt{2}+1)}{4}\right\}$$

(b) $x[n] = \sin(n\pi/4) + \cos(n\pi/4)$

$$= \left(\frac{e^{in\pi/4} - e^{-i\pi n/4}}{2i}\right) + \left(\frac{e^{i\pi n/4} + e^{-i\pi n/4}}{2}\right)$$

$$x[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j\left(\frac{2\pi}{N}\right)kn} \qquad N=8$$

$$= \sum_{n=0}^{7} \left(\sin(n\pi/4) + \cos(n\pi/4)\right) e^{-\frac{i\pi kn}{4}}$$

$[0, -4j, 0, 0, 0, 0, 0, 4j]$

$[0, 4, 0, 0, 0, 0, 0, 4]$

$x[0] = 0$

$x[1] = 4-4j$

$x[2] = 0$

$x[3] = 0$

$x[4] = 0$

$x[5] = 0$

$x[6] = 0$

$x[7] = 4+4j$

$$\boxed{x[k] = \{0, 4-4j, 0, 0, 0, 0, 0, 4+4j\}}$$

Calculations

$$x[0] = \sum_{n=0}^{7} \left(\sin(n\pi/4) + \cos(n\pi/4)\right)$$

$$= \sum_{n=0}^{3}\left[\left(\sin(n\pi/4)\right) + \cos(n\pi/4)\right] + \sum_{n=4}^{7}\left(\sin(n\pi/4) + \cos(n\pi/4)\right)$$

$$= \sum_{n=0}^{3}\left(\sin(n\pi/4) + \cos(n\pi/4) + \sin(n\pi + \pi/4) + \cos(n\pi + \pi/4)\right)$$

$$= \sum_{n=0}^{3}\left(\sin(n\pi/4) + \cos(n\pi/4) - \sin(n\pi/4) - \cos(n\pi/4)\right)$$

$$= 0$$

$$x[1] = \sum_{n=0}^{7}\left(\sin(n\pi/4) + \cos(n\pi/4)\right) \cdot e^{-j\pi n/4}$$

$$= 1 + \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right) + 0 - 1\cdot(-1) + 0 + 1\cdot(-j) + (-1)(j) + 0 + 1(1) - \sqrt{2}\left(-1/\sqrt{2} + j/\sqrt{2}\right)$$

$$= 4 - 4j$$

$$x[2] = \sum_{n=0}^{7} \sqrt{2}\cdot \sin((n+1)\pi/4)\cdot e^{-j\pi n/4\cdot(2)}$$

$$= \sum_{n=0}^{7} \sqrt{2}\cdot \sin((n+1)\pi/4)\cdot e^{-j\pi n/2}$$

$$= \cancel{1} + \cancel{\sqrt{2}(-i)} + \cancel{(-1)} + 0 + \cancel{(-1)(1)} + \cancel{(-\sqrt{2})(-i)}$$

$$+ \cancel{\sqrt{2}(-1/\sqrt{2})(-1)} = 0$$

$$x[3] = \sum_{n=0}^{7} \sqrt{2}\cdot \sin((n+1)\pi/4)\cdot e^{-3j\pi n/4}$$

$$= 1 + \sqrt{2}(-1/\sqrt{2} - j/\sqrt{2}) + i + 0 - (1)(-1)$$

$$+ \sqrt{2}(-1)(1/\sqrt{2} + j/\sqrt{2}) - (-i) = 0$$

$$x[4] = \sum_{n=0}^{7} \sqrt{2}\cdot \sin((n+1)\pi/4)\cdot e^{-j\pi(n)}$$

$$= \sum_{n=0}^{7} (-1)^n \cdot \sqrt{2}\cdot \sin((n+1)\pi/4)$$

$$= \sqrt{2}\cdot\left[+\cancel{1/\sqrt{2}} + \cancel{1(-1)} + \cancel{1/\sqrt{2}} + 0 - \cancel{1/\sqrt{2}} + \cancel{1} - \cancel{1/\sqrt{2}} + 0\right]$$

$$= 0$$

$$x[5] = \sum_{n=0}^{7} \sqrt{2}\cdot \sin((n+1)\pi/4)\cdot e^{-5j\pi n/4}$$

$$= 1 + \sqrt{2}(-1/\sqrt{2} + j/\sqrt{2}) + (-i) + 0 + 1 - \sqrt{2}\left(\frac{1}{\sqrt{2}} - j/\sqrt{2}\right)$$

$$-i = 0$$

$$x[6] = \sum_{n=0}^{7} \sqrt{2}\cdot \sin\left((n+1)\pi/4\right)\cdot e^{(\pi \cdot j n/2)}$$

$$= 1 + \sqrt{2}(i) - 1 + 0 - 1 + (-\sqrt{2})(i) + 1 = 0$$

$$x[7] = \sum_{n=0}^{7} \sqrt{2}\cdot \sin\left((n+1)\pi/4\right)\cdot e^{j\pi n/4}$$

$$= \sqrt{2}\cdot(1/\sqrt{2}) + \sqrt{2}(1/\sqrt{2} + i/\sqrt{2}) + \sqrt{2}\left(\tfrac{1}{\sqrt{2}}\right)(i)$$

$$+ 0 + \sqrt{2}\cdot(-1/\sqrt{2})\cdot(-1) + \sqrt{2}(-1)\left(\tfrac{-1}{\sqrt{2}} - \tfrac{j}{\sqrt{2}}\right)$$

$$+ \sqrt{2}(-1/\sqrt{2})(-i) + 0$$

$$= 4 + 4i$$

(c) $x[n] = \{1, -1-j, -1, -1+j\}$

$N=8$

$$x[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j\left(\frac{2\pi}{N}\right)kn}$$

$$= \sum_{n=0}^{N-1} x[n] \cdot e^{-jkn\pi/4}$$

$$x[k] = 1 + (-1-j) \cdot e^{-jk\pi/4} + (-1) \cdot e^{-jk\pi/2} + (-1+j) \cdot e^{-3kj\pi/4}$$

$x[0] = -2$

$x[1] = 1+j$

$x[2] = 0$

$x[3] = 1 + (2\sqrt{2}-1)j$

$x[4] = 2$

$x[5] = 1+j$

$x[6] = 4$

$x[7] = 1 - (2\sqrt{2}+1)j$

$$\boxed{x[k] = \{-2, 1+j, 0, 1+(2\sqrt{2}-1)j, 2, 1+j, 4, 1-(2\sqrt{2}+1)j\}}$$

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Calculations

$$x[0] = 1 + (-1-j) + (-1) + (-1+j) = -2$$

$$x[1] = 1 + (-1-j)\cdot\left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}} j\right) + (-1)\cdot(-i) + (-1+j)\cdot\left(-\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}} i\right)$$

$$= 1 - \frac{1}{\cancel{\sqrt{2}}}\cancel{(2)} + i + \frac{1}{\cancel{\sqrt{2}}}\cancel{(2)} = 1 + i$$

$$x[2] = 1 + (-1-j)\cdot(-i) + (-1)(-1) + (-1+j)(i)$$

$$= \cancel{1} + i \cancel{-1} \cancel{+1} - i \cancel{-1} = 0$$

$$x[3] = 1 + (-1-j)\cdot\left(-\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}} i\right) + (-1)(i) + (-1+j)\cdot\left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}} j\right)$$

$$= 1 + \frac{1}{\sqrt{2}}(1+i)^2 - i - \frac{1}{\sqrt{2}}(1-j)^2$$

$$= 1 + \frac{1}{\sqrt{2}}(2i) - i - \frac{1}{\sqrt{2}}(-2i)$$

$$= 1 + (2\sqrt{2}-1)i$$

$$x[4] = 1 + (-1-j)\cdot(i^2) + (-1) + (-1+j)(i^2)$$

$$= \cancel{1} - i^2 \cancel{+1} \cancel{-1} - i^2 \cancel{-1} = 2$$

$$x[5] = 1 + (-1-j)\cdot\left(\frac{-1}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right) + (-1)(-i) + (-1+j)\left(\frac{1+j}{\sqrt{2}}\right)$$

$$= 1 + \frac{1}{\cancel{\sqrt{2}}}\cancel{(2)} + i - \frac{1}{\cancel{\sqrt{2}}}\cancel{(2)} = 1 + i$$

$$x[6] = 1 + (-1-j)(-1) + (-1)\cdot 1 + (-1+j)(-1)$$

$$= 4$$

$$x[7] = 1 + (-1-j)\cdot\left(\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right) + (-1)\cdot(i) + (-1+j)\left(\frac{-1}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right)$$

$$= 1 - \frac{1}{\sqrt{2}}(2i) - i - \frac{1}{\sqrt{2}}(2i) = 1 - (2\sqrt{2}+1)i$$

(d) $x[n] = \{1,1,1,1,1,1,1,1\}$

$$x[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j\left(\frac{2\pi}{N}\right)kn} \qquad N=8$$

$$= \sum_{n=0}^{N-1} e^{-j\pi kn/4} = \sum_{n=0}^{7} e^{-j\pi kn/4}$$

$x[0] = 8$

$x[1] = 0$

$x[2] = 0$

$x[3] = 0$

$x[4] = 0$

$x[5] = 0$

$x[6] = 0$

$x[7] = 0$

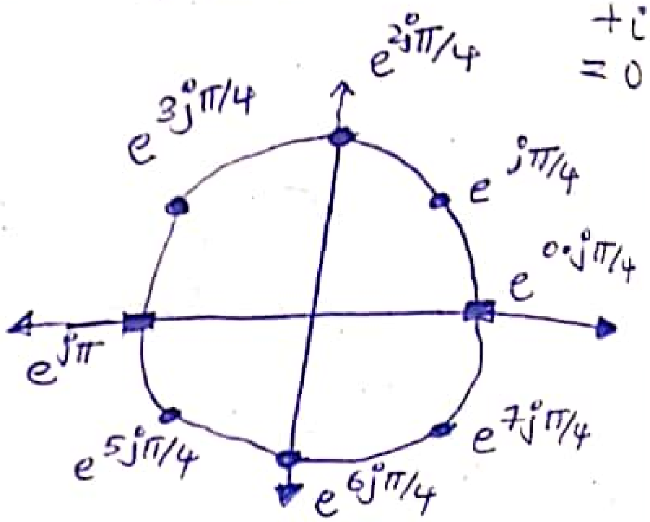
$$\boxed{x[k] = \{8,0,0,0,0,0,0,0\}}$$

Calculations

$$x[0] = \sum_{n=0}^{7} 1 = 8$$

$$x[1] = \sum_{n=0}^{7} e^{-j\pi n/4} = \sum_{n=0}^{7} e^{j\pi n/4} = 0$$

$$x[2] = \sum_{n=0}^{7} e^{-j\pi n/2} = \sum_{n=0}^{7} e^{3j\pi n/2} = 1 - i - 1 + i + 1 - i - 1 + i = 0$$



$$x[3] = \sum_{n=0}^{7} e^{-3j\pi n/2}$$

$$= 1 + i - 1 - i + 1 + i - 1 - i = 0$$

$$e^{3j\pi/4} + e^{7j\pi/4} = 0 \quad (1)$$

$$e^{j\pi/4} + e^{5j\pi/4} = 0 \quad (2)$$

$$e^{0} + e^{j\pi} = 0 \quad (3)$$

$$e^{2j\pi/4} + e^{6j\pi/4} = 0 \quad (4)$$

$$x[4] = \sum_{n=0}^{7} e^{-j\pi n}$$

$$= \cancel{1} - \cancel{1} + \cancel{1} - \cancel{1} + \cancel{1} - \cancel{1} + \cancel{1} - \cancel{1} = 0$$

$$x[5] = \sum_{n=0}^{7} e^{-5j\pi n/4}$$

$$= \sum_{n=0}^{7} e^{3j\pi n/4} = \sum_{n=0}^{7} e^{j\pi n/4}$$

$$= 0$$

Adding eqn's (1), (2), (3), (4)

$$\boxed{\sum_{n=0}^{7} e^{j\pi n/4} = 0}$$

$$x[6] = \sum_{n=0}^{7} e^{-3j\pi/2 \cdot n} = 1 + i - 1 - i + 1 + i - i - 1 = 0$$

$$x[7] = \sum_{n=0}^{7} e^{-7j\pi n/4} = \sum_{n=0}^{7} e^{j\pi n/4} = 0$$

2

a

$$X(e^{jw}) = \cos^3 w + \cos^2 w$$

$$= \left(\frac{\cos 3w + 3\cos w}{4}\right) + \left(\frac{1+\cos 2w}{2}\right)$$

$\cos 3w = 4\cos^3 w - 3\cos w$

$$= \frac{3}{4}\left(\frac{e^{jw}+e^{-jw}}{2}\right) + \frac{1}{4}\left(\frac{e^{3jw}+e^{-3jw}}{2}\right) + \frac{1}{2} + \frac{1}{2}\left(\frac{e^{2jw}+e^{-2jw}}{2}\right)$$

$$x(t) = \frac{\delta(t)}{2} + \frac{3}{4}\left(\frac{\delta(t+1)+\delta(t-1)}{2}\right) + 1/4\left(\frac{\delta(t+3)+\delta(t-3)}{2}\right) + \frac{1}{2}\left(\frac{\delta(t+2)+\delta(t-2)}{2}\right)$$

$$= \frac{1}{2}\delta(t) + \frac{3}{8}\delta(t+1) + \frac{3}{8}\delta(t-1) + \frac{1}{8}\delta(t+3) + \frac{1}{8}\delta(t-3) + \frac{1}{4}\delta(t+2) + \frac{1}{4}\delta(t-2)$$

where $\delta(n)$ is the Kronecker delta function which equals 1 when $n=0$ and 0 otherwise

$$\int \cos(aw) \cdot e^{jnw}\, dw = \frac{1}{2}(\delta(n-a) + \delta(n+a))$$

where a is +ve positive integer

(b) $$X(e^{j\omega}) = \frac{e^{-4j\omega} + e^{-3j\omega} - e^{-j\omega} - 1}{1+e^{-j\omega}}$$

$$= \frac{e^{-3j\omega}(1+e^{-j\omega})}{1+e^{-j\omega}} - \frac{(1+e^{-j\omega})}{1+e^{-j\omega}}$$

$$= e^{-3j\omega} - 1 = e^{-3j\omega} - e^{0j\omega}$$

$$\boxed{x(t) = \delta(t-3) - \delta(t)}$$

where $\delta(n)$ is the kronecker delta function which equals 1 when $n=0$ & 0 otherwise

(c) $$X(e^{j\omega}) = \frac{3e^{-j\omega} - 1}{3 - e^{-j\omega}} = e^{-j\omega}\left(\frac{1}{1-\frac{1}{3}e^{-j\omega}}\right) - \left(\frac{1}{3}\right)\left(\frac{1}{1-\frac{e^{-j\omega}}{3}}\right)$$

$$x[A-n_0] \longleftrightarrow e^{-j\omega n_0} X(e^{j\omega})$$

$$\alpha^n u[n] \longleftrightarrow \frac{1}{1-\alpha e^{-j\omega}}$$

$$x[n] = \left(\frac{1}{3}\right)^{n-1} u[n-1] - \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^n u[n]$$

$$\boxed{x[n] = \left(\frac{1}{3}\right)^{n-1} u[n-1] - \left(\frac{1}{3}\right)^{n+1} u[n]}$$

③ Given $X[k] = k^2$, $0 \le k \le 7$ be a 8-point dft of sequence $x[n]$

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j\left(\frac{2\pi kn}{N}\right)}$$ N=8

$$= \sum_{n=0}^{N-1} x[n] \cdot e^{-j\pi kn/4}$$

$$X[0] = 0 = x[0] + x[1] + x[2] + \cdots + x[7]$$

$$X[4] = 16 = x[0] - x[1] + x[2] + \cdots - x[7]$$

$$2 \cdot \left[x[1] + x[3] + x[5] + x[7]\right) = -16$$

$$x[1] + x[3] + x[5] + x[7] = -8$$

$$\Rightarrow \boxed{\sum_{n=0}^{3} x[2n+1] = -8}$$