Digital Signal Analysis
Assignment -3

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1) Given,

$x(n) = \{1,1,1,1\}$ and

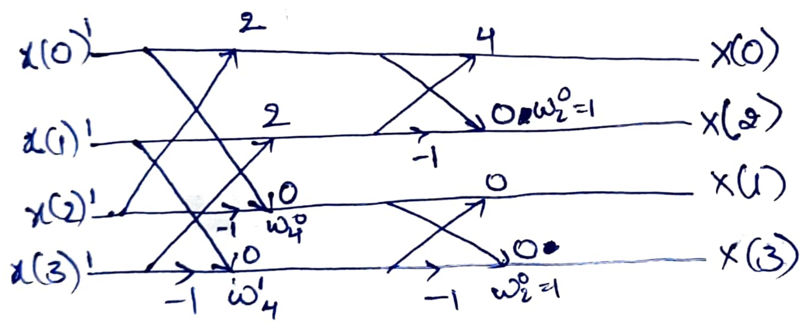
$h(n) = \{1,0,1,0\}$ and let circular convolution of $x(n)$ & $h(n)$ be $y(n)$

then, we know that

$$y(n) = x(n) \circledast h(n) \quad \text{and}$$

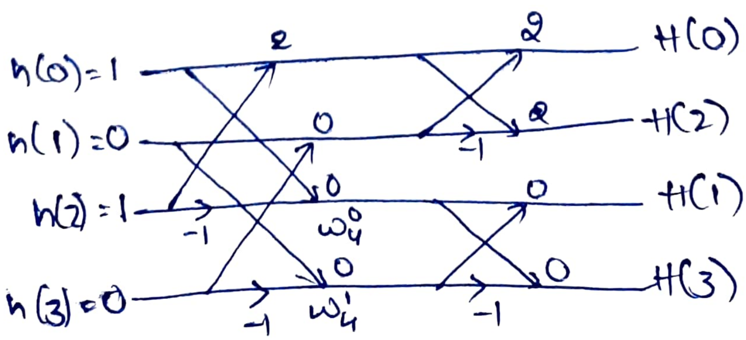
$$Y(k) = X(k) \cdot H(k) \quad \text{---} \quad (1)$$

Now, we can find $X(k)$ using fft as



$\Rightarrow \quad X(k) = \{4,0,0,0\}$

Now, applying FFT on $h(n)$ we get $H(k)$ as



$\Rightarrow H(k) = \{2,0,2,0\}$

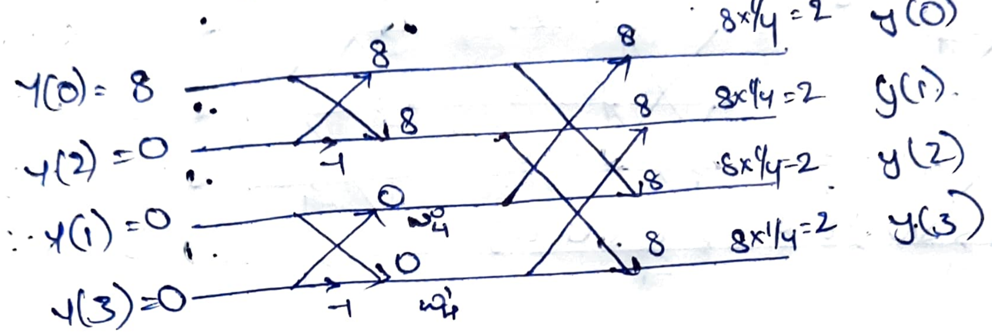
Now, from Eq ①

$$Y(k) = X(k) \cdot H(k)$$

$$\Rightarrow Y(k) = \{4,0,0,0\} \cdot \{2,0,2,0\}$$

$$\Rightarrow Y(k) = \{8,0,0,0\}$$

Now, we can get $y(n)$ by calculating the inverse as



$$\Rightarrow y(n) = \{2,2,2,2\}$$

Therefore, circular convolution of $x(n)$ & $h(n)$ is $\{2,2,2,2\}$

Q Given 8-point sequences is

$$x(n) = \{1, 2, -1, 2, 4, 2, -1, 2\}.$$

required to find $X(k)$ using decimation.

Now,

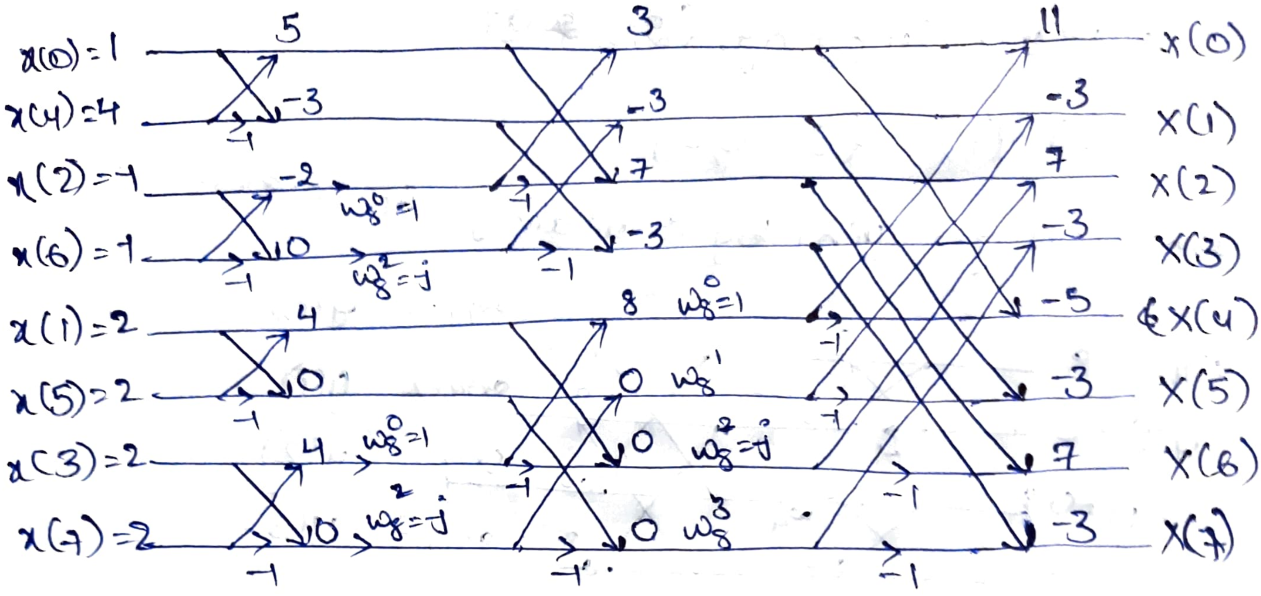
$$W_8^0 = 1$$

$$W_8^1 = e^{-j \cdot \frac{2 \cdot 1 \cdot \pi}{8}} = 1/\sqrt{2} - \frac{j}{\sqrt{2}}$$

$$W_8^2 = e^{-j \cdot \frac{2 \cdot 2 \cdot \pi}{8}} = -j$$

$$W_8^3 = e^{-j \cdot \frac{2 \cdot 3 \cdot \pi}{8}} = -1/\sqrt{2} - \frac{j}{\sqrt{2}}$$

$$W_8^4 = e^{-j\pi} = -1$$



Explanation:

$$X(0) = (x(0)+x(4)) + w_8^0(x(2)+x(6)) + w_8^0(x(1)+x(5) + w_8^0(x(3)+x(7)))$$

$$= (1+4) + 1(-1-1) + 2 + 2 + 2 + 2$$

$$X(0) = 11$$

$$X(1) = (x(0)-x(4)) + w_8^2(x(2)-x(6)) + w_8^1(x(1)-x(5) + w_8^2(x(3)-x(7)))$$

$$= (1-4) + (-j)(0) + w_8^1(0 + w_8^2(2-2))$$

$$X(1) = -3$$

$$X(2) = x(0)+x(4) - w_8^0(x(2)+x(6)) + w_8^2(x(1)+x(5) - w_8^0(x(3)+x(7)))$$

$$= 1+4 - 1(-1-1) + w_8^2(2+2-2-2)$$

$$X(2) = 7$$

$$X(3) = (x(0)-x(4)) - w_8^2(x(2)-x(6)) + w_8^3(x(1)-x(5) - w_8^2(x(3)-x(7)))$$

$$= 1-4 - w_8^2(-1+1) + w_8^3(2-2 - w_8^2(2-2))$$

$$X(3) = -3$$

$$X(4) = (x(0)+x(4)) + \omega_8^0(x(2)+x(6)) - \omega_8^0(x(1)+x(5) + \omega_8^0(x(3)+x(7)))$$

$$= (1+4) + (-1-1) - (2+2+2+2)$$

$$X(4) = -5$$

$$X(5) = (x(0)-x(4)) + \omega_8^2(x(2)-x(6)) - \omega_8^1(x(1)-x(5) + \omega_8^2(x(3)-x(7)))$$

$$= (1-4) + (-j)(0) - \omega_8^1(0 + \omega_8^2(2-2))$$

$$X(5) = -3$$

$$X(6) = x(0)+x(4) - \omega_8^2((x(2)+x(6)) - \omega_8^2(x(1)+x(5) - \omega_8^0(x(3)+x(7)))$$

$$= 1+4 - 1(-1-1) - \omega_8^2(2+2-2-2)$$

$$\Rightarrow X(6) = 7$$

$$X(7) = (x(0) - x(4)) - \omega_8^2(x(2)-x(6)) - \omega_8^3(x(1)-x(5) - \omega_8^2(x(3)-x(7)))$$

$$= 1-4 - \omega_8^2(-1+1) - \omega_8^3(2-2-\omega_8^2(2-2))$$

$$\Rightarrow X(7) = -3$$

$$\therefore X[k] = \{11, -3, 7, -3, -5, -3, 7, -3\}$$

3. i) Given,

$$x(n) = (1/4)^n u(n)$$

$$\Rightarrow \text{z transform of } x(n) = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^n u(n) z^{-n}$$

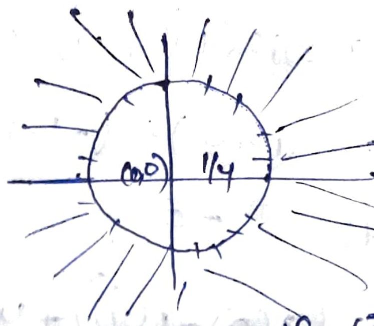
$$= \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \cdot 1 \cdot z^{-n} \quad \left(\because u(n) = 0 \text{ if } n<0,\ u(n) = 1 \text{ if } n \geq 0\right)$$

$$X(z) = \frac{1}{1 - 1/4z} \quad \left(\text{Sum of Infinite G.P}\right)$$

as $X(z)$ is finite value, $\left|\frac{1}{4z}\right| < 1$

$\Rightarrow |z| > 1/4$

$\Rightarrow$ ROC : $|z| > 1/4$



exterior of the circle.

ii, Given, $x(n) = (5(2^n) - 4(3^n))\,u(n)$

$\Rightarrow$ z transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} (5(2^n) - 4(3^n))\,u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} (5(2^n) - 4(3^n)) \cdot 1 \cdot z^{-n} \quad \left(\because u(n) = \begin{cases} 0 & n<0 \\ 1 & n \geq 1 \end{cases} \right.$$

$$= \sum_{n=0}^{\infty} 5(2z^{-1})^n - \sum_{n=0}^{\infty} 4 \cdot \left(\frac{3}{z}\right)^n$$

(Sum of infinite G.P's)

$$X(z) = 5 \cdot \frac{1}{1 - 2/z} - \frac{4}{1 - 3/z}$$

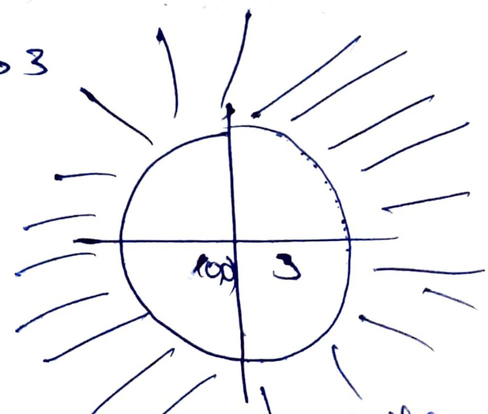
for $X(z)$ to be finite the common ratio's $\left(\frac{2}{z}, \frac{3}{z}\right)$ absolute value should be less than 1.

$\Rightarrow \left|\frac{2}{z}\right| < 1$ and $\left|\frac{3}{z}\right| < 1$

$\Rightarrow |z| > 2$ and $|z| > 3$

$\Rightarrow \boxed{|z| > 3}$

$\Rightarrow$ ROC: $|z| > 3$



exterior of the circle

iii) Given,

$x(n) = n a^n u(n)$

we know that, According to differentiation property

if $x(n) \rightarrow X(z)$ then

$n x(n) \rightarrow -z \frac{d}{dz}(X(z))$ --- ①

Now, let $y(n) = a^n u(n)$

then z-transform, $Y(z) = \sum_{n=-\infty}^{\infty} y(n) z^{-n}$

$= \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$

$= \sum_{n=0}^{\infty} a^n \cdot 1 \cdot z^{-n}$ ($\because u(n)$ is unit step)

$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$ (sum of infinite G.P)

$\Rightarrow Y(z) = \frac{1}{1-a/z} = \frac{z}{z-a}$ (here $\left|\frac{a}{z}\right| < 1$) --- ②

here $x(n) = n \cdot y(n) = n \cdot a^n \cdot u(n)$.

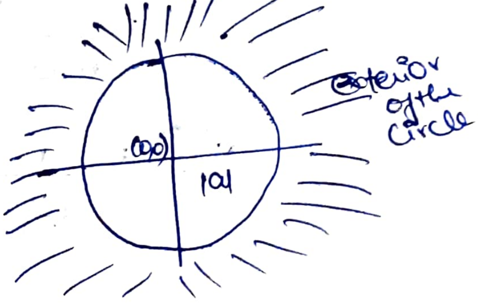
Now from Eq ①,

$X(z) = -z \cdot \frac{d}{dz}\left(\frac{z}{z-a}\right)$

$= -z \cdot \frac{(\cancel{z}-a) - \cancel{z}}{(z-a)^2} = \frac{az}{(z-a)^2}$

Here ROC: $\left|\frac{a}{z}\right| < 1$ (from ②)

$\Rightarrow \boxed{|z| > |a|}$



iv) Given,

$$x(n) = \{\underset{-2}{3}, \underset{-1}{4}, \underset{\uparrow}{\overset{0}{8}}, \overset{1}{7}, \overset{2}{0}, \overset{3}{4}\}$$

$\Rightarrow$ ~~z~~ transform $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$.

$$= 0 + \sum_{n=-2}^{3} x(n) z^{-n} + 0.$$

$$= 3 \cdot z^2 + 4 \cdot z^1 + 8 + 7 \cdot z^{-1} + 0 \cdot z^{-2} + 4z^{-3}$$

$$= 3z^2 + 4z + 8 + \frac{7}{z} + \frac{4}{z^3}$$

$\Rightarrow$ R.O.C : $\mathbb{C} - \{0\} - \{\infty\}$

$\{\because X(z)$ is finite for all ~~v~~alues of z expect 0 & $\infty\}$

v) Given,

$$x(n) = a^n u(n) + b^n u(-n-2)$$

z transform

$$X(z) = \sum_{n=-\infty}^{\infty} \left(a^n u(n) + b^n u(-n-2)\right) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} + \sum_{n=-\infty}^{\infty} b^n u(-n-2) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n \underbrace{u(n)}_{=1} z^{-n} + \sum_{n=-\infty}^{-2} b^n \cdot 1 \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=-\infty}^{-2} b^n z^{-n}$$

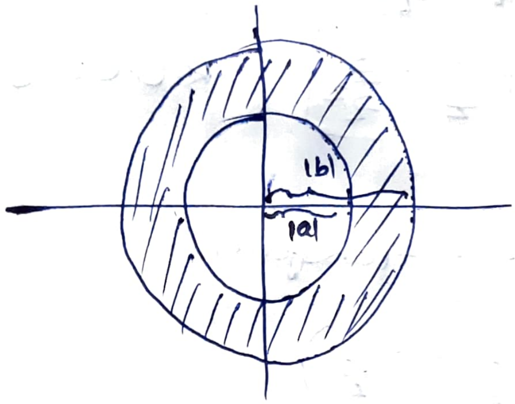
$$= \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{t=2}^{\infty} \frac{z^t}{b^t} \quad (\text{here } t=-n) \quad (\text{sum of infinite G.P.S})$$

$$X(z) = \frac{1}{1-(a/z)} + \frac{z^2}{b^2\left(1-(z/b)\right)} \quad \left\{ \boxed{\frac{z}{z-a} + \frac{z^2}{b(b-z)} = X(z)} \right.$$

here for $X(z)$ to finite $\left|\frac{a}{z}\right|<1$ and $\left|\frac{z}{b}\right|<1$

$\Rightarrow$ $|a|<|z|$ and $|z|>|b|$

$\Rightarrow$ R.O.C: $|a|<|z|<|b|$ (assuming $|b|>|a|$)



4) Given,

two sequences

$x_1(n) = 2\delta(n) - \delta(n-1)$ and

$x_2(n) = 4\delta(n) + 3\delta(n-1)$,

$X(z) = Z(x_1(n) * x_2(n))$ and $x(n) = x_1(n) * x_2(n)$

(i)

we know that,

if $x(n) \Rightarrow x_1(n) * x_2(n)$ then

$X(z) \rightarrow X_1(z) \cdot X_2(z)$.

$\Rightarrow$ Now we need to find $X_1(z)$ and $X_2(z)$

① $X_1(z) = \sum_{n=-\infty}^{\infty} (2\delta(n) - \delta(n-1)) z^{-n}$ $\left(\because X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}\right)$

$= (2(1) - 0) z^0 + (-1) z^{-1} = 2 - 1/z$.

② $X_2(z) = \sum_{n=-\infty}^{\infty} (4\delta(n) + 3\delta(n-1)) z^{-n}$ &

$= 4 \cdot z^0 + 3 \cdot z^{-1} = 4 + \frac{3}{z}$.

Here R.O.Z of $X_1(z)$ and $X_2(z)$ is $C - \{0\}$ ($\because$ expect 0 they take a finite value)

$\Rightarrow X(z) = X_1(z) \cdot X_2(z)$

$= (2 - 1/z) \cdot (4 + 3/z)$

$\Rightarrow X(z) = 8 - \frac{4}{z} + \frac{6}{z} - \frac{3}{z^2}$

$\Rightarrow$ (a) $\boxed{X(z) = 8 + \frac{2}{z} - \frac{3}{z^2}}$ R.O.C $= C - \{0\}$.

(ii) Now using inverse z-transform

$x(n) = \mathcal{I}z\left(8 + \frac{2}{z} - 3/z^2\right)$

$= 8\delta(n) + 2\delta(n-1) - 3\delta(n-2)$

($\because$ inverse z-transform of '1' is $\delta(n)$ and by applying shift we can get the other)

$\therefore \boxed{x(n) = 8\delta(n) + 2\delta(n-1) - 3\delta(n-2)}$

5. Given,

transfer function $H(z) = \frac{z+1}{z-0.5}$

$= \frac{z}{z-0.5} + \frac{1}{z-0.5}$ (dividing numerator & denominator by z we get)

$= \frac{1}{1-\frac{1}{2z}} + \frac{z^{-1}}{1-1/2z}$.

$= \left(1 + \frac{1}{2z} + \frac{1}{(2z)^2} + \cdots \infty\right) + 2\cdot\left(\frac{1}{2z} + \frac{1}{(2z)^2} + \cdots + \infty\right)$

$= \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n + 2\cdot\sum_{n=1}^{\infty}\left(\frac{1}{2z}\right)^n$ where $(|1/2z| < 1)$

$$= \sum_{n=-\infty}^{\infty} 2^{-n} \cdot u(n) z^{-n} + 2 \cdot \sum_{n=-\infty}^{\infty} 2^{-n} \cdot u(n-1) \cdot z^{-n}$$

$$H(z) = \sum_{n=-\infty}^{\infty} \left(2^{-n} \cdot u(n) + 2^{-n+1} \cdot u(n-1) \right) z^{-n}$$

i)

$$\Rightarrow \boxed{h(n) = 2^{-n} u(n) + 2^{-n+1} u(n-1)}$$

ii) Given, $y(n)$ be the step response

$$\Rightarrow y(n) = u(n) * h(n)$$

$$\Rightarrow Y(z) = U(z) \cdot H(z)$$

Now. $U(z) = \sum_{n=-\infty}^{\infty} u(n) \cdot z^{-n}$

$$= \sum_{n \geq 0}^{\infty} z^{-n} = \frac{1}{1-1/z}$$

$$\Rightarrow Y(z) = \frac{1}{1-1/z} \times \frac{z+1}{z-0.5}$$

$$\Rightarrow Y(z) = \frac{z \cdot z+1}{(z-1)(z-0.5)}$$

$$\Rightarrow \frac{Y(z)}{z} = \frac{z+1}{(z-1)(z-0.5)}$$

On splitting into partial fractions we get

$$\Rightarrow \frac{Y(z)}{z} = \frac{4}{z-1} - \frac{3}{z-0.5}$$

$$\Rightarrow Y(z) = 4\left(\frac{z}{z-1}\right) - 3\left(\frac{z}{z-0.5}\right)$$

$$\Rightarrow Y(z) = 4\left(\frac{1}{1-1/z}\right) - 3\left(\frac{1}{1-1/2z}\right)$$

$$\Rightarrow Y(z) = 4\left(1 + 1/z + 1/z^2 + \cdots\right) - 3\left(1 + \frac{1}{2z} + \frac{1}{(2z)^2} + \cdots \infty\right)$$

(where $|z| > 1$)

$$Y(z) = 4 \cdot \sum_{n=0}^{\infty} z^{-n} - 3 \sum_{n=0}^{\infty} 2^{-n} \cdot z^{-n}$$

$$Y(z) = \sum_{n=\infty}^{\infty} \left(4 \cdot u(n) - 3 \cdot 2^{-n} u(n)\right) z^{-n}$$

$$\left(\because Y(z) = \sum_{n=-\infty}^{\infty} y(n) z^{-n} \right)$$

$$\Rightarrow \boxed{y(n) = 4 \cdot u(n) - 3 \cdot 2^{-n} u(n)}$$

(iii) Given,

$$x(n) = (0.2)^n \cdot u(n) = \frac{1}{5^n} u(n)$$

$$\Rightarrow \text{z transform } X(z) = \frac{1}{1 - 1/5z}$$

now, $y(n) = x(n) * h(n)$

$$\Rightarrow Y(z) = X(z) \cdot H(z)$$

$$\Rightarrow Y(z) = \frac{1}{1 - 1/5z} \times \frac{z+1}{z - 0.5}$$

$$\cancel{\Rightarrow \frac{Y(z)}{5z} = \frac{5z+1}{(5z-1)(z-0.5)}} \quad \Rightarrow \frac{Y(z)}{z} = \frac{z+1}{(z-0.2)(z-0.5)}$$

On solving partial fractions we get

$$\Rightarrow \frac{Y(z)}{z} = \frac{-4}{z-0.2} + \frac{5}{z-0.5}$$

$$\Rightarrow \frac{Y(z)}{z} = \frac{-4}{z-0.2} + \frac{5}{z-0.5}$$

$$\Rightarrow Y(z) = \frac{-4}{1-0.2z^{-1}} + \frac{5}{1-0.5z^{-1}}$$

$$\Rightarrow \boxed{y(n) = -4 \cdot 5^{-n} u(n) + 5 \cdot 2^{-n} \cdot u(n)}$$

$$\left(\because \frac{1}{1-0.2z^{-1}} \text{ is z transform of } 5^{-n} u(n) \text{ — \&} \quad \frac{1}{1-0.5z^{-1}} \text{ is z transform of } 2^{-n} u(n) \right)$$

6, Given, shift invariant system is

$$y(n) = 0.2x(n) + x(n-1) + 0.3x(n-3) + 0.5\,x(n-4)$$

Now applying z transform we get

$$Y(z) = 0.2X(z) + z^{-1}X(z) + 0.3\cdot z^{-3}X(z) + 0.5\,z^{-4}X(z)$$

($\because$ by shift property)
if $x(n) \to X(z)$ then
$x(n-k) \to z^{-k}X(z)$

$$\Rightarrow \quad Y(z) = X(z)\left(\frac{1}{5} + z^{-1} + 0.3z^{-3} + 0.5z^{-4}\right) \quad \text{--} (1)$$

we know that,

$$Y(z) = X(z)\cdot H(z)$$

$$\Rightarrow \quad H(z) = \frac{Y(z)}{X(z)} \quad (\text{from } (1))$$

$$\Rightarrow \quad H(z) = 0.2 + z^{-1} + 0.3\,z^{-3} + 0.5\,z^{-4}$$

by applying inverse

$$\Rightarrow \quad h(n) = 0.2\,\delta(n) + \delta(n-1) + 0.3\,\delta(n-3) + 0.5\,\delta(n-4)$$

sketch: