

Digital Signal Analysis

* Dirichlet conditions:

- signal should have finite number of maxima & minima over the range of time period
- signal should have finite number of discontinuities over the range of time period
- signal should be absolutely integrable over the time period
- signal is periodic, single-valued, finite & piece-wise continuous

* Trigonometric Fourier Series

$$T = 2\pi/\omega_0$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) dt$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos(n\omega_0 t) dt$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin(n\omega_0 t) dt$$

* Exponential Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} F_n \cdot e^{j(n\omega_0 t)}$$

$$F_n = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) \cdot e^{-j(n\omega_0 t)} dt$$

$$* \quad X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt \rightarrow \text{Fourier Transform}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \cdot e^{j\omega t} d\omega \rightarrow \text{Inverse Fourier Transform}$$

* DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \cdot e^{j\omega n} d\omega$$

* DFT

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j \frac{2\pi n k}{N}}$$

$$X(j\omega) = X_R(j\omega) + j \cdot X_I(j\omega)$$

$$|X(\omega)| = \sqrt{X_R^2 + X_I^2}$$

$$X(\omega) = |X(\omega)| \cdot e^{-j\phi}$$

$$\text{phase } \phi = \tan^{-1} \left(\frac{X_I(\omega)}{X_R(\omega)} \right)$$

$\rightarrow X(e^{j\omega})$ is periodic version of $X(\omega)$ with period dependent on sampling rate

$$\Rightarrow x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) d\tau$$

$$\downarrow \text{FT}$$

$$X(j\omega) \cdot H(j\omega)$$

\Rightarrow Odd Fn: $f(-x) = -f(x)$ holds $\forall x \in D_f$

* Periodic — Fourier series

Non-periodic — Fourier Transform $\leftrightarrow X(\omega)$

$$\downarrow$$

$$\text{DTFT} \rightarrow X(e^{j\omega})$$

$$\downarrow$$

$$\text{DFT} \rightarrow X(k)$$

* Duality: $f(t) \rightarrow F(\omega)$
 $F(t) \rightarrow 2\pi f(-\omega)$

* Linearity: $ax_1(t) + bx_2(t) \rightarrow aX_1(\omega) + bX_2(\omega)$
 Homogeneity \swarrow Superposition \searrow

* Time-shifting:
 $x(t) \rightarrow X(\omega)$
 $x(t-t_1) \rightarrow X(\omega) \cdot e^{-j\omega t_1}$

* Scaling: $x(t) \rightarrow X(\omega)$
 $x(at) \rightarrow \frac{1}{|a|} X(\omega/a)$ (compression in 1 domain leads to expansion in other)

* $x(t) \rightarrow X(\omega)$
 $x(-t) \rightarrow X(-\omega)$

* $d/dt x(t) \rightarrow (j\omega) X(\omega)$ when $x(t) \rightarrow X(\omega)$

* Parseval's Theorem:

$$\int_{-\infty}^{\infty} x(t) \cdot \overline{x(t)} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot \overline{X(\omega)} d\omega$$

$$\Rightarrow \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

* $N \geq L$ → length of sequence
 N-point DFT

$$W_N = e^{-j(2\pi/N)}$$

* Decimation in Frequency

$$X(2m) = \sum_{n=0}^{N/2-1} (x(n) + x(n+N/2)) W_N^{mn}$$

$$X(2m+1) = \sum_{n=0}^{N/2-1} (x(n) - x(n+N/2)) \cdot W_N^{mn} \cdot W_N^n$$

* Nyquist Theorem: periodic signal must be sampled at more than twice the highest frequency component of the signal.

$$f_s \geq 2f_m$$

$$X_d(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(e^{j(\omega - k\omega_0)})$$

* Uniform-Quantizer:

$$\text{step-size} = \frac{V_{\max} - V_{\min}}{2^N}$$

$$\text{Max-quantization error} = \frac{V_{\max} - V_{\min}}{2^{N+1}}$$

* Energy of signal: $\sum_{n=-\infty}^{\infty} |x(n)|^2$

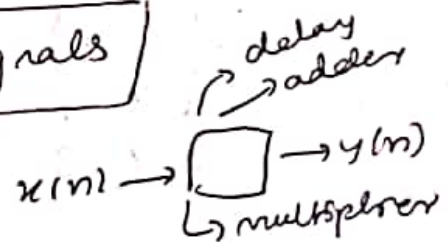
* Power of signal: $\lim_{N \rightarrow \infty} \frac{1}{2N+1} \left(\sum_{n=-N}^N |x(n)|^2 \right)$

* Energy signal: Energy = finite, power = 0

* Power signal: power = finite, energy = ∞

Periodic signals are power signals

* Types of Discrete systems:



→ Linear:

$$ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$$

→ Time-invariance:

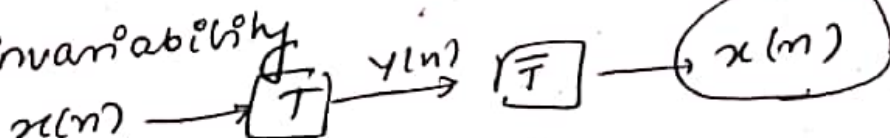
$$x(n) \rightarrow y(n)$$

$$x(n-d) \rightarrow y(n-d)$$

→ Stable-systems

$$x(n) \leq |M_x|, y(n) \leq |M_y|$$

→ Invariability



→ Causal

y(n)/output depends only on present, past inputs and past outputs

* Bit-rate: $f_s \times N$

* Memory-less: depends only on $x(n)$

*
$$x_1(n) * x_2(n) = \sum_{m=-\infty}^{\infty} x_1(m) \cdot x_2(n-m)$$

* Linear convolution:

$\{1, 2, 3, 4\} * \{4, 5, 6\} =$

	1	2	3	4
4	4	8	12	16
5	-	-	-	-
6	-	-	-	-

$\|y(n)\|$

$= \|x(n)\| + \|h(n)\| - 1$

* Circular convolution:

$\{1, 2, 3, 4\} * \{4, 5, 6\}$

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \\ 0 \end{bmatrix} =$$

* Circular convolution \longrightarrow Linear convolution

$\{1, 2, 3, 4, 0, 0\}$

* $\{4, 5, 6, 0, 0, 0\}$