1) Rational form of z-transform

2) Zeros

3) Poles

4) IIR (Infinite Impulse Response)

5) FIR (Finite Impulse Response)

$$x(n) \longrightarrow \boxed{\text{system}} \longrightarrow y(n)$$

$$y(n) = x(n) + 2x(n-1) + 3x(n-2) + y(n-1) + 2y(n-2)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

$$Y(z) = X(z) + 2z^{-1}X(z) + 3z^{-2}X(z) + z^{-1}Y(z) + 2z^{-2}Y(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1} + 3z^{-2}}{1 - z^{-1} - 2z^{-2}} = \frac{z^2 + 2z + 3}{z^2 - z - 2}$$

roots of this eqn are poles

roots of this eqn are zeroes

<u>**LTI**</u>

$$y(n) = -\sum_{k=1}^{N} a_k \cdot y(n-k) + \sum_{k=0}^{M} b_k \cdot x(n-k)$$

$$Y(z) = -\sum_{k=1}^{N} a_k \cdot z^{-k} Y(z) + \sum_{k=0}^{M} b_k \cdot z^{-k} \cdot X(z)$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{k=0}^{M} b_k \cdot z^{k}}{1 + \sum_{k=1}^{N} a_k \cdot z^{-k}} \longrightarrow \text{IIR}$$

$$\downarrow$$

FIR if sum = 0

For LTI systems, if

$$\begin{tikzcd} x(n) \arrow[r] \arrow[d, Rightarrow] & \boxed{\begin{matrix} LTI \\ h(n) \end{matrix}} \arrow[r] & y(n) \arrow[d, Rightarrow] \\ \delta(n) & & h(n) \end{tikzcd}$$

then,

$$y(n) = x(n) * h(n)$$
$$= X(z) \cdot H(z)$$

$$h(n) \leftarrow H(z) = \frac{Y(z)}{X(z)}$$

$h(n)$
- IIR
- FIR → No poles

All zero → FIR
All pole → IIR
Both → IIR

→ Speech production as All-pole system
↳ key idea in Mobile phones

* Inverse z-transform

Fourier Transform

$$x(t) = \frac{1}{2\pi} \int X(\omega) \cdot e^{j\omega t} \, dt$$

→ But for z-transform, we can't integrate at all points (it may not exist at some points)

- ↳ Contour Integration
- ↳ Long division
- ↳ partial fraction
- ↳ Tabular method

Example:

$$X(z) = \frac{1+z^{-1}}{1-z^{-1}} , |z|>1$$

$$= \left(\frac{1}{1-z^{-1}}\right) + \left(\frac{z^{-1}}{1-z^{-1}}\right)$$

$$\rightarrow u(n) + u(n-1)$$

if $|z|<1$

$\downarrow$

$$-u(n(-1)-1)$$
$$-u(-n-2)$$

Example:

$$X(z) = \frac{1-z^{-1}-z^{-2}}{1+2z^{-1}+3z^{-2}}$$

$$= \frac{z^2-z-1}{z^2+2z+3}$$

$$\begin{array}{r|l} & 1-3z^{-1}+2z^{-2} \\ \hline z^2+2z+3 & z^2-z-1 \\ & z^2+2z+3 \\ \hline & -3z-4 \\ & -3z-6-9z^{-1} \\ \hline & 2+9z^{-1} \end{array}$$

$$x(n) = \{1, -3, 2, --\}$$

Example

$$x(z) = \frac{z^2 + 3z - 12}{}$$

$$x(z) = \frac{1 - z^{-1}}{(1 - \frac{1}{2} z^{-1})(1 - \frac{1}{5} z^{-1})}$$

$$= \frac{\left(\frac{-1}{3/6}\right)}{1 - \frac{1}{2} z^{-1}} + \frac{\left(\frac{1-6}{-3/2}\right)}{1 - \frac{1}{5} z^{-1}}$$

$$= -6/3 \left(\frac{1}{1 - \frac{1}{2} z^{-1}}\right) + \left(\frac{8/3}{1 - \frac{1}{5} z^{-1}}\right)$$

$2^n u(n)$

Example

$$X(z) = \frac{3z^4 + 31z^3 + 122z^2 + 219z + 159}{z^3 + 9z^2 + 27z + 27} \quad + 21z + 63 \quad + 96$$

$$= \frac{3z^4 + 9z^3 + 22z^3 + 66z^2 + 66z^2 + 198z}{(z+3)^3}$$

$$= \cancel{\frac{3z^3}{(z+3)^2} + \frac{22z^2}{(z+3)^2}} + \frac{66z}{(z+3)^2} + \frac{21}{(z+3)^2} + \frac{96}{(z+3)^3}$$

$$= \frac{3z^3 + 22z^2 + 66z + 21}{(z+3)^2} + \frac{96}{(z+3)^3}$$

$$= \frac{3z^3 + 9z^2 + 13z^2 + 39z + 27z + 21}{(z+3)^2}$$

$$= \frac{3z^2}{z+3} + \frac{13z+27}{(z+3)} - \frac{60}{(z+3)^2} + \frac{96}{(z+3)^3}$$

$$= \frac{3z^2+9z+4z+27}{z+3}$$

$$= 3z + \frac{4z+12}{z+3} + \frac{15}{z+3} - \frac{60}{(z+3)^2} + \frac{96}{(z+3)^2}$$

$$= 4+3z+\frac{18}{z+3} - \frac{60}{(z+3)^2} + \frac{96}{(z+3)^2}$$

---

① Find the impulse response of difference eqn

$$y(z) = \frac{4}{3}y(n-1) - \frac{7}{12}y(n-2) + \frac{1}{12}y(n-3) + x(n) - x(n-3)$$

$$y(z) = \frac{4}{3}z^{-1}(y(z) - \frac{7}{12}z^{-2}(y(z) + \frac{1}{12}z^{-3}(y(z) + x(z) - z^{-3}x(z)$$

$$\frac{Y(z)}{X(z)} = H(z)$$

unit-step response

$x(n) = u(n)$

$y(n)$, $X(z) = \frac{z}{z-1}$

$$H(z) = \frac{1-z^{-3}}{(1-\frac{1}{2}z^{-1})^2(1-\frac{1}{3}z^{-1})}$$

Example

$x(n) = (-1)^n u(n) \rightarrow \frac{1}{1+z^{-1}}, |z|>1$

$y(n) = \frac{1}{2} y(n-1) + x(n)$

$Y(z) = \frac{1}{2} z^{-1} Y(z) + X(z)$

$X(z) \rightarrow x(n)$

$Y(z)\left[1 - \frac{1}{2} z^{-1}\right] = X(z)$

$\frac{Y(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}} \rightarrow 2^{-n} u(n)$

$Y(z) = \left(\frac{1}{1-\frac{1}{2}z^{-1}}\right) X(z)$ $\quad [|z|>1]$

$= \left(\frac{1}{1-\frac{1}{2}z^{-1}}\right)\left(\frac{1}{1+z^{-1}}\right)$

$= \frac{\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \frac{2/3}{1+z^{-1}}$

$= \frac{1}{3}(2^{-n}) u(n) + \frac{2}{3}(-1)^n u(n)$

$\frac{1}{3}(2^{-n}) u(n)$ → transient response → Natural response

$\frac{2}{3}(-1)^n u(n)$ → Forced response → Not there in initial input

$|z| >$ highest pole

If nothing mentioned, assume right sided signal

Contour

$$x(n) = \frac{1}{2\pi j} \oint X(z)\, z^{n-1}\, dz$$

* Analysis of LTI systems in z-domain

LTI

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

Impulse response

$y(n) = x(n) * h(n)$

$Y(z) = X(z) \cdot H(z)$

$H(z) = \dfrac{Y(z)}{X(z)}$

Filter

From $H(z)$ or $h(n)$
How can we say if system is
→ causal
→ stable (BIBO)

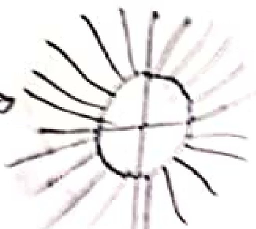
causal : $h(n) = 0,\ n < 0$

stable : $\sum |h(n)|^2 < \infty$

Given $H(z)$ with ROC, can we tell if its causal/stable

↓

ROC should be exterior including ∞

$\alpha^n u(n) \rightarrow$ 

↓

right sided signal

Example:

$y(n) = u(n+1)$

$z^{1}\left(\dfrac{1}{1-z^{-1}}\right)$

Timeshifting may cause removal of 0 or ∞ from previous ROC sometimes

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} h(n) \cdot r^{-n} \cdot e^{-j\omega n}$$

$$|H(z)|^2 = \left| \sum_{n=-\infty}^{\infty} h(n) \right|^2_{r=1} e^{-j\omega n} \quad \text{if } r=1$$

ROC should include unit circle → condition for stability

→ For $H(z)$ to be causal & stable, all poles should be within unit circle ☆

$H(z) \cdot H_I(z) = 1$

$x(n) \rightarrow \boxed{H(z)} \rightarrow y(n)$

$y(n) \rightarrow \boxed{H_I(z)} \rightarrow x(n)$

If both $H(z)$ and $H_I(z)$ are meant to be stable, then both zeroes & poles should be within the unit circle

Example

① $y(n) = 3y(n-1) + \frac{1}{3}x(n) - x(n-1)$

$Y(z) = 3z^{-1}Y(z) + \frac{1}{3}X(z) - z^{-1}X(z)$

$Y(z)(1-3z^{-1}) = X(z)\left(\frac{1}{3} - z^{-1}\right)$

$H(z) = \frac{Y(z)}{X(z)} = \frac{1-3z^{-1}}{3-9z^{-1}} \rightarrow \frac{1}{3}$

if quantized improperly, may not cancel

stable theoretically

② $y(n) = 3y(n-1) - x(n-1)$

$$Y(z) = 3z^{-1}Y(z) - z^{-1}X(z)$$

$$Y(z)(1-3z^{-1}) = -z^{-1}(X(z))$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-z^{-1}}{1-3z^{-1}} = \frac{-1}{z-3}$$

③ $y(n) = \sum_{k=0}^{M} b_k x(n-k) - \sum_{k=1}^{N} a_k y(n-k)$

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k \cdot z^{-k}}{1 + \sum_{k=1}^{N} a_k \cdot z^{-k}}$$

→ FIR is always stable

→ IIR is stable if all poles are within unit circle

<u>Filter design</u> → Designing a_k, b_k

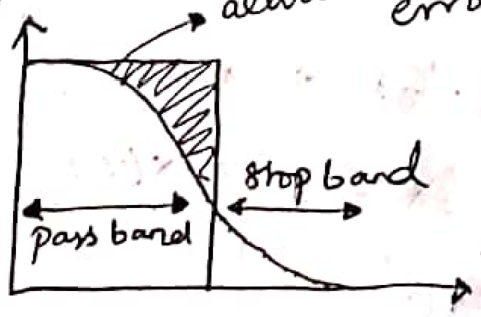
$$x(n) \rightarrow \boxed{} \rightarrow y(n)$$

→ But IIR is more flexible & controllable than FIR

→ if less error here, filter order ↑

complexity increases

allow some error here



depends on filter order

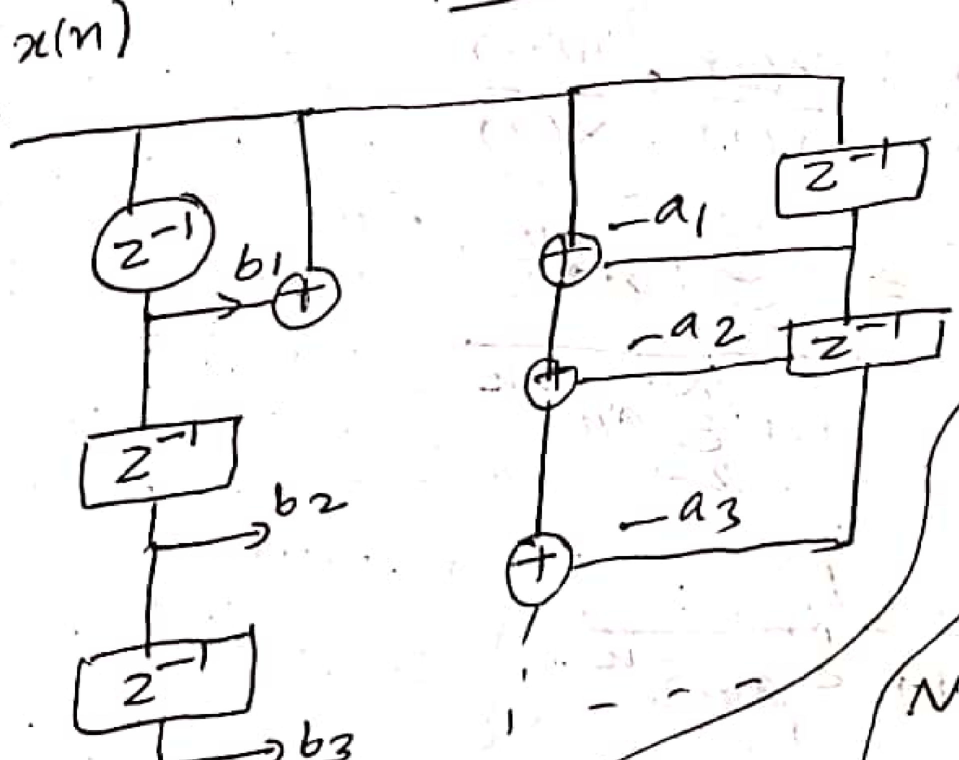
Lowpass band filter → Designing ideally is difficult

$$y(n) = \sum_{k=0}^{M} b_k x(n-k) - \sum_{k=1}^{N} a_k \cdot y(n-k)$$

To implement this we need

Delay
Multiplier
Adder

<u>D-I</u> $N+M \rightarrow$ delay elements



Number of delay elements is the order of filter used

We want to make the delay elements $= \max(M, N)$

<u>D-II</u>

Filter

- → stable, causal, LTI
- → $H(z)$ $[a_k, b_k]$
- → adder, multiplier, delay

D-II

$$H(z) = \frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$$

$$= \frac{\sum_{k=0}^{M} b_k \cdot z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 + \sum_{k=1}^{N} a_k \cdot z^{-k}}$$

$$\frac{Y(z)}{W(z)} = \sum_{k=0}^{M} b_k \cdot z^{-k}$$

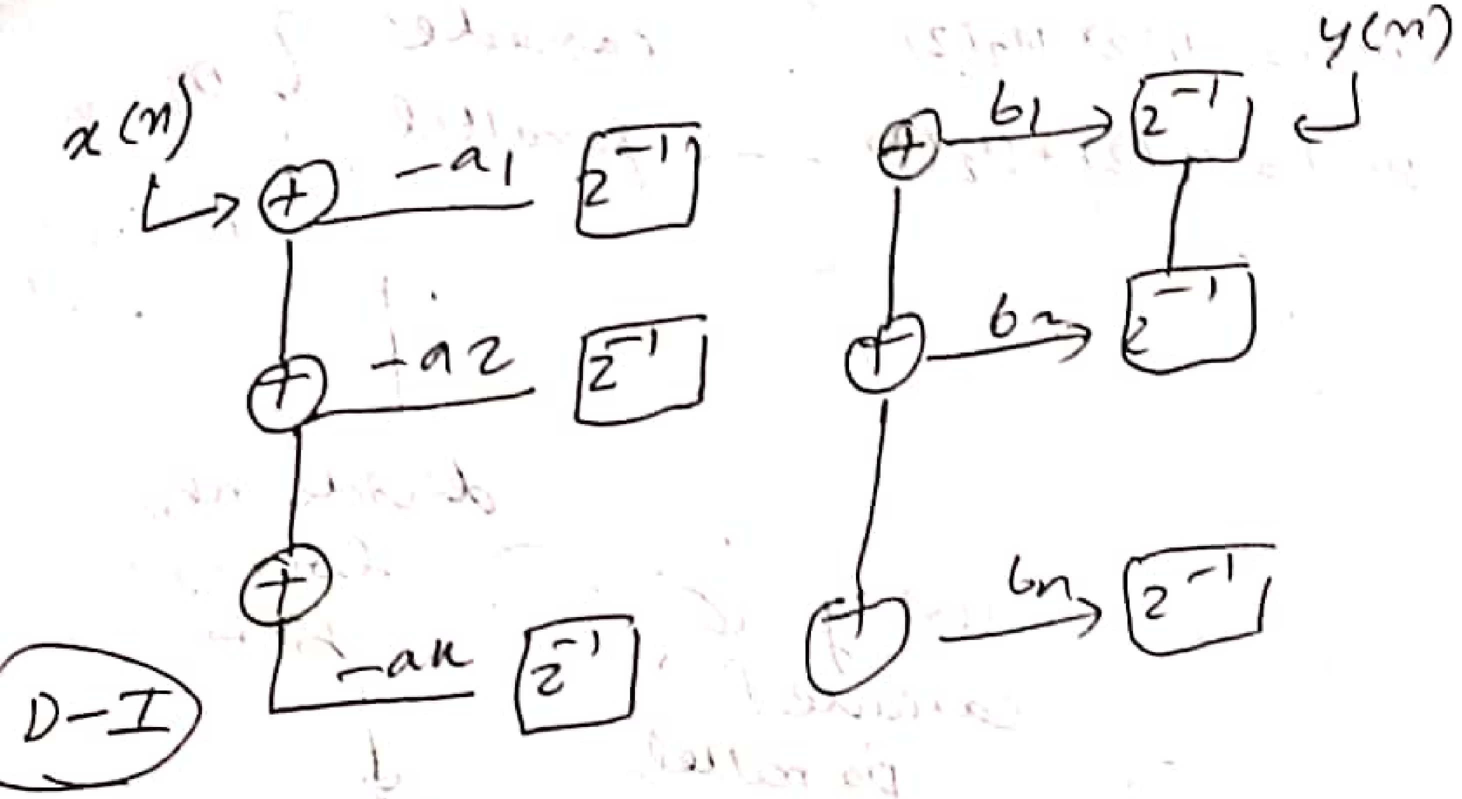
$$\Rightarrow W(z) = X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) + \cdots$$

$$Y(z) = b_0 W(z) + b_1 z^{-1} \cdot W(z) + b_2 z^{-2} W(z) + \cdots$$

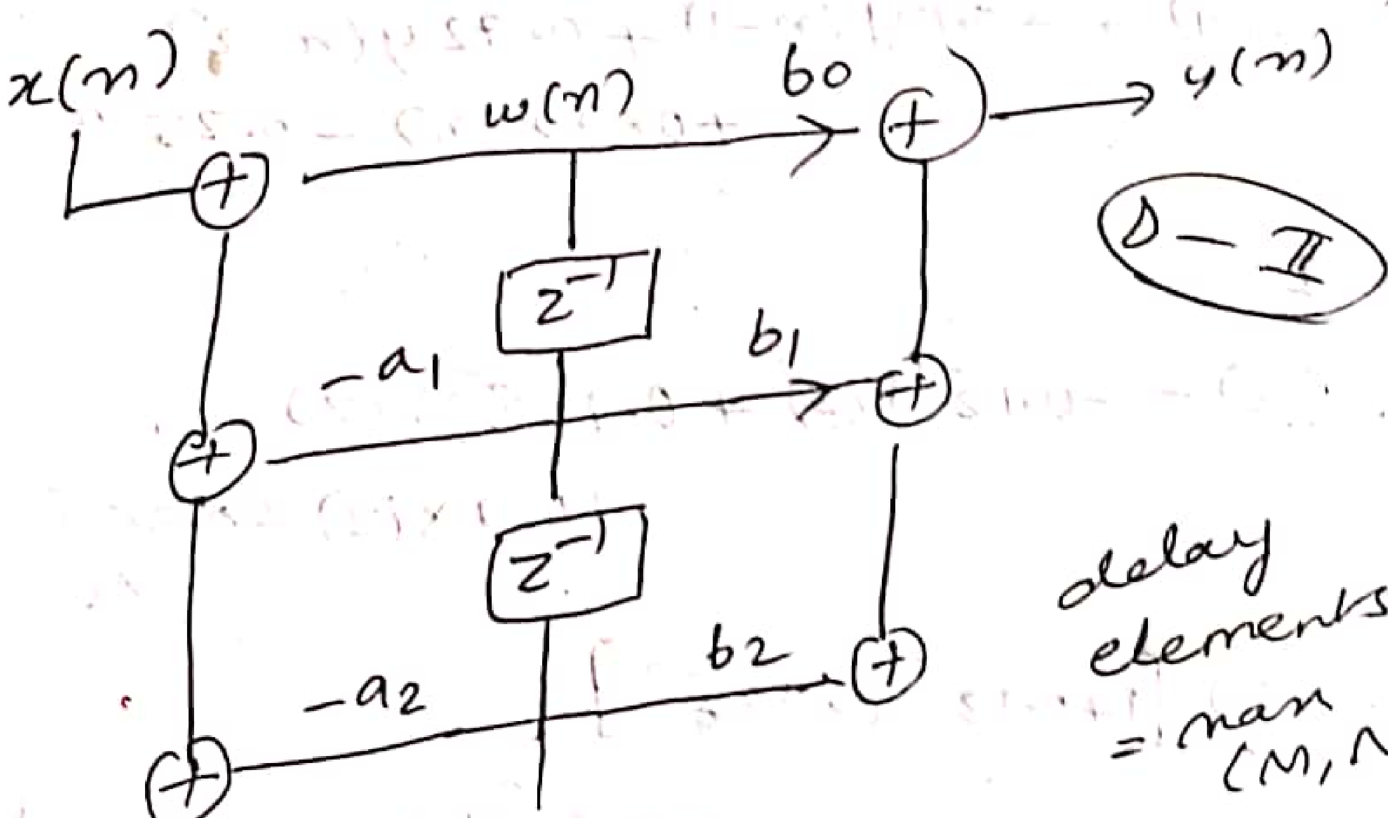
$$\Downarrow$$

$$w(n) = x(n) - a_1 w(n-1) - a_2 w(n-2) + \cdots$$

$$y(n) = b_0 w(n) + b_1 w(n-1) + b_2 w(n-2) + \cdots$$



⇓



delay elements = max (M, N)

$H(z) = H_1(z) \cdot H_2(z)$ --- cascade
$H(z) = H_1(z) + H_2(z)$ --- parallel

Designed using D – II

divide into low order filters

using cascade / parallel

Then apply D – II

Q) $y(n) = -0.1\, y(n-1) + 0.72\, y(n-2) + 0.7\, x(n) - 0.25\, x(n-2)$

D–I

$$Y(z) = -0.1 z^{-1} Y(z) + 0.72 z^{-2} Y(z) + 0.7 X(z) - 0.25 z^{-2} X(z)$$

$$Y(z)\left[1 + 0.1 z^{-1} - 0.72 z^{-2}\right] = X(z)\left[0.7 - 0.25 z^{-2}\right]$$

Digital Filter

Realization / Implementation

D-I / D-II, cascade,
parallel

Example

$$y(n) = 3x(n) - 3x(n-1) + 3y(n-1) + 5y(n-3)$$

order 3

↳ z Transform

↳ $H(z) = \dfrac{Y(z)}{X(z)}$

cascade 2^{nd} order system

Implement
in

Designing Design Filter

$$H(z) = \frac{\sum_{k=0}^{M} b_k \cdot z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$\left.\begin{matrix} a_n \\ b_n \end{matrix}\right\} \rightarrow IIR$

$b_k \rightarrow FIR$

FIR $\rightarrow$ stable

FIR $\rightarrow N\uparrow$

$\hookrightarrow$ Linear phase filters $\rightarrow$ not easy in IIR

IIR $\rightarrow$ indirect method

$\downarrow$

Analog form $H(s) \rightarrow H(z)$

Ex:- $s = \dfrac{1 - z^{-1}}{T}$

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

Linear-phase

$\theta = -\alpha\omega$

$\downarrow$

phase linearly varies with frequency

$$H(e^{j\omega}) = |H(e^{j\omega})| \, e^{j\theta(\omega)}$$

$\boxed{\tau_p = -\dfrac{\theta(\omega)}{\omega}} \rightarrow$ phase delay

$\boxed{\tau_g = -\dfrac{d\theta(\omega)}{d\omega}} \rightarrow$ group delay

Filter

multiple frequencies → Filter → introduces some delay

phase delay ensures this

Aim: To give same delay to all frequencies

For linear-phase systems, $\tau_p = \alpha$

$\tau_g = \alpha$

→ we need stable & linear phase filters

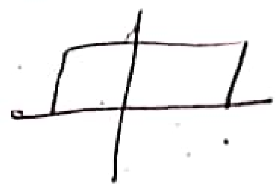
$$\boxed{\begin{aligned}\theta(\omega) &= -\alpha\omega+\beta \\ &\downarrow \\ \tau_p &= \alpha-\beta/\omega \\ \tau_g &= \alpha\end{aligned}}$$

→ h(n) is antisymmetric

→ $\theta(\omega) = -\alpha\omega$ → To ensure this, h(n) should be symmetric

$N = 10, 11$ → odd
↓
even

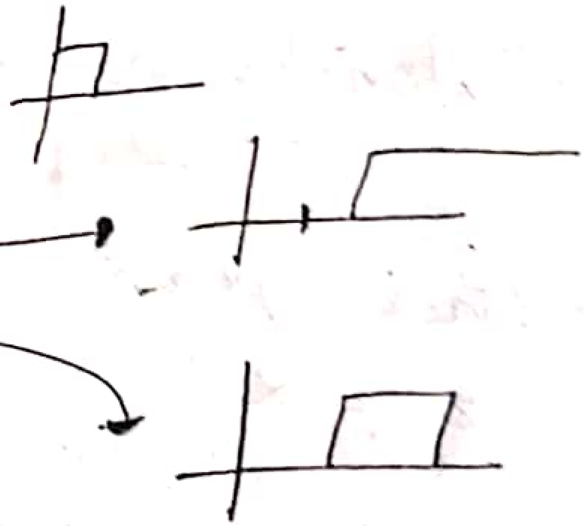
b_k values should be symmetric →

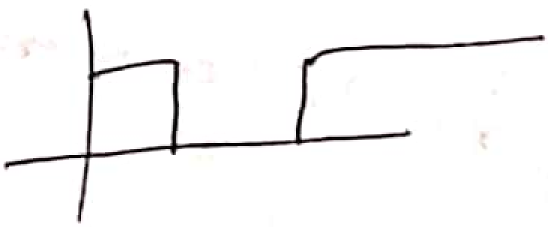


Symmetric
- N = odd
- N = even

Anti-symmetric
- N = odd
- N = even

Filter
- → Lowpass
- → Highpass
- → Bandpass
- → Band limited





1 - lowpass → Highpass

Bandpass, Band limited

⇒ If N is [redacted] (even), symmetric → we can't design high-pass

$h(n) = h(\underbrace{L}_{N-1} - n)$ $\quad N = 11$

condition for symmetric

$h(1) = h(9)$

$h(0) = h(10)$

$H(z) = z^{L} H(z^{-1})$

Assume N = even, $z = -1$

$H(-1) = (-1)^{L} H((-1)^{-1})$

$H(-1) = -H(-1)$

$H(-1) = 0$

Antisymmetric → can't design low pass

$h(n) = -h(L-N)$

$H(z) = z^{L} H(z^{-1})(-1)$

let $z = 1$, N = odd

$H(1) = -1 \cdot H(1)$

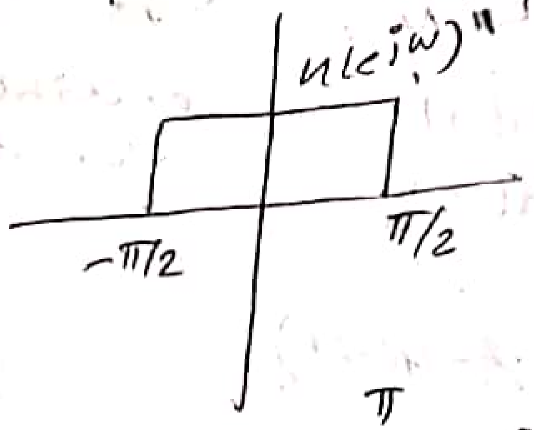
$H(1) = 0$ → 0 at $z = 1$

let $z = e^{jw}$

then

0 at $w = 0$

Example :



$$\text{DTFT} \longrightarrow h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) \cdot e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{jn\omega}\, d\omega$$

$$= \frac{1}{2\pi j n} \left(e^{j\omega n} \Big|_{-\pi/2}^{\pi/2} \right)$$

$$h(n) = \frac{\sin \frac{\pi}{2} n}{\pi n}, \quad -\infty \le n \le \infty$$

Take $N = 11$, $\longrightarrow \left(h(-5), h(-4), \text{---} \text{-----}, h(5) \right)$

$\downarrow$

$-5 \le n \le 5$

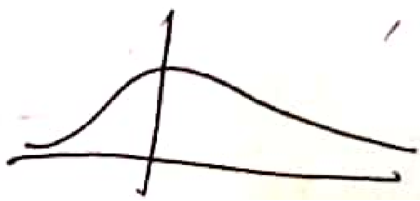
z^{-5}

$\downarrow$

$\left(h(0), \text{---}\ h(10) \right)$

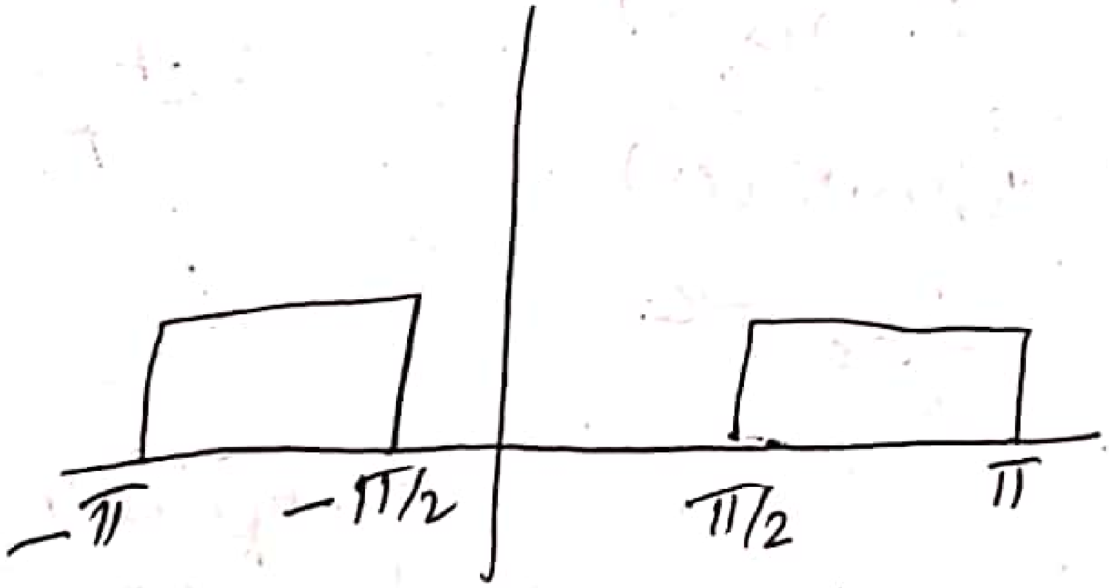
can't reconstruct signal

else use



$h(n) = h_d(n) \cdot w(n)$

window function



[Normalise to $(-\pi, \pi)$]

$\downarrow$

$\alpha = N-1/2$

$$H(z) = \frac{\Sigma b_k}{\Sigma a_k}$$

$$= \frac{s+1}{(s-1)(s-2)}$$

replace s with

$$s = \frac{1 - z^{-1}}{T}$$

$$h(n) = h_d(n) \cdot w(n)$$

$$H(z) = \sum_{k=0}^{M} b_k \cdot z^{-k}$$

OpenOrm Brookies

Speech production → for effective representation

ASR → Audio Speech Recognition
TTS → Text to Speech

vowels
consonants

pitch
↓
rate of vocal cord vibration

vocalcords energy ← ① Excitation ← input
② Vocal Track system
↓
the way you control your vocal track

Vowel | consonants
Voice | Voice /unvoiced

V ← Vocal Track system → Speech

UV (vocal cords not vibrating, like noise no Excitation)

→ LTI, All pole

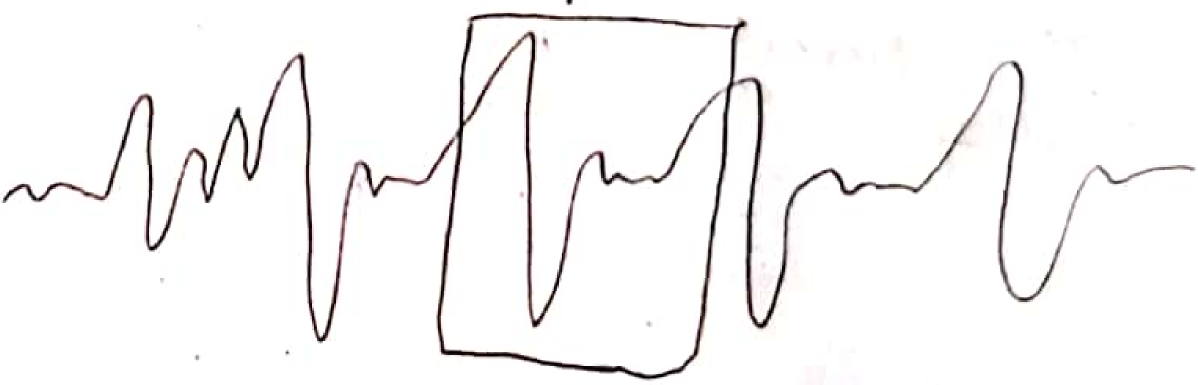
V	C
voiced	Voice/UV
open	closed / open (constriction)

↓
place of Articulation
↓
producing

→ syllabic – combination of phones

→ stationary signal – signal characteristic won't change with time

↓
sample the voice signal where vocal track is not changing (assuming stationary)



→ Formats → produced some frequencies with high energies

→ Model Vocal Track as All-pole filter

$$\boxed{\hat{s}(n) = \sum_{k=1}^{p} a_k s(n-k)}$$

8 or 16

→ Linear prediction

20ms

Excitation

$$e(n) = s(n) - \hat{s}(n)$$

$$= s(n) - \sum_{k=1}^{p} a_k s(n-k)$$

$e^2(n) \approx 0$ → prediction is proper

$$\frac{S(z)}{E(z)} = \frac{1}{1 - \sum_{k=1}^{p} a_k z^{-k}}$$

→ supplying z-transform on $e(n) = s(n) - \sum_{k=1}^{p} a_k s(n-k)$

Vocal Track

All pole system

Focus on a_k, approximate $e(n)$ is enough

$20\,ms \xrightarrow{8\,kHz}$ 160 samples

↓ 8 bits

64 kbps

represent with

8 (or) 10 a_k values

30 – 40 → $e(n)$

March-31st, 2023

*

$y(n) = x(n) * h(n)$

⇒ Circular convolution → DFT → $x(n)$

↳ $x_p(n)$
N

↓
N

$L_1 > L_2 \rightarrow N \geq L_1$
$L_2 > L_1 \rightarrow N \geq L_2$

N-DFT

⇒ Linear-convolution

$x_1(n) \rightarrow L_1$
$x_2(n) \rightarrow L_2$

$L_1 + L_2 - 1$

Circular convention to Linear convolution

8

4 — 4 0's
5 — 3 0's

* <u>Z-Transform</u>

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$z = re^{j\omega}$

ROC
↓
Region of convergence

if $r = 1 \rightarrow$ DTFT

<u>Example</u>

i) $x(n) = \delta(n)$
$X(z) = 1$ ⟶ ROC = entire z-plane

ii) $x(n) = \delta(n-k)$

$X(z) = z^{-k}$

$k=0 \rightarrow$ entire z plane

$k<0 \rightarrow \mathbb{C} - \{\infty\}$

$k>0 \rightarrow \mathbb{C} - \{0\}$

iii) $x(n) = p^n u(n)$

$$X(z) = 1 + pz^{-1} + p^2 z^{-2} + \cdots$$

$$= \frac{1}{1-pz^{-1}} \qquad |pz^{-1}| < 1$$

$$\Rightarrow |z| > |p|$$

$$= \frac{z}{z-p}$$

iv) $x(n) = -p^n u(-n-1)$

$$X(z) = -p^{-1} \cdot z + -p^{-2} z^2 + \cdots$$

$$= -(zp^{-1} + z^2 p^{-2} + \cdots)$$

$$= -\left(\frac{zp^{-1}}{1-zp^{-1}}\right) \qquad \text{if } |zp^{-1}| < 1$$

$$\hookrightarrow |z| < |p|$$

$$= \frac{z}{z-p}$$

v) $x(n) = u(n) + u(-n-1)$

$$= (1 + z^{-1} + z^{-2} + \cdots) + (z + z^2 + z^3 + \cdots)$$

$$= z^{-\infty} + \cdots + z^{-1} + 1 + z + z^2 + \cdots + z^{\infty}$$

No ROC exists

$|z| > 2$ $\quad$ $|z| < 1/2$

vi) $x(n) = 2^n u(n)$ [redacted] $\left(\frac{1}{2}\right)^n u(-n-1)$

$$X(z) = \left(1 + 2^1 \cdot z^{-1} + 2^2 \cdot z^{-2} + \cdots\right)$$
$$- \left(2 \cdot z^1 + 2^2 \cdot z^2 \cdots\right)$$

Z-transform doesn't exist

vii) $x(n) = \left(\frac{1}{3}\right)^n u(n) - 3^n u(-n-1)$

$$= \left(1 + \frac{1}{3} z^{-1} + \left(\frac{1}{3}\right)^2 z^{-2} + \cdots\right)$$
$$- \left(3^{-1} \cdot z + 3^{-2} \cdot z^2 + \cdots\right)$$

$|z| > 1/3$ $\qquad$ $|z| < 3$

$$\boxed{1/3 < |z| < 3}$$

$$a x_1(n) + b x_2(n) \xrightarrow{R_1} \ldots \xrightarrow{R_2}$$
$$\hookrightarrow a X_1(z) + b X_2(z)$$

$$\text{ROC} \longrightarrow R_1 \cap R_2$$

① $ax_1(n) + bx_2(n) \rightarrow aX_1(z) + bX_2(z)$

$\hookrightarrow R_1 \qquad \downarrow R_2 \qquad R_1 \cap R_2$

② Time-shifting

$$x'(n) = x(n-k)$$

$$x(n) \rightarrow X(z)$$

$$x(n-k) \rightarrow z^{-k} \cdot X(z)$$

$$y(n) = x(n) + 3x(n-1) + 2y(n-1)$$

$$Y(z) = X(z) + 3z^{-1}X(z) + 2z^{-1}(Y(z))$$

$$\frac{Y(z)}{X(z)} = \frac{1+3z^{-1}}{1-2z^{-1}}$$

$$\downarrow$$

$$H(z)$$

$$\downarrow$$

$$h(n)$$

$$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n)$$

③ Scaling in the z-domain

$$x(n) \rightarrow X(z)$$

$$a^n x(n) \rightarrow X(a^{-1}z)$$

Example

$$r_1 < |z| < r_2$$

$$|a|r_1 < |z| < |a|r_2$$

④ Time reversal

$x(n) \longrightarrow X(z) \longrightarrow$ Example

$x(-n) \longrightarrow X(z^{-1})$ $\qquad r_1 < |z| < r_2$

$\longrightarrow \frac{1}{r_2} < |z| < \frac{1}{r_1}$

⑤ Differentiation in z-domain

$x(n) \longrightarrow X(z)$

$nx(n) \longrightarrow -z\frac{d}{dz}X(z)$

Example

$$x(n) = n \cdot u(n)$$

$$= -z \cdot \frac{d}{dz}\left(\frac{1}{1-z^{-1}}\right)$$

$$= -z \cdot \frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= -z \cdot \frac{d}{dz}\left(1+\frac{1}{z-1}\right)$$

$$= \frac{z}{(z-1)^2}$$

$$* \; -z \cdot \frac{d}{dz}x(z) = -z \cdot \frac{d}{dz}\sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= -z \cdot \sum_{n=-\infty}^{\infty} x(n) \cdot (-n) \cdot z^{-n-1}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot n \cdot z^{-n}$$

Convolution

$$x_1(n) * x_2(n) \longrightarrow X_1(z) \cdot X_2(z)$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x_1(m) \cdot x_2(n-m) \cdot z^{-n}$$

$$= \cancel{\sum_{m=-\infty}^{\infty} x_1(m) \sum z}$$

$$= \sum_{m=-\infty}^{\infty} x_1(m) \cdot z^{-m} \, X_2(z)$$

$$= X_1(z) \cdot X_2(z)$$

Example

$$x(n) = u(n) * u(n-1)$$

$$X(z) = \left(\frac{1}{1-z^{-1}}\right) \cdot z^{-1} \cdot \left(\frac{1}{1-z^{-1}}\right) = \frac{z^{-1}}{(1-z^{-1})^2}$$

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$\hookrightarrow x(n) = \delta(n)$

$\hookrightarrow y(n) = h(n)$

$$y(n) = x(n) * h(n)$$

$$Y(z) = X(z) \cdot H(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

$$\int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt \longrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \, d\omega$$