Probability and Statistics
End Sem Solutions

Q1.

Using Law of Iterated Expectation

$$E[T] = E[E[T|N]]$$

$$E[T|N=n] = E\left[\sum_{i=1}^{n} X_i\right]$$

$$= nE[X] \quad (\because X_i\text{s are iids})$$

$$\Rightarrow E[T|N] = NE[X]$$

$$\Rightarrow E[T] = E[E[T|N]] = E[NE[X]] = E[N]E[X]$$

$(\because E[X] \text{ is a constant})$

Now if $N \sim \text{Geometric}(p)$

$$E[N] = \frac{1}{p}$$

if $X \sim \text{Exponential}(\lambda)$

$$E[X] = \frac{1}{\lambda}$$

$$\Rightarrow E[T] = E[N]E[X] = \frac{1}{p\lambda}$$

Q2.

2) $f_X(x) = \alpha_1 e^{-\alpha_1 x}$; $f_Y(y) = \alpha_2 e^{-\alpha_2 y}$

$Z = \min(X, Y)$

$F_Z(z) = P(Z \le z)$

$= 1 - P(\min(X,Y) > z)$

$= 1 - P(X > z \;\&\&\; Y > z)$

$= 1 - [1 - F_X(z)]\,[1 - F_Y(z)]$

$= 1 - e^{-(\alpha_1+\alpha_2) z}, \ \forall z \ge 0$

$F_W(w) = P(W \le w) = P(\max(X,Y) \le w)$

$= P(X \le w, Y \le w)$

$\therefore$ Independent, so:

$= P(X \le w)\, P(Y \le w)$

$= (1 - e^{-\alpha_1 w})(1 - e^{-\alpha_2 w})$

$F_W(w) = (1 - e^{-\alpha w})(1 - e^{-\alpha_2 w})$

Q3.

3) $F_Y(y) = P(X^2 \le Y) = P(X \le \sqrt{y}) - P(X \le -\sqrt{y})$

$\hookrightarrow (|X| \le \sqrt{y})$

$F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$

$f_Y(y) = \dfrac{d(F_Y(y))}{dy} = d\left(F_X(\sqrt{y}) - F_X(-\sqrt{y})\right)/dy$

$= \dfrac{f_Y(\sqrt{y})}{2\sqrt{y}} + \dfrac{f_Y(-\sqrt{y})}{2\sqrt{y}} =$

$\Rightarrow \dfrac{1}{2\sqrt{y}} \left(\dfrac{e^{-y/2}}{\sqrt{2\pi}} + \dfrac{e^{-y/2}}{\sqrt{2\pi}} \right)$

$f_Y(y) = \dfrac{1}{\sqrt{y}} \dfrac{e^{-y/2}}{\sqrt{2\pi}} \Rightarrow \begin{cases} 0, & y < 0 \\ \dfrac{e^{-y/2}}{\sqrt{2\pi y}}, & y \ge 0 \end{cases}$

Q4.

Uniform Distribution

From the definition of the continuous uniform distribution, X has probability density function:

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

From the definition of a moment generating function:

$$M_X(t) = \mathsf{E}\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f_X(x) \, \mathrm{d}x$$

where $\mathsf{E}(\cdot)$ denotes expectation.

First, consider the case $t \neq 0$.

Then:

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{a} 0 e^{tx} \, \mathrm{d}x + \int_{a}^{b} \frac{e^{tx}}{b-a} \, \mathrm{d}x + \int_{b}^{\infty} 0 e^{tx} \, \mathrm{d}x \\ &= \left[\frac{e^{tx}}{t(b-a)}\right]_a^b && \text{Primitive of } e^{ax}\text{, Fundamental Theorem of Calculus} \\ &= \frac{e^{tb} - e^{ta}}{t(b-a)} \end{aligned}$$

In the case $t = 0$, we have $\mathsf{E}\left(X^0\right) = \mathsf{E}(1) = 1$.

Exponential Distribution

From the definition of the Exponential distribution, X has probability density function:

$$f_X(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}$$

From the definition of a moment generating function:

$$M_X(t) = \mathsf{E}\left(e^{tX}\right) = \int_0^\infty e^{tx} f_X(x) \, \mathrm{d}x$$

Then:

$$\begin{aligned} M_X(t) &= \frac{1}{\beta} \int_0^\infty e^{x\left(-\frac{1}{\beta}+t\right)} \, \mathrm{d}x && \text{Exponential of Sum} \\ &= \frac{1}{\beta\left(-\frac{1}{\beta}+t\right)} \left[e^{x\left(-\frac{1}{\beta}+t\right)}\right]_0^\infty && \text{Primitive of Exponential Function} \end{aligned}$$

Note that if $t > \frac{1}{\beta}$, then $e^{x\left(-\frac{1}{\beta}+t\right)} \to \infty$ as $x \to \infty$ by Exponential Tends to Zero and Infinity, so the integral diverges in this case.

If $t = \frac{1}{\beta}$ then the integrand is identically 1, so the integral similarly diverges in this case.

If $t < \frac{1}{\beta}$, then $e^{x\left(-\frac{1}{\beta}+t\right)} \to 0$ as $x \to \infty$ from Exponential Tends to Zero and Infinity, so the integral converges in this case.

Therefore, the function is only well defined for $t < \frac{1}{\beta}$.

Proceeding:

$$\begin{aligned} \frac{1}{\beta\left(-\frac{1}{\beta}+t\right)} \left[e^{x\left(-\frac{1}{\beta}+t\right)}\right]_0^\infty &= \frac{1}{\beta\left(-\frac{1}{\beta}+t\right)}(0-1) && \text{Exponential Tends to Zero and Infinity, Exponential of Zero} \\ &= \frac{1}{\beta\left(\frac{1}{\beta}-t\right)} \\ &= \frac{1}{1-\beta t} \end{aligned}$$

Here, $\lambda = 1/\boldsymbol{\beta}$.

Q5.

There is an alternative way to get the result by applying the the Law of Total Probability:

$$P[W] = \int_Z P[W \mid Z = z] f_Z(z) dz$$

As others have done, let $X \sim \exp(\lambda)$ and $Y \sim \exp(\mu)$. What follows is the only slightly unintuitive step: instead of directly calculating the PDF of $Y - X$, first calculate the CDF: $P[X - Y \leq t]$ (we can then differentiate at the end).

$$P[Y - X \leq t] = P[Y \leq t + X]$$

This is where we'll apply total probability to get

$$= \int_0^\infty P[Y \leq t + X \mid X = x] f_X(x) dx$$

$$= \int_0^\infty P[Y \leq t + x] f_X(x) dx = \int_0^\infty F_Y(t + x) f_X(x) dx$$

Note substituting the CDF here is only valid if $t \geq 0$,

$$= \int_0^\infty (1 - e^{-\mu(t+x)}) \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{-\lambda x} dx - \lambda e^{-\mu t} \int_0^\infty e^{-(\lambda+\mu)x} dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^\infty - \lambda e^{-\mu t} \left[\frac{e^{-(\lambda+\mu)x}}{-(\lambda+\mu)} \right]_0^\infty = 1 - \frac{\lambda e^{-\mu t}}{\lambda + \mu}$$

Differentiating this last expression gives us the PDF:

$$f_{Y-X}(t) = \frac{\lambda \mu e^{-\mu t}}{\lambda + \mu} \quad \text{for } t \geq 0$$

Here, $\lambda = \mu$, Y = V, X = W, Y - X = V - W = Z.

Q6.

If $X_i \sim Exponential(\theta)$, then

$$f_{X_i}(x;\theta) = \theta e^{-\theta x} u(x),$$

where $u(x)$ is the unit step function, i.e., $u(x) = 1$ for $x \geq 0$ and $u(x) = 0$ for $x < 0$. Thus, for $x_i \geq 0$, we can write

$$\begin{aligned} L(x_1, x_2, x_3, x_4; \theta) &= f_{X_1 X_2 X_3 X_4}(x_1, x_2, x_3, x_4; \theta) \\ &= f_{X_1}(x_1;\theta) f_{X_2}(x_2;\theta) f_{X_3}(x_3;\theta) f_{X_4}(x_4;\theta) \\ &= \theta^4 e^{-(x_1+x_2+x_3+x_4)\theta}. \end{aligned}$$

Since we have observed $(x_1, x_2, x_3, x_4) = (1.23, 3.32, 1.98, 2.12)$, we have

$$L(1.23, 3.32, 1.98, 2.12; \theta) = \theta^4 e^{-8.65\theta}.$$

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Here, it is easier to work with the log likelihood function, $\ln L(1.23, 3.32, 1.98, 2.12; \theta)$. Specifically,

$$\ln L(1.23, 3.32, 1.98, 2.12; \theta) = 4 \ln \theta - 8.65\theta.$$

By differentiating, we obtain

$$\frac{4}{\theta} - 8.65 = 0,$$

which results in

$$\hat{\theta}_{ML} = 0.46$$

Q7.

(a) Here we define the test statistic as

$$\begin{aligned} W &= \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \\ &= \frac{5.96 - 5}{1/\sqrt{5}} \\ &\approx 2.15. \end{aligned}$$

Here, $\alpha = .05$, so $z_{\frac{\alpha}{2}} = z_{0.025} = 1.96$. Since $|W| > z_{\frac{\alpha}{2}}$, we reject H_0 and accept H_1.

(b) The 95% CI is given by

$$\left(5.96 - 1.96 * \frac{1}{\sqrt{(5)}}, 5.96 + 1.96 * \frac{1}{\sqrt{(5)}}\right) = (5.09, 6.84).$$

Since μ_0 is not included in the interval, we are able to reject the null hypothesis and conclude that μ is not 5.

Q8.

Q8: let X be a random variable showing the no. of observed heads. $X = 700$ (given)

let n be the total no. of times the coin is tossed.

$n = 1000$

Assume H_0 is true, W is a standard normal random variable $N(0,1)$

$$W = \frac{X - n\theta_0}{\sqrt{n\theta_0(1-\theta_0)}} = \frac{X - 1000 \times \frac{1}{2}}{\sqrt{1000 \times \frac{1}{2} \times \frac{1}{2}}}$$

$$= \frac{(X - 500)}{5\sqrt{10}}$$

If H_0 is true, then expect X to be close to 500, while if H_1 is true we expect X to be larger. Thus we can suggest the following test:

We choose a threshold c. If $W \leq c$, we accept H_0; otherwise we accept H_1.

1. If we require significance level $\alpha = 0.05$, then

$$C = Z_{0.05} = 1.645$$

$$W = \frac{700 - 500}{5\sqrt{10}} = \underline{12.65}$$

$W > c \rightarrow$ we reject H_0. and accept H_1.

2. If significance level $\alpha = 0.01$, then

$$C = Z_{0.01} = 2.33$$

$$W = 12.65$$

$W > c$, we reject H_0 and accept H_1.

3. P- value

Since here $W = 12.65$, we will reject H_0 iff $c < 12.65$. Let $z_\alpha = c$, then

$$\alpha = 1 - \phi(c)$$

$$\alpha = 1 - \phi(12.65) \qquad \boxed{c = 12.65}$$

From above eqn, we will get α.

$\therefore$ We reject H_0 for $\alpha > (1 - \phi(12.65))$

$\therefore$ P- value $= 1 - \phi(12.65)$

Q9.

Solution: For $0 \le x \le 1$, we have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy \\ &= \int_0^1 \left(x + \frac{3}{2}y^2\right) dy \\ &= \left[xy + \frac{1}{2}y^3\right]_0^1 \\ &= x + \frac{1}{2}. \end{aligned}$$

Thus,

$$f_X(x) = \begin{cases} x + \frac{1}{2} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Similarly, for $0 \le y \le 1$, we have

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dx \\ &= \int_0^1 \left(x + \frac{3}{2}y^2\right) dx \\ &= \left[\frac{1}{2}x^2 + \frac{3}{2}y^2 x\right]_0^1 \\ &= \frac{3}{2}y^2 + \frac{1}{2}. \end{aligned}$$

Thus,

$$f_Y(y) = \begin{cases} \frac{3}{2}y^2 + \frac{1}{2} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The MAP estimate of X, given $Y = y$, is the value of x that maximizes

$$f_{X|Y}(x|y) = \frac{x + \frac{3}{2}y^2}{\frac{3}{2}y^2 + \frac{1}{2}}, \qquad \text{for } 0 \le x, y \le 1.$$

For any $y \in [0,1]$, the above function is maximized at $x = 1$. Thus, we obtain the MAP estimate of x as

$$\hat{x}_{MAP} = 1.$$

The ML estimate of X, given $Y = y$, is the value of x that maximizes

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{x + \frac{3}{2}y^2}{x + \frac{1}{2}} \\ &= 1 + \frac{\frac{3}{2}y^2 - \frac{1}{2}}{x + \frac{1}{2}}, \qquad \text{for } 0 \le x, y \le 1. \end{aligned}$$

Therefore, we conclude

$$\hat{x}_{ML} = \begin{cases} 1 & 0 \le y \le \frac{1}{\sqrt{3}} \\ 0 & \text{otherwise} \end{cases}$$

Q10.

Solution: Since X and W are independent and normal, Y is also normal. The variance is

$$\begin{aligned} \mathrm{Cov}(X,Y) &= \mathrm{Cov}(X, X+W) \\ &= \mathrm{Cov}(X) + \mathrm{Cov}(X,W) \\ &= \mathrm{Var}(X) = \sigma_X^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \rho(X,Y) &= \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} \\ &= \frac{\sigma_X}{\sqrt{\sigma_X^2 + \sigma_W^2}}. \end{aligned}$$

(a) The MMSE estimator of X given Y is

$$\begin{aligned} \hat{X}_M &= E[X|Y] \\ &= \mu_X + \rho\sigma_X \frac{Y - \mu_Y}{\sigma_Y} \\ &= \frac{\sigma_X^2}{\sigma_X^2 + \sigma_W^2} Y. \end{aligned}$$

(b) The MSE of this estimator is given by

$$\begin{aligned} E[(X - \hat{X_M})^2] &= E\left[\tilde{X}^2\right] \\ &= E[X^2] - E[\hat{X_M}^2] \\ &= \sigma_X^2 - \left(\frac{\sigma_X^2}{\sigma_X^2 + \sigma_W^2}\right)^2 (\sigma_X^2 + \sigma_W^2) \\ &= \frac{\sigma_X^2 \sigma_W^2}{\sigma_X^2 + \sigma_W^2}. \end{aligned}$$