Quiz 1: Probability and Statistics

August 25, 2022

1 5 Marks

1. The CDF of a random variable X is defines as $F(x) = \mathcal{P}(\omega \in \Omega : \mathcal{X}(\omega) \leq x) = \sum_{x \leq x_1} p_{\mathcal{X}}(x)$ where p_X is the PMF. Prove that:
   $\mathcal{P}(a < X \leq b)= F_X(b) - F_X(a)$

   Solution: Since the outcomes for random variables define events in the event space, which is why we are able to assign probabilities to such outcomes. Let X be a random variable with cdf $F_X(x)$ For $a < b$, we can consider the following events:

   - $C = X \leq a$
   - $D = a < X \leq b$
   - $E = X \leq b$

   Then C and D are mutually exclusive and their union is the event E. By the third axiom of probability, we know that
   $P(E) = P(D) + P(C)$
   $P(X \leq b) = P(a < X \leq b) + P(X \leq a)$
   $P(a < X \leq b) = P(X \leq b) - P(X \leq a)$
   $P(a < X \leq b) = F_X(b) - F_X(a))$

2. A geometric random variable X with parameter p has PMF given by $p_x(k) = (1-p)^{k-1}p$. Derive the expression for its mean and variance
.

Solution: For the geometric distribution, the range $R_x = 1, 2, 3, ...$ and the PMF is given by $P_x(k) = (1-p)^{k-1}p, for\ k\ =\ 1, 2, ..,$ where $0 < p < 1$, Thus we can write

$E[X] = \sum_{x_k \in R_x} x_k P_x(x_k)$
$E[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1}p$
$E[X] = p\sum_{k=1}^{\infty} k(1-p)^{k-1}$

We know that:
$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} for |x| < 1$

Taking derivative of this equation with respect to x
$\frac{d}{dx}\sum_{k=0}^{\infty} x^k = \frac{d}{dx}\frac{1}{1-x}$
Thus we have
$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$

For the expectation we can write
$E[X] = p\frac{1}{(1-(1-p))^2}$
$E[X] = p\frac{1}{p^2}$
$E[X = \frac{1}{p}$
For Variance
$Var[X] = E[X^2] - E[X]^2$
$E[X^2] = \sum_{k=0}^{\infty} k^2 p_x(k)$
$E[X^2] = \sum_{k=1}^{\infty} k^2 p(1-p)^{k-1}$
$E[X^2] = p\sum_{k=1}^{\infty} k^2(1-p)^{k-1}$

Let $(1-p) = q$
$\sum_{x=1}^{\infty} x^2 q^{x-1} = \frac{1+q}{(1-q)^3}$

Substituting in $E[X^2]$
$E[X^2] = p * \frac{(1+(1-p))}{(1-(1-p))^3)}$
$E[X^2] = p * \frac{2-p}{p^3}$
$E[X^2] = \frac{2-p}{p^2}$

$Var[X] = \frac{2-p}{p^2} - \frac{1}{p^2}$
$Var[X] = \frac{1-p}{p^2}$

3. Two cards are chosen from a standard deck of 52 cards. Suppose that you win 2 Rs for each heart selected, and lose 1 Rs for each spade selected. Other suits (clubs or diamonds) bring neither win nor loss. Let X denote your winnings. Determine the probability mass function $p_X(x)$.

Solution: Considering all the possible cases for the Winnings

- Winnings $>$ 4 Cases: None
  - P(With replacement) $= 0$
  - P(Without replacement) $= 0$
- Winnings $=$ 4 Cases: Both hearts
  - P(With replacement) $= \frac{13}{52} * \frac{13}{52} = \frac{169}{2704} = \frac{1}{16}$
  - P(Without replacement) $= \frac{13}{52} * \frac{12}{51} = \frac{156}{2652} = \frac{1}{17}$
- Winnings $=$ 3 Cases: None
  - P(With replacement) $= 0$
  - P(Without replacement) $= 0$
- Winnings $=$ 2 Cases: One heart, other club or diamond
  - P(With replacement) $= 2 * (\frac{13}{52} * \frac{26}{52}) = \frac{676}{2704} = \frac{1}{4}$
  - P(Without replacement) $= \frac{13}{52} * \frac{26}{51} + \frac{26}{52} * \frac{13}{51} = \frac{676}{2652} = \frac{13}{51}$
- Winnings $=$ 1 Cases: One heart, One spade
  - P(With replacement) $= 2 * (\frac{13}{52} * \frac{13}{52}) = \frac{338}{2704} = \frac{1}{8}$
  - P(Without replacement) $= \frac{13}{52} * \frac{13}{51} + \frac{13}{52} * \frac{13}{51} = \frac{338}{2652} = \frac{13}{102}$
- Winnings $=$ 0 Cases: Both clubs, both diamonds, one club one diamond
  - P(With replacement) $= \frac{13}{52} * \frac{13}{52} + \frac{13}{52} * \frac{13}{52} + 2 * (\frac{13}{52} * \frac{13}{52}) = \frac{676}{2704} = \frac{1}{4}$
  - P(Without replacement) $= \frac{13}{52} * \frac{12}{51} + \frac{13}{52} * \frac{12}{51} + 2 * (\frac{13}{52} * \frac{13}{51}) = \frac{650}{2652} = \frac{25}{102}$
- Winnings $=$ -1 Cases: One spade, Other club or diamond
  - P(With replacement) $= 2 * (\frac{13}{52} * \frac{26}{52}) = \frac{676}{2704} = \frac{1}{4}$
  - P(Without replacement) $= \frac{13}{52} * \frac{26}{51} + \frac{26}{52} * \frac{13}{51} = \frac{676}{2652} = \frac{13}{51}$
- Winnings $=$ -2 Cases: Both spades
  - P(With replacement) $= \frac{13}{52} * \frac{13}{52} = \frac{169}{2704} = \frac{1}{16}$
  - P(Without replacement) $= \frac{13}{52} * \frac{12}{51} = \frac{156}{2652} = \frac{1}{17}$
- Winnings $<$ -2 Cases: None
  - P(With replacement) $= 0$
  - P(Without replacement) $= 0$

4. For a random variable X with mean μ, its variance $Var(X)$ is defined as $E[(X-\mu)^2]$. Prove that $Var(aX+b) = a^2Var(X)$ for arbitrary constants a and b

Solution:
$Var(X) = E[(X-\mu)^2]$
$Var(X) = E[X^2 + (\mu)^2 - 2 * \mu * X]$
$Var(X) = E[X^2] - 2E[X * \mu] + E[\mu^2]$
$Var(X) = E[X^2] - 2\mu * E[X] + \mu^2$
$Var(x) = E[X^2] - \mu^2$

$Var(aX+b) = E[(aX+b)^2] - (E[aX+b])^2$
$Var(aX+b) = E[a^2X^2 + b^2 + 2aXb] - (aE[X]+b)(aE[X]+b)$
$Var(aX+b) = a^2E[X^2] + b^2 + 2abE[X] - a^2E[X]^2 - b^2 - 2abE[X]$
$Var(aX+b) = a^2E[X^2] - a^2E[X]^2$
$Var(aX+b) = a^2(E[X^2] - E[X]^2)$
$Var(aX+b) = a^2(Var(X))$

2 10 Marks

1. Let random variable $\mathcal{X}$ denote the outcome of a dice. Plot the cumulative distribution function (CDF) of $\mathcal{X}$. Also find the mean and variance of $\mathcal{X}$. Additionally prove that(prove! don not numerically verify. Start with either RHS or LHS and prove the other side.)

$$\sum_{x\in\{1,2,\ldots 6\}} xp_X(x) = 1 + \sum_{x\in\{1,2,\ldots 6\}}(1 - F_X(x))$$

(Hint: Write the CDF on the rhs in terms of PMF). The RHS is an alternative formula to get the expectation of non-negative random variables using the CDF.)

10 marks

Q1. Let random variable X denote the outcome of a dice...

... prove that:

$$\sum_{x \in \{1,2,\dots 6\}} x \, P_X(x) = 1 + \sum_{x \in \{1,2,\dots 6\}} (1 - F_X(x))$$

Ans. Assuming we have a fair dice, the PMF of X is given by:

I

$$P_X(x=k) = \frac{1}{6}, \quad k \in \{1,2,3\dots 6\}$$

→ The CDF of X can be computed by sequentially summing up these probabilities:

$$F_X(x) = P_X(X \leq x)$$

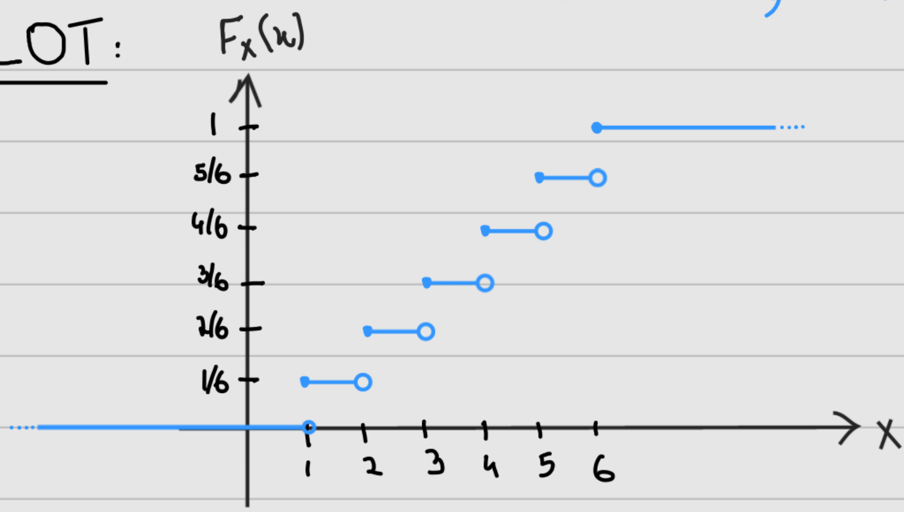
Since X is a discrete random variable;

$$F_X(x) = \sum_{k=1}^{n} P_X(X = x_k \mid x_k \leq x)$$

→ So, we have:

$$F_X(x) = \begin{cases} 0 & ; x < 1 \\ z/6 & ; z \leq x < z+1, z \in \{1,2,3\dots 5\} \\ 1 & ; x \geq 6 \end{cases}$$

PLOT:



II $\quad \text{mean}(\mu) = E[X] = \sum_{x \in \{1,2,\dots 6\}} x \, P_X(x)$

$$\Rightarrow \mu = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} = \frac{21}{6}$$

$$\Rightarrow \boxed{\mu = 3.5}$$

$$Var(X) = E[X^2] - \mu^2 = \sum_{x \in \{1,2,\dots 6\}} x^2 P_X(x) - \mu^2$$

$$\Rightarrow Var(X) = \frac{1}{6}[1^2 + 2^2 + \dots 6^2] - 3.5^2$$

$$\Rightarrow \boxed{Var(X) = \frac{35}{12} = 2.92}$$

III <u>TP</u>: $\sum_{x \in \{1,2,\dots 6\}} x\, P_X(x) = 1 + \sum_{x \in \{1,2,\dots 6\}} (1 - F_X(x))$ — ①

<u>Proof</u>: <u>Method-1</u>

Let $S = \sum_{x \in \{1,2,\dots 6\}} x\, P_X(x)$

In LHS of ①, we have:

$$\rightarrow S = \left[\begin{array}{l} 1 P_X(1) \\ +2 P_X(2) \\ +3 P_X(3) \\ +4 P_X(4) \\ +5 P_X(5) \\ +6 P_X(6) \end{array}\right] = \left[\begin{array}{llllll} P_X(1) & & & & & \\ +P_X(2) & +P_X(2) & & & & \\ +P_X(3) & +P_X(3) & +P_X(3) & & & \\ +P_X(4) & +P_X(4) & +P_X(4) & +P_X(4) & & \\ +P_X(5) & +P_X(5) & +P_X(5) & +P_X(5) & +P_X(5) & \\ +\underbrace{P_X(6)}_{1} & +\underbrace{P_X(6)}_{P_X(X>1)} & +\underbrace{P_X(6)}_{P_X(X>2)} & +\underbrace{P_X(6)}_{P_X(X>3)} & +\underbrace{P_X(6)}_{P_X(X>4)} & +\underbrace{P_X(6)}_{P_X(X>5)} \end{array}\right]$$

$$\rightarrow S = 1 + P_X(X>1) + P(X>2) + \dots + P(X>6) \quad [P(X>6) = 0]$$

$$\rightarrow S = 1 + (1 - P_X(X \le 1)) + (1 - P_X(X \le 2)) + \dots + (1 - P_X(X \le 6))$$

$$\rightarrow S = 1 + (1 - F_X(1)) + (1 - F_X(2)) + \dots + (1 - F_X(6))$$

$$\rightarrow S = 1 + \sum_{x \in \{1,2,\dots 6\}} (1 - F_X(x))$$

$$\Rightarrow S = RHS \qquad \Rightarrow \text{HP}$$

Method-2

→ In RHS,

$$S = 1 + \sum_{x \in \{1,2,\dots 6\}} (1 - F_X(x))$$

$$\rightarrow S = \sum_{x \in \{1,2,\dots 6\}} (P_X(x)) + \sum_{x \in \{1,2,\dots 6\}} (1 - F_X(x))$$

$$\rightarrow S = \sum_{x \in \{1,2,\dots 6\}} (P_X(x)) + \sum_{x \in \{1,2,\dots 6\}} P_X(X > x)$$

$$\rightarrow S = \sum_{x \in \{1,2,\dots 6\}} (P_X(x)) + \sum_{x \in \{1,2,\dots 6\}} \sum_{k \in \{1,2,\dots 6\}} P_X(X = x_k \mid x_k > x)$$

→ Now, in the term $\sum_{x \in \{1,2,\dots 6\}} \underbrace{\sum_{k \in \{1,2,\dots 6\}} P_X(X = x_k \mid x_k > x)}$;

for $x = 1$, we get terms $P_X(2), P_X(3), \dots, P_X(6)$

$x = 2$, we get terms $P_X(3), \dots, P_X(6)$

$\vdots$

$x = 6$, we get no term

→ So in this term, the weight of:

$P_X(1) : 0$

$P_X(2) : 1$ $\quad \Rightarrow P_X(x) : x - 1$

$\vdots$

$P_X(6) : 5$

$$\Rightarrow S = \sum_{x \in \{1,2,\dots 6\}} (P_X(x)) + \sum_{x \in \{1,2,\dots 6\}} (x-1)(P_X(x))$$

$$\Rightarrow \underline{S = \sum_{x \in \{1,2,\dots 6\}} x \, P_X(x)}$$

$$\Rightarrow \underline{S = LHS} \qquad \Rightarrow \underline{HP}$$