MA4.101 Real Analysis

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Assignment 2

Name : Gowlapalli Rohit

Roll No : 2021101113

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Theorem-1: Let f be a real-valued function defined on a subset S of the real-line. Let p be a limit point of S - that is, p is the limit of some sequence of elements of S distinct from p. The limit of f, as x approaches p from values in S, is L, if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $0 < |x-p| < \delta$ and $x \in S \Rightarrow |f(x) - L| < \epsilon$

$$L = \lim_{\substack{x \to p \\ x \in S}} f(x)$$

Theorem-2:- If $x, y \in \mathbb{R}$ and $x < y$, $\exists r \in \mathbb{Q}$ such that $x < r < y$ (or) $r \in (x, y)$

proof:- Let $\epsilon = y - x > 0$, By the Archimedean property $\exists n \in \mathbb{N}$ such that $0 < 1/n < \epsilon$ which implies that $ny - nx > 1$. Hence $\exists m \in \mathbb{Z}$ such that (by archimedean property as well)

$nx < m < ny$ which proves the result

$r = m/n$

Theorem-3: If $x, y \in \mathbb{R}$ and $x < y$, $\exists r \notin \mathbb{Q}$ such that $x < r < y$ (or) $r \in (x, y)$

$\Rightarrow$ Repeated application of Theorem-2 shows that there are, infact, infinitely many rational numbers b/w any pair of distinct real numbers, it can also be used to prove that the irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$

→ 4

Theorem: Let a, b be real no's with $a < b$. Then there is a sequence of rational numbers $\langle x_n \rangle, \langle w_n \rangle$ in (a, b) such that $\langle x_n \rangle$ converges to a and $\langle w_n \rangle$ converges to b

Proof:- The proof uses that fact that whenever $c, d \in \mathbb{R}$ with $c < d$ $\exists$ 1 rational & 1 irrational number in (c, d)

We want to find a sequence $\langle x_n \rangle$ of rational numbers in (a, b) with the property that $x_1 > x_2 - - -$ and such that $x_n \to a$ as $n \to \infty$

We begin, instead, by choosing ANY sequence of numbers $\langle b_n \rangle$ in (a, b) such that $b_1 > b_2 - -$ and such that $b_n \to a$ as $n \to \infty$. For ex; we could set $b_n = a + \left(\frac{b-a}{2n}\right)$. We have no idea at this point whether the numbers b_n are rational/irrational. However, since we always have $b_n > b_{n+1}$, we know that we can find atleast 1 rational numbers in (b_{n+1}, b_n). Choose such a rational number and call it x_n. This gives us a sequence of rational numbers x_n and we have

$$b > b_1 > x_1 > b_2 > x_2 > b_3 - - - > a$$

so certainly $\langle x_n \rangle \subseteq (a, b)$ and the sequence $\langle x_n \rangle$ is strictly decreasing. Finally, since

$$b_{n+1} < x_n < b_n$$

and

$$\lim_{n \to \infty} b_{n+1} = \lim_{n \to \infty} b_n = a$$

we may apply the sandwich theorem to deduce that $\lim_{n \to \infty} x_n = a$, also as required

Given the function

$$f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{R} \text{ and } x \text{ is rational} \\ 1, & \text{if } x \in \mathbb{R} \text{ and } x \text{ is irrational} \end{cases}$$

Method-1

$\Rightarrow$ We say that $\lim_{x \to a} f(x) = l$, iff for any $\epsilon > 0$, there must correspond $\delta > 0$ such that

$$x \in (a-\delta, a+\delta) \Rightarrow 0 < |x-a| < \delta$$

$$\Downarrow$$

$$|f(x) - l| < \epsilon \quad — \text{(☆)}$$

Fix $a \in \mathbb{R}$ such that $a > 0$

Assume $\epsilon = a'$ (if $a' < 0$, take $\epsilon = -a' = |a|$)

If $\lim_{x \to a} f(x)$ were to exist and equal l (say), we must exhibit a $\delta > 0$ for which

$$x \in (a-\delta, a+\delta) \setminus \{a\} \Rightarrow f(x) \in (l-\epsilon, l+\epsilon) \rightarrow (*)$$

$\Rightarrow$ There are only two possible values of l ($= \lim_{x \to a} f(x)$) i.e $l = 0$ (or) 1;

because, since otherwise, there is always a constant gap b/w the value of f_x & l and

(i) Hence $f_x - l$ wouldn't get arbitrarily small, where $l \neq 0, 1$ (we can find ϵ which doesn't follow (☆) & (*), if $L \neq 0, 1$ evidently)

(or)

(ii) Hence f_x wouldn't get arbitrarily close to l (the limit)

Case-1 : If $l = 1$, i.e $\lim_{x \to c} f(x) = 1$ for some $c \in \mathbb{R}$

From above hypothesis $L = 0$ (or) 1 if it happens to be exist

Assume $a' = \epsilon = 1/2$ & Eqn-(43)

let $\epsilon > 0$, then from Theorem-(2), $\exists$ a rational number q of the form a''/b'' (where $a'', b'' \in \mathbb{Z}$ and $b'' \neq 0$) in the interval $(a-\delta, a+\delta)$, let $x_0 = q$

$$\Rightarrow |f(x_0) - l| = |f(x_0) - 1| = |0 - 1| = 1 \nless \epsilon = 1/2$$

Hence $\lim_{x \to a} f(x) = l \neq 1$ — Eqn-(44) $\forall a \in \mathbb{R}$

(Because x_0/q (or) is a rational number & for $x \in \mathbb{Q}$, $f(x) = 0$ }

Case-2 : If $l = 0$ i.e $\lim_{x \to c} f(x) = 0$ for some $c \in \mathbb{R}$

Assume $a' = \epsilon = 1/2$ & let $\epsilon > 0$, then from Theorem-(2), $\exists$ a irrational number q' not of the form a''/b'' (where $a'', b'' \in \mathbb{Z}$ & $b'' \neq 0$) in the interval $(a-\delta, a+\delta)$, let $x_0 = q'$

$$\Rightarrow |f(x_0) - l| = |f(x_0) - 0| = |1 - 0| = 1 \nless \epsilon = 1/2$$

(Because x_0 (or) q' is an irrational number & for $x \in \mathbb{Q}'$, $f(x) = 1$)

$\Rightarrow$ Hence $\lim_{x \to a} f(x) = l \neq 0$ $\forall a \in \mathbb{R}$ — Eqn-(45)

From Eqn - 43, 44 & 48,

$\boxed{\lim_{x \to a} f(x) \text{ doesn't exist } \forall a \in \mathbb{R}}$

Method - 2

From Theorem - 4, $\exists$ a sequences of rational numbers and sequences of irrational numbers that converge to a real number $a \in \mathbb{R}$ $\longrightarrow$ 49

(From the Theorem of sequential criterion for Functional limits,

↓

Let $f : A \to \mathbb{R}$ and c be limit point of A, then the following two conditions are equivalent

i) $\lim_{x \to c} f(x) = L$

ii) For all sequences (x_n) satisfying $x_n \in A$, $x_n \neq c$ and $\langle x_n \rangle \to c$, it follows that $(f(x_n)) \to L$ $\longrightarrow$ Eqn - 47

(Since a is arbitrary, it applies $\forall a \in \mathbb{R}$)

From Eqn - 49 & 47,

Sequence 1 : $\langle c_n \rangle \Rightarrow$ (of rational numbers where $\lim_{n \to \infty} c_n = a$) $\longrightarrow \lim_{n \to \infty} f(\langle c_n \rangle) = 0$

Sequence - 2 : $\langle d_n \rangle \Rightarrow$ (of irrational numbers where $\lim_{n \to \infty} d_n = a$) $\longrightarrow \lim_{n \to \infty} f(\langle d_n \rangle) = 1$

$\lim_{n \to \infty} f(\langle c_n \rangle) \neq \lim_{n \to \infty} f(\langle d_n \rangle)$ (but $\langle c \rangle \to a$, $\langle d \rangle \to a$)

Hence from Eqn - 47, $\boxed{\lim_{x \to a} f(x) \text{ doesn't exist}}$

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(4)

Theorem: Let a, b be real no's with $a<b$. Then there is a sequence of rational numbers $\langle x_n \rangle, \langle w_n \rangle$ in (a,b) such that $\langle x_n \rangle$ converges to a and $\langle w_n \rangle$ converges to b

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$$b_{n+1} < x_n < b_n$$

and

$$\lim_{n\to\infty} b_{n+1} = \lim_{n\to\infty} b_n = a$$

we may apply the sandwich theorem to deduce that $\lim_{n\to\infty} x_n = a$, also as required

Theorem - 7: For any real number α and integer N, $\exists$ a $\delta > 0$ such that every rational number in the interval $(\alpha-\delta, \alpha+\delta)$, not equal to α, has denominator greater than N

proof:- We'll pick $\delta < 1$. Consider the rational numbers with denominator smaller than N, that is the denominator (which by our convention is +ve) is an integer from the set $\{1, 2, \ldots N\}$. Any such rational number has a bound

$$\frac{|m|}{N} < \left|\frac{m}{n}\right| < |m|$$

But the rational numbers have to be in the interval $(\alpha-\delta, \alpha+\delta)$ and so in particular in the interval $(\alpha-1, \alpha+1)$. That is,

$$-|\alpha|-1 < \left|\frac{m}{n}\right| < |\alpha|+1$$

Combining with the above inequalities,

$$-|\alpha|-1 < |m| < N(|\alpha|+1)$$

Since m is an integer, this only leaves a finitely many choices, say $m_1, m_2 \ldots m_k$. So in all we only have finitely many rational numbers $r_1, \ldots r_L$ such that

- If α is rational, then $r_k \neq \alpha$ $\forall k$
- Denominator of $r_k < N$
- $r_k \in (\alpha-1, \alpha+1)$

Let

$$\delta_0 = \frac{1}{2} \cdot \min(|r_1-\alpha|, \ldots |r_L-\alpha|)$$

and let $\delta = \min(\delta_0, 1)$

Then clearly $\delta > 0$. Also if r is rational in $(\alpha-\delta, \alpha+\delta)$, then the denominator of r has to be bigger than N

Given the function $f:[0,1] \to \mathbb{R}$ defined by

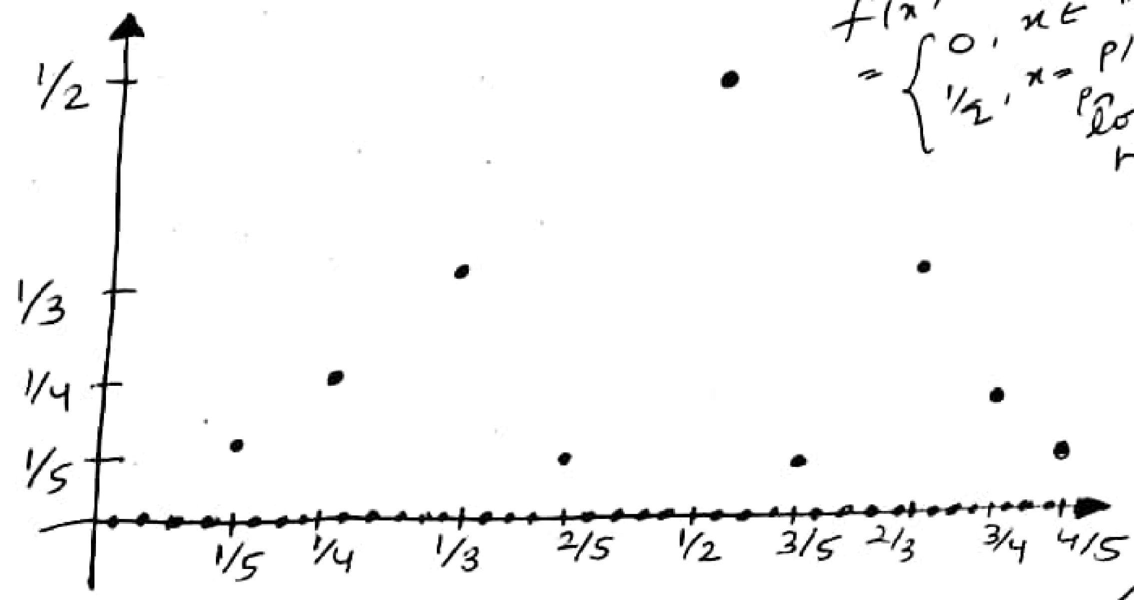
$$f(x) = \begin{cases} 1/q, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

where $x = p/q$ in lowest terms

RTP:- $\lim_{x \to a} f(x) = 0 \quad \forall\, a \in (0,1)$

The function f approaches the limit l near a means: for every $\epsilon > 0$ there is ~~some~~ $\delta > 0$ such that, for all x, if $0 < |x-a| < \delta$, then $|f(x) - l| < \epsilon$

$f(x) = \begin{cases} 0, & x \in \mathbb{R} - \mathbb{Q} \\ 1/q, & x = p/q \text{ in lowest term} \end{cases}$



$\Rightarrow$ Rough sketch of $f(x)$ in $x \in [0,1]$ (figure not to scale)

For any $a \in \mathbb{R}$ ($a \in [0,1]$), the function f approaches 0 at a

Consider any $\epsilon > 0$ (where ϵ belongs to $\mathbb{R}$)

From Archimedean property, $\exists\, n \in \mathbb{N}$ such that

$$1/n \le \epsilon$$

$\Rightarrow$ Let S be a set

$$S = \{x \mid |f(x) - 0| < \epsilon\}$$

Then S' (complement of S) can include real numbers $\in [0,1]$ like

$1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, \ldots \frac{1}{n}, \frac{n-1}{n}$

(If a is of the form p/q where $p, q \in \mathbb{Z}$ and $q \neq 0$ and $\gcd(p,q) = 1$

then a may belong to S')

$\Rightarrow$ However, many of these numbers there may be, there are, at any rate finitely many

$|S'|$ = finitely many $\rightarrow$ cardinality

(From cantor's theorem & diagonalization method, this can be proved)

$\Rightarrow$ Let $y \in S'$ such that $|y-a| \leq |x-a|$ $\forall x \in S'$

(minimum (where $y \neq a$))

$\Rightarrow$ Let $\delta = |y-a|$ (y is obtained from the above comparison)

Let $S'' = \{x \mid 0 < |x-a| < \delta\}$

Then $(S'')'$ (complement of S'') contains the numbers $1/2, \ldots\ldots, \frac{n-1}{n}$

$0 < |x-a| < \delta$

and therefore $\boxed{|f(x) - 0| < \epsilon \text{ is true } \forall x \in [a-\delta, a+\delta]}$

(Therefore, we have chosen δ such that $\forall x$. if $0<|x-a|<\delta$, then $|f(x)-0|<\epsilon$) for all $\epsilon>0$

(This happens because our choice of ϵ is arbitrary)

(We basically resolved all numbers to not belonging to S, by choosing necessary δ) and hence solved the functional limitation

$\Rightarrow$ (There is no need of establishing a relation b/w δ and ϵ, as that's not a requirement from the definition of functional limits)

$\Downarrow$

For every $\epsilon>0$, there is some $\delta>0$ such that, $\forall x$ if $0<|x-a|<\delta$, then $|f(x)-0|<\epsilon$

$\Downarrow$

$f(x)$ approaches the limit 0

$\Downarrow$

$\lim_{x\to a} f(x)=0$

since our choice of a was arbitrary in $[0,1]$

$\Downarrow$

$$\boxed{\lim_{x\to a} f(x)=0 \text{ for all } a\in[0,1)}$$

Alternate Solution

Given that $f:[0,1] \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 1/q, & \text{if } x \text{ is rational where } x = p/q \text{ in lowest terms} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

RTP:- For any real number α, $\lim_{t \to \alpha} f(t) = 0$

Let $\epsilon > 0$, then $\exists$ N such that N is an integer and $N > 1/\epsilon$ from Archimedian property

From Theorem-⑦, corresponding to this N, $\exists$ a δ such that for any rational number in $t = m/n$ such that $0 < |\alpha - t| < \delta$ satisfies $n > N$

But, then $0 < f(t) = 1/n < 1/N < \epsilon$

$$\downarrow$$

$$|f(t) - 0| < \epsilon \longrightarrow \lim_{t \to \alpha} f(t) = 0$$

On the other hand, for any irrational number t,

$f(t) = 0$ and so for any real number $t \neq \alpha$ such that $|t - \alpha| < \delta$, we have that $|f(t)| < \epsilon$

$$\downarrow$$

$$\lim_{t \to \alpha} f(t) = 0 \longleftarrow |f(t)| - 0| < \epsilon$$

$\Rightarrow$ Hence for any real number $\alpha \in (0,1)$

$$\lim_{t \to \alpha} f(\alpha) = 0$$

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Theorem-1: Let f be a real-valued function defined on a subset S of the real-line. Let p be a limit point of S — that is, p is the limit of some sequence of elements of S distinct from p. The limit of f, as x approaches p from values in S, is L, if $\forall \epsilon > 0, \exists \delta > 0$ such that $0 < |x-p| < \delta$ and $x \in S \Rightarrow |f(x) - L| < \epsilon$

$$L = \lim_{\substack{x \to p \\ x \in S}} f(x)$$

Theorem-2:- If $x, y \in \mathbb{R}$ and $x < y$, $\exists r \in \mathbb{Q}$ such that $x < r < y$ (or) $r \in (x, y)$

proof:- Let $\epsilon = y - x > 0$, By the Archimedian-property $\exists n \in \mathbb{N}$ such that $0 < 1/n < \epsilon$ which implies that $ny - nx > 1$. Hence $\exists m \in \mathbb{Z}$ such that (by archimedian property as well) $nx < m < ny$ which proves the result

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Theorem-3: If $x, y \in \mathbb{R}$ and $x < y$, $\exists r \notin \mathbb{Q}$ such that $x < r < y$ (or) $r \in (x, y)$

$\Rightarrow$ Repeated application of Theorem-2 shows that there are, infact, infinitely many irrational numbers b/w any pair of distinct real numbers, it can also be used to prove that the irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$

Theorem-(4):- If $a \in \mathbb{R}$ is such that $0 \leq a <$ ϵ for every $\epsilon > 0$, then $a = 0$

proof:- Suppose to the contrary that $a > 0$
Then, if we take $\epsilon_0 = \frac{1}{2}a$, we have $0 < \epsilon_0 < a$
Therefore, it is false that $a < \epsilon$ for every $\epsilon > 0$ and we conclude that $a = 0$

Given a periodic function $f(x)$.

RTP:- If $\lim_{x\to\infty} f(x)$ exists, then $f(x)$ is a constant function

Let the period of f be T

$\hookrightarrow \exists T \in \mathbb{R}^+, \forall x \in \mathbb{R}$,

$f(x+T) = f(x)$

Method-1

$\lim_{x\to\infty} f(x)$ exists and is equal to l

$\Rightarrow$ RTP:- f is a constant function

Assume that $f(x)$ is not a constant function

Then $\exists a, b \in \mathbb{R}$; $\boxed{f(a) \neq f(b)}$

$\hookrightarrow$ Eqn-(5)

$\lim_{x\to\infty} f(x) = l$

From the definition of functional limits,

$\exists N>0$ such that $|f(x)-l| < \epsilon$ $\forall x \in (\mathbb{N} - \{x \leq N\})$

$= \forall x > N$

for all $\epsilon > 0$

Given that $f(x)$ is periodic, hence

$\exists x > N$ such that $f(x) = f(a)$

(For that particular x) $\hookrightarrow x = a + kT$ ($k \in \mathbb{N}$) for some

$\Rightarrow$ Let $d = |l - f(a)|$

$0 \leq d < \epsilon$ and $d \in \mathbb{R}$ $\forall \epsilon > 0$ $\rightarrow$ arbitrary choice of ϵ.

[From Theorem-(4)]

$\Rightarrow$ $(d = 0)$

(Suppose to the contrary that $d > 0$, then if we take $\epsilon_0 = \frac{1}{2}d$, we have $0 < \epsilon_0 < d$. Therefore, it is false that $d < \epsilon$ for every $\epsilon > 0 \Rightarrow$ we conclude that $d = 0$)

$l = f(a)$ ⟶ Eqn – 53

Given that $f(x)$ is periodic, hence

[redacted] $\exists N > 0$ such that $|f(x) - l| < \epsilon$ $\forall x > N$ for all $\epsilon > 0$

$\exists x > N$ such that $f(x) = f(b)$

↳ $x = b + kT$ for some $(k \in \mathbb{N})$

From definition of Functional Limits

Let $h = |l - f(b)|$ (For particular x)

$0 \leq h < \epsilon$ and $h \in \mathbb{R}$ $\forall \epsilon > 0$ → arbitrary choice of ϵ

$\Rightarrow h = 0$ [From theorem – 4]

(Suppose to the contrary that $h > 0$, then if we take $\epsilon_0 = \frac{1}{2}h$, we have $0 < \epsilon_0 < h$. Therefore, it is false that $h < \epsilon$ for every $\epsilon > 0 \Rightarrow$ we conclude that $h = 0$)

$\Rightarrow \boxed{l = f(b)}$ — Eqn – 54

From eqn – 53 & 54

$\Rightarrow l = f(a)$ ⟶ $f(a) = f(b)$
$= f(b)$

But from Eqn – 51 $\Rightarrow f(a) \neq f(b)$

Contradiction

$\Rightarrow$ Contradiction $\longrightarrow$ Our assumption is wrong

(f(x) is not a constant function) is wrong

f(x) is a constant function

$\Rightarrow$ (If $\lim_{x\to\infty} f(x)$ exists, then f(x) is a constant function)

Eqn - 62

(when f(x) is periodic function)

(Basically, if f is not constant function, where $\exists\ x_1, x_2 \in [0,T)$ where $f(x_1) \neq f(x_2)$, $\lim_{n\to\infty} f(x_j + NT) = f(x_j)$ for $j = 1, 2$

Thus $\lim_{x\to\infty} f(x)$ doesn't exist)

Method-2

Let $\epsilon > 0$

Given that $\lim_{x\to\infty} f(x)$ exists and is equal to l

i.e.,

$$\boxed{\lim_{x\to\infty} f(x) = l}$$

Hence, there exists $M_{\epsilon/2} > 0$ such that $\forall x \geq M_{\epsilon/2}$, $|f(x) - l| < \frac{1}{2}\epsilon$

Since this holds $\forall x \geq M_{\epsilon/2}$, let $x_1, x_2 \geq M_{\epsilon/2}$

it is $|f(x_1) - f(x_2)|$

$$= |(f(x_1) - l) + (l - f(x_2))|$$

$$\leq |f(x_1) - l| + |f(x_2) - l| < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon$$

Assume that f is not constant, i.e $\exists\, x, y$ such that $|f(x) - f(y)| = \delta > 0$, (Given that T is the period of f)

Let $x, y \in [0, T]$ WLOG

Let $\epsilon = \delta/2$

$\Rightarrow$ There is a M_ϵ and $m \in \mathbb{N}$ such that

$x + MT,\ y + MT \geq M_\epsilon$ and

$$|f(x+MT) - f(y+MT)| = |f(x) - f(y)| = \delta$$

$$< \epsilon = \delta/2$$

Contradiction

$\downarrow$

$\boxed{f \text{ is constant function}}$

$\downarrow$

$\boxed{\text{If } \lim_{x \to \infty} f(x) \text{ exists, then } f(x) \text{ is a constant function}}$

Eqn - 63

(when f is a periodic function)

Let p be the statement that $\lim_{x \to \infty} f(x)$ exists

Let q be the statement that $f(x)$ is a constant function

From Eqn - 62 and 63,

$\boxed{\text{when } f \text{ is a periodic function, } p \rightarrow q}$

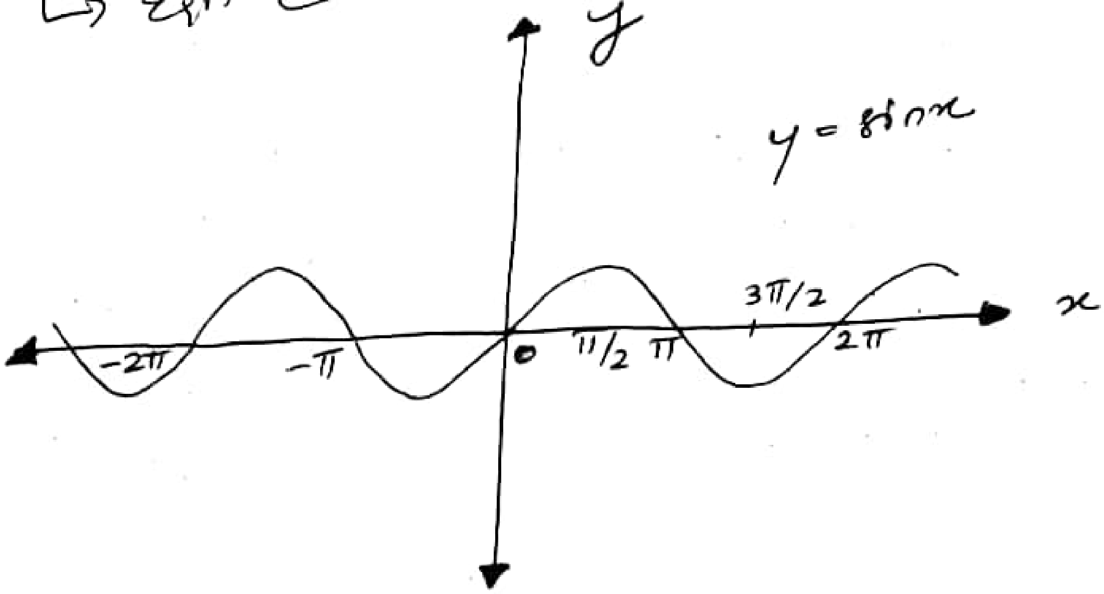
From Laws of Boolean algebra,

$\boxed{\text{If } p \rightarrow q, \text{ then } \sim q \rightarrow \sim p}$

When f is a periodic function,

$\boxed{\text{If } f \text{ is not a constant function, } \lim_{x\to\infty} f(x) \text{ doesn't exist}}$

↳ Eqn - 65



From Eqn - 65,

$y = \sin x = f(x)$ is <u>not</u> a constant function

$f(0) = \sin 0^c = 0$ $\Rightarrow f(0) \neq f(\pi/2)$

$f(\pi/2) = \sin \pi/2^c = 1$

$\Rightarrow$ $y = \sin x$ is a periodic function

$\sin(x) = \sin(2\pi + x)$ ($T = 2\pi$ is period of f)

↓

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \sin x$ doesn't exist

(because $\sin x$ is a periodic, non-constant function)

$\boxed{\lim_{x\to\infty} \sin x \text{ does not exist}}$

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Theorem-3: If $x, y \in \mathbb{R}$ and $x < y$, $\exists r \notin \mathbb{Q}$ such that $x < r < y$ (or) $r \in (x, y)$

$\Rightarrow$ Repeated application of Theorem-2 shows that there are, infact, infinitely many-irrational numbers b/w any pair of distinct real numbers, it can also be used to prove that the irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$

1 $f(x) = \dfrac{\sin x}{\sqrt{1-\cos x}} = \dfrac{\sin x}{(1-\cos x)^{1/2}}$

$\left(\cos x = 1 - 2\sin^2 x/2 \right)$

$$= \frac{\sin x}{\sqrt{1-(1-2\sin^2 x/2)}}$$

$$= \frac{\sin x}{\sqrt{2\sin^2 x/2}}$$

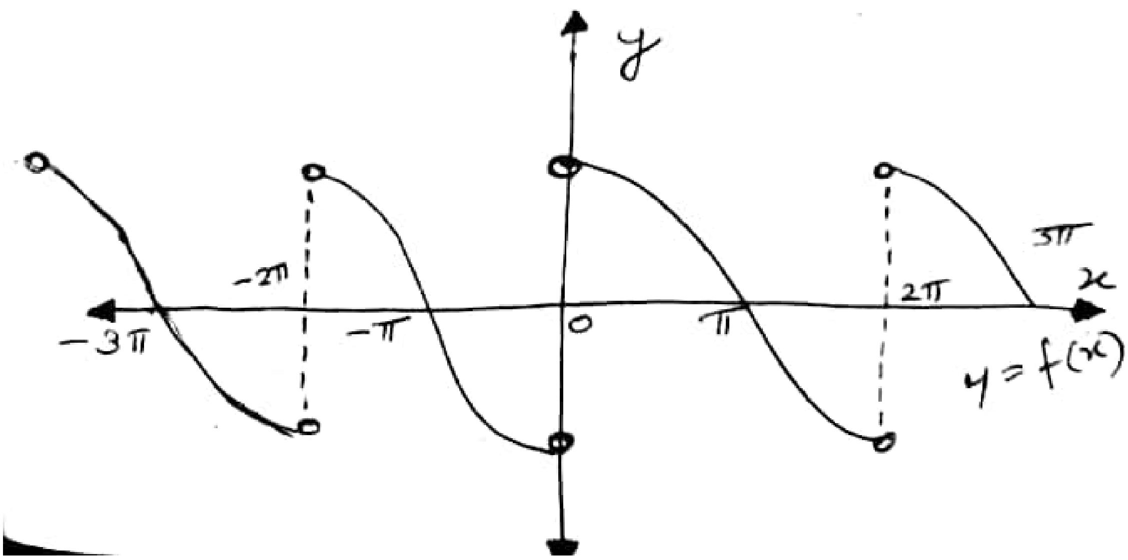
$$= \frac{1}{\sqrt{2}}\left(\frac{\sin x}{\sqrt{\sin^2 x/2}}\right)$$

$$= \frac{1}{\sqrt{2}}\left[\frac{\sin x}{|\sin x/2|}\right]$$

$$= \begin{cases} 1/\sqrt{2}\left(\dfrac{\sin x}{-\sin x/2}\right), & \sin\frac{x}{2} < 0 \\ 1/\sqrt{2}\left(\dfrac{\sin x}{\sin\frac{x}{2}}\right), & \sin\frac{x}{2} > 0 \end{cases}$$

$$= \begin{cases} \dfrac{1}{\sqrt{2}}\left(\dfrac{2\sin\frac{x}{2}\cos\frac{x}{2}}{-\sin x/2}\right), & \sin\frac{x}{2} < 0 \\ \dfrac{1}{\sqrt{2}}\left(\dfrac{2\sin x/2 \cos x/2}{\sin x/2}\right), & \sin\frac{x}{2} > 0 \end{cases}$$

$$\boxed{f(x) = \begin{cases} -\sqrt{2}\cos x/2, & \sin\frac{x}{2} < 0 \\ \sqrt{2}\cos x/2, & \sin\frac{x}{2} > 0 \end{cases}}$$



⇒ Let I be an open-interval containing c, and let f be a function defined on I, except possibly at c. The limit of $f(x)$, as x approaches c from the left, is L, or, the left-hand limit of f at c is L, denoted by

$$\lim_{x \to c^-} f(x) = L$$

means that given any $\epsilon > 0$, $\exists\ \delta > 0$ such that $\forall x < c$, if $|x-c| < \delta$, then $|f(x) - L| < \epsilon$

⇒ Let I be an open-interval containing c, and let f be a function defined on I, except possibly at c. The limit of $f(x)$, as x approaches c from the right, is L, or the Right-hand-limit of f at c is L, denoted by

$$\lim_{x \to c+} f(x) = L$$

means that given any $\epsilon > 0$, $\exists\ \delta > 0$ such that $\forall x > c$, if $|x-c| < \delta$, then $|f(x) - L| < \epsilon$

⇒ Limit exists precisely when the left and right-hand limits are equal

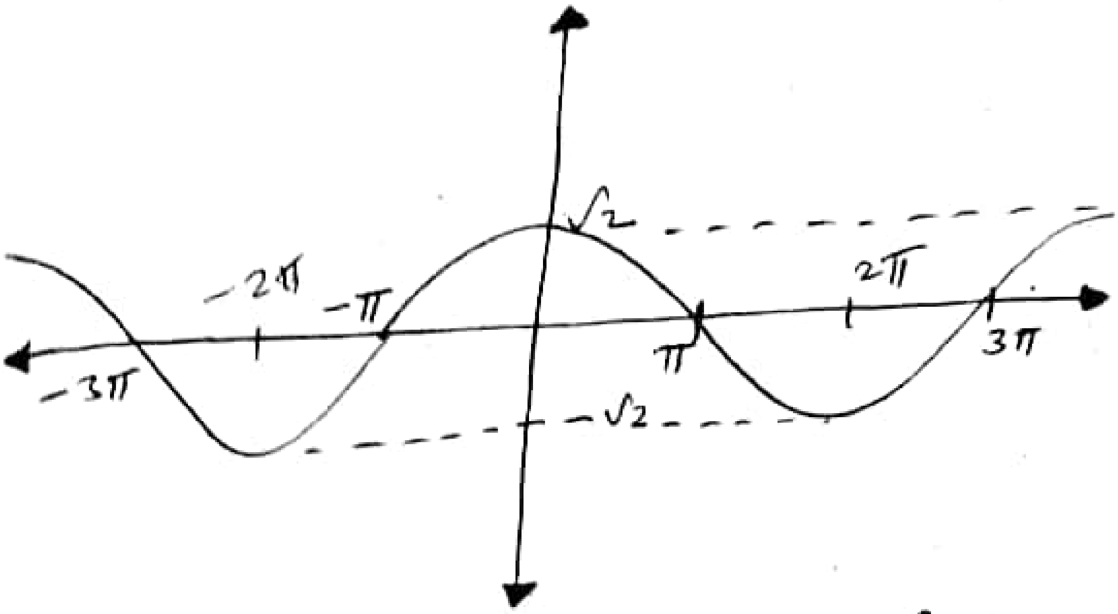
Let f be a function defined on an open-interval I containing c. Then

$$\lim_{x \to c} f(x) = L$$

if and only if,

$$\lim_{x \to c^-} f(x) = L \text{ and } \lim_{x \to c+} f(x) = L$$

$y_1 = \sqrt{2}\cos x/2$



$$L_1 = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sin x}{\sqrt{1-\cos x}}$$

$$= \frac{1}{\sqrt{2}} \lim_{x\to 0^+} \frac{\sin x}{|\sin \frac{x}{2}|}$$

$$= \frac{1}{\sqrt{2}} \lim_{x\to 0^+} \frac{\left(\frac{\sin x}{x}\right)^{\to 1}(2)}{\underbrace{\left(\frac{\sin x/2}{x/2}\right)}_{1=}}$$

$$\frac{1}{\sqrt{2}} \lim_{x\to 0^+} (2\cos \tfrac{x}{2})$$

$$= \frac{2\cos 0}{\sqrt{2}}$$

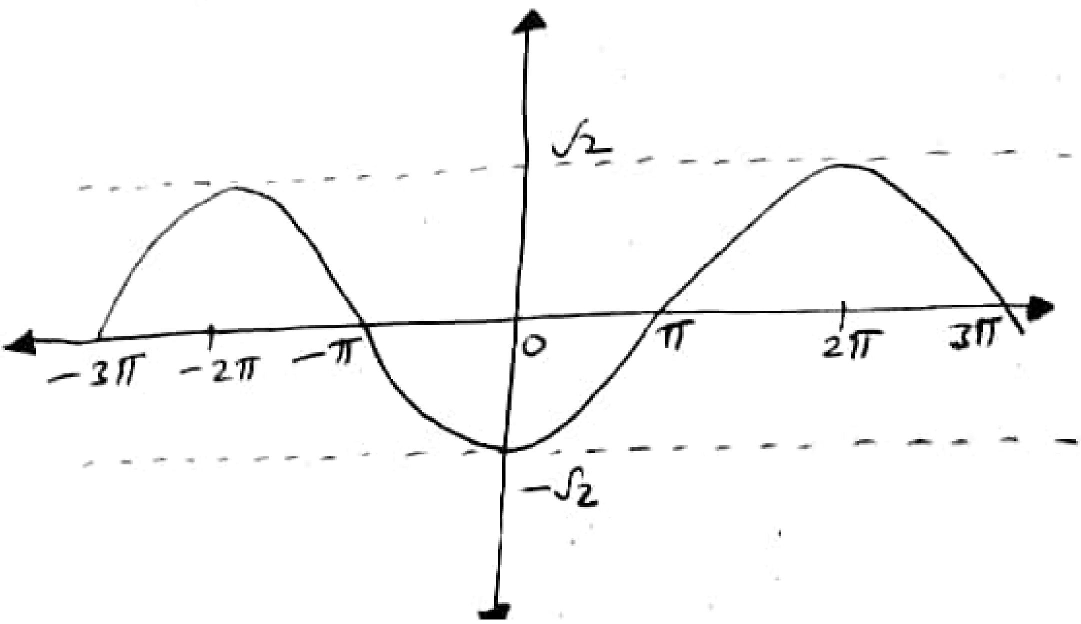
$$= \sqrt{2}$$

same as

$$= \sqrt{2}$$

$$\boxed{L_1 = \sqrt{2}}$$

$y_2 = -\sqrt{2}\cos x/2$



$$L_2 = \lim_{x \to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin x}{\sqrt{1-\cos x}}$$

$$= \lim_{x\to 0^-} \frac{\sin x}{|\sin x/2|}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{-1}{\sqrt{2}} \lim_{x\to 0^-} \frac{\sin x}{-\sin x/2}$$

$$= \frac{-1}{\sqrt{2}} \lim_{x\to 0^-} \frac{\left(\frac{\sin x}{x}\right)}{\left(\frac{\sin x/2}{x/2}\right)} (2)$$

$\left(\frac{\sin x}{x}\right) = 1$, $1 = \left(\frac{\sin x/2}{x/2}\right)$

same as

$-\frac{1}{\sqrt{2}} \lim_{x\to 0^-} (2\cos\frac{x}{2}) = -\frac{1}{\sqrt{2}} \lim_{x\to 0^-} (2\cos 0^\circ)$

$= -\sqrt{2}$

$$= -\sqrt{2}$$

$\Rightarrow \boxed{L_2 = -\sqrt{2}}$

From Eqn-(71), limit exists precisely when the left-hand and right-hand limits are equal

$\Rightarrow$ But $\underset{\text{RHL}}{\underset{\downarrow}{L_1}} = \sqrt{2}$ and $\underset{\text{LHL}}{\underset{\downarrow}{L_2}} = -\sqrt{2} \rightarrow L_1 \neq L_2$

Therefore, since the limits from different directions are different, the limit doesn't exist

$\hookrightarrow \lim_{x\to 0} f(x)$ doesn't exist

$\Downarrow$

$$\boxed{\lim_{x\to 0} \frac{\sin x}{\sqrt{1-\cos x}} \text{ doesn't exist}}$$

(5)

Theorem-5 : proof of L'Hôpital's rule

Suppose that f and g are continuously differentiable at a real number c that $f(c) = g(c) = 0$ and that $g'(c) \neq 0$. Then,

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f(x) - 0}{g(x) - 0} = \lim_{x \to c} \frac{f(x) - f(c)}{g(x) - g(c)}$$

$$= \lim_{x \to c} \frac{\left(\frac{f(x) - f(c)}{x - c}\right)}{\left(\frac{g(x) - g(c)}{x - c}\right)}$$

$$= \frac{\lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c}\right]}{\lim_{x \to c} \left[\frac{g(x) - g(c)}{x - c}\right]}.$$

$$= \frac{f'(c)}{g'(c)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

⇒ This follows from difference-quotient definition of the derivative

⇒ The last equality follows from the continuity of derivatives at c

⇒ The limit in the conclusion is not indeterminate because $g'(c) \neq 0$

(4)

Theorem: Let a, b be real no's with $a < b$. Then there is a sequence of rational numbers $\langle x_n \rangle, \langle w_n \rangle$ in (a,b) such that $\langle x_n \rangle$ converges to a and $\langle w_n \rangle$ converges to b

Proof:- The proof uses that fact that whenever $c, d \in \mathbb{R}$ with $c < d$ $\exists$ 1 rational & 1 irrational number in (c,d)

We want to find a sequence $\langle x_n \rangle$ of rational numbers in (a,b) with the property that $x_1 > x_2 - - -$ and such that $x_n \to a$ as $n \to \infty$
We begin, instead, by choosing ANY sequence of numbers $\langle b_n \rangle$ in (a,b) such that $b_1 > b_2 - -$ and such that $b_n \to a$ as $n \to \infty$. For ex; we could set $b_n = a + \left(\frac{b-a}{2n}\right)$. We have no idea at this point whether the numbers b_n are rational/irrational. However, since we always have $b_n > b_{n+1}$, we know that we can find atleast 1 rational numbers in (b_{n+1}, b_n). Choose such a rational number and call it x_n. This gives us a sequence of rational numbers x_n and we have

$$b > b_1 > x_1 > b_2 > x_2 > b_3 - - - - > a$$

so certainly $\langle x_n \rangle \subseteq (a,b)$ and the sequence $\langle x_n \rangle$ is strictly decreasing. Finally, since

$$b_{n+1} < x_n < b_n$$

and

$$\lim_{n \to \infty} b_{n+1} = \lim_{n \to \infty} b_n = a$$

we may apply the sandwich theorem to deduce that $\lim_{n \to \infty} x_n = a$, also as required

Given that $f: R \to R$ such that

$\boxed{f(x+y) = f(x) + f(y)}$ $\forall\ x, y \in R$

Also given that $\boxed{\lim_{x \to 0} f(x) = f(0)}$

Eqn-1

$f(x+y) = f(x) + f(y)$

Assume $x = y = 0$; then,

$f(x+y) = f(0+0) = f(0) + f(0)$

$= f(0) = 2 \times f(0)$

$\boxed{f(0) = 0}$

$\lim_{x \to 0} f(x) = f(0) = 0$

$\boxed{\lim_{x \to 0} f(x) = 0}$

Method-1

Derivative by the first principle refers to using algebra to find a general expression for the slope of a curve

The derivative is a measure of the instantaneous rate of change, which is equal to

$$f'(x) = dy/dx = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x}$$

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

$$\Rightarrow f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x) + f(h) - f(x)}{h} \quad \left[\text{from Eqn-1}\right]$$

$$= \lim_{h \to 0} \left(\frac{f(h)}{h}\right) = (0/0 \text{ form})$$

From Theorem-⑤, (L-Hôpital Rule)

$$= \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} \frac{f'(h)}{\frac{d}{dh}(h)} = \lim_{h\to 0} \frac{f'(h)}{1}$$

$$= \lim_{h\to 0} f'(h)$$

$$= f'(0)$$

say c

alternately,

$$\lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} \frac{f(0+h)-0}{h} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = f'(0)$$

$\Rightarrow \quad f'(x) = f'(0) = c$

$\frac{dy}{dx} = c$

$\Rightarrow \int dy = \int c\, dx$

$\Rightarrow \quad y = cx + e$

But $y(0) = 0$, $c(0) + e = 0 \Rightarrow e = 0$

$\boxed{y = cx}$

$\downarrow$

$\boxed{y = f'(0)(x)}$

$\downarrow$

$\boxed{f(x) = \underbrace{[f'(0)]}_{\text{say } k} x}$

$\downarrow$

$f(x) = kx \longrightarrow$ (where $k = f'(0)$)

$\longrightarrow (k \in \mathbb{R})$

Method-2

The equation $f(x+y) = f(x) + f(y)$ — Functional eqn

breaks down to

$f(2x) = f(x) + f(x) = 2 \times f(x)$

$f(3x) = f(2x+x) = f(2x) + f(x) = 2f(x) + f(x)$
$= 3f(x)$

$\vdots$

$\boxed{f(nx) = n f(x)} \longrightarrow$ (From induction on n)

Let r be of the form P/q i.e $r = P/q$

rational number

$f(qrx) = f(px) = pf(x)$ where $p, q \in \mathbb{Z}$ (and $q \neq 0$)

$f(qrx) = qf(rx)$

$\Rightarrow qf(rx) = pf(x) \Rightarrow f(rx) = P/q\, f(x)$

$$\boxed{f(rx) = rf(x)}$$

In particular, we have

$$\boxed{f(r) = rf(1)}$$

substitute x as 1

Let $x \in \mathbb{R}$. Then $\exists$ a sequence of rational numbers $\{r_n\}$ such that

$$\lim_{n \to \infty} r_n = x$$

$$f(x) = f(x - r_n + r_n)$$
$$= f(x - r_n) + f(r_n)$$
$$= f(x - r_n) + r_n f(1)$$

same as $f(x)$

$$\lim_{n\to\infty} f(x) = \lim_{n\to\infty} (x - r_n) + \lim_{n\to\infty} (r_n f(1))$$

$$= f(0) + xf(1) \quad \left[\text{Because } \lim_{n\to 0} f(x) = f(0)\right]$$

But we know that $f(0) = 0$ (put $x, y = 0$ in functional eqn)

$$= 0 + xf(1)$$

$$f(x) = xf(1)$$

Let $m = f(1)$

$$f(x) = mx$$ where $m = f(1)$

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