MA4.101 Real Analysis

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Assignment 1

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(x be an element of set)

* Let A, B be arbitrary sets (properties)

1. If $x \in A$ (or) $x \in B \Leftrightarrow x \in A \cup B$
2. If $x \in A$ and $x \in B \Leftrightarrow x \in A \cap B$
3. If $x \notin A \Leftrightarrow x \in A^c$

* If f is invertible and $f(x) = y$, then $y = f^{-1}(x)$

* A function f is called one-one function if $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$

* A function f is called onto function if codomain = range of f
  (Every element in codomain has pre-image in domain)

* The Cartesian product $X \times Y$ between two sets X and Y is the set of all possible ordered pairs with first element from X and second element from Y.

$$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}$$

* A function f is called bijective if it's both one-one & onto function.

* $\boxed{A \times \phi = \phi \times A = \phi}$

* ($x \in A$ and $y \in C$) and ($x \in B$ and $y \in C$)

$$\equiv (x \in A \text{ and } x \in B) \text{ and } y \in C$$

* ($x \in A$ and $z \in C$) (or) ($x \in B$ and $z \in C$)

$$\equiv [(x \in A \text{ (or) } x \in B)] \text{ and } z \in C$$

<u>Theorem-1</u>: Consider two arbitrary sets A and B. Then, $A \subseteq A \cup B$ and $B \subseteq A \cup B$

<u>proof</u>:- Suppose $x \in A$. That is, x is an element of (A set is well defined collection of objects) A. Then certainly x is either an element of A or an element of B (or both). Thus x is an element of $A \cup B$, i.e $x \in A \cup B$. Since, we have shown that any element of A is an element of $A \cup B$, $A \subseteq A \cup B$ (similarly for B)

<u>Theorem-2</u>:- Consider Two arbitrary sets A and B. Then $A \cap B \subseteq A$ and $A \cap B \subseteq B$

<u>proof</u>:

Let $x \in A \cap B$

From the definition of intersection of sets,

$x \in A$ and $x \in B$

$\Rightarrow$ Thus in particular, $x \in A$

If $x \in A \cap B$, then $x \in A \Rightarrow A \cap B \subseteq A$

(Similarly for B)

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* $A \cup \emptyset = A$ * $A \cap \emptyset = \emptyset$ * $A \cup B = B \cup A$
* $A \cap B = B \cap A$ * $(A \cup B) \cup C = A \cup (B \cup C)$
* $(A \cap B) \cap C = A \cap (B \cap C)$

associative laws

* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Distributive laws

* $A \cup A = A \cap A = A$
* $\underbrace{(A \cup B)^c = A^c \cap B^c \quad * \ (A \cap B)^c = A^c \cup B^c}_{\text{De-Morgan's Laws}}$

Given two sets A and B

RTP:- The following statements are equivalent.

ⓐ $A \subseteq B$

ⓑ $A \cup B = B$

($\Leftarrow$) From Theorem-1,

$A \subseteq A \cup B$

But $A \cup B = B$

$\Rightarrow \boxed{A \subseteq B}$

$\boxed{\text{If } A \cup B = B \text{, then } A \subseteq B}$

($\Rightarrow$) Given that $A \subseteq B$

Let $x \in A$

From ⓐ, $A \subseteq B$

$\boxed{\text{If } x \in A \text{, then } x \in B}$ — (34)

Let $y \in A \cup B$

then, $y \in A$ (or) $y \in B \Rightarrow y \in A \cup B$

From eqn-(34),

$y \in A$ (or) $y \in B \equiv y \in B$ (or) $y \in B$

$\equiv y \in B$

If $y \in A \cup B$, then $y \in B \Rightarrow \boxed{A \cup B \subseteq B}$ — (61)

From Theorem-(1), we know that

$\boxed{B \subseteq A \cup B}$ — (62)

From Eqns, (61) and (62);

$\boxed{A \cup B = B}$

$\boxed{\text{If } A \subseteq B \text{, then } A \cup B = B}$

$\boxed{\text{Hence ⓐ and ⓑ are equivalent}}$ — (41)

RTP:- The following statements are equivalent

(c) $A \cap B = A$

(a) $A \subseteq B$

($\Rightarrow$) Given that $A \cap B = A$, We know that

$A \cap B \subseteq B$

But from (c), $A \cap B = A$

$\boxed{A \subseteq B}$

$\boxed{\text{If } A \cap B = A \text{ , then } A \subseteq B}$

($\Leftarrow$) Given that $A \subseteq B$

Let $x \in A$

From (a), $A \subseteq B$

$\boxed{\text{If } x \in A, \text{ then } x \in B}$ - (34)

Let $y \in A \cap B$

then, $y \in A$ and $y \in B$

From eqn - (34),

$y \in A$ and $y \in B \equiv y \in B$ and $y \in A$

$\equiv y \in A$ and $y \in A$

$\equiv y \in A$

$\Rightarrow$ If $y \in A \cap B$, then $y \in A$, $\boxed{A \cap B \subseteq A}$ - (63)

From Theorem - (2),

Let $z \in A$, then $z \in B \Rightarrow z \in A$ and $z \in B$

$\Rightarrow z \in A \cap B$

If $z \in A$, then $z \in A \cap B \Rightarrow \boxed{A \subseteq A \cap B}$ - (64)

From Eqns (63) & (64), $\Rightarrow \boxed{A = A \cap B}$

$\Rightarrow$ $\boxed{\text{(a) \& (c) are equivalent}}$ - (43)

RTP:- The following statements are equivalent

(b) $A \cup B = B$

(c) $A \cap B = A$

($\Rightarrow$) Given that $A \cup B = B$.

$A \cap B = A \cap (A \cup B)$ (From (b))

$= (A \cap A) \cup (A \cap B)$

$= A \cup (A \cap B)$

Let $x \in A \cap B$

$x \in A$ & $x \in B$

$\equiv$ ($x \in A$ and $x \in A$)

(or)

($x \in A$ and $x \in B$)

$\equiv$ (($x \in A$) (or) ($x \in A$ & $x \in B$))

$\equiv$ $x \in A$

From the previous proof. (a & b)

$A \cap B \subseteq A$

Hence, $A \cup (A \cap B) = A$

$\Rightarrow A \cap B = A$

$\boxed{\text{If } A \cup B = B \text{, then } A \cap B = A}$

($\Leftarrow$) Given that $A \cap B = A$

$A \cup B = (A \cap B) \cup B$ [From (c)]

$= B \cup (A \cap B)$

$= (B \cup A) \cap (B \cup B)$

$= (A \cup B) \cap (B \cup B)$

$= [(A \cup B)] \cap B$

Let $x \in A \cap B$

Let $y \in A \cup B$

$\equiv$ ($y \in A$ (or) $y \in B$)

$\equiv$ ($y \in B$ (or) ($y \in A$ & $y \in B$))

$\equiv$ [($y \in A$ (or) $y \in B$) and $y \in B$]

$\equiv y \in B$

From the previous proof, (c & a)

$(A \cup B) \supseteq B$

Hence $(A \cup B) \cap B = B$

$\Rightarrow \boxed{A \cup B = B}$

$\boxed{\text{If } A \cap B = A \text{, then } A \cup B = B}$

$\boxed{\text{(b) and (c) are equivalent}}$ — 42

From (A1), (A2), (A3) $\Rightarrow$ (a), (b) & (c) are equivalent
(similar to zeroth law of Thermodynamics)

$\Rightarrow$ $(A \subseteq B)$ & & $(A \cup B = B)$ & & $(A \cap B = A)$ are equivalent where A & B are arbitrary sets

(2)

(x be an element of set)

* Let A, B be arbitrary sets (properties)

(1) If $x \in A$ (or) $x \in B \Leftrightarrow x \in A \cup B$

(2) If $x \in A$ and $x \in B \Leftrightarrow x \in A \cap B$

(3) If $x \notin A \Leftrightarrow x \in A^c$

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* A function f is called bijective if it's both one-one & onto function.

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* $(x \in A \text{ and } y \in C)$ and $(x \in B \text{ and } y \in C)$
$\equiv (x \in A \text{ and } x \in B)$ and $y \in C$

* $(x \in A \text{ and } z \in C)$ (or) $(x \in B \text{ and } z \in C)$
$\equiv [(x \in A \text{ (or) } x \in B)]$ and $z \in C$

Theorem-1: Consider two arbitrary sets A and B. Then, $A \subseteq A \cup B$ and $B \subseteq A \cup B$

proof:- Suppose $x \in A$. That is, x is an element of (A set is well defined collection of objects) A. Then certainly x is either an element of A or an element of B (or both). Thus x is an element of $A \cup B$, i.e $x \in A \cup B$. Since, we have shown that any element of A is an element of $A \cup B$, $A \subseteq A \cup B$ (similarly for B)

Theorem-2:- Consider Two arbitrary sets A and B. Then $A \cap B \subseteq A$ and $A \cap B \subseteq B$

proof:

Let $x \in A \cap B$

From the definition of intersection of sets,

$x \in A$ and $x \in B$

$\Rightarrow$ Thus in particular, $x \in A$

If $x \in A \cap B$, then $x \in A \Rightarrow A \cap B \subseteq A$

(Similarly for B)

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* $A \cap B = B \cap A$ * $(A \cup B) \cup C = A \cup (B \cup C)$
* $(A \cap B) \cap C = A \cap (B \cap C)$

associative laws

* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
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Distributive laws

* $A \cup A = A \cap A = A$
* $(A \cup B)^c = A^c \cap B^c$ * $(A \cap B)^c = A^c \cup B^c$

De-Morgan's Laws

(a) Given two sets A and B

RTP:- $A \cup B = B \cup A$ (commutativity on union of sets)

Let $x \in A \cup B$, Then
$\Rightarrow x \in A$ (or) $x \in B$ (by definition of union of sets)

Then,
$\Rightarrow x \in B$ (or) $x \in A$ (property)
$\Rightarrow x \in B \cup A$ (by definition of union of sets)

Therefore, if $x \in A \cup B$, then $x \in B \cup A$
$\Rightarrow \boxed{(A \cup B) \subseteq B \cup A}$ (From property of sets)
$\rightarrow$ Eqn - 13

Let $y \in B \cup A$, Then
$\Rightarrow y \in B$ (or) $y \in A$ (by definition of union of sets)

Then,
$\Rightarrow y \in A$ (or) $y \in B$ (property)
$\Rightarrow y \in A \cup B$ (by definition of union of sets)

Therefore, if $y \in B \cup A$, then $y \in A \cup B$
$\Rightarrow \boxed{(B \cup A) \subseteq (A \cup B)}$ (From property of sets)
$\rightarrow$ Eqn - 14

Therefore, From
Eqn 13 & 14,

$\boxed{A \cup B = B \cup A}$ by proposition.

RTP:- $A \cap B = B \cap A$ (commutativity on intersection of sets)

Let $x \in A \cap B$, Then

$\Rightarrow x \in A$ and $x \in B$ (by definition of intersection of sets)

Then,

$\Rightarrow x \in B$ and $x \in A$ (property)

$\Rightarrow x \in B \cap A$ (by definition of intersection of sets)

Therefore, if $x \in A \cap B$, then $x \in B \cap A$ (From property of sets)

$\Rightarrow \boxed{A \cap B \subseteq B \cap A}$ → Eqn - 15

Let $y \in B \cap A$, Then

$\Rightarrow y \in B$ and $y \in A$ (by definition of intersection of sets)

Then,

$\Rightarrow y \in A$ and $y \in B$ (property)

$\Rightarrow y \in A \cap B$ (by definition of intersection of sets)

Therefore, if $x \in B \cap A$, then $y \in A \cap B$ (From property of sets)

$\Rightarrow \boxed{B \cap A \subseteq A \cap B}$ → Eqn - 16

From Eqns 15 & 16,

$\boxed{A \cap B = B \cap A}$ (by proposition)

(b) Given $A, B, C \rightarrow$ sets

<u>RTP</u>:- $A \cup (B \cup C) = (A \cup B) \cup C$

Let $x \in A \cup (B \cup C)$

$\Rightarrow x \in A$ (or) $x \in (B \cup C)$

$\Rightarrow x \in A$ (or) $[x \in B$ (or) $x \in C]$ (Definition of union)

$\Rightarrow (x \in A$ (or) $x \in B)$ (or) $x \in C$ (property)

$\Rightarrow (x \in A \cup B)$ (or) $(x \in C)$ (Definition of union)

$\Rightarrow x \in (A \cup B) \cup C$ (Definition of union)

Therefore, if $x \in A \cup (B \cup C)$, then $x \in (A \cup B) \cup C$ (From property of sets)

$\Rightarrow \boxed{A \cup (B \cup C) \subseteq (A \cup B) \cup C} \rightarrow$ Eqn - (18)

Let $y \in (A \cup B) \cup C$

$\Rightarrow y \in (A \cup B)$ (or) $y \in C$

$\Rightarrow [y \in A$ (or) $y \in B]$ (or) $y \in C$ (Definition of union)

$\Rightarrow y \in A$ (or) $y \in B$ (or) $y \in C$

$\Rightarrow y \in A$ (or) $(y \in B$ (or) $y \in C)$ (property)

$\Rightarrow y \in A$ (or) $y \in B \cup C$ (Definition of union)

$\Rightarrow y \in A \cup (B \cup C)$ (Definition of union)

Therefore, if $y \in (A \cup B) \cup C$, then $y \in A \cup (B \cup C)$

$\Rightarrow \boxed{(A \cup B) \cup C \subseteq A \cup (B \cup C)} \rightarrow$ Eqn - (19)

From Eqn's - (18) and (19),

$\boxed{A \cup (B \cup C) = (A \cup B) \cup C}$ (by proposition)

RTP:- $A \cap (B \cap C) = (A \cap B) \cap C$

Let $x \in A \cap (B \cap C)$

$\Rightarrow x \in A$ and $x \in B \cap C$

$\Rightarrow x \in A$ and ($x \in B$ and $x \in C$) [Defn of intersection]

$\Rightarrow x \in A$ and $x \in B$ and $x \in C$

$\Rightarrow$ ($x \in A$ and $x \in B$) and $x \in C$ (property)

$\Rightarrow x \in A \cap B$ and $x \in C$ [Defn of intersection]

$\Rightarrow x \in (A \cap B) \cap C$ [Defn of intersection]

Therefore, if $x \in A \cap (B \cap C)$,
then $x \in (A \cap B) \cap C$ (From property of sets)

$\Rightarrow \boxed{A \cap (B \cap C) \subseteq (A \cap B) \cap C}$ → Eqn - 20

Let $y \in (A \cap B) \cap C$

$\Rightarrow y \in A \cap B$ and $y \in C$

$\Rightarrow$ ($y \in A$ and $y \in B$) and $y \in C$ (Defns of intersection)

$\Rightarrow y \in A$ and $y \in B$ and $y \in C$

$\Rightarrow y \in A$ and ($y \in B$ and $y \in C$) (property)

$\Rightarrow y \in A$ and $y \in B \cap C$ (Defn of intersection)

$\Rightarrow y \in A \cap (B \cap C)$ (Defn of intersection)

Therefore, if $y \in (A \cap B) \cap C$,
then $y \in A \cap (B \cap C)$ (from property of sets)

$\Rightarrow \boxed{(A \cap B) \cap C \subseteq A \cap (B \cap C)}$ → Eqn - 21

From Eqns 20 and 21,

$\boxed{A \cap (B \cap C) = (A \cap B) \cap C}$ (by proposition)

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(2) Given A, B, C → sets

RTP:- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Let $x \in A \cup (B \cap C)$

$\Rightarrow x \in A$ (or) $x \in B \cap C$

$\Rightarrow x \in A$ (or) ($x \in B$ and $x \in C$) (Defn of intersection)

$\Rightarrow$ ($x \in A$ (or) $x \in B$) and ($x \in A$ (or) $x \in C$) (property)

$\Rightarrow$ ($x \in A \cup B$) and ($x \in A \cup C$) (Defn of union)

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$ (Defn of intersection)

If $x \in A \cup (B \cap C)$, then $x \in ((A \cup B) \cap (A \cup C))$

$\Rightarrow \boxed{A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)}$ (property of sets)

↳ Eqn - (22)

Let $y \in (A \cup B) \cap (A \cup C)$

$\Rightarrow y \in (A \cup B)$ and $y \in (A \cup C)$

$\Rightarrow$ ($y \in A$ (or) $y \in B$) and ($y \in A$ (or) $y \in C$)

$\Rightarrow y \in A$ (or) ($y \in B$ and $y \in C$)

$\Rightarrow y \in A$ (or) $y \in B \cap C$

$\Rightarrow y \in A \cup (B \cap C)$

If $y \in (A \cup B) \cap (A \cup C)$, then $y \in A \cup (B \cap C)$

$\Rightarrow \boxed{(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)}$ (property of sets)

↳ Eqn - (23)

From Eqns (22) and (23),

$\boxed{(A \cup B) \cap (A \cup C) = A \cup (B \cap C)}$ (by proposition)

RTP: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Let $x \in A \cap (B \cup C)$

$\Rightarrow x \in A$ and $x \in B \cup C$

$\Rightarrow x \in A$ and $(x \in B$ (or) $x \in C)$ (definition of union)

$\Rightarrow (x \in A$ and $x \in B)$ (or) $(x \in A$ and $x \in C)$

$\Rightarrow x \in A \cap B$ (or) $x \in A \cap C$ (definition of intersection)

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$ (defn of union)

If $x \in A \cap (B \cup C)$, then $x \in (A \cap B) \cup (A \cap C)$

$\Rightarrow \boxed{A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)}$ — Eqn - 24

Let $y \in (A \cap B) \cup (A \cap C)$

$\Rightarrow y \in A \cap B$ (or) $y \in A \cap C$, (defn of union)

$\Rightarrow (y \in A$ and $y \in B)$ (or) $(y \in A$ and $y \in C)$

$\Rightarrow y \in A$ and $(y \in B$ (or) $y \in C)$

$\Rightarrow y \in A$ and $y \in B \cup C$ (defn of union)

$\Rightarrow y \in A \cap (B \cup C)$ (defn of intersection)

If $y \in (A \cap B) \cup (A \cap C)$, then $y \in A \cap (B \cup C)$

$\Rightarrow \boxed{(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)}$

— Eqn - 25

From Eqns 24 and 25,

$\boxed{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$

(by proposition)

(d) Given $A, B \to$ sets

RTP:- $A \subseteq B \Leftrightarrow B^c \subseteq A^c$

($\Rightarrow$) Assuming $A \subseteq B$, (If $y \in A$, then $y \in B$)

Let $x \in B^c$

$\Rightarrow x \notin B$.

But since $A \subseteq B$ and $x \notin B$, we must have $x \notin A$

(A can only contains elements in set B as A is subset of B)

$\Rightarrow x \in A^c$

If $x \in B^c$, then $x \in A^c$

$\Rightarrow \boxed{B^c \subseteq A^c}$

($\Leftarrow$) Assuming $B^c \subseteq A^c$ (If $z \in B^c$, then $z \in A^c$)

Let $w \in A$.

$\Rightarrow w \notin A^c$

But since $B^c \subseteq A^c$ and $w \notin A^c$, we must have $w \notin B^c$

(B^c can only contain elements in set A^c as B^c is a subset of A^c)

$\Rightarrow \boxed{w \in B}$

If $w \in A$, then $w \in B$

$\Rightarrow \boxed{A \subseteq B}$

$\boxed{\therefore A \subseteq B \text{ if and only if } B^c \subseteq A^c}$

(e) Given two sets A & B;

RTP :- $A \setminus B = A \cap B^c$

Let $x \in A \setminus B$ ($A \setminus B$ is same as $A - B$)

$\Rightarrow x \in A$ and $x \notin B$ (From definition of difference of sets)

$\Rightarrow x \in A$ and $x \in B^c$ (complement)

$\Rightarrow x \in A \cap B^c$ (From definition of intersection of sets)

If $x \in A \setminus B$, then $x \in A \cap B^c$ (From property of sets)

$\Rightarrow \boxed{A \setminus B \subseteq A \cap B^c}$ (Eqn - 27)

Let $y \in A \cap B^c$ (B^c is same as B')

$\Rightarrow y \in A$ and $y \in B^c$ (From definition of intersection)

$\Rightarrow y \in A$ and $y \notin B$ (defn of complement)

$\Rightarrow y \in A \setminus B$ (From definition of difference of sets)

If $y \in A \cap B^c$, then $y \in A \setminus B$

$\Rightarrow \boxed{A \cap B^c \subseteq A \setminus B}$ (From property of sets)

(Eqn - 28)

From Eqns - 27 and 28,

$\Rightarrow \boxed{A \setminus B = A \cap B^c}$ (by proposition)

(f) Let A, B be two sets

<u>RTP:-</u> $(A \cup B)^c = A^c \cap B^c$

Let $x \in (A \cup B)^c$

$\Rightarrow x \notin (A \cup B)$ (Definition of complement)

$\Rightarrow x \notin A$ and $x \notin B$ (property)

$\Rightarrow x \in A^c$ and $x \in B^c$ (Definition of complement)

$\Rightarrow x \in A^c \cap B^c$ (Defn of intersection)

If $x \in (A \cup B)^c$, then $x \in A^c \cap B^c$

$\Rightarrow \boxed{(A \cup B)^c \subseteq A^c \cap B^c}$ — Eqn - 29

Let $y \in A^c \cap B^c$

$\Rightarrow y \in A^c$ and $y \in B^c$ (Defn of intersection)

$\Rightarrow y \notin A$ and $y \notin B$ (Defn of complement)

$\Rightarrow y \notin (A \cup B)$ (property)

$\Rightarrow y \in (A \cup B)^c$ (Defn of complement)

If $y \in A^c \cap B^c$, then $y \in (A \cup B)^c$

$\Rightarrow \boxed{A^c \cap B^c \subseteq (A \cup B)^c}$ — Eqn - 30

From Eqns - 29 and 30

$\Rightarrow \boxed{A^c \cap B^c = (A \cup B)^c}$ → Demorgans Law

↳ (by proposition)

RTP:- Let A, B be two sets, $(A \cap B)^c = A^c \cup B^c$

Let $x \in (A \cap B)^c$

$\Rightarrow x \notin (A \cap B)$ (Defn of complement)

$\Rightarrow x \notin A$ (or) $x \notin B$ (property)

$\Rightarrow x \in A^c$ (or) $x \in B^c$ (Defn of complement)

$\Rightarrow x \in A^c \cup B^c$ (Defn of union)

If $x \in (A \cap B)^c$, then $x \in A^c \cup B^c$

$\Rightarrow \boxed{(A \cap B)^c \subseteq A^c \cup B^c}$ — Eqn - 37

Let $y \in A^c \cup B^c$

$\Rightarrow y \in A^c$ (or) $y \in B^c$ (Defn of union)

$\Rightarrow y \notin A$ (or) $y \notin B$ (Defn of complement)

$\Rightarrow y \notin (A \cap B)$ (property)

$\Rightarrow y \in (A \cap B)^c$ (Defn of complement)

If $y \in A^c \cup B^c$, then $y \in (A \cap B)^c$

$\Rightarrow \boxed{A^c \cup B^c \subseteq (A \cap B)^c}$ — Eqn - 38

From Eqn's - 37 and 38,

$\boxed{A^c \cup B^c = (A \cap B)^c}$ (by proposition)

↳ Demorgan's Law

(x be an element of set)

* Let A, B be arbitrary sets (properties)

① If $x \in A$ (or) $x \in B \Leftrightarrow x \in A \cup B$

② If $x \in A$ and $x \in B \Leftrightarrow x \in A \cap B$

③ If $x \notin A \Leftrightarrow x \in A^c$

* If f is invertible and $f(x) = y$, then $y = f^{-1}(x)$

* A function f is called one-one function if $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$

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$$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}$$

* A function f is called bijective if it's both one-one & onto function.

* $\boxed{A \times \phi = \phi \times A = \phi}$

* $(x \in A \text{ and } y \in C)$ and $(x \in B \text{ and } y \in C)$
$\equiv (x \in A \text{ and } x \in B)$ and $y \in C$

* $(x \in A \text{ and } z \in C)$ (or) $(x \in B \text{ and } z \in C)$
$\equiv [(x \in A \text{ (or) } x \in B)]$ and $z \in C$

Theorem-1 : Consider two arbitrary sets A and B.
Then, $A \subseteq A \cup B$ and $B \subseteq A \cup B$

proof:- Suppose $x \in A$. That is, x is an element of (A set is well defined collection of objects) A. Then certainly x is either an element of A or an element of B (or both). Thus x is an element of $A \cup B$, i.e $x \in A \cup B$. Since, we have shown that any element of A is an element of $A \cup B$, $A \subseteq A \cup B$
(similarly for B)

Theorem-2 :- Consider Two arbitrary sets A and B.
Then $A \cap B \subseteq A$ and $A \cap B \subseteq B$

proof:

Let $x \in A \cap B$
From the definition of intersection of sets,
$x \in A$ and $x \in B$

$\Rightarrow$ Thus in particular, $x \in A$

If $x \in A \cap B$, then $x \in A \Rightarrow A \cap B \subseteq A$

(Similarly for B)

---

* $A \cup \emptyset = A$ * $A \cap \emptyset = \emptyset$ * $A \cup B = B \cup A$
* $A \cap B = B \cap A$ * $(A \cup B) \cup C = A \cup (B \cup C)$
* $(A \cap B) \cap C = A \cap (B \cap C)$

associative laws

* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Distributive laws

* $A \cup A = A \cap A = A$
* $\underbrace{(A \cup B)^c = A^c \cap B^c \quad * \ (A \cap B)^c = A^c \cup B^c}_{\text{De-Morgan's Laws}}$

RTF :- Necessary & sufficient conditions for $\boxed{A \times B = B \times A}$

Given are two sets A & B namely,

Case-1 :-

A and B are non-empty sets

Given that $A \times B = B \times A$

$\Rightarrow$ Let $y \in A$

since, it's assumed that B are non-empty set,

$B \neq \phi$ $\rightarrow$ then there exists an element in B

Let $z \in B$ such that $(y, z) \in A \times B$

since, it's given that $B \times A = A \times B$, we have that $(y, z) \in B \times A$

From the properties of cartesian product,

$y \in B$

$\Rightarrow$ If $y \in A$ then $y \in B \Rightarrow \boxed{A \subseteq B}$

$\rightarrow$ Eqn - (44)

$\Rightarrow$ Let $w \in B$

since, it's assumed that A is a non-empty set,

$A \neq \phi$ $\rightarrow$ then there exists an element in A

Let $x \in A$ such that

$(w, x) \in B \times A$

Since, it's given that $B \times A = A \times B$, we have that $(w, x) \in A \times B$

From the properties of Cartesian product,

$w \in A$

$\Rightarrow$ If $w \in B$, then $w \in A \Rightarrow \boxed{B \subseteq A}$ $\rightarrow$ Eqn - (45)

From Eqn's - (44) and (45),

$\boxed{A = B}$

Case-2:

If atleast one of the sets of A and B is empty i.e, ϕ

RTP:- $A' \times \phi = \phi \times A' = \phi$ (Assuming B is empty)

Exchanging these both eqn's covers the case when A is empty

Suppose, let's try to prove this by contradiction,

Let us assume, $A' \times \phi \neq \phi$, then there exists an ordered pair $(x, y) \in A' \times \phi$

Then from the properties of Cartesian-product, $y \in \phi$

Not possible (ϕ is empty)

Hence, contradiction $\Rightarrow$ Assumption is wrong

$\boxed{A' \times \phi = \phi}$

Now, let's try to prove the other equality by contradiction

Let us assume, $\phi \times A' = \phi$, then there exists an ordered pair $(w, z) \in \phi \times A'$

Then from the properties of Cartesian product, $w \in \phi$

Not possible (ϕ is empty)

Hence, contradiction $\Rightarrow$ Assumption is wrong

$\boxed{\phi \times A' = \phi}$

Hence $\phi \times A' = A' \times \phi = \phi$

In this case also $A \times B = B \times A$ holds and is equal to ϕ

$\Rightarrow$ Sufficient conditions

$\hookrightarrow$ $A = B$, $A = \phi$, $B = \phi$

$\Rightarrow$ Necessary condition

$\hookrightarrow$ $[(A = B)]$ (or) $[(A = \phi)]$ (or) $[(B = \phi)]$

(4)

(x be an element of set)

* Let A, B be arbitrary sets (properties)

(1) If $x \in A$ (or) $x \in B \Leftrightarrow x \in A \cup B$

(2) If $x \in A$ and $x \in B \Leftrightarrow x \in A \cap B$

(3) If $x \notin A \Leftrightarrow x \in A^c$

* If f is invertible and $f(x) = y$, then $y = f^{-1}(x)$

* A function f is called one-one function if $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$

* A function f is called onto function if codomain = range of f
  (Every element in codomain has pre-image in domain)

* The Cartesian product $X \times Y$ between two sets X and Y is the set of all possible ordered pairs with first element from X and second element from Y.

$$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}$$

* A function f is called bijective if it's both one-one & onto function.

* $\boxed{A \times \phi = \phi \times A = \phi}$

* ($x \in A$ and $y \in C$) and ($x \in B$ and $y \in C$)
  $\equiv$ ($x \in A$ and $x \in B$) and $y \in C$

* ($x \in A$ and $z \in C$) (or) ($x \in B$ and $z \in C$)
  $\equiv$ [($x \in A$ (or) $x \in B$)] and $z \in C$

Theorem-1 : Consider two arbitrary sets A and B. Then, $A \subseteq A \cup B$ and $B \subseteq A \cup B$

proof :- Suppose $x \in A$. That is, x is an element of (A set is well defined collection of objects) A. Then certainly x is either an element of A or an element of B (or both). Thus x is an element of $A \cup B$, i.e $x \in A \cup B$. Since, we have shown that any element of A is an element of $A \cup B$, $A \subseteq A \cup B$ (similarly for B)

Theorem-2 :- Consider Two arbitrary sets A and B. Then $A \cap B \subseteq A$ and $A \cap B \subseteq B$

proof :

Let $x \in A \cap B$

From the definition of intersection of sets,

$x \in A$ and $x \in B$

$\Rightarrow$ Thus in particular, $x \in A$

If $x \in A \cap B$, then $x \in A \Rightarrow A \cap B \subseteq A$

(Similarly for B)

---

* $A \cup \emptyset = A$ * $A \cap \emptyset = \emptyset$ * $A \cup B = B \cup A$
* $A \cap B = B \cap A$ * $(A \cup B) \cup C = A \cup (B \cup C)$
* $(A \cap B) \cap C = A \cap (B \cap C)$

associative laws

* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Distributive laws

* $A \cup A = A \cap A = A$
* $\underbrace{(A \cup B)^c = A^c \cap B^c \quad * \ (A \cap B)^c = A^c \cup B^c}_{\text{De-Morgan's Laws}}$

(a)

RTP:- $A \times B = \phi \iff A = \phi$ (or) $B = \phi$

($\Rightarrow$) Assume $A \times B = \phi$;

Method-1:

From theorem of cardinality of cartesian product,

(We know that $|\{\phi\}| = 0$} $\rightarrow$ cardinality of null set is 0

$|A \times B| = |A| \times |B|$ $\rightarrow$ Eqn - (A4)

multiplication

But given that $A \times B = \phi$,

From Eqn - (A4),

$|A| \times |B| = |\{\phi\}| = 0$

$\Rightarrow |A| \times |B| = 0$

Either of $|A|, |B|$ is 0

$\Downarrow$

Either $\boxed{A = \phi \text{ or } B = \phi}$

Method-2:-

Assuming that $A \times B = \phi$,

Also assume that both A & B are non-empty,

Let $x \in A$ and $y \in B$

then from properties of cartesian product,

$(x, y) \in A \times B$

But given that $A \times B = \phi$

Contradiction $\leftarrow$ our assumption is wrong

Atleast one of the sets A, B is empty

$\Rightarrow$ $\boxed{A = \phi \text{ (or) } B = \phi}$

$\boxed{\text{If } A \times B = \phi \text{, then } A = \phi \text{ or } B = \phi}$

($\Leftarrow$)

Given that $A = \phi$ (or) $B = \phi$

<u>RTP</u> :- $A \times B = \phi$

Since either A or B can be ϕ, we have to prove that $\underbrace{A \times \phi = \phi \text{ and } \phi \times B = \phi}$

Since in both cases A & B are general sets, we replace them with A' for easeness in proof

If atleast one of the sets A and B is empty i.e, ϕ

RTP:- $A' \times \phi = \phi \times A' = \phi$ (Assuming B is empty)

Exchanging these both cases covers the case when A is empty

Suppose, let's try to prove this by contradiction,

Let us assume, $A' \times \phi \neq \phi$, then there exists an ordered pair $(x, y) \in A' \times \phi$

Then from the properties of Cartesian-product, $y \in \phi$

Not possible (ϕ is empty)

Hence, contradiction $\Rightarrow$ Assumption is wrong

$\boxed{A' \times \phi = \phi}$

Now, let's try to prove the other equality by contradiction

Let us assume, $\phi \times A' = \phi$, then there exists an ordered pair $(w, z) \in \phi \times A'$

Then from the properties of Cartesian product,

$w \in \phi$

Not possible (ϕ is empty)

Hence, contradiction $\Rightarrow$ Assumption is wrong

$\boxed{\phi \times A' = \phi}$

$\Rightarrow$ Hence from the above contradictions,

$\therefore$ $\boxed{A' \times \phi = \phi \times A' = \phi}$

If $A = \phi \Rightarrow A \times B = \phi \times B \equiv \phi \times A' = \phi$

If $B = \phi \Rightarrow A \times B = A \times \phi \equiv A' \times \phi = \phi$

(Both $A = B = \phi$ can be included in any one of the two cases above)

$\Rightarrow$ $\boxed{\text{If } A = \phi \text{ (or) } B = \phi \text{, then } A \times B = \phi}$

$\therefore$ $\boxed{A \times B = \phi \text{ if and only if } A = \phi \text{ (or) } B = \phi}$

(b) Given that A, B, C are sets

RTP:- $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Case - 1 :-

Let $A \cup B = D$,

Assuming C and D are non-empty sets

Let $x \in D$ and $y \in C$ (distributive Law)

$(x, y) \in D \times C$ (From properties of cartesian product)

$\Rightarrow (x, y) \in (A \cup B) \times C$

From properties of cartesian product,

$\Rightarrow x \in A \cup B$ and $y \in C$

$\Rightarrow (x \in A$ (or) $x \in B)$ and $y \in C$ (property of union)

$\Rightarrow [x \in A$ and $y \in C]$ (or) $[x \in B$ and $y \in C]$

($\Rightarrow$ From properties of cartesian product)

$\Rightarrow$ [redacted]

$[(x, y) \in A \times C]$ (or) $[(x, y) \in B \times C]$

$\Rightarrow \boxed{(x, y) \in (A \times C) \cup (B \times C)}$ (property of union)

$\hookrightarrow$ Eqn - (A3)

If $(x,y) \in D \times C$, then $(x,y) \in (A \times C) \cup (B \times C)$

$$\boxed{(A \cup B) \times C \subseteq (A \times C) \cup (B \times C)}$$ (From property of sets)

Since C and D are non-empty, both A & B are not-empty (at a time)

Let $(w,z) \in (A \times C) \cup (B \times C)$

$\Rightarrow ((w,z) \in (A \times C))$ (or) $((w,z) \in (B \times C))$

$\Rightarrow (w \in A$ and $z \in C)$ (or) $(w \in B$ and $z \in C)$

$\Rightarrow (w \in A$ (or) $w \in B)$ and $z \in C$ (Distributive Law)

$\Rightarrow (w \in A \cup B)$ and $z \in C$ (property of union)

From properties of cartesian product,

$\Rightarrow (w,z) \in (A \cup B) \times C$

If $(w,z) \in (A \times C) \cup (B \times C)$, then $(w,z) \in (A \cup B) \times C$

$$\Rightarrow \boxed{(A \times C) \cup (B \times C) \subseteq (A \cup B) \times C} \rightarrow \text{Eqn-}74$$

From Eqn-73 and 74,

$$\boxed{(A \times C) \cup (B \times C) = (A \cup B) \times C}$$

Case-2:-

If $A \cup B = \phi$ (or) $C = \phi$

Assume $A \cup B = D$

$(A \cup B) \times C = D \times C$

To prove the main-equation, lets prove that

$$\boxed{A' \times \phi = \phi \times A' = \phi}$$ (A' be an arbitrary set)

If atleast one of the sets A and B is empty i.e. ϕ

RTP:- $A' \times \phi = \phi \times A' = \phi$ (Assuming B is empty)

Exchanging these both eqns covers the case when A is empty

Suppose, let's try to prove this by contradiction,

Let us assume, $A' \times \phi \neq \phi$, then there exists an ordered pair $(x, y) \in A' \times \phi$

Then from the properties of Cartesian-product, $y \in \phi$

Not possible (ϕ is empty)

Hence, contradiction $\Rightarrow$ Assumption is wrong

$\boxed{A' \times \phi = \phi}$

Now, let's try to prove the other equality by contradiction

Let us assume, $\phi \times A' = \phi$, then there exists an ordered pair $(w, z) \in \phi \times A'$

Then from the properties of Cartesian product,

$w \in \phi$

Not possible (ϕ is empty)

Hence, contradiction $\Rightarrow$ Assumption is wrong

$\boxed{\phi \times A' = \phi}$

$\Rightarrow$ Hence from above contradictions,

$$\boxed{A' \times \phi = \phi \times A' = \phi} \longrightarrow \text{Eq}^n - 78$$

If $C = \phi$:-

$(A \cup B) \times C = (A \cup B) \times \phi = \phi$ (From Eqⁿ - 78)

$A \times C = A \times \phi = \phi$ (From Eqⁿ - 78)

$B \times C = B \times \phi = \phi$ (From Eqⁿ - 78)

$\Rightarrow (A \times C) \cup (B \times C) = \phi \cup \phi = \phi = A \times \phi = A \times C$

$= (A \cup B) \times \phi$

$= (A \cup B) \times C$

$(A \times C) \cup (B \times C) = (A \times \phi) \cup (B \times \phi)$

$= (A \cup B) \times \phi$

$$\boxed{(A \times C) \cup (B \times C) = (A \cup B) \times C}$$

If $A \cup B = \phi$

If $A \cup B = \phi \Rightarrow A = \phi$ and $B = \phi$ (property)

$(A \cup B) \times C = \phi \times C = \phi$ (From Eqⁿ - 78)

$A \times C = \phi \times C = \phi$
$B \times C = \phi \times C = \phi$ (From Eqⁿ - 78)

$\Rightarrow (A \cup B) \times C = \phi = \phi \cup \phi = (A \times C) \cup (B \times C)$

$$\boxed{(A \cup B) \times C = (A \times C) \cup (B \times C)}$$

$\therefore$ In both cases, in accordance to the proofs

$$\boxed{(A \cup B) \times C = (A \times C) \cup (B \times C)}$$

(c) Given A, B, C are sets

RTP:- $(A \cap B) \times C = (A \times C) \cap (B \times C)$

Case-1 :-

Let $A \cap B = D$

Assuming C and D are non-empty sets,

Let $(x, y) \in D \times C$

($\Rightarrow$ From properties of cartesian product,)

$\Rightarrow x \in D$ and $y \in C$

$\Rightarrow (x \in A \cap B)$ and $y \in C$

$\Rightarrow (x \in A$ and $x \in B)$ and $y \in C$

$\Rightarrow (x \in A$ and $y \in C)$ and $(x \in B$ and $y \in C)$

($\Rightarrow$ From properties of cartesian product,)

$\Rightarrow$ $x \in A$ & $y \in C \equiv (x, y) \in A \times C$

$x \in B$ & $y \in C \equiv (x, y) \in B \times C$

$\Rightarrow ((x, y) \in A \times C)$ and $((x, y) \in B \times C)$

$\Rightarrow (x, y) \in (A \times C) \cap (B \times C)$

If $(x, y) \in D \times C$, then $(x, y) \in (A \times C) \cap (B \times C)$.

$D \times C \subseteq (A \times C) \cap (B \times C)$

$\boxed{(A \cap B) \times C \subseteq (A \times C) \cap (B \times C)}$ — Eqn - (A)

Let $(w, z) \in (A \times C) \cap (B \times C)$ (since $A \cap B \neq \phi$ There exist element belonging to both sets)

From properties of cartesian product,

$\Rightarrow ((w, z) \in (A \times C))$ and $((w, z) \in B \times C)$

(From properties of Cartesian product,)

$\Rightarrow (w \in A \text{ and } z \in C) \text{ and } (w \in B \text{ and } z \in C)$

$\Rightarrow (w \in A \text{ and } w \in B) \text{ and } (z \in C)$

$\Rightarrow (w \in A \cap B) \text{ and } z \in C$

(From properties of Cartesian product,)

$\Rightarrow (w, z) \in (A \cap B) \times C$.

If $(w, z) \in (A \times C) \cap (B \times C)$, then $(w, z) \in (A \cap B) \times C$

$\boxed{(A \times C) \cap (B \times C) \subseteq (A \cap B) \times C}$ – Eqn – 78

From Eqns – 77 & 78,

$\boxed{(A \times C) \cap (B \times C) = (A \cap B) \times C}$ (by proposition)

Case – 2:

If $A \cap B = \phi$ (or $C = \phi$,

Assume $A \cap B = D$

RTP:- $(A \cap B) \times C = (A \times C) \cap (B \times C)$

$(A \cap B) \times C = D \times C$

To prove the main equation, let's prove that

$\boxed{A' \times \phi = \phi \times A' = \phi}$

(A' be any arbitrary set)

If atleast one of the sets A and B is empty i.e, ϕ

RTP:- $A' \times \phi = \phi \times A' = \phi$ (Assuming B is empty)

Exchanging these both case covers the case when A is empty

Suppose, let's try to prove this by contradiction,

Let us assume, $A' \times \phi \neq \phi$, then there exists an ordered pair $(x, y) \in A' \times \phi$

Then from the properties of Cartesian-product, $y \in \phi$

Not possible (ϕ is empty)

Hence, contradiction $\Rightarrow$ Assumption is wrong

$\boxed{A' \times \phi = \phi}$

Now, let's try to prove the other equality by contradiction

Let us assume, $\phi \times A' = \phi$, then there exists an ordered pair $(w, z) \in \phi \times A'$

Then from the properties of Cartesian product, $w \in \phi$

Not possible (ϕ is empty)

Hence, contradiction $\Rightarrow$ Assumption is wrong

$\boxed{\phi \times A' = \phi}$

$\Rightarrow$ Hence from above contradictions,

$$\boxed{A' \times \phi = \phi \times A' = \phi} \rightarrow \text{Eqn} - (83)$$

If $C = \phi$:-

$(A \cap B) \times C = (A \cap B) \times \phi = \phi$ (From Eqn-(83))

$A \times C = A \times \phi = \phi$ (From Eqn-(83))

$B \times C = B \times \phi = \phi$ (From Eqn-(83))

$(A \times C) \cap (B \times C) = \phi \cap \phi = \phi = (A \cap B) \times C$

$$\boxed{(A \times C) \cap (B \times C) = (A \cap B) \times C}$$

If $A \cap B = \phi$

Let $A \cap B = \phi$

Assume that $(A \times C) \cap (B \times C) \neq \phi$

Then let there exist an ordered pair

$(x, y) \in (A \times C) \cap (B \times C)$

$(x, y) \in A \times C$ and $(x, y) \in B \times C$

$\Rightarrow$ $(x \in A$ and $y \in C)$ and $(x \in B$ and $y \in C)$

$x \in A$ and $x \in B$

$\downarrow$

$x \in A \cap B$ $\times$ (But $A \cap B = \phi$)

Contradiction $\rightarrow$ Our assumption is wrong

$\boxed{(A \times C) \cap (B \times C) = \phi} \rightarrow$ (if $A \cap B = \phi$)

$(A \cap B) \times C = \phi \times C = \phi$ (From Eqn-(83))

$$\boxed{(A \cap B) \times C = (A \times C) \cap (B \times C)}$$

$\Rightarrow \therefore$ In both cases, in accordance with the proofs $\boxed{(A \cap B) \times C = (A \times C) \cap (B \times C)}$

⑤ (x be an element of set)

* Let A, B be arbitrary sets (properties)

(1) If $x \in A$ (or) $x \in B \Leftrightarrow x \in A \cup B$

(2) If $x \in A$ and $x \in B \Leftrightarrow x \in A \cap B$

(3) If $x \notin A \Leftrightarrow x \in A^c$

* If f is invertible and $f(x) = y$, then $y = f^{-1}(x)$

* A function f is called one-one function if $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$

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* A function f is called bijective if it's both one-one & onto function.

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* ($x \in A$ and $y \in C$) and ($x \in B$ and $y \in C$)
  $\equiv$ ($x \in A$ and $x \in B$) and $y \in C$

* ($x \in A$ and $z \in C$) (or) ($x \in B$ and $z \in C$)
  $\equiv$ [($x \in A$ (or) $x \in B$)] and $z \in C$

Theorem-1 : Consider two arbitrary sets A and B. Then, $A \subseteq A \cup B$ and $B \subseteq A \cup B$

proof:- Suppose $x \in A$. That is, x is an element of (A set is well defined collection of objects) A. Then certainly x is either an element of A or an element of B (or both). Thus x is an element of $A \cup B$, i.e $x \in A \cup B$. Since, we have shown that any element of A is an element of $A \cup B$, $A \subseteq A \cup B$ (similarly for B)

Theorem-2 :- Consider Two arbitrary sets A and B. Then $A \cap B \subseteq A$ and $A \cap B \subseteq B$

proof:

Let $x \in A \cap B$

From the definition of intersection of sets,

$x \in A$ and $x \in B$

$\Rightarrow$ Thus in particular, $x \in A$

If $x \in A \cap B$, then $x \in A \Rightarrow A \cap B \subseteq A$

(similarly for B)

---

* $A \cup \phi = A$ * $A \cap \phi = \phi$ * $A \cup B = B \cup A$
* $A \cap B = B \cap A$ * $(A \cup B) \cup C = A \cup (B \cup C)$
* $(A \cap B) \cap C = A \cap (B \cap C)$

associative laws

* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Distributive laws

* $A \cup A = A \cap A = A$
* $(A \cup B)^c = A^c \cap B^c$ * $(A \cap B)^c = A^c \cup B^c$

De-Morgan's Laws

Given that $f: A \to B$ and $g: B \to C$ are functions

(a)

RTP:- If both f and g are one to one functions, then $g \circ f$ is a one-one function

$\Rightarrow$ From the definitions of one-one functions

(A function k is said to be one-one (or) injective if $k(x_1) = k(x_2)$ implies $x_1 = x_2$)

$\Rightarrow$ Given that f is a one-one function,

let $a_1, a_2 \in A$

$f(a_1) = f(a_2)$ implies $a_1 = a_2$

$\Rightarrow$ Given that g is a one-one function,

let $b_1, b_2 \in B$

$g(b_1) = g(b_2)$ implies $b_1 = b_2$

Assume that $b_1', b_2' \in B$ & $a_1', a_2' \in A$

Let $g\circ f(a_1') = g\circ f(a_2')$ and assume that

$a_1' \neq a_2'$

$\Rightarrow g\circ f(a_1') = g\circ f(a_2')$

$\Rightarrow g(f(a_1')) = g(f(a_2'))$

$\Rightarrow f(a_1') = f(a_2')$ (because g is one-one)

$\Rightarrow a_1' = a_2'$ (because f is one-one)

Contradiction! $\Rightarrow$ Our assumption is wrong

$a_1' = a_2'$

$\Rightarrow \boxed{g\circ f(a_1') = g\circ f(a_2') \text{ implies } a_1' = a_2'}$

Hence $g\circ f$ is a one-one function (if f, g are one-one)

Given that $f: A \to B$ and $g: B \to C$ are functions,

(6) RTP:- If both f and g are onto functions, then $g \circ f$ is an onto-function

⇒ From the definition of onto-functions

(A function $k: L \to M$ is onto if, for every element $m \in M$, there exists a pre-image $l \in L$ such that $k(l) = m$)

⇒ Given that g is an onto-function
$g: B \to C$

Let $c \in C$, then there exists a pre-image in B → (let that pre-image be b) such that
$\boxed{g(b) = c}$ → $(b \in B)$

⇒ Given that f is an onto function
$f: A \to B$

Since $b \in B$ and f is onto, there exists a preimage in A [let the pre-image be a] such that
$\boxed{f(a) = b}$ → $(a \in A)$

⇒ $g \circ f: A \to C$

$g(b) = g(f(a)) = c = g \circ f(a)$

Since c was an arbitrary element of C, then for all $c \in C$, there exists $a \in A$ such that $g \circ f(a) = c$

⇒ For every element $c \in C$, there exists pre-image $a \in A$ such that $g \circ f(a) = c$

$g \circ f$ is an onto function → (If f, g are onto)

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Given that $f: A \to B$ and $g: B \to C$ are functions, we have

(c)

RTP :- If both f and g are bijective functions, then gof is a bijective function

⇒ From the definition of bijective function,

(A function $k: A' \to B'$ is said to be a bijective function, if $k: A' \to B'$ is a one-one function as well as $k: A' \to B'$ is a onto function)

Given that $f: A \to B$ and $g: B \to C$ are bijective functions, We show that gof is a bijective function by showing that gof is a one-one function (given $\underbrace{f, g}_{\text{as bijective}}$ are one-one) and then by showing that gof is an onto function (given $\underbrace{f, g}_{\text{as bijective}}$ are onto)

RTP:- If both f and g are one to one functions, then gof is a one-one function

$\Rightarrow$ From the definitions of one-one functions

(A function k is said to be one-one (or) injective if $k(x_1)=k(x_2)$ implies $x_1=x_2$)

$\Rightarrow$ Given that f is a one-one function,
let $a_1, a_2 \in A$

$f(a_1)=f(a_2)$ implies $a_1=a_2$

$\Rightarrow$ Given that g is a one-one function,
let $b_1, b_2 \in B$

$g(b_1)=g(b_2)$ implies $b_1=b_2$

Assume that $b_1', b_2' \in B$ & $a_1', a_2' \in A$

Let $gof(a_1')=gof(a_2')$ and assume that

$a_1' \neq a_2'$

$\Rightarrow$ $gof(a_1')=gof(a_2')$

$\Rightarrow$ $g(f(a_1'))=g(f(a_2'))$

$\Rightarrow$ $f(a_1')=f(a_2')$ (because g is one-one)

$\Rightarrow$ $a_1'=a_2'$ (because f is one-one)

Contradiction! $\Rightarrow$ Our assumption is wrong

$a_1'=a_2'$

$\Rightarrow$ $\boxed{gof(a_1')=gof(a_2') \text{ implies } a_1'=a_2'}$

Hence gof is a one-one function (if f, g are one-one)

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RTP:- If both f and g are onto functions, then gof is an onto-function

⇒ From the definition of onto-functions

(A function $k: L \to M$ is onto if, for every element $m \in M$, there exists a pre-image $l \in L$ such that $k(l) = m$)

⇒ Given that g is an onto-function
$g: B \to C$

Let $c \in C$, then there exists a pre-image in B → (let that pre-image be b) such that

$\boxed{g(b) = c}$ → $(b \in B)$

⇒ Given that f is an onto function
$f: A \to B$

Since $b \in B$ and f is onto, there exists a preimage in A [let the pre-image be a]
such that

$\boxed{f(a) = b}$ → $(a \in A)$

⇒ $gof: A \to C$

$g(b) = g(f(a)) = c = gof(a)$

Since c was an arbitrary element of C, then for all $c \in C$, there exists $a \in A$ such that $gof(a) = c$

⇒ For every element $c \in C$, there exists pre-image $a \in A$ such that $gof(a) = c$

gof is an onto function → (If f, g are onto)

⇒ From the preceding proofs, we have proved that gof is both one-one function and onto-function (if f, g → bijective function)

From the definition of bijective function,

⇒ gof is a bijective function

∴ If both f and g are bijective functions, then gof is a bijective function

⑥

(x be an element of set)

* Let A, B be arbitrary sets (properties)

① If $x \in A$ (or) $x \in B \Leftrightarrow x \in A \cup B$

② If $x \in A$ and $x \in B \Leftrightarrow x \in A \cap B$

③ If $x \notin A \Leftrightarrow x \in A^c$

* If f is invertible and $f(x) = y$, then
$y = f^{-1}(x)$

* A function f is called one-one function if
$f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$

* A function f is called onto function if
codomain = range of f
(Every element in codomain has pre-image in domain)

* The Cartesian product $X \times Y$ between two sets X and Y is the set of all possible ordered pairs with first element from X and second element from Y.

$$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}$$

* A function f is called bijective if it's both one-one & onto function.

* $\boxed{A \times \phi = \phi \times A = \phi}$

* ($x \in A$ and $y \in C$) and ($x \in B$ and $y \in C$)
$\equiv$ ($x \in A$ and $x \in B$) and $y \in C$

* ($x \in A$ and $z \in C$) (or) ($x \in B$ and $z \in C$)
$\equiv$ [($x \in A$ (or) $x \in B$)] and $z \in C$

Theorem-1 : Consider two arbitrary sets A and B. Then, $A \subseteq A \cup B$ and $B \subseteq A \cup B$

proof:- Suppose $x \in A$. That is, x is an element of (A set is well defined collection of objects) A. Then certainly x is either an element of A or an element of B (or both). Thus x is an element of $A \cup B$, i.e $x \in A \cup B$. Since, we have shown that any element of A is an element of $A \cup B$, $A \subseteq A \cup B$ (similarly for B)

Theorem-2 :- Consider Two arbitrary sets A and B. Then $A \cap B \subseteq A$ and $A \cap B \subseteq B$

proof:

Let $x \in A \cap B$

From the definition of intersection of sets,

$x \in A$ and $x \in B$

$\Rightarrow$ Thus in particular, $x \in A$

If $x \in A \cap B$, then $x \in A \Rightarrow A \cap B \subseteq A$

(Similarly for B)

---

* $A \cup \emptyset = A$ * $A \cap \emptyset = \emptyset$ * $A \cup B = B \cup A$
* $A \cap B = B \cap A$ * $(A \cup B) \cup C = A \cup (B \cup C)$
* $(A \cap B) \cap C = A \cap (B \cap C)$

associative laws

* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Distributive laws

* $A \cup A = A \cap A = A$
* $(A \cup B)^c = A^c \cap B^c$ * $(A \cap B)^c = A^c \cup B^c$

De-Morgan's Laws

Given a function $f: X \to Y$

RTP:- The following statements are equivalent

(a) f is a one-one function

(b) $f(A \cap B) = f(A) \cap f(B)$ holds for all $A, B \in P(X)$

(a) $\Rightarrow$ (b)

Given that f is a one-one function

(A function K is said to be one-one (or) injective if $K(x_1) = K(x_2)$ implies $x_1 = x_2$)

$\Rightarrow$ Let $y \in f(A) \cap f(B)$

$\Rightarrow$ $y \in f(A)$ and $y \in f(B)$

Let $a_1 \in A$ such that $f(a_1) = y$

Let $b_1 \in B$ such that $f(b_1) = y$

$\Rightarrow$ Given that f is an injective-function

$f(a_1) = y = f(b_1)$

$\Rightarrow$ $f(a_1) = f(b_1)$ $\Rightarrow$ $a_1 = b_1$ (Because f is a one-one function)

But $a_1 \in A$ and $b_1 \in B$ (Let $a_1 = b_1 = h$)

$\Rightarrow$ $h \in A$ and $h \in B$ $\Rightarrow$ $h \in A \cap B$ ($h \in X$)

$y = f(h)$ for some $h \in A \cap B$

$\Rightarrow$ $y \in f(A \cap B)$

If $y \in f(A) \cap f(B)$, then $y \in f(A \cap B)$

$\boxed{f(A) \cap f(B) \subseteq f(A \cap B)}$ — Eqn (1)

Let $z \in f(A \cap B)$

Assume that $f(z)$

Let $z \in f(A \cap B)$

Assume that $f(x) = z$, where $x \in A \cap B$

$\Rightarrow x \in A \cap B$

$\Rightarrow x \in A$ and $x \in B$ (property of intersection)

$\Rightarrow f(x) \in f(A)$ and $f(x) \in f(B)$

$\Rightarrow f(x) \in f(A) \cap f(B)$

$\Rightarrow \boxed{z \in f(A) \cap f(B)}$

If $z \in f(A \cap B)$, then $z \in f(A) \cap f(B)$

$\boxed{f(A \cap B) \subseteq f(A) \cap f(B)}$ — Eqn (112)

From Eqn's (111) and (112),

$\boxed{f(A \cap B) = f(A) \cap f(B)}$ (If f is one-one function)

$\Rightarrow$ If f is a one-one function, $f(A \cap B) = f(A) \cap f(B)$ holds for all $A, B \in P(X)$ (where $f: X \to Y$)

(b) ⇒ (a)

Given that $f: X \to Y$ be a function

⇒ Given that $f(A \cap B) = f(A) \cap f(B)$ holds for all $A, B \in P(X)$

RTP → f is a one-one function

To prove (b) ⇒ (a), lets use proof by contraposition and prove $\sim$(a) ⇒ ($\sim$b)

RTP:- If f is a many-one function, then $f(A \cap B) \neq f(A) \cap f(B)$ holds for some $A, B \in P(X)$

Let $\{x_1\}, \{x_2\} \in X$ such that

$f(\{x_1\}) = f(\{x_2\})$ and $x_1 \neq x_2$

since f is a many-one function

$A = \{x_1\}$ $B = \{x_2\}$

$A \cap B = \{x_1\} \cap \{x_2\} = \phi$ (As $x_1 \neq x_2$)

Given that $f(A) = \{f(a), \forall a \in A\}$

$f(A)$ is the set of images of all elements of A

Let $f(\{x_1\}) = f(\{x_2\}) = c$

$f(A \cap B) = f(\{x_1\} \cap \{x_2\}) = f(\phi)$

$f(A) = f(\{x_1\}) = c$

$f(B) = f(\{x_2\}) = c$

$f(A) \cap f(B) = c \neq f(\phi)$

$\boxed{f(A) \cap f(B) \neq f(A \cap B)}$ for some $A, B \in P(X)$

$\{x_1\}$ $\{x_2\}$

Having proved the contrapositive, we can now infer that the original statement is true.

$\Rightarrow$ If $f(A \cap B) = f(A) \cap f(B)$ holds for all $A, B \in P(X)$, then f is a one to one function

$\therefore$ Since $\underline{a \Rightarrow b}$ & $\underline{b \Rightarrow a}$ are true, the statements (a), (b) are equivalent

$\Rightarrow$ Iff f is a one to one-function, then $f(A \cap B) = f(A) \cap f(B)$ holds $\forall A, B \in P(X)$

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(x be an element of set)

* Let A, B be arbitrary sets (properties)

① If $x \in A$ (or) $x \in B \Leftrightarrow x \in A \cup B$

② If $x \in A$ and $x \in B \Leftrightarrow x \in A \cap B$

③ If $x \notin A \Leftrightarrow x \in A^c$

* If f is invertible and $f(x) = y$, then
  $y = f^{-1}(x)$

* A function f is called one-one function if
  $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$

* A function f is called onto function if
  codomain = range of f
  (Every element in codomain has pre-image in domain)

* The Cartesian product $X \times Y$ between two sets X and Y is the set of all possible ordered pairs with first element from X and second element from Y.

$$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}$$

* A function f is called bijective if it's both one-one & onto function.

* $\boxed{A \times \phi = \phi \times A = \phi}$

* ($x \in A$ and $y \in C$) and ($x \in B$ and $y \in C$)
  $\equiv$ ($x \in A$ and $x \in B$) and $y \in C$

* ($x \in A$ and $z \in C$) (or) ($x \in B$ and $z \in C$)
  $\equiv$ [($x \in A$ (or) $x \in B$)] and $z \in C$

Theorem-1 : Consider two arbitrary sets A and B. Then, $A \subseteq A \cup B$ and $B \subseteq A \cup B$

proof :- Suppose $x \in A$. That is, x is an element of (A set is well defined collection of objects) A. Then certainly x is either an element of A or an element of B (or both). Thus x is an element of $A \cup B$, i.e $x \in A \cup B$. Since, we have shown that any element of A is an element of $A \cup B$, $A \subseteq A \cup B$ (similarly for B)

Theorem-2 :- Consider Two arbitrary sets A and B. Then $A \cap B \subseteq A$ and $A \cap B \subseteq B$

proof :

Let $x \in A \cap B$

From the definition of intersection of sets,

$x \in A$ and $x \in B$

$\Rightarrow$ Thus in particular, $x \in A$

If $x \in A \cap B$, then $x \in A \Rightarrow A \cap B \subseteq A$

(Similarly for B)

---

* $A \cup \emptyset = A$ * $A \cap \emptyset = \emptyset$ * $A \cup B = B \cup A$
* $A \cap B = B \cap A$ * $(A \cup B) \cup C = A \cup (B \cup C)$
* $(A \cap B) \cap C = A \cap (B \cap C)$

associative laws

* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Distributive laws

* $A \cup A = A \cap A = A$
* $(A \cup B)^c = A^c \cap B^c$ * $(A \cap B)^c = A^c \cup B^c$

De-Morgan's Laws

Given an arbitrary function $f: X \to Y$

(a)

RTP:- $f^{-1}(\bigcup_{i \in I} B_i) = \bigcup_{i \in I} f^{-1}(B_i)$

Case-1

$\bigcup_{i \in I} B_i \neq \phi$ (Union of sets of the form B_i)

(Atleast one of B_i is non-empty)

$\Rightarrow$ Let $y \in \bigcup_{i \in I} f^{-1}(B_i)$

It means that $y \in f^{-1}(B_i)$ for some $i \in I$

$y \in X$

$\Rightarrow$ $f(y) \in B_i$ for some $i \in I$

$\Rightarrow$ $f(y) \in B_{i_1}$ (or) $f(y) \in B_{i_2}$ --- (or) $f(y) \in B_{i_k}$

(suppose numbering possible values of i)

$\Rightarrow$ $f(y) \in \bigcup_{i \in I} B_i$ $(i_1, i_2, \ldots i_k \in I)$

$\Rightarrow$ $\boxed{y \in f^{-1}(\bigcup_{i \in I} B_i)}$

If $y \in \bigcup_{i \in I} f^{-1}(B_i)$, then $y \in f^{-1}(\bigcup_{i \in I} B_i)$

$\boxed{\bigcup_{i \in I} f^{-1}(B_i) \subseteq f^{-1}(\bigcup_{i \in I} B_i)}$ — Eqn — (10)

$\Rightarrow$ Let $z \in f^{-1}(\bigcup_{i \in I} B_i)$ $z \in X$

which means that

$f(z) \in \bigcup_{i \in I} B_i$

(Suppose numbering possible values of i)

$\Rightarrow$ $f(z) \in B_{i_1}$ (or) $f(z) \in B_{i_2}$ --- (or) $f(z) \in B_{i_k}$ $(i_1, i_2, \ldots i_k \in I)$

$\Rightarrow f(z) \in B_i$ for some $i \in I$

It means that $z \in f^{-1}(B_i)$ for some $i \in I$

$\Rightarrow$

$z \in f^{-1}(B_{i_1})$ (or) $z \in f^{-1}(B_{i_2})$ (or) ---- $z \in f^{-1}(B_{i_k})$

(suppose numbering possible values of i)

$(i_1, i_2, \ldots i_k \in I)$

$\Rightarrow \boxed{z \in \bigcup_{i \in I} f^{-1}(B_i)}$

If $z \in f^{-1}(\bigcup_{i \in I} B_i)$, then $z \in \bigcup_{i \in I} f^{-1}(B_i)$

$\boxed{f^{-1}(\bigcup_{i \in I} B_i) \subseteq \bigcup_{i \in I} f^{-1}(B_i)}$ — Eqn — (102)

From Eqn's (101) & (102),

$\boxed{f^{-1}(\bigcup_{i \in I} B_i) = \bigcup_{i \in I} f^{-1}(B_i)}$

__Case-2__

If $\bigcup_{i \in I} B_i = \phi$

(union of sets of the form B_i) → All sets of form B_i are empty

$B_i = \phi$ for $i \in I$

$f^{-1}(\bigcup_{i \in I} B_i) = f^{-1}(\phi)$

$= \bigcup_{i \in I} f^{-1}(\phi)$

$= \bigcup_{i \in I} f^{-1}(B_i)$

($A \cup A = A$ in sets)

$\Rightarrow \boxed{f^{-1}(\bigcup_{i \in I} B_i) = \bigcup_{i \in I} f^{-1}(B_i)}$

In both cases, in accordance to the proofs

$\boxed{\bigcup_{i \in I} f^{-1}(B_i) = f^{-1}(\bigcup_{i \in I} B_i)}$

Given an arbitrary function $f: X \to Y$

(b)

RTP:- $f^{-1}(\bigcap_{i \in I} B_i) = \bigcap_{i \in I} f^{-1}(B_i)$

Case-1

$\bigcap_{i \in I} B_i \neq \emptyset$ (Intersection of sets of the form B_i)

$\Rightarrow$ Let $y \in \bigcap_{i \in I} f^{-1}(B_i)$ (No element common in all B_i's)

It means that $y \in f^{-1}(B_i)\ \forall i \in I$

$\Rightarrow f(y) \in B_i$ for all $i \in I$

$\Rightarrow f(y) \in B_{i_1}$ and $f(y) \in B_{i_2}$ --- and $f(y) \in B_{i_k}$ (suppose numbering the possible values of i $(i_1, i_2, i_3 - - i_k \in I)$)

$\Rightarrow f(y) \in \bigcap_{i \in I} B_i$

$\Rightarrow \boxed{y \in f^{-1}(\bigcap_{i \in I} B_i)}$

If $y \in \bigcap_{i \in I} f^{-1}(B_i)$, then $y \in f^{-1}(\bigcap_{i \in I} B_i)$

$$\boxed{\bigcap_{i \in I} f^{-1}(B_i) \subseteq f^{-1}(\bigcap_{i \in I} B_i)} \text{ — Eqn-(103)}$$

$\Rightarrow$ Let $z \in f^{-1}(\bigcap_{i \in I} B_i)$

which means that

$f(z) \in \bigcap_{i \in I} B_i$ (suppose numbering the possible values of i)

$\Rightarrow f(z) \in B_{i_1}$ and $f(z) \in B_{i_2}$ and --- and $f(z) \in B_{i_k}$ $(i_1, i_2, -- i_k \in I)$

$\Rightarrow f(z) \in B_i \; \forall i \in I$

It means that $z \in f^{-1}(B_i)$ for all $i \in I$

$\Rightarrow z \in f^{-1}(B_{i_1})$ and $z \in f^{-1}(B_{i_2})$ and --- $z \in f^{-1}(B_{i_k})$ (suppose, numbering the possible values of i $(i_1, i_2, \ldots i_k \in I)$)

$\Rightarrow \boxed{z \in \bigcap_{i \in I} f^{-1}(B_i)}$

If $z \in f^{-1}(\bigcap_{i \in I} B_i)$, then $z \in \bigcap_{i \in I} f^{-1}(B_i)$

$$\boxed{f^{-1}\left(\bigcap_{i \in I} B_i\right) \subseteq \bigcap_{i \in I} f^{-1}(B_i)} \quad \text{— Eqn — (104)}$$

From Eqns — (103) & (104),

$$\boxed{f^{-1}\left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} f^{-1}(B_i)}$$

Case-2

If $\bigcap_{i \in I} B_i = \phi$ (Intersection of the sets of the form B_i)

Assume that

let $x \in \bigcap_{i \in I} f^{-1}(B_i)$

$\Rightarrow f(x) \in B_i$ for all $i \in I$

$\Rightarrow f(x) \in B_{i_1}$ and $f(x) \in B_{i_2}$ --- and $f(x) \in B_{i_k}$ (suppose numbering possible values of i)

$\Rightarrow f(x) \in \bigcap_{i \in I} B_i \; = \phi$ $\quad (i_1, i_2 \ldots i_k \in I)$

Contradiction! Assumption is wrong

$\Rightarrow X = f^{-1}(\phi)$

$\Rightarrow$ From both cases, in accordance with proofs, $\left(= f^{-1}\left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} f^{-1}(B_i)\right)$

$$\boxed{f^{-1}\left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} f^{-1}(B_i)}$$

(c) Given an arbitrary function $f: X \to Y$

RTP:- $f^{-1}(B^c) = [f^{-1}(B)]^c$

Let $y \in f^{-1}(B^c)$

$\Rightarrow f(y) \in B^c$ (property of inverse) as $y \in X$ and since f is a function

$\Rightarrow f(y) \notin B$ (but $f(y) \in Y$)

$\Rightarrow y \notin f^{-1}(B)$ such that $y \in X$

$\Rightarrow y \in (f^{-1}(B))^c$ (property of sets)

If $y \in f^{-1}(B^c)$, then $y \in (f^{-1}(B))^c$

$\boxed{f^{-1}(B^c) \subseteq (f^{-1}(B))^c}$ — Eqn — (111)

Let $z \in [f^{-1}(B)]^c$

Then $z \notin (f^{-1}(B))$ (and $z \in X$) (property of complement)

$f(z) \notin B$, but since $z \in X$

$f(z) \in Y$

which means that

$f(z) \in B^c$

$\boxed{z \in f^{-1}(B^c)}$ (property of inverse)

$\Rightarrow$ If $z \in [f^{-1}(B)]^c$, then $z \in f^{-1}(B^c)$

$\boxed{[f^{-1}(B)]^c \subseteq f^{-1}(B^c)}$ - Eqn (112)

From Eqns (111) and (112), (property of sets)

$\boxed{[f^{-1}(B)]^c = f^{-1}(B^c)}$ (by proposition)