

Assignment - 2

1. Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \\ & \text{and } x \text{ is rational} \\ 1 & \text{if } x \in \mathbb{R} \\ & \text{and } x \text{ is irrational} \end{cases}$$

Show that $\lim_{x \rightarrow a} f(x)$ does not exist
 $\forall a \in \mathbb{R}$.

2. Consider the function

$f : [0, 1] \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1/q & \text{if } x \text{ is rational} \\ & \text{and } x = p/q \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

s.t. $\lim_{x \rightarrow a} f(x) = 0$ for all $a \in (0, 1)$.

3. Let $f(x)$ be a ~~is~~ periodic function. Show that if $\lim_{x \rightarrow a} f(x)$ exists, then $f(x)$ is a constant function. Deduce from this that $\lim_{x \rightarrow a} \sin x$ does not exist.

4. Evaluate

$$\lim_{x \rightarrow 0} \frac{\sin x}{\sqrt{1 - \cos x}}$$

5. Let $f: \mathbb{R} \rightarrow \mathbb{R}$.

$$\text{s.t. } f(x+y) = f(x) + f(y).$$

$$\forall (x, y) \in \mathbb{R}.$$

$$\text{g.f. } \lim_{x \rightarrow 0} f(x) = f(0).$$

Find $f(x)$.