Real Analysis

End-Sem 2022

Full marks 100 (10 × 10) Time - 3 hours

1. Prove that $\sqrt{2}$ is not rational.

2. Consider the Fibonacci numbers $\{F_n\}$ defined by $F_1 = 1,\ \ F_2 = 1,\ \ and\ \ F_{n+2} = F_{n+1} + F_n$. Show that

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}, \quad n = 1, 2, 3, \ldots.$$

3. Show that the sequence $\{x_n\}$ defined by $x_n = \int_1^n \frac{\cos t}{t^2} dt$ is Cauchy.

4. Discuss the convergence or divergence of

$$x_n = \frac{[\alpha] + [2\alpha] + [3\alpha] + \ldots + [n\alpha]}{n^2}, \quad n \in \mathbb{N},$$

where $[x]$ represents the greatest integer less that or equal to the x and α is an arbitrary real number.

5. Given $x \geq 1$, show that $\lim_{n\to\infty} \left(2x^{1/n} - 1\right)^n = x^2$.

6. Let $f(x) = [x]$ and $g(x) = x - [x]$. Sketch the plots for f and g. Find the points at which they are continuous.

7. Show that any function continuous and periodic on $\mathbb{R}$ must be uniformly continuous.

8. Show that there exists a continuous function $F : [0,1] \to \mathbb{R}$ whose derivative exists and equals zero almost everywhere but which is not constant.

9. Let $f(x)$ is differentiable at a. Then find

$$\lim_{n\to\infty} \frac{a^n f(x) - x^n f(a)}{x - a}, \quad n \in \mathbb{N}.$$

10. Consider a function $f(x)$, whoose second derivative $f''(x)$ exists and continuous on (a, b) with $c \in (a, b)$. Show that

$$\lim_{h\to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c).$$

Is the existence of the second derivative necessary to prove the existence of the above limit?