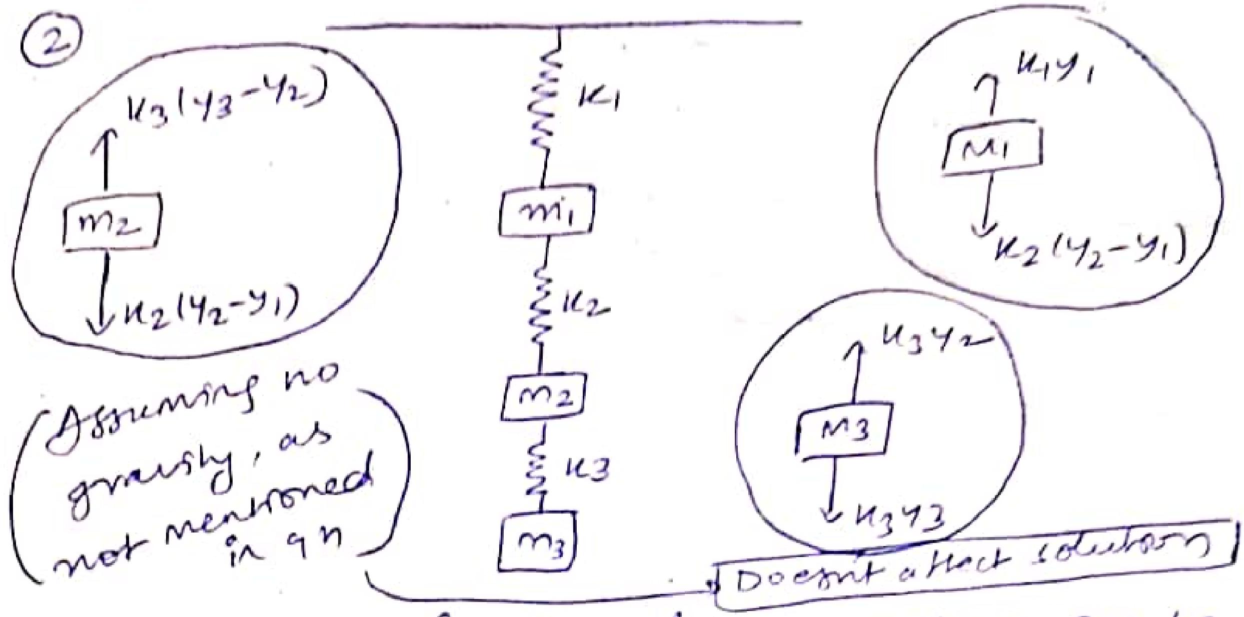
②



The equations of motion for the system can be derived using Newton's 2nd law, which states that net force acting on an object is equal to the mass of object multiplied by it's acceleration

For mass m_1:

$$(F_{net})_{m_1} = m_1 * \left(\underbrace{\frac{d^2y_1}{dt^2}}_{a_1}\right) = k_2(y_2-y_1) - k_1(y_1) \quad —①$$

For mass m_2:

$$(F_{net})_{m_2} = m_2\left(\underbrace{\frac{d^2y_2}{dt^2}}_{a_2}\right) = k_2(y_2-y_1) - k_3(y_3-y_2) \quad —②$$

For mass m_3:

$$(F_{net})_{m_3} = m_3\left(\underbrace{\frac{d^2y_3}{dt^2}}_{a_3}\right) = k_3(y_3-y_2) \quad —③$$

→ Motion of the system is governed by the system of ordinary differential equations

$$My'' + ky = 0$$

where $M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$ and $k = \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$

and $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

→ We will find complex solutions $z(t) = ae^{i\omega t}$, but there could possibly be more than one frequency of oscillation, since we have 3m's and 3k's

→ It turns out that we only need to assume one frequency $\omega = \sqrt{k/m}$ initially, but we arrive at an equation for ω that is satisfied by more than one frequency

$\boxed{\text{BONUS}}$

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} e^{i\omega t} = Ae^{i\omega t}$$

$y_1 = A_1 e^{i\omega t} \to y_1(t)$

$y_2 = A_2 e^{i\omega t} \to y_2(t)$

$y_3 = A_3 e^{i\omega t} \to y_3(t)$

→ Here we will keep only the real part of the solution, i.e, $y(t) = Re[z(t)]$

→ When we substitute $z(t)$ into $My'' + Ky = 0$, we find the relation $-\omega^2 MAe^{i\omega t} = -KAe^{i\omega t}$

so the following equation must be satisfied

$$(K - \omega^2 M)a = 0 \qquad \left(\text{assume } a = A\right)$$

$$\downarrow$$

$$(K - \omega^2 M)A = 0.$$

→ Clearly, if we ignore the trivial solution $A = 0$ (no motion at all), we must have $\det(K - \omega^2 M) = 0$

→ Solving this equation for ω^2 will give natural frequencies of the system. The eigen-values of the matrix equation are the square of the natural frequencies of the system

→ To solve the characteristic equation, we need to find the determinant of the matrix $(K - \omega^2 M)$

$$|K-\omega^2 M| = \begin{vmatrix} (K_1+K_2-m_1\omega^2) & -K_2 & 0 \\ -K_2 & (K_2+K_3-m_2\omega^2) & -K_3 \\ 0 & -K_3 & K_3-\omega^2 m_3 \end{vmatrix} = 0$$

Given that $K_1 = K_2 = K_3 = 1$, $m_1 = 2$, $m_2 = 2$, $m_3 = 4$

$$|K-\omega^2 M| \Rightarrow \begin{vmatrix} 2-2\omega^2 & -1 & 0 \\ -1 & 2-2\omega^2 & -1 \\ 0 & -1 & 1-4\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow (2)(1-\omega^2)\left[2(1-\omega^2)(1-4\omega^2)-1\right] - (1-4\omega^2) = 0$$

$$\Rightarrow 2(1-\omega^2)\left[2(4\omega^4-5\omega^2+1)-1\right] - 1 + 4\omega^2 = 0$$

$$\Rightarrow 2(1-\omega^2)\left[8\omega^4-10\omega^2+1\right] - 1 + 4\omega^2 = 0$$

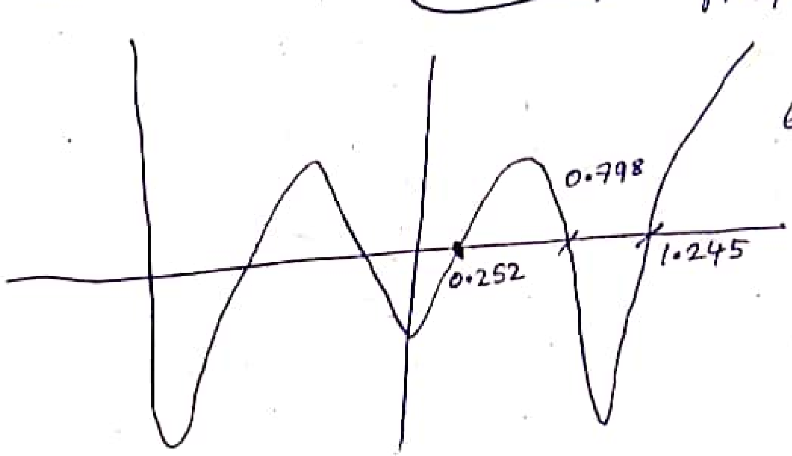
$$\Rightarrow 16\omega^4 - 20\omega^2 + 2 - 16\omega^6 + 20\omega^4 - 2\omega^2 - 1 + 4\omega^2 = 0$$

$$\Rightarrow -16\omega^6 + 36\omega^4 - 18\omega^2 + 1 = 0$$

$$\Rightarrow 16\omega^6 - 36\omega^4 + 18\omega^2 - 1 = 0$$

$$\boxed{\omega = 0.252,\ 0.798,\ 1.245}$$

Natural frequencies



→ Let $\omega_1 = 0.282$ [First normal mode]

The solutions are then

$$(K-\omega_1^2 M)A = \begin{bmatrix} 2-2\omega_1^2 & -1 & 0 \\ -1 & 2-2\omega_1^2 & -1 \\ 0 & -1 & 1-4\omega_1^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = 0$$

$A_2 = A_1(2-2\omega_1^2) \Rightarrow$ Let $A_1 = Ae^{-i\delta}$

$A_2 = A_3(1-4\omega_1^2) \Rightarrow$ $A_2 = A(1.87)e^{-i\delta}$

$A_3 = 2.51Ae^{-i\delta}$

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} e^{i\omega_1 t} = \begin{bmatrix} A \\ 1.87A \\ 2.51A \end{bmatrix} e^{i(\omega_1 t-\delta)}$$

real part

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} A \\ 1.87A \\ 2.51A \end{bmatrix} \cos(\omega_1 t-\delta)$$

→ All masses m_1, m_2, m_3 move in unison

$y_1(t) = A\cos(\omega_1 t-\delta)$

$y_2(t) = 1.87A\cos(\omega_1 t-\delta)$

$y_3(t) = 2.51A\cos(\omega_1 t-\delta)$

[First Normal mode]

Let $\omega_2 = 0.798$ [Second normal mode]

The solutions are then

$$(K-\omega_2^2 M)A = \begin{bmatrix} 2-2\omega_2^2 & -1 & 0 \\ -1 & 2-2\omega_2^2 & -1 \\ 0 & -1 & 1-4\omega_2^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

$$(2-2\omega_2^2)A_1 = A_2$$

$$A_3(1-4\omega_2^2) = A_2$$

→ Let $A_1 = A e^{-i\delta}$, then

$$A_2 = A(2-2\omega_2^2)e^{-i\delta}$$

$$A_3 = \frac{A(2-2\omega_2^2)}{1-4\omega_2^2} e^{-i\delta}$$

Hence, we finally have,

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} e^{i\omega_2 t} = \begin{bmatrix} A \\ 0.72A \\ -0.47A \end{bmatrix} e^{i(\omega_2 t - \delta)}$$

real part

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} A \\ 0.72A \\ -0.47A \end{bmatrix} \cos(\omega_2 t - \delta)$$

→ m_1 and m_2 move in same direction whereas m_3 moves in the opposite direction

$$y_1(t) = A\cos(\omega_2 t - \delta)$$

$$y_2(t) = A(0.72)\cos(\omega_2 t - \delta)$$

$$y_3(t) = -0.47A\cos(\omega_2 t - \delta)$$

Let $w_3 = 1.245$ [Third Normal mode]

The solutions are then,

$$(K - w_3^2 M) A = \begin{vmatrix} 2-2w_3^2 & -1 & 0 \\ -1 & 2-2w_3^2 & -1 \\ 0 & -1 & 1-4w_3^2 \end{vmatrix} A = 0$$

$$= \begin{vmatrix} 2-2w_3^2 & -1 & 0 \\ -1 & 2-2w_3^2 & -1 \\ 0 & -1 & 1-4w_3^2 \end{vmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = 0$$

$$A_2 = A_1 (2 - 2w_3^2)$$

$$A_3 (1 - 4w_3^2) = A_2$$

→ Then, let $A_1 = A e^{-i\delta}$

$$A_2 = A (2 - 2w_3^2) e^{-i\delta} = -1.1 A e^{-i\delta}$$

$$A_3 = \frac{A (2 - 2w_3^2)}{1 - 4w_3^2} e^{-i\delta} = 0.21 A e^{-i\delta}$$

Hence we finally have,

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} e^{iw_3 t} = \begin{bmatrix} A \\ -1.1A \\ 0.21A \end{bmatrix} e^{i(w_3 t - \delta)}$$

real part

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} A \\ -1.1A \\ 0.21A \end{bmatrix} \cos(w_3 t - \delta)$$

→ m_1 and m_3 move opposite in direction to m_2.

$$y_1(t) = A \cos(w_3 t - \delta)$$

$$y_2(t) = -1.1 A \cos(w_3 t - \delta)$$

$$y_3(t) = 0.21 A \cos(w_3 t - \delta)$$

General motion → It is important to realize that these are the only three normal modes for oscillation, general oscillation is a combination of these 3 modes with possibly different amplitudes & phases based on initial conditions

$$y(t) = A_1 \begin{bmatrix} 1 \\ 1.87 \\ 2.51 \end{bmatrix} \cos(\omega_1 t - \delta_1) + A_2 \begin{bmatrix} 1 \\ 0.72 \\ -0.47 \end{bmatrix} \cos(\omega_2 t - \delta_2)$$

$$+ A_3 \begin{bmatrix} 1 \\ -1.1 \\ 0.21 \end{bmatrix} \cos(\omega_3 t - \delta_3)$$