① Given the polynomial $p(t) = a_1t^2 + a_2t + a_3 = y$

Given a set of 5 data points,

	t_1	t_2	t_3	t_4	t_5
t	-1.0	-0.5	0.0	0.5	1.0
y	1.0	0.5	0.0	0.5	2.0
	y_1	y_2	y_3	y_4	y_5

Least squares solution x minimizes squared Euclidean norm of residual vector, $r = b - Ax$

$$\min_x \|r\|_2^2 = \min_x \|b - Ax\|_2^2$$

$$\phi = \|r\|_2^2 = r^Tr = (b-Ax)^T(b-Ax)$$
$$= b^Tb - 2x^TA^Tb + x^TA^TAx$$

$$d\phi/dx = 0 \longrightarrow 2A^TAx - 2A^Tb$$
$$x = (A^TA)^{-1}(A^Tb)$$

→ System can be treated as linear, it's overdetermined

$$Ax = \begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \\ 1 & t_4 & t_4^2 \\ 1 & t_5 & t_5^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cong \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = b$$

$$x = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$ → We find least squares solution

$$Ax = \begin{bmatrix} 1 & -1.0 & 1.0 \\ 1 & -0.5 & 0.25 \\ 1 & 0.0 & 0.0 \\ 1 & 0.5 & 0.25 \\ 1 & 1.0 & 1.0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cong \begin{bmatrix} 1.0 \\ 0.5 \\ 0.0 \\ 0.5 \\ 2.0 \end{bmatrix} = b$$

$$x = (A^TA)^{-1}A^Tb$$

$$= \left(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -1/2 & 0 & 1/2 & 1 \\ 1 & 1/4 & 0 & 1/4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1/2 & 1/4 \\ 1 & 0 & 0 \\ 1 & 1/2 & 1/4 \\ 1 & 1 & 1 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -1/2 & 0 & 1/2 & 1 \\ 1 & 1/4 & 0 & 1/4 & 1 \end{bmatrix} \right)$$

$$\left(\begin{bmatrix} 1 \\ 1/2 \\ 0 \\ 1/2 \\ 2 \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} 5 & 0 & 5/2 \\ 0 & 5/2 & 0 \\ 5/2 & 0 & 17/8 \end{bmatrix} \right)^{-1} \begin{bmatrix} 4 \\ 1 \\ 13/4 \end{bmatrix}$$

$$= \begin{pmatrix} 17/35 & 0 & -4/7 \\ 0 & 2/5 & 0 \\ -4/7 & 0 & 8/7 \end{pmatrix} \begin{bmatrix} 4 \\ 1 \\ 13/4 \end{bmatrix}$$

$$= \begin{pmatrix} 68/35 - 13/7 \\ 2/5 \\ -16/7 + \frac{104}{28} \end{pmatrix} = \begin{pmatrix} 3/35 \\ 2/5 \\ 10/7 \end{pmatrix} = \begin{pmatrix} 0.086 \\ 0.4 \\ 1.4 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$a_1 = 0.086$

$a_2 = 0.4$

$a_3 = 1.4$

$$p(t) = 0.086 + 0.4t + 1.4t^2$$