Introduction to the stability of nonlinear systems

Objective. display some tools used the design of robust nonlinear control laws

- sliding mode control
- backstepping control

Idea used in these both control strategies.

- deterministic approach
- Lyapunov functions

Equilibrium point. Consider the nonlinear system $\dot{x} = f(x) + g(x).u$

➡ all point (x_0,u_0) solution of $\dot{x} = 0$

Stability (Lyapunov sense). The equilibrium point x_0 is said to be stable if

$$\forall R > 0,\ \exists r > 0 : \|x(0)\| < r \Rightarrow \|x(t)\| < R, \forall t > 0 .$$

Asymptotic stability (Lyapunov sense). The equilibrium point x_0 is said to be asymptotically stable if

$$\exists r > 0 : \|x(0)\| < r \Rightarrow \|x(t)\| \to 0 \text{ quand } t \to \infty$$

Sufficient conditions for stability around an equilibrium point. Consider the linear approximation of a nonlinear system around the equilibrium point

$$\{\dot{x} = f(x) \quad \to \quad \dot{\eta} = \left[\frac{\partial f}{\partial x}\right]_{x = x_0} \eta, \quad \dot{\eta} := A\eta$$

Theorem.

- If the linearized system is asymptotically stable (real part of A-eigenvalues < 0), then the nonlinear system is asymptotically stable;
- if the linearized system is instable (there is at least one eigenvalue of A whose its real part >0), then the nonlinear system is unstable;
- if there is an eigenvalue of A =0, then one can not conclude.

Examples. Analyze the stability of $\dot{x} = \pm x + x^2$ around x = 0.

$\dot{x} = kx^2 \;\; k > 0$ around x = 0.

Stability analysis thanks to Lyapunov functions.

Definition. A continuously differentiable V(x) is said to be a Lyapunov function if

$$V(0)=0 \qquad V(x)>0 \quad \forall x \neq 0 \quad \dot{V}(x) \leq 0 \qquad \forall x \neq 0$$

Then, the equilibrium point 0 is **stable**.

Theorem. Consider a nonlinear system with an equilibrium point at 0. If there exists a function V(x) such that it is definite positive, and its time derivative is semi definite negative,

$$V(x)>0 \qquad \dot{V}(x)<0$$

then the equilibrium point is **asymptotically stable**.

Remarks.

- Stability in the sense of Lyapunov is a mathematical traduction of the following feature: if the total amount of energy of a system continuously dissipates, then the system tends to its equilibrium point.
- Lyapunov function is not unique.
- This approach is conservative.

Analysis of stability thanks two different Lyapunov functions.

$$\begin{cases} \dot{x}_1 = 2x_1(x_2^2 - 1) \\ \dot{x}_2 = -x_2(x_1^2 + 1) \end{cases} \Rightarrow \quad V_1(x_1, x_2) = \frac{x_1^2 + x_2^2}{2} \qquad V_2(x_1, x_2) = \frac{x_1^2 + 2x_2^2}{2}$$

Analyze the stability of the following systems

$$\dot{x} = -x^3 \qquad \dot{x} = -x^2$$