Sliding mode control

Objectives of this class of <u>nonlinear</u> control ?

- Robustness versus uncertainties/perturbations
- Finite time convergence towards the objectives

Features of this class of control ?

- Discontinuous control law
- For standart sliding mode (*first order*) : chattering effect, robustness
- For high order sliding mode : accuracy, finite time convergence, robustness

This class of controllers has been studied

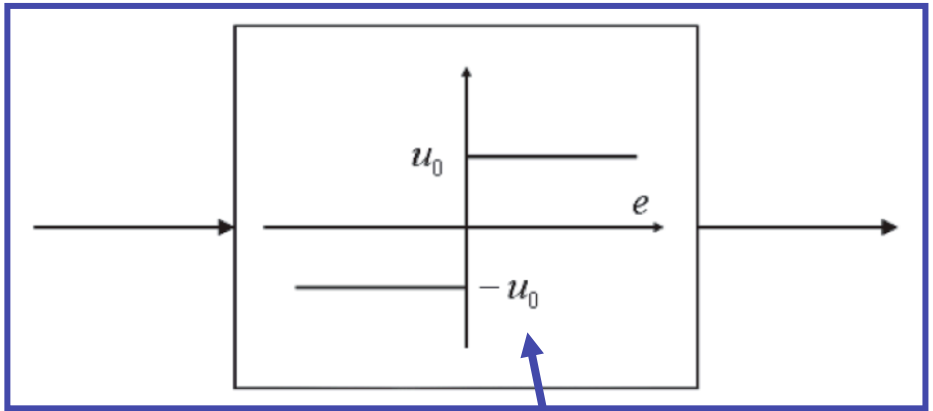
- from 40's in the former USSR
- intensively since 20 years

Sliding modes as a phenomenon may appear in a dynamic system governed by ordinary differential equations with discontinuous right hand sides.

The term « **sliding mode** » first appeared in the context of relay systems.

→ It may happen that the control as a function of the system state switches at high (theoretically infinite) frequency

→ This motion is referred to as « sliding mode ».



Nonlinear system (uncertain)

$$\dot{x} = f(x) + u \qquad |f(x)| < f_0$$

How to stabilize, in a finite time, this system in spite of uncertainties ?

Tracking error $e = r(t) - x$

Reference

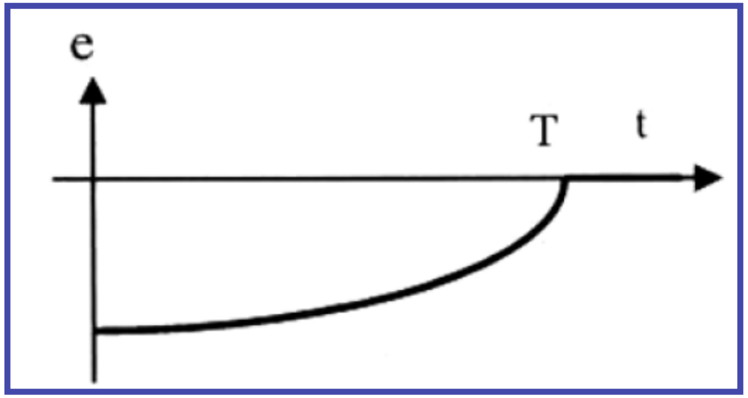
Switched control input

$$u = \begin{cases} u_0 & \text{if} \quad e > 0 \\ -u_0 & \text{if} \quad e < 0 \end{cases} \Leftrightarrow \boxed{u = u_0 \ sign(e)}$$

$$\Rightarrow \quad \dot{e} = \dot{r} - f(x) - u_0 \ sign(e)$$

$$\boxed{u_0 > f_0 + |\dot{r}|}$$

- The magnitude of the tracking error decays at a finite rate.
- The error is equal to 0 after a finite time.



At the convergence time, the tracking error is equal to 0 → ***Discontinuity point***.

→ Real-applications : the control switches at high frequency

→ The motion for *t* > *T* is called « **Sliding mode** ».

Consider the following system

$$\boxed{\ddot{x} = u}$$

Fictive output $s = cx + \dot{x}$

(sliding variable)

Control input $u = -M sign(s)$

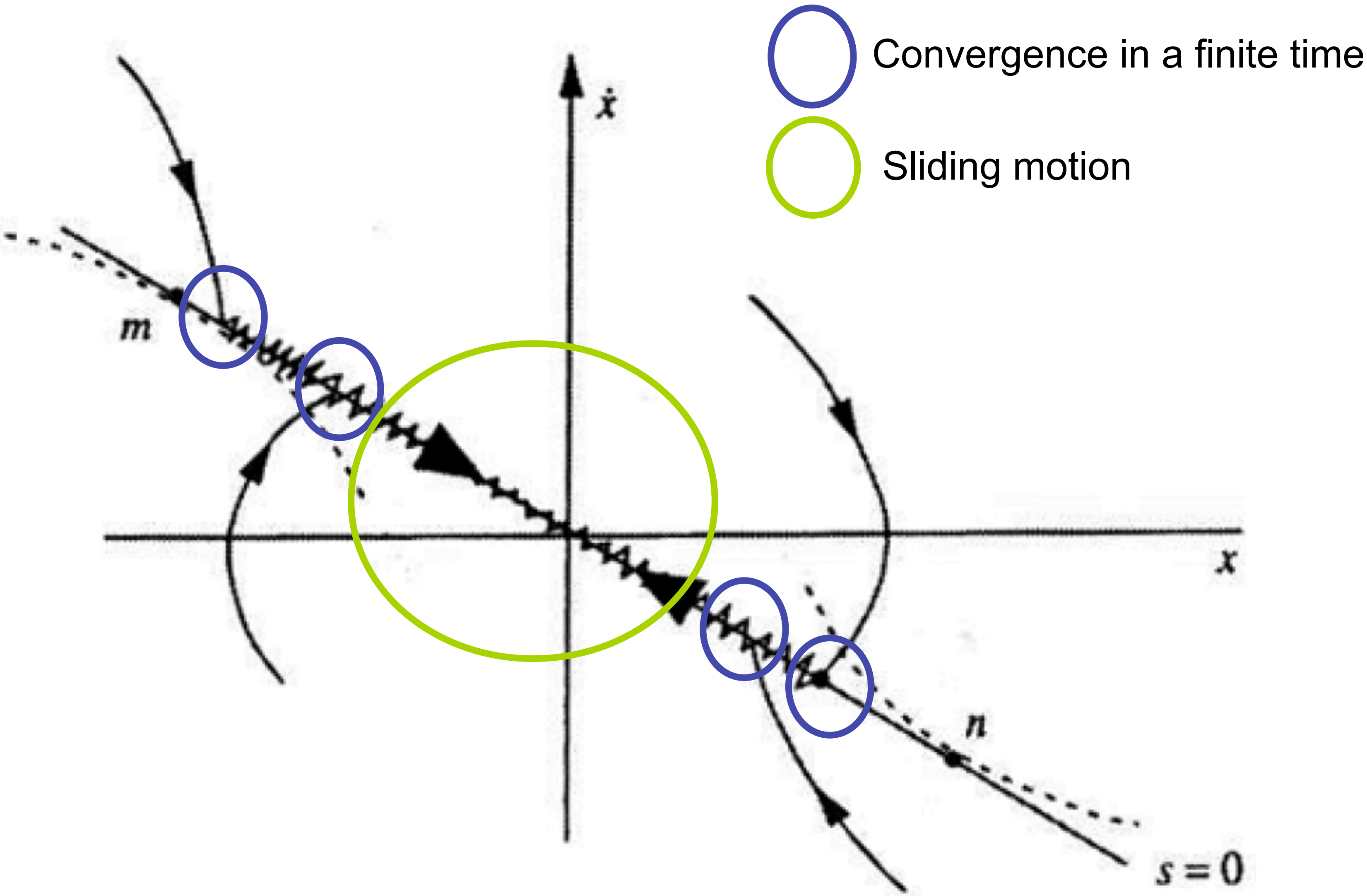
M sufficiently large

$c > 0$

In a **finite time** $t > t_1$, one has $s = 0$ (sliding surface).

Once the control objective is reached, one has

$$\dot{x} + cx = 0 \quad \Rightarrow \quad \boxed{x(t) = x(t_1)e^{-c(t-t_1)}}$$



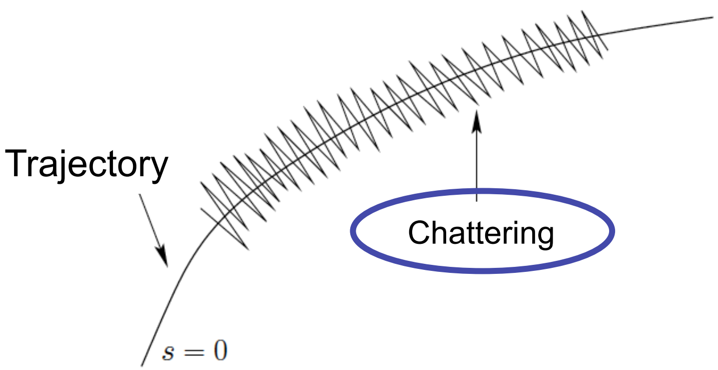
Convergence in a finite time
Sliding motion
$\dot{x}$
m
x
n
$s = 0$

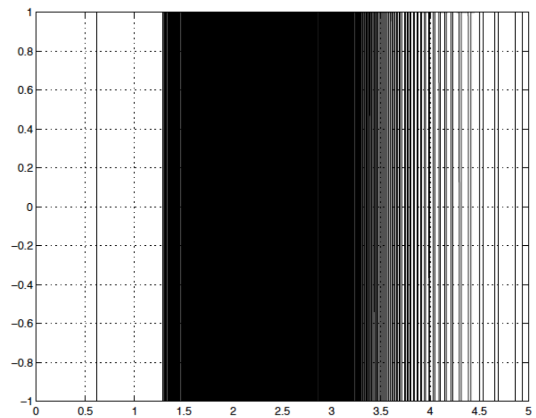
• The motion on the sliding surface depends neither on the plant parameters, nor the disturbances/uncertainties.

• This so-called « invariance » property looks promising for designing feedback control for the dynamic plants operating under uncertainty conditions.

Real implementation

- **Finite frequency** of the control commutation → it is not possible to reach *exactly* the sliding surface → notion of **real sliding mode**.

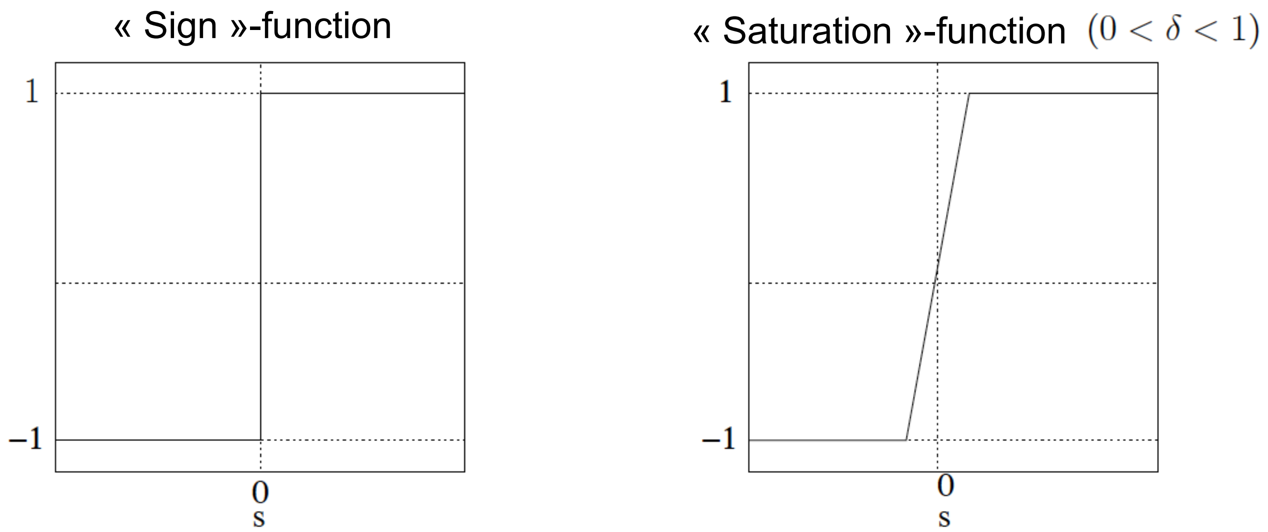




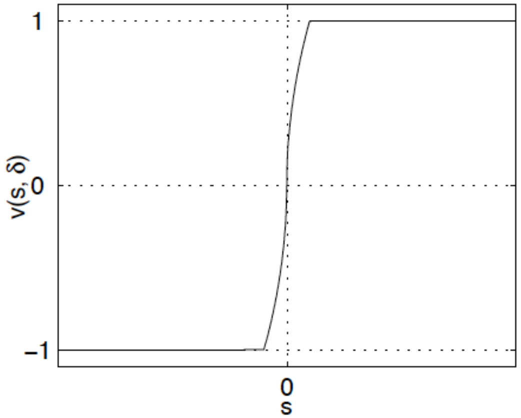
***Chattering* effect** : oscillations around the sliding surface (inducing commutation of the control input) → dangerous for the actuators.

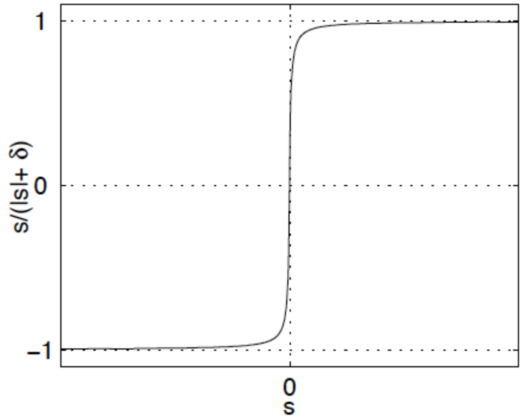
How to decrease ?

- **Modification of « sign »-function (approximation)**



$$\mathrm{sat}(s,\delta) = \begin{cases} \mathrm{sign}\,(s) & \text{si } |s| > \delta \\ \frac{s}{\delta} & \text{si } |s| \leq \delta \end{cases}$$

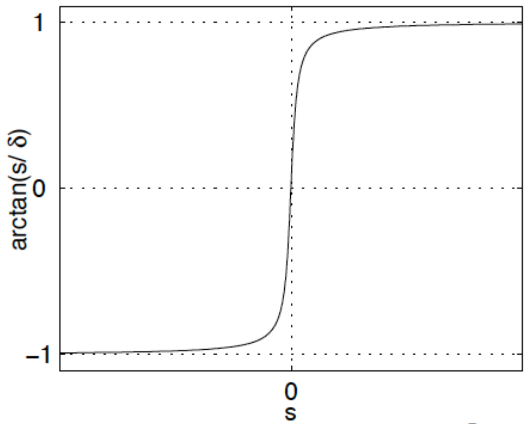


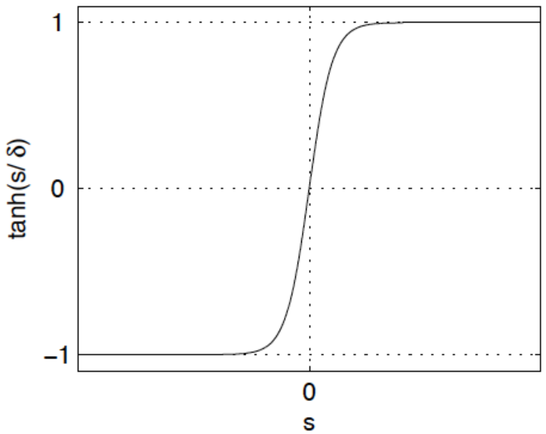


$0 < \delta < 1$

$$v(s,\delta) = \begin{cases} \text{sign } (s) & \text{si } |s| > \delta \\ (\delta/|s|)^{(q-1)} \text{ sign } (s) & \text{si } 0 < |s| \leq \delta \quad q \in [0.\ 1) \\ 0 \quad \text{si } s = 0 \end{cases}$$

$$v(s,\ \delta) = \frac{s}{|s| + \delta}$$





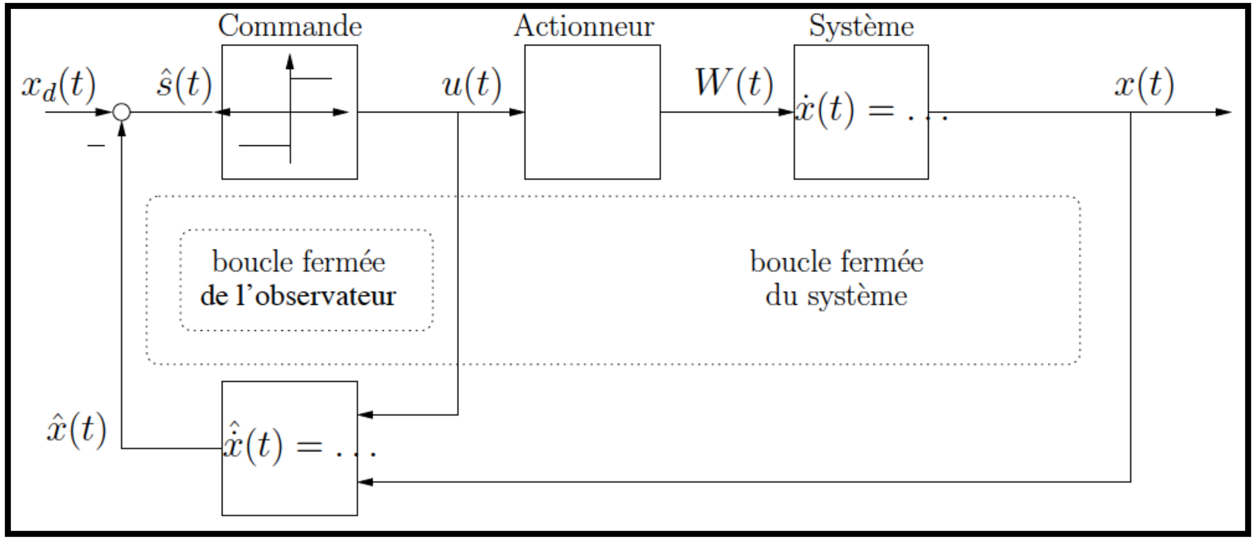
$0 < \delta < 1$

$$v(s,\ \delta) = k.\arctan(\frac{s}{\delta})$$

$$v(s,\ \delta) = \tanh(\frac{s}{\delta})$$

Drawback of these approximations. Loss of accuracy and robustness.

- **Use of observers (as a filter ?)**



Drawback. The observer « needs » to be a finite time convergence one.

$\longrightarrow$ Solutions based on sliding mode oberver.

Concept of *High Order Sliding Mode Control*.

Standart sliding mode control.

Objective

$$s = 0$$

in a finite time

The discontinuous function *sign(s)* is acting on the control input (relative degree of s = 1)

High sliding mode control.

Objective

$$s = \dot{s} = \cdots = s^{(r-1)} = 0$$

in a finite time

The discontinuous function *sign(s)* can act on the control input high order time derivative (for all relative degree of s)

Concept of *Adaptive Sliding Mode Control*.

Principle. The gain is *dynamically* adapted w.r.t. the uncertainties/perturbations magnitude.

→ The tuning of the gain is made *online*

→ The gain is reduced : reduction of *chattering*

→ No knowledge required on the uncertainties/pertubations bounds

→ Simplification of identification

→ Great interest by a practical point-of-view.

Some references (books)

Slotine *et al*., Applied nonlinear control, Prentice-Hall, 1997.
Utkin, Sliding modes in control and optimization, Springer, 1992
Utkin *et al*., Sliding mode control in electromechanical systems, Taylor&Francis, 1999.
Utkin *et al*., Sliding mode control in electromechanical systems (2nd Edition), CRC
Shtessel *et al*., Sliidng mode control and observation, Birkhauser, 2013

Desgin of a sliding mode controller.

Principles of sliding mode control

- To force the system to reach *in a finite time* a set named *surface*, which is defined from the control objectives
- Once the system has reached this surface, it evolves on it in spite of uncertainties and perturbations.

Synthesis in two steps

- Design of the *sliding variable* (connected to the surface) ***w.r.t.* to the control objectives** and the desired (static and dynamic) features of the closed-loop system.

- Design of the discontinuous control law in order
  - to constraint the system trajectories ro reach the surface
  - and, then, to be maintained on it in spite of uncertainties/perturbations, …

Consider the SISO nonlinear system

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

Control objective : the output must track a reference trajectory $y_{ref}(t)$ (sufficiently differentiable).

Assumption: the output y has a relative degree equal to r.

Definition: the **sliding variable** is defined as (coefficients λ_i chosen in order to have a stable behavior)

$$S(t,x) = \sum_{i=0}^{r-1} \lambda_i \cdot \left(y_{ref}(t) - y(x)\right)^{(i)}$$

➡ Relative degree of the sliding variable = 1.

Definition: Sliding surface $\mathcal{S} = \{x \in \mathcal{X} \mid S(x) = 0\}$

Control objective: force the system trajectories to evolve, in a finite time, on the sliding surface, in spite of uncertainties and perturbation. Once it is obtained, the dynamics of the closed-loop system is defined by the definition of the sliding variable.

Given the definition of the sliding variable, once S(x)=0, the tracking error $e = y - y_R(t)$ is converging asymptotically to 0.

Example: system with 2 state variables, and $S = x_2 + cx_1$ $(c>0)$

Definition. Consider the nonlinear system (1), and let the system be closed by some possibly dynamical discontinuous feedback. The motion on S(x)=0 is called « Sliding Mode » with respect to the sliding variable S(x).

$$\dot{S} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x}\left[f(x) + g(x)u\right]$$
$$:= S_1(x,t) + S_2(x)u$$

Define the control input as $u = u_{eq} + v_n$

$$u_{eq} = -\frac{S_1(x)}{S_2(x)} \qquad v_n = \frac{u_n}{S_2(x)}$$

u_{eq} : equivalent control – this control law linearizes the system by an input-output point-of-view

How to choose u_n in order to ensure finite time convergence ? → **Lyapunov approach**.

Lyapunov candidate function $V(t,x) = \frac{1}{2} S^2$ ➡ $\dot{V} = S\, u_n$

$$\Rightarrow \quad \dot{V} = s\dot{s} \leq -\eta|s|, \qquad \eta > 0$$

$$\Rightarrow \quad u_n = -k.Sign\ (S) \quad \Rightarrow \quad t_e \leq \frac{|s(0)|}{\eta}$$

Suppose now that the system is not well-known (uncertainties, perturbations)

$$\boxed{\begin{aligned}\dot{S} &= \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x}\left[f(x) + g(x)u\right] \\ &:= S_1(x,t) + S_2(x)u\end{aligned}}$$

$$S_1(x,t) := S_{1n}(x,t) + \Delta S_1(x,t)$$
$$S_2(x) := S_{2n}(x) + \Delta S_2(x)$$

Hypothesis. $\left|\Delta S_1(x,t)\right| << \left|S_{1n}(x,t)\right| \quad \left|\Delta S_2(x)\right| << \left|S_{2n}(x)\right|$

$$\Rightarrow \boxed{\dot{S} = S_{1n}(x,t) + \Delta S_1(x,t) + (S_{2n}(x) + \Delta S_2(x))u}$$

$$\Rightarrow u = -\frac{S_{1n}}{S_{2n}} - \frac{K\,Sign\ (S)}{S_{2n}} \quad \Rightarrow \boxed{\dot{S} = \Delta_1 - (\Delta_2 + 1)\,K\ sign(S)}$$

$$\left|\Delta_1\right| = \left|\Delta S_1(x) - \Delta S_2(x) S_{1n}(x) / S_{2n}(x)\right|$$
$$\left|\Delta_2\right| = \left|\Delta S_2(x) / S_{2n}(x)\right|$$

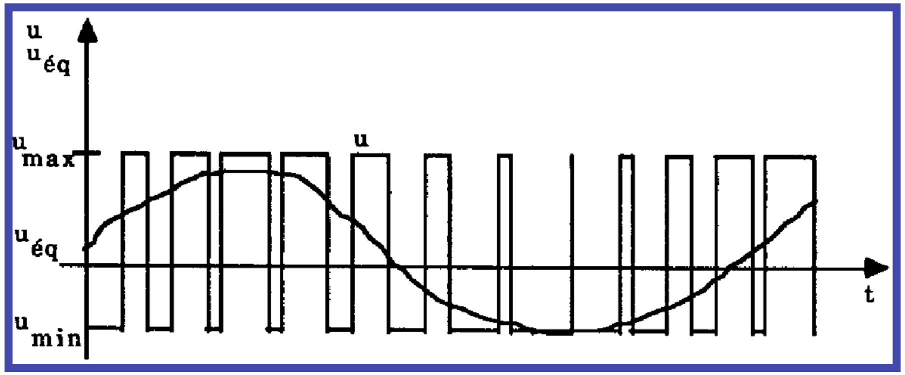
Tuning of K: large enough in order to ensure the establishment of a sliding mode.

Problem : dynamical systems with a discontinuous second member

$\rightarrow$ Existence / unicity of the solution ? In case of sliding motion ?

<u>Fillipov theory</u>

Physically, the equivalent control is the average value of the successive commutations of the control



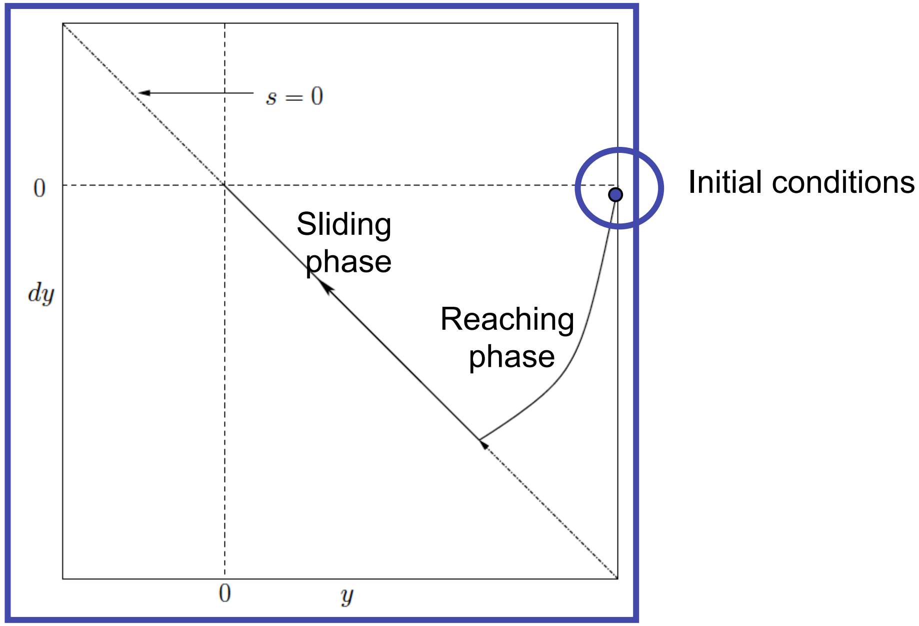
A first « ideal » example (no uncertainty)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \\ y &= x_1 \end{aligned}$$

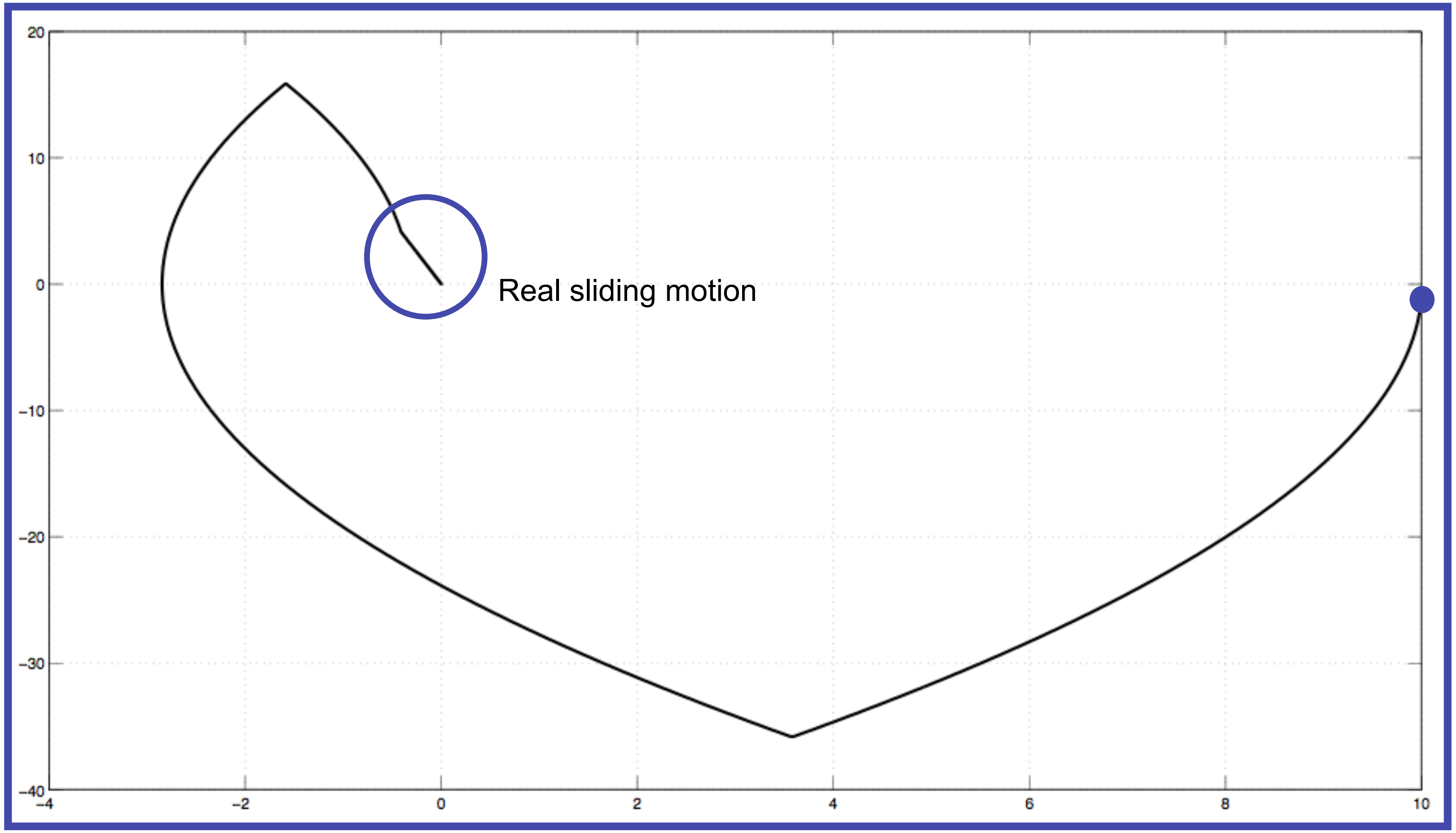
Objective : force y and $\dot{y}$ to zero.

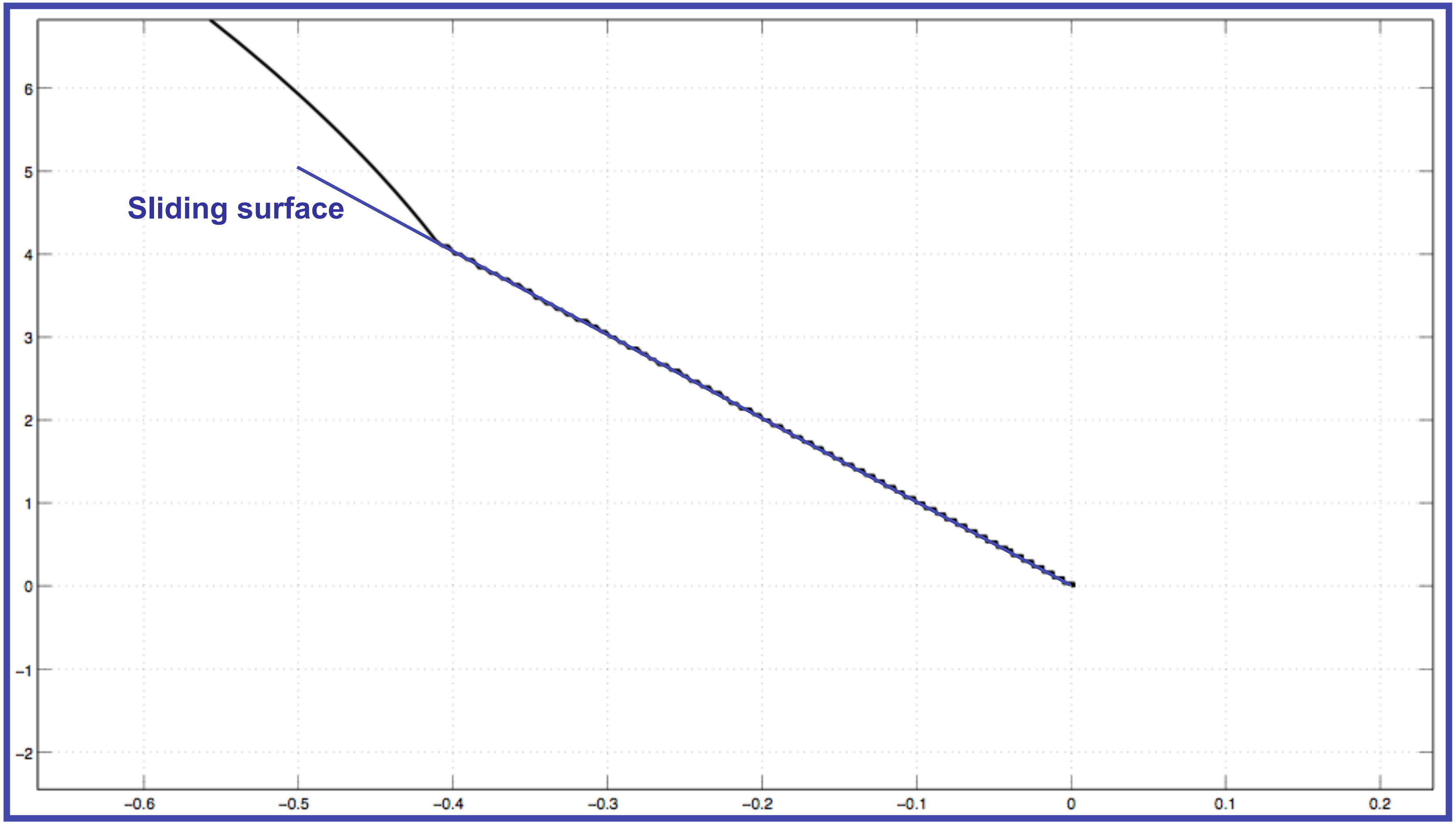
➡ Define a control input based on the sliding mode approach

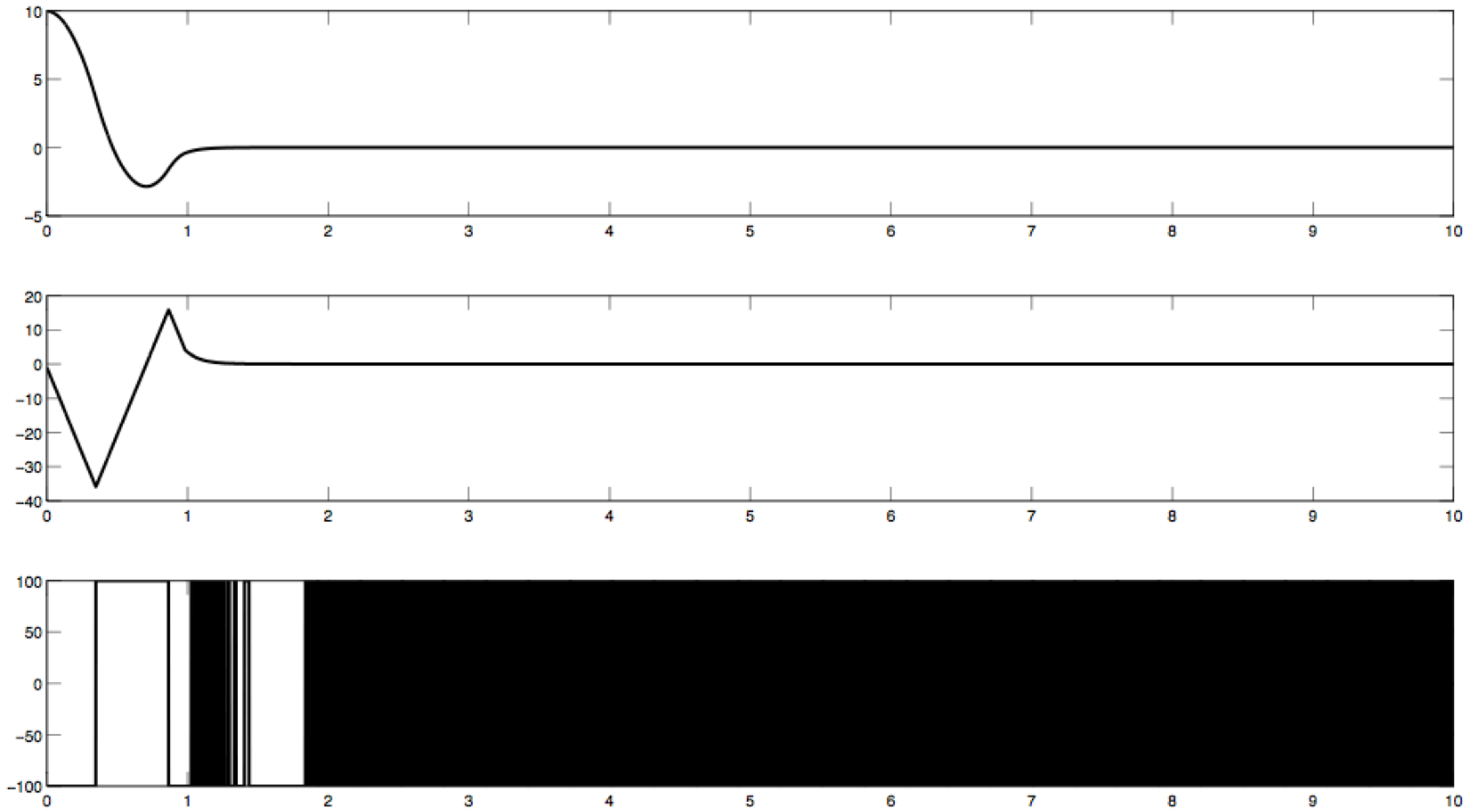
Suppose that ***u = -K sign(s)*** : what is the condition on gain K ? What is the value of the equivalent control ? (viewed as the mean value once the sliding mode is established !)



Real behaviour : due to sampling period, the system is not reaching exactly the sliding surface. However, it will concergence in a neighborhood of the origin, in a finite time.




Sliding surface
6
5
4
3
2
1
0
−1
−2
−0.6
−0.5
−0.4
−0.3
−0.2
−0.1
0
0.1
0.2


10
5
0
−5
0
1
2
3
4
5
6
7
8
9
10
20
10
0
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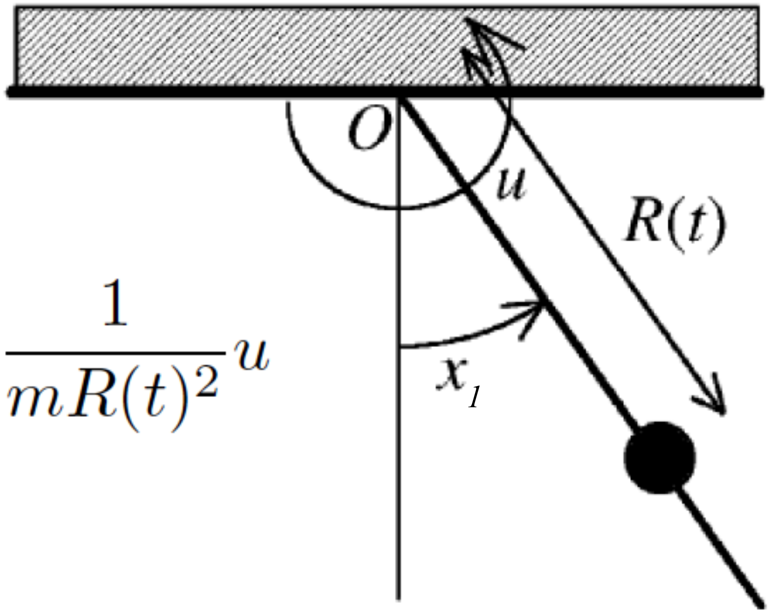
−20
−30
−40
100
50
0
−50
−100

Control of a variable-length pendulum



$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2\frac{\dot{R}(t)}{R(t)}x_2 - \frac{g}{R(t)}\sin(x_1) + \frac{1}{mR(t)^2}u$$

Nominal value of R(t) $= 1\ m$.

Nominal value of dR/dt(t) $= 1\ m/s$.

$$\boxed{R(t) \;=\; 0.8 + 0.1\sin(8t) + 0.3\cos(4t)}$$

1 - Determine the sliding variable in order to control the angle and its velocity.
2 - By using no equivalent term in the control law, what is the minimal value of the control gain *K* ensuring the establishment of a sliding motion ?

MIMO case : commande d'une machine asynchrone

$$\dot{x} = f(x) + gu + \boxed{\xi} \qquad x = \left[\Omega, \quad \phi_{r\alpha}, \quad \phi_{r\beta}, \quad i_{s\alpha}, \quad i_{s\beta}\right]^T$$

Couple de charge non mesuré

$$u = \left[u_{s\alpha}, \quad u_{s\beta}\right]^T$$

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \\ f_5(x) \end{bmatrix} = \begin{bmatrix} (pM_{sr}/JL_r)(\phi_{r\alpha} i_{s\beta} - \phi_{r\beta} i_{s\alpha}) - (f_v/J)\Omega \\ -(R_r/L_r)\phi_{r\alpha} - p\Omega\phi_{r\beta} + (R_r/L_r)M_{sr} i_{s\alpha} \\ +p\Omega\phi_{r\alpha} - (R_r/L_r)\phi_{r\beta} + (R_r/L_r)M_{sr} i_{s\beta} \\ (M_{sr}/\sigma L_s L_r)\left((R_r/L_r)\phi_{r\alpha} - p\Omega\phi_{r\beta}\right) - \gamma i_{s\alpha} \\ (M_{sr}/\sigma L_s L_r)\left((R_r/L_r)\phi_{r\beta} - p\Omega\phi_{r\alpha}\right) - \gamma i_{s\beta} \end{bmatrix} \qquad g = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1/\sigma L_s & 0 \\ 0 & 1/\sigma L_s \end{bmatrix}, \quad \xi = \begin{bmatrix} -T_l/J \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

1 - Design a controller of Ω

2 - Design a controller of $\phi^2 = \phi_{r\alpha}^2 + \phi_{r\beta}^2$

$$\sigma := 1 - \frac{M_{sr}^2}{L_s L_r}, \quad \gamma := \frac{L_r^2 R_s + M_{sr}^2 R_r}{\sigma L_s L_r^2}$$

$$\boxed{\begin{aligned} S_1 &= -\dot{y}_1 - l_1(y_{1,ref} - y_1) = -\dot{\Omega} - l_1\left(\Omega_{ref} - \Omega\right) \\ S_2 &= (\dot{y}_{2,ref} - \dot{y}_2) - l_2(y_{2,ref} - y_2) = \left(\dot{\phi}^2{}_{ref} - \dot{\phi}^2\right) - l_2\left(\phi^2{}_{ref} - \phi^2\right) \end{aligned}}$$

$$\dot{S}_1 = -\ddot{y}_1 - l_1(\dot{\Omega}_{ref} - \dot{\Omega})$$

$$\Rightarrow$$

$$\begin{aligned} \dot{S}_1(x,u,t) &= \ddot{\Omega}_{ref} - \dot{f}_1(x,u) - l_1(\dot{\Omega}_{ref} - f_1(x)) \\ &:= a_1(x,t) + b_{11}(x)u_{s\alpha} + b_{12}(x)u_{s\beta} \end{aligned}$$

$$\begin{aligned} \dot{S}_2(x,u,t) = \; & \ddot{\phi}^2{}_{ref} - 2(\phi_{r\alpha}\dot{f}_2(x,u) + (f_2(x))^2 + \phi_{r\beta}\dot{f}_3(x,u) + (f_3(x))^2) \\ & - l_2(\dot{\phi}^2{}_{ref} - 2(\phi_{r\alpha}f_2(x) + \phi_{r\beta}f_3(x))) \end{aligned}$$

$$\dot{S}_2(x,u,t) := a_2(x,t) + b_{21}(x)u_{s\alpha} + b_{22}(x)u_{s\beta}$$

$$\boxed{u = \begin{bmatrix} u_{s\alpha} \\ u_{s\beta} \end{bmatrix} = \begin{bmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{bmatrix}^{-1} \left[-\begin{bmatrix} a_1(x,t) \\ a_2(x,t) \end{bmatrix} + \begin{bmatrix} k_1 * signe(S_1) \\ k_2 * signe(S_2) \end{bmatrix} \right]}$$