Some preliminaries

Nonlinear system under interest

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x) \end{cases} \quad \text{or} \quad \begin{array}{l} \dot{x} = f(x) + g(x)u \\ x \in \Re^n, u \in \Re^m, y \in \Re^p \end{array}$$

An intuitive way to analyze/control/estimate such nonlinear systems can be made through the linearization of the previous system around a point

$$\dot{z} = \left(\frac{\partial f}{\partial x}\right)_{x_0, u_0} z + \left(\frac{\partial f}{\partial u}\right)_{x_0, u_0} v$$

$$y = \left(\frac{\partial h}{\partial x}\right)_{x_0} x$$

<u>Problem</u> : the linearization can modify the structural features of the system.

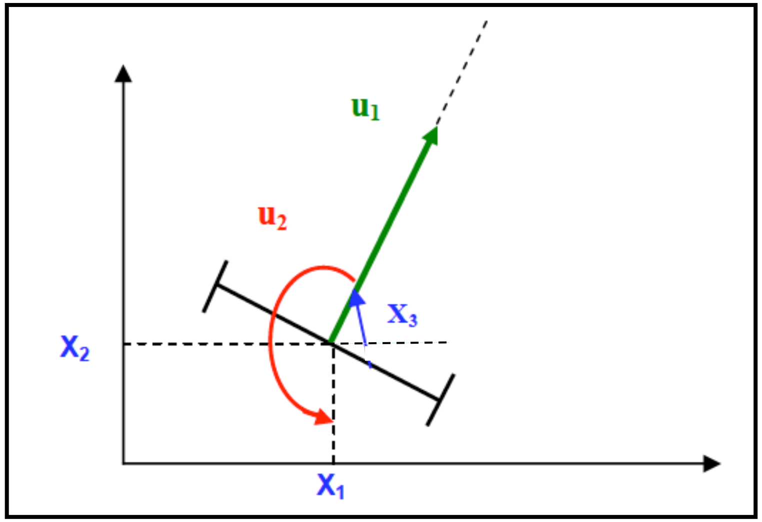
Example

$$\boxed{\begin{aligned}\dot{x}_1 &= x_2{}^3\\ \dot{x}_2 &= u\end{aligned}}$$

➡ Controllable system ➡ Linearization around $(x_0,\ u_0) = ([0,\ 0]^T, 0)$

➡ $$\begin{aligned}\dot{Z}_1 &= 0\\ \dot{Z}_2 &= v\end{aligned}$$ ➡ Loss of controllability

Mobile robot



$$\dot{x}1 = \cos x_3.u_1$$

$$\dot{x}2 = \sin x_3.u_1$$

$$\dot{x}3 = u_2$$

u_1 : longitudinal velocity
u_2 : angular velocity
(x_1, x_2): longitudinal coordinates

Let us define $(X_0, u_0) = \left([0,0,0]^T, [0,0]^T\right)$ ⇒

$$\dot{z}_1 = u_1$$
$$\dot{z}_2 = 0 * u_1 = 0$$
$$\dot{z}_3 = u_2$$

The system is not controllable ⇐

Conclusion : there is a real interest to study/analyze the nonlinear system under their nonlinear representation.

Analytic function. A function $f : \Re \to \Re$ is analytic if and only if it is equal to its Taylor series in some neighborhood of every point.

Features.
The sums, products, … of analytic functions are analytic.
Any analytic function is smooth, that is, infinitely differentiable.
Analytic functions admit isolated zeros.
Some examples : trigonometric functions, polynomial functions.

Property. If $f : \Re \to \Re$ is an analytic function, then

- either $f \equiv 0$
- or its zeros are isolated.

Corollary.

- If f_1 is analytic and f_2 is analytic ($f_2 \neq 0$), then f_1/f_2 is analytic.
- If $f_1.f_2 = 0$, then $f_1 = 0$ or $f_2 = 0$.

Meromorphic functions. $f : \Re \to \Re$ is said meromorphic if $\exists\, f_1$ analytic and $\exists \quad f_2 \neq 0$ analytic such $f = \dfrac{f_1}{f_2}$

Class of nonlinear systems under consideration.

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

Meromorphic functions

State	$x \in \Re^n$
Input	$u \in \Re^m$
Output	$y \in \Re^p$

A particular case : linear systems $\boxed{\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}}$

$$f(x) = Ax$$
$$g(x) = B$$
$$h(x) = Cx$$

➡ All the results givenin the sequel can be applied to linear systems.

Consider the set of variables

$$\left\{x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m, \dot{u}_1, \dot{u}_2, \ldots, \dot{u}_m, \ldots, u_1^{(k)}, u_2^{(k)}, \ldots, u_m^{(k)}\right\}$$

Let K denote the field of meromorphic functions of

$$\left\{x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m, \dot{u}_1, \dot{u}_2, \ldots, \dot{u}_m, \ldots, u_1^{(k)}, u_2^{(k)}, \ldots, u_m^{(k)}\right\}$$

Example

$$\boxed{f(x) = \frac{\left(x_2.\sin(x_2)\right).u^2 + \ddot{u}}{\dot{u} + u.\ddot{u}.tg(x)}}$$

$$\begin{array}{c} \left(x_2.\sin(x_2)\right) \in K \\ tg(x) \in K \end{array} \Rightarrow f(x) \in K$$

Space. The space $\mathcal{E}$ on the field K is defined such that the unit vectors of this space on K read as

$$dx_1, dx_2, \ldots, dx_n, du_1, \ldots, du_m, d\dot{u}_1, \ldots, d\dot{u}_m^{(n-1)}$$

Interest of this notation. $y_1 = h_1(x)$

$$\boxed{dy_1 = \frac{\partial h_1}{\partial x_1} dx_1 + \frac{\partial h_1}{\partial x_2} dx_2 + \ldots + \frac{\partial h_1}{\partial x_n} dx_n = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \ldots & \frac{\partial h_1}{\partial x_n} \end{bmatrix} . \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}}$$

$$\dot{y}_1 = \frac{dy_1}{dt} = \frac{\partial h_1}{\partial x} \cdot \frac{dx}{dt} = \frac{\partial h_1}{\partial x} \dot{x} = \frac{\partial h_1}{\partial x} \left(f(x) + g(x) u \right) = h_1^{\,1}(x, u)$$

$$\boxed{d\dot{y}_1 = \underbrace{\sum_{i=1}^{n} \frac{\partial h_1^{\,1}}{\partial x_i}}_{\substack{\text{Fonctions}(x,u) \\ \in K}} dx_i + \underbrace{\sum_{j=1}^{m} \frac{\partial h_1^{\,1}}{\partial u_j}}_{\substack{\text{Fonctions}(x) \\ \in K}} du_j \in \mathcal{E}}$$

Remark: contains the differential of all function of K.

An important problem in nonlinear control systems (for structure analysis).

$$dx_1 + x_3.dx_2 = (1, x_3, 0)\begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

Does a function $\varphi \in K$ exist such that

$$d\varphi = dx_1 + X_3.dx_2 \ ?$$

1-form. Given a function $\varphi \in K$ (**called 0-form**). $\omega \in \varepsilon$ is a 1-form and reads as

$$\omega = \sum_{i=1}^{n} \underbrace{\alpha_i(.)}_{\in K} dx_i + \sum_{j=0}^{n-1} \sum_{k=1}^{m} \underbrace{\beta_{kj}(.)}_{\in K} du_k^{(j)}$$

Exact 1-form. Consider the 1-form $\omega \in \varepsilon$: it is exact if

$$\exists \ \varphi \in K \text{ such that } \omega = d\varphi$$

Some examples.

$$\omega_1 = dx_1 + x_2 dx_3 + X_3 dx_2 \qquad \omega_2 = x_1 dx_2 - x_2 dx_1$$

2-form. Ω is a 2-form $\Leftrightarrow$ $\Omega = \sum_{i,j} \alpha_{ij} e_{ij}$

Exterior product

$$\alpha_{ij} \in K \qquad e_{ij} = dx_i \wedge dx_j$$

Methodology. The function $\varphi \in K$ is a 0-form

- One differentiates $d\varphi = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i} dx_i + \sum_{j,k}^{m} \frac{\partial \varphi}{\partial u_j^{(k)}} du_j^{(k)}$

- One differentiates a second time …

Properties.

$$1)\quad dx_i \wedge dx_j = -dx_j \wedge dx_i$$

$$2)\quad dx_i \wedge dx_i = 0$$

$$3)\quad d^2 \equiv 0, \text{i.e.}: d(dx) \equiv 0$$

$$\boxed{\begin{aligned} d\omega &= \left(\sum_{i=1}^{n} \frac{\partial \alpha_1}{\partial x_i} dx_i\right) \wedge dx_1 + \ldots + \left(\sum_{i=1}^{n} \frac{\partial \alpha_n}{\partial x_i} dx_i\right) \wedge dx_n \\ &= \sum_{j<i} \sum_{i=1}^{n} \left(\frac{\partial \alpha_j}{\partial x_i} - \frac{\partial \alpha_i}{\partial x_j}\right) dx_i \wedge dx_j \end{aligned}}$$

Exercice. Compute the 1-form and 2-form of $\varphi = x_1 x_2$

Problem of integration. Given a 1-form $\omega = \sum_{i=1}^{n} \alpha_i dx_i$

does it exist $\varphi \in K$ such that $\omega = d\varphi$? (In fact, is ω exact ?)

One needs to solve

$$\alpha_1 = \frac{\partial \varphi}{\partial x_1} \quad \alpha_2 = \frac{\partial \varphi}{\partial x_2} \quad \cdots \quad \alpha_n = \frac{\partial \varphi}{\partial x_n}$$

Poincaré's lemma. Given a 1-form $\omega \in \varepsilon$, then ω is an exact 1-form

$$\Leftrightarrow \quad \exists \varphi \text{ such that } \omega = d\varphi \quad \Leftrightarrow \quad d\omega = 0$$

$$d\omega = \sum_{i=1}^{n} d\alpha_i \wedge dx_i$$

Example. $\omega = dx_1 + x_2 dx_3 + x_3 dx_2$

Proof. Consider the 1-form $\omega = \sum_{i=1}^{n} \alpha_i dx_i$

$$d\omega = \left(\frac{\partial \alpha_1}{\partial x_1} dx_1 + \ldots + \frac{\partial \alpha_1}{\partial x_n} dx_n\right)_n dx_1 + \left(\frac{\partial \alpha_2}{\partial x_1} dx_1 + \ldots + \frac{\partial \alpha_2}{\partial x_n} dx_n\right)_n dx_2$$

$$+ \ldots + \left(\frac{\partial \alpha_n}{\partial x_1} dx_1 + \ldots + \frac{\partial \alpha_n}{\partial x_n} dx_n\right)_n$$

It yields $d\omega = \sum_{i,j} \frac{\partial \alpha_i}{\partial x_j} dx_j \wedge dx_i$

From $dx_i \wedge dx_j = -dx_j \wedge dx_i$ ➡ $$\boxed{d\omega = \sum_{i,j} \left(\frac{\partial \alpha_i}{\partial x_j} - \frac{\partial \alpha_j}{\partial x_i} \right) dx_j \wedge dx_i}$$

➡ $d\omega = 0$

Exercice. $\omega = x_2 dx_1 - x_1 dx_2$

$$\omega = \frac{x_2}{x_1^2} dx_1 - \frac{1}{x_1} dx_2$$

$$\omega = \frac{x_2}{x_1^2 + x_2^2} dX_1 - \frac{x_1}{x_1^2 + x_2^2} dx_2$$

➡ Integration ?
Computation ?

Given a 1-form $\omega \in \varepsilon$, does it exist $\varphi \in K$ and $\lambda \in K$ such that

$$\omega = \lambda \, \mathrm{d} \varphi \; ?$$

Frobenius' theorem. (First version) Given a 1-form $\omega \in \varepsilon$. $\exists \varphi \in \mathrm{K}$ and $\exists \lambda \in \mathrm{K}$ such that $\omega = \lambda \mathrm{d}\varphi \Leftrightarrow \mathrm{d}\omega \wedge \omega = 0$

Exercice. $\omega = x_2 dx_1 - x_1 dx_2$

$$\omega = dx_1 + x_1 dx_2 + x_2 dx_3$$

$$\omega = x_3 dx_1 + dx_3$$