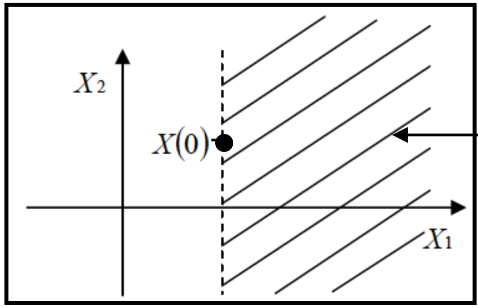
Controllability - Accessibility

Some examples.

EX1. Consider the nonlinear system

$$\dot{x}_1 = x_2^2$$
$$\dot{x}_2 = u$$



Area reachable from the initial condition.

➡ Problem of controllability (it is not possible to reach a point located in the neighborhood of the initial point).

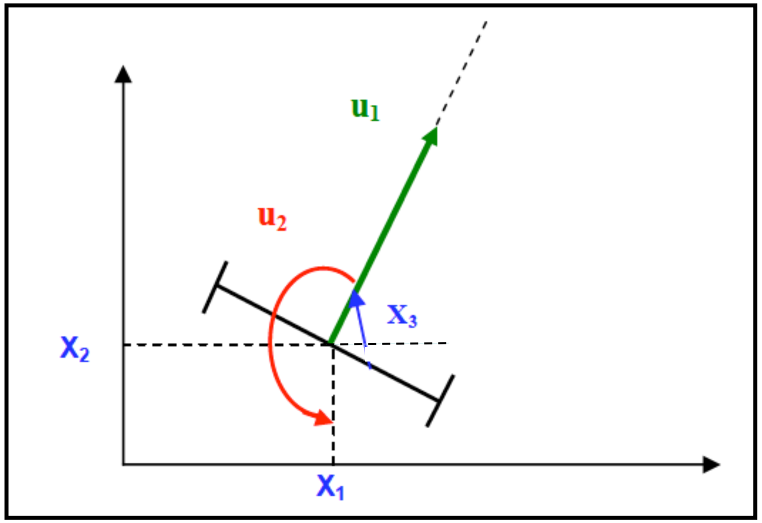
EX2. Consider the nonlinear system

$$\dot{X}_1 = \cos(X_2)$$
$$\dot{X}_2 = u$$

➡ For all value of u, dynamics of x_1 can *only* be between -1 and 1 : x_1 is *constrained.*

➡ Problem of controllability

EX3. Mobile robot.



$$\dot{x}_1 = \cos x_3 . u_1$$

$$\dot{x}_2 = \sin x_3 . u_1$$

$$\dot{x}_3 = u_2$$

Singularities for $X3 = 0$

$X3 = \pi/2$

The system is ***generically controllable*** : controllable except at the singularities.

What is …

Controllability : possibility of steering a system from a point to an other point

- **Linear systems** : controllability = structural property [any system can be divided into a controllable part and an autonomous one.]

- **Nonlinear systems** : structural property : **accessibility**.

Definition. For a nonlinear system, a state x_1 is **reachable** from an initial state x_0 if there exists a finite time *T* and control input $u(t)$ such that $x(T) = x_1$.

Definition. A system is said to be **controllable** at x_0 if there exists a neighborhood V of x_0 such that any state x_1 in V is reachable from x_0.

Definition. A system is said to be **accessible** at x_0 if the set of reachable points from x_0 contains an open subset of $\Re^n$.

Some recalls (of linear theory).

$$\dot{x} = Ax + Bu$$

Definition. The ***relative degree*** of the output y=C.x is equal to the degree of denominator of the transfer function minus the degree of its numerator.

<u>Other definition</u>. The relative degree is the minimal differentiation order k of the output such that

$$\frac{\partial y^{(k)}}{\partial u} \neq 0$$

Controllability matrix. The controllability matrix reads as

$$\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

Theorem. Controllability criteria

$$\mathrm{Rang}\left(\begin{bmatrix} B & AB & .. & A^{n-1}B \end{bmatrix}\right) = n$$

$\Leftrightarrow$ No output with a relative degree $\geq$ $n+1$ $\quad \left(\dfrac{\partial y^{(n)}}{\partial u} \neq 0 \right)$

$\Leftrightarrow$ No output with an infinite relative degree

$\Leftrightarrow$ There exists no output as a solution of an equation independent of u.

To summarize, if the system is not controllable, then there exists at least $w \neq 0$ such that an output $y = w.x$ has a relative degree greater or equal to *n+1*.

- The relative degree of y is said infinite
- The output y is solution of a linear and autonomous (not depending on u) differential equation

Consider the linear system $\dot{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$ ➡ Rank criteria not fulfilled

The output $y = -2x_1 + x_2$ admits an infinite relative degree. $\dot{y} = 0$

$(\dot{y} = 0, \ldots, y^{(k)} = 0)$

Nonlinear systems.

Definition. The ***relative degree*** of the function $\varphi(x) \in K$ is defined as

$$r = \min\left\{ k \in N \left| \frac{\partial \varphi^{(k)}}{\partial u} \neq 0 \right.\right\}$$

Example. Consider a mechanical system whose state = (position,velocity) and input = (force ou torque)

➡ Relative degree = 2.

Definition. Relative degree of a 1-form $\omega \in \mathcal{E}$.

Consider $\omega := \sum_{i=1}^{n} \alpha_i dx_i$ with $\forall i,\ \alpha_i \in K$

The relative degree equals $r = \min\left\{k \in N \,\middle|\, \omega^{(k)} \notin \mathrm{span}\{dx\}\right\}$ with the time derivative of ω,

$$\dot{\omega} = \sum_{i=1}^{n} \dot{\alpha}_i dx_i + \alpha_i d\dot{x}_i$$

Proposition. $\varphi(x) \in K$ is an autonomous element

$\Leftrightarrow$ The relative degree of φ is infinite : $\forall k \in N, \dfrac{\partial \varphi^{(k)}}{\partial u} \equiv 0$

Definition. Controllability of a nonlinear system.

Given the nonlinear system $\dot{x} = f(x) + g(x)u$. This system is said « controllable » if $\forall x_0$ (the initial state) and $\forall x_1$ (forall state), $\exists\, u(t)$ and $T < \infty$ such that

$$x\left(x_0, u(t), T\right) = x_1$$

No way to characterize the controllability of nonlinear systems

Definition. Accessibility of a nonlinear system.

The nonlinear system $\dot{x} = f(x) + g(x)u$ is said « accessible » **if there exists no autonomous element.**

Property. For the linear systems *only*.

Controllable $\Leftrightarrow$ Accessible

Example. Consider the nonlinear system

$$\dot{x}_1 = x_2^{\,2}$$
$$\dot{x}_2 = u$$

This system is not controllable (x_1 cannot decrease), but is accessible $\forall \varphi(x_1, x_2),\ dr(\varphi) < \infty$

Property. For nonlinear systems, a nonlinear system is accessible

$$\Leftrightarrow\ \forall\ \varphi(x) \neq cte\ ,\ dr(\varphi) < \infty$$

Algorithm allowing to evaluate the accessibility.

- Caracterization of the 1-forms $\omega \in \mathcal{E}$ such that the relative degree of ω is ≥ 2.

  - $H_0 := \mathrm{span}\{dx, du\} \quad \Rightarrow \quad \omega \in H_0 \Leftrightarrow$ d.r. ≥ 0
  - $H_1 := \mathrm{span}\{dx\} \subset H_0 \Rightarrow \quad \omega \in H_1 \Leftrightarrow$ d.r. ≥ 1
  - H_2 is the space of the 1-forms ω $(\in H_1)$ such that d.r.$(\omega) \geq 2$. $\Rightarrow H_2 \subset H_1$

- Computation of H_2. Consider the 1-form $\omega \in H_1$

$$\boxed{\omega = \sum_{i=1}^{n} \alpha_i dx_i \quad \Rightarrow \quad \dot{\omega} = \sum_{i=1}^{n} \left(\dot{\alpha}_i dx_i + \alpha_i d\dot{x}_i\right)}$$

$\Rightarrow$ Compute α_i s.t $\dot{\omega} \in H_1$

$$\Rightarrow \dot{\omega} = \sum_{i=1}^{n} \alpha_i d\dot{x}_i + \sum_{i=1}^{n} \dot{\alpha}_i dx_i$$

$$= [\alpha_1, \ldots, \alpha_n].d\left(f(x) + g(x)u\right) + ("\in H_1")$$

Then $\dot{\omega} \in H_1 \quad \Leftrightarrow \quad [\alpha_1, \dots, \alpha_n].[g(x)]du = 0$

$$\Leftrightarrow \quad [\alpha_1, \dots, \alpha_n].[g(x)] = 0$$

Result. $\boxed{H_2 = g(x)^{\perp} = \operatorname{span}\{\omega \in H_1 \mid w.g \equiv 0\}}$

Definition. Chain of subspaces of $\mathcal{E}$.

$$H_0 \supset H_1 \supset H_2 \supset \dots \supset H_k \supset H_{k+1} \supset \dots H_\infty$$

$$H_0 = \operatorname{span}\{dx, du\}$$

$$H_1 = \operatorname{span}\{dx\}$$

$$H_2 = g^{\perp}$$

$$\dots$$

$H_k =$ Subspaces of all the linear 1-forms with a d.r.≥ k.

$$H_{k+1} = \operatorname{span}\{\omega \in H_k \mid \dot{\omega} \in H_k\}$$

$\exists k^*$ limit s.t. $H_{k^*-2} = H_{k^*-1} := H_\infty$

Theorem. Accessibility of a nonlinear system.

A nonlinear system satisfies the accessibility condition if and only if $H_\infty = 0$

i.e. there is no autonomous element.

Example. Consider the nonlinear system $\dot{x} = \begin{bmatrix} -x_2{}^3 \\ u \end{bmatrix}$. Is it accessible ?

Example. Consider the nonlinear system $\dot{x} = \begin{bmatrix} x_2(1-u) \\ x3 \\ x_2 u \end{bmatrix}$. Is it accessible ?

Linear. Given the linear system $\dot{x} = Ax + Bu$

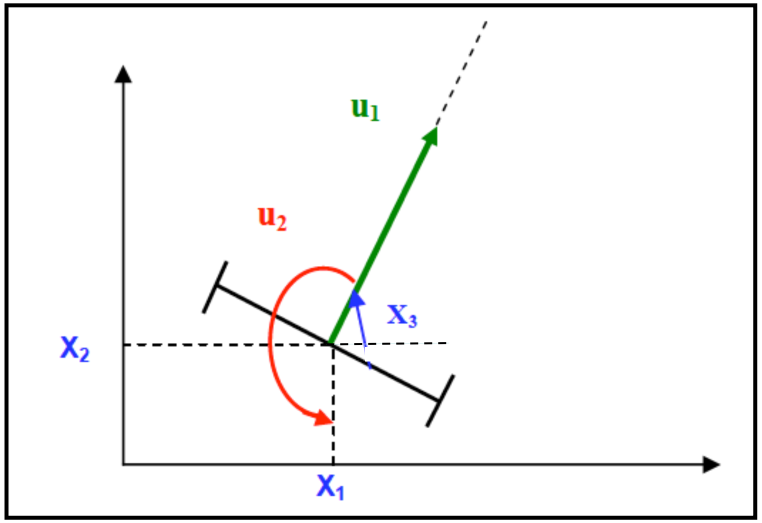
$$H_2 = B^{\perp}$$

$$H_3 = [B\ AB]^{\perp}$$

$$\vdots$$

$$H_k = \left[B \cdots A^{k-2}B\right]^{\perp}$$

Example. Mobile robot.



$$\dot{x} = \begin{bmatrix} \cos(x_3) & 0 \\ \sin x_3 & 0 \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad f(x) \equiv 0$$