Input-output linearization by state feedback and static decoupling

Example.
(monovariable)

$$\begin{cases} \dot{x} = -x^2 + u \\ y = x \end{cases}$$

By stating $u = x^2 + v$, the closed-loop system reads as

$$\boxed{\dot{x} = v}$$

- The system is transformed into a linear one controlled by a « new » control input v.
- No approximation ; linear control theory can then be used.
- A solution $v = -\frac{1}{T}[x - y_c]$

Example.
(multivariable)

$$\begin{cases} \dot{x}_1 = -x_2 + u_1 \\ \dot{x}_2 = -2x_2 + u_2 \\ y_1 = x_1 \\ y_2 = x_2 \end{cases}$$

There is a coupling :

- u_1 is acting on x_1, and then on y_1
- u_2 is acting on x_2, and then on y_2 and y_1

Does it exist a control law which allows a decoupling ?

A **static** solution : $u = F(x) + G(x)v$

Problem statement: in a general framework (multivariable), how is it possible to compute a pre-feedback (static) which will give **an exact input-output linearization** of the system ?

Two ways for the linearization

- Invertible static pre-feedback (G being invertible square matrix)

$$u = F(x) + G(x)v$$

- State coordinates transformation (locally invertible) $z = \varphi(x)$

Example.

$$\dot{x}_1 = x_3$$

$$\dot{x}_2 = \sin(x_3)$$

$$\dot{x}_3 = u$$

$$y = x_2$$

Determination of a linearizing control law

⬇

Sructural analysis (relative degree)

Problem statement. Given the nonlinear system,

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

does it exist

- a state feedback $u = F(x) + G(x)v$
- a bijective state coordinates transformation $z = \varphi(x)$

such that, after feedback and state transformation, the nonlinear system reads as

$$\boxed{\dot{z}_1 = A.z_1 + B.v} \quad \text{Linear dynamics}$$
$$\boxed{\dot{z}_2 = f_2(z_1, z_2) + g_2(z_1, z_2).v} \quad \text{Internal dynamics}$$
$$y = C.z_1$$

with $z = (z_1^T, z_2^T)^T,\ v = (v_1^T, v_2^T)^T$

Monovariable case (r = relative degree of the output y).

$$A = \begin{bmatrix} 0 & 1 & & 0 \\ & & & \\ & & & 1 \\ 0 & & & 0 \end{bmatrix}_{r*r}, \quad B = \begin{bmatrix} 0 \\ \\ 0 \\ 1 \end{bmatrix}_{r*1}$$

Multivariable case (r_i = relative degree of the output y_i).

$$A = \begin{bmatrix} A_{11} & & 0 \\ & A_{12} & \\ 0 & & \end{bmatrix} \qquad A_{11} = \begin{bmatrix} 0 & 1 & & 0 \\ & & & \\ & & & 1 \\ 0 & & & 0 \end{bmatrix}_{r_1*r_1}, .$$

$$B = \begin{bmatrix} B_1 & B_2 & & \end{bmatrix} \qquad B_1 = \begin{bmatrix} 0 \\ \\ 0 \\ 1 \end{bmatrix}_{r_1*1}$$

After state coordinates transformation, and static state feedback, the input-output relations read as

$$\boxed{y_1^{(r_1)} = v_1, \dots, y_p^{(r_p)} = v_p}$$

➡ *p* chains of integrators (length: r_i)

$$\boxed{\dot{z}_1 = A.z_1 + B.v}$$

z_2 : zero dynamics

$$\boxed{\dot{z}_2 = f_2(z_1, z_2) + g_2(z_1, z_2).v}$$

$$y = C.z_1$$

➡ The transfer function reads as, once the state feedback is applied,

$$\boxed{\frac{Y_i(s)}{V_i(s)} = C_i(sI - A_i)^{-1}B_i}$$

➡ It yields that a linear state feedback *v* allows now to control/stabilize the linear system.

LIMITS

- Loss of controllability (accessibility, singularities)
- Zero dynamics can be instable
- This approach is *strongly* connected to the model : the obtained system is linear if there is NO uncertainty/perturbation.

➡ **WHAT HAPPENS IF IT IS NOT THE CASE ? ROBUST CONTROL**

Input-output linearization - Single Input-Single Output (SISO) (dim (y) = p =1))

> **Theorem.** The input-output linearization of an accessible nonlinear system admits a solution ⇔ the relative degree of y is finite, *i.e.*
>
> $$r = dr(y) < \infty$$

Proof.

- **Sufficient condition**. If the relative degree of y is finite,then

$$y^{(r)} = F'(x) + G'(x)u := v$$

$$\boxed{u = -G'^{-1}(x)F'(x) + G'^{-1}(x)v}$$ is a I/O linearizing control law with

$$z_1 = \left[y, \ldots, y^{(r-1)} \right]^T$$

z_2, arbitrary function of x s.t. $z = \varphi(x)$ locally invertible.

- **Necessary condition**. Every accessible system has a finite time relative degree.

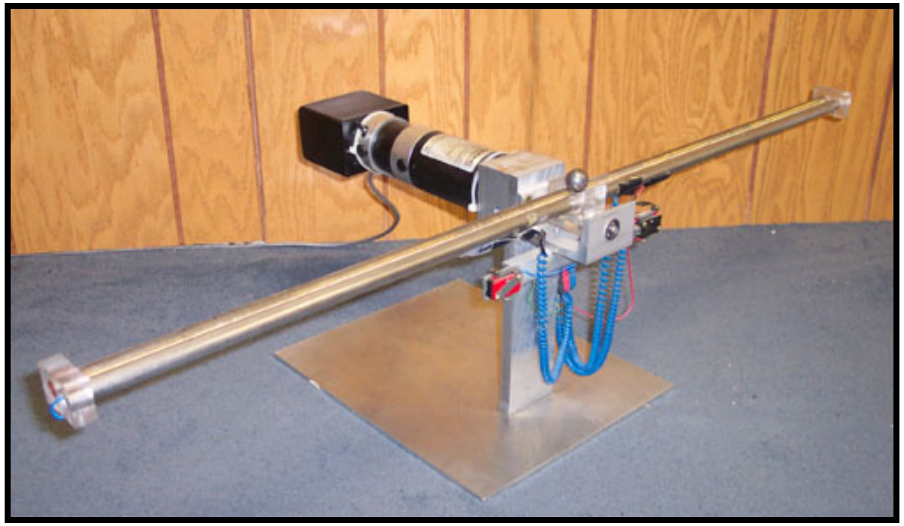
- **Example**.

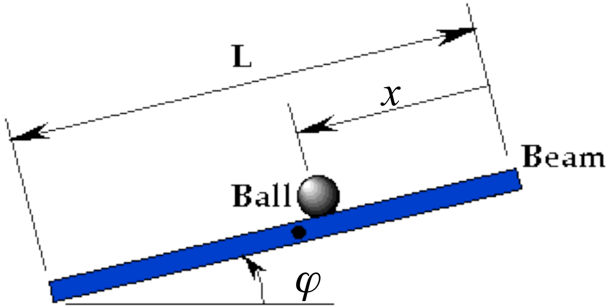
$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \\ y &= x_2 - x_1 \end{aligned}}$$

Pay attention to the zero dynamics !

One can change the output.

• **Example : Ball and beam.**





$$\dot{x} = v$$

$$\dot{v} = -g\sin\varphi + x\omega^2$$

$$\dot{\varphi} = \omega$$

$$\dot{\omega} = -\frac{2mxv\omega}{J + mx^2} - \frac{mgx\cos\varphi}{J + mx^2} + u$$

$$y = x$$

$$\bar{x}^T = [x \ v \ \varphi \ \omega]^T$$

1. Compute d.r.(y)
2. Determine the linearizing state feedback

$$u = \alpha(\bar{x}) + \beta(\bar{x})\bar{u}$$

3. Compute linearizing state coordinates

$$(z_1, \ z_2, \ z_3, \ z_4)$$

Input-output linearization - Multi Input-Multi Output (MIMO) (dim (y) = p =1))

Suppose that the relative degree vector reads as $(r_1, \cdots, r_p)$.

$$\Rightarrow \boxed{\begin{bmatrix} y_1^{(r_1)} \\ \cdots \\ y_p^{(r_p)} \end{bmatrix} = A_0(x) + B_0(x)u = v}$$

Theorem. The input-output linearization problem admits a solution if

$$\text{Rank } \frac{\partial\left(y_1^{(r_1)}, \ldots, y_p^{(r_p)}\right)}{\partial(u_1, \ldots, u_m)} := \text{Rank } B_o = p$$

Proof. ***Sufficiency*** - An input-output linearizing controller $\Rightarrow u = -G^{-1}(x)F(x) + G^{-1}(x)v$

Proof. ***Necessity*** - through a counter-example.

Consider a linear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ x_3 + u_1 \\ u_2 \end{bmatrix} \qquad y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \boxed{\begin{array}{llcl} \dot{y}_1 = u_1 & \Rightarrow r_1 & = & 1 \\ \dot{y}_2 = x_3 + u_1 & \Rightarrow r_2 & = & 1 \end{array}}$$

$$\text{rank } g \frac{\partial\left(y_1^{(r_1)}, \dots, y_p^{(r_p)}\right)}{\partial\left(u_1, \dots, u_m\right)} = \text{rank} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = 1 < p$$

Static decoupling and input-output linearization - MIMO case

Objective - Find, if possible, a state coordinates transformation $z = \varphi(x)$
and a state feedback $u = F(x) + G(x)v$
in order to get p independant sub-systems (p control inputs, p outputs)

$$\dot{z}_1 = A.z_1 + B.v$$
$$\dot{z}_2 = f_2(z_1, z_2) + g_2(z_1, z_2).v$$
$$y = C.z_1$$

System composed by p decoupled subsystems

$$A = \begin{bmatrix} A_{11} & & 0 \\ & A_{12} & \\ 0 & & \end{bmatrix}, \quad A_{11} = \begin{bmatrix} 0 & 1 & 0 \\ & & 1 \\ 0 & & 0 \end{bmatrix}_{r_1 * r_1} \quad B_1 = \begin{bmatrix} 0 \\ \\ 0 \\ 1 \end{bmatrix}_{r_1 * 1} \quad B = \begin{bmatrix} B_1 & B_2 & \end{bmatrix}$$

Theorem. The nonlinear system can be decoupled by static state feedback

$$\Leftrightarrow \text{rank} \frac{\partial\left(y_1^{(r_1)}, \ldots, y_p^{(r_p)}\right)}{\partial(u_1, \ldots, u_m)} = p$$

Definition. The jacobian matrix

$$\frac{\partial\left(y_1^{(r_1)},\dots,y_p^{(r_p)}\right)}{\partial(u_1,\dots,u_m)} \quad (:=B_0(x))$$

is called « decoupling matrix ».

If the decoupling matrix $B_0(x)$ is invertible, one has

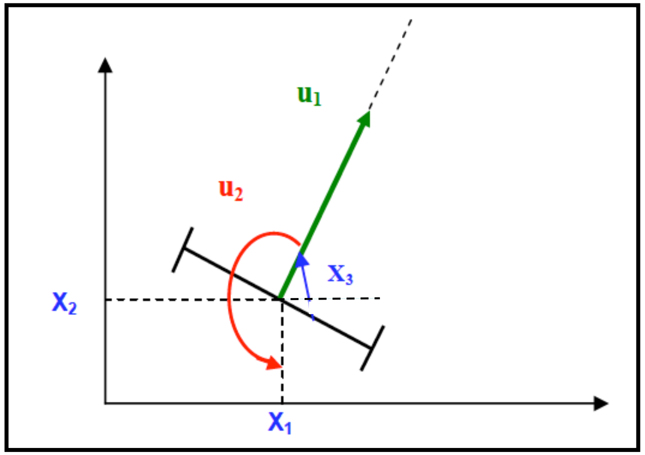
$$\begin{bmatrix} y_1^{(r_1)} \\ \dots \\ y_p^{(r_p)} \end{bmatrix} = A_0(x) + B_0(x)u = v$$

$$u = B_0^{-1}(x)\left[-A_0(x) + v\right]$$

➡ One gets p chains of r_i-integrators.

➡ $p.r_i$ controlled variables : then, not-controlled dynamics

Example



$$\dot{x}_1 = \cos x_3 . u_1$$

$$\dot{x}_2 = \sin x_3 . u_1$$

$$\dot{x}_3 = u_2$$

Two kinds of output

1. $y_1 = x_1,\ y_2 = x_3$
2. $y_1 = x_1,\ y_2 = x_2$

Decoupable ?
Control design ?

An alternative : dynamical compensator

Consider the following transformation $u_1 = x_4,\quad \dot{x}_4 = v_1,\quad u_2 = v_2$

Longitudinal acceleration

$$\begin{aligned}\dot{x}_1 &= (\cos x_3) x_4 \\ \dot{x}_2 &= (\sin x_3) x_4 \\ \dot{x}_3 &= v_2 \\ \dot{x}_4 &= v_1\end{aligned} \qquad \begin{aligned} y_1 &= x_1 \\ y_2 &= x_2 \end{aligned}$$

Decoupable ?
Control design ?

Application to trajectory tracking: consider the SISO nonlinear system

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

Control objective: $y_{ref}(t) - y(t) \rightarrow 0$

$$e = y_{ref}(t) - y(x)$$

Relative degree of $y(t)$: r

Control solution

$$\boxed{u = \frac{1}{b(x)}\left[-a(x) + {y_{ref}}^{(r)}(t) + \sum_k \lambda_k e^{(k)}(t)\right]}$$

$e(t) \rightarrow 0$ if coefficients λ_k are well-chosen.