Finite difference method for numerical computation of discontinuous solutions of the equations of fluid dynamics

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Introduction

The method of characteristics used for numerical computation of solutions of fluid dynamical equations is characterized by a large degree of nonstandardness and therefore is not suitable for automatic computation on electronic computing machines, especially for problems with a large number of shock waves and contact discontinuities.

In 1950 v. Neumann and Richtmyer (1) proposed to use, for the solution of fluid dynamics equations, difference equations into which viscosity was introduced artificially; this has the effect of smearing out the shock wave over several mesh points. Then it was proposed to proceed with the computations across the shock waves in the ordinary manner.

In 1954 Lax (2) published the "triangle" scheme suitable for computation across the shock waves. A deficiency of this scheme is that it does not allow computation with arbitrarily fine time steps (as compared with the space steps divided by the sound speed) because it then transforms any initial data into linear functions. In addition this scheme smears out contact discontinuities.

The purpose of this paper is to choose a scheme which is in some sense best and which still allows computation across the shock waves. This choice is made for linear equations and then by analogy the scheme is applied to the general equations of fluid dynamics.

Following this scheme we carried out a large number of computations on Soviet electronic computers. For a check, some of these computations were compared with the computations carried out by the method of characteristics. The agreement of results was fully satisfactory.

I have found out through the courtesy of N. N. Yanenko that he has also investigated a scheme for the solution of equations of fluid dynamics which is closely related to the scheme proposed in this paper.

Chapter I Finite Difference Schemes for Linear Equations

§ 1. A new requirement on difference schemes

To solve the differential equations of mathematical physics one often uses the method of finite differences. It is natural to require of the solution obtained by an approximate method that its qualitative behavior should be similar to the behavior of the exact solution of the differential equation. Such a requirement, however, is not always satisfied.

For example, consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

If initially the temperature u is a monotonic function of x then, clearly, it will remain such for all later times. When solving this equation by a finite difference scheme, even though it be stable and sufficiently accurate, it may happen that the temperature u which is monotonic initially will develop a maximum or a minimum at some later time.

As an example consider the scheme:

$$u_m^{n+1} = u_m^n + \frac{\tau}{2h^2}\left(u_{m+1}^{n+1} - 2u_m^n + u_{m-1}^{n+1}\right) + \frac{\tau}{2h^2}\left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right),$$

where u_m^n is the value of temperature u at the point whose coordinates are $x = mh$, $t = n\tau$. This scheme is stable for all positive $r = \tau/h^2$. Prescribe the following initial conditions:

$$u_m^0 = 0 \text{ for } m > 0,$$
$$u_m^0 = 1 \text{ for } m \leqslant 0.$$

After the first time step we obtain for the quantity u_m' an infinite system of equations which when solved yields:

$$u_m' = 1 - \frac{2r}{2r+1+\sqrt{2r+1}}\left(\frac{1+r-\sqrt{2r+1}}{r}\right)^{-m} \quad \text{for } m < 0,$$

$$u_m' = \frac{2r}{2r+1+\sqrt{2r+1}}\left(\frac{1+r-\sqrt{2r+1}}{r}\right)^{m-1} \quad \text{for } m > 0.$$

For m tending to $+\infty$, u_m' tends to 0, and for m tending to $-\infty$, u_m' tends to 1. It is not difficult to show by an analysis of the above solution that its monotonicity will be always violated for $r > 3/2$.

It is natural that for $r > 3/2$ this scheme should not be considered as a satisfactory one. However it must be noted that the effects connected with nonmonotonicity will appear only in the solution of problems with sharply varying initial conditions. Smooth solutions will be computed by this scheme with sufficient accuracy with a sufficiently fine mesh.

Analogous facts obtain also for difference schemes devised to solve the equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}.$$

It is well known that the solution of this equation has the form of a stationary wave* $u = u(x+t)$, and if u was monotonic for $t = 0$ it will remain so afterwards.

Let us examine examples of difference schemes for this equation and verify whether they preserve monotonicity of solution.

1. The "triangle" scheme of first-order accuracy:

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* A "stationary" waves is defined as one which is stationary in a coordinate system moving with the wave velocity.

$$u^0 = \frac{u_1 + u_{-1}}{2} + \frac{\tau}{2h} \cdot (u_1 - u_{-1}).^{*}$$

It clearly can be rewritten as follows:

$$u^0 = \frac{1+r}{2} \cdot u_1 + \frac{1-r}{2} \cdot u_{-1},$$

where $r = \tau/h$ (the stability condition for this scheme: $r \leqslant 1$) . Consider the initial conditions for $t = 0$ in the form of a step function:

$$u_k = 0 \text{ for } k \leqslant 0,$$

$$u_k = 1 \text{ for } k \geqslant 1,$$

and compute u for $t = \tau$. We obtain

$$u^k = 0 \text{ for } k \leqslant -1,$$

$$u^0 = \frac{1+r}{2},$$

$$u^1 = \frac{1+r}{2}.$$

$$u^k = 1 \text{ for } k \geqslant 2.$$

Since for $r \leqslant 1$, $\frac{1+r}{2} \leqslant 1$, we conclude that the monotonicity in this case is not violated.

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* Here and in the following we shall denote $u_0 = u(t_0, x_0)$, $u^0 = u(t_0 + \tau, x_0)$, $u_1 = u(t_0, x_0 + h)$, $u_{-1} = u(t_0, x - h)$ etc.; τ and h = time and space steps respectively.

An arbitrary monotonic function on the mesh of size h can be represented as a sum of step functions each of which changes its value only in one mesh interval and such step functions are either increasing or decreasing. Using this fact we may conclude that the "triangle" scheme transforms an arbitrary monotonic function into another monotonic function.

2. The scheme "tripod" of second order accuracy:

$$u^0 = u_0 + \frac{r}{2}(u_1 - u_{-1}) + \frac{r^2}{2}(u_1 - 2u_0 + u_{-1}).$$

This scheme is stable for $r \leqslant 1$. If again we take the step function

$$u_k = 0 \text{ for } k \leqslant 0,$$

$$u_k = 1 \text{ for } k \geqslant 1$$

for initial data at $t = 0$ then from this scheme we obtain at $t = \tau$

$$u^k = 0 \text{ for } k < -1,$$

$$u^0 = \frac{r + r^2}{2},$$

$$u^1 = 1 + \frac{r - r^2}{2},$$

$$u^k = 1 \text{ for } k \geqslant 2.$$

Since $r > r^2$ for $r < 1$, then $u^1 > 1$ and the monotonicity is violated.

Note that the scheme of second order accuracy expressing the value u^0 in terms of u_1, u_0, u_{-1} is unique; i.e., among these schemes there are none which would transform every monotonic function into other monotonic ones.

§ 2. Criterion to verify the monotonicity condition

We begin by noting that difference schemes can be either explicit or implicit.

An explicit scheme expresses the value of u at the desired point only in terms of known values of u at the preceding time interval. For a linear equation with constant coefficients such a scheme has the form

$$u^k = \sum c_{n-k} u_n .$$

Here the sum can be either finite or infinite. In the latter case the difference scheme will be defined not for all mesh functions $\{u_m\}$ but only for those which do not increase very rapidly with the increase of m ; the allowable rate of growth is determined by the rate at which the coefficients c_j decrease: It is necessary that the sum $\sum c_{n-k} u_n$ should converge.

An implicit scheme is a system of equations for the determination of the unknowns u^m , i.e., it has the form

$$\sum a_{m-k} u^m = \sum b_{n-k} u_n .$$

We assume that the left hand sum is finite.

An example of an implicit scheme is the difference scheme for the heat equation examined at the beginning of Section 1 of this chapter. Implicit schemes are of value to us only because they determine u^k uniquely.

We shall seek $\{u^k\}$ in the class of sequences bounded for $|k| \to \infty$. In this class uniqueness holds obviously for all schemes for which the difference equations

$$\sum a_{m-k} z_m = 0$$

do not have a nontrivial bounded solution. As is well known the general solution of these difference equations has the form

$$z_m = \sum_i \left(A_{i,k_i} m^{k_i-1} + A_{i,k_i-1} m^{k_i-2} + \dots + A_{i,k_i-(k_i-1)}\right)\lambda_i^m,$$

where λ_i are the roots with multiplicity k_i of the equation

$$\sum a_j \lambda^j = 0$$

From the examination of the expression for the general solution it is clear that in order to ensure uniqueness it is necessary and sufficient that the equation

$$\sum a_j \lambda^j = 0$$

not have roots of modulus one. In the following we shall assume that all difference schemes with which we will deal satisfy this condition.

It is not difficult to show that each such difference scheme can be solved for u^k and written in the form

$$u^k = \sum c_{n-k} u_n,$$

nd thus converted into an explicit scheme. Therefore, even though in this and the following paragraph we will consider only explicit schemes, the results obtained can be applied directly to implicit schemes.

We shall not consider schemes which connect more than two layers.

We shall now give a simple criterion allowing one to verify easily whether an arbitrary difference scheme transforms monotonic functions into monotonic ones or not.

In order that the difference scheme $u^k = \sum c_{n-k} u_n$ should transform all monotonic functions into monotonic ones with the same sense of growth it is necessary and sufficient that all c_m be nonnegative.

Proof: Suppose $c_m \geq 0$ and $\{u_n\}$ monotonic. For the sake of definiteness assume that $\{u_n\}$ increases, i.e., that all $u_n - u_{n-1}$ are nonnegative.

Then

$$u^k - u^{k-1} = \sum c_{n-k} u_n - \sum c_{n-k+1} u_n = \sum c_{n-k} u_n - \sum c_{n-k} u_{n-1} =$$
$$= \sum c_{n-k} (u_n - u_{n-1}),$$

i.e., $u^k - u^{k-1} \geqslant 0$. In this way the sufficiency of the condition is established.

We now prove the necessity. Suppose for example, $c_{m_0} < 0$. Let

$$u_k = 1 \text{ for } k \geqslant m_0,$$

$$u_k = 0 \text{ for } k < m_0 - 1.$$

Then $u^0 - u^{-1} = c_{m_0} < 0$, which is not possible because of the hypothesis that the scheme transforms monotonic sequences into monotonic ones with the same direction of growth. Thus the necessity is demonstrated.

It is not difficult to show that if all $c_m \geqslant 0$ and $\sum c_m = 1$, then the difference scheme is necessarily stable. Indeed

$$\max|u^k| \leqslant \sum |c_{n-k}| \cdot \max|u_n|.$$

But because of our assumptions $\max|c_m| \leqslant 1$; therefore $\sum|u^m| \leqslant \sum|u_m|$ and this means that the scheme is stable.

The condition $\sum c_m = 1$ appears to be quite natural for the schemes devised to solve, for example, the following equations:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

and means that the solution of these equations u = const. is also a solution of the difference equations.

As an application of the monotonicity criterion listed above we shall now give the derivation of a most accurate scheme of first order accuracy for the equation $\partial u/\partial t = \partial u/\partial x$, which expresses the value u^0 in terms of u_0, u_1, u_{-1} and satisfies the monotonicity condition (as we remarked at the end of Section 1, there are no such second order schemes).

It is easily verified that the general form of a first order scheme -- connecting only the above listed points -- is the following:

$$u^0 = u_0 + \frac{r}{2}(u_1 - u_{-1}) + k(u_1 - 2u_0 + u_{-1}) =$$

$$= \left(\frac{r}{2} + k\right) u_1 + (1 - 2k) u_0 + \left(k - \frac{r}{2}\right) u_{-1}.$$

For $k = r^2/2$ this scheme is of second order accuracy and for an arbitrary k its last term is $\left(k - \frac{r^2}{2}\right) h^2 u_{xx}$.

In this way the problem is reduced to the determination of k which differs least from $r^2/2$ and such that all coefficients of the scheme

$$u_0 = \left(\frac{r}{2} + k\right) u_1 + (1 - 2k) u_0 + \left(k - \frac{r}{2}\right) u_{-1}$$

are nonnegative (this last requirement is necessary in order that the scheme satisfy the monotonicity condition). Clearly it is necessary to take $k = r/2$. Then the scheme becomes

$$u^0 = u_0 + \frac{r}{2}(u_1 - u_{-1}) + \frac{r}{2}(u_1 - 2u_0 + u_{-1}) = r u_1 + (1 - r) u_0.$$

As one can easily check, the stability condition for this scheme is $r < 1$. It is of interest to note another way of obtaining this formula. If from the point $(t_0 + \tau, 0)$ at which we seek u^0 we draw a straight line which is the characteristic of the equation $\partial u/\partial t = \partial u/\partial x$, then it will intersect the initial layer $t = t_0$ at the point (t_0, rh) which lies (for $r < 1$) between points $(t_0, 0)$ and (t_0, h) at which the values of u_0 and u_1 are given. The value of u at this point is obviously u^0 since u remains constant along the characteristic.

Consequently we will obtain our scheme if we compute u at the point (t_0, rh) by a linear interpolation between the values u_0 and u_1 at the points $(t_0, 0)$ and (t_0, h) and then transfer this value along the characteristic to the point $(t_0 + \tau, 0)$.

§ 3. <u>Among schemes of second order accuracy for the equation $\partial u/\partial t = \partial u/\partial x$ there is none which satisfied the monotonicity condition</u>

In Section 1 we remarked that for the equation $\partial u/\partial t = \partial u/\partial x$ there are no difference schemes of second order accuracy expressing α^0 in terms of u_1, u_0, u_{-1} and transforming monotonic functions into monotonic ones. Now we shall generalize this statement and prove that for this equation with $r = \tau/h \neq 0, 1, 2, \ldots$ in general there are no explicit or implicit schemes of second order accuracy connecting an arbitrary number of points at two successive time steps and transforming monotonic functions into other monotonic ones.

As we noted at the beginning of Section 2 it is sufficient without loss of generality to consider only schemes of the form

$$u^k = \sum c_{n-k} u_n.$$

We shall say that this scheme is of second order accuracy if it is exact for initial data that are a polynominal of second degree, i.e., if for such initial conditions the result of computation according to the scheme agrees with the solution of the differential equation at the point considered. Prescribe

$$u(0, x) = \left(\frac{x}{h} - \frac{1}{2}\right)^2 - \frac{1}{4}.$$

Then at integer points

$$u_n = u(0, nh) = \left(n - \frac{1}{2}\right)^2 - \frac{1}{4}.$$

The solution of equation $\partial u/\partial t = \partial u/\partial x$ with these initial conditions is

$$u(t,x) = \left(\frac{x+t}{h} - \frac{1}{2}\right)^2 - \frac{1}{4}.$$

Suppose now we wish to compute the value $u^p = u(\tau, ph)$ by the difference scheme. Since we assume that the scheme has second order accuracy we should obtain the exact value of the differential equation because the initial function is a second degree polynomial; i.e., we obtain

$$u^p = \left(p + r - \frac{1}{2}\right)^2 - \frac{1}{4}.$$

Using the difference scheme we arrive at the equation

$$\left(p + r - \frac{1}{2}\right)^2 - \frac{1}{4} = \sum c_{n-p}\left[\left(n - \frac{1}{2}\right)^2 - \frac{1}{4}\right].$$

If this scheme satisfied the monotonicity condition then all c_{n-p} would be nonnegative and since $(n-1/2)^2 - 1/4 \geqslant 0$ we would obtain that for all p, $(p + r - 1/2)^2 - 1/4 \geqslant 0$. Actually it is not so. Indeed if $\ell > -r > \ell - 1$, where ℓ is an integer, then

$$u^\ell = \left(\ell + r - \frac{1}{2}\right)^2 - \frac{1}{4} < 0.$$

This contradiction proves the original statement.

§ 4. Construction of the best scheme for a system of two equations

We shall now investigate the system of equations

$$\frac{\partial u}{\partial t} = A\frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial t} = B\frac{\partial u}{\partial x} \tag{1}$$

(Coefficients A and B will be assumed constant.) Multiplying the second equation by λ and adding it to the first we obtain

$$\frac{\partial(u + \lambda v)}{\partial t} = \frac{A}{\lambda}\cdot\frac{\partial(\lambda v)}{\partial x} + \lambda B\frac{\partial u}{\partial x}.$$

Choosing $\lambda = \pm\sqrt{\frac{A}{B}}$ we obtain

$$\frac{\partial\left(u+\sqrt{\frac{A}{B}}v\right)}{\partial t} = \sqrt{AB}\,\frac{\partial\left(u+\sqrt{\frac{A}{B}}v\right)}{\partial x},$$

$$\frac{\partial\left(u-\sqrt{\frac{A}{B}}v\right)}{\partial t} = -\sqrt{AB}\,\frac{\partial\left(u-\sqrt{\frac{A}{B}}v\right)}{\partial x}.$$

Each of the above equations has a general solution in the form of a stationary wave.

$$u + \sqrt{\frac{A}{B}}v = F_+(x+\sqrt{AB}\,t),$$

$$u - \sqrt{\frac{A}{B}}v = F_-(x-\sqrt{AB}\,t).$$

Obviously if $u+\sqrt{A/B}\,v$ or $u-\sqrt{A/B}\,v$ were monotonic initially then they would remain so for all later times. It is natural therefore to impose on the difference scheme for equations (1) the requirement that it preserve this monotonicity.

It is not difficult to verify that any linear difference scheme for the system (1) expressing the values u^0, v^0 in terms of u_1, v_1, u_0, v_0, u_{-1}, v_{-1} will have the form

$$\begin{aligned} u^0 &= u_0 + \frac{\tau A}{2h}(v_1 - v_{-1}) + K(v_1 - 2v_0 + v_{-1}) + L(u_1 - 2u_0 + u_{-1}),\\ v^0 &= v_0 + \frac{\tau B}{2h}(u_1 - u_{-1}) + M(v_1 - 2v_0 + v_{-1}) + N(u_1 - 2u_0 + u_{-1}). \end{aligned} \qquad (2)$$

We shall not consider schemes which use for the computation of u^0, v^0 values of u and v at the initial time in more than three points because in solving problems with boundary values such schemes require considerable modification near the boundary; this is awkward when standard machine computations are used.

Multiplying the second of equations (2) by $\pm\sqrt{A/B}$ and adding the result to the first we obtain

$$\begin{aligned}\left(u\pm\sqrt{\frac{A}{B}}v\right)^0 &= \left(u\pm\sqrt{\frac{A}{B}}v\right)_0 \pm \frac{\tau\sqrt{AB}}{2h}\left[\left(u\pm\sqrt{\frac{A}{B}}v\right)_1 - \left(u\pm\sqrt{\frac{A}{B}}v\right)_{-1}\right] + \\ &+ \frac{L+M\pm N\sqrt{\frac{A}{B}}\pm K\sqrt{\frac{B}{A}}}{2}\left[\left(u\pm\sqrt{\frac{A}{B}}v\right)_1 - 2\left(u\pm\sqrt{\frac{A}{B}}v\right)_0 + \right. \\ &\left. + \left(u\pm\sqrt{\frac{A}{B}}v\right)_{-1}\right] + \frac{L-M\pm N\sqrt{\frac{A}{B}}\mp K\sqrt{\frac{B}{A}}}{2}\left[\left(u\mp\sqrt{\frac{A}{B}}v\right)_1 - \right. \\ &\left. - 2\left(u\mp\sqrt{\frac{A}{B}}v\right)_0 + \left(u\mp\sqrt{\frac{A}{B}}v\right)_{-1}\right]\end{aligned} \tag{3}$$

In these formulae consider first the upper sign. Suppose initially $u+\sqrt{A/B}\,v=0$ everywhere and $u-\sqrt{A/B}\,v=1$ everywhere except at one point where $u-\sqrt{A/B}\,v\neq 1$. Obviously if $L-M+N\sqrt{A/B}-K\sqrt{A/B}\neq 0$ then the values of $u+\sqrt{A/B}\,v$ will be different from zero at three points and therefore the monotonicity of $u+\sqrt{A/B}\,v$ will be violated.

From this we conclude that necessarily

$$L-M+N\sqrt{\frac{A}{B}}-K\sqrt{\frac{B}{A}}=0$$

Choosing in (3) the lower sign and carrying out analogous considerations we obtain that, also necessarily,

$$L+M-N\sqrt{\frac{A}{B}}-K\sqrt{\frac{B}{A}}=0.$$

Introducing the notation

$$L+M+N\sqrt{\frac{A}{B}}+K\sqrt{\frac{B}{A}}=g,$$

$$L-M+N\sqrt{\frac{A}{B}}-K\sqrt{\frac{B}{A}}=G,$$

equations (3) assume the form:

$$\left(u+\sqrt{\frac{A}{B}}v\right)^0=\left(u+\sqrt{\frac{A}{B}}v\right)_0+\frac{\tau\sqrt{AB}}{2h}\left[\left(u+\sqrt{\frac{A}{B}}v\right)_1-\right.$$
$$\left.-\left(u+\sqrt{\frac{A}{B}}v\right)_{-1}\right]+g\left[\left(u+\sqrt{\frac{A}{B}}v\right)_1-2\left(u+\sqrt{\frac{A}{B}}v\right)_0+\right.$$
$$\left.+\left(u+\sqrt{\frac{A}{B}}v\right)_{-1}\right], \tag{3a}$$
$$\left(u-\sqrt{\frac{A}{B}}v\right)^0=\left(u-\sqrt{\frac{A}{B}}v\right)_0-\frac{\tau\sqrt{AB}}{2h}\left[\left(u-\sqrt{\frac{A}{B}}v\right)_1-\right.$$
$$\left.-\left(u-\sqrt{\frac{A}{B}}v\right)_{-1}\right]+G\left[\left(u-\sqrt{\frac{A}{B}}v\right)_1-2\left(u-\sqrt{\frac{A}{B}}v\right)_0+\right.$$
$$\left.+\left(u-\sqrt{\frac{A}{B}}v\right)_{-1}\right].$$

As we have shown at the beginning of this section $u+\sqrt{A/B}\,v$ satisfies the equation

$$\frac{\partial\left(u+\sqrt{\frac{A}{B}}v\right)}{\partial t}=\sqrt{AB}\,\frac{\partial\left(u+\sqrt{\frac{A}{B}}v\right)}{\partial x}.$$

In the same way in which we chose the most accurate scheme for the equation $\partial u/\partial t=\partial u/\partial x$ (see § 2), which transforms monotonic functions into monotonic ones, we can convince ourselves that for $u+\sqrt{A/B}\,v$ the most accurate scheme satisfying the monotonicity condition will be one with $g=\frac{\tau\sqrt{AB}}{2h}$; for $u-\sqrt{A/B}\,v$, one with $G=\frac{\tau\sqrt{AB}}{2h}$. By substituting these expressions for g and G in (3a) and adding the equations we obtain the expression for u^0; by subtracting one second from the first and multiplying the difference by $\sqrt{B/A}$ we get the expression for v^0:

$$\begin{aligned} u^0&=u_0-\frac{\tau A}{2h}(v_1-v_{-1})+\frac{\tau\sqrt{AB}}{2h}(u_1-2u_0+u_{-1}),\\ v^0&=v_0+\frac{\tau B}{2h}(u_1-u_{-1})+\frac{\tau\sqrt{AB}}{2h}(v_1-2v_0+v_{-1}). \end{aligned} \tag{4}$$

§ 5. Physical interpretation of the constructed scheme

We shall now give a physical interpretation for the difference scheme (4). Consider the equations of fluid dynamics in Lagrangian coordinates:

$$\begin{aligned} \frac{\partial u}{\partial t} + B\frac{\partial p(v)}{\partial x} &= 0, \\ \frac{\partial v}{\partial t} - B\frac{\partial u}{\partial x} &= 0. \end{aligned} \tag{5}$$

Here u = velocity, p = pressure, v = specific volume.

In the case when $p(v)$ is a linear function

$$p(v) = -\frac{A}{B}(v - v_0) + p_0 \tag{6}$$

(This may be assumed for the case of acoustic waves.) System (5) is identical with (1).

$$\begin{aligned} \frac{\partial u}{\partial t} &= A\frac{\partial v}{\partial x}, \\ \frac{\partial v}{\partial t} &= B\frac{\partial u}{\partial x}, \end{aligned} \tag{1}$$

for which the scheme (4) was constructed. Using the equation of state (6) this scheme can be rewritten as follows

$$u^0 = u_0 - \frac{\tau B}{h}\left[\left(\frac{p_1 + p_0}{2} - \sqrt{\frac{A}{B}}\,\frac{u_1 - u_0}{2}\right) - \left(\frac{p_0 + p_{-1}}{2} - \sqrt{\frac{A}{B}}\cdot\frac{u_0 - u_{-1}}{2}\right)\right],$$

$$v^0 = v_0 + \frac{\tau B}{h}\left[\left(\frac{u_1 + u_0}{2} - \frac{p_1 - p_0}{2\sqrt{\frac{A}{B}}}\right) - \left(\frac{u_0 + u_{-1}}{2} - \frac{p_0 - p_{-1}}{2\sqrt{\frac{A}{B}}}\right)\right].$$

Introducing the notation:

$$\begin{aligned} P_{m+\frac{1}{2}} &= \frac{p_{m+1} + p_m}{2} - \sqrt{\frac{A}{B}}\cdot\frac{u_{m+1} - u_m}{2}, \\ U_{m+\frac{1}{2}} &= \frac{u_{m+1} + u_m}{2} - \frac{p_{m+1} - p_m}{2\sqrt{\frac{A}{B}}}. \end{aligned} \tag{7}$$

this scheme assumes the form:

$$u^0 = u_0 - \frac{\tau B}{h}\left(P_{\frac{1}{2}} - P_{-\frac{1}{2}}\right),$$
$$v^0 = v_0 + \frac{\tau B}{h}\left(U_{\frac{1}{2}} - U_{-\frac{1}{2}}\right), \tag{8}$$

It turns out that the quantities $P_{\frac{1}{2}}$ and $U_{\frac{1}{2}}$ have a definite physical meaning. Let us imagine that in the interval between the points $\frac{1}{2}$ and $\frac{3}{2}$ (i.e. between the points $x = \frac{1}{2}h$ and $x = \frac{3}{2}h$ the values of u and p are initially constant and equal u_1, p_1 ; between points $\frac{1}{2}$ and $-\frac{1}{2}$ they are also constant and equal u_0, p_0 . Since the point $\frac{1}{2}$ is a point of contact for two regions occupied by a gas with, in general, different velocities and pressures, the so-called resolution of the discontinuity will take place at this point. Namely, sound waves will spread to the right and left of the point $\frac{1}{2}$ with the velocity $\frac{dx}{dt} = \pm\sqrt{AB}$ (this is the equation of characteristics for the system (1)). In front of these waves u and p will remain constant and equal to u_1, p_1 before the right-travelling wave and to u_0, p_0 before the left-travelling wave. (Obviously such a state will be preserved only until the waves generated by the resolution of discontinuity at the point $\frac{1}{2}$ collide with those waves generated at the points $\frac{3}{2}$ and $-\frac{1}{2}$). Between the waves spreading from the point $\frac{1}{2}$ the values of u and p will be constants which can be computed using relations satisfied on a sound wave. Consider the first of our equations

$$\frac{\partial u}{\partial t} + B\frac{\partial p(v)}{\partial x} = 0.$$

From which it follows that for an arbitrary contour

$$\oint u\,dx - Bp(v)\,dt = 0.$$

It is not difficult to obtain as a consequence of this integral identity that the discontinuities in u and p must satisfy the condition

$$(u)\,dx - B(p)\,dt = 0.$$

On the wave propagating to the right, $\frac{dx}{dt} = \sqrt{AB}$ and we obtain

$$(u)\sqrt{\frac{A}{B}} - (p) = 0,$$

and on the wave propagating to the left, $\frac{dx}{dt} = -\sqrt{AB}$ and

$$(u)\sqrt{\frac{A}{B}} + (p) = 0.$$

Denoting the values of u and p between the propagating waves by U and P respectively we arrive at the system of equations

$$\left(U - u_{\frac{1}{2}}\right)\sqrt{\frac{A}{B}} - \left(P - p_{\frac{1}{2}}\right) = 0,$$

$$\left(U - u_{\frac{1}{2}}\right)\sqrt{\frac{A}{B}} + \left(P - p_{-\frac{1}{2}}\right) = 0.$$

Solving it we find

$$P = \frac{p_1 + p_0}{2} - \sqrt{\frac{A}{B}} \cdot \frac{u_1 - u_0}{2},$$

$$U = \frac{u_1 + u_0}{2} - \frac{p_1 - p_0}{2\sqrt{\frac{A}{B}}}.$$

We observe that U and P agree with $U_{\frac{1}{2}}$ and $P_{\frac{1}{2}}$, determined from (7) and entering into our difference scheme (8). In this way we see that $U_{\frac{1}{2}}$ and $P_{\frac{1}{2}}$ are the values of velocity and pressure obtained as a result of the resolution of the discontinuity in the region between propagating waves and consequently also at the point $\frac{1}{2}$ from which the waves emanated.

It is of interest to note that the obtained values $U_{\frac{1}{2}}$ and $P_{\frac{1}{2}}$ will remain constant until the boundary considered is reached by the waves generated from the resolution of the discontinuities on the neighboring boundaries, i.e., at the points $-\frac{1}{2}$ and $\frac{3}{2}$. For the system considered the disturbances propagate with the sound velocity $\sqrt{AB}$. Therefore if during the time interval τ the values $U_{\frac{1}{2}}$ and $P_{\frac{1}{2}}$ are to remain constant it is necessary to have $\tau < h/\sqrt{AB}$. Fortunately this inequality agrees with the stability condition for the scheme (4).

Clearly, after the time interval τ the values of u and v between the points $\frac{1}{2}$ and $-\frac{1}{2}$ will no longer be constant. Let us denote their mean values by u^0 and v^0. For their computation we shall use the law of conservation of momentum which yields the first of equations (8) and the conservation of volume which gives the second one.

The physical interpretation of the difference scheme (4) will serve as a basis for the construction of a computation scheme for the general system of equations of fluid mechanics.

Chapter II An Approximate Scheme for the Computation of Generalized Solutions of the Equations of Fluid Mechanics

§ 1. Formulation of the problem

Our aim will be to construct the difference scheme for plane (non-axisymmetric) one-dimensional unsteady equations of fluid mechanics (in Lagrangian form)

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + B\frac{\partial p(v,E)}{\partial x} = 0, \\ &\frac{\partial v}{\partial t} - B\frac{\partial u}{\partial x} = 0, \\ &\frac{\partial\left(E + \frac{u^2}{2}\right)}{\partial t} + B\frac{\partial pu}{\partial x} = 0. \end{aligned}\right\} \tag{1}$$

As is well known this system of equations does not always have a smooth solution even for smooth initial data. Therefore one also must consider generalized solutions with discontinuities -- shock waves.

After S. L. Sobolev, we shall call the system of functions (u, v, E) a generalized solution of the system (1) if for each infinitely differential function $\varphi(x,t)$ which differs from zero only on a finite subdomain of the domain G on which the functions u, v, E are defined, the following equalities are satisfied:

$$\iint_G \left[u \frac{\partial \varphi}{\partial t} + Bp(v, E) \frac{\partial \varphi}{\partial x} \right] dx\,dt = 0,$$

$$\iint_G \left[v \frac{\partial \varphi}{\partial t} - Bu \frac{\partial \varphi}{\partial x} \right] dx\,dt = 0,$$

$$\iint_G \left[\left(E + \frac{u^2}{2} \right) \frac{\partial \varphi}{\partial t} + Bp(v, E) \cdot u \frac{\partial \varphi}{\partial x} \right] dx\,dt = 0.$$

If functions u, v, E are pie~~course~~wise continuous then these requirements are equivalent to the fact that around an arbitrary contour

$$\left. \begin{aligned} &\oint u\,dx - Bp\,dt = 0, \\ &\oint v\,dx + Bu\,dt = 0, \\ &\oint \left(E + \frac{u^2}{2} \right) dx - Bpu\,dt = 0, \end{aligned} \right\} \qquad (2)$$

and this is the usual formulation of the conservation laws. From the conservation laws it is easy to establish, as is done in every course of gas dynamics, the relations across the discontinuities (shock waves):

$$\left. \begin{aligned} &[wu - Bp] = 0, \\ &[wv + Bu] = 0, \\ &\left[w\left(E + \frac{u^2}{2} \right) - Bpu \right] = 0. \end{aligned} \right\} \qquad (3)$$

Here $w = \frac{dx}{dt}$ is the velocity of the shock wave and $[\ ]$ means the jump of the quantity (the difference of its values on the right and left of the wave).

We must note that in order to ensure uniqueness it is necessary to exclude rarefaction shock waves; for this it is sufficient to require that around any contour the following integral inequality is satisfied:

$$\oint S\,dx \geqslant 0,$$

where S is the entropy determined by the known methods of thermodynamics as a certain function of p and v.

We propose to construct for system (1) an approximate scheme with the property that as the size of steps diminishes the solution obtained by this scheme will converge to the generalized solution of the system.

We shall construct the difference scheme in such a way that for the linear case of sound waves it will coincide with the scheme considered in the previous chapter which transforms monotonic waves into other monotonic ones. The application of schemes which do not possess this property does not appear to be intelligent since the effect of nonmonotonicity appears precisely in regions where the solution varies sharply which are the shock waves. In attempts to compute shock waves using schemes which do not satisfy the monotonicity condition one obtains for them "humped" profiles and the humps pulsate from one time step to the next (see, for example, the graph in Application 1).

Sometimes instead of (1) one has to consider the following system of equations of fluid dynamics:

$$\frac{\partial u}{\partial t} + B\frac{\partial p(v)}{\partial x} = 0,$$
$$\frac{\partial v}{\partial t} - B\frac{\partial u}{\partial x} = 0 \qquad (1a)$$

(Such equations describe, for example, the flow of water in shallow channels.)

The reader will have no difficulty in transferring all our considerations to this simpler case. Let us state only that as the uniqueness condition, the law of increase of entropy $\oint S dx \geqslant 0$, in this case should be replaced by the law of dissipation of energy

$$\oint\left(\frac{u^2}{2} - \int p dv\right)dx - Bpudt \geqslant 0.$$

§ 2. The description of the computational scheme

We will now describe the proposed scheme.

Let us imagine that the gas whose behavior we wish to compute is divided into a sequence of layers by points with integer labels 0, 1, 2, 3, 4, ... and the layers themselves are numbered by 'half integers" $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\ldots$ We shall assume that the quantities $u, v, E, p = p(v, E)$ are initially constant inside each layer. At the boundary m between the two adjacent layers $m-\frac{1}{2}$ and $m+\frac{1}{2}$ the discontinuity is resolved, in consequence of which at the point m the pressure and velocity become P_m and U_m (unlike § 5 of Chapter I, points at which the discontinuity is resolved are now labeled by integers and layers between them by half integers).

The rules for computation of P and U are derived in any course of gas dynamics (see e.g. (3)). We shall list the formulae for P_m and U_m in a form convenient for us in Section 4 below.

After the values of P_m and U_m are determined at all integer points we determine the values which u, v, E will assume when the time interval τ has elapsed by formulae analogous to (8) of Chapter I:

$$u^{m+\frac{1}{2}} = u_{m+\frac{1}{2}} - \frac{\tau B}{h}\left(P_{m+1} - P_m\right),$$

$$v^{m+\frac{1}{2}} = v_{m+\frac{1}{2}} + \frac{\tau B}{h}\left(U_{m+1} - U_m\right),$$

$$E^{m+\frac{1}{2}} = E_{m+\frac{1}{2}} + \frac{\left(u_{m+\frac{1}{2}}\right)^2}{2} - \frac{\left(u^{m+\frac{1}{2}}\right)^2}{2} - \frac{\tau B}{h}\left(P_{m+1}U_{m+1} - P_m U_m\right). \tag{4}$$

Here h denotes the mesh size of the scheme, i.e., the difference in Lagrangian coordinates for any two adjacent integer points.

As in Chapter I we must note that after the time interval τ has elapsed the values between two successive integer points will no longer be constant and the computed values $u^{m+\frac{1}{2}}$, $v^{m+\frac{1}{2}}$, $E^{m+\frac{1}{2}}$ represent only averages over the layer which replace their true distribution with a certain accuracy that is characteristic of the approximate method described above.

§ 3. <u>If with the diminishing mesh size the difference solution converges, then it converges to the generalized solution of the differential equation</u>

Now assuming that as the mesh size decreases, u, v, E computed by the difference scheme (4) converge to certain piecemeal smooth limit functions, we shall show that for these limit functions the conservation laws (2) are satisfied, i.e., these limit functions are generalized solutions of (1) or (1a).*

Let us consider simple rectlinear contours of the form represented in Figure 1. On this figure, crosses denote the half-integer points located inside the layers and circles, integer points.

From (4) it follows that

$$h\sum_{A_1}^{A_2} u - h\sum_{A_3}^{A_4} u - B\sum_{A_4}^{A_2} P\tau + B\sum_{A_3}^{A_1} P\tau. \tag{5a}$$

If as the mesh size decreases the mesh functions u, v, E, P, U converge to some limit functions defined in the plane (these limit functions we shall denote by the same letters as the corresponding mesh functions), then from the difference conservation law (5a) it follows that for the limit functions around an arbitrary rectlinear contour,

$$\oint u\,dx + BP\,dt = 0. \tag{5}$$

From the fact that (5) is satisfied for an arbitrary rectlinear contour it follows that (5) is satisfied for any contour.

From the formulae for P and U derived in the following section it follows that if in the regions where the solution of the differential equation is smooth the mesh functions, u, v, E converge to these solutions, then in these regions the limit functions for P and p coincide. Using this and the fact that the discontinuity lines cannot influence the values of the integrals, we arrive from (5) at the equation

$$\oint u\,dx + Bp\,dt = 0.$$

---

* We shall prove in § 4 that for the limit functions the law of increase of entropy (if the system considered is (1)) or of dissipation of energy (if the system considered is (1a)) hold.

Analogously one can demonstrate that the remaining two equations (2) are satisfied.

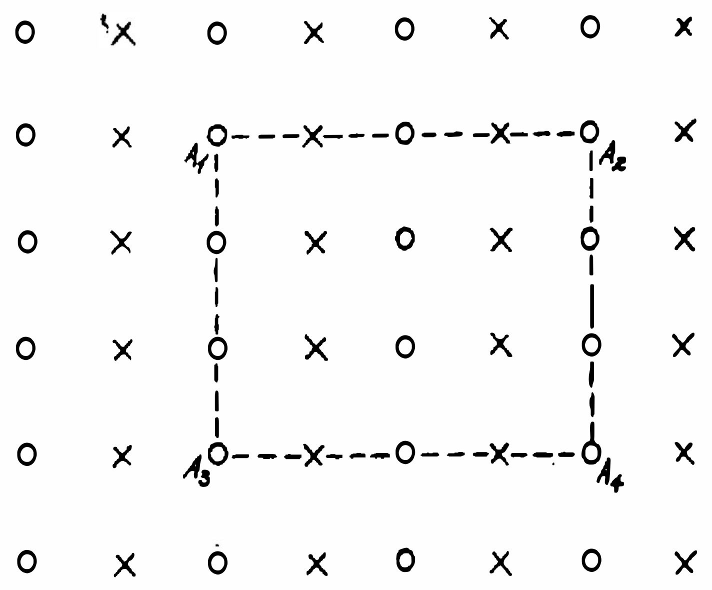

Figure 1

In this way we have proved the statement formulated in the title of this section.

§ 4. Formulae for computating the resolution of a discontinuity

We shall now describe the derivation of formulae for P and U ; for simplicity we limit ourselves to the case of a gas with the equation of state

$$E = \frac{1}{(\gamma - 1)} pv.$$

Suppose that to the right of the point O the gas has specific volume $v_{\frac{1}{2}}$, internal energy (per unit mass) $E_{\frac{1}{2}}$ and velocity $u_{\frac{1}{2}}$.

The pressure in this gas is given by

$$p_{\frac{1}{2}} = \frac{(\gamma - 1) E_{\frac{1}{2}}}{v_{\frac{1}{2}}} .$$

Suppose the state of the gas to the left of O is defined by the values

$$v_{-\frac{1}{2}},\ E_{-\frac{1}{2}},\ u_{-\frac{1}{2}},\ p_{-\frac{1}{2}} = \frac{(\gamma - 1)E_{-\frac{1}{2}}}{v_{-\frac{1}{2}}}.$$

We assume to begin with that the pressure P_0 which obtains after the resolution of the discontinuity is greater than $p_{-\frac{1}{2}}$ and $p_{\frac{1}{2}}$. In this case shock waves will propagate to the right and left of point O.

As we have already noted (see formulae (3) of § 1 of this chapter), on the shock wave the following conditions are satisfied:

$$\begin{aligned} [wu - Bp] &= 0, \\ [wv + Bu] &= 0, \\ \left[w\left(E + \frac{u^2}{2}\right) - Bpu\right] &= 0. \end{aligned}$$

Introducing the definitions $\frac{w_r}{B} = b_0$, $\frac{w_l}{B} = -a_0$ (w_r = velocity of the wave propagating to the right, w_l = velocity of the wave propagating to the left) we can rewrite these relations as

$$\left.\begin{aligned} a_0[u] + [p] &= 0, \\ a_0[v] - [u] &= 0. \\ a_0\left[E + \frac{u^2}{2}\right] + [pu] &= 0 \end{aligned}\right\} \quad \text{on the left wave} \qquad (6)$$

$$\left.\begin{aligned} b_0[u] - [p] &= 0, \\ b_0[v] + [u] &= 0, \\ b_0\left[E + \frac{u^2}{2}\right] - [pu] &= 0 \end{aligned}\right\} \quad \text{on the right wave} \qquad (7)$$

As is well known, in the region between the two waves u and p will be constant and equal to U and P -- the values on the contact discontinuity originating at the point O. The values of the specific volume will be constants between the contact discontinuity and the waves but these constants will be different to the right and left of this discontinuity. We shall denote by v_r the specific volume between the contact discontinuity and the right shock wave and by v_l the specific volume between the contact discontinuity and the left shock wave.

The first of equations (6) and (7) can be rewritten in the following way:

$$b_0 u_{\frac{1}{2}} - p_{\frac{1}{2}} = b_0 U_0 - P_0 ,$$

$$a_0 u_{-\frac{1}{2}} + p_{-\frac{1}{2}} = a_0 U_0 + P_0 .$$

Assuming a_0 and b_0 known, U_0 and P_0 can be determined from the above system:

$$P_0 = \frac{b_0 p_{-\frac{1}{2}} + a_0 p_{\frac{1}{2}} + a_0 b_0 \left(u_{-\frac{1}{2}} - u_{\frac{1}{2}}\right)}{a_0 + b_0} ,$$

$$U_0 = \frac{a_0 u_{-\frac{1}{2}} + b_0 u_{\frac{1}{2}} + p_{-\frac{1}{2}} - p_{\frac{1}{2}}}{a_0 + b_0} .$$

On the other hand, if we knew P_0 then for the determination of a_0 we could use the equation obtained from the first two relations (6) after the velocity is eliminated between them:

$$a_0 = \sqrt{-\frac{P_0 - p_{-\frac{1}{2}}}{v_{0l} - v_{-\frac{1}{2}}}} .$$

The value v_{0l} can be eliminated from this formula with the aid of the Hugoniot curve obtained from (6) by the method described in any course on gas dynamics (see e. g. (3)):

$$\frac{v_{0l}}{v_{-\frac{1}{2}}} = \frac{(\gamma-1)P_0 + (\gamma+1)p_{-\frac{1}{2}}}{(\gamma+1)P_0 + (\gamma-1)p_{-\frac{1}{2}}} .$$

After this expression for v_{0l} is substituted into the formula for a_0 we obtain

$$a_0 = \sqrt{\frac{(\gamma+1)P_0 + (\gamma-1)p_{-\frac{1}{2}}}{2 v_{-\frac{1}{2}}}} .$$

Completely analogously we can derive

$$b_0 = \sqrt{\frac{(\gamma+1)P_0 + (\gamma-1)p_{\frac{1}{2}}}{2v_{\frac{1}{2}}}} .$$

To determine P_0 we can now use the following iteration process: beginning with an arbitrary P_0 we determine a_0 and b_0 and then compute the new value for P_0. Substituting it into the formulae for a_0 and b_0 we find P_0 again, and so on until the process converges. After this we determine U_0.

So far we have considered only the case when simultaneously $P_0 > p_{-\frac{1}{2}}$ and $P_0 > p_{\frac{1}{2}}$, i.e., when no rarefaction wave occurs in the resolution of the discontinuity. It turns out that when such waves occur the process of resolution can be computed in the same manner only changing the formulae for a_0 and b_0.

Imagine, for example, that $P_0 < p_{\frac{1}{2}}$, that a rarefaction wave propagates to the right. As is well known, across a rarefaction wave the following relations hold:

$$u_{\frac{1}{2}} - \frac{2c_{\frac{1}{2}}}{\gamma-1} = U_0 - \frac{2c_{0r}}{\gamma-1} ,$$

$$p_{\frac{1}{2}} v_{\frac{1}{2}}^{\gamma} = P_0 v_{0r}^{\gamma} ,$$

where $c = \sqrt{\gamma p v}$ = sound speed, and c_{0r} and v_{0r} = the values of c and v to the right of the contact discontinuity. The first of the above equalities can be rewritten in the following way:

$$\frac{\frac{\gamma-1}{2}\left(p_{\frac{1}{2}} - P_0\right)}{c_{\frac{1}{2}} - c_{0r}} \cdot u_{\frac{1}{2}} - p_{\frac{1}{2}} = \frac{\frac{\gamma-1}{2}\left(p_{\frac{1}{2}} - P_0\right)}{c_{\frac{1}{2}} - c_{0r}} \cdot U_0 - P_0 ;$$

Denoting

$$b_0 = \frac{\gamma - 1}{2} \cdot \frac{p_{\frac{1}{2}} - P_0}{c_{\frac{1}{2}} - c_{0r}} ,$$

we obtain the relation completely analogous to the one obtaining across the shock wave.

If in the formula for b_0 , c is expressed in terms of p and v and v_{0r} is eliminated with the aid of the Poisson adiabat (the second of our equalities holding across the rarefaction wave(, the following obtains:

$$b_0 = \frac{\gamma - 1}{2\gamma} \sqrt{\frac{\gamma p_{\frac{1}{2}}}{v_{\frac{1}{2}}}} \cdot \frac{1 - \frac{P_0}{p_{\frac{1}{2}}}}{\left(1 - \frac{P_0}{p_{\frac{1}{2}}}\right)^{\frac{\gamma-1}{2\gamma}}} .$$

In the case when the rarefaction wave propagates to the left we should have set

$$a_0 = \frac{\gamma - 1}{2\gamma} \sqrt{\frac{\gamma p_{-\frac{1}{2}}}{v_{-\frac{1}{2}}}} \cdot \frac{1 - \frac{P_0}{p_{-\frac{1}{2}}}}{1 - \left(\frac{P_0}{p_{-\frac{1}{2}}}\right)^{\frac{\gamma-1}{2\gamma}}} .$$

In this way we arrive at the result that in order to determine P_0 we must solve by iterations the following system:

$$a_0 = \begin{cases} \sqrt{\dfrac{(\gamma+1) P_0^{(i-1)} + (\gamma - 1) p_{-\frac{1}{2}}}{2 v_{-\frac{1}{2}}}} & \text{for } P_0^{(i-1)} \geqslant p_{-\frac{1}{2}} , \\ \dfrac{\gamma - 1}{2\gamma} \sqrt{\dfrac{\gamma p_{-\frac{1}{2}}}{v_{-\frac{1}{2}}}} \cdot \dfrac{1 - \frac{P_0^{(i-1)}}{p_{-\frac{1}{2}}}}{1 - \left(\frac{P_0^{(i-1)}}{p_{-\frac{1}{2}}}\right)^{\frac{\gamma-1}{2\gamma}}} & \text{for } P_0^{(i-1)} < p_{-\frac{1}{2}} , \end{cases} \tag{8}$$

$$b_0 = \begin{cases} \sqrt{\dfrac{(\gamma+1)P_0^{(i-1)} + (\gamma-1)p_{\frac{1}{2}}}{2v_{\frac{1}{2}}}} & \text{for } P_0^{(i-1)} \geqslant p_{\frac{1}{2}}, \\[2ex] \dfrac{\gamma-1}{2\gamma}\sqrt{\dfrac{\gamma p_{\frac{1}{2}}}{v_{\frac{1}{2}}}} \cdot \dfrac{1-\dfrac{P_0^{(i-1)}}{p_{\frac{1}{2}}}}{1-\left(\dfrac{P_0^{(i-1)}}{p_{\frac{1}{2}}}\right)^{\frac{\gamma-1}{2\gamma}}} & \text{for } P_0^{(i-1)} < p_{\frac{1}{2}}, \end{cases}$$

$$P_0^{(i)} = \varphi\left(P_0^{(i-1)}\right) = \frac{b_0^{(i-1)}p_{-\frac{1}{2}} + a_0^{(i-1)}p_{\frac{1}{2}} + a_0^{(i-1)}b_0^{(i-1)}\left(u_{-\frac{1}{2}} - u_{\frac{1}{2}}\right)}{a_0^{(i-1)} + b_0^{(i-1)}} .$$

After the iterations have converged and we have determined the final values for P_0, a_0, b_0 we find U_0 from the formula

$$U_0 = \frac{a_0 u_{-\frac{1}{2}} + b_0 u_{\frac{1}{2}} + p_{-\frac{1}{2}} - p_{\frac{1}{2}}}{a_0 + b_0} .$$

Detailed investigation of the convergence of the iterations $P_0^{(i)} = \varphi\left(P_0^{(i-1)}\right)$ shows that this process converges if in the resolution of the discontinuity the resulting rarefaction wave is not of excessive strength. In order to make it convergent in all cases, it is necessary to carry it out with somewhat modified formulae, e. g.

$$P_0^{(i)} = \frac{\alpha^{(i-1)}p_0^{(i-1)} + \varphi\left(P_0^{(i-1)}\right)}{\alpha^{(i-1)} + 1} ,$$

where

$$\alpha^{(i-1)} = \begin{cases} \dfrac{\gamma-1}{3\gamma} \cdot \dfrac{1 - z_{i-1}}{z_{i-1}^{\frac{\gamma+1}{2\gamma}}\left(1 - z_{i-1}^{\frac{\gamma-1}{2\gamma}}\right)} - 1, & \text{if this expression exceeds } 0 \\[2ex] 0 & \text{otherwise} \end{cases}$$

$$z_{i-1} = \frac{P_0^{(i-1)}}{p_{\frac{1}{2}} + p_{-\frac{1}{2}}} .$$

We omit the investigation of this convergence because it follows standard methods for investigating the convergence of iteration processes of the type $x^i = f(x^{i-1})$ which reduce to computation and investigation of the complicated expressions for the derivatives.

From formulae (8) it is seen that if u and p converge with diminishing mesh size to bounded continuous functions. Then U and P converge to the same limits. We have already used this fact to prove that the limit functions are generalized solutions of the equations of fluid dynamics.

In Chapter I we derived formulae (7) for the resolution of a discontinuity using sound waves. They agree with the expression of this paragraph if we let

$$a_0 = b_0 \doteq \sqrt{\frac{A}{B}} \,.$$

In the computation with sound waves the time steps had to be limited by the stability condition

$$\tau \leqslant \frac{h}{\sqrt{AB}} = \frac{h}{B\sqrt{\frac{B}{A}}} \,.$$

It seems natural to us to use in the present nonlinear case the following bound on the time step

$$\tau \leqslant \frac{h}{B \max(a_m, b_m)} \,.$$

It is true the above defined τ only approximately equals the time necessary for the waves obtained from the resolution at one integer point to reach the adjacent one and change the values of U and P obtained there after the resolution of the discontinuity. However, a large number of different computations using this condition shows convincingly that with such a bound on τ the computation is stable. In addition, this condition coincides with the one given above for the linear scheme when weak (sound) waves are being computed.

Let us add that if we wished to know the true distribution of quantities u, v, E at the end of time interval τ after the resolution of the discontinuity we could obtain it by solving the elementary problem of gas dynamics inside each layer. (It is only necessary that τ should not exceed the time necessary for the wave from one integer point to reach the adjacent one.)

An especially simple case and the one which always admits closed-form solution is the case when τ is smaller than the time necessary for the waves emitted from the two adjacent points to collide.

As is known from elementary gas dynamics the entropy S which obtains inside the layer after the time interval τ will be for all larger than the initial value: $S(x) > S^n_{\frac{1}{2}}$ (if we examine the layer numbered $\frac{1}{2}$). Recalling that

$$S = c_v \ln E v^{\gamma-1} = c_v \left[\ln E + (\gamma - 1) \ln v\right],$$

and using the following simple inequalities:

$$\frac{\int_{x_1}^{x_2} \ln z(x)\,dx}{x_2 - x_1} \leqslant \ln \frac{\int_{x_1}^{x_2} z(x)\,dx}{x_2 - x_1}$$ follows from convexity of the curve $\ln z$

$$\left(\frac{\int_{x_1}^{x_2} u\,dx}{x_2 - x_1}\right)^2 \leqslant \frac{\int_{x_1}^{x_2} u^2\,dx}{x_2 - x_1}$$ follows from concavity of the curve u^2

we conclude that

$$S^n_{\frac{1}{2}} \leqslant \frac{1}{h}\int_0^h S(x)\,dx \leqslant c_v\left[\ln \frac{1}{h}\int_0^h E(x)\,dx + (\gamma-1) \ln \frac{1}{h}\int_0^h v(x)\,dx\right] \leqslant$$

$$\leqslant c_v\left\{\ln\left[\frac{1}{h}\int_0^h \left(E + \frac{u^2}{2}\right)dx - \frac{1}{2}\left(\frac{1}{h}\int_0^h u\,dx\right)^2\right] + (\gamma-1) \ln \frac{1}{h}\int_0^x v(x)\,dx\right.$$

Now we note that $\frac{1}{h}\int_0^h\left(E+\frac{u^2}{2}\right)dx-\frac{1}{2}\left[\frac{1}{h}\int_0^h u\,dx\right]^2,\ \frac{1}{h}\int_0^h v(x)\,dx$

are the mean values $E, v;$ which in computing by our scheme we assign to the "point" $\frac{1}{2}$ during the time interval τ and denote by $E_{\frac{1}{2}}^{n+1},\ v_{\frac{1}{2}}^{n+1}$
In this way we arrive at the inequality

$$S_{\frac{1}{2}}^{n+1}=S\left(E_{\frac{1}{2}}^{n+1}, v_{\frac{1}{2}}^{n+1}\right)=c_v\left[\ln E_{\frac{1}{2}}^{n+1}+(\gamma-1)\ln v_{\frac{1}{2}}^{n+1}\right]\geq S_{\frac{1}{2}}^{n}. \qquad (9)$$

Applying the reasoning analogous to that used in 3 in the proof of the integral conservation laws we can show, using the fact that at each mesh point the inequality of the type (9) holds, that as the mesh size tends to zero the limit solution satisfies for every closed contour the integral inequality

$$\oint S\,dx \geq 0$$

(condition guaranteeing uniqueness).

Using similar arguments we can show that for the system (1a) also, the corresponding uniqueness condition is satisfied.

§ 5. Computation of Euler coordinates

Usually after solving the system (1)

$$\frac{\partial u}{\partial t}+B\frac{\partial p(v,E)}{\partial x}=0,$$

$$\frac{\partial v}{\partial t}-B\frac{\partial u}{\partial x}=0,$$

$$\frac{\partial\left(E+\frac{u^2}{2}\right)^2}{\partial t}+B\frac{\partial pu}{\partial x}=0$$

one still has to solve the equation for the Euler coordinates of the gas particles

$$\frac{\partial r}{\partial t}=u.$$

We propose to determine r at integer points by the formula

$$r^m=r_m+\tau U_m.$$

It is of interest to note that from the preceding formula and (4) it follows that if initially

$$v_{m+\frac{1}{2}} = \frac{B}{h}\left(r_{m+1} - r_m\right), \tag{10}$$

then this equation will be satisfied for all later times. It means that the volume of the gas layer can be determined from the knowledge of its boundaries.

Equation (10) can be used to determine v in place of the second of formulae (4). When computations are carried out on an electronic computer it is more convenient to use formula (10) because it decreases the number of quantities to be stored at each step.

§ 6. Some results of numerical computations

In Application II we list the results of computation of a steady-state travelling shock wave carried out using formulae (4) and (8). It is seen from the curves that if the computation is begun from a step function which satisfies the shock conditions at the point O, then after several time steps the profile of each quantity settles down to a steady-state shape which propagates in time with the velocity equal to the velocity of the shock wave with the same jumps in pressures and velocities. Only near the point O there remains a bump in the curve of v. This is explained by the fact that in the process of reaching the steady state the scheme "erred" in entropy, which is conserved in the smooth region behind the front of the wave. In the smooth region the scheme is sufficiently accurate to show this conservation of entropy. After the steady state is reached the pressure behind the front of the wave equalizes and since pv^{γ} is not correct there this leads to the appearance of the bump on the curve of v.

Analogous entropy traces remain also after the computation of other unsteady processes, for example the process of formation of a shock wave during the impact of a moving gas onto a rigid wall (see Application III). These entropy traces usually cover two or three mesh points and therefore do not influence the results of computation for a sufficiently small mesh size.

§ 7. A certain effect obtained in the computation of contact discontinuities

All considerations which we adduced in arriving at our scheme were obtained by considering the case of constant x-steps and with the assumption that the entire computational process occurs in an infinite gas; however, the numerical scheme obtained has such a clear physical meaning that it is difficult to resist the desire to apply it also at the boundaries between two media -- contact discontinuities. For this it is sufficient to include the contact discontinuity among the number of integer points and in computing a and b at this point use for a, constants characterizing the gas located to the left of the separation line and for b, constants referring to the gas located to the right.

The results of our computations show that the application of the scheme so constructed on the contact discontinuity is allowable, but, as is not difficult to verify, it leads to a decrease in accuracy.

In this paragraph we wish to describe one effect which is a consequence of the decrease in accuracy and which was observed during an analysis of computations near contact discontinuities. This effect appeared in the computation of smooth solutions, it bears no relation to the shock waves and therefore it is natural to attempt to explain it starting with the assumption that our system of equations can be approximated by a linear system. Computations based on such a linearized system of equations yielded the magnitude of the effect, which agreed with the one observed in computation of gas dynamical problems.

Suppose processes in a certain gas are described by the system

$$\begin{aligned} \frac{\partial u}{\partial t} + B\frac{\partial p}{\partial x} &= 0, \\ \frac{\partial v}{\partial t} - B\frac{\partial u}{\partial x} &= 0. \end{aligned} \tag{11}$$

The equation of state of a gas in case of small variations in pressure admits the following linearized representation:

$$p = p_0 - \frac{A}{B}(v - v_0).$$

Using this equation of state the system (11) can be rewritten

$$\frac{\partial u}{\partial t} + B\frac{\partial p}{\partial x} = 0,$$

$$\frac{\partial p}{\partial t} + A\frac{\partial u}{\partial x} = 0.$$

Let the contact discontinuity be at $x = 0$, i.e., let the coefficients be different for $x > 0$ than for $x < 0$. Set

$$A = \begin{cases} A_+ \text{ for } x > 0, \\ A_- \text{ for } x < 0, \end{cases}$$

$$B = \begin{cases} B_+ \text{ for } x > 0. \\ B_- \text{ for } x < 0, \end{cases}$$

The system of equations

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + B_+\frac{\partial p}{\partial x} = 0, \\ &\frac{\partial p}{\partial t} + A_+\frac{\partial u}{\partial x} = 0 \end{aligned}\right\} \text{ for } x > 0$$

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + B_-\frac{\partial p}{\partial y} = 0, \\ &\frac{\partial p}{\partial t} + A_-\frac{\partial u}{\partial x} = 0 \end{aligned}\right\} \text{ for } x < 0$$

with the continuity condition on u and p at $x = 0$ admits the following solution

$$\left.\begin{aligned} u &= \frac{\alpha}{A_+}x - \beta t + \gamma, \\ p &= \frac{\beta}{B_+}x - \alpha t + \pi \end{aligned}\right\} \quad \text{for } x > 0,$$

$$\left.\begin{aligned} u &= \frac{\alpha}{A_-}x - \beta t + \gamma, \\ p &= \frac{\beta}{B_-}x - \alpha t + \pi \end{aligned}\right\} \quad \text{for } x < 0, \tag{12}$$

which at $t = 0$ satisfies the following initial conditions:

$$\left.\begin{aligned} u &= \frac{\alpha}{A_+}x + \gamma, \\ p &= \frac{\beta}{B_+}x + \pi \end{aligned}\right\} \quad \text{for } x > 0,$$

$$\left.\begin{aligned} u &= \frac{\alpha}{A_-}x + \gamma, \\ p &= \frac{\beta}{B_-}x + \pi \end{aligned}\right\} \quad \text{for } x < 0. \tag{13}$$

We shall investigate what the solution of the difference equations is for the same initial data. It is more interesting because any smooth solution of our system near the point $x = 0$, $t = t_0$ can be represented in the form

$$u = \frac{\alpha}{A_+} x - \beta (t - t_0) + \gamma + O\left[x^2 + (t - t_0)^2\right],$$

for $x > 0$,

$$p = \frac{\beta}{B_+} x - \alpha (t - t_0) + \pi + O\left[x^2 + (t - t_0)^2\right]$$

$$u = \frac{\alpha}{A_-} x - \beta (t - t_0) + \gamma + O\left[x^2 + (t - t_0)^2\right],$$

for $x < 0$.

$$p = \frac{\beta}{B_-} x - \alpha (t - t_0) + \pi + O\left[x^2 + (t - t_0)^2\right]$$

Therefore the behavior of the difference solution near $x = 0$ will characterize the behavior near the contact discontinuity of the quantities obtained as the result of numerican computation of any smooth solution of our system.

We begin by giving the explicit expressions for the difference scheme for the present case (we have explained at the beginning of this paragraph how to obtain these formulae). We will assume that the step h equal to the difference of x coordinates of the two neighboring integer points, may be different in regions to the right and left of $x = 0$. Namely, for $x > 0$ $h = h_+$ and for $x < 0$ $h = h_-$. The computation formulae are:

$$
\left.
\begin{aligned}
P_m &= \frac{p_{m-\frac{1}{2}} + p_{m+\frac{1}{2}}}{2} - \sqrt{\frac{A_+}{B_+}} \cdot \frac{u_{m+\frac{1}{2}} - u_{m-\frac{1}{2}}}{2}, \\
U_m &= \frac{u_{m-\frac{1}{2}} + u_{m+\frac{1}{2}}}{2} - \frac{p_{m+\frac{1}{2}} - p_{m-\frac{1}{2}}}{2\sqrt{\frac{A_+}{B_+}}}
\end{aligned}
\right\} \text{for } m > 0,
$$

$$
\left.
\begin{aligned}
P_m &= \frac{p_{m-\frac{1}{2}} + p_{m+\frac{1}{2}}}{2} - \sqrt{\frac{A_-}{B_-}} \cdot \frac{u_{m+\frac{1}{2}} - u_{m-\frac{1}{2}}}{2}, \\
U_m &= \frac{u_{m-\frac{1}{2}} + u_{m+\frac{1}{2}}}{2} - \frac{p_{m+\frac{1}{2}} - p_{m-\frac{1}{2}}}{2\sqrt{\frac{A_-}{B_-}}}
\end{aligned}
\right\} \text{for } m < 0,
$$

$$
P_0 = \frac{\sqrt{\frac{A_+}{B_+}}\, p_{-\frac{1}{2}} + \sqrt{\frac{A_-}{B_-}}\, p_{\frac{1}{2}} + \sqrt{\frac{A_+}{B_+}}\sqrt{\frac{A_-}{B_-}}\left(u_{-\frac{1}{2}} - u_{\frac{1}{2}}\right)}{\sqrt{\frac{A_+}{B_+}} + \sqrt{\frac{A_-}{B_-}}},
$$

$$
U_0 = \frac{\sqrt{\frac{A_-}{B_-}} \cdot u_{-\frac{1}{2}} + \sqrt{\frac{A_+}{B_+}} \cdot u_{\frac{1}{2}} + p_{-\frac{1}{2}} - p_{\frac{1}{2}}}{\sqrt{\frac{A_+}{B_+}} + \sqrt{\frac{A_-}{B_-}}},
$$

$$
\left.
\begin{aligned}
u^{m+\frac{1}{2}} &= u_{m+\frac{1}{2}} - \frac{\tau B_+}{h_+}\left(P_{m+1} - P_m\right), \\
p^{m+\frac{1}{2}} &= p_{m+\frac{1}{2}} - \frac{\tau A_+}{h_+}\left(U_{m+1} - U_m\right)
\end{aligned}
\right\} \text{for } m \geqslant 0,
$$

$$
\left.
\begin{aligned}
u^{m+\frac{1}{2}} &= u_{m+\frac{1}{2}} - \frac{\tau B_-}{h_-}\left(P_{m+1} - P_m\right), \\
p^{m+\frac{1}{2}} &= p_{m+\frac{1}{2}} - \frac{\tau A_-}{h_-}\left(U_{m+1} - U_m\right)
\end{aligned}
\right\} \text{for } m < 0. \qquad (14)
$$

For our difference equations one can find a solution which, like (12), is a linear function of x and t in each of the regions $x > 0$ and $x < 0$ and has in these regions gradients identical with (12). Namely, it turns out that such solution will be

$$\left.\begin{aligned} u &= \frac{\alpha}{A_+} x - \beta t + \frac{1}{2} \cdot \frac{\beta h_+}{\sqrt{A_+ B_+}} + \delta, \\ p &= \frac{\beta}{B_+} x - \alpha t + \frac{1}{2} \cdot \frac{\alpha h_+}{\sqrt{A_+ B_+}} + \theta \end{aligned}\right\} \text{ for } x > 0,$$

$$\left.\begin{aligned} u &= \frac{\alpha}{A_-} x - \beta t + \frac{1}{2} \cdot \frac{\beta h_-}{\sqrt{A_- B_-}} + \delta, \\ p &= \frac{\beta}{B_-} x - \alpha t + \frac{1}{2} \cdot \frac{\alpha h_-}{\sqrt{A_- B_-}} + \theta \end{aligned}\right\} \text{ for } x < 0. \qquad (15)$$

We leave to the reader the completely elementary verification of this fact. To compute by these formulae $u^{m+\frac{1}{2}}$ and $p^{m+\frac{1}{2}}$ for an arbitrary, e.g. n^{th}, time step it is necessary to let $t = h\tau$, $x = (m+\frac{1}{2})h$

If we begin the computation by our scheme from the initial conditions (13) then experimental computations show that near the contact discontinuity the solution of the difference equations (15) tends to a steady state with certain δ and θ obtained in the process.

If we compute the values of u and p by formulae (15) at $x = 0$ we will see that these quantities assume at this point different values to the left and right; differences between them are:

$$[u] = \frac{1}{2} \beta \left(\frac{h_+}{\sqrt{A_+ B_+}} - \frac{h_-}{\sqrt{A_- B_-}} \right),$$

$$[p] = \frac{1}{2} \alpha \left(\frac{h_+}{\sqrt{A_+ B_+}} - \frac{h_-}{\sqrt{A_- B_-}} \right).$$

...ed, when solving the difference equations the values of u and p are computed only at half-integer points $\left(-\frac{3}{2}h_-, -\frac{1}{2}h_-, \frac{1}{2}h_+, \frac{3}{2}h_+, \ldots\right)$, and therefore on the contact discontinuity we do not find any real discontinuities in pressure and velocity. If, however, the pressure and velocity are linearly extrapolated to the point $x = 0$ we obtain exactly the values computed at $x = 0$ by formulae (15).

From what we have said so far it follows that the values of pressures and velocities extrapolated from the right and left will in general differ on the contact discontinuity and the differences between them will be determined from the formulae

$$u_{r,\,extrap} - u_{l,\,extrap.} = \frac{1}{2}\beta\left(\frac{h_+}{\sqrt{A_+B_+}} - \frac{h_-}{\sqrt{A_-B_-}}\right),$$

$$p_{r,\,extrap.} - p_{l,\,extrap.} = \frac{1}{2}\alpha\left(\frac{h_+}{\sqrt{A_+B_+}} - \frac{h_-}{\sqrt{A_-B_-}}\right). \qquad (16)$$

This disagreement of velocities and pressures on the contact discontinuity is especially noticeable on the graphs of u and p and obviously characterizes the inaccuracy of our scheme. Indeed, if our scheme were exact for the linear functions it would compute solution (12) exactly and we would not observe any discrepancies in u and p.

In order to counteract this effect we should as one can see from (16) choose the steps h such that as closely as possible

$$\frac{h_+}{\sqrt{A_+B_+}} \approx \frac{h_-}{\sqrt{A_-B_-}}.$$

We have already explained that $h/\sqrt{AB}$ represents the largest allowable time step which does not violate the stability of the difference scheme. Thus we should attempt to choose the space steps in such a way that the largest time steps consistent with the stability requirement are if possible equal or approximately equal for the gases on both sides of the contact discontinuity.

It is not possible to satisfy this condition exactly for nonlinear problems of gas dynamics because the speed of sound which determines the time step is different at different stages* of the problem.

If in computations we succeeded in choosing the steps in different regions in such a way that the above condition was not strongly violated, the effect being studied on the interior boundaries was almost absent which signified an increase in accuracy. The greater accuracy in those cases was also noted by comparison with the method of characteristics.

§ 8. The stability of our difference scheme on the contact discontinuities

In the preceeding paragraph we have given formulae by which one can compute solutions to our equations near a contact discontinuity. Now we shall investigate the stability of these formulae. This investigation will be carried out on the difference scheme for the linear system.

$$\frac{\partial u}{\partial t} + B\frac{\partial p}{\partial x} = 0,$$

$$\frac{\partial p}{\partial t} + A\frac{\partial u}{\partial x} = 0$$

with coefficients A and B which are constant in each of the regions $x > 0$ or $x < 0$ (in the preceding paragraph we studied the computational phenomenon described there by considering just such a system).

After A. F. Filippov (see (4)) by stability we will mean uniformly continuous dependence (with decreasing mesh size) of solutions to the difference equations on their right hand sides and on the initial data.

In order to prove stability it suffices to define for the solutions of difference equations a norm which in the limit as the mesh size tends to zero goes over into a certain norm for the solutions of differential equations such that

$$\|\vec{u}^{n+1}\| \leqslant \|\vec{u}^{n}\|.$$

---

* We have determined in § 4 of Chapter II that the admissible time step in the solution of gasdynamical problems is $\tau = h/B_{max}(a_m, b_m)$. For the smooth solutions considered in this section and sufficiently small h, a_m, and b_m equal the convected velocity of sound.

By $\vec{u}^n$ we understand here an infinitely dimensional vector defined by the values of the solution $(u^n_{m+\frac{1}{2}},\ p^n_{m+\frac{1}{2}})$ to the difference equations for the n^{th} time step.

We shall assume that the time step is chosen by the stability conditions inside of each region. As we noted earlier this implies that the following inequalities are satisfied

$$r_+ = \frac{\tau\sqrt{A_+B_+}}{h_+} \leqslant 1, \quad r_- = \frac{\tau\sqrt{A_-B_-}}{h_-} \leqslant 1.$$

We introduce the notation

$$q^n_{m+\frac{1}{2}} = u^n_{m+\frac{1}{2}} + \frac{p^n_{m+\frac{1}{2}}}{\sqrt{\frac{A}{B}}}, \quad s^n_{m+\frac{1}{2}} = u^n_{m+\frac{1}{2}} - \frac{p^n_{m-\frac{1}{2}}}{\sqrt{\frac{A}{B}}}.$$

From (14) of the preceeding paragraph one can conclude without difficulty that the following equalities are satisfied

$$q^{n+1}_{m+\frac{1}{2}} = (1-r_+)q^n_{m+\frac{1}{2}} + r_+ q^n_{m-\frac{1}{2}} \quad \text{for } m \geqslant 1,$$

$$q^{n+1}_{\frac{1}{2}} = (1-r_+)q^n_{\frac{1}{2}} + r_+\frac{2\sqrt{\frac{A_-}{B_-}}}{\sqrt{\frac{A_+}{B_+}}+\sqrt{\frac{A_-}{B_-}}}\cdot q^n_{-\frac{1}{2}} + r_+\frac{\sqrt{\frac{A_+}{B_+}}-\sqrt{\frac{A_-}{B_-}}}{\sqrt{\frac{A_+}{B_+}}+\sqrt{\frac{A_-}{B_-}}}\cdot s^n_{\frac{1}{2}},$$

$$q^{n+1}_{m+\frac{1}{2}} = (1-r_-)q^n_{m+\frac{1}{2}} + r_- q^n_{m-\frac{1}{2}} \quad \text{for } m \leqslant -1,$$

$$s^{n+1}_{m+\frac{1}{2}} = (1-r_+)s^n_{m+\frac{1}{2}} + r_+ s^n_{m+\frac{3}{2}} \quad \text{for } m \geqslant 0,$$

$$s^{n+1}_{-\frac{1}{2}} = (1-r_-)s^n_{-\frac{1}{2}} + r_-\frac{2\sqrt{\frac{A_+}{B_+}}}{\sqrt{\frac{A_+}{B_+}}+\sqrt{\frac{A_-}{B_-}}}s^n_{\frac{1}{2}} + r_-\frac{\sqrt{\frac{A_-}{B_-}}-\sqrt{\frac{A_+}{B_+}}}{\sqrt{\frac{A_+}{B_+}}+\sqrt{\frac{A_-}{B_-}}}q^n_{-\frac{1}{2}},$$

$$s^{n+1}_{m+\frac{1}{2}} = (1-r_-)s^n_{m+\frac{1}{2}} + r_- s^n_{m+\frac{3}{2}} \quad \text{for } m \leqslant -2.$$

All subsequent considerations will be carried out with the assumption that $\sqrt{A_+/B_+} > \sqrt{A_-/B_-}$. We leave to the reader the entirely analogous considerations in the case when $\sqrt{A_+/B_+} < \sqrt{A_-/B_-}$

Set

$$\bar{q}^k_{m+\frac{1}{2}} = q^k_{m+\frac{1}{2}} \quad \text{for all } m,$$

$$\bar{s}^k_{m+\frac{1}{2}} = s^k_{m+\frac{1}{2}} \quad \text{for } m \geqslant 0,$$

$$\bar{s}_{m+\frac{1}{2}} = \frac{\sqrt{\frac{A_+}{B_+}} + \sqrt{\frac{A_-}{B_-}}}{3\sqrt{\frac{A_+}{B_+}} - \sqrt{\frac{A_-}{B_-}}} \cdot s^k_{m+\frac{1}{2}} \quad \text{for } m < 0.$$

From the expressions for $q^{n+1}_{m+\frac{1}{2}}$ and $s^{n+1}_{m+\frac{1}{2}}$ analogous formulae for $\bar{q}^{n+1}_{m+\frac{1}{2}}$ and $\bar{s}^{n+1}_{m+\frac{1}{2}}$ follow. We list them below

$$\bar{q}^{n+1}_{m+\frac{1}{2}} = (1-r_+)\,\bar{q}^n_{m+\frac{1}{2}} + r_+\,\bar{q}^n_{m-\frac{1}{2}} \quad \text{for } m \geqslant 1,$$

$$\bar{q}^{n+1}_{\frac{1}{2}} = (1-r_+)\,\bar{q}^n_{\frac{1}{2}} + r_+ \frac{2\sqrt{\frac{A_-}{B_-}}}{\sqrt{\frac{A_+}{B_+}} + \sqrt{\frac{A_-}{B_-}}}\,\bar{q}^n_{-\frac{1}{2}} + r_+ \frac{\sqrt{\frac{A_+}{B_+}} - \sqrt{\frac{A_-}{B_-}}}{\sqrt{\frac{A_+}{B_+}} + \sqrt{\frac{A_-}{B_-}}}\,\bar{s}^n_{\frac{1}{2}},$$

$$\bar{q}^{n+1}_{m+\frac{1}{2}} = (1-r_-)\,\bar{q}^n_{m+\frac{1}{2}} + r_-\,\bar{q}^n_{m-\frac{1}{2}} \quad \text{for } m \leqslant -1,$$

$$\bar{s}^{n+1}_{m+\frac{1}{2}} = (1-r_+)\,\bar{s}^n_{m+\frac{1}{2}} + r_+\,\bar{s}^n_{m+\frac{3}{2}} \quad \text{for } m \geqslant 0,$$

$$\bar{s}^{n+1}_{-\frac{1}{2}} = (1-r_-)\,\bar{s}^n_{-\frac{1}{2}} + r_- \frac{2\sqrt{\frac{A_+}{B_+}}}{3\sqrt{\frac{A_+}{B_+}} - \sqrt{\frac{A_-}{B_-}}}\,\bar{s}^n_{\frac{1}{2}} - r_- \frac{\sqrt{\frac{A_+}{B_+}} - \sqrt{\frac{A_-}{B_-}}}{3\sqrt{\frac{A_+}{B_+}} - \sqrt{\frac{A_-}{B_-}}}\,\bar{q}^n_{-\frac{1}{2}},$$

$$\bar{s}^{n+1}_{m+\frac{1}{2}} = (1-r_-)\,\bar{s}^n_{m+\frac{1}{2}} + r_-\,\bar{s}^n_{m+\frac{3}{2}} \quad \text{for } m \leqslant -2.$$

Since the sum of the absolute values of the coefficients of $\bar{g}$ and $\bar{s}$ in the right hand side of each of these equations equals unity (recall that k_+ and k_- are less than 1) then

$$\max_m \left(\left| \bar{g}^{n+1}_{m+\frac{1}{2}} \right|, \left| \bar{s}^{n+1}_{m+\frac{1}{2}} \right| \right) \leqslant \max_m \left(\left| \bar{g}^{n}_{m+\frac{1}{2}} \right|, \left| \bar{s}^{n}_{m+\frac{1}{2}} \right| \right).$$

This inequality proves stability if we choose for the norm

$$\| \vec{u}^n \| = \max_m \left(\left| \bar{g}^{n}_{m+\frac{1}{2}} \right|, \left| \bar{s}^{n}_{m+\frac{1}{2}} \right| \right).$$

Thus established stability of the difference scheme for the linear system may serve as some kind of a justification for its application in the case of a nonlinear system. In addition, let us state once more that in all the numerous computations, using our scheme, carried out with consideration of our bound on the time step, the computations were always stable.

Application I

Below (Figure 2) is the graph of pressure in the steady-state shock wave for the system

$$\frac{\partial u}{\partial t} + \frac{\partial p(v)}{\partial x} = 0,$$

$$\frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} = 0,$$

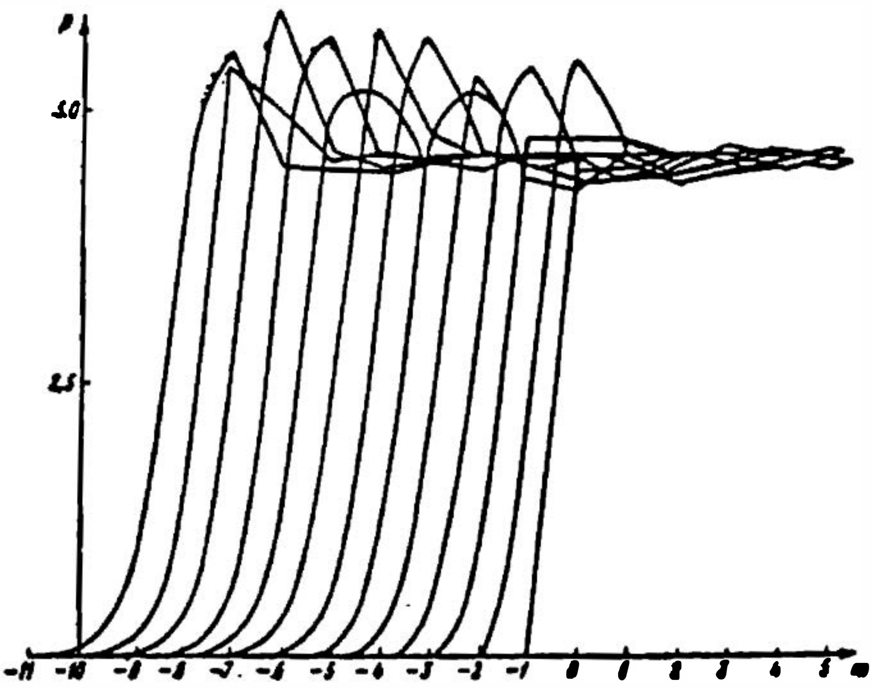

Figure 2

computed with the scheme of second order accuracy

$$u^0 = u_0 - \frac{\tau}{2h}(p_1 - p_{-1}) + \frac{\tau^2}{2h^2}\left[\frac{A_1 + A_0}{2}(u_1 - u_0) - \frac{A_0 + A_{-1}}{2}(u_0 - u_{-1})\right],$$

$$v^0 = v_0 + \frac{\tau}{2h}(u_1 - u_{-1}) + \frac{\tau^2}{2h^2}(p_1 - 2p_0 + p_{-1}),$$

where

$$A = -\frac{dp}{dv}.$$

Application II

Computation for the steady-state shock wave for the gas with $\gamma = 5/3$. Numbers near curves (Figures 3, 4, 5) indicate the layer number.

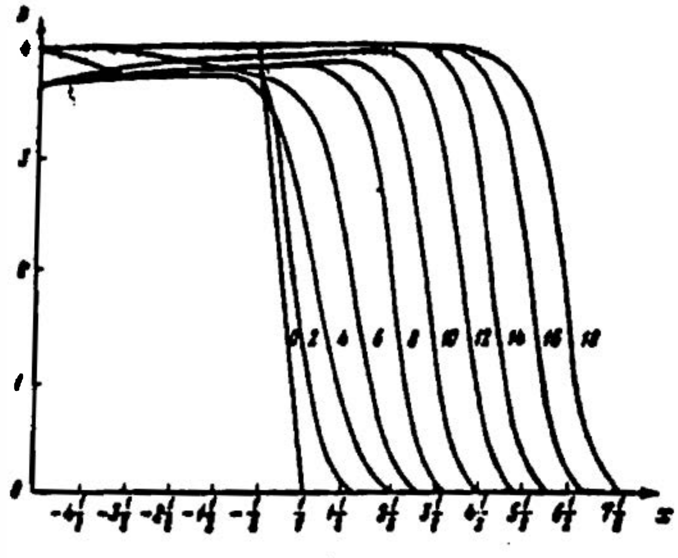

Figure 3

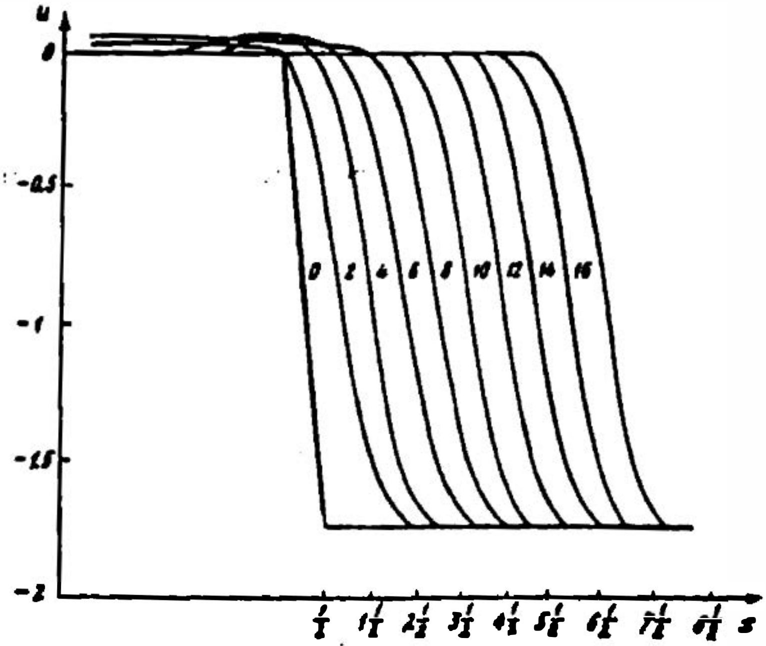

Figure 4

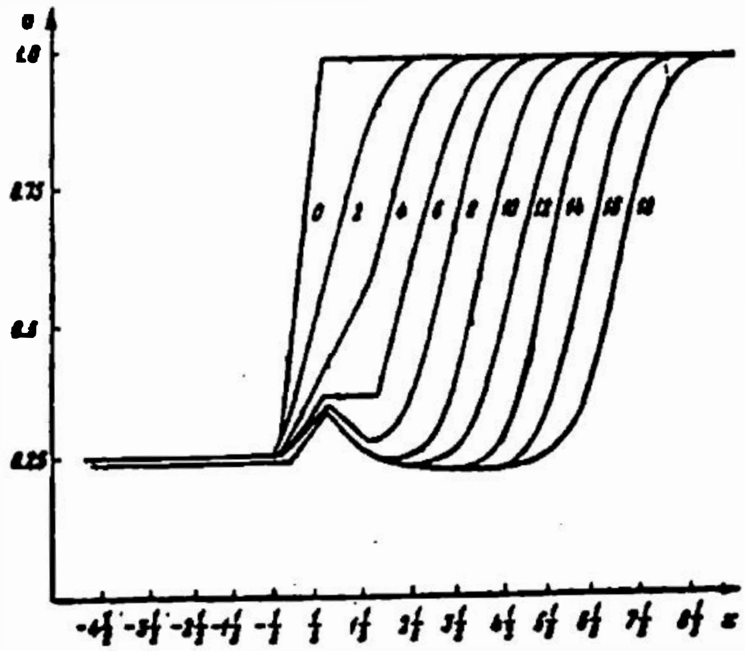

Figure 5

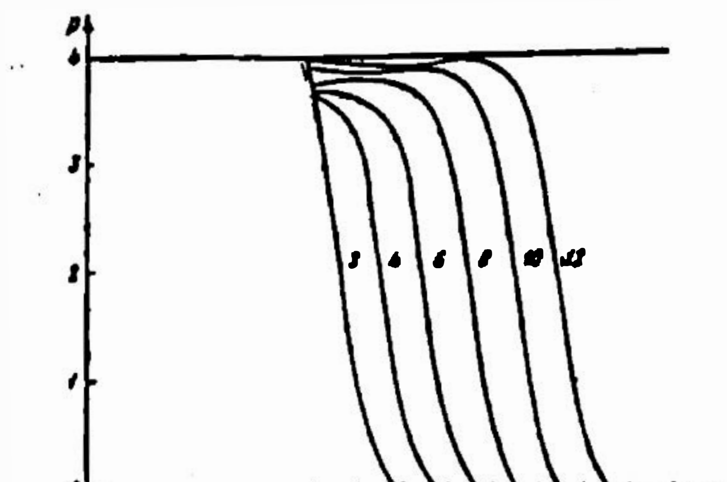

Figure 6

Application III

The impact of an absolutely cold gas ($\gamma = 5/3$) moving into a wall. At the initial moment we prescribed $v = 1$, $p = 0$ for $x > 0$. For $x = 0$ we specified the boundary condition $U = 0$ which was taken into account in computations of resolutions of discontinuities at this point.

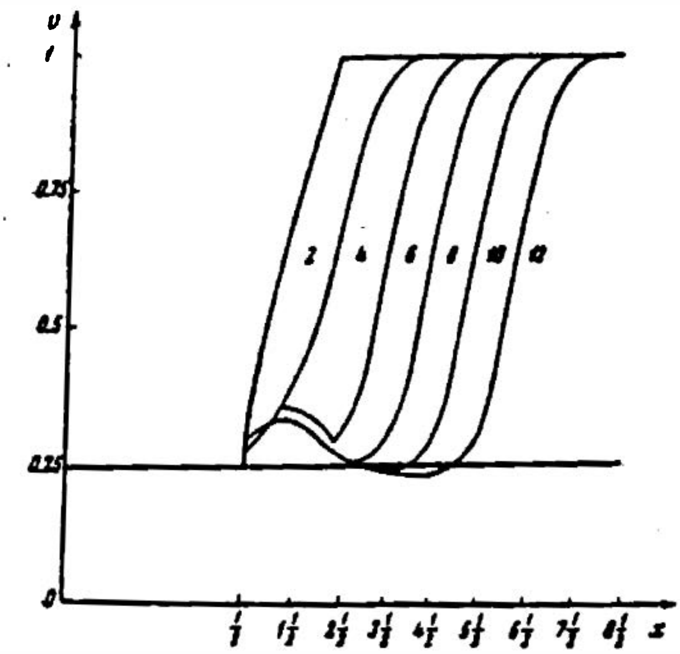

Figure 7

On the curves (Figures 6, 7) now the right travelling shock wave forms.

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