Project 1

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Introdution

The main objective of this project is to use Monte Carlo simulation to investigate whether the empirical $Type\ I$ error rate of the one sample $t - test$ is approximately equal to the nominal significance level α, when the sampled population is non-normal. The $t - test$ is robust to mild departures from normality.

We tested this for the cases when the sample is (i) $\chi^2(1)$, (ii) $Uniform(0, 2)$, and (iii) $Exponential(rate = 1)$.

In each case, we tested $H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$, where μ_0 is the mean of $\chi^2(1)$, $Uniform(0, 2)$, and $Exponential(1)$, respectively.

Definitions

Null hypothesis (H_0): $\theta \in \theta_0$ the status quo or that nothing interesting is happening.

Alternative hypothesis(H_a or H_1): $\theta \in \theta_1$ the claim for which we seek significance evidence.

One sample t-test:
A t-test is used when the population parameters (mean and standard deviation) are not known. One sample t-test which tests the mean of a single group against a known mean. If the sample distribution is known we can compare the sample mean against the theoretical mean.

Empirical Type I Error: A Type I error occurs if the null hypothesis is rejected when in fact the null hypothesis is true. The probability of Type I error is the conditional probability that the null hypothesis is rejected given that H_0 is true. Thus, if the test procedure is replicated a large number of times under the conditions of the null hypothesis, the observed Type I error rate should be at most (approximately) α.

An empirical Type I error rate can be computed by a Monte Carlo experiment. The test procedure is replicated a large number of times under the conditions of the null hypothesis. The empirical Type I error rate for the Monte Carlo experiment is the sample proportion of significant test statistics among the replicates.

Monte Carlo experiment to asses Type I error rate:

1. For each replicate, indexed by $j = 1, ..., m$:
   (a) Generate the j^{th} random sample $x_1^{(j)}, ..., x_n^{(j)}$ from the null distribution.
   (b) Compute the test statistic T_j from the j^{th} sample.
   (c) Record the test decision $I_j = 1$ if H_0 is rejected at significance level α and otherwise $I_j = 0$.
2. Compute the proportion of significant tests $\frac{1}{m} \sum_{j=1}^{m} I_j$. This proportion is the observed Type I error rate.

Monte Carlo Simulation

First, I used the Monte Carlo Simulation for $10,000$ replicates of samples of size 20 with significance level $\alpha = 0.05$ for the three different distributions.

With this, I performed a one sample t-test for each one and then recorded the sample mean, the p-value, and the test result I_j (1 if H_0 was rejected, 0 otherwise).

The sample means to make sure that the observed mean of the $10,000$ replicates' sample means $\hat{\mu} \approx \mu$. Where μ is the theoretical mean of each distribution.

The p-value is then used to decide whether or not to accept or reject the null Hypothesis H_0. If $p-value < \alpha$, then we reject H_0, else we accept H_0. This means that if the p-value is small enough, we can say, with a certain degree of confidence, that the sample mean $\mu \neq 1$.

I then calculated the percentage of rejection of the null hypothesis H_0 by counting the number of replicates that rejected H_0 and dividing that number by the total number of replicates.

Results

Distribution	$\hat{\mu}$	Percentage of Rejection of H_0
$\chi^2(1)$	0.9976	10.74%
$Uniform(0,2)$	0.9999	5.15%
$Exponential(1)$	0.999	8.01%

A t-test works best when sampling from a Normal distribution, however, this test is robust enough for distributions that deviate from it. The less they deviate, the better it works. After performing the analysis, I found that the percentage rejection rate or the Empirical Type I error for the $Uniform(0,2)$ was the lowest, meaning that this specific test works better for this distribution in comparison to the $\chi^2(1)$ and the $Exponential(1)$.

By this analysis we can conclude that the t-test is a better fit for the $Uniform(0,2)$, then for the $Exponential(1)$, and finally for the $\chi^2(1)$. The empirical Type I Error means that any sample assigned from each of these distributions has a 10.74%, 5.15% , and 8.01% of being wrong, respectively.

Appendix 1 - R Code.

```r
#Set Seed for identical reproduction.
set.seed(110421)

#Assumptions
n <- 20 #sample size
m <- 10000 #number of replicates for MC simulation
alpha <- 0.05 #significance level
mu0 <- 1 #theoretical mean of the three distributions

#Empirical Type I Error Rate for Chisquare(1)
df1 <- data.frame(matrix(ncol = 3, nrow = m))
colnames(df1) <- c('Sample Mean', 'p-value', 'Rejects H0')
for (j in 1:m){
  x <- rchisq(n, 1)
  df1[j,1] <- mean(x)
  ttest <- t.test(x, alternative = "two.sided", mu=mu0, conf.level = 1-alpha)
  df1[j,2] <- ttest$p.value
  if (ttest$p.value < alpha) df1[j,3] = 1 else df1[j,3] = 0
}
p.hat1 <- sum(df1[,3])/m #percentage of rejection of H0
mu.hat1 <- round(mean(df1[,1]),4) #mean of sample means


#Empirical Type I Error Rate for Uniform(0,2)
df2 <- data.frame(matrix(ncol = 3, nrow = m))
colnames(df2) <- c('Sample Mean', 'p-value', 'Rejects H0')
for (j in 1:m){
  x <- runif(n, 0, 2)
  df2[j,1] <- mean(x)
  ttest <- t.test(x, alternative = "two.sided", mu=mu0, conf.level = 1-alpha)
  df2[j,2] <- ttest$p.value
  if (ttest$p.value < alpha) df2[j,3] = 1 else df2[j,3] = 0
}
p.hat2 <- sum(df2[,3])/m #percentage of rejection of H0
mu.hat2 <- round(mean(df2[,1]),4) #mean of sample means

#Empirical Type I Error Rate for Exponential(1)
df3 <- data.frame(matrix(ncol = 3, nrow = m))
colnames(df3) <- c('Sample Mean', 'p-value', 'Rejects H0')
for (j in 1:m){
  x <- rexp(n, 1)
  df3[j,1] <- mean(x)
  ttest <- t.test(x, alternative = "two.sided", mu=mu0, conf.level = 1-alpha)
  df3[j,2] <- ttest$p.value
  if (ttest$p.value < alpha) df3[j,3] = 1 else df3[j,3] = 0
}
p.hat3 <- sum(df3[,3])/m #percentage of rejection of H0
mu.hat3 <- round(mean(df3[,1]),4) #mean of sample means
```