Homework 6

10/14/21

Question 1

Compute a Monte Carlo estimate $\hat{\theta}$ of

$$\theta = \int_0^{0.5} e^{-x} dx$$

by sampling from Uniform(0, 0.5). Find another Monte Carlo estimator θ^* by sampling from the exponential distribution. Use simulations to estimate the variance of $\hat{\theta}$ and $\hat{\theta}^*$, which estimator has smaller variance?

```r
g1 <- function(x){
  exp(-x)*I(x>=0)*I(x<=0.5)
}
n1 <- 10000

#Sampling from Uniform (0,0.5)
u1 <- runif(n1, 0, 0.5)
gx1 <- g1(u1)
theta.hat1 <- 0.5*mean(gx1)

#Sampling from Exponential
e1 <- rexp(n1)
fgx1 <- g1(e1)/exp(-e1)
theta.star1 <- mean(fgx1)

#Simulations for Variance
theta1_1 <- integer(n1)
theta1_2 <- integer(n1)
for(i in 1:n1){
  U <- runif(n1, 0, 0.5)
  Y1_1 <- g1(U)
  theta1_1[i] <- mean(Y1_1)*0.5

  E <- rexp(n1)
  Y1_2 <- g1(E)/exp(-E)
  theta1_2[i] <- mean(Y1_2)
}

v1_1 <- var(theta1_1)
v1_2 <- var(theta1_2)
```

-Ans

$\hat{\theta}$	$\theta*$	Var($\hat{\theta}$)	Var($\theta*$)
0.3932406	0.3856	3.2224673×10^{-7}	2.3783509×10^{-5}

The variance for $\hat{\theta}$ is smaller than the variance for $\theta*$.

Question 2

Use a Monte Carlo simulation to estimate

$$\theta = \int_0^1 e^x dx$$

by the antithetic variate approach and by the simple Monte Carlo method. Compute an empirical estimate of the percent reduction in variance using the antithetic variate.

```
n2 <- 10000
theta2_1 <- integer(n2)
theta2_2 <- integer(n2)
for (i in 1:n2){
  #Basic MC method
  U1 <- runif(n2)
  theta2_1[i] <- mean(exp(U1))

  #Antithetic method
  U2 <- runif(n2/2)
  Y1 <- exp(U2)
  Y2 <- exp(1-U2)
  theta2_2[i] <- mean((Y1+Y2)/2)
}
```

-Ans

Estimate	MC Method	Antithetic Method
$\hat{\theta}$	1.7182217	1.7182653
$Var(\hat{\theta})$	2.376288×10^{-5}	7.9839246×10^{-7}

Percent reduction in Variance is 96.6401696%.

Question 3 - 6.10

Use Monte Carlo integration with antithetic variables to estimate

$$\int_0^1 \frac{e^{-x}}{1+x^2} dx$$

and find the approximate reduction in variance as a percentage of the variance without variance reduction.

```
n3 <- 10000
theta3 <- numeric(n3)
for (i in 1:n3){
  #Antithetic method
  U3 <- runif(n3/2)
  Y1 <- exp(-U3)/(1+U3^2)
  Y2 <- exp(-(1-U3)/(1+(1-U3)^2))
  theta3[i] <- mean((Y1+Y2)/2)
}
```

Approximate percent reduction in Variance is 99.990001%.

Quesiton 4 - 6.13

Find two importance functions f_1 and f_2 that are supported on $(1, \infty)$ and are "close" to

$$g(x) = \frac{x^2}{\sqrt{2\pi}} e^{-x^2/2}, \qquad x > 1$$

Which of your two importance functions should produce the smaller variance in estimating

$$\int_1^\infty \frac{x^2}{\sqrt{2\pi}} e^{-x^2/2} dx$$

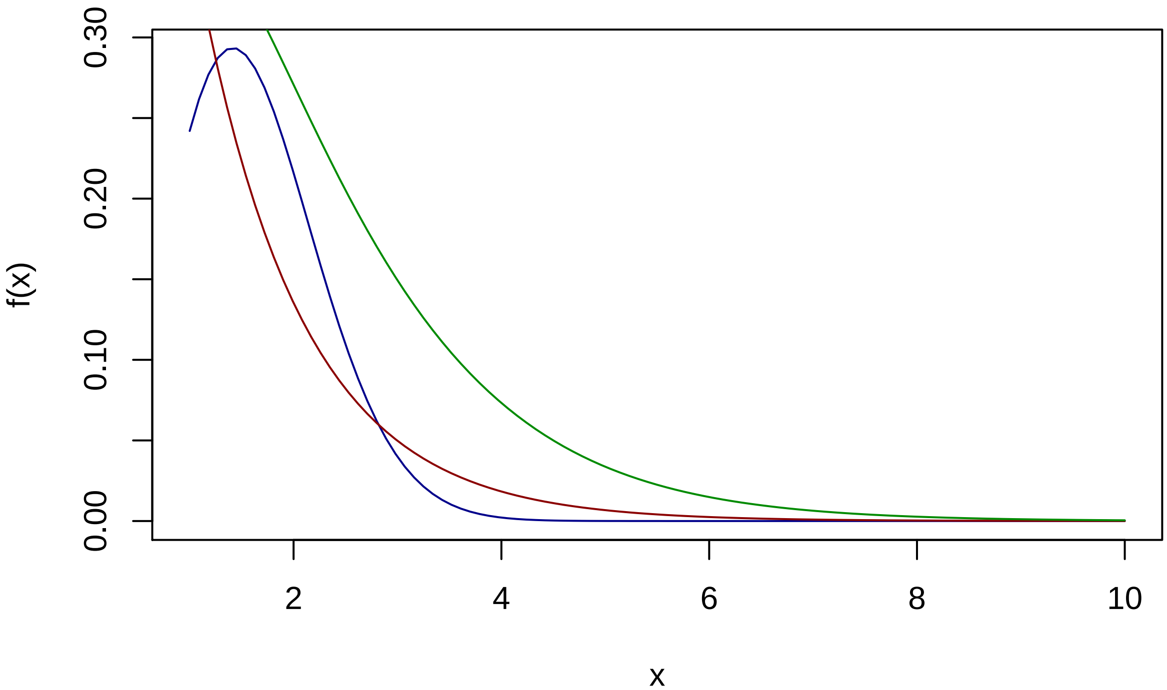
by importance sampling? Explain.

```
gx4 <- function(x){
  ((x^2)/sqrt(2*pi))*exp(-(x^2)/2)*I(x>=1)
}

#Importance function 1
f4_1 <- function(x){
  exp(-x)
}

#Importance function 2
f4_2 <- function(x){
  1/x^2
}
```

```
curve(((x^2)/sqrt(2*pi))*exp(-(x^2)/2)*I(x>=1), col='blue4', xlim=c(1,10), ylab="f(x)")
curve(exp(-x), add=T, col='red4', xlim=c(1,10))
curve(dgamma(x,2,1), add=T, col='green4', xlim=c(1,10))
```




```
n4 <- 10000
theta.hat4 <- numeric(2)
se4 <- numeric(2)

#Using f1
x4_1 <- rexp(n4)
fg4_1 <- gx4(x4_1)/f4_1(x4_1)
theta.hat4[1] <- mean(fg4_1)
se4[1] <- sd(fg4_1)

#Using f2
x4_2 <- rgamma(n4, 2, 1)
fg4_2 <- gx4(x4_2)/dgamma(x4_2, 2,1)
theta.hat4[2] <- mean(fg4_2)
se4[2] <- sd(fg4_2)


rbind(theta.hat4,se4/sqrt(n4))
```

```
##                   [,1]        [,2]
## theta.hat4 0.411741631 0.397809992
##            0.005937153 0.003690821
```

-Ans
The $f_2 = Gamma(2, 1)$ has a smaller variance. It is "closer" than f_1, therefore achieves a smaller variance.