Representation of polynomials

- Coefficient Representation
  - $A(x)= \sum a_j x^j$
- Point Value representation
  - $< y_0, y_1,\ldots,y_{n-1} >$ evaluated at $< x_0, x_1,\ldots,x_{n-1} >$
- Evaluation at given x
  - $A(x)= a_0+ x(a_1+ x(a_2+\ldots))\ldots) = \sum a_j x^j$
  - Choose $< x_0, x_1,\ldots,x_{n-1} >$ as the 2n-th roots of unity
  - $\omega^k{}_n = \exp(2\pi i\ k/n) = \cos(2\pi\ k/n) + i \sin(2\pi\ k/n)$

Operation on polynomials

Coefficient representation

- Addition - O(n)
  - C(x)=A(x)+B(x)
    - C[j]=a[j]+b[j]
- Multiplication - $O(n^2)$
  - C(x)=A(x) o B(x)
    - C[j] = ∑ a[k]b[j-k]
    - convolution
- Transform to point value
    - > y = V .a

Point value representation

- Addition - O(n)
  - C(x)=A(x)+B(x)
    - $< y_{c,i} > \; = \; < y_{a,i} + y_{b,i} >$
- Multiplication - O(n)
  - C(x)=A(x) B(x)
    - $< y_{c,i} > \; = \; < y_{a,i} \, . \, y_{b,i} >$
    - element wise
- Transform to coefficient
    - > $a = V^{-1} \, . \, y$

Properties of roots of unity

- Group under multiplication: $\omega^{k}_{n}\, \omega^{j}_{n} = \omega^{k+j}_{n}$
- Cancellation: $\omega^{dk}_{dn} = \omega^{k}_{n}$
- Squaring: $(\omega^{k+n/2}_{n})^2 = \omega^{2k}_{n}\omega^{n}_{n} = (\omega^{k}_{n})^2 = (\omega^{k}_{n/2})$
  - Squares of n complex n-th roots = n/2 complex n/2-th roots
- Summing all roots: $\sum(\omega^{k}_{n})^{j} = ((\omega^{k}_{n})^{n} - 1)/(\omega^{k}_{n} - 1)) = 0$
- (k,j) th entry of V is (ω^{kj}_{n})
- (j,k) th entry of V^{-1} has to be $(\omega^{-kj}_{n})/n$, shown below
- $[V^{-1} V]_{jj'}$ is $\sum (\omega^{-kj}_{n}/n)(\omega^{kj'}_{n}) = \sum (\omega^{k(j'-j)}_{n}/n)$
- When j=j', $[V^{-1} V]_{jj'} = 1$; 0 otherwise so that $[V^{-1} V] = I$

Discrete Fourier Transform

- $< y_0, y_1,\ldots,y_{n-1} > = \mathrm{DFT}\ (a_0, a_1,\ldots,a_{n-1})$
- $y_k = \sum a_J\ (\omega^{kj}{}_n)$ with $A(x)= \sum a_j\ x^j$ and $x= \omega^{kj}{}_n$
- $A^{[0]}(x)= a_0 + a_2\ x+ a_4\ x^2+ \ldots + a_{n-2}\ x^{n/2-1}$
- $A^{[1]}(x)= a_1 + a_3\ x+ a_5\ x^2+ \ldots + a_{n-1}\ x^{n/2-1}$
- $A(x) = A^{[0]}(x^2) + x\ A^{[1]}(x^2) \rightarrow$ divide and conquer
- Evaluating $A^{[0]}(x^2)$ at $\omega^k{}_n$ $\Rightarrow$ Evaluating $A^{[0]}(x)$ at $\omega^k{}_{n/2}$
- Therefore problem splits into two equal subproblems
- $T(n)=2\ T(n/2) + O(n) \rightarrow T(n) = O(n \lg n)$

Recursive FFT algorithm (a)

- Basis: if n==1 return a // n=length[a] = power of 2
- Initialize: ω_n=exp(2πi/n) and ω=1
- Recursive DFT:
  - $y^{[0]}$= RFFT($a^{[0]}$) ➔ $y^{[0]}_k = A^{[0]}(\omega^k_{n/2}) = A^{[0]}(\omega^{2k}_n)$
  - $y^{[1]}$= RFFT($a^{[1]}$) ➔ $y^{[1]}_k = A^{[1]}(\omega^k_{n/2}) = A^{[1]}(\omega^{2k}_n)$
- Combine results
  - For k= 0 to n/2 -1
  - $y_k = y^{[0]}_k + \omega y^{[1]}_k$; $y_{k+n/2} = y^{[0]}_k - \omega y^{[1]}_k$
- Update $\omega = \omega\ \omega_n$
- Return column vector y
- Inverse DFT is same problem with y replacing a