EE-559 - HW-12

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1 a) $p(\underline{x}|S_i) = N(\underline{x}, m_i, \underline{\underline{\Sigma}}_i)$, $i=1,2$

$P(S_1) = P(S_2) = 0.5$

$$p(\underline{x}|s_i) = \frac{1}{(2\pi)^{D/2} \|\underline{\underline{\Sigma}}_i\|^{1/2}} \exp\left\{-\frac{1}{2}(\underline{x}-\underline{m}_i)^T \underline{\underline{\Sigma}}_i^{-1}(\underline{x}-\underline{m}_i)\right\}$$

$$g_i(\underline{x}) = \ln\left\{ p(\underline{x}|S_i)\, P(S_i)\right\}$$

$$g_i(\underline{x}) = -\frac{D}{2}\ln(2\pi) - \frac{1}{2}\ln\|\underline{\underline{\Sigma}}_i\| - \frac{1}{2}(\underline{x}-\underline{m}_i)^T \underline{\underline{\Sigma}}_i^{-1}(\underline{x}-\underline{m}_i)$$
$$+ \ln P(S_i)$$

$$g_i(\underline{x}) = -\frac{1}{2}\ln(\|\underline{\underline{\Sigma}}_i\|) - \frac{1}{2}\left(\underline{x}^T\underline{\underline{\Sigma}}_i^{-1} - m_i^T\underline{\underline{\Sigma}}_i^{-1}\right)(\underline{x}-\underline{m}_i)$$

$$= -\frac{1}{2}\ln\|\underline{\underline{\Sigma}}_i\| - \frac{1}{2}\left(\underline{x}^T\underline{\underline{\Sigma}}_i^{-1}\underline{x} - \underline{x}^T\underline{\underline{\Sigma}}_i^{-1}\underline{m}_i - m_i^T\underline{\underline{\Sigma}}_i^{-1}\right.$$
$$\left. + m_i^T\underline{\underline{\Sigma}}_i^{-1} m_i\right)$$

$$\boxed{g_i(x) = -\frac{1}{2}\ln\|\underline{\underline{\Sigma}}_i\| - \frac{1}{2}\left(x^T\underline{\underline{\Sigma}}_i^{-1}\underline{x} - 2\underline{x}^T\underline{\underline{\Sigma}}_i^{-1} m_i + m_i^T\underline{\underline{\Sigma}}_i^{-1} m_i\right)}$$

1)

b) $g_1(x) = -\frac{1}{2} \ln \|\Sigma_1\| - \frac{1}{2}\left(\underline{x}^T \Sigma_1^{-1} \underline{x} - 2\underline{x}^T \Sigma_1^{-1} m_1 + m_1^T \Sigma_1^{-1} m_1\right)$

$\|\Sigma_1\| = \|\Sigma_2\| = 2 \qquad \Sigma_1^{-1} = \Sigma_2^{-1} = \frac{1}{2}\begin{bmatrix} 2.25 & -0.5 \\ -0.5 & 1 \end{bmatrix}$

$$= -\frac{1}{2} \ln \|\Sigma_1\| - \frac{1}{4}\left[(x_1 \ x_2)\begin{bmatrix} 2.25 & -0.5 \\ -0.5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 2\left[(x_1 \ x_2)\begin{bmatrix} 2.25 & -0.5 \\ -0.5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right] + \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 2.25 & -0.5 \\ -0.5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

$$= \cancel{-\frac{1}{2} \ln \|\Sigma_1\| - \frac{1}{2}\left[\begin{bmatrix} 2.25x_1 - 0.5x_2 & -0.5x_1 + x_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 2\left[\begin{bmatrix} 2.25x_1 - 0.5x_2 & -0.5x_1 + x_2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right] + [2.25}$$

$$\boxed{g_1(x) = 0.875x_1 + 0.25x_2 - 0.5625}$$

$$g_2(x) = -\frac{1}{2} \ln \|\Sigma_2\| - \frac{1}{2}\left[(x_1 \ x_2)\begin{bmatrix} 2.25 & -0.5 \\ -0.5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 2\left[(x_1 \ x_2)\begin{bmatrix} 2.25 & -0.5 \\ -0.5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right] + \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 2.25 & -0.5 \\ -0.5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right]$$

$$\boxed{g_2(x) = 1.375x_1 - 0.75x_2 - 0.3693}$$

By Max value method,

$$\boxed{\begin{aligned} g_1(x) &= g_2(x) \\ g_1(x) - g_2(x) &= 0 \end{aligned}}$$

$$= \cancel{-\frac{1}{2} \ln \|\Sigma_1\|} - \frac{1}{2}\left[(x_1 \ x_2)\begin{bmatrix} 2.25 & -0.5 \\ -0.5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right] - 2\left[(x_1 \ x_2)\begin{bmatrix} 2.25 & -0.5 \\ -0.5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right] + \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 2.25 & -0.5 \\ -0.5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$+\frac{1}{2}\ln \cancel{\|\Sigma_2\|} + \frac{1}{2}\left[\cancel{[x_1 \; x_2]\begin{bmatrix}2.25 & -0.5\\ -0.5 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}}\right]$$

$$+\frac{2}{2}\left[[x_1 \; x_2]\begin{bmatrix}2.25 & -0.5\\ -0.5 & 1\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix} - [1 \; -1]\begin{bmatrix}2.25 & -0.5\\ -0.5 & 1\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix}\right.$$

$$\frac{1}{2}(-3.5x_1 - x_2 + 2.25 + 5.5x_1 + 3x_2 - 4.25) = 0$$

$$2x_1 - 4x_2 = 2$$

$$x_1 - 2x_2 = 1$$

$$-2x_2 = -x_1 + 1$$

$$\boxed{x_2 = \frac{x_1}{2} - \frac{1}{2}}$$

$$\boxed{\text{Decision Rule} :\to \quad g_1(x) \underset{S_2}{\overset{S_1}{\gtrless}} g_2(x)}$$

$$\Rightarrow \boxed{x_2 \underset{S_2}{\overset{S_1}{\gtrless}} \frac{x_1}{2} - \frac{1}{2}}$$

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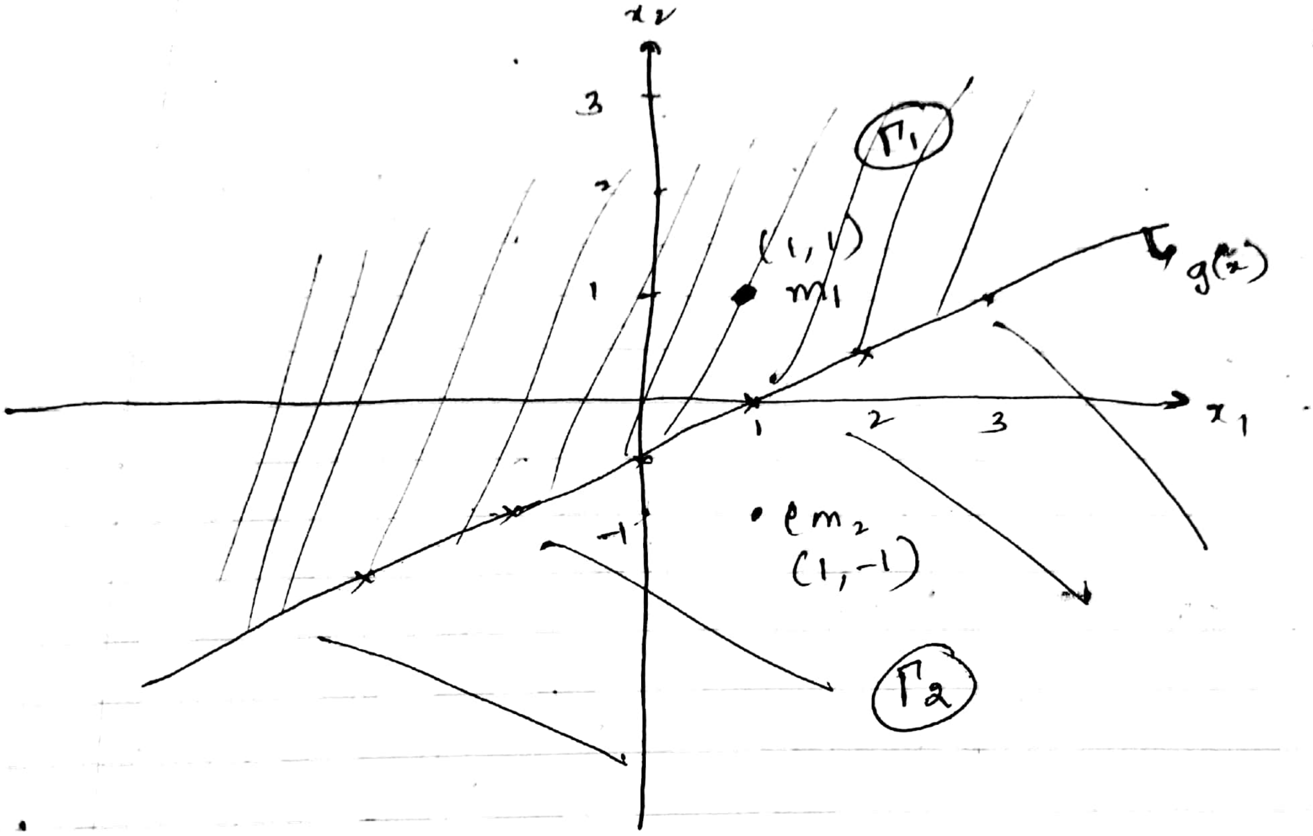

$$x_2 = \frac{1}{2}x_1 - \frac{1}{2}$$

x_1	0	1	2	3	4	-2	-3
x_2	$-1/2$	0	$1/2$	1	-1	$-3/2$	-2

(2)

$$\Sigma_1^{-1} = \Sigma_2^{-1} = \frac{1}{2.25}\begin{bmatrix} 2.25 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0.44 \end{bmatrix}$$

$$g_1(x) = -\frac{1}{2}\ln[2.25] - \frac{1}{2}\Bigg[[x_1\ x_2]\begin{bmatrix} 1 & 0 \\ 0 & 0.44 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$-2\,[x_1\ x_2]\begin{bmatrix} 1 & 0 \\ 0 & 0.44 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + [1\ 1]\begin{bmatrix} 1 & 0 \\ 0 & 0.44 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\Bigg]$$

$$= -\frac{1}{2}\ln[2.25] - \frac{1}{2}\Bigg[[x_1\ x_2]\begin{bmatrix} 1 & 0 \\ 0 & 0.44 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\Bigg]$$

$$+ [x_1\ \ 0.44x_2]\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}[1\ \ 0.44]\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= -\frac{1}{2}\ln[2.25] - \frac{1}{2}\Bigg[[x_1\ x_2]\begin{bmatrix} 1 & 0 \\ 0 & 0.44 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\Bigg]$$

$$+ x_1 + 0.44x_2 - 0.5 - 0.22$$

$$\boxed{g_1(x) = -\frac{1}{2}\ln[2.25] - \frac{1}{2}\Bigg[[x_1\ x_2]\begin{bmatrix} 1 & 0 \\ 0 & 0.44 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\Bigg] + x_1 + 0.44x_2 - 0.72}$$

$$g_2(x) = -\frac{1}{2}\ln[2.25] - \frac{1}{2}\Bigg[[x_1\ x_2]\begin{bmatrix} 1 & 0 \\ 0 & 0.44 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\Bigg]$$

$$-[x_1\ x_2]\begin{bmatrix} 1 & 0 \\ 0 & 0.44 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \frac{1}{2}\Bigg[[1\ -1]\begin{bmatrix} 1 & 0 \\ 0 & 0.44 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= -\frac{1}{2}\ln[2.25] - \frac{1}{2}\Bigg[[x_1\ x_2]\begin{bmatrix} 1 & 0 \\ 0 & 0.44 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\Bigg]$$

$$+ [x_1\ -0.44x_2]\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \frac{1}{2}\Bigg[[1\ -0.44]\begin{bmatrix} 1 \\ -1 \end{bmatrix}\Bigg]$$

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$$\boxed{g_2(x) = -\frac{1}{2}\ln[0.25] - \frac{1}{2}\left[[x_1\ x_2]\begin{bmatrix}1 & 0\\ 0 & 0.44\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\right] + [x_1 - 0.44x_2] - \frac{1}{2}[1 + 0.44]}$$

max value, $g_1(x) = g_2(x)$

$$\cancel{x_1} + 0.44x_2 - \cancel{0.72} = \cancel{x_1} - 0.44x_2 - \cancel{0.72}$$

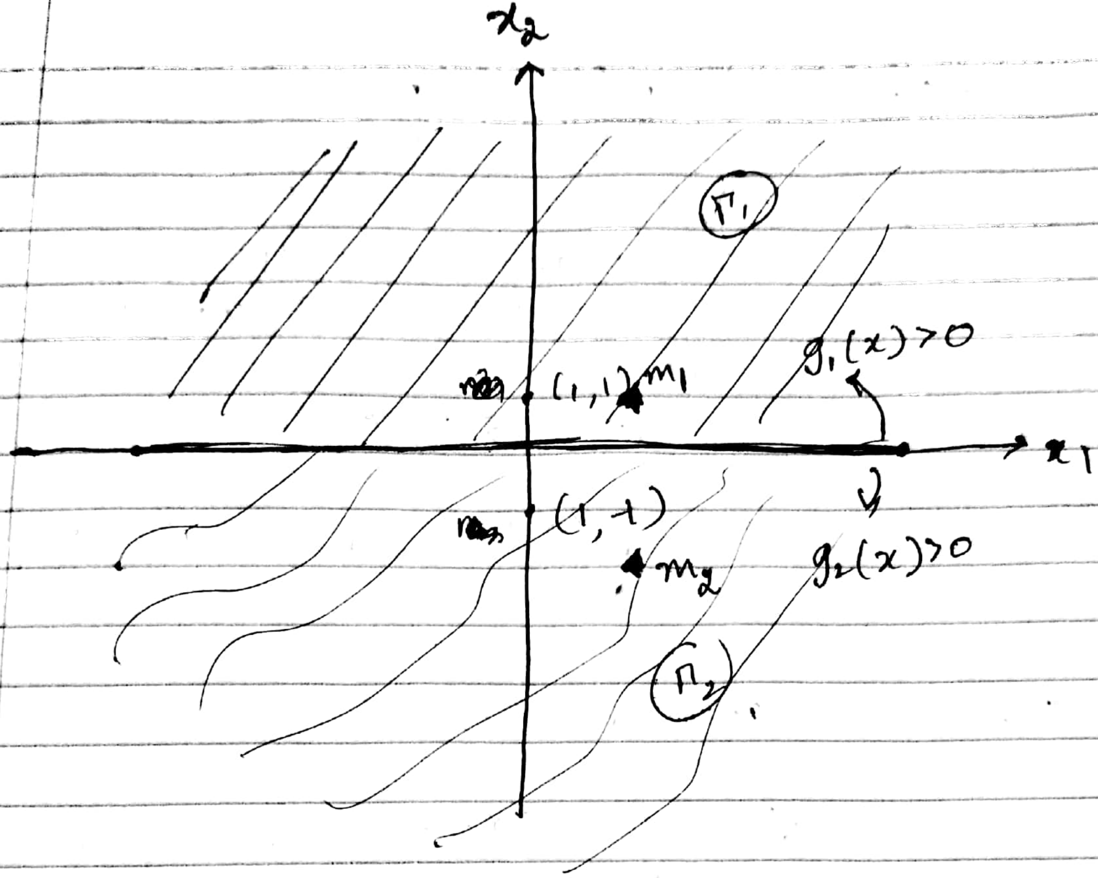
$$0.88x_2 = 0$$

$$\cancel{0.88x_2 = 0.45}$$

$$\cancel{x_2 = -\frac{0.45}{0.88}}$$

Decision boundary $\rightarrow$ $\boxed{x_2 = 0}$

Decision rule $\rightarrow$ $\boxed{x_2 \overset{S_1}{\gtrless} 0}$



(3) <u>Decision rule:</u>

$$(\lambda_{21} - \lambda_{11}) P(S_1 | \underline{x}) \underset{S_2}{\overset{S_1}{\gtrless}} (\lambda_{12} - \lambda_{22}) P(S_2 | \underline{x})$$

w.r.t λ_{11} & $\lambda_{22} = 0$
and $\lambda_{12} = 2\lambda_{21}$

$$g_1(x) = \ln \{ P(S_1 | \underline{x}) \}$$

$$\boxed{g_1(x) = 3.5x_1 + x_2 - 2.25}$$

$$g_2(x) = \ln \{ 2P(S_2 | \underline{x}) \}$$

$$g_2(x) = \ln 2 + \ln P(S_2 | \underline{x})$$

$$g_2(x) = 0.69 + 5.5x_1 + 3x_2 - 4.25$$

$$\boxed{g_2(x) = 5.5x_1 + 3x_2 - 4.25 + 0.693}$$

$g_1(x) = 0.875x_1 + 0.25x_2 - 0.5625$

$g_2(x) = 1.375x_1 - 0.75x_2 - 1.0625 + \ln(2)$

$= 1.375x_1 - 0.75x_2 - 0.3693$

$\boxed{g_1(x) = g_2(x)}$

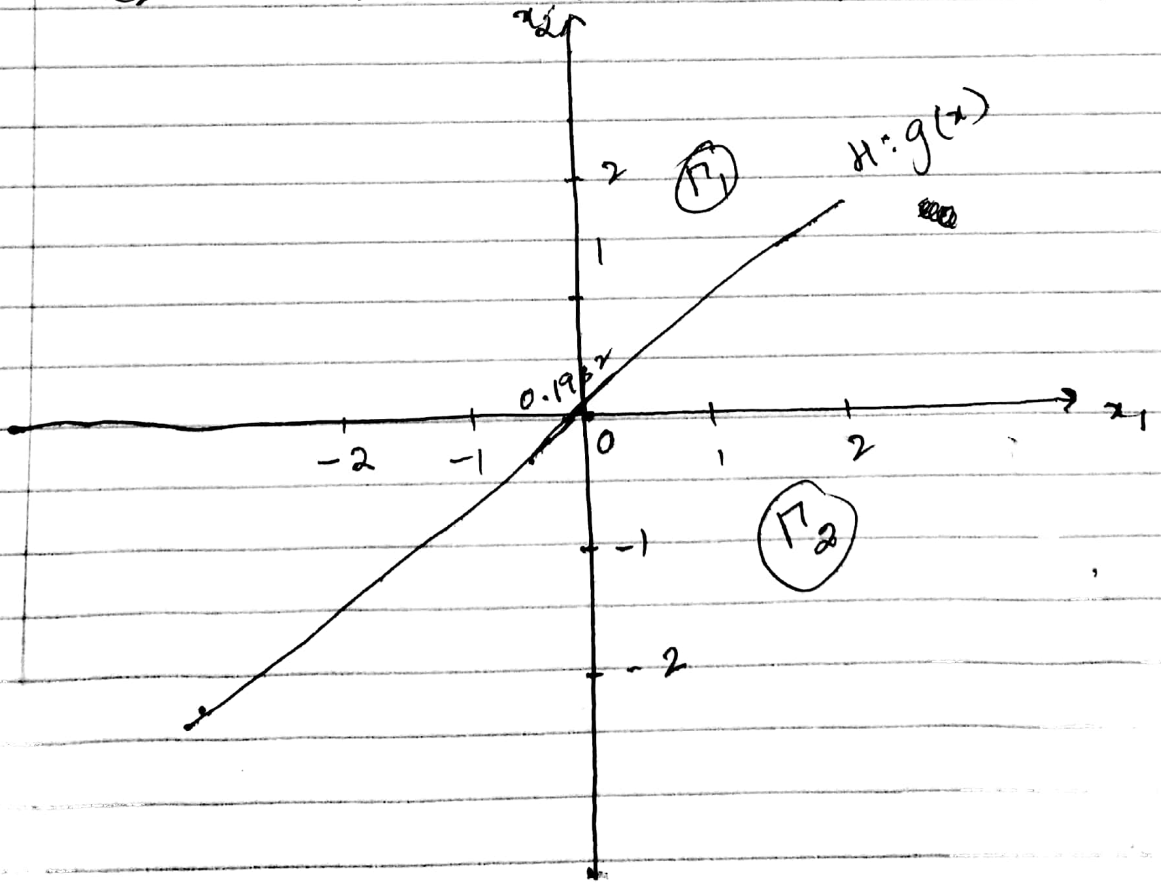
$0.875x_1 + 0.25x_2 - 0.5625 = 1.375x_1 - 0.75x_2 - 0.3693$

$0.875x_1 - 1.375x_1 + 0.25x_2 + 0.75x_2 = 0.5625 - 0.3693$

$-0.5x_1 + x_2 = 0.1932$

$\boxed{x_2 = 0.5x_1 + 0.1932}$

x_1	0	1	2	3
x_2	0.1932	0.6932	1.1932	1.6932



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→ The Decision region r_2 has grown ~~be~~ more due to the losses.

→ Deciding s_1 when the class is s_2 incurs more loss.