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# CE-559-HW-14

1) Normalized of distribution,

$$p(x|\theta) = \begin{cases} 1/\theta & 0 \le x \le \theta \\ 0 & \text{otherwise} \end{cases}$$

a) The likelihood function is $I(\cdot)$ is given by,

$$p(D|\theta) = \prod_{k=1}^{n} p(x_k|\theta)$$

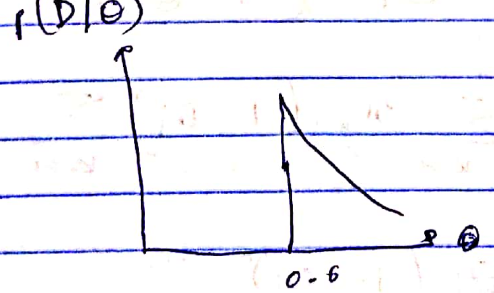
$$= \prod_{k=1}^{n} \frac{1}{\theta} I(0 \le x_k \le \theta)$$

$$= \frac{1}{\theta^n} I\left(\theta \ge \max_k x_k\right) I(\min x_k \ge 0)$$

$1/\theta^n$ decreases monotonically as $\theta$ increases ~~but~~ since $I(\theta \ge \max_k x_k)$ is 0 if $\theta$ is less than the maximum value of $x_k$.

$\therefore$ The ~~lik~~ likelihood function is maximized at $\hat{\theta} = \max_k x_k$.

b)



(2) w.r.t

The Bernoulli distribution is →

$$p(x|\theta) = \prod_{i=1}^{d} \theta_i^{x_i} (1-\theta_i)^{1-x_i}$$

$$D = \{x_1 \dots x_n\}$$

a) $s = (s_1, s_2 \dots s_d)^t$ → sum of the 'n' samples.

$$x_k = (x_{k1}, x_{k2} \dots x_{kd})^t$$

then,

$$s_i = \sum_{k=1}^{n} x_{ki}, \quad i = 1 \dots d.$$

The likelihood is given by,

$$P(D|\theta) = P(x_1 \dots x_n|\theta) = \prod_{k=1}^{n} P(x_k|\theta)$$

$$= \prod_{k=1}^{n} \sum_{i=1}^{d} \theta_i^{x_{ki}} (1-\theta_i)^{1-x_{ki}}$$

$$= \prod_{i=1}^{d} \theta_i \sum_{k=1}^{n} x_{ki} (1-\theta_i) \sum_{k=1}^{n} (1-x_{ki})$$

$$= \prod_{i=1}^{d} \theta_i^{s_i} (1-\theta_i)^{n-s_i}$$

b) Assuming uniform prior for $\theta$, $P(\theta) = 1$ for $0 \le \theta_i \le 1$

$$P(\theta|D) = \frac{P(D|\theta) P(\theta)}{P(D)}$$

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w.k.t $p(D|\theta) = \prod_{i=1}^{d} \theta_i^{s_i} (1-\theta_i)^{n-s_i}$

$\therefore$ p.d.f is

$$p(D) = \int p(D|\theta)\, p(\theta)\, d\theta$$

$$= \int \prod_{i=1}^{d} \theta_i^{s_i} (1-\theta_i)^{n-s_i} d\theta$$

$$= \prod_{i=1}^{d} \int_0^1 \theta_i^{s_i} (1-\theta_i)^{n-s_i} d\theta_i$$

$s_i = \sum_{k=1}^{n} x_{ki}$ takes values in set $(0 \ldots n)$

$$\int_0^1 \theta^m (1-\theta)^n d\theta = \frac{m!\,n!}{(m+n+1)!}$$

substituting,

$$P(D) = \prod_{i=1}^{d} \frac{s_i!\,(n-s_i)!}{(n+1)!}$$

$$\therefore \quad p(\theta|D) = \frac{p(D|\theta)\,.\,p(\theta)}{p(D)}$$

$$= \frac{\prod_{i=1}^{d} \theta_i^{s_i} (1-\theta_i)^{n-s_i}}{\prod_{i=1}^{d} s_i!\,(n-s_i)!\,/(n+1)!}$$

$$= \prod_{i=1}^{d} \frac{(n+1)!}{s_i!\,(n-s_i)!} \theta_i^{s_i} (1-\theta_i)^{n-s_i}$$

c. $d = 1$, $n = 1$

$$p(\theta_1|D) = \frac{2!}{s_1!\,(n-s_1)!}\,\theta_1^{s_1}(1-\theta_1)^{(n-s_1)}$$

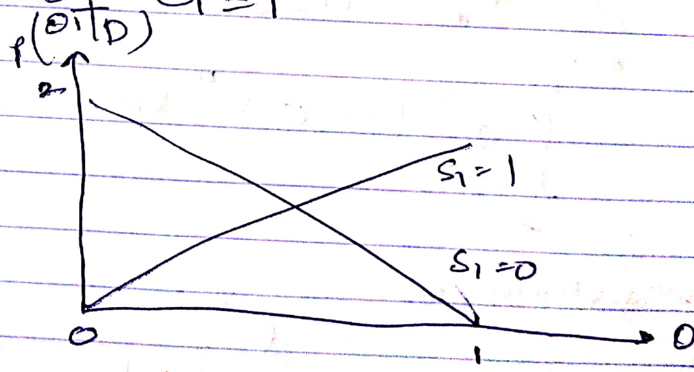
$$= \frac{2}{s_1!\,(1-s_1)!}\,\theta_1^{s_1}(1-\theta_1)^{1-s_1}$$

Noting, $s_1$ takes values between 0 and 1,

$s_1 = 0$ ; $p(\theta_1|D) = 2(1-\theta_1)$

$s_1 = 1$ ; $p(\theta_1|D) = 2\theta_1$

for $0 \le \theta_1 \le 1$



3)

a)

Given $s_1 = 1, 4$

$s_2 = 3, 6.$

$P(x, s_1) \longrightarrow k_n = 2, \ n = 2$

$$P(x|s_1) = \frac{\frac{k_n}{n}}{V_n} = \frac{\frac{2}{2}}{V_n} = \frac{1}{V_n} = \frac{1}{2(x-P_1)}$$

$$= \frac{1}{2(x-1)}$$

At $x = 2.5 \Rightarrow \frac{1}{2(2.5-1)} = \frac{1}{3} = 0.333$

At $x = 6 \Rightarrow \frac{1}{2(6-1)} = \frac{1}{10} = 0.1$

At $x = 4 \Rightarrow \frac{1}{2(4-1)} = \frac{1}{6} = 0.166$

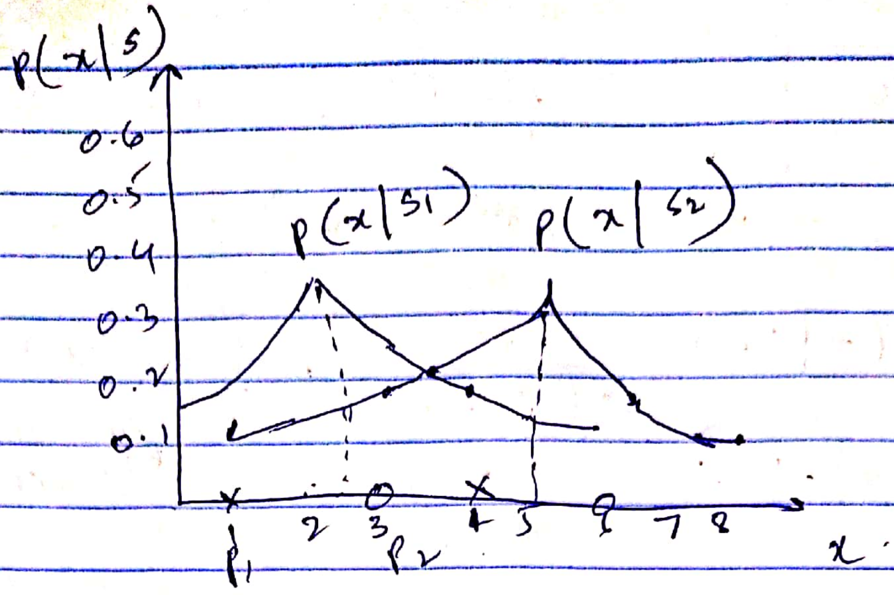
$P(x, s_2) \quad k_n = 2, \ n = 2$

$$P(x, s_2) = \frac{k_n/n}{V_n} = \frac{1}{V_n} = \frac{1}{2(x-P)}$$

$$= \frac{1}{2(x-3)}$$

At $x = 4.5 \Rightarrow \frac{1}{2(4.5-3)} = \frac{1}{3}$

At $x = 6 \Rightarrow \frac{1}{2(6-3)} = \frac{1}{6} = 0.166$




b) $n^{(1)} = 2$

$n^{(2)} = 2$

$n = 4$

$$P(s_1) = \frac{n^{(1)}}{n} = \frac{1}{2}$$

$$P(s_2) = \frac{n^{(2)}}{n} = \frac{1}{2}$$

c) Baye Minimum error decision rule,

$$P(x|s_1)\,P(s_1) \underset{s_2}{\overset{s_1}{\gtrless}} P(x|s_2)\,P(s_2)$$

Since $P(s_1) = P(s_2)$

$$P(x|s_1) > P(x|s_2)$$

From graph, a good decision boundary is a vertical line at $x = 3.5$

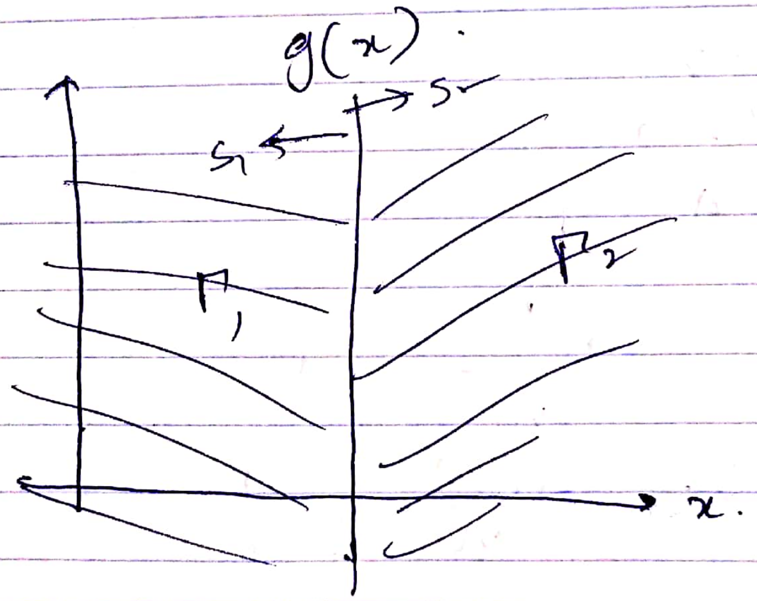
considering a point,

$x_1 = 3.5$ with $k_n = 2$

$$P(x|s_1) = \frac{1}{2(3.5-1)} = \frac{1}{5} = 0.2$$

$$P(x|s_2) = \frac{1}{2(6-3.5)} = \frac{1}{5} = 0.2$$

Since $P(x|S_2) > P(x|s_1)$ = $x = 4 \in S_2$




$$\boxed{\begin{gathered} g(x) = 3.5 - x \\ g(x) \underset{S_2}{\overset{S_1}{\gtrless}} 0 \end{gathered}}$$

2d) $p(x|D) = \int p(x|\theta)\, p(\theta|D)\, d\theta$

$$\int_0^1 \int_0^1 \theta_1^{x_1} (1-\theta_1)^{1-x_1} \frac{(n+1)!}{s_1!\,(n-s_1)!} \theta_1^{s_1} (1-\theta_1)^{n-s_1}\, d\theta_1\, d\theta_2$$

$$= \prod_{k=1}^{d} \frac{(n+1)!}{(s_k!)(n-s_k)!} \frac{(x_k+s_k)!\,(n+1-x_k-s_k)!}{(n+2)!}$$

$$\prod_{k=1}^{d} \frac{(n+1)!}{s_k!\,(n-s_k)!} \quad \frac{(x_k+s_k)!\,(n+1-x_k-s_k)!}{(n+2)\,\cancel{(n+1)}!}$$

$$P(x|D) = \prod_{k=1}^{d} \frac{(x_k+s_k)!}{s_k!\,(n-s_k)!} \quad \frac{(n+1-x_k-s_k)}{(n+2)}$$

when $x_k = 0$

$$P(x|D) = \prod_{k=1}^{d} \frac{s_k!}{s_k!\,(n-s_k)!} \quad \frac{(n+1-s_k)!}{(n+2)}$$

$$= \prod_{k=1}^{d} \frac{(n+1-s_k)}{(n+2)}$$

when $x_k = 1$

$$P(x|D) = \prod_{k=1}^{d} \frac{(1+s_k)!}{s_k!\,(n-s_k)!} \quad \frac{(n-s_k)!}{(n+2)}$$

$$\therefore P(x|D) = \prod_{k=1}^{d} \left(\frac{s_k+1}{n+2}\right)^{x_k} \left(1 - \frac{s_k+1}{n+2}\right)^{1-x_k}$$

2) e) $p(x|\hat{\theta}) = \prod_{i=1}^{d} \hat{\theta}_i^{x_i} (1-\hat{\theta}_i)^{1-x_i}$

$$P(x|D) = \int_0^1 P(x|\hat{\theta})\, P(\theta|D)\, d\theta$$

$$= \int_0^1 \cdots \int_0^1 \left( \prod_{i=1}^{d} \hat{\theta}_i^{x_i} (1-\theta_i)^{1-x_i} \right) \prod_{i=1}^{d} \frac{(n+1)!}{s_i!\,(n-s_i)!} \theta_i^{s_i}\, d\theta_i$$

$$= \prod_{i=1}^{d} \hat{\theta}_i^{x_i} (1-\hat{\theta}_i)^{1-x_i} \frac{(n+1)!}{s_i!\,(n-s_i)!} \left[ \int_0^1 \theta_i^{s_i} (1-\theta_i)^{n-s_i}\, d\theta_i \right]$$

$$= \prod_{i=1}^{d} \hat{\theta}_i^{x_i} (1-\hat{\theta}_i)^{1-x_i}$$

$$\therefore \boxed{\hat{\theta}_i = \frac{s_i+1}{n+2}}$$

2) f) $s^{(k)} = [\, s_1^{(k)} \quad s_2^{(k)} \quad \ldots \quad s_d^{(k)} \,]^T$

Baye's Minimum error decision rule

$$P(\underline{x} \mid s_i)\, P(s_i) > P(\underline{x} \mid s_k)\, P(s_k) \quad \text{—} (1)$$

$\forall\, i \neq k$

$$P(s_k) = \frac{n_k}{N}$$

$$P(\underline{x} \mid s_k) = P(x \mid s_k, z_k)$$

$$= \prod_{i=1}^{d} \left( \frac{s_i^{(k)} + 1}{n_k + 2} \right)^{x_i} \left( 1 - \frac{s_i + 1}{n_k + 2} \right)$$

Sub. $P(s_k)$ & $P(\underline{x} \mid k_s)$ in eqn (1)