Game Architecture

# Linear Algebra 2

Welcome to the Matrix

# Today’s Agenda

- Anatomy of a Matrix
- Matrix Operations
- Linear Transforms
- Other Transforms

# Anatomy of a matrix

- A matrix is a rectangular array of numbers. Usually shown surrounded by square brackets.
- The *dimensions* of the matrix are expressed as number of rows x number of columns (or *m* x *n*).

A 2x3 matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

# Anatomy of a matrix

- The entries (or elements) are indexed first by row and then column.
  - An entry at row *i* and column *j* in matrix **A** is indexed with $\mathbf{A}_{ij}$
- Indices begin at 1, not 0, at the top-left and increase to the right and down.

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{33} \end{bmatrix}$$

# Square Matrix

- A matrix where the number of rows equals the number of columns is called a *square* matrix.
- Typically we use 2x2 matrices when working in 2D and 3x3 matrices when working in 3D.
- We'll also find uses for 4x4 matrices. But more on that later.

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$$\begin{array}{c} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{array} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

# Identity Matrix

- A *diagonal* matrix is a square matrix that has non-zero elements only in the diagonal.
- A diagonal matrix whose diagonal elements are all equal to one is the *identity* matrix.
- The identity matrix in *n* dimensions is denoted as $\mathbf{I}_n$ (or just $\mathbf{I}$ if the dimensions can be inferred).

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$\begin{array}{c} x \\ y \\ z \end{array} \quad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Transpose

- The transpose of a matrix **M** with dimensions *m*, *n* is the matrix $\mathbf{M}^T$ with dimensions *n*, *m*.
- Substitute the the $n^{th}$ row with the $n^{th}$ column.
    - Alternatively, think of flipping the matrix along the diagonal.

Properties:

- The transpose is its own inverse
    - $(\mathbf{M}^T)^T = \mathbf{M}$
- Transposing a diagonal matrix is a no-op.
    - $\mathbf{D}^T = \mathbf{D}$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

# Scalar Multiplication

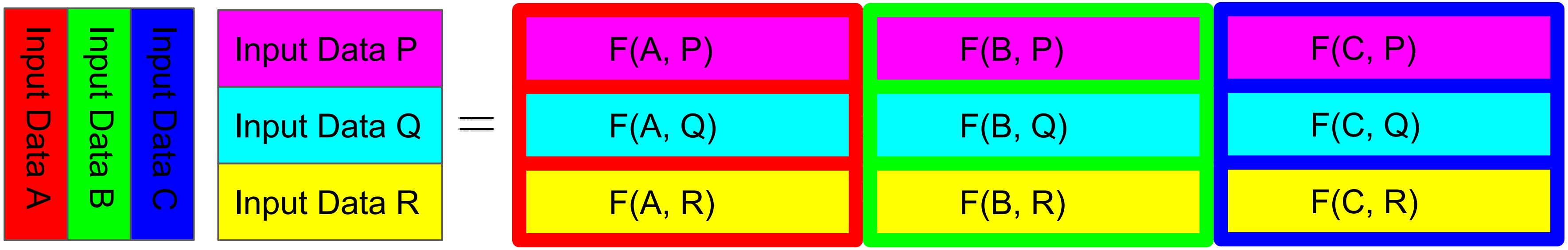
- You can multiply a matrix with a scalar.
- Just multiply each component by the scalar.
- Identical to scalar multiplication with vectors.

$$k\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} ka & kb & kc \\ kd & ke & kf \\ kg & kh & ki \end{bmatrix}$$

# Matrix Multiplication

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} aj + bm + cp & ak + bn + cq & al + ba + cr \\ dh + em + fp & dk + en + fq & dl + eo + fr \\ gj + hm + ip & gk + hn + iq & gl + ho + ir \end{bmatrix}$$

# Matrices for Transforming Data



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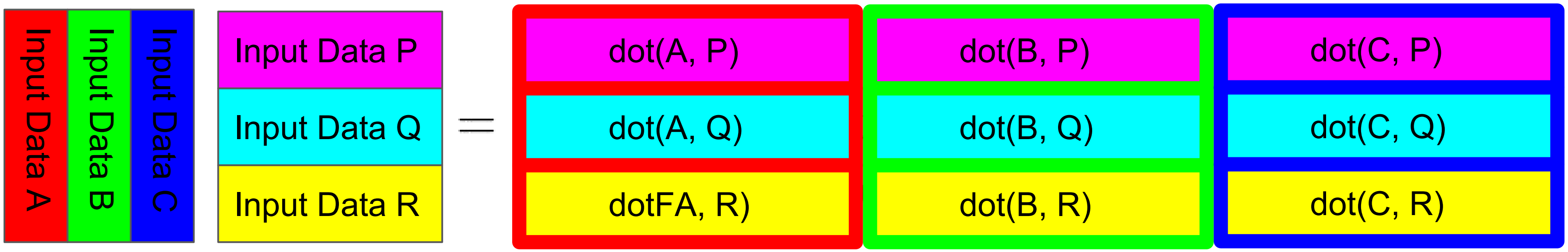

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$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}_x \mathbf{b}_x + \mathbf{a}_y \mathbf{b}_y + \mathbf{a}_z \mathbf{b}_z$$

# Matrix Multiplication

- A vector is a 1xn matrix.
- Multiplying two matrices transforms one vector space into another.
- Multiplying a vector with a matrix is just a special case of transforming several vectors at once to only one.

# Linear Transforms

- A linear transform is a mapping between vector spaces that preserves the operations of addition and scalar multiplication.
- More intuitively, a linear transformation is any transformation that preserves parallel lines.

# Linear Transforms

- There is a one to one correspondence between matrices and linear transformations.
    - All linear transformations are represented by a matrix.
    - All matrices represent a linear transformation.

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# Example

What does this 2D matrix do?

$$\mathbf{M} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

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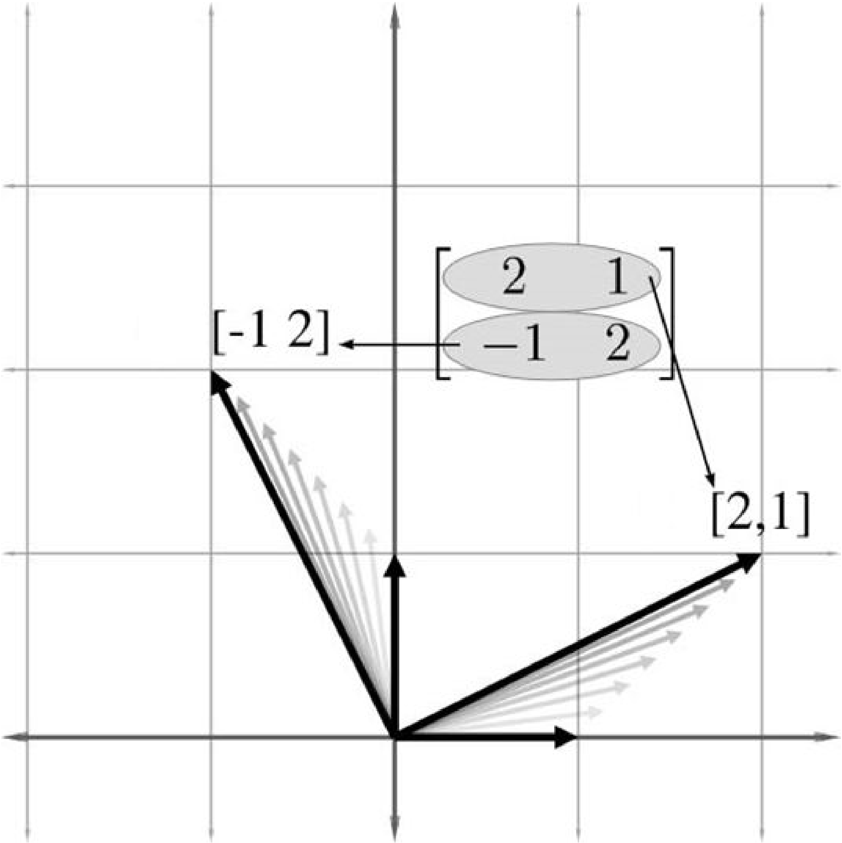

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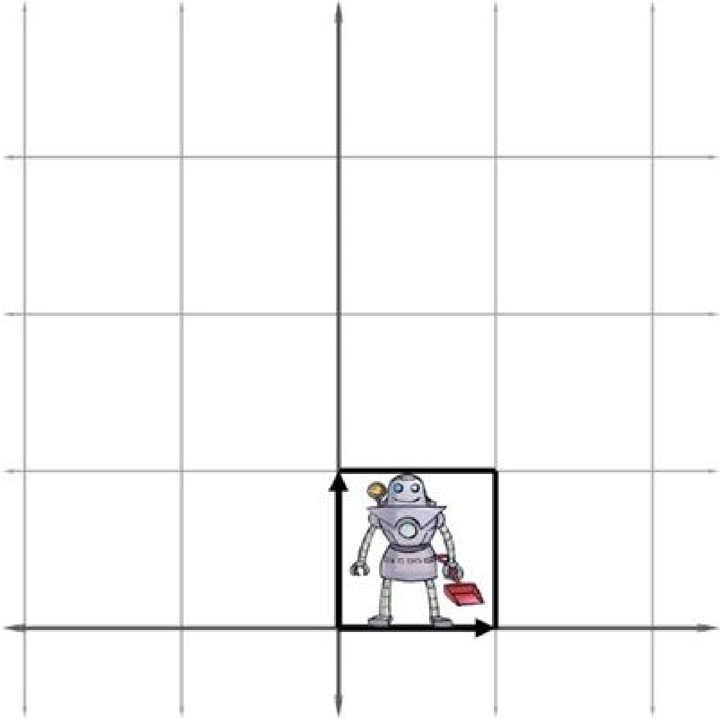
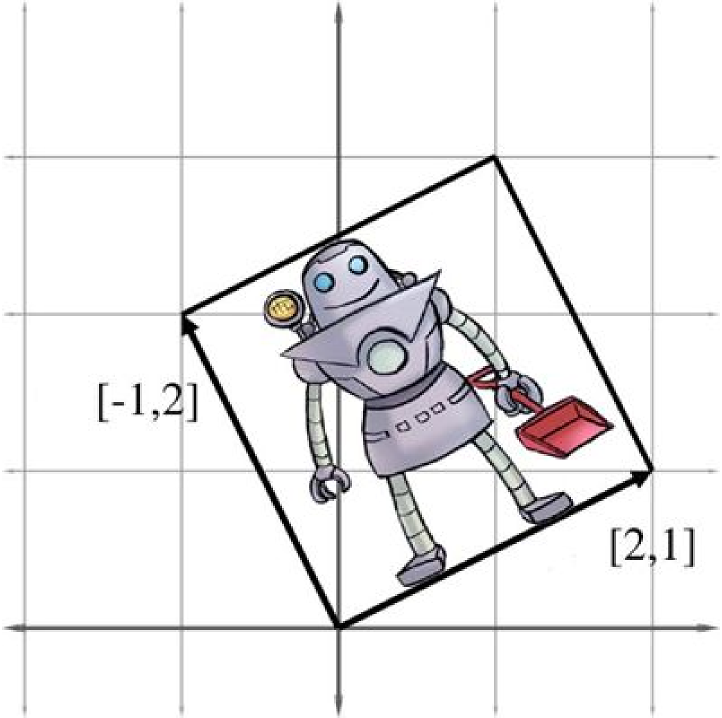
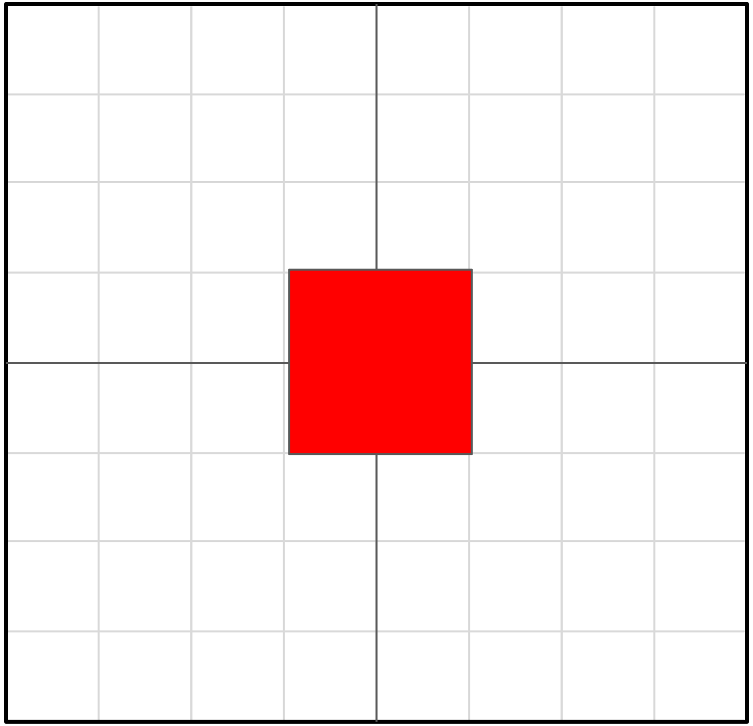




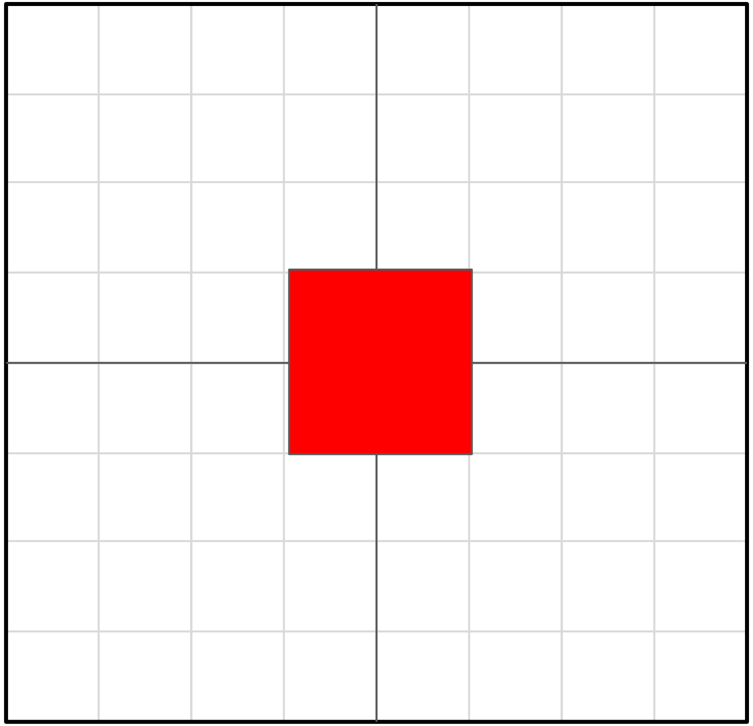
# Scale

- What are our basis vectors?
  - ***i*** =
  - ***j*** =



# Scale

- What are our basis vectors?
  - ***i*** = [1 0]
  - ***j*** = [0 1]



# Scale

- What are our transformed basis vectors?
  - ***i*** =
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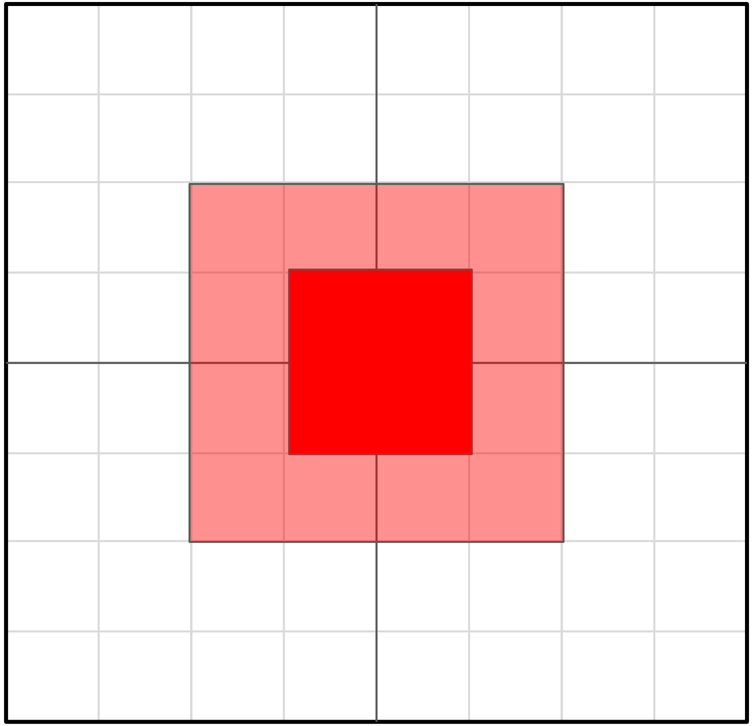

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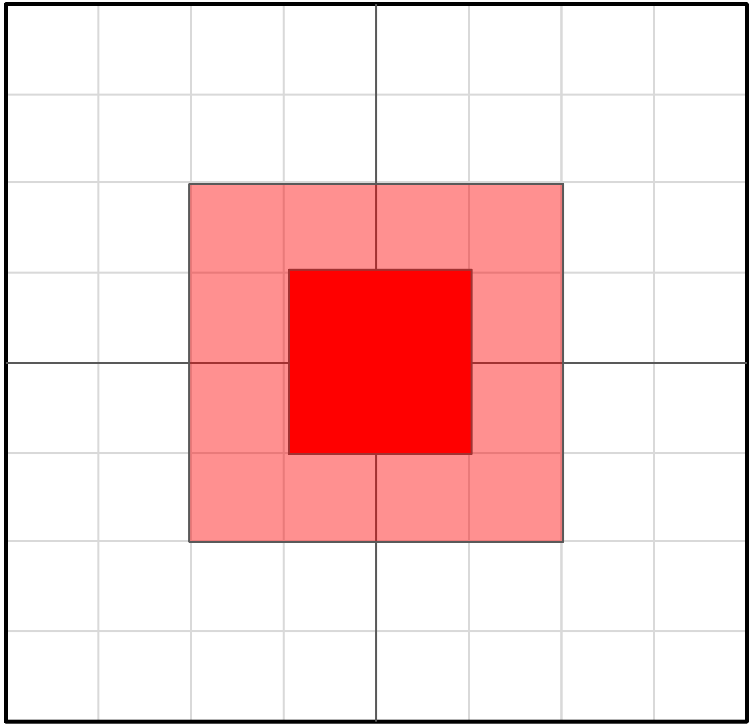

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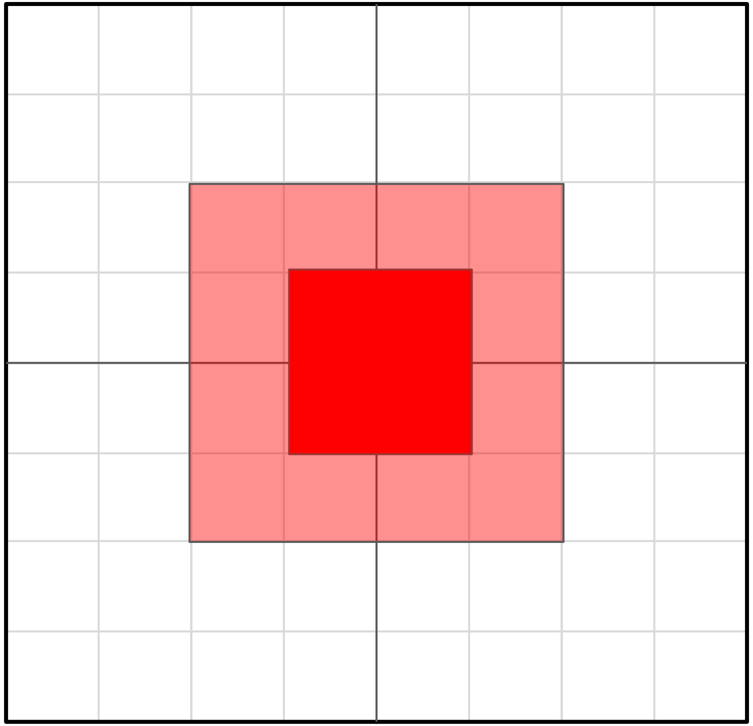

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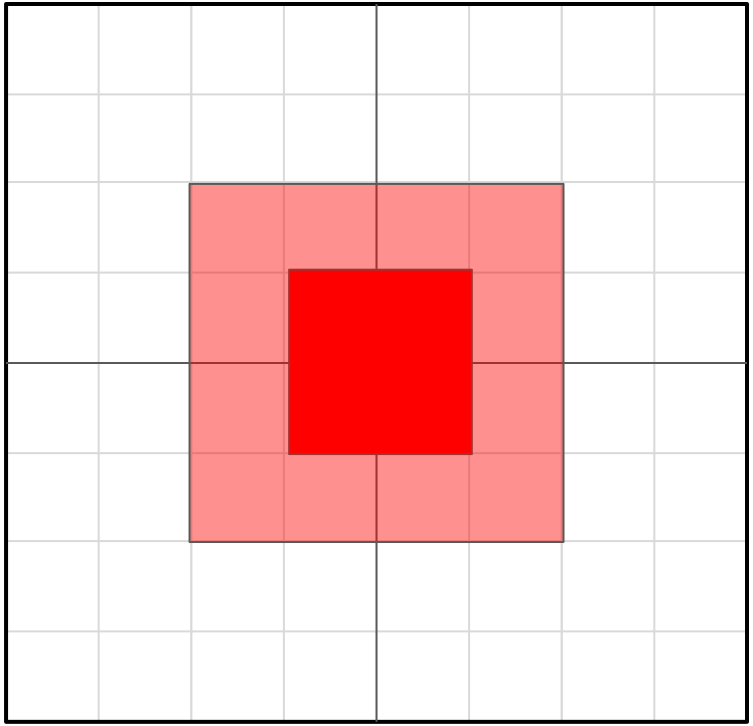
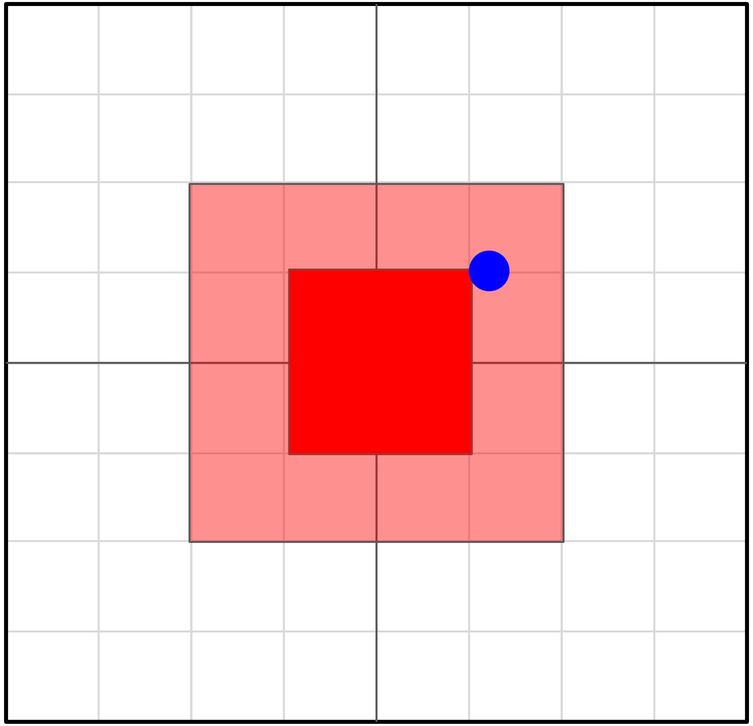
$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$



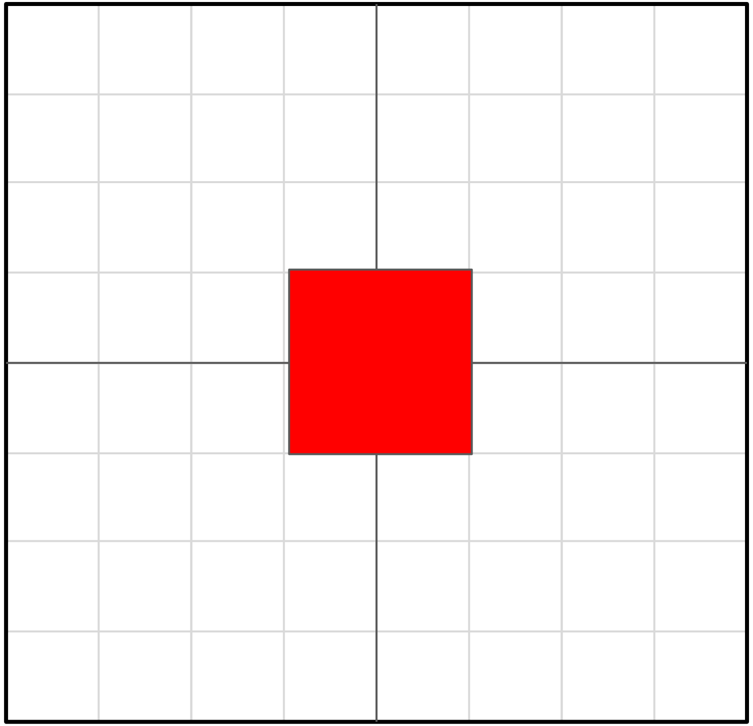
# Scale

$$\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix} = \begin{bmatrix}? & ?\end{bmatrix}$$



# Rotation

- Let’s try a 45 degree rotation counter-clockwise.



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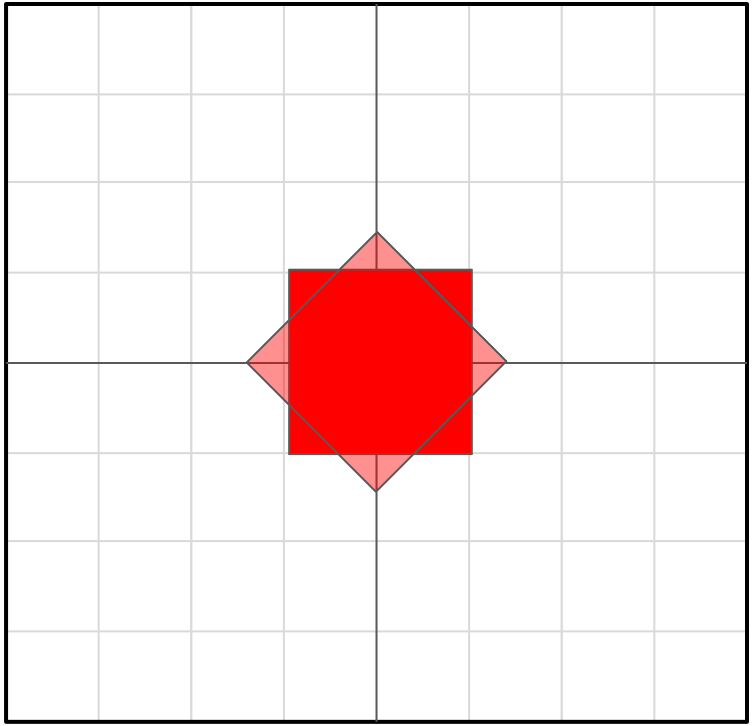

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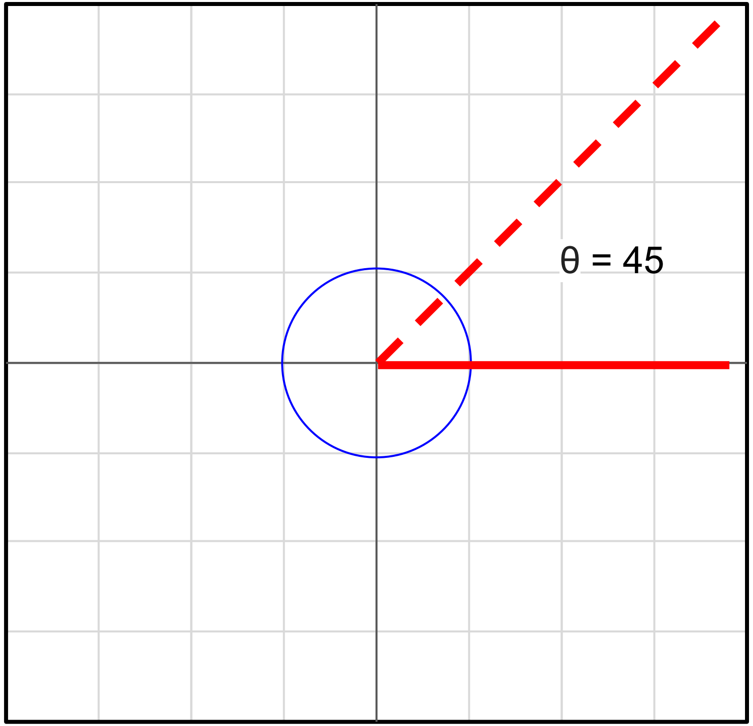

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- Let's try a 45 degree rotation counter-clockwise.
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  - ***i*** = [cos(45) sin(45)]
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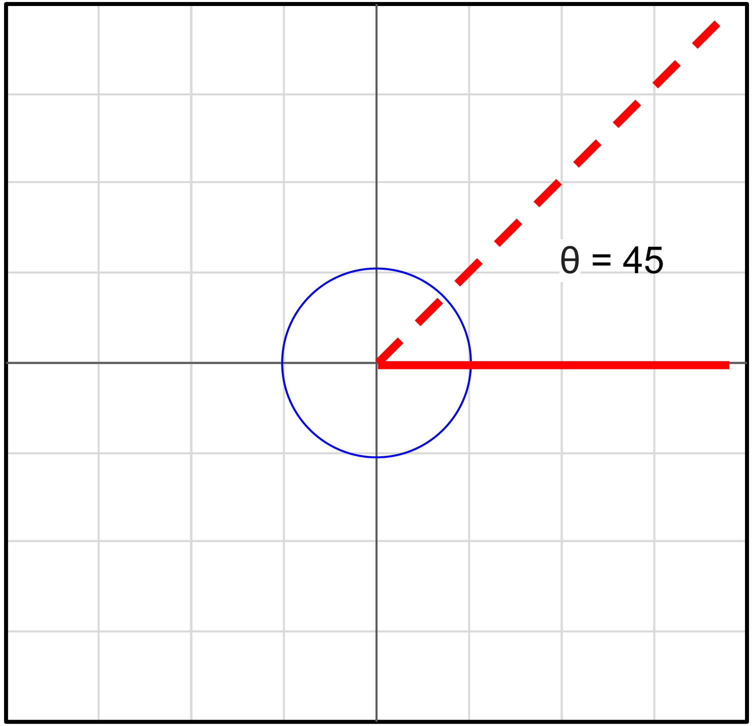

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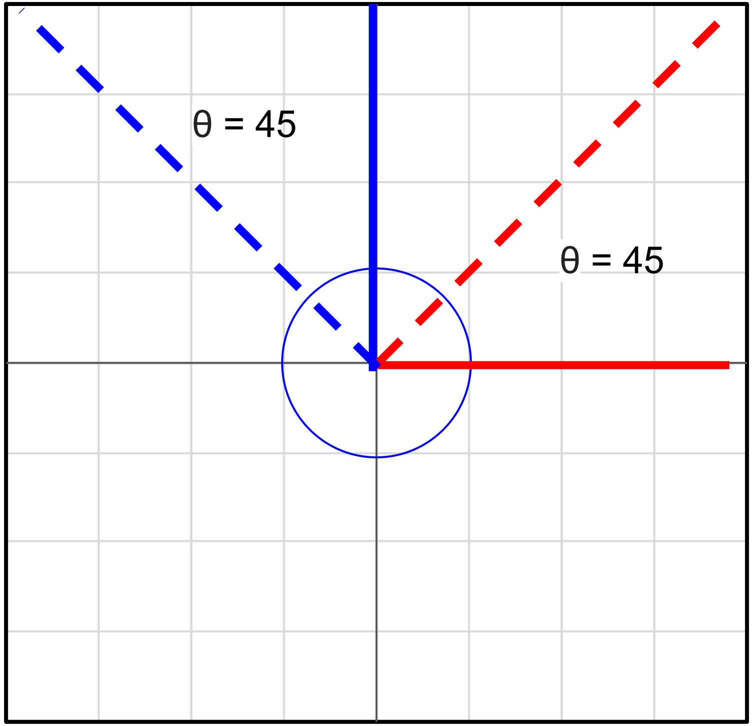
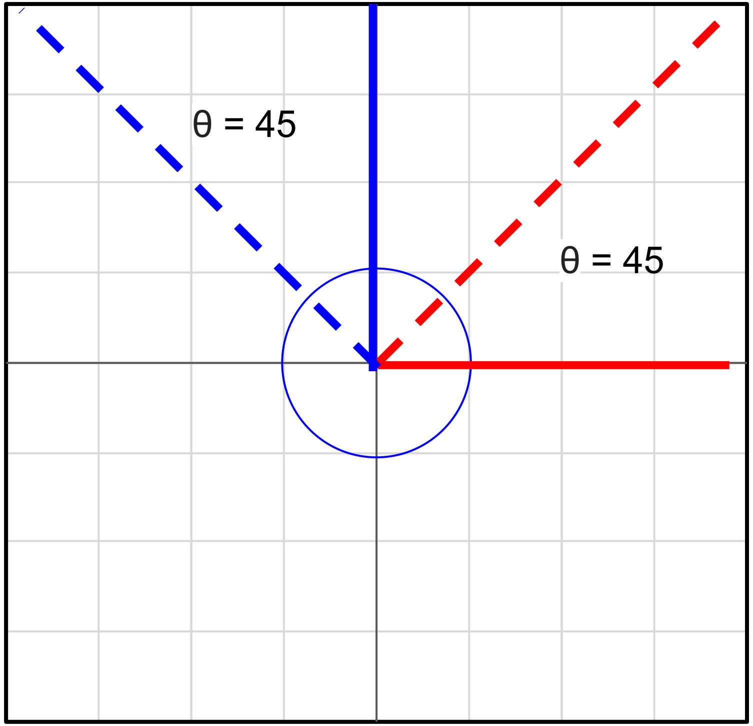


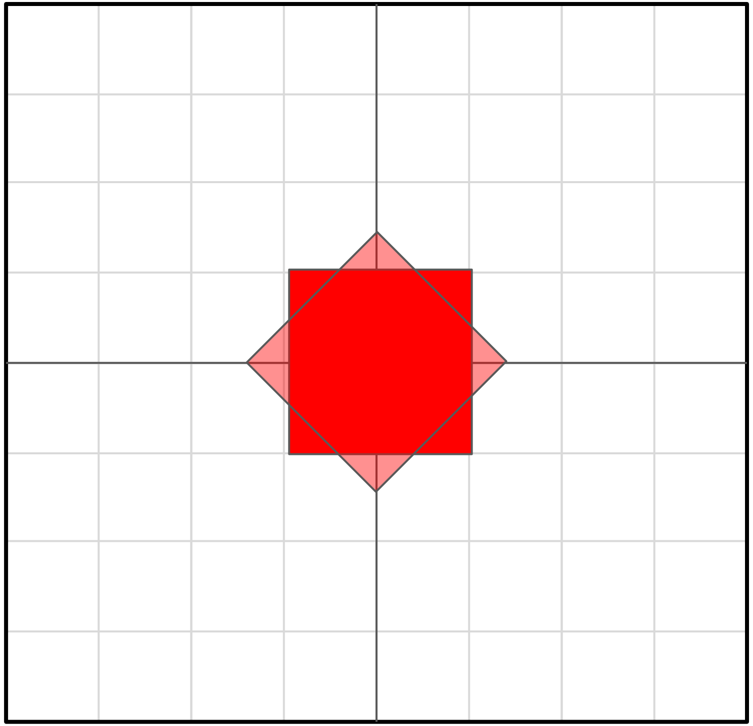
# Rotation

- Let's try a 45 degree rotation counter-clockwise.
- What are our transformed basis vectors?
  - $\boldsymbol{i}$ = [cos(45) sin(45)]
  - $\boldsymbol{j}$ = [cos(90 + 45) sin(90 + 45)]
    = [-sin(45) cos(45)]



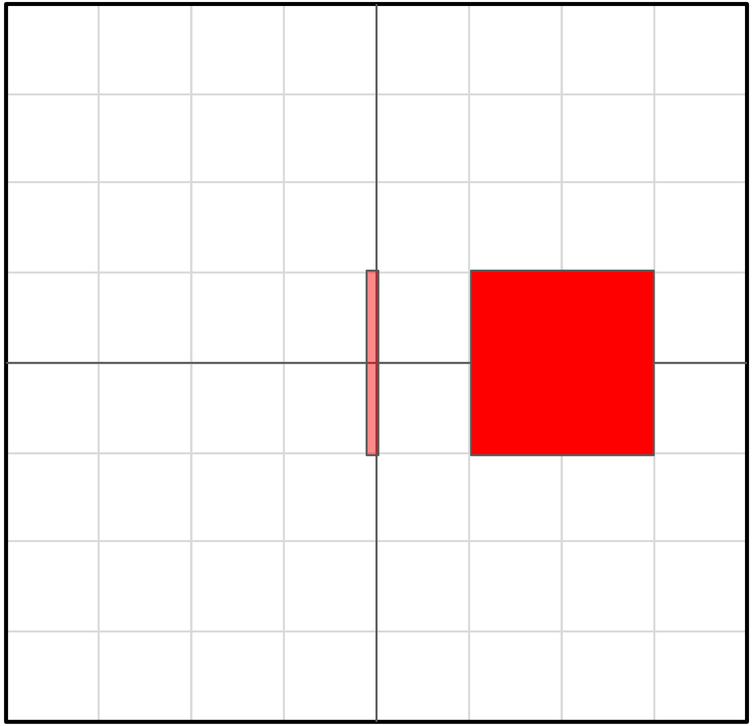
# Rotation

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} cos(45) & sin(45) \\ -sin(45) & cos(45) \end{bmatrix} = \begin{bmatrix} ? & ? \end{bmatrix}$$



# Orthographic Projection

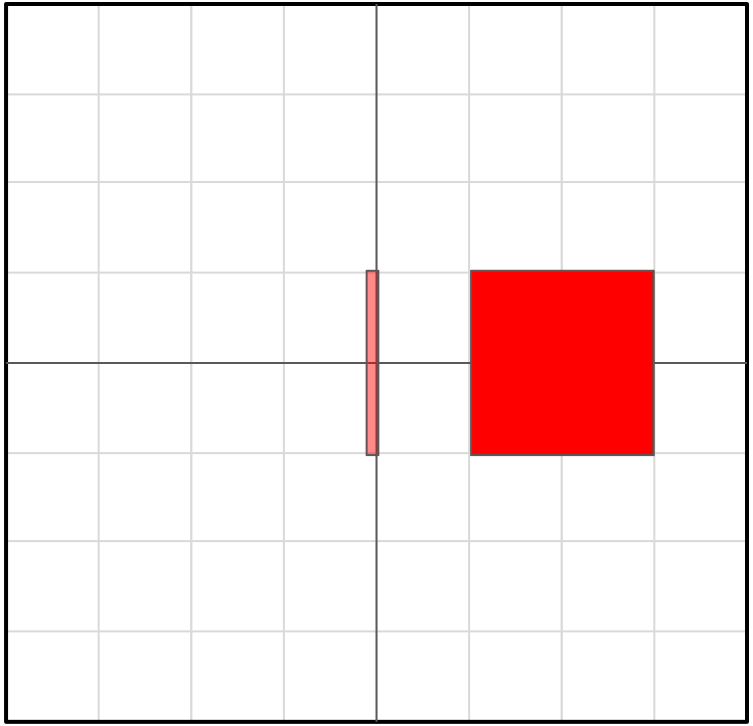
- Allows you to discard a dimension.
- So far the transformations we've looked at are all reversible. But here it is not.



# Orthographic Projection

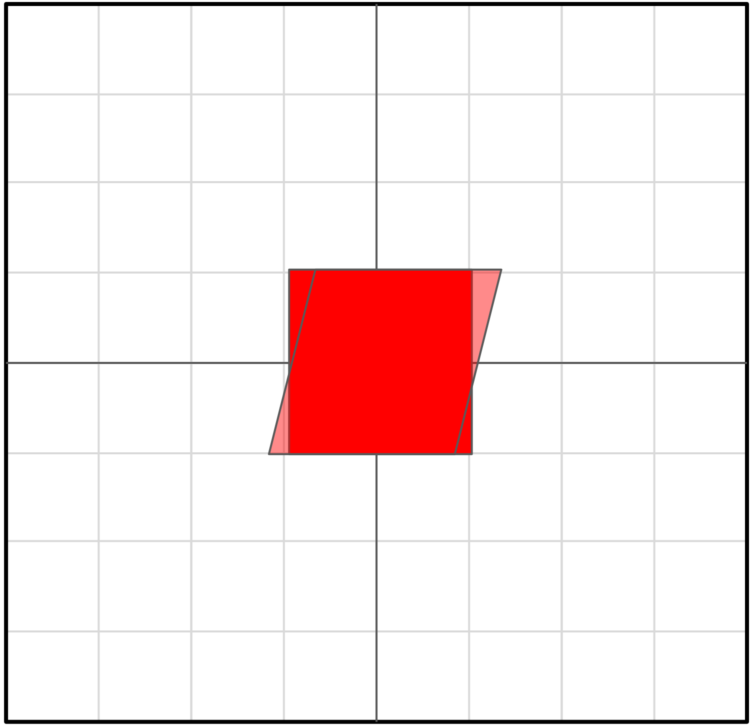
$$\mathbf{P}_x = \mathbf{S}\left(\begin{bmatrix}0 & 1\end{bmatrix}, 0\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{P}_y = \mathbf{S}\left(\begin{bmatrix}1 & 0\end{bmatrix}, 0\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$



# Shear

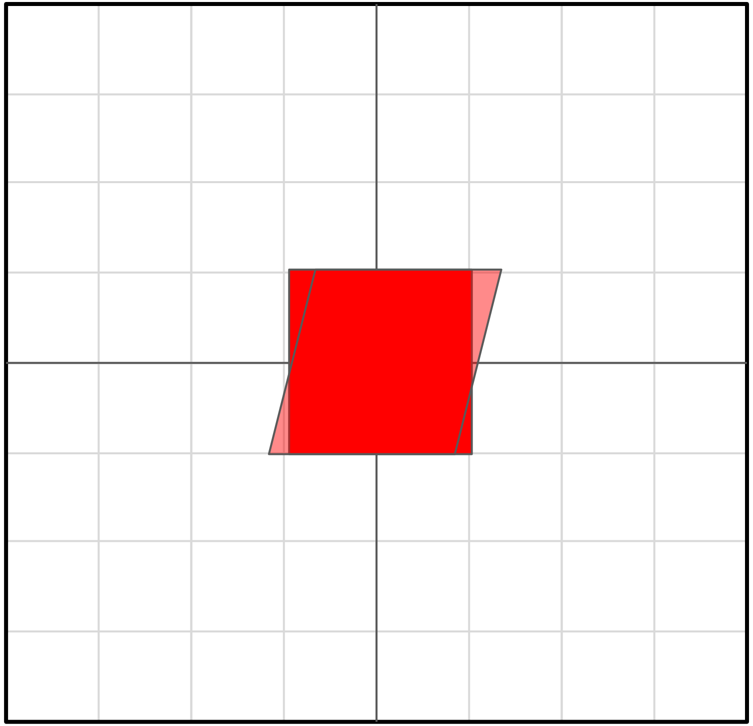
- Allows us to skew our vector space.
- Basis vectors no longer orthogonal.
- Angles are not preserved but area and volume is.



# Shear

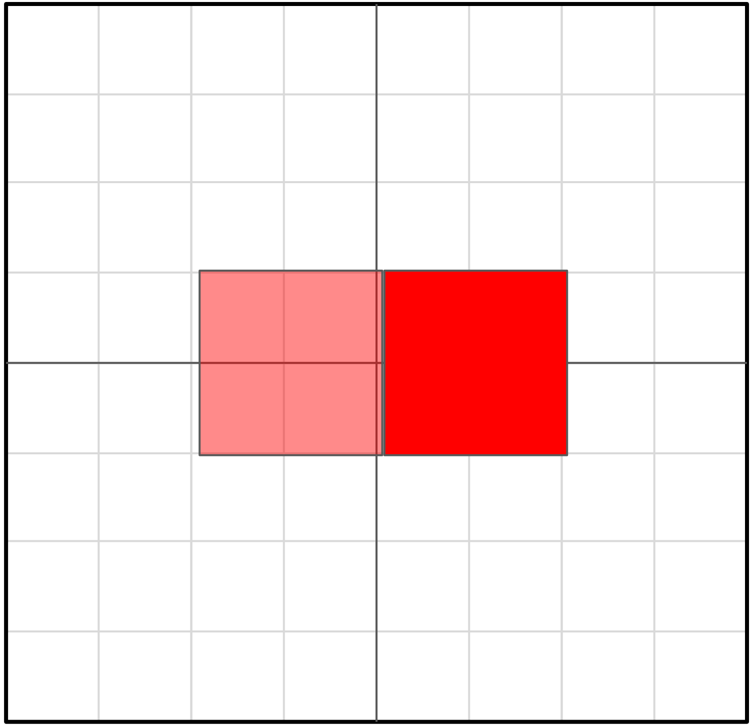
$$\mathbf{H}_x(s) = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix},$$

$$\mathbf{H}_y(s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}.$$



# Reflection

- Reflection is just a scaling by a negative factor.



# Affine Transforms

- So far all of our transformations have been fixed around the origin.
- An affine transformation is the composition of a linear transform followed by a translation.
- All linear transforms are also affine transformations with translation $c = 0$.

# Affine Transforms

- 

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{bmatrix}$$

# Affine Transforms

- 

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\1 & 2 & 3 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\2 & 4 & 8 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\3 & 6 & 11 & 1\end{bmatrix}$$

# Affine Transforms

- 

$$[x \;\; y \;\; z \;\; w] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{bmatrix} = [? \;\; ? \;\; ? \;\; ?]$$

# Transforming Point

- 

$$\begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{bmatrix} = \begin{bmatrix} (x + t_x) & (y + t_y) & (z + t_z) & 1 \end{bmatrix}$$

# Transforming Vector

- 

$$[x \;\; y \;\; z \;\; 0] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{bmatrix} = [x \;\; y \;\; z \;\; 0]$$

# Homogenous Space