Quantum properties

- **Discretization** - Energy is discrete in atoms and molecules. This means that only some values of energy are allowed. These allowed values of energy are also called energy levels. Discretization allows quantum objects to be used for computing. Their energy levels can be used as 0 and 1.

- **Superposition** - Quantum objects can be in a combination of multiple possible states. For example, in an atom, electrons are in a superposition of many possible positions.

- **Interference** - The possible states of quantum objects can add up or cancel out.
    - Ex: noise cancelling headphones, which produce sound waves that cancel out external noise through interference.

- **Entanglement** - Two quantum objects are entangled if the state of one object depends on the state of another. If you know the state of one quantum object, you know the state of the other. Entanglement is unaffected by distance. Entangled quantum particles remain entangled even if they are separated by millions of miles.

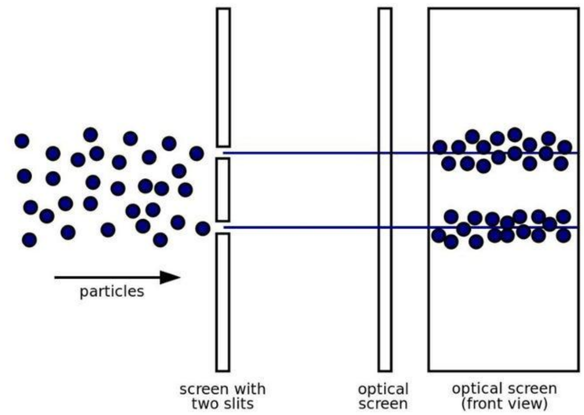
- **Measurement** - The results of measurements on quantum objects can be random. Phenomena like superposition, interference, and entanglement and the reason and make it extremely difficult to predict the exact outcome of the measurement. In addition, the state of the quantum object being measured can change as a result of the measurement.

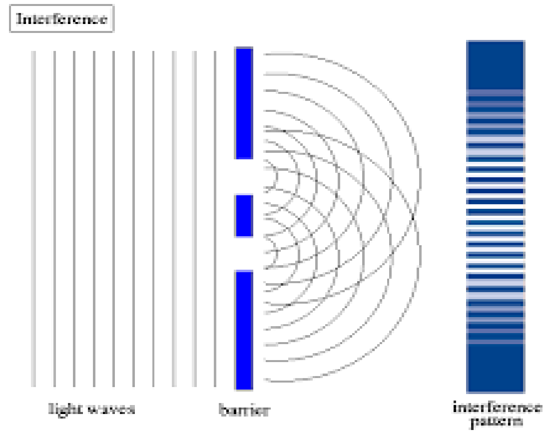
What makes Quantum properties possible?

Wave properties: Waves, such as light and water waves, travel with a unique velocity. Further, waves interact with each other to form complex patterns in a process called interference. Sometimes waves can add onto each other to create a bigger wave (constructive interference), and sometimes waves can cancel each other out (destructive interference)

Particle properties: Particles have mass, a definite, discrete location, and also travel with a certain velocity. Anything that can be measured consistently in a particle is called a particle property.

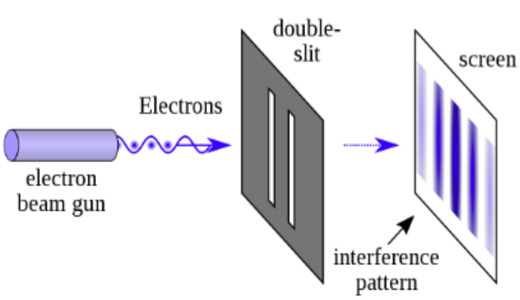
Double-slit experiment: The double-slit experiment helps visualize the differences between waves and particles. In this experiment, the wave or particle is aimed at two slits, behind which is a plain wall. Particles pass through either the left or the right slit, and create two lines of discrete spots on the wall. Waves passing through the slits interfere with each other and create a pattern of bright and dark lines on the wall.





Explanation:

Wave-particle duality: Quantum objects show both wave-like and particle-like properties. When quantum objects, such as photons and electrons, are used in a double slit experiment, they create discrete spots (like particles) but the spots are arranged in an interference pattern (like waves). Because of wave-particle duality, we can think of qubits as both waves and particles.



Superposition with waves: Superposition stems from wave-particle duality. Using the wave nature of qubits, we can represent the two states of the qubit (0 and 1) with two waves. To create a superposition state, we can combine these waves.

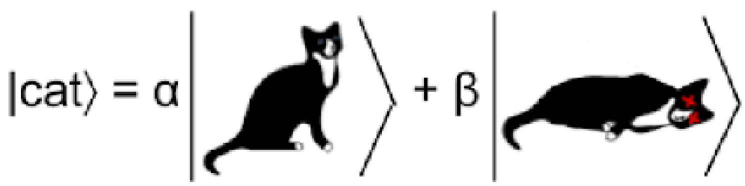
Interference with waves: Interference also stems from wave-particle duality, and can be described as the addition or subtraction of the waves representing qubit states. Both superposition and interference involve overlap between waves.

Discretization with waves: The discreteness of the quantum world comes into play when quantum objects are confined. Confinement of waves forces them to only take certain shapes or energies. In the quantum world, confinement can exist naturally or artificially.

- Ex. in a trapped ion qubit, where the negatively charged electron is confined by the positively charged nucleus.
- Ex. such as through electric circuits in superconducting qubits.

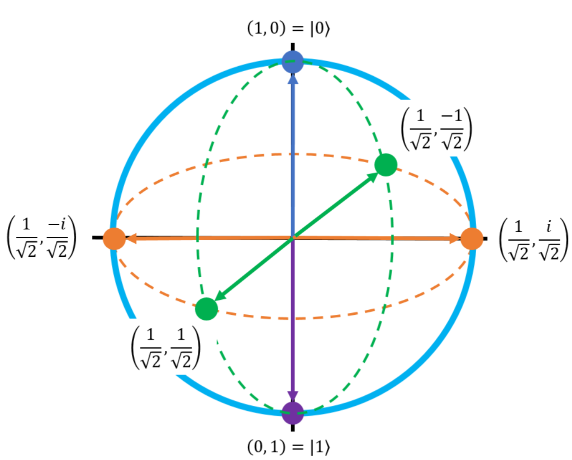
Representing Qubits

Ket notation: The ket notation is used to represent the state of qubits. Putting a "0" or a "1" inside a ket shows that it represents a quantum state. Implicitly, the state is a vector but more on that later.



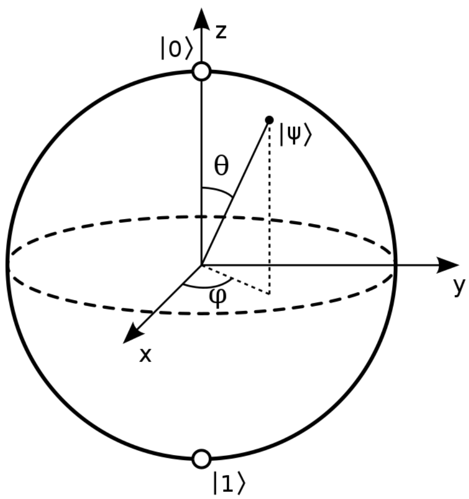
Bloch sphere: The Bloch sphere is a way to visually represent qubit states. Any individual qubit state can be represented on the Bloch sphere. However, entangled states are a notable exception.

- The |0> state is located at the top of the Bloch sphere, and the |1> state at the bottom.
- Any other state on the Bloch sphere represents a superposition of |0> and |1>. A superposition can be equal, meaning that |0> and |1> contribute equally to the state, or unequal, meaning that either |0> contributes more or |1> does. If the state is closer to |0>, it has a greater contribution from |0>. If it is closer to |1>, it has a greater contribution from |1>.



Quantum Gates: Quantum gates manipulate or change the state of qubits. Gates are how we create superposition, interference, and entanglement. The operation of gates on qubits can be visualized as rotations on the Bloch sphere for a single qubit. Quantum Gates are a generalization of classical logic gates.

To visualize these rotations, we need to associate a coordinate system with the Bloch sphere. Here is the conventional coordinate system:



The X gate: The X gate can be visualized as a 180 degree rotation about the X axis.

- | 0 > →| 1 > or | 1 > → | 0 >

The H gate: The H gate creates superposition. It is a uniquely quantum gate. Here, |+> and |-> represent two superposition states.

- | 0 > →| + > or | 1 > → | - >

The Z gate: The Z gate performs a 180 degree rotation about the Z axis.

- | + > →| - > or | - >→| + >

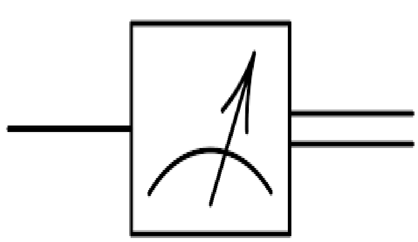
The | 1 > and | 0 > states are unaffected by the Z gate since it points in the vertical direction as well. However, the Z gate gives the |1> state a negative sign. This however, does not change where the state lies on the Bloch sphere.

- | 1 > → - | 1 >

How to find out what state the qubit was in?

The Quantum Circuit model captures the 3 major parts of any quantum circuit: the qubit states, gates, and measurements.

a. Measurement is the final step of any circuit. It is how we extract information about the state of our qubits in the actual circuit. Without measurements, we would never know what state our qubits were in and we would not get the results of our computations.



When we make measurements on a circuit, we get one of two answers, either |0> or |1>. Some circuits have a definite answer to this question. For other circuits there isn’t a definite answer to this question. When dealing with qubits in superposition, the measurement we get is a random result. Sometimes we will get |0>, and sometimes we will get |1>.

b. After the measurement, the state of the qubit changes to the state that it was measured in. Therefore, if the result of the measurement was |1>, the qubit will be in state |1> after the measurement. Measurement can change the state of the qubit. This change is also known as collapse.

c. We can visualize this collapse by thinking about the wave representation of qubit states. If the qubit is in a superposition state before measurement, its wave will collapse to the wave representing the |0> state or the |1> state after measurement.

How to find state of qubit with superposition

- We can run the circuit and measure the state of the qubit many, many times. Repeated measurements on an identically prepared qubit increase our knowledge about the qubit by revealing the statistical distribution it emerges from.

- Although we cannot always predict what the exact result of an individual measurement will be, we can always predict the probability of different possible results of a measurement, however doing so is not always computationally efficient.

Measuring with Different Bases

In the Z basis, these two possible answers are |0> and |1>. In the X basis, the two possible answers are |+> and |->. In other bases, the two answers would be two other states. Luckily, as all vectors in space can be expressed with a complete spanning basis, no one particular basis is superior to the others. Physicists choose the Z basis for historic reasons, mainly because projections along the Z-axis in spherical coordinates are especially convenient.

The randomness of a quantum measurement is determined partially by both the state being measured, and the basis that the measurement is performed in.

Math of Quantum Measurements

Born rule: The probability of measuring a qubit in the |0> state is given by the square of the contribution of |0> to that qubit's normalized state vector. Similarly, the probability of measuring a qubit in the |1> state is given by the square of the contribution of |1> to that qubit's normalized state vector.

Interesting Fact: Physicists do not fully know if the Born rule works. However, it correctly predicts the measurement probabilities of every quantum experiment ever done and is the most consistent assumption to our expectations from physics.

The Born rule is used to get the probability from the state coefficient, called the amplitude. The probabilities of the state being measured as |0> or |1> must add up to 1. Therefore, the squares of the contributions of |0> and |1> to the quantum state must add up to 1, which is exactly what we ensure by normalizing the state.

Vector Notation

Vector notation: The vector notation for qubit states allows us to represent the component of |0> and |1> in a superposition state mathematically. The vector for a single-qubit state contains two numbers in a column - the top number is the contribution of |0>, and the bottom number is the contribution of |1>.

Equal vs unequal superpositions: In an equal superposition, the contributions of |0> and |1> are equal up to something called a global phase, more on that later. One can think of this as the two numbers in the vector for that quantum state being equal. In an unequal superposition, either |0> or |1> contributes more to the superposition state. Therefore, the number corresponding to the state that contributes more is greater in the vector representation of the superposition state.

Vector addition: To get the corresponding vector form of a superposition state, we can add the vectors for the |0> and |1> state, scaled by their contributions in a process called normalization which is important to ensure the probability does not go above 100%. For example, to get the vector form of the |+> state, we just add the vectors for |0> and |1>, since their contributions are equal:

Rules for vector addition: To add vectors, we add the numbers in each row of the vector. The two vectors must have the same number of rows; if they have different numbers of rows, they cannot be added.

Normalizing a Vector

Vector normalization: The Bloch sphere has a radius of 1 unit. Since all quantum states lie on the Bloch sphere, they must have a length of 1 unit as well. Therefore, we must ensure that the vector for any quantum state we write down has a length of 1. This process is known as normalization. Here are the steps for normalizing quantum states:

Step 1: Write the state in the vector form $\begin{bmatrix} a \\ b \end{bmatrix}$

Step 2: Find the length of the state as $\sqrt{a^2 + b^2}$

Step 3: If the length is 1, the state is normalized. If not, divide the state by its length to get the normalized state

Normalized form of the state $= \frac{1}{\sqrt{a^2+b^2}} \begin{bmatrix} a \\ b \end{bmatrix}$

Phase: To obtain the |+> state, we added the contributions of the |0> and |1> states together. Another way to make an equal superposition is to subtract these contributions. This process leads to the |-> state

Matrices

Matrix notation: In any quantum circuit, we apply quantum gates to the initial quantum state to find the final quantum state. The transformation of the input state to the output is achieved using a matrix.

For single-qubit gates, a matrix consists of two columns of numbers.

Gate	Notation	Matrix
NOT (Pauli-X)	$-\boxed{X}-$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Pauli-Z	$-\boxed{Z}-$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Hadamard	$-\boxed{H}-$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Using matrices: We can use the matrix for a quantum gate to calculate the result of applying that gate to a quantum state. To find this result, we multiply the matrix for the quantum gate with the vector for the quantum state.

Rules for Multiplying Matrices: The first matrix must have the same number of columns as the second matrix has rows. The number of rows of the resulting matrix equals the number of rows of the first matrix, and the number of columns of the resulting matrix equals the number of columns of the second matrix.

Applying multiple gates to quantum states: Most quantum circuits will have more than one gate. We can find the result of applying multiple gates to an initial quantum state by applying one gate at a time. So, in a circuit with two gates, we first find the result of applying the first gate using matrix-vector multiplication. Then, we find the result of applying the second gate, again using matrix vector multiplication.

Generating superposition: Superposition is an important quantum resource used by all quantum algorithms. To generate superposition in the circuit, we use the H gate. Using the matrix of the H gate, we can prove how applying the H gate to the |0> states results in the |+> state, which is an equal superposition of |0> and |1>.

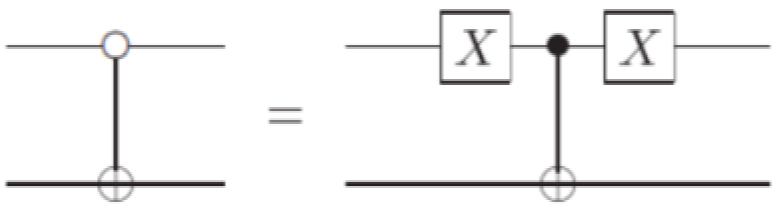
Multi Qubit Circuits

Multi-qubit circuits: A multi-qubit circuit is a circuit with more than one qubit. Each qubit can have any number of gates on it

Types of Gates:

- Single qubit gates: These are all the gates we are familiar with - X, Z, H, Rx, Rz. They all act on a single qubit at a time. However, any 2x2 Unitary Matrix is a valid single qubit gate composable with this gate set, i.e. the gate set given is computationally complete.
- Control gates: These gates act on multiple qubits and only apply their operation if their control qubits are in the one state. Note, controls can be in a superposition leading to a situation where the gate is both applied and not applied, pending on which basis state one wishes to consider.

Controlled Gates: The control qubit is not affected by the gate - its state remains unchanged. The target qubit is dependent on the control qubit. If the control qubit is in state |0>, no gate is applied to the target qubit. If the control qubit is in state |1>, a gate is applied to the target qubit. The specific gate that gets applied to the target qubit determines the name of the gate. If the gate that gets applied is an X gate, we call it a controlled X (CX) gate. If it is a Z gate, we call it a controlled Z (CZ) gate.



Representing Multi Qubits

Kets for multi-qubit circuits: We can express multi-qubit states using the ket notation by combining the kets for the single qubits together.

- Ex… a qubit in |1 > and another in |0> would be |10>

Vector notation for multi-qubit states: For a single qubit, the vector had two numbers, corresponding to the contributions of |0> and |1> to the qubit's state. For 2 qubits, the vector will have 4 numbers, corresponding to the contributions of |00>, |01>, |10>, and |11>.

Finding final state of multi-qubit circuits:

a. Find the ket representation of the final state by applying the gates in the circuit. Here, we follow the usual rules for applying gates to kets that we are familiar with.

b. Fill in the vector based on the ket.

Measurement on Multi Qubit States

Measurement on multi-qubit circuits: With two qubits, we have four possible results of measurement in the Z basis. The two qubits could be measured to be in the state |00>, |01>, |10>, or |11>.

Born rule - The probability of measuring one of these possible states (|00>, |01>, |10>, and |11>) is equal to the square of the contribution of that state to the overall state vector of the qubits being measured, officially found by taking an inner product of the state with the measurement result and squaring its magnitude.

Predicting the outcome of measurement in multi-qubit circuits:

- Find the final state of the circuit, using the rules of single-qubit and controlled gates that we have studied over the last few weeks

- Using the final state and the Born rule, predict the probability of measuring the circuit in each of the possible states.

Quantum Key Distribution

One area that is seeing an influx in applications of quantum properties is cybersecurity, and to be more specific,