$$\alpha x_{\xi\xi}+2\beta x_{\xi\eta}+\gamma x_{\eta\eta}=\varphi$$
$$\alpha y_{\xi\xi}+2\beta y_{\xi\eta}+\gamma y_{\eta\eta}=\psi$$
$$\varphi=-\mathrm{J}^{-2}\left(\mathrm{Px}_{\xi}+\mathrm{Q}x_{\eta}\right)$$
$$\psi=-\mathrm{J}^{-2}\left(\mathrm{Py}_{\xi}+\mathrm{Qy}_{\eta}\right)$$

$$\alpha=x_{\eta}^{2}+y_{\eta}^{2}\text{与}\beta=-\left(x_{\xi}x_{\eta}+y_{\xi}y_{\eta}\right)\text{与}\gamma=x_{\xi}^{2}+y_{\xi}^{2}$$

得到:

$$\alpha x_{\xi\xi}+2\beta x_{\xi\eta}+\gamma x_{\eta\eta}=\left(x_{\eta}^{2}+y_{\eta}^{2}\right)*x_{\xi\xi}-2*\left(x_{\xi}x_{\eta}+y_{\xi}y_{\eta}\right)*x_{\xi\eta}+\left(x_{\xi}^{2}+y_{\xi}^{2}\right)*x_{\eta\eta}=-\mathrm{J}^{-2}\left(\mathrm{Px}_{\xi}+\mathrm{Q}x_{\eta}\right).$$

$$\alpha y_{\xi\xi}+2\beta y_{\xi\eta}+\gamma y_{\eta\eta}=(x_{\eta}^{2}+y_{\eta}^{2})*y_{\xi\xi}-2*\left(x_{\xi}x_{\eta}+y_{\xi}y_{\eta}\right)*y_{\xi\eta}+\left(x_{\xi}^{2}+y_{\xi}^{2}\right)*y_{\eta\eta}=-\mathrm{J}^{-2}\left(\mathrm{Py}_{\xi}+\mathrm{Qy}_{\eta}\right).$$

对于 x 由离散方式:

$$x_{\xi}=\frac{x_{i+1,j}-x_{i-1,j}}{2}\quad x_{\xi\xi}=x_{i+1,j}+x_{i-1,j}-2x_{i,j}$$
$$x_{\eta}=\frac{x_{i,j+1}-x_{i,j-1}}{2}\quad x_{\eta\eta}=x_{i,j+1}+x_{i,j-1}-2x_{i,j}$$
$$x_{\xi\eta}=x_{i+1,j}+x_{i-1,j}+x_{i,j+1}+x_{i,j-1}-4x_{i,j}$$

对于 y 有离散方式（将上述的 x 替换成 y):

$$y_{\xi}=\frac{y_{i+1,j}-y_{i-1,j}}{2}\quad y_{\xi\xi}=y_{i+1,j}+y_{i-1,j}-2y_{i,j}$$
$$y_{\eta}=\frac{y_{i,j+1}-y_{i,j-1}}{2}\quad y_{\eta\eta}=y_{i,j+1}+y_{i,j-1}-2y_{i,j}$$
$$y_{\xi\eta}=y_{i+1,j}+y_{i-1,j}+y_{i,j+1}+y_{i,j-1}-4y_{i,j}$$

$\left(x_\eta^2+y_\eta^2\right)*x_{\xi\xi}-2*\left(x_\xi x_\eta+y_\xi y_\eta\right)*x_{\xi\eta}+\left(x_\xi^2+y_\xi^2\right)*x_{\eta\eta}=-\mathrm{J}^{-2}\left(\mathrm{P}\mathrm{x}_\xi+\mathrm{Q}x_\eta\right)$.对应离散方程每一项有:

$\alpha=x_\eta^2+y_\eta^2=(\frac{x_{i,j+1}-x_{i,j-1}}{2})^2+(\frac{y_{i,j+1}-y_{i,j-1}}{2})^2$

$\beta=-(x_\xi x_\eta+y_\xi y_\eta)=-\left(\frac{x_{i+1,j}-x_{i-1,j}}{2}\right)*\left(\frac{x_{i,j+1}-x_{i,j-1}}{2}\right)-\left(\frac{y_{i+1,j}-y_{i-1,j}}{2}\right)*\left(\frac{y_{i,j+1}-y_{i,j-1}}{2}\right)$.

$\gamma=x_\xi^2+y_\xi^2=(\frac{x_{i+1,j}-x_{i-1,j}}{2})^2+(\frac{y_{i+1,j}-y_{i-1,j}}{2})^2$

原式有:

$$\left((\frac{x_{i,j+1}-x_{i,j-1}}{2})^2+(\frac{y_{i,j+1}-y_{i,j-1}}{2})^2\right)*(x_{i+1,j}+x_{i-1,j}-2x_{i,j})-2*\left(\left(\frac{x_{i+1,j}-x_{i-1,j}}{2}\right)*\left(\frac{x_{i,j+1}-x_{i,j-1}}{2}\right)+\left(\frac{y_{i+1,j}-y_{i-1,j}}{2}\right)*\left(\frac{y_{i,j+1}-y_{i,j-1}}{2}\right)\right)*(x_{i+1,j}+x_{i-1,j}+x_{i,j+1}+x_{i,j-1}-4x_{i,j})+\left((\frac{x_{i+1,j}-x_{i-1,j}}{2})^2+\left(\frac{y_{i+1,j}-y_{i-1,j}}{2}\right)^2\right)*(x_{i,j+1}+x_{i,j-1}-2x_{i,j})=-\mathrm{J}^{-2}\left(P*\left(\frac{x_{i+1,j}-x_{i-1,j}}{2}\right)+\mathrm{Q}*(\frac{x_{i,j+1}-x_{i,j-1}}{2})\right)$$

改写成$C_{i-1,j}x_{i-1,j}+C_{i+1,j}x_{i+1,j}+C_{i,j+1}x_{i,j+1}+C_{i,j-1}x_{i,j-1}+C_{i,j}x_{i,j}=\varphi_{ij}$的形式可得每一项系数

$$\varphi_{ij}=-\mathrm{J}^{-2}\left(P*\left(\frac{x_{i+1,j}-x_{i-1,j}}{2}\right)+\mathrm{Q}*(\frac{x_{i,j+1}-x_{i,j-1}}{2})\right).$$

$$C_{i,j}=-2*(\left(\frac{x_{i,j+1}-x_{i,j-1}}{2}\right)^2+\left(\frac{y_{i,j+1}-y_{i,j-1}}{2}\right)^2+\left(\left(\frac{x_{i+1,j}-x_{i-1,j}}{2}\right)^2+\left(\frac{y_{i+1,j}-y_{i-1,j}}{2}\right)^2\right)+8(\frac{x_{i+1,j}-x_{i-1,j}}{2}*\frac{x_{i,j+1}-x_{i,j-1}}{2}+\frac{y_{i+1,j}-y_{i-1,j}}{2}*\frac{y_{i,j+1}-y_{i,j-1}}{2}).$$

$$C_{i,j+1}=C_{i,j-1}=(\frac{x_{i+1,j}-x_{i-1,j}}{2})^2+\left(\frac{y_{i+1,j}-y_{i-1,j}}{2}\right)^2-2*(\frac{x_{i+1,j}-x_{i-1,j}}{2}*\frac{x_{i,j+1}-x_{i,j-1}}{2}+\frac{y_{i+1,j}-y_{i-1,j}}{2}*\frac{y_{i,j+1}-y_{i,j-1}}{2}).$$

$$C_{i+1,j}=C_{i-1,j}=(\frac{x_{i,j+1}-x_{i,j-1}}{2})^2+(\frac{y_{i,j+1}-y_{i,j-1}}{2})^2-2*(\frac{x_{i+1,j}-x_{i-1,j}}{2}*\frac{x_{i,j+1}-x_{i,j-1}}{2}+\frac{y_{i+1,j}-y_{i-1,j}}{2}*\frac{y_{i,j+1}-y_{i,j-1}}{2}).$$

$$C_{i,j}=-2*(\alpha+\gamma)-8*\beta$$

$$C_{i,j+1}=C_{i,j-1}=\gamma+2*\beta$$

$$C_{i+1,j}=C_{i-1,j}=\alpha+2*\beta$$

$\left(x_\eta^2+y_\eta^2\right)*y_{\xi\xi}-2*\left(x_\xi x_\eta+y_\xi y_\eta\right)*y_{\xi\eta}+\left(x_\xi^2+y_\xi^2\right)*y_{\eta\eta}=-J^{-2}\left(Py_\xi+Qy_\eta\right)$. *对应离散方程*

$(x_\xi^2+y_\xi^2)=\ (\frac{x_{i+1,j}-x_{i-1,j}}{2})^2+(\frac{y_{i+1,j}-y_{i-1,j}}{2})^2$

$\left(x_\xi x_\eta+y_\xi y_\eta\right)=\left(\frac{x_{i+1,j}-x_{i-1,j}}{2}\right)*\left(\frac{x_{i,j+1}-x_{i,j-1}}{2}\right)+\left(\frac{y_{i+1,j}-y_{i-1,j}}{2}\right)*\left(\frac{y_{i,j+1}-y_{i,j-1}}{2}\right)$.

$x_\xi^2+y_\xi^2=(\frac{x_{i+1,j}-x_{i-1,j}}{2})^2+(\frac{y_{i+1,j}-y_{i-1,j}}{2})^2$

原式有:

$$\left((\frac{x_{i,j+1}-x_{i,j-1}}{2})^2+(\frac{y_{i,j+1}-y_{i,j-1}}{2})^2\right)*\left(y_{i+1,j}+y_{i-1,j}-2y_{i,j}\right)-2*\left(\left(\frac{x_{i+1,j}-x_{i-1,j}}{2}\right)*\left(\frac{x_{i,j+1}-x_{i,j-1}}{2}\right)+\left(\frac{y_{i+1,j}-y_{i-1,j}}{2}\right)*\left(\frac{y_{i,j+1}-y_{i,j-1}}{2}\right)\right)*\left(y_{i+1,j}+y_{i-1,j}+y_{i,j+1}+y_{i,j-1}-4y_{i,j}\right)+\left(\left(\frac{x_{i+1,j}-x_{i-1,j}}{2}\right)^2+\left(\frac{y_{i+1,j}-y_{i-1,j}}{2}\right)^2\right)*\left(y_{i,j+1}+y_{i,j-1}-2y_{i,j}\right)=-J^{-2}\left(P*\frac{y_{i+1,j}-y_{i-1,j}}{2}+Q*\frac{y_{i,j+1}-y_{i,j-1}}{2}\right)$$

改写成$D_{i-1,j}y_{i-1,j}+D_{i+1,j}y_{i+1,j}+D_{i,j+1}y_{i,j+1}+D_{i,j-1}y_{i,j-1}+D_{i,j}y_{i,j}=\psi_{ij}$的形式可得每一项系数

$$\psi_{ij}=-J^{-2}\left(P*\frac{y_{i+1,j}-y_{i-1,j}}{2}+Q*\frac{y_{i,j+1}-y_{i,j-1}}{2}\right)$$

$D_{i,j}$ =-2*$\left(\left((\frac{x_{i,j+1}-x_{i,j-1}}{2})^2+(\frac{y_{i,j+1}-y_{i,j-1}}{2})^2\right)+\left(\frac{x_{i+1,j}-x_{i-1,j}}{2}\right)^2+\left(\frac{y_{i+1,j}-y_{i-1,j}}{2}\right)^2\right)+8*(\left(\frac{x_{i+1,j}-x_{i-1,j}}{2}\right)*\left(\frac{x_{i,j+1}-x_{i,j-1}}{2}\right)+\left(\frac{y_{i+1,j}-y_{i-1,j}}{2}\right)*\left(\frac{y_{i,j+1}-y_{i,j-1}}{2}\right))$.

$$D_{i,j+1}=D_{i,j-1}=\left(\frac{x_{i+1,j}-x_{i-1,j}}{2}\right)^2+\left(\frac{y_{i+1,j}-y_{i-1,j}}{2}\right)^2-2*\left(\left(\frac{x_{i+1,j}-x_{i-1,j}}{2}\right)*\left(\frac{x_{i,j+1}-x_{i,j-1}}{2}\right)+\left(\frac{y_{i+1,j}-y_{i-1,j}}{2}\right)*\left(\frac{y_{i,j+1}-y_{i,j-1}}{2}\right)\right)$$

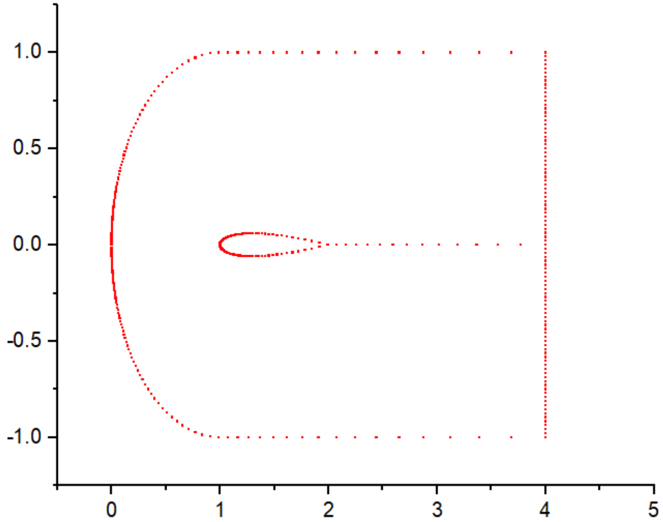
$$D_{i+1,j}=D_{i-1,j}=\left((\frac{x_{i,j+1}-x_{i,j-1}}{2})^2+(\frac{y_{i,j+1}-y_{i,j-1}}{2})^2\right)-2*\left(\left(\frac{x_{i+1,j}-x_{i-1,j}}{2}\right)*\left(\frac{x_{i,j+1}-x_{i,j-1}}{2}\right)+\left(\frac{y_{i+1,j}-y_{i-1,j}}{2}\right)*\left(\frac{y_{i,j+1}-y_{i,j-1}}{2}\right)\right)$$

$$D_{i,j}=-2*(\alpha+\gamma)-8*\beta$$

$$D_{i,j+1}=D_{i,j-1}=\gamma+2*\beta$$

$$D_{i+1,j}=D_{i-1,j}=\alpha+2*\beta$$

根据等差数列关系，首先生成初始网格边界值，并且再翼型头部附近加密，同时再外计算域加密得到的初始结果如下图所示，加密所用的等比数列系数通过二分法求解：



采用 LU-SGS 对方程进行迭代求解无源项离散方程，xy 坐标方程形式一致，仅仅只有坐标形式上的差距，因此有：

$$D_{i-1,j}y_{i-1,j}+D_{i+1,j}y_{i+1,j}+D_{i,j+1}y_{i,j+1}+D_{i,j-1}y_{i,j-1}+D_{i,j}y_{i,j}=0$$

$$D_{i,j}\tilde{y}_{i,j}+D_{i-1,j}\tilde{y}_{i-1,j}+D_{i,j-1}\tilde{y}_{i,j-1}=0$$

上扫方程：$\tilde{y}_{i,j}=-\frac{D_{i-1,j}\tilde{y}_{i-1,j}+D_{i,j-1}\tilde{y}_{i,j-1}}{D_{i,j}}$

$$D_{i,j}y_{i,j}+D_{i+1,j}y_{i+1,j}+D_{i,j+1}y_{i,j+1}=D_{i,j}\tilde{y}_{i,j}$$

$$\text{下扫方程：} y_{i,j} = \frac{D_{i,j}\tilde{y}_{i,j} - D_{i+1,j}y_{i+1,j} - D_{i,j+1}y_{i,j+1}}{D_{i,j}}$$

由于 LU-SGS 只能上下扫一次给出一个近似解，难以用来将求解网格，因此考虑其他迭代方法

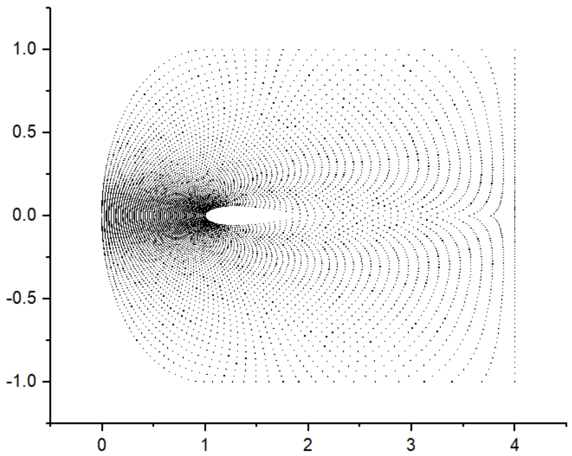
$$D_{i-1,j}y_{i-1,j} + D_{i+1,j}y_{i+1,j} + D_{i,j+1}y_{i,j+1} + D_{i,j-1}y_{i,j-1} + D_{i,j}y_{i,j} = 0$$

采用 JKB 迭代法，对方程进行化简可得如下所示的迭代公式

$$y_{i,j} = -\frac{(D_{i-1,j}y_{i-1,j} + D_{i+1,j}y_{i+1,j} + D_{i,j+1}y_{i,j+1} + D_{i,j-1}y_{i,j-1})}{D_{i,j}}.$$

$$x_{i,j} = -\frac{(C_{i-1,j}x_{i-1,j} + C_{i+1,j}x_{i+1,j} + C_{i,j+1}x_{i,j+1} + C_{i,j-1}x_{i,j-1})}{C_{i,j}}.$$

最后得到的结果如下图所示，再尾延部分网格有所弯曲是由上下边界的网格比例不合适导致的，除此之外头部的网格汇聚可能是由于设定网格初始值的时候对外层网格同样也使用了等比数列加密



通过略微调整代码参数还可以得到另类的网格：