作业 12.1

试推导不可压缩湍动能 k 所满足的控制方程。

其中： $k=\frac{1}{2}\overline{u_i'u_i'}$

要求： 必须给出详细的推导过程，切勿只照抄最终公式

提示：
step 1) 写出脉动量满足的方程 $\frac{\partial u_i'}{\partial t}+\ldots\ldots=\ldots\ldots$ (1)
step 2) 两端乘以 u_i' 并平均，即可的k满足的方程

推导以$\langle * \rangle$代表平均$\overline{*}$

1、N-S 方程：

$$\nabla V=0.$$

$$\rho\frac{DV}{Dt}=\rho f-\nabla p+\mu\nabla^2 V.$$

不考虑体积力，可以采用爱因斯坦求和约定表述，有：

$$\frac{\partial u_i}{\partial x_i}=0$$

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_j\partial x_j}$$

2、采用系综平均：

$$\left\langle\frac{\partial u_i}{\partial x_i}\right\rangle=\frac{\langle\partial u_i\rangle}{\partial x_i}=0.$$

$$\left\langle\frac{\partial u_i}{\partial t}\right\rangle+\left\langle u_j\frac{\partial u_i}{\partial x_j}\right\rangle=\left\langle-\frac{1}{\rho}\frac{\partial p}{\partial x_i}\right\rangle+\left\langle\nu\frac{\partial^2 u_i}{\partial x_j\partial x_j}\right\rangle.$$

3、对动量方程进行平均，并利用输运方程的性质有：

$$\left\langle\frac{\partial u_i}{\partial t}\right\rangle=\frac{\partial\langle u_i\rangle}{\partial t}.$$

$$\left\langle-\frac{1}{\rho}\frac{\partial p}{\partial x_i}\right\rangle=-\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_i}.$$

$$\left\langle\nu\frac{\partial^2 u_i}{\partial x_j\partial x_j}\right\rangle=\nu\frac{\partial^2\langle u_i\rangle}{\partial x_j\partial x_j}.$$

$$\left\langle u_j\frac{\partial u_i}{\partial x_j}\right\rangle=\left\langle\frac{\partial u_iu_j}{\partial x_j}-u_i\frac{\partial u_j}{\partial x_j}\right\rangle=\left\langle\frac{\partial u_iu_j}{\partial x_j}\right\rangle-\left\langle u_i\frac{\partial u_j}{\partial x_j}\right\rangle=\left\langle\frac{\partial u_iu_j}{\partial x_j}\right\rangle=\frac{\partial\langle u_iu_j\rangle}{\partial x_j}=\langle u_j\rangle\frac{\partial\langle u_i\rangle}{\partial x_j}+\frac{\partial\langle u_i'u_j'\rangle}{\partial x_j}.$$

4、替换原先的动量方程来获得 RANS 方程有：

$$\frac{\partial\langle u_i\rangle}{\partial t}+\langle u_j\rangle\frac{\partial\langle u_i\rangle}{\partial x_j}+\frac{\partial\langle u_i'u_j'\rangle}{\partial x_j}=-\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_i}+\nu\frac{\partial^2\langle u_i\rangle}{\partial x_j\partial x_j}.$$

或写为↓

$$\frac{\partial\langle u_i\rangle}{\partial t}+\langle u_j\rangle\frac{\partial\langle u_i\rangle}{\partial x_j}=-\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_i}+\nu\frac{\partial^2\langle u_i\rangle}{\partial x_j\partial x_j}-\frac{\partial\langle u_i'u_j'\rangle}{\partial x_j}.$$

5、将原始N—S方程减去RANS方程可以获得脉动方程有：

对于输运方程：

$$\frac{\partial u_i}{\partial x_i}-\frac{\partial\langle u_i\rangle}{\partial x_i}=\frac{\partial(u_i-\langle u_i\rangle)}{\partial x_i}=\frac{\partial u_i'}{\partial x_i}=0$$

对于运动方程中的各项有：

$$\frac{\partial u_i}{\partial t}-\frac{\partial\langle u_i\rangle}{\partial t}=\frac{\partial(u_i-\langle u_i\rangle)}{\partial t}=\frac{\partial u_i'}{\partial t}.$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_i}=-\frac{1}{\rho}\frac{\partial(p-\langle p\rangle)}{\partial x_i}=-\frac{1}{\rho}\frac{\partial p'}{\partial x_i}.$$

$$\nu\frac{\partial^2 u_i}{\partial x_j\partial x_j}-\nu\frac{\partial^2\langle u_i\rangle}{\partial x_j\partial x_j}=\nu\frac{\partial^2(u_i-\langle u_i\rangle)}{\partial x_j\partial x_j}=\nu\frac{\partial^2 u_i'}{\partial x_j\partial x_j}.$$

对于生成项，可以作如下推导，这里有$u_j=\langle u_j\rangle+u_j'$既平均与脉动之和，且$\frac{\partial u_j'}{\partial x_j}=0$，因此可以写为：

$$\begin{aligned}u_j\frac{\partial u_i}{\partial x_j}-\langle u_j\rangle\frac{\partial\langle u_i\rangle}{\partial x_j}&=\left(\langle u_j\rangle+u_j'\right)\frac{\partial(\langle u_i\rangle+u_i')}{\partial x_j}-\langle u_j\rangle\frac{\partial\langle u_i\rangle}{\partial x_j}\\&=\langle u_j\rangle\frac{\partial\langle u_i\rangle}{\partial x_j}-\langle u_j\rangle\frac{\partial\langle u_i\rangle}{\partial x_j}+\langle u_j\rangle\frac{\partial u_i'}{\partial x_j}+u_j'\frac{\partial\langle u_i\rangle}{\partial x_j}+u_j'\frac{\partial u_i'}{\partial x_j}\\&=\langle u_j\rangle\frac{\partial u_i'}{\partial x_j}+u_j'\frac{\partial\langle u_i\rangle}{\partial x_j}+\frac{\partial u_i'u_j'}{\partial x_j}-u_i'\frac{\partial u_j'}{\partial x_j}=\langle u_j\rangle\frac{\partial u_i'}{\partial x_j}+u_j'\frac{\partial\langle u_i\rangle}{\partial x_j}+\frac{\partial u_i'u_j'}{\partial x_j}\end{aligned}$$

对于$\frac{\partial\left\langle u_i'u_j'\right\rangle}{\partial x_j}$项则不做改变

6、整理上述各项有：

$$\frac{\partial u_i'}{\partial t}+\langle u_j\rangle\frac{\partial u_i'}{\partial x_j}+u_j'\frac{\partial\langle u_i\rangle}{\partial x_j}+\frac{\partial u_i'u_j'}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p'}{\partial x_i}+\nu\frac{\partial^2 u_i'}{\partial x_j\partial x_j}+\frac{\partial\langle u_i'u_j'\rangle}{\partial x_j}.$$

7、两侧同乘脉动u_i'，两侧取平均，且令$k=\frac{1}{2}\overline{u_i'u_i'}$

$$u_i'\frac{\partial u_i'}{\partial t}+u_i'\langle u_j\rangle\frac{\partial u_i'}{\partial x_j}+u_i'u_j'\frac{\partial\langle u_i\rangle}{\partial x_j}+u_i'\frac{\partial u_i'u_j'}{\partial x_j}=-u_i'\frac{1}{\rho}\frac{\partial p'}{\partial x_i}+u_i'\nu\frac{\partial^2 u_i'}{\partial x_j\partial x_j}+u_i'\frac{\partial\left\langle u_i'u_j'\right\rangle}{\partial x_j}$$

对于其中各项有：

$$\left\langle u_i'\nu\frac{\partial^2 u_i'}{\partial x_j\partial x_j}\right\rangle=\left\langle\nu\left(\frac{\partial\left(u_i'\frac{\partial u_i'}{\partial x_j}\right)}{\partial x_j}-\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}\right)\right\rangle=\nu\left(\left\langle\frac{\partial\left(u_i'\frac{\partial u_i'}{\partial x_j}\right)}{\partial x_j}\right\rangle-\left\langle\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}\right\rangle\right)=\frac{\partial}{\partial x_j}\nu\frac{\partial k}{\partial x_j}-\nu\left\langle\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}\right\rangle=\frac{\partial}{\partial x_j}\nu\frac{\partial k}{\partial x_j}-\varepsilon.$$

$$\left\langle u_i'\frac{\partial u_i'}{\partial t}\right\rangle=\frac{\partial k}{\partial t}.$$

$$\left\langle u_i'\langle u_j\rangle\frac{\partial u_i'}{\partial x_j}\right\rangle=\langle u_j\rangle\frac{\partial k}{\partial x_j}.$$

$$\left\langle u_i'\frac{1}{\rho}\frac{\partial p'}{\partial x_i}\right\rangle=\left\langle\frac{1}{\rho}\left(\frac{\partial(u_i'p')}{\partial x_i}-p'\frac{\partial(u_i')}{\partial x_i}\right)\right\rangle=\left\langle\frac{1}{\rho}\frac{\partial(u_i'p')}{\partial x_i}\right\rangle.$$

$$\left\langle u_i'u_j'\frac{\partial\langle u_i\rangle}{\partial x_j}\right\rangle=-\langle u_i'u_j'\rangle\frac{\partial\langle u_i\rangle}{\partial x_j}.$$

8、最后化简可得：

$$\frac{\partial k}{\partial t}+\langle u_j\rangle\frac{\partial k}{\partial x_j}=-\overline{u_i'u_j'}\frac{\partial\langle u_i\rangle}{\partial x_j}-\varepsilon+\frac{\partial}{\partial x_j}\nu\frac{\partial k}{\partial x_j}+\frac{\partial}{\partial x_j}\left(-\frac{1}{2}\overline{u_i'u_i'u_j'}-\frac{1}{\rho}\langle p'u_j'\rangle\right).$$