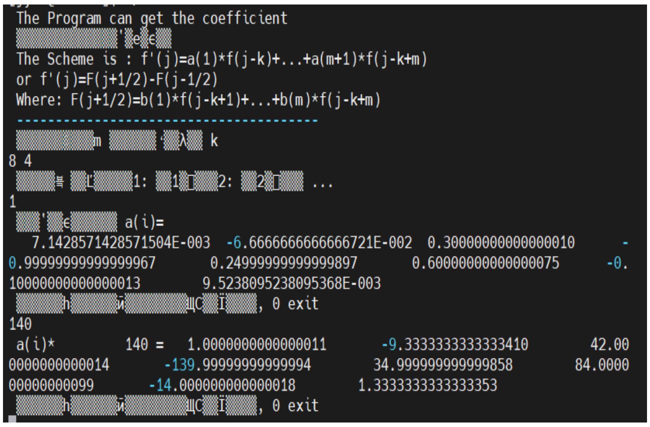
本次作业针对右侧通量$f_{j+1/2}$进行求解

对于七阶迎风格式有表达式采用 Cofficient-f 化简计算，八个基架点，最左侧为 4 号点，计算得到系数，由于格式原因未能显示中文



实际表达式如下所示：

$$f=-\frac{1}{140}f_{j-4}-\frac{1}{15}f_{j-3}+\frac{3}{10}f_{j-2}-f_{j-1}+\frac{1}{4}f_j+\frac{3}{5}f_{j+1}-\frac{1}{10}f_{j+2}+\frac{1}{105}f_{j+3}$$

选择基架点

$$j-3,j-2,j-1,j$$
$$j-2,j-1,j,j+1$$
$$j-1,j,j+1,j+2$$
$$j,j+1,j+2,j+3$$

采用公式

$$h_{j+1/2}=f_{j+1/2}+a_2\Delta x^2\frac{\partial^2 f}{\partial x^2}|_{j+1/2}+a_4\Delta x^4\frac{\partial^4 f}{\partial x^4}|_{j+1/2}+\ldots+a_{2m+1}\Delta x^{2m+1}\frac{\partial^{2m+1}f}{\partial x^{2m+1}}|_{j+1/2}+O(\Delta x^{2m+2}).$$

对方程进行构造以保证精度可以得到如下所示的四个模板对应的四种基架点上的重构的通量

$$\begin{cases} f_{j+1/2}^{(0)}=-\frac{1}{4}f_{j-3}+\frac{13}{12}f_{j-2}-\frac{23}{12}f_{j-1}+\frac{25}{12}f_j \\ f_{j+\frac{1}{2}}^{(1)}=\frac{1}{12}f_{j-2}-\frac{5}{12}f_{j-1}+\frac{13}{12}f_j+\frac{1}{4}f_{j+1} \\ f_{j+\frac{1}{2}}^{(2)}=-\frac{1}{12}f_{j-1}+\frac{7}{12}f_j+\frac{7}{12}f_{j+1}-\frac{1}{12}f_{j+2} \\ f_{j+\frac{1}{2}}^{(3)}=\frac{1}{4}f_j+\frac{13}{12}f_{j+1}-\frac{5}{12}f_{j+2}+\frac{1}{12}f_{j+3} \end{cases}$$

对理想权重进行计算可得$f_{j+\frac{1}{2}}=C_1f_{j+1/2}^{(0)}+C_2f_{j+1/2}^{(1)}+C_3f_{j+1/2}^{(2)}+C_4f_{j+1/2}^{(3)}$由上面的式子和该表达式进行联立，可以解得各个理想权重的值如下所示

$$\begin{cases} C_0 = \frac{1}{35} \\ C_1 = \frac{12}{35} \\ C_2 = \frac{18}{35} \\ C_3 = \frac{4}{35} \end{cases}$$

其次在构建 WENO 通量的加权表达式$\begin{cases} \alpha_0 = \frac{C_0}{(\varepsilon+IS_0)^2} \\ \alpha_1 = \frac{C_2}{(\varepsilon+IS_1)^2} \\ \alpha_2 = \frac{C_2}{(\varepsilon+IS_2)^2} \\ \alpha_3 = \frac{C_3}{(\varepsilon+IS_2)^2} \end{cases}$和$\begin{cases} \omega_0 = \frac{\alpha_0}{\alpha_0+\alpha_1+\alpha_2+\alpha_3} \\ \omega_1 = \frac{\alpha_1}{\alpha_0+\alpha_1+\alpha_2+\alpha_3} \\ \omega_2 = \frac{\alpha_2}{\alpha_0+\alpha_1+\alpha_2+\alpha_3} \\ \omega_3 = \frac{\alpha_3}{\alpha_0+\alpha_1+\alpha_2+\alpha_3} \end{cases}$时候需要用到光滑因子 IS，定义如下所示

$$IS_k = \sum_{l=1}^{3} \int_{x_{j-1/2}}^{x_{j+1/2}} \Delta x^{2l-1} \left(\frac{\partial^l}{\partial x^l} q_k(x)\right)^2 dx$$

令 k=0，此时将上述表达式进行展开

$$IS_0 = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \Delta x^1 \left(\frac{\partial^l}{\partial x^l} q_k(x)\right)^2 dx + \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \Delta x^3 \left(\frac{\partial^2}{\partial x^2} q_k(x)\right)^2 dx + \int_{x_{j-1/2}}^{x_{j+1/2}} \Delta x^5 \left(\frac{\partial^3}{\partial x^3} q_k(x)\right)^2 dx$$

其中$q_k(x)$为插值多项式，同 PPT 中的 h(x)

可以令$q_k(x) = ax^3 + bx^2 + cx + d$，假如在 J 点对差值多项式进行构成，则有

$$q_1(x) = \frac{a}{6}\left(x - x_j\right)^3 + \frac{b}{2}\left(x - x_j\right)^2 + c\left(x - x_j\right) + d$$

对上式子求各阶导数，$\frac{\partial^2}{\partial x^2} q_k(x)$，和$\frac{\partial^3}{\partial x^3} q_k(x)$并带入光滑因子$IS_0$的表达式

$$IS_0 = \frac{1043}{960} * (6\Delta x^3 a)^2 + \frac{13}{12} * (2\Delta x^2 b)^2 + \frac{1}{12} * c + c^2$$

由于此时的 abc 为各个模板的函数关系，因此可以得到

$$6\Delta x^3 a = -1f_{j-3} + 3f_{j-2} - 3f_{j-1} + f_j$$
$$2\Delta x^2 b = -1f_{j-3} + 4f_{j-2} - 5f_{j-1} + 2f_j$$
$$c = -\frac{1}{3}f_{j-3} + \frac{3}{2}f_{j-2} - 3f_{j-1} + \frac{11}{6}f_j$$

对IS_0代入所需要的式子并且合并同类项可以得到

$$IS_0 = \frac{13}{12}\left(fr_{j+1} - 2f_j + f_{j-1}\right)^2 + \frac{1}{4}\left(3f_{j+1} - 4f_j + f_{j-1}\right)^2$$

同理对于其他IS_1，IS_2，IS_3进行求解可以得到如下表达式，其中α_0等如右侧所示

$$
\begin{cases}
IS_0=\frac{13}{12}\left(fr_{j+1}-2f_j+f_{j-1}\right)^2+\frac{1}{4}\left(3f_{j+1}-4f_j+f_{j-1}\right)^2\\
IS_1=\frac{13}{12}\left(f_{j+2}-2f_{j+1}+f_j\right)^2+\frac{1}{4}\left(f_{j+2}-f_j\right)^2\\
IS_2=\frac{13}{12}\left(f_{j+3}-2f_{j+2}+f_{j+1}\right)^2+\frac{1}{4}\left(f_{j+3}-4f_{j+2}+3f_{j+1}\right)^2\\
IS_3=\frac{13}{12}\left(f_{j+3}-2f_{j+2}+f_{j+1}\right)^2+\frac{1}{4}\left(f_{j+3}-4f_{j+2}+3f_{j+1}\right)^2
\end{cases}.
\begin{cases}
\alpha_0=\frac{C_0}{(\varepsilon+IS_0)^2}=\frac{0.05}{(10^{-6}+IS_0)^2}\\
\alpha_1=\frac{C_2}{(\varepsilon+IS_1)^2}=\frac{0.45}{(10^{-6}+IS_1)^2}\\
\alpha_2=\frac{C_2}{(\varepsilon+IS_2)^2}=\frac{0.45}{(10^{-6}+IS_2)^2}\\
\alpha_3=\frac{C_3}{(\varepsilon+IS_2)^2}=\frac{0.05}{(10^{-6}+IS_3)^2}
\end{cases}
$$

将得到的模板带入$f_{j+\frac{1}{2}}=\omega_0 f_{j+1/2}^{(1)}+\omega_1 f_{j+1/2}^{(2)}+\omega_2 f_{j+1/2}^{(3)}+\omega_3 f_{j+1/2}^{(4)}$中可以得到七阶

weno 的通量格式