CONTENTS

MATHEMATICS - XI

CHAPTER 4

MEASUREMENT OF ANGLES

4.1 INTRODUCTION

The word *'Trigonometry'* is derived from two Greek words : (i) trigonon and, (ii) metron. The word trigonon means a triangle and the word metron means a measure. Hence, trigonometry means the science of measuring triangles. In broader sense it is that branch of Mathematics which deals with the measurement of the sides and the angles of a triangle and the problems *allied* with angles.

4.2 ANGLES

ANGLE *Consider a ray $\vec{OA}$. If this ray rotates about its end point O and takes the position OB , then we say that the angle $\angle AOB$ has been generated.*

Thus, an angle is considered as the figure obtained by rotating a given ray about its end-point.

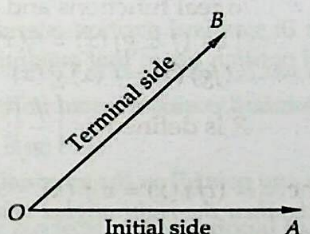


Fig. 4.1

The revolving ray is called the generating line of the angle. The initial position *OA* is called the *initial side* and the final position *OB* is called *terminal side* of the angle. The end point O about which the ray rotates is called the vertex of the angle.

MEASURE OF AN ANGLE *The measure of an angle is the amount of rotation from the initial side to the terminal side.*

SENSE OF AN ANGLE *The sense of an angle is determined by the direction of rotation of the initial side into the terminal side. The sense of an angle is said to be positive or negative according as the initial side rotates in anticlockwise or clockwise direction to get to the terminal side.*

RIGHT ANGLE *If the revolving ray starting from its initial position to final position describes one quarter of a circle, then we say that the measure of the angle formed is a right angle.*

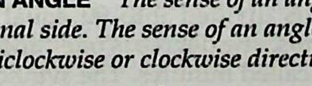


Fig. 4.2

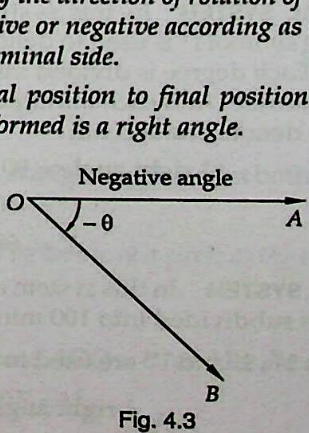


Fig. 4.3

4.3 SOME USEFUL TERMS

QUADRANTS *Let X'OX and YOY' be two lines at right angles in the plane of the paper. These lines divide the plane of the paper into four equal parts which are known as quadrants. The lines X'OX and YOY' are known as x-axis and y-axis respectively. These two lines taken together are known as the coordinate axes. The regions XOY, YOX', X' OY' and Y' OX are known as the first, the second, the third and the fourth quadrant respectively.*

ANGLE IN STANDARD POSITION *An angle is said to be in standard position if its vertex coincides with the origin O and the initial side coincides with OX i.e. the positive direction of x-axis.*

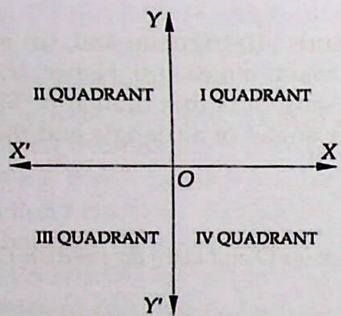


Fig. 4.4

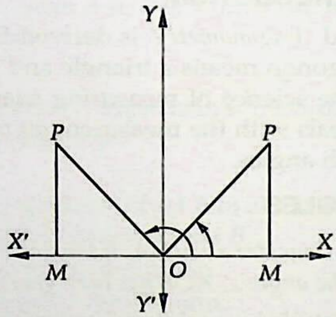


Fig. 4.5

ANGLE IN A QUADRANT *An angle in standard position is said to be in a particular quadrant, if the terminal side of the angle in standard position lies in that quadrant.*

QUADRANT ANGLE *An angle in standard position is said to be a quadrant angle, if the terminal side coincides with one of the axes.*

TRIANGLE OF REFERENCE *If from any point P on the terminal side of an angle in standard position a perpendicular PM is drawn on x-axis, then the right angled triangle OMP, thus formed, is called the triangle of reference of the $\angle XOP$. (See Fig. 4.5)*

CO-TERMINAL ANGLES *Two angles with different measures but having the same initial sides and the same terminal sides are known as co-terminal angles.*

4.4 SYSTEMS OF MEASUREMENT OF ANGLES

There are three systems for measuring angles, viz. (i) Sexagesimal or English system, (ii) Centesimal or French system, (iii) Circular system.

SEXAGESIMAL SYSTEM In this system a right angle is divided into 90 equal parts, called degrees. The symbol $1°$ is used to denote one degree. Thus, one degree is one-ninetieth part of a right angle. Each degree is divided into 60 equal parts, called minutes. The symbol $1'$ is used to denote one minute. And each minute is divided into 60 equal parts, called seconds. The symbol $1''$ is used to denote one second.

Thus, 1 right angle = 90 degrees ($90°$)

$$1° = 60 \text{ minutes } (= 60')$$

$$1' = 60 \text{ seconds } (= 60'')$$

CENTESIMAL SYSTEM In this system a right angle is divided into 100 equal parts, called grades; each grade is subdivided into 100 minutes, and each minute into 100 seconds.

The symbols 1^g, $1'$ and $1''$ are used to denote a grade, a minute, and a second respectively.

Thus, 1 right angle = 100 grades ($= 100^g$)

$$1 \text{ grade} = 100 \text{ minutes } (=100')$$
$$1 \text{ minute} = 100 \text{ seconds } (=100'')$$

CIRCULAR SYSTEM In this system the unit of measurement is radian as defined below.

RADIAN *One radian, written as 1^c, is the measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.*

Consider a circle of radius r having centre at O. Let A be a point on the circle. Now, cut off an arc AP whose length is equal to the radius r of the circle. Then by the definition the measure of $\angle AOP$ is 1 radian $(=1^c)$.

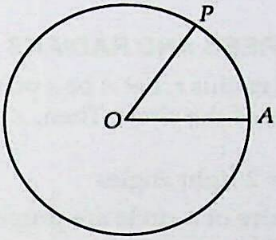


Fig. 4.6

Since a radian is chosen as the unit of measurement of an angle, therefore it should be a constant quantity. This we shall show in the following two theorems.

THEOREM 1 *Radian is a constant angle.*

PROOF Consider a circle with centre O and radius r. Take a point A on the circle and cut off an arc AP whose length is equal to the radius r. Join OA and OP. Then, by definition $\angle AOP = 1^c$.

Produce AO to meet the circle at B so that

$$\angle AOB = \text{a straight angle} = 2 \text{ right angles.}$$

Since the angles at the centre of a circle are proportional to the arcs subtending them.

$$\therefore \quad \frac{\angle AOP}{\angle AOB} = \frac{\text{arc } AP}{\text{arc } APB}$$

$$\Rightarrow \quad \frac{\angle AOP}{\angle AOB} = \frac{r}{\pi r} \qquad \left[\because \text{ arc } APB = \frac{1}{2}(2\pi r) = \pi r\right]$$

$$\Rightarrow \quad \frac{\angle AOP}{\angle AOB} = \frac{1}{\pi}$$

$$\Rightarrow \quad \angle AOP = \frac{1}{\pi}\angle AOB = \frac{\text{a straight angle}}{\pi}$$

$$\Rightarrow \quad 1^c = \frac{\text{a straight angle}}{\pi} \qquad [\because \angle AOP = 1^c]$$

$$\Rightarrow \quad 1^c = \text{constant} \qquad [\because \text{ A straight angle and } \pi \text{ both are constants}]$$

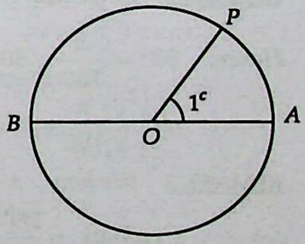


Fig. 4.7

Hence, radian is a constant angle **Q.E.D.**

THEOREM 2 *The number of radians in an angle subtended by an arc of a circle at the centre is equal to* $\frac{arc}{radius}$.

PROOF Consider a circle with centre O and radius r. Let $\angle AOQ = \theta^c$ and let arc $AQ = s$. Let P be a point on the arc AQ such that arc $AP = r$. Then, $\angle AOP = 1^c$.

Since angles at the centre of a circle are proportional to the arcs subtending them.

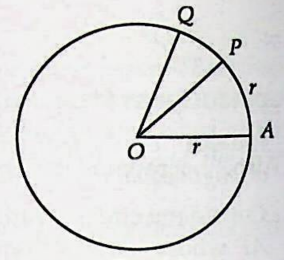


Fig. 4.8

$\therefore \quad \dfrac{\angle AOQ}{\angle AOP} = \dfrac{\text{arc } AQ}{\text{arc } AP}$

$\Rightarrow \quad AOQ = \left(\dfrac{\text{arc } AQ}{\text{arc } AP} \times 1\right)^c \qquad [\because \angle AOP = 1^c]$

$\Rightarrow \quad \theta = \dfrac{s}{r}$ radians. **Q.E.D.**

4.5 RELATION BETWEEN DEGREES AND RADIANS

Consider a circle with centre O and radius r. Let A be a point on the circle. JoinOA and cut off an arc OP of length equal to the radius of the circle. Then, $\angle AOP = 1$ radian. Produce AO to meet the circle at B.

$\therefore \quad \angle AOB =$ a straight angle $=$ 2 right angles

We know that the angles at the centre of a circle are proportional to the arcs subtending them.

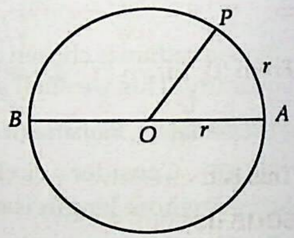


Fig. 4.9

$\therefore \quad \dfrac{\angle AOP}{\angle AOB} = \dfrac{\text{arc } AP}{\text{arc } APB}$

$\Rightarrow \quad \dfrac{\angle AOP}{2 \text{ right angles}} = \dfrac{r}{\pi r} \qquad \left[\because \text{ arc } APB = \dfrac{1}{2}(\text{Circumference})\right]$

$\Rightarrow \quad 2 = \dfrac{\text{rightangles}}{\pi}$

$\Rightarrow \quad 1^c = \dfrac{180^\circ}{\pi}$

Hence, One radian $= \dfrac{180^\circ}{\pi} \Rightarrow \pi$ radians $= 180^\circ$.

<u>REMARK 1</u> *When an angle is expressed in radians, the word radian is generally omitted.*

<u>REMARK 2</u> *Since* $180^\circ = \pi$ *radians. Therefore,* $1^\circ = \dfrac{\pi}{180}$ *radian.*

Hence, $30^\circ = \dfrac{\pi}{180} \times 30 = \dfrac{\pi}{6}$ *radians,* $45^\circ = \dfrac{\pi}{180} \times 45 = \dfrac{\pi}{4}$ *radians,*

$60^\circ = \dfrac{\pi}{180} \times 60 = \dfrac{\pi}{3}$ *radians,* $90^\circ = \dfrac{\pi}{180} \times 90 = \dfrac{\pi}{2}$ *radians etc.*

<u>REMARK 3</u> *We have,* π *radians* $= 180^\circ$

$\therefore \quad 1 \text{ radian} = \dfrac{180^\circ}{\pi} = \left(\dfrac{180}{22} \times 7\right)^\circ = 57^\circ 16' 22''$ *(approx).*

<u>REMARK 4</u> *We have,* $180^\circ = \pi$ radians

$\therefore \quad 1^\circ = \dfrac{\pi}{180}$ radian $= \left(\dfrac{22}{7 \times 180}\right)$ radian $= 0.01746$ radian.

4.6 RELATION BETWEEN THREE SYSTEMS OF MEASUREMENT OF AN ANGLE

Let D be the number of degrees, R be the number of radians and G be the number of grades in an angle θ.

$\because \quad 90^\circ = 1$ right angle

$\Rightarrow \quad 1^\circ = \dfrac{1}{90}$ right angle

$\Rightarrow \quad D^\circ = \dfrac{D}{90}$ right angles

$\Rightarrow \quad \theta = \dfrac{D}{90}$ right angles ... (ii)

Also, π radians = 2 right angles

$\Rightarrow \quad 1$ radian $= \dfrac{2}{\pi}$ right angles

$\Rightarrow \quad R$ radians $= \dfrac{2R}{\pi}$ right angles

$\Rightarrow \quad \theta = \dfrac{2R}{\pi}$ right angles ... (ii)

And, 100 grades = 1 right angle

$\Rightarrow \quad 1$ grade $= \dfrac{1}{100}$ right angle

$\Rightarrow \quad G$ grades $= \dfrac{G}{100}$ right angles

$\Rightarrow \quad \theta = \dfrac{G}{100}$ right angles ... (iii)

From (i), (ii) and (iii), we get

$$\frac{D}{90} = \frac{G}{100} = \frac{2R}{\pi}$$

This is the required relation between the three systems of measurement of an angle.

SOME USEFUL POINTS

(i) The angle between two consecutive digits in a clock is $30^\circ (= \pi / 6$ radians).
(ii) The hour hand rotates through an angle of 30° in one hour i.e. $(1/2)^\circ$ in one minute.
(iii) The minute hand rotates through an angle of 6° in one minute.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the degree measure corresponding to the following radian measures:*

(i) $\left(\dfrac{2\pi}{15}\right)^c$ (ii) $\left(\dfrac{\pi}{8}\right)^c$ (iii) $\left(\dfrac{1}{4}\right)^c$ (iv) -2^c (v) 6^c (vi) $\left(\dfrac{11}{16}\right)^c$

SOLUTION We have, π radians $= 180^\circ$

$\therefore \quad 1^c = \left(\dfrac{180}{\pi}\right)^\circ$

(i) $\left(\dfrac{2\pi}{15}\right)^c = \left(\dfrac{2\pi}{15} \times \dfrac{180}{\pi}\right)^\circ = 24^\circ$

(ii) $\left(\dfrac{\pi}{8}\right)^c = \left(\dfrac{\pi}{8} \times \dfrac{180}{\pi}\right)^\circ = \left(\dfrac{45}{2}\right)^\circ = \left(22\dfrac{1}{2}\right)^\circ = 22^\circ \left(\dfrac{1}{2} \times 60\right)' = 22^\circ\ 30'$

(iii) $\left(\dfrac{1}{4}\right)^c = \left(\dfrac{1}{4} \times \dfrac{180}{\pi}\right)^\circ = \left(\dfrac{1}{4} \times \dfrac{180}{22} \times 7\right)^\circ = \left(\dfrac{315}{22}\right)^\circ = \left(14\dfrac{7}{22}\right)^\circ$

$= 14^\circ \left(\dfrac{7}{22} \times 60\right)' = 14^\circ \left(19\dfrac{1}{11}\right)' = 14^\circ\ 19' \left(\dfrac{1}{11} \times 60\right)'' = 14^\circ 19'\ 5''$

(iv) $$(-2)^c = \left(\frac{180}{\pi} \times -2\right)^\circ = \left(\frac{180}{22} \times 7 \times (-2)\right)^\circ = \left(-114\frac{6}{11}\right)^\circ = \left\{-114^\circ\left(\frac{6}{11} \times 60\right)'\right\}$$

$$= -\left[114^\circ\left(32\frac{8}{11}\right)'\right] = -\left[114^\circ\, 32'\left(\frac{8}{11} \times 60\right)''\right] = -(114^\circ\, 32'\, 44'')$$

(v) $$6^c = \left(\frac{180}{\pi} \times 6\right)^\circ = \left(\frac{180}{22} \times 7 \times 6\right)^\circ = \left(\frac{90 \times 7 \times 6}{11}\right)^\circ = \left(\frac{3780}{11}\right)^\circ = \left(343\frac{7}{11}\right)^\circ$$

$$= 343^\circ\left(\frac{7}{11} \times 60\right)' = 343^\circ\left(\frac{420}{11}\right)' = 343'\, 38'\left(\frac{2}{11} \times 60\right)'' = 343^\circ\, 38'\, 11''$$

(vi) $$\left(\frac{11}{16}\right)^c = \left(\frac{180}{\pi} \times \frac{11}{16}\right)^\circ = \left(\frac{180}{22} \times 7 \times \frac{11}{16}\right)^\circ = \left(\frac{315}{8}\right)^\circ = \left(39\frac{3}{8}\right)^\circ$$

$$= 39^\circ\left(\frac{3}{8} \times 60\right)' = 39^\circ\, 22'\left(\frac{1}{2} \times 60\right)'' = 39^\circ\, 22'\, 30''$$

EXAMPLE 2 *Find the radian measures corresponding to the following degree measures :*

(i) 340° (ii) 75° (iii) $-37^\circ\, 30'$ (iv) $5^\circ\, 37'\, 30''$ (v) $40^\circ\, 20'$ (vi) 520°

SOLUTION We have,

$$180^\circ = \pi^c. \text{ Therefore, } 1^\circ = \left(\frac{\pi}{180}\right)^c$$

(i) $$340^\circ = \left(340 \times \frac{\pi}{180}\right)^c = \left(\frac{17\pi}{9}\right)^c$$

(ii) $$75^\circ = \left(75 \times \frac{\pi}{180}\right)^c = \left(\frac{5\pi}{12}\right)^c$$

(iii) Clearly, $30' = \left(\frac{30}{60}\right)^\circ = \frac{1}{2}^\circ$

$\therefore$ $$-37^\circ\, 30' = -\left(37\frac{1}{2}\right)^\circ = -\left(\frac{75}{2}\right)^\circ = -\left(\frac{75}{2} \times \frac{\pi}{180}\right)^c = -\left(\frac{5\pi}{24}\right)^c$$

(iv) Clearly, $30'' = \left(\frac{30}{60}\right)' = \left(\frac{1}{2}\right)'$

$\therefore$ $$37'\, 30'' = \left(37\frac{1}{2}\right)' = \left(\frac{75}{2}\right)' = \left(\frac{75}{2} \times \frac{1}{60}\right)^\circ = \left(\frac{5}{8}\right)^\circ$$

So, $$5^\circ\, 37'\, 30'' = \left(5\frac{5}{8}\right)^\circ = \left(\frac{45}{8}\right)^\circ = \left(\frac{45}{8} \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{32}\right)^c$$

(v) Clearly, $20' = \left(\frac{20}{60}\right)^\circ = \frac{1}{3}^\circ$

$\therefore$ $$40^\circ\, 20' = \left(40\frac{1}{3}\right)^\circ = \left(\frac{121}{3}\right)^\circ = \left(\frac{121}{3} \times \frac{\pi}{180}\right)^c = \left(\frac{121\pi}{540}\right)^c$$

(vi) Clearly, $520^\circ = \left(520 \times \frac{\pi}{180}\right)^c = \left(\frac{26\pi}{9}\right)^c$

EXAMPLE 3 *Find the length of an arc of a circle of radius 5 cm subtending a central angle measuring 15°.*

SOLUTION Let s be the length of the arc subtending an angle θ^c at the centre of a circle of radius r.

Then, $\theta = \frac{s}{r}$. Here, $r = 5$ cm and $\theta = 15° = \left(15 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{12}\right)^c$

$$\therefore \quad \theta = \frac{s}{r} \Rightarrow \frac{\pi}{12} = \frac{s}{5} \Rightarrow s = \frac{5\pi}{12} \text{ cm.}$$

EXAMPLE 4 *Find in degrees the angle subtended at the centre of a circle of diameter 50 cm by an arc of length 11 cm.*

SOLUTION Here, $r = 25$ cm and $s = 11$ cm

$$\therefore \quad \theta = \left(\frac{s}{r}\right)^c$$

$$\Rightarrow \quad \theta = \left(\frac{11}{25}\right)^c = \left(\frac{11}{25} \times \frac{180}{\pi}\right)^\circ = \left(\frac{11}{25} \times \frac{180}{22} \times 7\right)^\circ = \left(\frac{126}{5}\right)^\circ = \left(25\frac{1}{5}\right)^\circ = 25°\left(\frac{1}{5} \times 60\right)' = 25° 12'$$

EXAMPLE 5 *In a circle of diameter 40 cm the length of a chord is 20 cm. Find the length of minor arc corresponding to the chord.*

SOLUTION Let arc $AB = s$. It is given that $OA = 20$ cm and chord $AB = 20$ cm. Therefore, ΔOAB is an equilateral triangle. Hence,

$$\angle AOB = 60° = \left(60 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c$$

$$\therefore \quad \theta = \frac{\text{arc}}{\text{radius}}$$

$$\Rightarrow \quad \frac{\pi}{3} = \frac{s}{20} \Rightarrow s = \frac{20\pi}{3} \text{ cm.}$$

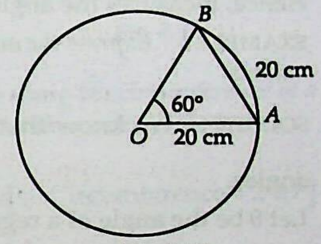


Fig. 4.10

EXAMPLE 6 *The angles of a triangle are in A.P. The number of degrees in the least is to the number of radians in the greatest as 60 : π. Find the angles in degrees.*

SOLUTION Let the angles of the triangle be $(a-d)°$, $a°$ and $(a+d)°$ Then,

$$(a-d) + a + (a+d) = 180° \Rightarrow 3a = 180° \Rightarrow a = 60°$$

So, the angles are $(60-d)°$, $60°$, $(60+d)°$.

Clearly, $(60-d)°$ is the least angle and $(60+d)°$ is the greatest angle.

Now, Greatest angle $= (60+d)° = \left\{(60+d)\frac{\pi}{180}\right\}^c$

It is given that:

$$\frac{\text{Number of degrees in the least angle}}{\text{Number of radians in the greatest angle}} = \frac{60}{\pi}$$

$$\Rightarrow \quad \frac{(60-d)}{(60+d)\frac{\pi}{180}} = \frac{60}{\pi} \Rightarrow 3(60-d) = (60+d) \Rightarrow 120 = 4d \Rightarrow d = 30.$$

Hence, measures of the angles are $(60-30)°$, $60°$, $(60+30)°$ i.e. $30°$, $60°$, $90°$.

EXAMPLE 7 *The angles of a triangle are in A.P. The number of grades in the least, is to the number of radians in the greatest as 40 : π. Find the angles in degrees.*

SOLUTION Let measures of the angles of the triangle in degrees be $(a-d)°$, $a°$ and $(a+d)°$. Then,

$$(a-d) + a + (a+d) = 180 \Rightarrow 3a = 180 \Rightarrow a = 60$$

So, measures of the angles are $(60-d)°$, $60°$ and $(60+d)°$.

Clearly, measure of the least angle is $(60° - d)°$ and that of the greatest angle is $(60 + d)°$.
Now,

Measure of the least angle $= (60 - d)°$

$$= \left\{(60-d)\times\frac{100}{90}\right\}^g = \left\{(60-d)\times\frac{10}{9}\right\}^g \qquad [\because 90° = 100^g]$$

Measure of the greatest angle $= (60+d)° = \left\{(60+d)\times\frac{\pi}{180}\right\}^c$

It is given that:

$$\frac{\text{Number of grades in the least angle}}{\text{Number of radians in the greatest angle}} = \frac{40}{\pi}$$

$$\Rightarrow \quad \frac{(60-d)\times\frac{10}{9}}{(60+d)\frac{\pi}{180}} = \frac{40}{\pi}$$

$$\Rightarrow \quad \frac{600-10d}{9}\times\frac{180}{(60+d)\,\pi} = \frac{40}{\pi}$$

$$\Rightarrow \quad 600 - 10d = 120 + 2d \Rightarrow 12d = 480 \Rightarrow d = 40.$$

Hence, measures the angles of the triangle are 20°, 60° and 100°.

EXAMPLE 8 *Express the angular measurement of the angle of a regular decagon in degrees, grades and radians.*

SOLUTION We know that the angle of an n sided regular polygon is equal to $\left(\frac{2n-4}{n}\right)$ right angles.

Let θ be the angle of a regular decagon. Then,

$$\theta = \left(\frac{2\times 10-4}{10}\right) = \frac{8}{5}\text{ right angles} = \left(\frac{8}{5}\times 90\right)^\circ = 144° \qquad [\because 1 \text{ right angle} = 90°]$$

$$\text{Again, } \theta = \frac{8}{5}\text{ right angles} = \left(\frac{8}{5}\times 100\right)^g = 160^g \qquad [\because 1 \text{ right angle} = 100^g]$$

$$\text{And, } \quad \theta = \frac{8}{5}\text{ right angles} = \left(\frac{8}{5}\times\frac{\pi}{2}\right)^c = \left(\frac{4\pi}{5}\right)^c \qquad \left[\because 1 \text{ right angle} = \frac{\pi^c}{2}\right]$$

EXAMPLE 9 *If the arcs of same length in two circles subtend angles of 60° and 75° at their centres. Find the ratio of their radii.*

SOLUTION Let r_1 and r_2 be the radii of the given circles and let their arcs of same length s subtend angles of 60° and 75° at their centres.

$$\text{Now, } \quad 60° = \left(60\times\frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c, \text{ and } 75° = \left(75\times\frac{\pi}{180}\right)^c = \left(\frac{5\pi}{12}\right)^c$$

$$\therefore \quad \frac{\pi}{3} = \frac{s}{r_1}, \text{ and } \frac{5\pi}{12} = \frac{s}{r_2} \qquad \left[\because \theta = \left(\frac{s}{r}\right)^c\right]$$

$$\Rightarrow \quad \frac{\pi}{3}r_1 = s, \text{ and } \frac{5\pi}{12}r_2 = s$$

$$\Rightarrow \quad \frac{\pi}{3}r_1 = \frac{5\pi}{12}r_2 \Rightarrow 4r_1 = 5r_2 \Rightarrow r_1 : r_2 = 5:4$$

Hence, $r_1 : r_2 = 5:4$

EXAMPLE 10 *Find in degrees the angle through which a pendulum swings if its length is 50 cm and the tip describes an arc of length 10 cm.*

SOLUTION Here, $r = 50$ cm and $s = 10$ cm.

$$\therefore \quad \theta = \left(\frac{s}{r}\right)^c$$

$$\Rightarrow \quad \theta = \left(\frac{10}{50}\right)^c = \left(\frac{1}{5}\right)^c = \left(\frac{1}{5} \times \frac{180}{\pi}\right)^\circ = \left(\frac{36}{22} \times 7\right)^\circ = \left(\frac{126}{11}\right)^\circ = \left(11\,\frac{5}{11}\right)^\circ = 11^\circ \left(\frac{5}{11} \times 60\right)' = 11^\circ\, 27'\, 16''$$

EXAMPLE 11 *A horse is tied to a post by a rope. If the horse moves along a circular path always keeping the rope tight and describes 88 metres when it has traced out 72° at the centre, find the length of the rope.*

SOLUTION Let the post be at point P and let PA be the length of the rope in tight position. Suppose the horse moves along the arc AB so that $\angle APB = 72^\circ$ and arc $AB = 88$ m. Let r be the length of the rope i.e. $PA = r$ metres.

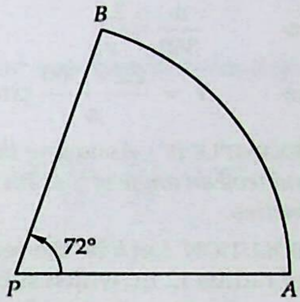


Fig. 4.11

$$\text{Here,} \quad \theta = 72^\circ = \left(72 \times \frac{\pi}{180}\right)^c = \left(\frac{2\pi}{5}\right)^c \quad \text{and } s = 88 \text{ m}$$

$$\therefore \quad \theta = \frac{\text{arc}}{\text{radius}}$$

$$\Rightarrow \quad \frac{2\pi}{5} = \frac{88}{r} \Rightarrow r = 88 \times \frac{5}{2\pi} = 70 \text{ metres.}$$

EXAMPLE 12 *A circular wire of radius 7.5 cm is cut and bent so as to lie along the circumference of a hoop whose radius is 120 cm. Find in degrees the angle which is subtended at the centre of the hoop.*

SOLUTION It is given that the radius of the circular wire is 7.5 cm.

$\therefore$ Length of the circular wire $= 2\pi \times 7.5 = 15\,\pi$ cm [$\because$ Circumference $= 2\,\pi r$]

Radius of the hoop = 120 cm.

Let θ be the angle subtended by the wire at the centre of the hoop. Then,

$$\theta = \frac{\text{arc}}{\text{radius}} \Rightarrow \theta = \left(\frac{15\,\pi}{120}\right)^c = \left(\frac{\pi}{8}\right)^c = \left(\frac{\pi}{8} \times \frac{180}{\pi}\right)^\circ = 22^\circ\, 30'$$

EXAMPLE 13 *The moon's distance from the earth is 360,000 kms and its diameter subtends an angle of 31′ at the eye of the observer. Find the diameter of the moon.*

SOLUTION Let AB be the diameter of the moon and let E be the eye of the observer. Since the distance between the earth and the moon is quite large, so we take diameter AB as arc AB. Let d be the diameter of the moon. Then, d = arc AB.

We have,

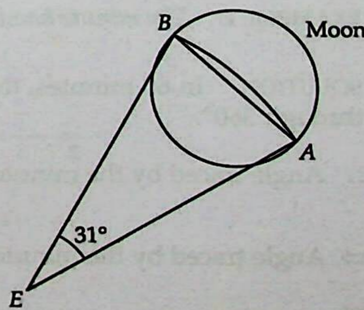


Fig. 4.12

$$\theta = 31' = \left(\frac{31}{60}\right)^\circ = \left(\frac{31}{60} \times \frac{\pi}{180}\right)^c, \text{ and } r = 360000 \text{ kms}$$

$$\therefore \quad \theta = \frac{\text{arc}}{\text{radius}}$$

$$\Rightarrow \quad \frac{31}{60} \times \frac{\pi}{180} = \frac{d}{360000}$$

$$\Rightarrow \quad d = \left(\frac{31}{60} \times \frac{\pi}{180} \times 360000\right) \text{km}$$

$$\Rightarrow \quad d = 3247.62 \text{ kms}$$

Hence, the diameter of the moon is 3247.62 km.

EXAMPLE 14 *If the angular diameter of the moon be 30′, how far from the eye a coin of diameter 2.2 cm be kept to hide the moon ?*

SOLUTION Suppose the coin is kept at a distance r from the eye to hide the moon completely. Let E be the eye of the observer and let AB be the diameter of the coin.

Then, arc AB = diameter AB = 2.2 cm.

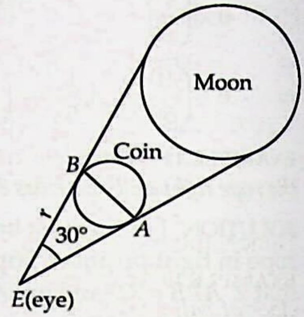


Fig. 4.13

We have, $\theta = 30' = \left(\frac{30}{60}\right)^{\circ} = \left(\frac{1}{2} \times \frac{\pi}{180}\right)^{c} = \left(\frac{\pi}{360}\right)^{c}$

$\therefore \quad \theta = \frac{\text{arc}}{\text{radius}}$

$\Rightarrow \quad \frac{\pi}{360} = \frac{2.2}{r}$

$\Rightarrow \quad r = \frac{2.2 \times 360}{\pi}$ cm $\Rightarrow r = \frac{2.2 \times 360 \times 7}{22} = 252$ cm.

EXAMPLE 15 *Assuming that a person of normal sight can read print at such a distance that the letters subtend an angle of 5′ at his eye, find what is the height of the letters that he can read at a distance of 12 metres.*

SOLUTION Let h be the required height in metres. Here h can be considered as the arc of a circle of radius 12 m, which subtends an angle of 5′ at its centre.

Here, $\theta = 5' = \left(\frac{5}{60}\right)^{\circ} = \left(\frac{1}{12} \times \frac{\pi}{180}\right)^{c}$, and $r = 12$ m.

$\therefore \quad \theta = \frac{\text{arc}}{\text{radius}} \Rightarrow \frac{\pi}{12 \times 180} = \frac{h}{12} \Rightarrow h = \left(\frac{\pi}{180}\right)$ metre = 1.7 cm.

EXAMPLE 16 *The perimeter of a certain sector of a circle is equal to the length of the arc of a semi-circle having the same radius. Express the angle of the sector in degrees, minutes and seconds.*

SOLUTION Let r be the radius of the circle and θ be the sector angle. Then,

Perimeter of the sector $= 2r + r\,\theta$

Length of the arc of a semi-circle of radius $r = \pi r$

It is given that

$2r + r\theta = \pi r$

$\Rightarrow \quad 2 + \theta = \pi$

$\Rightarrow \quad \theta = (\pi - 2)$ radians $= \left\{(\pi - 2) \times \frac{180}{\pi}\right\}^{\circ} = 180° - \left(\frac{360}{\pi}\right)^{\circ} = 180° - 114°\ 32'\ 44'' = 65°\ 27'\ 16''$

EXAMPLE 17 *The minute hand of a watch is 1.5 cm long. How far does its tip move in 40 minutes?*
(Use $\pi = 3.14$)

SOLUTION In 60 minutes, the minute hand of a watch completes one rotation i.e., it rotates through 360°.

$\therefore$ Angle traced by the minute hand in 1 minute $= \left(\frac{360}{60}\right)^{\circ} = 6^{\circ}$

$\Rightarrow$ Angle traced by the minute hand in 40 minutes $= (40 \times 6)° = 240° = \left(240 \times \frac{\pi}{180}\right)^{c} = \left(\frac{4\pi}{3}\right)^{c}$

Now,

$\theta = \frac{\text{arc}}{\text{radius}} \Rightarrow \frac{4\pi}{3} = \frac{\text{arc}}{1.5} \Rightarrow \text{arc} = \left(\frac{4\pi}{3} \times 1.5\right)$ cm $= 2\,\pi$ cm $= 2 \times 3.14$ cm $= 6.28$ cm

Hence, the tip of the minute hand travels 6.28 cm in 40 minutes.

EXAMPLE 18 *Find the angle between the minute hand of a clock and the hour hand when the time is 7 : 20 AM.*

SOLUTION We know that the hour hand completes one rotation in 12 hours while the minute hand completes one rotation in 60 minutes.

$\therefore$ Angle traced by the hour hand in 12 hours = 360°

$\Rightarrow$ Angle traced by the hour hand in 7 hrs 20 min. i.e $\frac{22}{3}$ hrs $= \left(\frac{360}{12} \times \frac{22}{3}\right)^\circ = 220°$.

Also, the angle traced by the minute hand in 60 min = 360°.

$\Rightarrow$ Angle traced by the minute hand in 20 min $= \left(\frac{360}{60} \times 20\right)^\circ = 120°$

Hence, the required angle between two hands = 220° − 120° = 100°.

EXAMPLE 19 *Find in degrees and radians the angle between the hour hand and the minute-hand of a clock at half past three.*

SOLUTION The angle traced by the hour hand in 12 hours = 360°

$\therefore$ The angle traced by the hour hand in 3 hrs 30 min. i.e. $\frac{7}{2}$ hrs $= \left(\frac{360}{12} \times \frac{7}{2}\right)^\circ = 105°$

The angle traced by the minute hand in 60 min = 360°

$\Rightarrow$ The angle traced by the minute hand in 30 min $= \left(\frac{360}{60} \times 30\right) = 180°$

Hence, the required angle between two hands $= 180° - 105° = 75° = \left(75 \times \frac{\pi}{180}\right) = \frac{5\pi}{12}$ radians.

LEVEL-2

EXAMPLE 20 *For each natural number k, let C_k denote the circle with radius k centimetres and centre at the origin. On the circle C_k, a particle moves k centimetres in the counter-clockwise direction. After completing its motion on C_k, the particle moves on C_{k+1} in the radial direction. The motion of the particle continues in this manner. The particle starts at (1, 0). If the particle crosses the positive direction of the x-axis for the first time on the circle C_n, then find the value of n.*

SOLUTION The path of the particle is shown by bold line segments and arcs. It is given that on the circle C_k of radius k centimetres the particle moves k centimetres. Therefore, angular displacement on kth circle is given by

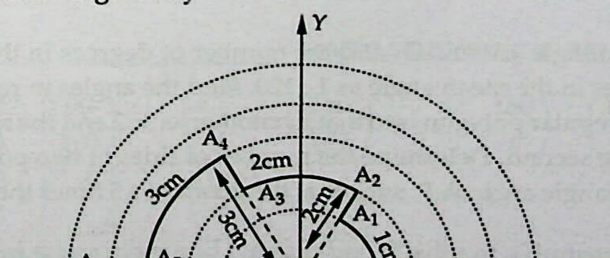


Fig. 4.14

$$\theta = \frac{k}{k} \text{ radian } = 1 \text{ radian.}$$

Thus, angular displacement on each circle is 1 radian.

If the particle crosses the x-axis for the first time on circle C_n then

Total angular displacement $= n$ radians.

As the particle crosses the positive direction of the x-axis for the first time on the n^{th} circle C_n.

Total angular displacement $> 2\pi$ radians

$\Rightarrow \quad n > 2\pi$

$\Rightarrow \quad n = 7$ [$\because$ n is the natural number such that $n > 2\pi$]

EXERCISE 4.1

LEVEL-1

1. Find the degree measure corresponding to the following radian measures (Use $\pi = 22/7$:
   (i) $\frac{9\pi}{5}$ (ii) $-\frac{5\pi}{6}$ (iii) $\left(\frac{18\pi}{5}\right)^c$ (iv) $(-3)^c$ (v) 11^c (vi) 1^c
2. Find the radian measure corresponding to the following degree measures:
   (i) 300° (ii) 35° (iii) −56° (iv) 135° (v) − 300° (vi) 7° 30′
   (vii) 125° 30′ (viii) − 47° 30′
3. The difference between the two acute angles of a right-angled triangle is $\frac{2\pi}{5}$ radians. Express the angles in degrees.
4. One angle of a triangle is $\frac{2}{3}x$ grades and another is $\frac{3}{2}x$ degrees while the third is $\frac{\pi x}{75}$ radians. Express all the angles in degrees.
5. Find the magnitude, in radians and degrees, of the interior angle of a regular
   (i) pentagon (ii) octagon (iii) heptagon (iv) duodecagon.
6. The angles of a quadrilateral are in A.P. and the greatest angle is 120°. Express the angles in radians.
7. The angles of a triangle are in *A.P.* and the number of degrees in the least angle is to the number of degrees in the mean angle as 1 : 120. Find the angles in radians.
8. The angle in one regular polygon is to that in another as 3 : 2 and the number of sides in first is twice that in the second. Determine the number of sides of two polygons.
9. The angles of a triangle are in A.P. such that the greatest is 5 times the least. Find the angles in radians.
10. The number of sides of two regular polygons are as 5 : 4 and the difference between their angles is 9°. Find the number of sides of the polygons.
11. A rail road curve is to be laid out on a circle. What radius should be used if the track is to change direction by 25° in a distance of 40 metres?
12. Find the length which at a distance of 5280 m will subtend an angle of 1′ at the eye.
13. A wheel makes 360 revolutions per minute. Through how many radians does it turn in 1 second?
14. Find the angle in radians through which a pendulum swings if its length is 75 cm and the tip describes an arc of length (i) 10 cm (ii) 15 cm (iii) 21 cm.

15. The radius of a circle is 30 cm. Find the length of an arc of this circle, if the length of the chord of the arc is 30 cm.
16. A railway train is travelling on a circular curve of 1500 metres radius at the rate of 66 km/hr. Through what angle has it turned in 10 seconds?
17. Find the distance from the eye at which a coin of 2 cm diameter should be held so as to conceal the full moon whose angular diameter is 31′.
18. Find the diameter of the sun in km supposing that it subtends an angle of 32′ at the eye of an observer. Given that the distance of the sun is 91×10^6 km.
19. If the arcs of the same length in two circles subtend angles 65° and 110° at the centre, find the ratio of their radii.
20. Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm (Use $\pi = 22/7$).

ANSWERS

1. (i) 324° (ii) −150° (iii) 648° (iv) −171° 49′ 5″ (v) 630° (vi) 57° 16′ 21″
2. (i) $\frac{5\pi}{3}$ (ii) $\frac{7\pi}{36}$ (iii) $-\frac{14\pi}{45}$ (iv) $\frac{3\pi}{4}$ (v) $-\frac{5\pi}{3}$ (vi) $\frac{\pi}{24}$ (vii) $\frac{251\pi}{360}$ (viii) $-\frac{19\pi}{72}$
3. 81°, 9°
4. 24°, 60°, 96°
5. (i) $\left(\frac{3\pi}{5}\right)^c$; 108° (ii) $\left(\frac{3\pi}{4}\right)^c$; 135° (iii) $\left(\frac{5\pi}{7}\right)^c$, 128° 34′ 17″ (iv) $\left(\frac{5\pi}{6}\right)^c$; 150°
6. $\frac{\pi}{3}, \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{2\pi}{3}$
7. $\frac{\pi}{360}, \frac{\pi}{3}, \frac{239\pi}{360}$
8. 8, 4
9. $\frac{\pi}{9}, \frac{\pi}{3}, \frac{5\pi}{9}$
10. 10, 8
11. 91.64 m
12. 1.5365
13. 12π
14. (i) $\left(\frac{11}{90}\right)^c$ (ii) $\frac{1}{5}$ (iii) $\frac{7}{25}$
15. 10π cm
16. $\left(\frac{11}{90}\right)^c$
17. 2.217m
18. 847407.4 km
19. 22 : 13
20. 12°, 36′

HINTS TO NCERT & SELECTED PROBLEMS

4. $$\left(\frac{2}{3}x\right)^g = \left(\frac{2}{3}x \times \frac{90}{100}\right)^\circ = \left(\frac{3x}{5}\right)^\circ \text{ and, } \left(\frac{\pi x}{75}\right)^c = \left(\frac{\pi}{75} \times \frac{180}{\pi}\right)^c = \left(\frac{12x}{5}\right)^\circ$$
$$\therefore \quad \left(\frac{3}{5}x\right)^\circ + \left(\frac{3}{2}x\right)^\circ + \left(\frac{12x}{5}\right)^\circ = 180^\circ \Rightarrow x = 40^\circ$$
5. A heptagon has seven sides and the number of sides of a dudecagon is twelve .
6. Let the measures of angles in degrees be $a - 3d, a - d, a + d, a + 3d$. Then,
Sum of the angles = 360°°⇒ $4a = 360° \Rightarrow a = 90°$.
Also, Greatest angle = 120° ⇒ $a + 3d = 120° \Rightarrow d = 10°$.
11. Here, $\theta = 25° = \left(25 \times \frac{\pi}{180}\right)^c$ and arc = 40 meters.

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If D, G and R denote respectively the number of degrees, grades and radians in an angle, then

(a) $\frac{D}{100} = \frac{G}{90} = \frac{2R}{\pi}$ (b) $\frac{D}{90} = \frac{G}{100} = \frac{R}{\pi}$

(c) $\frac{D}{90} = \frac{G}{100} = \frac{2R}{\pi}$ (d) $\frac{D}{90} = \frac{G}{100} = \frac{R}{2\pi}$

2. If the angles of a triangle are in A.P., then the measures of one of the angles in radians is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

3. The angle between the minute and hour hands of a clock at 8:30 is

(a) 80° (b) 75° (c) 60° (d) 105°

4. At 3:40, the hour and minute hands of a clock are inclined at

(a) $\frac{2\pi^c}{3}$ (b) $\frac{7\pi^c}{12}$ (c) $\frac{13\pi^c}{18}$ (d) $\frac{3\pi^c}{4}$

5. If the arcs of the same length in two circles subtend angles 65° and 110° at the centre, then the ratio of the radii of the circles is

(a) 22 : 13 (b) 11 : 13 (c) 22 : 15 (d) 21 : 13

6. If OP makes 4 revolutions in one second, the angular velocity in radians per second is

(a) π (b) 2π (c) 4π (d) 8π

7. A circular wire of radius 7 cm is cut and bent again into an arc of a circle of radius 12 cm. The angle subtended by the arc at the centre is

(a) 50° (b) 210° (c) 100° (d) 60° (e) 195°

8. The radius of the circle whose arc of length 15π cm makes an angle of $3\pi/4$ radian at the centre is

(a) 10 cm (b) 20 cm (c) $11\frac{1}{4}$ cm (d) $22\frac{1}{2}$ cm

ANSWERS

1. (c) 2. (b) 3. (b) 4. (c) 5. (a) 6. (d) 7. (b) 8. (b)

SUMMARY

1. The measure of an angle is the amount of rotation from the initial side to the terminal side.
2. The sense of an angle is positive or negative according as the initial side rotates in anti-clockwise or clockwise direction to get the terminal side.
3. Three systems of measuring angles are:

   (i) Sexagesimal system (ii) Centesimal system (iii) Circular system

   In sexagesimal system:

   1 right angle = 90 degrees (= 90°)

   1° = 60 minutes (= 60′)

   1′ = 60 seconds (= 60″)

   In centesimal system:

   1 right angle = 100 grades (= 100^g)

   1^g = 100 minutes (= 100′)

   1′ = 100 seconds (= 100″)

   In circular system, the unit of measurement is radian. One radian is the measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.

   π radians = 180°

4. The relation between three systems of measurement of an angle is

$$\frac{D}{90^\circ} = \frac{G}{100} = \frac{2R}{\pi}$$

CHAPTER 5

TRIGONOMETRIC FUNCTIONS

5.1 INTRODUCTION

In the present chapter, we will first introduce trigonometric ratios which are also known as trigonometric functions and then the identities involving them.

5.2 TRIGONOMETRIC RATIOS OR FUNCTIONS

Consider an angle $\theta = \angle XOA$ as shown in Fig. 5.1. Let P be any point other than O on its terminal side OA and let PM be perpendicular from P on x-axis. Let length $OP = r$, $OM = x$ and $MP = y$. We take the length $OP = r$ always positive while x and y can be positive or negative depending upon the position of the terminal side OA of $\angle XOA$.

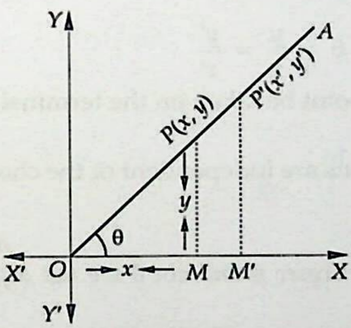


Fig. 5.1

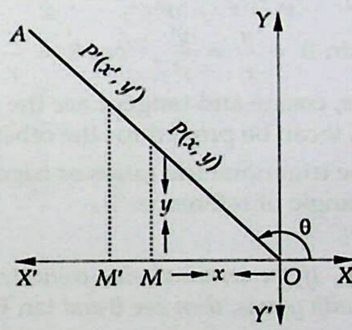


Fig. 5.2

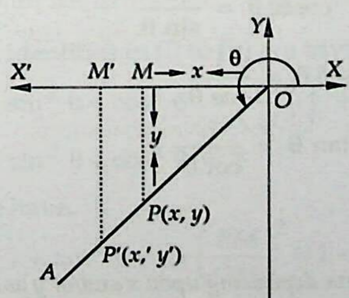


Fig. 5.3

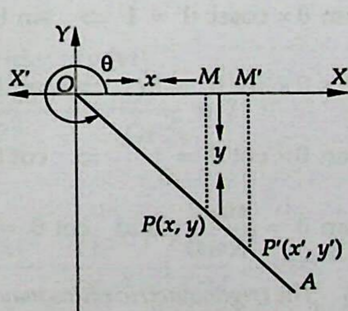


Fig. 5.4

In the right angled triangle OMP, we have

Base $= OM = x$, Perpendicular $= PM = y$, and Hypotenues $= OP = r$

We define the following trigonometric ratios which are also known as trgonometric functions.

$$\text{Sine } \theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{y}{r}, \text{ and is written as } \sin\theta$$

$$\text{Cosine } \theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{x}{r}, \text{ and is written as } \cos\theta$$

$$\text{Tangent } \theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{y}{x}, \text{ and is written as } \tan\theta$$

$$\text{Cosecant } \theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{r}{y}, \text{ and is written as } \operatorname{cosec}\theta$$

$$\text{Secant } \theta = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{r}{x}, \text{ and is written as } \sec\theta$$

$$\text{Cotangent } \theta = \frac{\text{Base}}{\text{Perpendicular}} = \frac{x}{y}, \text{ and is written as } \cot\theta$$

NOTE 1 *The triangle OMP is known as the triangle of reference.*

NOTE 2 *It should be noted that* $\sin\theta$ *does not mean the product of sin and* θ. *The* $\sin\theta$ *is correctly read as sine of angle* θ.

These functions depend only on the value of the angle θ and not on the position of the point P chosen on the terminal side of the angle θ as proved in the following theorem.

THEOREM *The trigonometric ratios are same for the same angle.*

PROOF Let $P'(x', y')$ be any point other than $P(x, y)$ on the terminal side OA with $OP' = r'$. Let $P'M'$ be perpendicular on x-axis. Clearly, triangles OMP and $OM'P'$ are similar. Therefore,

$$\frac{y}{r} = \frac{y'}{r'}, \quad \frac{x}{r} = \frac{x'}{r'} \quad \text{and} \quad \frac{y}{x} = \frac{y'}{x'}$$

$$\Rightarrow \quad \sin\theta = \frac{y}{r} = \frac{y'}{r'}, \quad \cos\theta = \frac{x}{r} = \frac{x'}{r'} \quad \text{and} \quad \tan\theta = \frac{y}{x} = \frac{y'}{x'}$$

Thus, sine, cosine and tangent are the same whatever point be taken on the terminal side OA. Similarly, it can be proved for the other ratios.

Hence, the trigonometric ratios or trigonometric functions are independent of the choice of the size of triangle of reference.

Q.E.D.

REMARK 1 *If the terminal side coincides with x-axis, then* $\operatorname{cosec}\theta$ *and* $\cot\theta$ *are not defined. If it coincides with y-axis, then* $\sec\theta$ *and* $\tan\theta$ *are not defined.*

REMARK 2 *The following relations are obvious from the definitions of trigonometric ratios :*

(i) $\sin\theta \times \operatorname{cosec}\theta = 1 \Rightarrow \sin\theta = \dfrac{1}{\operatorname{cosec}\theta}$ and $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

(ii) $\cos\theta \times \sec\theta = 1 \Rightarrow \cos\theta = \dfrac{1}{\sec\theta}$ and $\sec\theta = \dfrac{1}{\cos\theta}$

(iii) $\tan\theta \times \cot\theta = 1 \Rightarrow \cot\theta = \dfrac{1}{\tan\theta}$ and $\tan\theta = \dfrac{1}{\cot\theta}$

(iv) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

REMARK 3 *The trigonometric ratios may be positive or negative depending upon x and/or y as discussed in section 5.4.*

5.3 TRIGONOMETRIC IDENTITIES

IDENTITY *An equation involving trigonometric functions which is true for all those angles for which the functions are defined is called a trigonometric identity.*

For example, $\sec\theta = \dfrac{1}{\cos\theta}$, $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ etc. are trigonometric identities as they hold for all those values of θ except those values for which $\sec\theta$ and $\operatorname{cosec}\theta$ are not defined.

But, $\sin\theta = \cos\theta$ is a trigonometric equation not a trigonometric identity because it does not hold for all values of θ.

5.3.1 FUNDAMENTAL TRIGONOMETRIC IDENTITIES

In this subsection, we shall state and prove some fundamental trigonometrical identities.

THEOREM *Prove that:*

(i) $\sin\theta = \dfrac{1}{\operatorname{cosec}\theta}$ *or,* $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ (ii) $\cos\theta = \dfrac{1}{\sec\theta}$ *or,* $\sec\theta = \dfrac{1}{\cos\theta}$

(iii) $\cot\theta = \dfrac{1}{\tan\theta}$ *or,* $\tan\theta = \dfrac{1}{\cot\theta}$ (iv) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

(v) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ (vi) $\sin^2\theta + \cos^2\theta = 1$

(vii) $1 + \tan^2\theta = \sec^2\theta$ (viii) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

PROOF Let a revolving ray start from OX and revolve into the position OP to trace out any angle θ in any of the four quadrants. From P drawn PM perpendicular to x-axis. In the right angled triangle OMP, we have

$OP^2 = OM^2 + PM^2,$

$\sin\theta = \dfrac{PM}{OP}$, $\cos\theta = \dfrac{OM}{OP}$, $\tan\theta = \dfrac{PM}{OM}$, cosec $= \dfrac{OP}{PM}$, $\sec\theta = \dfrac{OP}{OM}$ and, $\cot\theta = \dfrac{OM}{PM}$.

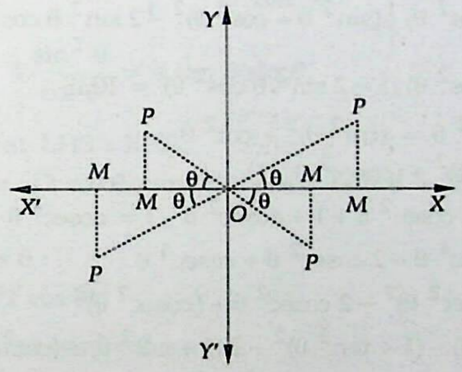


Fig. 5.5

Clearly, identities in (i) to (v) are trivial So, let us prove identity (vi).

(vi) $$\sin^2\theta + \cos^2\theta = \left(\frac{PM}{OP}\right)^2 + \left(\frac{OM}{OP}\right)^2 = \frac{PM^2 + OM^2}{OP^2} = \frac{OP^2}{OP^2} = 1$$

Hence, $\sin^2\theta + \cos^2\theta = 1$

(vii) We have,

$$1 + \tan^2\theta = 1 + \left(\frac{PM}{OM}\right)^2 = 1 + \frac{PM^2}{OM^2} = \frac{OM^2 + PM^2}{OM^2} = \frac{OP^2}{OM^2} = \left(\frac{OP}{OM}\right)^2 = \sec^2\theta$$

Hence, $1 + \tan^2\theta = \sec^2\theta$.

(viii) We have,

$$1 + \cot^2\theta = 1 + \left(\frac{OM}{PM}\right)^2 = 1 + \frac{OM^2}{PM^2} = \frac{PM^2 + OM^2}{PM^2} = \frac{OP^2}{PM^2} = \left(\frac{OP}{PM}\right)^2 = \operatorname{cosec}^2\theta$$

Hence, $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ **Q.E.D.**

NOTE *It should be noted that* $(\sin\theta)^2$ *is written as* $\sin^2\theta$, $(\cos\theta)^2$ *is written as* $\cos^2\theta$ *etc.*

We shall now discuss more identities involving the trigonometrical functions in the following examples.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Prove the following identities:*

(i) $\sin^8\theta - \cos^8\theta = (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta)$

(ii) $\cot^4\theta + \cot^2\theta = \text{cosec}^4\theta - \text{cosec}^2\theta$

(iii) $2\sec^2\theta - \sec^4\theta - 2\,\text{cosec}^2\theta + \text{cosec}^4\theta = \cot^4\theta - \tan^4\theta$

(iv) $(\sin\theta + \text{cosec}\,\theta)^2 + (\cos\theta + \sec\theta)^2 = \tan^2\theta + \cot^2\theta + 7$

SOLUTION (i) We have,

$$\begin{aligned}
\text{LHS} &= (\sin^8\theta - \cos^8\theta) = (\sin^4\theta)^2 - (\cos^4\theta)^2\\
&= (\sin^4\theta - \cos^4\theta)(\sin^4\theta + \cos^4\theta)\\
&= (\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)(\sin^4\theta + \cos^4\theta)\\
&= (\sin^2\theta - \cos^2\theta)(\sin^4\theta + \cos^4\theta)\\
&= (\sin^2\theta - \cos^2\theta)(\sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta - 2\sin^2\theta\cos^2\theta)\\
&= (\sin^2\theta - \cos^2\theta)\left\{(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta\right\}\\
&= (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta) = \text{RHS}
\end{aligned}$$

(ii)
$$\begin{aligned}
\text{LHS} &= \cot^4\theta + \cot^2\theta = (\cot^2\theta)^2 + \cot^2\theta\\
&= (\text{cosec}^2\theta - 1)^2 + (\text{cosec}^2\theta - 1) \qquad [\because 1 + \cot^2\theta = \text{cosec}^2\theta]\\
&= \text{cosec}^4\theta - 2\,\text{cosec}^2\theta + 1 + \text{cosec}^2\theta - 1 = \text{cosec}^4\theta - \text{cosec}^2\theta = \text{RHS}
\end{aligned}$$

(iii)
$$\begin{aligned}
\text{LHS} &= 2\sec^2\theta - \sec^4\theta - 2\,\text{cosec}^2\theta + \text{cosec}^4\theta\\
&= 2\sec^2\theta - (\sec^2\theta)^2 - 2\,\text{cosec}^2\theta + (\text{cosec}^2\theta)^2\\
&= 2(1 + \tan^2\theta) - (1 + \tan^2\theta)^2 - 2(1 + \cot^2\theta) + (\cot^2\theta + 1)^2\\
&= 2 + 2\tan^2\theta - (1 + \tan^4\theta + 2\tan^2\theta) - 2 - 2\cot^2\theta + (\cot^4\theta - 2\cot^2\theta + 1)\\
&= \cot^4\theta - \tan^4\theta = \text{RHS}
\end{aligned}$$

(iv)
$$\begin{aligned}
\text{LHS} &= (\sin\theta + \text{cosec}\,\theta)^2 + (\cos\theta + \sec\theta)^2\\
&= \sin^2\theta + \text{cosec}^2\theta + 2\sin\theta\,\text{cosec}\,\theta + \cos^2\theta + \sec^2\theta + 2\cos\theta\sec\theta\\
&= (\sin^2\theta + \cos^2\theta) + (\text{cosec}^2\theta + \sec^2\theta) + 2 + 2\\
&= 1 + (1 + \cot^2\theta) + (1 + \tan^2\theta) + 4 = \tan^2\theta + \cot^2\theta + 7 = \text{RHS}
\end{aligned}$$

EXAMPLE 2 *Prove the following identities:*

(i) $(1 + \cot\theta - \text{cosec}\,\theta)(1 + \tan\theta + \sec\theta) = 2$

(ii) $\dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \dfrac{1 + \sin\theta}{\cos\theta}$ **[NCERT EXEMPLAR]**

SOLUTION (i) We have,

$$\begin{aligned}
\text{LHS} &= (1 + \cot\theta - \text{cosec}\,\theta)(1 + \tan\theta + \sec\theta)\\
&= \left(1 + \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}\right)\left(1 + \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)\\
&= \frac{(\sin\theta + \cos\theta - 1)}{\sin\theta}\,\frac{(\sin\theta + \cos\theta + 1)}{\cos\theta}
\end{aligned}$$

$$= \frac{(\sin\theta + \cos\theta)^2 - 1}{\sin\theta\cos\theta} = \frac{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - 1}{\sin\theta\cos\theta} = \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} = 2 = \text{RHS}$$

(ii) $$\text{LHS} = \frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \frac{(\tan\theta + \sec\theta) - (\sec^2\theta - \tan^2\theta)}{\tan\theta - \sec\theta + 1} \quad [\because \sec^2\theta - \tan^2\theta = 1]$$

$$= \frac{(\sec\theta + \tan\theta)\{1 - (\sec\theta - \tan\theta)\}}{\tan\theta - \sec\theta + 1} = \frac{(\sec\theta + \tan\theta)(\tan\theta - \sec\theta + 1)}{\tan\theta - \sec\theta + 1}$$

$$= \sec\theta + \tan\theta = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = \frac{1 + \sin\theta}{\cos\theta} = \text{RHS}$$

EXAMPLE 3 *If* $\tan\theta + \sin\theta = m$ *and* $\tan\theta - \sin\theta = n$, *show that* $m^2 - n^2 = 4\sqrt{mn}$.

SOLUTION We have, **[NCERT EXEMPLAR]**

$$\text{LHS} = m^2 - n^2 = (\tan\theta + \sin\theta)^2 - (\tan\theta - \sin\theta)^2 = 4\tan\theta\sin\theta \quad \text{...(i)}$$

$$\text{RHS} = 4\sqrt{mn} = 4\sqrt{(\tan\theta + \sin\theta)(\tan\theta - \sin\theta)} = 4\sqrt{\tan^2\theta - \sin^2\theta}$$

$$\Rightarrow \quad \text{RHS} = 4\sqrt{\frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta} = 4\sqrt{\frac{\sin^2\theta - \sin^2\theta\cos^2\theta}{\cos^2\theta}} = 4\sqrt{\frac{\sin^2\theta(1 - \cos^2\theta)}{\cos^2\theta}}$$

$$\Rightarrow \quad \text{RHS} = 4\sqrt{\frac{\sin^4\theta}{\cos^2\theta}} = 4\frac{\sin^2\theta}{\cos\theta} = 4\tan\theta\sin\theta \quad \text{...(ii)}$$

From (i) and (ii), we obtain that LHS = RHS.

EXAMPLE 4 *If* $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, *prove that* $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$.

SOLUTION We have,

$$\cos\theta + \sin\theta = \sqrt{2}\cos\theta$$

$$\Rightarrow \quad (\cos\theta + \sin\theta)^2 = (\sqrt{2}\cos\theta)^2$$

$$\Rightarrow \quad \cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta = 2\cos^2\theta$$

$$\Rightarrow \quad \cos^2\theta - \sin^2\theta = 2\sin\theta\cos\theta$$

$$\Rightarrow \quad (\cos\theta + \sin\theta)(\cos\theta - \sin\theta) = 2\sin\theta\cos\theta$$

$$\Rightarrow \quad \cos\theta - \sin\theta = \frac{2\sin\theta\cos\theta}{\cos\theta + \sin\theta}$$

$$\Rightarrow \quad \cos\theta - \sin\theta = \frac{2\sin\theta\cos\theta}{\sqrt{2}\cos\theta} \quad [\because \cos\theta + \sin\theta = \sqrt{2}\cos\theta]$$

$$\Rightarrow \quad \cos\theta - \sin\theta = \sqrt{2}\sin\theta$$

ALITER We know that

$$(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2 = 2$$

$$\Rightarrow \quad (\sqrt{2}\cos\theta)^2 + (\cos - \sin\theta)^2 = 2 \quad [\because \cos\theta + \sin\theta = \sqrt{2}\cos\theta]$$

$$\Rightarrow \quad (\cos\theta - \sin\theta)^2 = 2 - 2\cos^2\theta$$

$$\Rightarrow \quad (\cos\theta - \sin\theta)^2 = 2\sin^2\theta \Rightarrow \cos\theta - \sin\theta = \sqrt{2}\sin\theta.$$

EXAMPLE 5 *If* $a\cos\theta + b\sin\theta = x$ *and* $a\sin\theta - b\cos\theta = y$, *prove that* $a^2 + b^2 = x^2 + y^2$.

[NCERT EXEMPLAR]

SOLUTION We have,

$$x = a\cos\theta + b\sin\theta \text{ and } y = a\sin\theta - b\cos\theta.$$

$\therefore \quad x^2 + y^2 = (a \cos\theta + b \sin\theta)^2 + (a \sin\theta - b\cos\theta)^2$

$= (a^2 \cos^2\theta + b^2 \sin^2\theta + 2ab \sin\theta\cos\theta) + (a^2 \sin^2\theta + b^2\cos^2\theta - 2ab\sin\theta\cos\theta)$

$= a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta) = a^2 + b^2$

EXAMPLE 6 *If* $a\cos\theta - b\sin\theta = c$, *show that* $a\sin\theta + b\cos\theta = \pm\sqrt{a^2+b^2-c^2}$.

SOLUTION Clearly,

$(a\cos\theta - b\sin\theta)^2 + (a\sin\theta + b\cos\theta)^2$

$= a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta) - 2ab\sin\theta\cos\theta + 2ab\sin\theta\cos\theta$

$= a^2 + b^2$

$\therefore \quad (a\sin\theta + b\cos\theta)^2 = a^2 + b^2 - (a\cos\theta - b\sin\theta)^2$

$\Rightarrow \quad (a\sin\theta + b\cos\theta)^2 = a^2 + b^2 - c^2 \qquad [\because a\cos\theta - b\sin\theta = c]$

$\Rightarrow \quad a\sin\theta + b\cos\theta = \pm\sqrt{a^2+b^2-c^2}$

EXAMPLE 7 *If* $\sec\theta + \tan\theta = p$, *obtain the values of* $\sec\theta$, $\tan\theta$ *and* $\sin\theta$ *in terms of* p.

SOLUTION We know that: $\sec^2\theta - \tan^2\theta = 1$.

$\therefore \quad (\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1 \Rightarrow p(\sec\theta - \tan\theta) = 1 \Rightarrow \sec\theta - \tan\theta = \dfrac{1}{p}$

Now, $\sec\theta + \tan\theta = p$ and, $\sec\theta - \tan\theta = \dfrac{1}{p}$

$\Rightarrow \quad (\sec\theta + \tan\theta) + (\sec\theta - \tan\theta) = p + \dfrac{1}{p}$

$\Rightarrow \quad 2\sec\theta = p + \dfrac{1}{p} \Rightarrow \sec\theta = \dfrac{p^2+1}{2p} \qquad \text{...(i)}$

Again, $\sec\theta + \tan\theta = p$ and $\sec\theta - \tan\theta = \dfrac{1}{p}$

$\Rightarrow \quad (\sec\theta + \tan\theta) - (\sec\theta - \tan\theta) = p - \dfrac{1}{p}$

$\Rightarrow \quad 2\tan\theta = p - \dfrac{1}{p} \Rightarrow \tan\theta = \dfrac{p^2-1}{2p} \qquad \text{...(ii)}$

Dividing (ii) by (i), we get: $\sin\theta = \dfrac{p^2-1}{p^2+1}$

EXAMPLE 8 *Prove that:* $2\sec^2\theta - \sec^4\theta - 2\,\text{cosec}^2\theta + \text{cosec}^4\theta = \dfrac{1-\tan^8\theta}{\tan^4\theta}$.

SOLUTION We have,

$2\sec^2\theta - \sec^4\theta - 2\,\text{cosec}^2\theta + \text{cosec}^4\theta$

$= 2(1+\tan^2\theta) - (1+\tan^2\theta)^2 - 2(1+\cot^2\theta) + (1+\cot^2\theta)^2$

$= 2(1+\tan^2\theta - 1 - \cot^2\theta) + (1 + 2\cot^2\theta + \cot^4\theta) - (1 + 2\tan^2\theta + \tan^4\theta)$

$= 2(\tan^2\theta - \cot^2\theta) + (2\cot^2\theta - 2\tan^2\theta) + \cot^4\theta - \tan^4\theta$

$= \cot^4\theta - \tan^4\theta = \dfrac{1}{\tan^4\theta} - \tan^4\theta = \dfrac{1-\tan^8\theta}{\tan^4\theta}$

EXAMPLE 9 *Prove that:* $3(\sin\theta - \cos\theta)^4 + 6(\sin\theta + \cos\theta)^2 + 4(\sin^6\theta + \cos^6\theta) - 13 = 0$.

SOLUTION We have,

$$3(\sin\theta-\cos\theta)^4+6(\sin\theta+\cos\theta)^2+4(\sin^6\theta+\cos^6\theta)-13$$

$$=3\left\{(\sin\theta-\cos\theta)^2\right\}^2+6(\sin\theta+\cos\theta)^2$$
$$+4\left\{(\sin^2\theta+\cos^2\theta)^3-3\sin^2\theta\cos^2\theta(\sin^2\theta+\cos^2\theta)\right\}-13$$

$$=3(1-2\sin\theta\cos\theta)^2+6(1+2\sin\theta\cos\theta)+4(1-3\sin^2\theta\cos^2\theta)-13$$

$$=3(1-4\sin\theta\cos\theta+4\sin^2\theta\cos^2\theta)+6(1+2\sin\theta\cos\theta)+4(1-3\sin^2\theta\cos^2\theta)-13$$

$$=3+6+4-13=0$$

EXAMPLE 10 *Given that:* $(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)=(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)$. *Show that one of the values of each member of this equality is* $\sin\alpha\sin\beta\sin\gamma$.

SOLUTION We have,

$$(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)=(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)$$

Multiplying both sides by $(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)$, we get

$$(1+\cos\alpha)^2(1+\cos\beta)^2(1+\cos\gamma)^2=(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)$$

$$\Rightarrow\quad(1+\cos\alpha)^2(1+\cos\beta)^2(1+\cos\gamma)^2=(1-\cos^2\alpha)(1-\cos^2\beta)(1-\cos^2\gamma)$$

$$\Rightarrow\quad(1+\cos\alpha)^2(1+\cos\beta)^2(1+\cos\gamma)^2=\sin^2\alpha\sin^2\beta\sin^2\gamma$$

$$\Rightarrow\quad(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)=\pm\sin\alpha\sin\beta\sin\gamma$$

Hence, one of the values of $(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)$ is $\sin\alpha\sin\beta\sin\gamma$.

Similarly, by multiplying both sides by $(1-\cos\alpha)(1-\cos\beta)\ (1-\cos\gamma)$, we find that one of the values of $(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)$ is also $\sin\alpha\sin\beta\sin\gamma$.

LEVEL-2

EXAMPLE 11 *Prove that:* $\sec^2\theta+\mathrm{cosec}^2\theta\geq 4$.

SOLUTION We have,

$$\sec^2\theta+\mathrm{cosec}^2\theta=(1+\tan^2\theta)+1+(\cot^2\theta)=2+\tan^2\theta+\cot^2\theta$$
$$=2+\tan^2\theta+\cot^2\theta-2\tan\theta\cot\theta+2\tan\theta\cot\theta$$
$$=2+(\tan\theta-\cot\theta)^2+2$$
$$=4+(\tan\theta-\cot\theta)^2\geq 4\qquad[\because\ (\tan\theta-\cot\theta)^2\geq 0]$$

EXAMPLE 12 *If* $10\sin^4\alpha+15\cos^4\alpha=6$, *find the value of* $27\,\mathrm{cosec}^6\alpha+8\sec^6\alpha$.

SOLUTION We have,

$$10\sin^4\alpha+15\cos^4\alpha=6$$

$$\Rightarrow\quad 10\sin^4\alpha+15\cos^4\alpha=6(\sin^2\alpha+\cos^2\alpha)^2$$

$$\Rightarrow\quad 10\tan^4\alpha+15=6(\tan^2\alpha+1)^2\qquad\text{[Dividing boths sides by }\cos^4\alpha]$$

$$\Rightarrow\quad(2\tan^2\alpha-3)^2=0\Rightarrow\tan^2\alpha=\frac{3}{2}$$

$$\therefore\quad 27\,\mathrm{cosec}^6\alpha+8\sec^6\alpha=27(1+\cot^2\alpha)^3+8(1+\tan^2\alpha)^3$$
$$=27\left(1+\frac{2}{3}\right)^3+8\left(1+\frac{3}{2}\right)^3=27\times\frac{125}{27}+8\times\frac{125}{8}=250.$$

EXAMPLE 13 *If* $\dfrac{\sin A}{\sin B} = p$ *and* $\dfrac{\cos A}{\cos B} = q$, *find* $\tan A$ *and* $\tan B$.

SOLUTION We have,

$$\frac{\sin A}{\sin B} = p \text{ and } \frac{\cos A}{\cos B} = q$$

$$\Rightarrow \quad \frac{\sin A}{\sin B} \cdot \frac{\cos B}{\cos A} = \frac{p}{q}$$

$$\Rightarrow \quad \frac{\tan A}{\tan B} = \frac{p}{q} \Rightarrow \frac{\tan A}{p} = \frac{\tan B}{q} = \lambda \text{ (say)} \Rightarrow \tan A = p\lambda \text{ and } \tan B = q\lambda \quad \text{...(i)}$$

Now, $\sin A = p \sin B$

$$\Rightarrow \quad \frac{\tan A}{\sqrt{1+\tan^2 A}} = p \frac{\tan B}{\sqrt{1+\tan^2 B}}$$

$$\Rightarrow \quad \frac{p\lambda}{\sqrt{1+p^2\lambda^2}} = p \frac{q\lambda}{\sqrt{1+q^2\lambda^2}}$$

$$\Rightarrow \quad p^2(1+q^2\lambda^2) = p^2q^2(1+p^2\lambda^2)$$

$$\Rightarrow \quad \lambda^2(q^2 - p^2q^2) = q^2 - 1$$

$$\Rightarrow \quad \lambda^2 = \frac{q^2-1}{q^2(1-p^2)} \Rightarrow \lambda = \pm \frac{1}{q}\sqrt{\frac{q^2-1}{1-p^2}}$$

$$\therefore \quad \tan A = \pm \frac{p}{q}\sqrt{\frac{q^2-1}{1-p^2}} \text{ and, } \tan B = \pm\sqrt{\frac{q^2-1}{1-p^2}} \quad \text{[Using (i)]}$$

EXAMPLE 14 *If* $\tan^2\theta = 1 - a^2$, *prove that* $\sec\theta + \tan^3\theta \operatorname{cosec}\theta = (2-a^2)^{3/2}$. *Also, find the values of* a *for which the above result holds true.*

SOLUTION We have,

$$\begin{aligned}
\sec\theta + \tan^3\theta \operatorname{cosec}\theta &= \sec\theta \left\{1 + \tan^3\theta \frac{\operatorname{cosec}\theta}{\sec\theta}\right\} \\
&= \sqrt{1+\tan^2\theta}\left\{1 + \tan^3\theta \times \cot\theta\right\} \\
&= (1+\tan^2\theta)^{3/2} \\
&= (1+1-a^2)^{3/2} = (2-a^2)^{3/2} \quad [\because \tan^2\theta = 1-a^2]
\end{aligned}$$

Now,

$$\tan^2\theta \geq 0 \text{ for all } \theta \Rightarrow 1-a^2 \geq 0 \Rightarrow a^2 - 1 \leq 0 \Rightarrow -1 \leq a \leq 1 \quad \text{...(i)}$$

Since LHS of $\sec\theta + \tan^3\theta \operatorname{cosec}\theta = (2-a^2)^{3/2}$ is real for all $\theta \in R$. So, RHS must also be real.

$$\therefore \quad 2 - a^2 \geq 0 \Rightarrow a^2 - 2 \leq 0 \Rightarrow -\sqrt{2} \leq a \leq \sqrt{2} \quad \text{...(ii)}$$

From (i) and (ii), we find that the given relation holds true for all $a \in [-1, 1]$.

EXAMPLE 15 *If* $a\cos^3\theta + 3a\cos\theta\sin^2\theta = m$ *and* $a\sin^3\theta + 3a\cos^2\theta\sin\theta = n$, *then prove that:* $(m+n)^{2/3} + (m-n)^{2/3} = 2a^{2/3}$.

SOLUTION We have,

$$a\cos^3\theta + 3a\cos\theta\sin^2\theta = m \quad \text{and} \quad a\sin^3\theta + 3a\cos^2\theta\sin\theta = n$$

$$\Rightarrow \quad a\cos^3\theta + 3a\cos\theta\sin^2\theta + a\sin^3\theta + 3a\cos^2\theta\sin\theta = m+n$$

and, $\quad a\cos^3\theta + 3a\cos\theta\sin^2\theta - a\sin^3\theta - 3a\cos^2\theta\sin\theta = m - n$

$\Rightarrow \quad a(\cos\theta + \sin\theta)^3 = m + n$ and, $a(\cos\theta - \sin\theta)^3 = m - n$

$\Rightarrow \quad \cos\theta + \sin\theta = \left(\dfrac{m+n}{a}\right)^{1/3}$ and, $\cos\theta - \sin\theta = \left(\dfrac{m-n}{a}\right)^{1/3}$

$$\Rightarrow \quad (\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2 = \left(\frac{m+n}{a}\right)^{2/3} + \left(\frac{m-n}{a}\right)^{2/3}$$

$$\Rightarrow \quad 2(\cos^2\theta + \sin^2\theta) = \frac{(m+n)^{2/3}}{a^{2/3}} + \frac{(m-n)^{2/3}}{a^{2/3}}$$

$$\Rightarrow \quad (m+n)^{2/3} + (m-n)^{2/3} = 2a^{2/3}$$

EXAMPLE 16 *If* $2\tan^2\alpha\tan^2\beta\tan^2\gamma + \tan^2\alpha\tan^2\beta + \tan^2\beta\tan^2\gamma + \tan^2\gamma\tan^2\alpha = 1$, *prove that* $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 1$.

SOLUTION We have,

$$2\tan^2\alpha\tan^2\beta\tan^2\gamma + \tan^2\alpha\tan^2\beta + \tan^2\beta\tan^2\gamma + \tan^2\gamma\tan^2\alpha = 1$$

Dividing throughout by $\tan^2\alpha\tan^2\beta\tan^2\gamma$, we get

$\Rightarrow \quad 2 + \cot^2\gamma + \cot^2\alpha + \cot^2\beta = \cot^2\alpha\cot^2\beta\cot^2\gamma$

$\Rightarrow \quad 2 + \operatorname{cosec}^2\gamma - 1 + \operatorname{cosec}^2\alpha - 1 + \operatorname{cosec}^2\beta - 1 = (\operatorname{cosec}^2\alpha - 1)(\operatorname{cosec}^2\beta - 1)(\operatorname{cosec}^2\gamma - 1)$

$\Rightarrow \quad \operatorname{cosec}^2\alpha + \operatorname{cosec}^2\beta + \operatorname{cosec}^2\gamma - 1$

$\quad = \operatorname{cosec}^2\alpha\operatorname{cosec}^2\beta\operatorname{cosec}^2\gamma - \operatorname{cosec}^2\alpha\operatorname{cosec}^2\beta - \operatorname{cosec}^2\beta\operatorname{cosec}^2\gamma$

$\qquad - \operatorname{cosec}^2\gamma\operatorname{cosec}^2\alpha + \operatorname{cosec}^2\alpha + \operatorname{cosec}^2\beta + \operatorname{cosec}^2\gamma - 1$

$\Rightarrow \quad \operatorname{cosec}^2\alpha\operatorname{cosec}^2\beta\operatorname{cosec}^2\gamma = \operatorname{cosec}^2\alpha\operatorname{cosec}^2\beta + \operatorname{cosec}^2\beta\operatorname{cosec}^2\gamma + \operatorname{cosec}^2\gamma\operatorname{cosec}^2\alpha$

$\Rightarrow \quad 1 = \sin^2\gamma + \sin^2\alpha + \sin^2\beta$ [Multiplying throughout by $\sin^2\alpha\sin^2\beta\sin^2\gamma$]

$\Rightarrow \quad \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 1$

EXAMPLE 17 *If* $\dfrac{ax}{\cos\theta} + \dfrac{by}{\sin\theta} = a^2 - b^2$, *and* $\dfrac{ax\sin\theta}{\cos^2\theta} - \dfrac{by\cos\theta}{\sin^2\theta} = 0$, *prove that*

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$$

SOLUTION We have,

$$\frac{ax\sin\theta}{\cos^2\theta} - \frac{by\cos\theta}{\sin^2\theta} = 0$$

$$\Rightarrow \quad ax\sin^3\theta - by\cos^3\theta = 0$$

$$\Rightarrow \quad \frac{\sin^3\theta}{by} = \frac{\cos^3\theta}{ax}$$

$$\Rightarrow \quad \left(\frac{\sin^3\theta}{by}\right)^{2/3} = \left(\frac{\cos^3\theta}{ax}\right)^{2/3}$$

$$\Rightarrow \quad \frac{\sin^2\theta}{(by)^{2/3}} = \frac{\cos^2\theta}{(ax)^{2/3}}$$

$\Rightarrow \quad \dfrac{\sin^2\theta}{(by)^{2/3}} = \dfrac{\cos^2\theta}{(ax)^{2/3}} = \dfrac{\sin^2\theta + \cos^2\theta}{(by)^{2/3} + (ax)^{2/3}}$ [Using ratio and proportions]

$$\Rightarrow \quad \frac{\sin^2\theta}{(by)^{2/3}} = \frac{\cos^2\theta}{(ax)^{2/3}} = \frac{1}{(ax)^{2/3}+(by)^{2/3}}$$

$$\Rightarrow \quad \sin^2\theta = \frac{(by)^{2/3}}{(ax)^{2/3}+(by)^{2/3}} \text{ and, } \cos^2\theta = \frac{(ax)^{2/3}}{(ax)^{2/3}+(by)^{2/3}}$$

$$\Rightarrow \quad \sin\theta = \frac{(by)^{1/3}}{\sqrt{(ax)^{2/3}+(by)^{2/3}}} \text{ and, } \cos\theta = \frac{(ax)^{1/3}}{\sqrt{(ax)^{2/3}+(by)^{2/3}}}$$

Substituting these values in $\frac{ax}{\cos\theta}+\frac{by}{\sin\theta}=a^2-b^2$, we get

$$(ax)^{2/3}\sqrt{(ax)^{2/3}+(by)^{2/3}}+(by)^{2/3}\sqrt{(ax)^{2/3}+(by)^{2/3}} = a^2-b^2$$

$$\Rightarrow \quad \left\{\sqrt{(ax)^{2/3}+(by)^{2/3}}\right\}\left\{(ax)^{2/3}+(by)^{2/3}\right\} = a^2-b^2$$

$$\Rightarrow \quad \left\{(ax)^{2/3}+(by)^{2/3}\right\}^{3/2} = a^2-b^2$$

$$\Rightarrow \quad (ax)^{2/3}+(by)^{2/3} = (a^2-b^2)^{2/3}$$

EXAMPLE 18 *If* $m^2+m'^2+2mm'\cos\theta=1, n^2+n'^2+2nn'\cos\theta=1$, *and* $mn+m'n'+(mn'+m'n)\cos\theta=0$, *prove that* (i) $m^2+n^2=\operatorname{cosec}^2\theta$ (ii) $m'^2+n'^2=\operatorname{cosec}^2\theta$

SOLUTION (i) We have,

$$m^2+m'^2+2mm'\cos\theta = 1 \text{ and } n^2+n'^2+2nn'\cos\theta = 1$$

$$\Rightarrow \quad m'^2+2mm'\cos\theta+m^2\cos^2\theta-m^2\cos^2\theta+m^2 = 1$$

and, $\quad n'^2+2nn'\cos\theta+n^2\cos^2\theta-n^2\cos^2\theta+n^2=1$

$$\Rightarrow \quad (m'+m\cos\theta)^2+m^2(1-\cos^2\theta)=1 \text{ and } (n'+n\cos\theta)^2+n^2(1-\cos^2\theta)=1$$

$$\Rightarrow \quad (m'+m\cos\theta)^2 = 1-m^2\sin^2\theta \quad ...(i) \quad \text{and,} \quad (n'+n\cos\theta)^2 = 1-n^2\sin^2\theta \quad ...(ii)$$

Now, $\quad (m'+m\cos\theta)(n'+n\cos\theta) = m'n'+(mn'+m'n)\cos\theta+mn\cos^2\theta$

$$\Rightarrow \quad (m'+m\cos\theta)(n'+n\cos\theta) = -mn+mn\cos^2\theta \quad [\because mn+m'n'+(mn'+m'n)\cos\theta=0]$$

$$\Rightarrow \quad (m'+m\cos\theta)(n'+n\cos\theta) = -mn(1-\cos^2\theta)$$

$$\Rightarrow \quad (m'+m\cos\theta)(n'+n\cos\theta) = -mn\sin^2\theta$$

$$\Rightarrow \quad (m'+m\cos\theta)^2(n'+n\cos\theta)^2 = m^2n^2\sin^4\theta \quad \text{[On squaring both sides]}$$

$$\Rightarrow \quad (1-m^2\sin^2\theta)(1-n^2\sin^2\theta) = m^2n^2\sin^4\theta \quad \text{[Using (i) and (ii)]}$$

$$\Rightarrow \quad 1-(m^2+n^2)\sin^2\theta+m^2n^2\sin^4\theta = m^2n^2\sin^4\theta$$

$$\Rightarrow \quad 1 = (m^2+n^2)\sin^2\theta$$

$$\Rightarrow \quad m^2+n^2 = \operatorname{cosec}^2\theta$$

(ii) As the given relations do not alter by replacing m by m' and n by n'. Therefore, on replacing m by m' and n by n' in $m^2+n^2=\operatorname{cosec}^2\theta$, we get $m'+n'^2=\operatorname{cosec}^2\theta$.

EXAMPLE 19 *If* $\frac{\sin^4\theta}{a}+\frac{\cos^4\theta}{b}=\frac{1}{a+b}$, *prove that*

(i) $\frac{\sin^8\theta}{a^3}+\frac{\cos^8\theta}{b^3}=\frac{1}{(a+b)^3}$ (ii) $\frac{\sin^{4n}\theta}{a^{2n-1}}+\frac{\cos^{4n}\theta}{b^{2n-1}}=\frac{1}{(a+b)^{2n-1}}, n\in N$

SOLUTION We have,

$$\frac{\sin^4\theta}{a}+\frac{\cos^4\theta}{b} = \frac{1}{a+b}$$

$$\Rightarrow \quad (a+b)\left(\frac{\sin^4\theta}{a}+\frac{\cos^4\theta}{b}\right) = 1$$

$$\Rightarrow \quad (a+b)\left(\frac{\sin^4\theta}{a}+\frac{\cos^4\theta}{b}\right) = (\sin^2\theta+\cos^2\theta)^2$$

$$\Rightarrow \quad \sin^4\theta+\cos^4\theta+\frac{b}{a}\sin^4\theta+\frac{a}{b}\cos^4\theta = \sin^4\theta+\cos^4\theta+2\sin^2\theta\cos^2\theta$$

$$\Rightarrow \quad \frac{b}{a}\sin^4\theta+\frac{a}{b}\cos^4\theta-2\sin^2\theta\cos^2\theta = 0$$

$$\Rightarrow \quad \left(\sqrt{\frac{b}{a}}\sin^2\theta-\sqrt{\frac{a}{b}}\cos^2\theta\right)^2 = 0$$

$$\Rightarrow \quad \sqrt{\frac{b}{a}}\sin^2\theta = \sqrt{\frac{a}{b}}\cos^2\theta$$

$$\Rightarrow \quad \tan^2\theta = \frac{a}{b}$$

$$\Rightarrow \quad \frac{\sin^2\theta}{a} = \frac{\cos^2\theta}{b}$$

$$\Rightarrow \quad \frac{\sin^2\theta}{a} = \frac{\cos^2\theta}{b} = \frac{\sin^2\theta+\cos^2\theta}{a+b}$$

$$\Rightarrow \quad \frac{\sin^2\theta}{a} = \frac{\cos^2\theta}{b} = \frac{1}{a+b}$$

$$\Rightarrow \quad \sin^2\theta = \frac{a}{a+b},\ \cos^2\theta = \frac{b}{a+b} \qquad \text{...(i)}$$

(i) $$\frac{\sin^8\theta}{a^3}+\frac{\cos^8\theta}{b^3} = \frac{1}{a^3}(\sin^2\theta)^4+\frac{1}{b^3}(\cos^2\theta)^4$$

$$= \frac{1}{a^3}\left(\frac{a}{a+b}\right)^4+\frac{1}{b^3}\left(\frac{b}{a+b}\right)^4 \qquad \text{[Using (i)]}$$

$$= \frac{a}{(a+b)^4}+\frac{b}{(a+b)^4} = \frac{a+b}{(a+b)^4} = \frac{1}{(a+b)^3}$$

(ii) $$\frac{\sin^{4n}\theta}{a^{2n-1}}+\frac{\cos^{4n}\theta}{b^{2n-1}} = \frac{(\sin^2\theta)^{2n}}{a^{2n-1}}+\frac{(\cos^2\theta)^{2n}}{b^{2n-1}}$$

$$= \frac{1}{a^{2n-1}}\left(\frac{a}{a+b}\right)^{2n}+\frac{1}{b^{2n-1}}\left(\frac{b}{a+b}\right)^{2n}$$

$$= \frac{a}{(a+b)^{2n}}+\frac{b}{(a+b)^{2n}} = \frac{a+b}{(a+b)^{2n}} = \frac{1}{(a+b)^{2n-1}}$$

EXAMPLE 20 *If* $\dfrac{\cos^4\alpha}{\cos^2\beta}+\dfrac{\sin^4\alpha}{\sin^2\beta} = 1$, *prove that*

(i) $\sin^4\alpha+\sin^4\beta = 2\sin^2\alpha\sin^2\beta$ (ii) $\dfrac{\cos^4\beta}{\cos^2\alpha}+\dfrac{\sin^4\beta}{\sin^2\alpha} = 1$

SOLUTION We have,

$$\frac{\cos^4\alpha}{\cos^2\beta}+\frac{\sin^4\alpha}{\sin^2\beta}=1$$

$\Rightarrow$ $\cos^4\alpha\sin^2\beta+\sin^4\alpha\cos^2\beta=\cos^2\beta\sin^2\beta$

$\Rightarrow$ $\cos^4\alpha(1-\cos^2\beta)+\cos^2\beta(1-\cos^2\alpha)^2=\cos^2\beta(1-\cos^2\beta)$

$\Rightarrow$ $\cos^4\alpha-\cos^4\alpha\cos^2\beta+\cos^2\beta-2\cos^2\alpha\cos^2\beta+\cos^4\alpha\cos^2\beta=\cos^2\beta-\cos^4\beta$

$\Rightarrow$ $\cos^4\alpha-2\cos^2\alpha\cos^2\beta+\cos^4\beta=0$

$\Rightarrow$ $(\cos^2\alpha-\cos^2\beta)^2=0$

$\Rightarrow$ $\cos^2\alpha-\cos^2\beta=0$

$\Rightarrow$ $\cos^2\alpha=\cos^2\beta$ ...(i)

$\Rightarrow$ $1-\sin^2\alpha=1-\sin^2\beta$

$\Rightarrow$ $\sin^2\alpha=\sin^2\beta$ ... (ii)

(i) $\sin^4\alpha+\sin^4\beta=(\sin^2\alpha-\sin^2\beta)^2+2\sin^2\alpha\sin^2\beta$

$=2\sin^2\alpha\sin^2\beta$ $[\because \sin^2\alpha=\sin^2\beta]$

(ii) $\frac{\cos^4\beta}{\cos^2\alpha}+\frac{\sin^4\beta}{\sin^2\alpha}=\frac{\cos^2\beta\cos^2\beta}{\cos^2\alpha}+\frac{\sin^2\beta\sin^2\beta}{\sin^2\alpha}$

$=\frac{\cos^2\beta\cos^2\alpha}{\cos^2\alpha}+\frac{\sin^2\beta\sin^2\alpha}{\sin^2\alpha}$ [Using (i)and (ii)]

$=\cos^2\beta+\sin^2\beta$

EXAMPLE 21 *If x is any non-zero real number, show that* $\cos\theta$ *and* $\sin\theta$ *can never be equal to* $x+\frac{1}{x}$.

SOLUTION We have following cases:

<u>CASE I</u> *When* $x>0$: In this case, we have

$$x+\frac{1}{x}=(\sqrt{x})^2+\left(\frac{1}{\sqrt{x}}\right)^2-2\times\sqrt{x}\times\frac{1}{\sqrt{x}}+2\times\sqrt{x}\times\frac{1}{\sqrt{x}}$$

$\Rightarrow$ $x+\frac{1}{x}=\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2+2\geq 2$ $\left[\because \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2\geq 0\right]$

<u>CASE II</u> *When* $x<0$: Let $x=-y$. Then, $y>0$

$\therefore$ $x+\frac{1}{x}=-y-\frac{1}{y}=-\left(y+\frac{1}{y}\right)$

But, $y+\frac{1}{y}\geq 2$ [From Case I]

$\Rightarrow$ $-\left(y+\frac{1}{y}\right)\leq -2\Rightarrow x+\frac{1}{x}\leq -2$

$\therefore$ $x+\frac{1}{x}\geq 2$ for $x>0$ and, $x+\frac{1}{x}\leq -2$ for $x<0$

But, $-1\leq\sin\theta\leq 1$ and $-1\leq\cos\theta\leq 1$ for all θ.

Hence, $\sin\theta$ and $\cos\theta$ cannot be equal to $x+\frac{1}{x}$ for any non-zero x.

EXAMPLE 22 *If $A = \cos^2\theta + \sin^4\theta$, prove that $\frac{3}{4} \le A \le 1$ for all values of θ.*

SOLUTION We have

$$A = \cos^2\theta + \sin^4\theta = \cos^2\theta + (\sin^2\theta)^2$$

Now, $-1 \le \sin\theta \le 1$ for all θ

$\Rightarrow$ $0 \le \sin^2\theta \le 1$ for all θ

$\Rightarrow$ $(\sin^2\theta)^2 \le \sin^2\theta$ [For $0 < x < 1$, $x^n < x$ for all $n \in N - \{1\}$]

$\Rightarrow$ $\cos^2\theta + (\sin^2\theta)^2 \le \cos^2\theta + \sin^2\theta$ for all θ

$\Rightarrow$ $A \le 1$ for all θ ...(i)

Again,

$$A = \cos^2\theta + \sin^4\theta$$

$$\Rightarrow A = 1 - \sin^2\theta + (\sin^2\theta)^2$$

$$\Rightarrow A = 1 - \frac{1}{4} + \left\{\frac{1}{4} - \sin^2\theta + (\sin^2\theta)^2\right\}$$

$$\Rightarrow A = \frac{3}{4} + \left(\frac{1}{2} - \sin^2\theta\right)^2$$

Now,

$$\left(\frac{1}{2} - \sin^2\theta\right)^2 \ge 0 \quad \text{for all } \theta$$

$$\Rightarrow \frac{3}{4} + \left(\frac{1}{2} - \sin^2\theta\right)^2 \ge \frac{3}{4} \text{ for all } \theta$$

$\Rightarrow$ $A \ge \frac{3}{4}$ for all θ ...(ii)

From (i) and (ii), we obtain $\frac{3}{4} \le A \le 1$ for all θ

EXERCISE 5.1

LEVEL-1

Prove the following identities (1-16)

1. $\sec^4\theta - \sec^2\theta = \tan^4\theta + \tan^2\theta$
2. $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$
3. $(\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta) = 1$
4. $\operatorname{cosec}\theta(\sec\theta - 1) - \cot\theta(1 - \cos\theta) = \tan\theta - \sin\theta$
5. $\dfrac{1 - \sin A\cos A}{\cos A(\sec A - \operatorname{cosec} A)} \cdot \dfrac{\sin^2 A - \cos^2 A}{\sin^3 A + \cos^3 A} = \sin A$
6. $\dfrac{\tan A}{1 - \cot A} + \dfrac{\cot A}{1 - \tan A} = (\sec A\operatorname{cosec} A + 1)$
7. $\dfrac{\sin^3 A + \cos^3 A}{\sin A + \cos A} + \dfrac{\sin^3 A - \cos^3 A}{\sin A - \cos A} = 2$
8. $(\sec A\sec B + \tan A\tan B)^2 - (\sec A\tan B + \tan A\sec B)^2 = 1$
9. $\dfrac{\cos\theta}{1 - \sin\theta} = \dfrac{1 + \cos\theta + \sin\theta}{1 + \cos\theta - \sin\theta}$

10. $\dfrac{\tan^3\theta}{1+\tan^2\theta}+\dfrac{\cot^3\theta}{1+\cot^2\theta}=\dfrac{1-2\sin^2\theta\cos^2\theta}{\sin\theta\cos\theta}$

11. $1-\dfrac{\sin^2\theta}{1+\cot\theta}+\dfrac{\cos^2\theta}{1+\tan\theta}=\sin\theta\cos\theta$

12. $\left(\dfrac{1}{\sec^2\theta-\cos^2\theta}+\dfrac{1}{\operatorname{cosec}^2\theta-\sin^2\theta}\right)\sin^2\theta\cos^2\theta=\dfrac{1-\sin^2\theta\cos^2\theta}{2+\sin^2\theta\cos^2\theta}$

13. $(1+\tan\alpha\tan\beta)^2+(\tan\alpha-\tan\beta)^2=\sec^2\alpha\sec^2\beta$

14. $\dfrac{(1+\cot\theta+\tan\theta)(\sin\theta-\cos\theta)}{\sec^3\theta-\operatorname{cosec}^3\theta}=\sin^2\theta\cos^2\theta$

15. $\dfrac{2\sin\theta\cos\theta-\cos\theta}{1-\sin\theta+\sin^2\theta-\cos^2\theta}=\cot\theta$

16. $\cos\theta(\tan\theta+2)(2\tan\theta+1)=2\sec\theta+5\sin\theta$

17. If $x=\dfrac{2\sin\theta}{1+\cos\theta+\sin\theta}$, then prove that $\dfrac{1-\cos\theta+\sin\theta}{1+\sin\theta}$ is also equal to x.

[NCERT EXEMPLAR]

18. If $\sin\theta=\dfrac{a^2-b^2}{a^2+b^2}$, find the values of $\tan\theta$, $\sec\theta$ and $\operatorname{cosec}\theta$

19. If $\tan x=\dfrac{b}{a}$, then find the value of $\sqrt{\dfrac{a+b}{a-b}}+\sqrt{\dfrac{a-b}{a+b}}$. **[NCERT EXEMPLAR]**

20. If $\tan\theta=\dfrac{a}{b}$, show that $\dfrac{a\sin\theta-b\cos\theta}{a\sin\theta+b\cos\theta}=\dfrac{a^2-b^2}{a^2+b^2}$

LEVEL-2

21. If $\operatorname{cosec}\theta-\sin\theta=a^3$, $\sec\theta-\cos\theta=b^3$, then prove that $a^2b^2(a^2+b^2)=1$.

22. If $\cot\theta(1+\sin\theta)=4m$ and $\cot\theta(1-\sin\theta)=4n$, prove that $(m^2-n^2)^2=mn$.

23. If $\sin\theta+\cos\theta=m$, then prove that

$$\sin^6\theta+\cos^6\theta=\frac{4-3(m^2-1)^2}{4},\ \text{where } m^2\le 2$$

24. If $a=\sec\theta-\tan\theta$ and $b=\operatorname{cosec}\theta+\cot\theta$, then show that $ab+a-b+1=0$.

[NCERT EXEMPLAR]

25. Prove that: $\left|\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}+\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}\right|=-\dfrac{2}{\cos\theta}$, where $\dfrac{\pi}{2}<\theta<\pi$

26. If $T_n=\sin^n\theta+\cos^n\theta$, prove that

(i) $\dfrac{T_3-T_5}{T_1}=\dfrac{T_5-T_7}{T_3}$ (ii) $2T_6-3T_4+1=0$ (iii) $6T_{10}-15T_8+10T_6-1=0$

ANSWERS

18. $\tan\theta=\dfrac{a^2-b^2}{2ab}$, $\sec\theta=\dfrac{a^2+b^2}{2ab}$, $\operatorname{cosec}\theta=\dfrac{a^2+b^2}{a^2-b^2}$ 19. $\dfrac{2\cos x}{\sqrt{\cos^2x-\sin^2x}}$.

HINTS TO SELECTED PROBLEMS

21. We have, $\text{cosec}\,\theta - \sin\theta = a^3$, $\sec\theta - \cos\theta = b^3$

$\Rightarrow \quad \dfrac{1-\sin^2\theta}{\sin\theta} = a^3, \dfrac{1-\cos^2\theta}{\cos\theta} = b^3$

$\Rightarrow \quad \dfrac{\cos^2\theta}{\sin\theta} = a^3, \dfrac{\sin^2\theta}{\cos\theta} = b^3$

$\Rightarrow \quad \dfrac{\sin^2\theta}{\cos\theta} \div \dfrac{\cos^2\theta}{\sin\theta} = \dfrac{b^3}{a^3}$

$\Rightarrow \quad \tan^3\theta = \dfrac{b^3}{a^3} \Rightarrow \tan\theta = \dfrac{b}{a} \Rightarrow \sin\theta = \dfrac{b}{\sqrt{a^2+b^2}}$ and $\cos\theta = \dfrac{a}{\sqrt{a^2+b^2}}$

Substituting these values of $\sin\theta$ and $\cos\theta$ in $\dfrac{\cos^2\theta}{\sin\theta} = a^3$, we obtain

$$\frac{a^2}{b\sqrt{(a^2+b^2)}} = a^3 \Rightarrow ab\sqrt{a^2+b^2} = 1 \Rightarrow a^2b^2(a^2+b^2) = 1$$

22. We have, $\cot\theta\,(1+\sin\theta) = 4m$ and $\cot\theta\,(1-\sin\theta) = 4n$

$\Rightarrow \quad \cot\theta + \cos\theta = 4m$ and $\cot\theta - \cos\theta = 4n$

$\Rightarrow \quad (\cot\theta+\cos\theta)^2 - (\cot\theta-\cos\theta)^2 = 16m^2 - 16n^2$ and $(\cot\theta+\cos\theta)(\cot\theta-\cos\theta) = 16\,mn$

$\Rightarrow \quad 4\cot\theta\cos\theta = 16\,(m^2-n^2)$ and $\cot^2\theta - \cos^2\theta = 16\,mn$

$\Rightarrow \quad \dfrac{\cos^2\theta}{\sin\theta} = 4\,(m^2-n^2)$ and $\dfrac{\cos^4\theta}{\sin^2\theta} = 16\,mn$

$\Rightarrow \quad \dfrac{\cos^4\theta}{\sin^2\theta} = 16\,(m^2-n^2)^2$ and $\dfrac{\cos^4\theta}{\sin^2\theta} = 16\,mn$

$\Rightarrow \quad 16\,(m^2-n^2)^2 = 16\,mn$

$\Rightarrow \quad (m^2-n^2) = mn$

5.4 SIGNS OF THE TRIGONOMETRIC RATIOS OR FUNCTIONS

In the previous section, we have introduced six trigonometric ratios. Their signs depend on the quadrant in which the terminal side of the angle lies. We always take the length $OP = r$ (see Fig. 5.1 — 5.4) to be positive.

Thus, $\sin\theta = \dfrac{y}{r}$ has the sign of y, $\cos\theta = \dfrac{x}{r}$ has the sign of x. The sign of $\tan\theta$ depends on the signs of x and y and similarly the signs of other trigonometric ratios are decided by the signs of x and/or y. Thus, we have the following :

In first quadrant: We have, $x > 0$ and $y > 0$

$\therefore \quad \sin\theta = \dfrac{y}{r} > 0$, $\cos\theta = \dfrac{x}{r} > 0$, $\tan\theta = \dfrac{y}{x} > 0$, $\text{cosec}\,\theta = \dfrac{r}{y} > 0$, $\sec\theta = \dfrac{r}{x} > 0$ and $\cot\theta = \dfrac{x}{y} > 0$.

Thus, in the first quadrant all trigonometric functions are positive.

In second quadrant: We have, $x < 0$ and $y > 0$

$\therefore \quad \sin\theta = \dfrac{y}{r} > 0$, $\cos\theta = \dfrac{x}{r} < 0$, $\tan\theta = \dfrac{y}{x} < 0$, $\text{cosec}\,\theta = \dfrac{r}{y} > 0$, $\sec\theta = \dfrac{r}{x} < 0$ and $\cot\theta = \dfrac{x}{y} < 0$.

Thus, in the second quadrant sine and cosecant functions are positive and all others are negative.

In third quadrant: We have, $x < 0$ and $y < 0$

$\therefore\ \sin\theta = \frac{y}{r} < 0,\ \cos\theta = \frac{x}{r} < 0,\ \tan\theta = \frac{y}{x} > 0,\ \text{cosec}\,\theta = \frac{r}{y} < 0,\ \sec\theta = \frac{r}{x} < 0$ and $\cot\theta = \frac{x}{y} > 0.$

Thus, in the third quadrant all trigonometric functions are negative except tangent and its reciprocal cotangent.

In fourth quadrant: We have, $x > 0$ and $y < 0$

$\therefore\ \sin\theta = \frac{y}{r} > 0,\ \cos\theta = \frac{x}{r} > 0,\ \tan\theta = \frac{y}{x} > 0,\ \text{cosec}\,\theta = \frac{r}{y} < 0,\ \sec\theta = \frac{r}{x} > 0$ and $\cot\theta = \frac{x}{y} < 0.$

Thus, in the fourth quadrant all trigonometric functions are negative except cosine and its reciprocal secant.

It follows from the above discussion that the signs of the trigonometric ratios in different quadrants are as under :

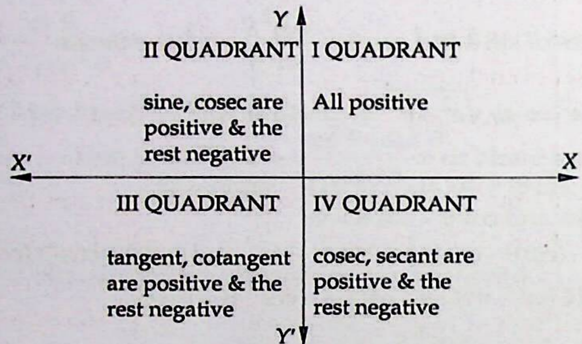


Fig. 5.6

SIMPLE RULE TO REMEMBER *A crude aid to memorise the signs of trigonometrical ratios in different quadrants is the four-word phrase "ALL SCHOOL TO COLLEGE". The first letter of the first word in this phrase is 'A'. This may be taken to indicate that all trigonometric ratios are positive in the first quadrant. The first letter of the second word is 'S'. This indicates that sine and its reciprocal are positive in the second quadrant. The first letter of third word is 'T'. This may be taken as to indicate that tangent and its reciprocal are positive in the third quadrant. The first letter of the fourth word in the phrase is 'C' which may be taken as to indicate that only cosine and its reciprocal are positive in the fourth quadrant.*

5.5 VARIATIONS IN VALUES OF TRIGONOMETRIC FUNCTIONS IN DIFFERENT QUADRANTS

DIFFERENT QUADRANTS Let $X'OX$ and YOY' be the coordinate axes. Draw a circle with centre at origin O and radius unity. Suppose the circle cuts the coordinate axes at A, B, A' and B' as shown in Fig. 5.7. Let $P(x, y)$ be a point on the circle such that $\angle AOP = \theta$. Then, $x = \cos\theta$ and $y = \sin\theta$.

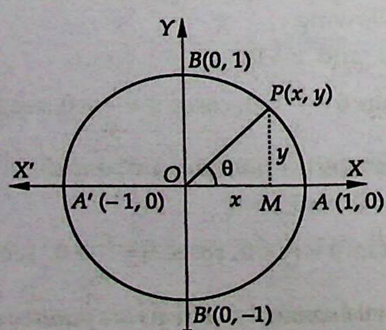


Fig. 5.7

It is evident from the Fig. 5.7 that

$$-1 \le x \le 1 \text{ and } -1 \le y \le 1$$

$$\Rightarrow \quad -1 \le \cos\theta \le 1 \text{ and } -1 \le \sin\theta \le 1 \text{ for all values of } \theta.$$

In the first quadrant as the angle θ increases from $0°$ to $90°$, we observe that the values of y increase from 0 to 1. Consequently $\sin\theta$ increases from 0 to 1. In the second quadrant as θ increases from $90°$ to $180°$, y decreases from 1 to 0. Consequently, $\sin\theta$ decreases from 1 to 0. In the third quadrant as θ increases from $180°$ to $270°$, $\sin\theta\ (= y)$ decreases from 0 to –1 and finally in the fourth quadrant $\sin\theta$ increases from –1 to 0 as θ increases from $270°$ to $360°$. Similarly, we can observe the variations in the values of other trigonometric functions. The following table exhibits the variations in the values of all trigonometrical ratios.

	I QUADRANT		II QUADRANT
$\sin\theta$	increases from 0 to 1	$\sin\theta$	decreases from 1 to 0
$\cos\theta$	decreases from 1 to 0	$\cos\theta$	decreases from 0 to –1
$\tan\theta$	increases from 0 to ∞	$\tan\theta$	increases from $-\infty$ to 0
$\cot\theta$	decreases from ∞ to 0	$\cot\theta$	decreases from 0 to $-\infty$
$\sec\theta$	increases from 1 to ∞	$\sec\theta$	increases from $-\infty$ to –1
$\operatorname{cosec}\theta$	decreases from ∞ to 1	$\operatorname{cosec}\theta$	decreases from 1 to ∞

	III QUADRANT		IV QUADRANT
$\sin\theta$	decreases from 0 to –1	$\sin\theta$	increases from –1 to 0
$\cos\theta$	decreases from –1 to 0	$\cos\theta$	increases from 0 to 1
$\tan\theta$	increases from 0 to ∞	$\tan\theta$	increases from $-\infty$ to 0
$\cot\theta$	decreases from ∞ to 0	$\cot\theta$	decreases from 0 to $-\infty$
$\sec\theta$	decreases from -1 to $-\infty$	$\sec\theta$	decreases from ∞ to 1
$\operatorname{cosec}\theta$	decreases from $-\infty$ to –1	$\operatorname{cosec}\theta$	decreases from -1 to ∞

REMARK *Note that* $+\infty$ and $-\infty$ *are two symbols. These are not real numbers. When we say that* $\tan\theta$ *increases from 0 to* ∞ *as* θ *varies from 0 to* $\pi/2$, *it means that* $\tan\theta$ *increases in the interval* $(0, \pi/2)$ *and it attains arbitrarily large positive values as* θ *tends to* $\pi/2$. *Similarly, we interpret for other trigonometrical functions.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find* $\sin\theta$ and $\tan\theta$ *if* $\cos\theta = -\dfrac{12}{13}$ *and* θ *lies in the third quadrant.*

SOLUTION We have,

$$\cos^2\theta + \sin^2\theta = 1 \Rightarrow \sin\theta = \pm\sqrt{1-\cos^2\theta}$$

In third quadrant $\sin\theta$ is negative.

$$\therefore \quad \sin\theta = -\sqrt{1-\cos^2\theta} \Rightarrow \sin\theta = -\sqrt{1-\left(-\frac{12}{13}\right)^2} = -\frac{5}{13}$$

$$\text{Now,} \quad \tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = -\frac{5}{13} \times \frac{13}{-12} = \frac{5}{12}$$

EXAMPLE 2 *Find the values of* $\cos\theta$ *and* $\tan\theta$, *if* $\sin\theta = -\dfrac{3}{5}$ *and* $\pi < \theta < \dfrac{3\pi}{2}$.

SOLUTION We have,

$$\cos^2\theta + \sin^2\theta = 1 \Rightarrow \cos\theta = \pm\sqrt{1-\sin^2\theta}$$

In the third quadrant $\cos\theta$ is negative and $\tan\theta$ is positive.

$\therefore \quad \cos\theta = -\sqrt{1-\sin^2\theta} \Rightarrow \cos\theta = -\sqrt{1-\frac{9}{25}} = -\frac{4}{5}$.

and, $\quad \tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = -\frac{3}{5}\times -\frac{5}{4} = \frac{3}{4}$

EXAMPLE 3 *Find all other trigonometrical ratios if* $\sin\theta = -\frac{2\sqrt{6}}{5}$ *and* θ *lies in quadrant III.*

SOLUTION We have,

$$\cos^2\theta + \sin^2\theta = 1 \Rightarrow \cos\theta = \pm\sqrt{1-\sin^2\theta}$$

In the third quadrant $\cos\theta$ is negative.

$\therefore \quad \cos\theta = -\sqrt{1-\sin^2\theta} \Rightarrow \cos\theta = -\sqrt{1-\frac{24}{25}} = -\frac{1}{5}$

In the third quadrant $\tan\theta$ is positive.

$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = -\frac{2\sqrt{6}}{5}\times -\frac{5}{1} = 2\sqrt{6}$

Now, $\quad \operatorname{cosec}\theta = \frac{1}{\sin\theta} \Rightarrow \operatorname{cosec}\theta = -\frac{5}{2\sqrt{6}}$

$\sec\theta = \frac{1}{\cos\theta} \Rightarrow \sec\theta = -5$

and, $\quad \cot\theta = \frac{1}{\tan\theta} \Rightarrow \cot\theta = \frac{1}{2\sqrt{6}}$.

EXAMPLE 4 *If* $\cos\theta = -\frac{1}{2}$ *and* $\pi < \theta < \frac{3\pi}{2}$, *find the value of* $4\tan^2\theta - 3\operatorname{cosec}^2\theta$.

SOLUTION Since θ lies in the third quadrant. Therefore, $\sin\theta$ is negative and $\tan\theta$ is positive.

Now, $\quad \sin\theta = \pm\sqrt{1-\cos^2\theta} \Rightarrow \sin\theta = -\sqrt{1-\frac{1}{4}} = -\frac{\sqrt{3}}{2} \Rightarrow \operatorname{cosec}\theta = \frac{-2}{\sqrt{3}}$

And, $\quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \sqrt{3}$.

Hence, $\quad 4\tan^2\theta - 3\operatorname{cosec}^2\theta = 4\times 3 - 3\times\frac{4}{3} = 8$.

EXAMPLE 5 *If* $\sec\theta = \sqrt{2}$ *and* $\frac{3\pi}{2} < \theta < 2\pi$, *find the value of* $\frac{1+\tan\theta+\operatorname{cosec}\theta}{1+\cot\theta-\operatorname{cosec}\theta}$.

SOLUTION We have,

$$\sec\theta = \sqrt{2} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$$

$\therefore \quad \sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\sqrt{1-\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$

But, θ lies in the fourth quadrant in which $\sin\theta$ is negative.

$\therefore \quad \sin\theta = -\frac{1}{\sqrt{2}} \Rightarrow \operatorname{cosec}\theta = -\sqrt{2}$

Now, $\tan\theta = \dfrac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = -\dfrac{1}{\sqrt{2}} \times \dfrac{\sqrt{2}}{1} = -1 \Rightarrow \cot\theta = -1$

$$\therefore \quad \frac{1+\tan\theta+\operatorname{cosec}\theta}{1+\cot\theta-\operatorname{cosec}\theta} = \frac{1-1-\sqrt{2}}{1-1+\sqrt{2}} = -1$$

LEVEL-2

EXAMPLE 6 *Prove that :*

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \begin{cases} \sec\theta - \tan\theta & , \text{if } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ -\sec\theta + \tan\theta & , \text{if } \frac{\pi}{2} < \theta < \frac{3\pi}{2} \end{cases}$$

SOLUTION We have,

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}} = \frac{1-\sin\theta}{\sqrt{\cos^2\theta}} = \frac{1-\sin\theta}{|\cos\theta|}$$

$$\Rightarrow \quad \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \begin{cases} \dfrac{1-\sin\theta}{\cos\theta} & , \text{if } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \dfrac{1-\sin\theta}{-\cos\theta} & , \text{if } \frac{\pi}{2} < \theta < \frac{3\pi}{2} \end{cases}$$

$$\Rightarrow \quad \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \begin{cases} \sec\theta - \tan\theta & , \text{if } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ -\sec\theta + \tan\theta & , \text{if } \frac{\pi}{2} < \theta < \frac{3\pi}{2} \end{cases}$$

EXAMPLE 7 Prove that : $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}} = \begin{cases} \operatorname{cosec}\theta + \cot\theta & , \text{if } 0 < \theta < \pi \\ -\operatorname{cosec}\theta - \cot\theta & , \text{if } \pi < \theta < 2\pi \end{cases}$

SOLUTION We have,

$$\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}}$$

$$\Rightarrow \quad \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \frac{1+\cos\theta}{\sqrt{\sin^2\theta}} = \frac{1+\cos\theta}{|\sin\theta|}$$

$$\Rightarrow \quad \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \begin{cases} \dfrac{1+\cos\theta}{\sin\theta} & , \text{if } 0 < \theta < \pi \\ \dfrac{1+\cos\theta}{-\sin\theta} & , \text{if } \pi < \theta < \theta < 2\pi \end{cases}$$

$$\Rightarrow \quad \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \begin{cases} \operatorname{cosec}\theta + \cot\theta & , \text{if } 0 < \theta < \pi \\ -\operatorname{cosec}\theta - \cot\theta & , \text{if } \pi < \theta < 2\pi \end{cases}$$

EXAMPLE 8 Prove that :

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} + \sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \begin{cases} \dfrac{2}{\cos\theta} & , \text{if } 0 \le \theta < \frac{\pi}{2} \\ -\dfrac{2}{\cos\theta} & , \text{if } \frac{\pi}{2} < \theta \le \pi \end{cases}$$

SOLUTION We have,

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} + \sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \frac{(1-\sin\theta)+(1+\sin\theta)}{\sqrt{1-\sin^2\theta}}$$

$$\Rightarrow \quad \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} + \sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \frac{2}{\sqrt{\cos^2\theta}} = \frac{2}{|\cos\theta|} \qquad [\because \sqrt{x^2} = |x|]$$

$$\Rightarrow \quad \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} + \sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \begin{cases} \dfrac{2}{\cos\theta}, \text{ if } 0 \le \theta < \dfrac{\pi}{2} \\ -\dfrac{2}{\cos\theta}, \text{ if } \dfrac{\pi}{2} < \theta \le \pi \end{cases}$$

EXERCISE 5.2

LEVEL-1

1. Find the values of the other five trigonometric functions in each of the following:
   (i) $\cot\theta = \frac{12}{5}$, θ in quadrant III
   (ii) $\cos\theta = -\frac{1}{2}$, θ in quadrant II
   (iii) $\tan\theta = \frac{3}{4}$, θ in quadrant III
   (iv) $\sin\theta = \frac{3}{5}$, θ in quadrant I
2. If $\sin\theta = \frac{12}{13}$ and θ lies in the second quadrant, find the value of $\sec\theta + \tan\theta$.
3. If $\sin\theta = \frac{3}{5}$, $\tan\phi = \frac{1}{2}$ and $\frac{\pi}{2} < \theta < \pi < \phi < \frac{3\pi}{2}$, find the value of $8\tan\theta - \sqrt{5}\sec\phi$.
4. If $\sin\theta + \cos\theta = 0$ and θ lies in the fourth quadrant, find $\sin\theta$ and $\cos\theta$.
5. If $\cos\theta = -\frac{3}{5}$ and $\pi < \theta < \frac{3\pi}{2}$, find the values of other five trigonometric functions and hence evaluate $\frac{\text{cosec}\,\theta + \cot\theta}{\sec\theta - \tan\theta}$.

ANSWERS

1. (i) $\sin\theta = -\frac{5}{13}$, $\cos\theta = -\frac{12}{13}$, $\tan\theta = \frac{5}{12}$, $\text{cosec}\,\theta = -\frac{13}{5}$, $\sec\theta = -\frac{13}{12}$
   (ii) $\sin\theta = \frac{\sqrt{3}}{2}$, $\tan\theta = -\sqrt{3}$, $\text{cosec}\,\theta = \frac{2}{\sqrt{3}}$, $\cot\theta = \frac{-1}{\sqrt{3}}$, $\sec\theta = -2$
   (iii) $\sin\theta = -\frac{3}{5}$, $\cos\theta = -\frac{4}{5}$, $\text{cosec}\,\theta = -\frac{5}{3}$, $\sec\theta = -\frac{5}{4}$, $\cot\theta = \frac{4}{3}$
   (iv) $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$, $\sec\theta = \frac{5}{4}$, $\cot\theta = \frac{4}{3}$, $\text{cosec}\,\theta = \frac{5}{3}$
2. –5
3. $-\frac{7}{2}$
4. $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$
5. $\frac{1}{6}$

HINTS TO SELECTED PROBLEM

4. We have,
$$\sin\theta + \cos\theta = 0 \Rightarrow \sin\theta = -\cos\theta \Rightarrow \tan\theta = -1$$
$$\therefore \sec^2\theta = 1 + \tan^2\theta \Rightarrow \sec^2\theta = 1 + (-1)^2 = 2 \Rightarrow \sec\theta = \sqrt{2} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$$

5.6 TRIGONOMETRIC RATIOS OF ALLIED ANGLES

Two angles are said to be allied when their sum or difference is either zero or a multiple of 90°.

The angles $-\theta$, $90° \pm \theta$, $180° \pm \theta$, $360° \pm \theta$ etc are angles allied to the angle θ if θ is measured in degrees. However, if θ is measured in radians, then the angles allied to θ are $-\theta$, $\frac{\pi}{2} \pm \theta$, $\pi \pm \theta$, $2\pi \pm \theta$ etc. Using trigonometric ratios of allied angles we can find the trigonometric ratios of angles of any magnitude.

5.6.1 TRIGONOMETRIC RATIOS OF $(-\theta)$ IN TERMS OF THAT OF θ

Let a revolving ray starting from its initial position OX, trace out an angle $\angle XOA = \theta$. Let $P(x, y)$ be a point on OA such that $OP = r$. Draw PM perpendicular from P on x-axis. Let there be another revolving ray OA' which starts from the initial position OX and describes an angle $\angle XOA' = -\theta$ in the clockwise sense. Let P' be a point on OA' such that $OP' = OP$. Draw $P'M'$ perpendicular from P' on x-axis. Clearly, M and M' coincide and ΔOMP is congruent to $\Delta OMP'$. Then the coordinates of P' are $(x, -y)$.

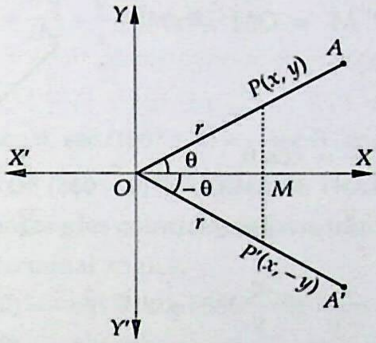


Fig. 5.8

$$\therefore \quad \sin(-\theta) = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta,\ \cos(-\theta) = \frac{x}{r} = \cos\theta,\ \tan(-\theta) = -\frac{y}{x} = -\tan\theta.$$

Taking the reciprocals of these trigonometric ratios, we have

$$\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta,\ \sec(-\theta) = \sec\theta \text{ and } \cot(-\theta) = -\cot\theta.$$

5.6.2 TRIGONOMETRIC RATIOS OF $(90° - \theta)$ IN TERMS OF THAT OF θ

Let a revolving ray starting from its initial position OX, trace out an angle $\angle XOA = \theta$. Let $P(x, y)$ be a point on OA such that $OP = r$. Draw PM perpendicular from P on x-axis. Let another revolving ray OA' starting from the initial position OX, trace out angle of $90°$ to coincide with OY and then it rotates in the clockwise direction through an angle θ. Then, OA' in its final position traces out an angle $\angle XOA' = 90° - \theta$. Let P' be a point on OA' such that $OP' = OP = r$. From P' draw perpendicular $P'M'$ on x-axis. Then, ΔOMP and $OM'P'$ are congruent.

Clearly, $M'P' = OM = x$ and $OM' = MP = y$

So, the coordinates of P' are (y, x).

$$\therefore \quad \sin(90 - \theta) = \frac{P'M'}{OP'} = \frac{x}{r} = \cos\theta.$$

$$\cos(90 - \theta) = \frac{OM'}{OP'} = \frac{y}{r} = \sin\theta$$

$$\tan(90 - \theta) = \frac{P'M'}{OM'} = \frac{x}{y} = \cot\theta$$

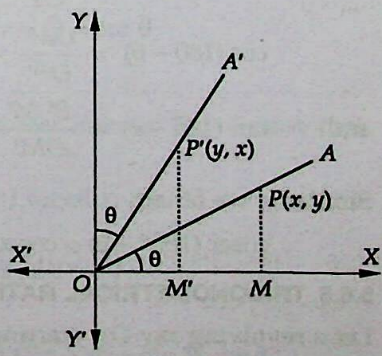


Fig. 5.9

Similarly, we obtain that

$$\operatorname{cosec}(90 - \theta) = \sec\theta,\ \sec(90 - \theta) = \operatorname{cosec}\theta \text{ and, } \cot(90 - \theta) = \tan\theta.$$

5.6.3 TRIGONOMETRIC RATIOS OF (90° + θ) IN TERMS OF THAT OF θ

Let a revolving ray OA starting from its initial position OX, trace out an angle $\angle XOA = \theta$ and let another revolving ray OA' starting from the same initial position OX, first trace out an angle θ so as to coincide with OA and the it revolves through an angle of 90° in anticlockwise direction to form an angle $\angle XOA' = 90° + \theta$. Let P and P' be points on OA and OA' respectively such that $OP = OP' = r$. Draw perpendiculars PM and $P'M'$ from P and P' respectively on OX. Let the coordinates of P be (x, y). Then, $OM = x$ and $P'M' = y$.

Clearly, $\Delta P'M'O$ is congruent to ΔOMP.

$\therefore \quad OM' = PM = y$ and $P'M' = OM = x$.

So, the coordinates of P' are $(-y, x)$.

Hence,

$$\sin(90+\theta) = \frac{M'P'}{OP'} = \frac{x}{r} = \cos\theta.$$

$$\cos(90+\theta) = \frac{OM'}{OP'} = \frac{-y}{r} = -\sin\theta$$

$$\tan(90+\theta) = \frac{M'P'}{OM'} = \frac{x}{-y} = -\frac{x}{y} = -\cot\theta$$

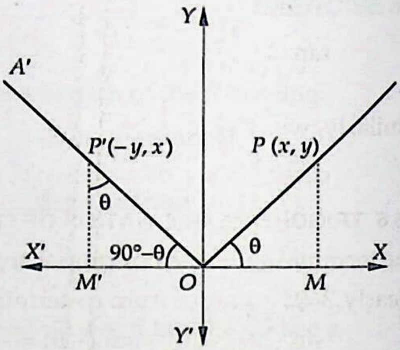


Fig. 5.10

Similarly, we obtain

$\operatorname{cosec}(90° + \theta) = \sec\theta$, $\sec(90° + \theta) = -\operatorname{cosec}\theta$ and $\cot(90° + \theta) = -\tan\theta$.

5.6.4 TRIGONOMETRIC RATIOS OF (180° − θ) IN TERMS OF THAT OF θ

Let a revolving ray OA starting from its initial position OX, trace out an angle $\angle XOA = \theta$. Let another revolving ray OA' starting from its initial position OX, trace out angle of 180° to coincide with OX' and then rotates in clockwise sense through an angle θ. Let P and P' be points on OA and OA' respectively such that $OP = OP' = r$. Draw perpendiculars PM and $P'M'$ from P and P' on OX and OX' respectively.

Clearly, triangles OMP and $OM'P'$ are congruent.

$\therefore \quad M'P' = MP = y$ and $OM' = OM$.

Hence, the coordinates of P' are $(-x, y)$.

Now, $$\sin(180-\theta) = \frac{P'M'}{OP'} = \frac{y}{r} = \sin\theta.$$

$$\cos(180-\theta) = \frac{OM'}{OP'} = -\frac{x}{r} = -\cos\theta$$

and, $$\tan(180-\theta) = \frac{P'M'}{OM'} = \frac{y}{-x} = -\tan\theta$$

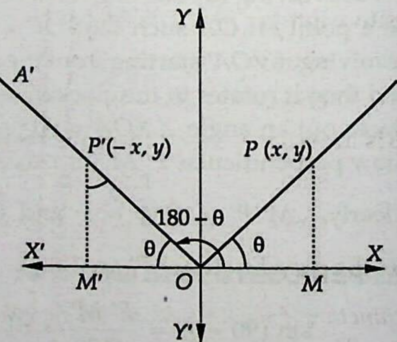


Fig. 5.11

Similarly, we obtain

$\operatorname{cosec}(180° - \theta) = \operatorname{cosec}\theta$, $\sec(180° - \theta) = -\sec\theta$ and, $\cot(180° - \theta) = -\cot\theta$

5.6.5 TRIGONOMETRICAL RATIOS OF (180° + θ) IN TERMS OF THAT OF θ

Let a revolving ray OA starting from its initial position OX, trace out an angle $\angle XOA = \theta$. Let another revolving line OA' starting from its position OX, trace out an angle $\angle XOA' = 180° + \theta$. Let P and P' be points on OA and OA' respectively such that $OP = OP' = r$. Draw perpendiculars PM and $P'M'$ from P and P' respectively on OX and OX'. Clearly, triangles OPM and $OP'M'$ are congruent.

$\therefore \quad OM' = OM = x, \; PM = P'M' = y$

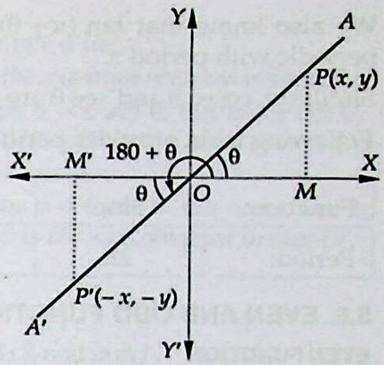


Fig. 5.12

Hence, the coordiantes of P' are $(-x, -y)$.

Clearly, $\sin(180+\theta) = \dfrac{P'M'}{OP'} = \dfrac{-y}{r} = -\dfrac{y}{r} = -\sin\theta$

$\cos(180+\theta) = \dfrac{OM'}{OP'} = \dfrac{-x}{r} = -\dfrac{x}{r} = -\cos\theta$

$\tan(180+\theta) = \dfrac{P'M'}{OM'} = \dfrac{-y}{-x} = \dfrac{y}{x} = \tan\theta$

Similarly, we obtain

$\operatorname{cosec}(180°+\theta) = -\operatorname{cosec}\theta$, $\sec(180°+\theta) = -\sec\theta$ and $\cot(180°+\theta) = \cot\theta$..

5.6.6 TRIGONOMETRIC RATIOS OF $(360° - \theta)$ IN TERMS OF THAT OF θ

The terminal sides of co-terminal angles coincide, so their trigonometrical ratios are same.

Clearly, $360° - \theta$ and $-\theta$ are co-terminal angles.

$\therefore \quad \sin(360° - \theta) = \sin(-\theta) = -\sin\theta$, $\cos(360° - \theta) = \cos(-\theta) = \cos\theta$,

and, $\quad \tan(360° - \theta) = \tan(-\theta) = -\tan\theta$

Similarly, we have

$\operatorname{cosec}(360° - \theta) = -\operatorname{cosec}\theta$, $\sec(360° - \theta) = \sec\theta$ and $\cot(360° - \theta) = -\cot\theta$.

We know that the terminal sides of co-terminal angles always coincide and θ and $360° + \theta$ are co-terminal angles. Therefore,

$\sin(360°+\theta) = \sin\theta$, $\cos(360°+\theta) = \cos\theta$, $\tan(360°+\theta) = \tan\theta$,

$\sec(360°+\theta) = \sec\theta$, $\operatorname{cosec}(360°+\theta) = \operatorname{cosec}\theta$ and $\cot(360°+\theta) = \cot\theta$

In fact, for any positive integer n, angle $(360° \times n + \theta)$ is coterminal to angle θ. Therefore, for any positive integer n, we have

$\sin(360° \times n + \theta) = \sin\theta$, $\cos(360° \times n + \theta) = \cos\theta$, $\tan(360° \times n + \theta) = \tan\theta$,

$\operatorname{cosec}(360° \times n + \theta) = \operatorname{cosec}\theta$, $\sec(360° \times n + \theta) = \sec\theta$ and, $\cot(360° \times n + \theta) = \cot\theta$.

If θ is in radians, then the above results may be written as

$\sin(2n\pi + \theta) = \sin\theta$, $\cos(2n\pi + \theta) = \cos\theta$, $\tan(2n\pi + \theta) = \tan\theta$,

$\operatorname{cosec}(2n\pi + \theta) = \operatorname{cosec}\theta$, $\sec(2n\pi + \theta) = \sec\theta$ and $\cot(2n\pi + \theta) = \cot\theta$

5.7 PERIODIC FUNCTION

A function $f(x)$ is said to be a periodic function if there exists a real number $T > 0$ such that $f(x+T) = f(x)$ for all x.

If T is the smallest positive real number such that $f(x+T) = f(x)$ for all x, then T is called the fundamental period of $f(x)$.

Since $\sin(2n\pi + \theta) = \sin\theta$, $\cos(2n\pi + \theta) = \cos\theta$ for all values of θ and $n \in N$.

Therefore, sine and cosine functions are periodic functions.

We find that 2π is the smallest positive real number such that $\sin(2\pi + \theta) = \sin\theta$, and $\cos(2\pi + \theta) = \cos\theta$ for all values of θ.

So, sine and cosine functions are periodic with period 2π.

We also know that tan $(\pi + \theta) = \tan \theta$ and $\cot(\pi + \theta) = \cot \theta$. Therefore, tan θ and cot θ are periodic with period π.

Similarly, cosec θ and sec θ are periodic functions with period 2 π.

Following table provides periods of all trigonometric functions as a ready reference.

Function:	Sine	Cosine	Tangent	Cosecant	Secant	Cotangent
Period:	2π	2π	π	2π	2π	π

5.8 EVEN AND ODD FUNCTIONS

EVEN FUNCTION *A function f(x) is said to be an even function, if* $f(-x) = f(x)$ *for all x in its domain.*

ODD FUNCTION *A function f(x) is an odd function, if* $f(-x) = -f(x)$ *for all x in its domain.*

We have seen that $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$ and $\tan(-\theta) = -\tan\theta$.

Therefore, sin θ and tan θ and their reciprocals cosec θ and cot θ are odd functions whereas cos θ and its reciprocal sec θ are even functions.

In earlier classes, we have learnt about the values of trigonometrical ratios for 0°, 30°, 45°, 60° and 90°. The values of trigonometric functions for these angles are same as that of trigonometrical ratios studied in earlier classes. In fact, the value of a trigonometrical ratio for any angle is same as the value of trigonometrical function for the same angle. Thus, we have the following table:

Angle / Trigonometric Function	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	0	not defined	0
cosec	not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	not defined	-1	not defined
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined	-1	not defined	1
cot	not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	not defined	0	not defined

In order to find the values of trigonometrical functions for any angle in terms of those of positive acute angle, we may follow the following algorithm:

ALGORITHM

STEP I *See whether the given angle α is positive or negative if it is negative, make it positive by using the following :*

$$\sin(-\theta) = -\sin\theta,\ \cos(-\theta) = \cos\theta,\ \tan(-\theta) = -\tan\theta \text{ etc.}$$

STEP II *Express the positive angle α obtained in step I in the form* $\alpha = 90° \times n \pm \theta$, *where θ is an acute angle.*

STEP III *Determine the quadrant in which the terminal side of the angle* α *lies.*

STEP IV *Determine the sign of the given trigonometrical function in the quadrant obtained in step III.*

STEP V *If n in step II is an odd integer, then* $\sin\alpha = \pm\cos\theta$, $\cos\alpha = \pm\sin\theta$, $\tan\alpha = \pm\cot\theta$, $\sec\alpha = \pm\operatorname{cosec}\theta$ *and* $\operatorname{cosec}\theta = \pm\sec\theta$. *The sign on RHS will be the sign obtained in step IV.*

If n in step II is an even integer, then $\sin\alpha = \pm\sin\theta$, $\cos\alpha = \pm\cos\theta$ $\tan\alpha = \pm\tan\theta$, $\sec\alpha = \pm\sec\theta$ *and* $\operatorname{cosec}\alpha = \pm\operatorname{cosec}\theta$. *The sign on RHS is the sign obtained in step IV.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the values of the following trigonometric ratios:*

(i) $\sin 315°$ (ii) $\cos 210°$ (iii) $\cos(-480°)$ (iv) $\sin(-1125°)$

SOLUTION (i) Clearly,

$$\sin 315° = \sin(90° \times 3 + 45°)$$

Since 315° lies in the IVth quadrant in which sine function is negative and 3 is an odd integer.

$$\therefore \quad \sin 315° = \sin(90° \times 3 + 45) = -\cos 45° = -\frac{1}{\sqrt{2}}$$

(ii) Clearly,

$$\cos 210° = \cos(90° \times 2 + 30°).$$

Since 210° is in the III quadrant in which cosine function is negative. Also the multiple of 90° is even.

$$\therefore \quad \cos 210° = \cos(90° \times 2 + 30°) = -\cos 30° = -\frac{\sqrt{3}}{2}$$

(iii) Clearly,

$$\cos(-480°) = \cos 480° = \cos(90° \times 5 + 30°)$$

Since 480° is in the II quadrant in which cosine function is negative. Also the multiple of 90° is odd.

$$\therefore \quad \cos(-480°) = \cos 480° = \cos(90° \times 5 + 30°) = -\sin 30° = -\frac{1}{2}$$

(iv) We have,

$$\sin(-1125°) = -\sin 1125° = -\sin(90° \times 12 + 45°)$$

Clearly, 1125° lies in the first quadrant. The multiple of 90° in this expression is even.

$$\therefore \quad \sin(-1125°) = -\sin(90° \times 12 + 45°) = -\sin 45° = -\frac{1}{\sqrt{2}}$$

EXAMPLE 2 *Find the values of the following trigonometric ratios.*

(i) $\operatorname{cosec} 390°$ (ii) $\cot 570°$ (iii) $\tan 480°$ (iv) $\cos 270°$

(v) $\tan\frac{19\pi}{3}$ (vi) $\sin\left(\frac{-11\pi}{3}\right)$ (vii) $\cot\left(\frac{-15\pi}{4}\right)$

SOLUTION (i) We have,

$$390° = 90° \times 4 + 30°$$

Clearly, 390° is in I quadrant and the multiple of 90° is even.

$$\therefore \quad \operatorname{cosec} 390° = \operatorname{cosec}(90° \times 4 + 30°) = \operatorname{cosec} 30° = 2.$$

(ii) We have,

$$570° = 90° \times 6 + 30°$$

Clearly, 570° is in the IIIrd quadrant and the multiple of 90° is even.

$\therefore \quad \cot 570° = \cot(90° \times 6 + 30°) = \cot 30° = \sqrt{3}$

(iii) We have,

$480° = 90° \times 5 + 30°$

Clearly, 480° is in the second quadrant and the multiple of 90° is odd.

$\therefore \quad \tan 480° = \tan(90° \times 5 + 30°) = -\cot 30° = -\sqrt{3}$

(iv) We have,

$270° = 90° \times 3 + 0°$

Clearly, 270° is in the negative direction of y-axis i.e. on the boundary line of II and III quadrant. Also, the multiple of 90° is an odd integer.

$\therefore \quad \cos 270° = \cos(90° \times 3 + 0°) = \pm \sin 0° = 0$

(v) We have,

$$\frac{19\pi}{3} = \left(\frac{19}{3} \times 180\right)^{\circ} = 1140° = 90° \times 12 + 60°$$

Clearly, this angle lies in first quadrant.

$$\therefore \quad \tan \frac{19\pi}{3} = \tan(90° \times 12 + 60°) = \tan 60° = \sqrt{3}$$

(vi) We have,

$$\frac{11\pi}{3} = \left(\frac{11 \times 180°}{3}\right)^{\circ} = 660° = 90° \times 7 + 30°$$

$$\therefore \quad \sin\left(-\frac{11\pi}{3}\right) = -\sin\frac{11\pi}{3} = -\sin(90° \times 7 + 30°) = -(-\cos 30°) = -\left(-\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}$$

(vii) We have,

$$\frac{15\pi}{4} = \left(\frac{15 \times 180}{4}\right)^{\circ} = 675° = 90° \times 7 + 45°$$

$$\therefore \quad \cot\left(-\frac{15\pi}{4}\right) = -\cot\left(\frac{15\pi}{4}\right) = -\cot(90° \times 7 + 45°) = -(-\cot 45°) = -(-1) = 1.$$

EXAMPLE 3 *Prove that:* $\cos 510° \cos 330° + \sin 390° \cos 120° = -1$.

SOLUTION We have,

$$\begin{aligned} \text{LHS} &= \cos 510° \cos 330° + \sin 390° \cos 120° \\ &= \cos(90° \times 5 + 60°)\cos(90° \times 3 + 60°) + \sin(90° \times 4 + 30°°)\cos(90° \times 1 + 30°) \\ &= (-\sin 60°)(\sin 60°) + (\sin 30°)(-\sin 30°) \\ &= -\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) = -\frac{3}{4} - \frac{1}{4} = -1 = \text{RHS} \end{aligned}$$

EXAMPLE 4 *Prove that :* $\sin(-420°)(\cos 390°) + \cos(-660°)(\sin 330°) = -1$.

SOLUTION We have,

$$\begin{aligned} \text{LHS} &= \sin(-420°)(\cos 390°) + \cos(-660°)(\sin 330°) \\ &= -\sin 420° \cos 390° + \cos 660° \sin 330° \qquad [\because \sin(-\theta) = -\sin\theta, \cos(-\theta) = \cos\theta] \\ &= -\sin(90° \times 4 + 60°)\cos(90° \times 4 + 30°) + \cos(90° \times 7 + 30°)\sin(90° \times 3 + 60°) \\ &= -(\sin 60°)(\cos 30°) + (\sin 30°)(-\cos 60°) = -\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2}\left(-\frac{1}{2}\right) = -1 = \text{RHS} \end{aligned}$$

EXAMPLE 5 *Prove that:*

(i) $\sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{3} - \tan^2\frac{\pi}{4} = -\frac{1}{2}$ (ii) $2\sin^2\frac{\pi}{6} + \operatorname{cosec}^2\frac{7\pi}{6}\cos^2\frac{\pi}{3} = \frac{3}{2}$

(iii) $\cot^2\frac{\pi}{6} + \operatorname{cosec}\frac{5\pi}{6} + 3\tan^2\frac{\pi}{6} = 6$ (iv) $2\sin^2\frac{3\pi}{4} + 2\cos^2\frac{\pi}{4} + 2\sec^2\frac{\pi}{3} = 10$

SOLUTION (i) $\text{LHS} = \sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{3} - \tan^2\frac{\pi}{4}$

$$= \left(\sin\frac{\pi}{6}\right)^2 + \left(\cos\frac{\pi}{3}\right)^2 - \left(\tan\frac{\pi}{4}\right)^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - (1)^2 = \frac{1}{4} + \frac{1}{4} - 1 = -\frac{1}{2}$$

(ii) $\text{LHS} = 2\sin^2\frac{\pi}{6} + \operatorname{cosec}^2\frac{7\pi}{6}\cos^2\frac{\pi}{3}$

$$= 2\left(\sin\frac{\pi}{6}\right)^2 + \left(\operatorname{cosec}\frac{7\pi}{6}\right)^2\left(\cos\frac{\pi}{3}\right)^2$$

$$= 2\left(\sin\frac{\pi}{6}\right)^2 + \left\{\operatorname{cosec}\left(\pi + \frac{\pi}{6}\right)\right\}^2\left(\cos\frac{\pi}{3}\right)^2$$

$$= 2\left(\sin\frac{\pi}{6}\right)^2 + \left\{-\operatorname{cosec}\frac{\pi}{6}\right\}^2\left(\cos\frac{\pi}{3}\right)^2 \qquad [\because \operatorname{cosec}(\pi + \theta) = -\operatorname{cosec}\theta]$$

$$= 2\left(\frac{1}{2}\right)^2 + (-2)^2 \times \left(\frac{1}{2}\right)^2 = \frac{1}{2} + 1 = \frac{3}{2} = \text{RHS}$$

(iii) $\text{LHS} = \cot^2\frac{\pi}{6} + \operatorname{cosec}\frac{5\pi}{6} + 3\tan^2\frac{\pi}{6}$

$$= \left(\cot\frac{\pi}{6}\right)^2 + \operatorname{cosec}\left(\pi - \frac{\pi}{6}\right) + 3\left(\tan\frac{\pi}{6}\right)^2 = (\sqrt{3})^2 + 2 + 3\left(\frac{1}{\sqrt{3}}\right)^2 = 3 + 2 + 1 = 6 = \text{RHS}$$

(iv) $\text{LHS} = 2\sin^2\frac{3\pi}{4} + 2\cos^2\frac{\pi}{4} + 2\sec^2\frac{\pi}{3}$

$$= 2\left(\sin\frac{3\pi}{4}\right)^2 + 2\left(\cos\frac{\pi}{4}\right)^2 + 2\left(\sec\frac{\pi}{3}\right)^2$$

$$= 2\left(\sin\frac{\pi}{4}\right)^2 + 2\left(\cos\frac{\pi}{4}\right)^2 + 2\left(\sec\frac{\pi}{3}\right)^2 \qquad \left[\because \sin\frac{3\pi}{4} = \sin\left(\pi - \frac{\pi}{4}\right) = \sin\frac{\pi}{4}\right]$$

$$= 2\left(\frac{1}{\sqrt{2}}\right)^2 + 2\left(\frac{1}{\sqrt{2}}\right)^2 + 2(2)^2 = 1 + 1 + 8 = 10 = \text{RHS}$$

EXAMPLE 6 *Prove that:* $\dfrac{\cos(90° + \theta)\sec(-\theta)\tan(180° - \theta)}{\sec(360° - \theta)\sin(180° + \theta)\cot(90° - \theta)} = -1.$

SOLUTION We have,

$$\text{LHS} = \frac{\cos(90° + \theta)\sec(-\theta)\tan(180° - \theta)}{\sec(360° - \theta)\sin(180° + \theta)\cot(90° - \theta)} = \frac{(-\sin\theta)(\sec\theta)(-\tan\theta)}{(\sec\theta)(-\sin\theta)(\tan\theta)} = -1 = \text{RHS}$$

EXAMPLE 7 *Prove that :*

(i) $\dfrac{\cos(\pi + x)\cos(-x)}{\sin(\pi - x)\cos\left(\dfrac{\pi}{2} + x\right)} = \cot^2 x$

(ii) $\cos\left(\dfrac{3\pi}{2} + x\right)\cos(2\pi + x)\left\{\cot\left(\dfrac{3\pi}{2} - x\right) + \cot(2\pi + x)\right\} = 1$

SOLUTION (i) $\text{LHS} = \dfrac{\cos(\pi + x)\cos(-x)}{\sin(\pi - x)\cos\left(\dfrac{\pi}{2} + x\right)} = \dfrac{(-\cos x) \times (\cos x)}{(\sin x)(-\sin x)} = \dfrac{-\cos^2 x}{-\sin^2 x} = \cot^2 x = \text{RHS}$

(ii) $\text{LHS} = \cos\left(\dfrac{3\pi}{2} + x\right)\cos(2\pi + x)\left\{\cot\left(\dfrac{3\pi}{2} - x\right) + \cot(2\pi + x)\right\}$

$$= (\sin x)(\cos x)(\tan x + \cot x) \qquad \left[\begin{array}{l} \because \cos\left(\dfrac{3\pi}{2}+x\right) = \sin x,\ \cos(2\pi + x) = \cos x \\ \cot\left(\dfrac{3\pi}{2}-x\right) = \tan x \text{ and } \cot(2\pi + x) = \cot x \end{array}\right]$$

$$= \sin x \cos x \left\{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right\}$$

$$= \sin x \cos x \left\{\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}\right\} = \sin x \cos x \times \frac{1}{\sin x \cos x} = 1$$

EXAMPLE 8 *If A, B, C, D are angles of a cyclic quadrilateral, prove that*

$$\cos A + \cos B + \cos C + \cos D = 0.$$

SOLUTION We know that the opposite angles of a cyclic quadrilateral are supplementary i.e. $A + C = \pi$ and $B + D = \pi$.

$\therefore \quad A = \pi - C$ and $B = \pi - D$

$\Rightarrow \quad \cos A = \cos(\pi - C) = -\cos C$ and, $\cos B = \cos(\pi - D) = -\cos D$

$\therefore \quad \cos A + \cos B + \cos C + \cos D = -\cos C - \cos D + \cos C + \cos D = 0$

EXAMPLE 9 *In any quadrilateral ABCD, prove that*

(i) $\sin(A + B) + \sin(C + D) = 0$ (ii) $\cos(A + B) = \cos(C + D)$

SOLUTION We have,

(i) $A + B + C + D = 2\pi$

$\Rightarrow \quad A + B = 2\pi - (C + D)$

$\Rightarrow \quad \sin(A + B) = \sin\{2\pi - (C + D)\}$

$\Rightarrow \quad \sin(A + B) = -\sin(C + D)$

$\Rightarrow \quad \sin(A + B) + \sin(C + D) = 0 \qquad [\because \sin(2\pi - \theta) = -\sin\theta]$

(ii) We have,

$A + B + C + D = 2\pi$

$\Rightarrow \quad A + B = 2\pi - (C + D)$

$\Rightarrow \quad \cos(A + B) = \cos\{2\pi - (C + D)\}$

$\Rightarrow \quad \cos(A + B) = \cos(C + D) \qquad [\because \cos(2\pi - \theta) = \cos\theta]$

LEVEL-2

EXAMPLE 10 *Find the value of the expression*

Imp.

$$3\left\{\sin^4\left(\frac{3\pi}{2}-\theta\right)+\sin^4(3\pi+\theta)\right\}-2\left\{\sin^6\left(\frac{\pi}{2}+\theta\right)+\sin^6(5\pi-\theta)\right\}$$

SOLUTION The given expression is **[NCERT EXEMPLAR]**

$$3\left\{\sin^4\left(\frac{3\pi}{2}-\theta\right)+\sin^4(3\pi+\theta)\right\}-2\left\{\sin^6\left(\frac{\pi}{2}+\theta\right)+\sin^6(5\pi-\theta)\right\}$$

$$= 3\left\{(-\cos\theta)^4 + (-\sin\theta)^4\right\} - 2\left\{(\cos\theta)^6 + (\sin\theta)^6\right\}$$

$$= 3(\cos^4\theta + \sin^4\theta) - 2(\cos^6\theta + \sin^6\theta)$$

$$= 3\left\{(\cos^2\theta + \sin^2\theta)^2 - 2\sin^2\theta\cos^2\theta\right\}$$

$$-2\left\{(\cos^2\theta + \sin^2\theta)^3 - 3\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)\right\}$$

$$= 3(1 - 2\sin^2\theta\cos^2\theta) - 2(1 - 3\cos^2\theta\sin^2\theta)$$

$$= 3 - 6\sin^2\theta\cos^2\theta - 2 + 6\sin^2\theta\cos^2\theta = 1.$$

EXERCISE 5.3

LEVEL-1

1. Find the values of the following trigonometric ratios:

(i) $\sin \frac{5\pi}{3}$ (ii) $\sin 3060°$ (iii) $\tan \frac{11\pi}{6}$ (iv) $\cos(-1125°)$

(v) $\tan 315°$ (vi) $\sin 510°$ (vii) $\cos 570°$ (viii) $\sin(-330°)$

(ix) $\operatorname{cosec}(-1200°)$ (x) $\tan(-585°)$ (xi) $\cos 855°$ (xii) $\sin 1845°$

(xiii) $\cos 1755°$ (xiv) $\sin 4530°$

2. Prove that:

(i) $\tan 225° \cot 405° + \tan 765° \cot 675° = 0$

(ii) $\sin \frac{8\pi}{3} \cos \frac{23\pi}{6} + \cos \frac{13\pi}{3} \sin \frac{35\pi}{6} = \frac{1}{2}$

(iii) $\cos 24° + \cos 55° + \cos 125° + \cos 204° + \cos 300° = \frac{1}{2}$

(iv) $\tan(-225°)\cot(-405°) - \tan(-765°)\cot(675°) = 0$

(v) $\cos 570° \sin 510° + \sin(-330°)\cos(-390°) = 0$

(vi) $\tan \frac{11\pi}{3} - 2\sin \frac{4\pi}{6} - \frac{3}{4}\operatorname{cosec}^2 \frac{\pi}{4} + 4\cos^2 \frac{17\pi}{6} = \frac{3-4\sqrt{3}}{2}$

(vii) $3\sin \frac{\pi}{6} \sec \frac{\pi}{3} - 4\sin \frac{5\pi}{6} \cot \frac{\pi}{4} = 1$

3. Prove that:

(i) $\dfrac{\cos(2\pi+\theta)\operatorname{cosec}(2\pi+\theta)\tan(\pi/2+\theta)}{\sec(\pi/2+\theta)\cos\theta\cot(\pi+\theta)} = 1$

(ii) $\dfrac{\operatorname{cosec}(90°+\theta)+\cot(450°+\theta)}{\operatorname{cosec}(90°-\theta)+\tan(180°-\theta)} + \dfrac{\tan(180°+\theta)+\sec(180°-\theta)}{\tan(360°+\theta)-\sec(-\theta)} = 2$

(iii) $\dfrac{\sin(180°+\theta)\cos(90°+\theta)\tan(270°-\theta)\cot(360°-\theta)}{\sin(360°-\theta)\cos(360°+\theta)\operatorname{cosec}(-\theta)\sin(270°+\theta)} = 1$

(iv) $\left\{1+\cot\theta-\sec\left(\frac{\pi}{2}+\theta\right)\right\}\left\{1+\cot\theta+\sec\left(\frac{\pi}{2}+\theta\right)\right\} = 2\cot\theta$

(v) $\dfrac{\tan(90°-\theta)\sec(180°-\theta)\sin(-\theta)}{\sin(180°+\theta)\cot(360°-\theta)\operatorname{cosec}(90°-\theta)} = 1$

4. Prove that : $\sin^2 \frac{\pi}{18} + \sin^2 \frac{\pi}{9} + \sin^2 \frac{7\pi}{18} + \sin^2 \frac{4\pi}{9} = 2$

5. Prove that : $\sec\left(\frac{3\pi}{2}-\theta\right)\sec\left(\theta-\frac{5\pi}{2}\right)+\tan\left(\frac{5\pi}{2}+\theta\right)\tan\left(\theta-\frac{3\pi}{2}\right) = -1.$

6. In a ΔABC, prove that :

(i) $\cos(A+B)+\cos C = 0$ (ii) $\cos\left(\frac{A+B}{2}\right) = \sin\frac{C}{2}$ (iii) $\tan\frac{A+B}{2} = \cot\frac{C}{2}$

7. If A, B, C, D be the angles of a cyclic quadrilateral, taken in order, prove that :

$\cos(180°-A)+\cos(180°+B)+\cos(180°+C)-\sin(90°+D) = 0$

8. Find x from the following equations :

(i) $\operatorname{cosec}(90°+\theta)+x\cos\theta\cot(90°+\theta) = \sin(90°+\theta)$

(ii) $x\cot(90°+\theta)+\tan(90°+\theta)\sin\theta+\operatorname{cosec}(90°+\theta) = 0$

9. Prove that:

(i) $\tan 720° - \cos 270° - \sin 150° \cos 120° = \frac{1}{4}$

(ii) $\sin 780° \sin 480° + \cos 120° \sin 150° = \frac{1}{2}$

(iii) $\sin 780° \sin 120° + \cos 240° \sin 390° = \frac{1}{2}$

(iv) $\sin 600° \cos 390° + \cos 480° \sin 150° = -1$

(v) $\tan 225° \cot 405° + \tan 765° \cot 675° = 0$

ANSWERS

1. (i) $-\frac{\sqrt{3}}{2}$ (ii) 0 (iii) $-\frac{1}{\sqrt{3}}$ (iv) $\frac{1}{\sqrt{2}}$ (v) -1 (vi) $\frac{1}{2}$
(vii) $-\frac{\sqrt{3}}{2}$ (viii) $\frac{1}{2}$ (ix) $-\frac{2}{\sqrt{3}}$ (x) -1 (xi) $-\frac{1}{\sqrt{2}}$ (xii) $\frac{1}{\sqrt{2}}$
(xiii) $1/\sqrt{2}$ (xiv) $-1/2$

8. (i) $\tan\theta$ (ii) $\sin\theta$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the maximum and minimum values of $\cos(\cos x)$.
2. Write the maximum and minimum values of $\sin(\sin x)$.
3. Write the maximum value of $\sin(\cos x)$.
4. If $\sin x = \cos^2 x$, then write the value of $\cos^2 x(1 + \cos^2 x)$.
5. If $\sin x + \operatorname{cosec} x = 2$, then write the value of $\sin^n x + \operatorname{cosec}^n x$.
6. If $\sin x + \sin^2 x = 1$, then write the value of $\cos^{12} x + 3\cos^{10} x + 3\cos^8 x + \cos^6 x$.
7. If $\sin x + \sin^2 x = 1$, then write the value of $\cos^8 x + 2\cos^6 x + \cos^4 x$.
8. If $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 3$, then write the value of $\cos\theta_1 + \cos\theta_2 + \cos\theta_3$.
9. Write the value of $\sin 10° + \sin 20° + \sin 30° + \ldots + \sin 360°$.
10. A circular wire of radius 15 cm is cut and bent so as to lie along the circumference of a loop of radius 120 cm. Write the measure of the angle subtended by it at the centre of the loop.
11. Write the value of $2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1$.
12. Write the value of $\cos 1° + \cos 2° + \cos 3° + \ldots + \cos 180°$.
13. If $\cot(\alpha + \beta) = 0$, then write the value of $\sin(\alpha + 2\beta)$.
14. If $\tan A + \cot A = 4$, then write the value of $\tan^4 A + \cot^4 A$.
15. Write the least value of $\cos^2\theta + \sec^2\theta$.
16. If $x = \sin^{14}\theta + \cos^{20}\theta$, then write the smallest interval in which the value of x lie.
17. If $3\sin\theta + 5\cos\theta = 5$, then write the value of $5\sin\theta - 3\cos\theta$.

ANSWERS

1. 1, cos 1 2. sin 1, − sin 1 3. sin 1 4. 1 5. 2 6. 1
7. 1 8. 0 9. 0 10. 45° 11. 0 12. 0 13. $\sin\alpha$ or $\cos\beta$
14. 194 15. 2 16. (0, 1] 17. 3 or − 3

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If $\tan\theta = x - \frac{1}{4x}$, then $\sec\theta - \tan\theta$ is equal to

(a) $-2x, \frac{1}{2x}$ (b) $-\frac{1}{2x}, 2x$ (c) $2x$ (d) $2x, \frac{1}{2x}$

2. If $\sec\theta = x + \frac{1}{4x}$, then $\sec\theta + \tan\theta =$

(a) $x, \frac{1}{x}$ (b) $2x, \frac{1}{2x}$ (c) $-2x, \frac{1}{2x}$ (d) $-\frac{1}{x}, x$

3. If $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, then $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$ is equal to

(a) $\sec\theta - \tan\theta$ (b) $\sec\theta + \tan\theta$ (c) $\tan\theta - \sec\theta$ (d) none of these

4. If $\pi < \theta < 2\pi$, then $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$ is equal to

(a) $\operatorname{cosec}\theta + \cot\theta$ (b) $\operatorname{cosec}\theta - \cot\theta$ (c) $-\operatorname{cosec}\theta + \cot\theta$ (d) $-\operatorname{cosec}\theta - \cot\theta$

5. If $0 < \theta < \frac{\pi}{2}$, and if $\frac{y+1}{1-y} = \sqrt{\frac{1+\sin\theta}{1-\sin\theta}}$, then y is equal to

(a) $\cot\frac{\theta}{2}$ (b) $\tan\frac{\theta}{2}$ (c) $\cot\frac{\theta}{2} + \tan\frac{\theta}{2}$ (d) $\cot\frac{\theta}{2} - \tan\frac{\theta}{2}$

6. If $\frac{\pi}{2} < \theta < \pi$, then $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} + \sqrt{\frac{1+\sin\theta}{1-\sin\theta}}$ *is equal to*

(a) $2\sec\theta$ (b) $-2\sec\theta$ (c) $\sec\theta$ (d) $-\sec\theta$

7. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$, then $x^2 + y^2 + z^2$ is independent of

(a) θ, ϕ (b) r, θ (c) r, ϕ (d) r

8. If $\tan\theta + \sec\theta = \sqrt{3}$, $0 < \theta < \pi$, then θ is equal to

(a) $\frac{5\pi}{6}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$

9. If $\tan\theta = -\frac{1}{\sqrt{5}}$ and θ lies in the IV quadrant, then the value of $\cos\theta$ is

(a) $\frac{\sqrt{5}}{\sqrt{6}}$ (b) $\frac{2}{\sqrt{6}}$ (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{6}}$

10. If $\frac{3\pi}{4} < \alpha < \pi$, then $\sqrt{2\cot\alpha + \frac{1}{\sin^2\alpha}}$ is equal to

(a) $1 - \cot\alpha$ (b) $1 + \cot\alpha$ (c) $-1 + \cot\alpha$ (d) $-1 - \cot\alpha$

11. $\sin^6 A + \cos^6 A + 3\sin^2 A\cos^2 A =$

(a) 0 (b) 1 (c) 2 (d) 3

12. If $\operatorname{cosec}\theta - \cot\theta = \frac{1}{2}$, $0 < \theta < \frac{\pi}{2}$, then $\cos\theta$ is equal to

(a) $\frac{5}{3}$ (b) $\frac{3}{5}$ (c) $-\frac{3}{5}$ (d) $-\frac{5}{3}$

13. If $\operatorname{cosec}\theta + \cot\theta = \frac{11}{2}$, then $\tan\theta =$

(a) $\frac{21}{22}$ (b) $\frac{15}{16}$ (c) $\frac{44}{117}$ (d) $\frac{117}{44}$

14. $\sec^2\theta = \frac{4xy}{(x+y)^2}$ is true if and only if

(a) $x + y \neq 0$ (b) $x = y, x \neq 0$ (c) $x = y$ (d) $x \neq 0, y \neq 0$

15. If θ is an acute angle and $\tan\theta = \frac{1}{\sqrt{7}}$, then the value of $\frac{\operatorname{cosec}^2\theta - \sec^2\theta}{\operatorname{cosec}^2\theta + \sec^2\theta}$ is

(a) 3/4 (b) 1/2 (c) 2 (d) 5/4

16. The value of $\sin^2 5° + \sin^2 10° + \sin^2 15° + \ldots + \sin^2 85° + \sin^2 90°$ is

(a) 7 (b) 8 (c) 9.5 (d) 10

17. $\sin^2\frac{\pi}{18} + \sin^2\frac{\pi}{9} + \sin^2\frac{7\pi}{18} + \sin^2\frac{4\pi}{9} =$

(a) 1 (b) 4 (c) 2 (d) 0

18. If $\tan A + \cot A = 4$, then $\tan^4 A + \cot^4 A$ is equal to

(a) 110 (b) 191 (c) 80 (d) 194

19. If $x \sin 45° \cos^2 60° = \frac{\tan^2 60° \operatorname{cosec} 30°}{\sec 45° \cot^{2°} 30°}$, then $x =$

(a) 2 (b) 4 (c) 8 (d) 16

20. If $\operatorname{cosec}\theta - \cot\theta = \frac{1}{2}$, $0 < \theta < \frac{\pi}{2}$, then $\cos\theta$ is equal to

(a) –3/5 (b) –5/3 (c) 5/3 (d) 3/5

21. If $\operatorname{cosec} A + \cot A = \frac{11}{2}$, then $\tan A =$

(a) 21/22 (b) 15/16 (c) 44/117 (d) 117/43

22. If $\tan\theta + \sec\theta = e^x$, then $\cos\theta$ equals

(a) $\frac{e^x + e^{-x}}{2}$ (b) $\frac{2}{e^x + e^{-x}}$ (c) $\frac{e^x - e^{-x}}{2}$ (d) $\frac{e^x - e^{-x}}{e^x + e^{-x}}$

23. If $\sec\theta + \tan\theta = k$, $\cos\theta =$

(a) $\frac{k^2+1}{2k}$ (b) $\frac{2k}{k^2+1}$ (c) $\frac{k}{k^2+1}$ (d) $\frac{k}{k^2-1}$

24. If $f(x) = \cos^2 x + \sec^2 x$, then

(a) $f(x) < 1$ (b) $f(x) = 1$ (c) $2 < f(x) < 1$ (d) $f(x) \geq 2$

25. Which of the following is incorrect?

(a) $\sin\theta = -1/5$ (b) $\cos\theta = 1$ (c) $\sec\theta = 1/2$ (d) $\tan\theta = 20$

26. The value of $\cos 1° \cos 2° \cos 3° \ldots \cos 179°$ is

(a) $1/\sqrt{2}$ (b) 0 (c) 1 (d) –1

27. The value of $\tan 1° \tan 2° \tan 3° \ldots \tan 89°$ is

(a) 0 (b) 1 (c) 1/2 (d) not defined

28. Which of the following is correct?

(a) $\sin 1° > \sin 1$ (b) $\sin 1° < \sin 1$ (c) $\sin 1° = \sin 1$ (d) $\sin 1° = \frac{\pi}{180}\sin 1$

29. If A lies in second quadrant and $3\tan A + 4 = 0$, then the value of $2\cot A - 5\cos A + \sin A$ is equal to

(a) –53/10 (b) 23/10 (c) 37/10 (d) 7/10

ANSWERS

1. (a)	2. (b)	3. (c)	4. (d)	5. (b)	6. (b)	7. (a)	8. (c)
9. (a)	10. (d)	11. (b)	12. (b)	13. (c)	14. (b)	15. (a)	16. (c)
17. (c)	18. (d)	19. (c)	20. (d)	21. (c)	22. (b)	23. (b)	24. (d)
25. (c)	26. (b)	27. (b)	28. (b)	29. (b)			

SUMMARY

1. Following are some of the fundamental trigonometric identities:

(i) $\sin\theta = \dfrac{1}{\text{cosec}\,\theta}$ or, $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$

(ii) $\cos\theta = \dfrac{1}{\sec\theta}$ or, $\sec\theta = \dfrac{1}{\cos\theta}$ (iii) $\cot\theta = \dfrac{1}{\tan\theta}$ or, $\tan\theta = \dfrac{1}{\cot\theta}$

(iv) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ or, $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ (v) $\sin^2\theta + \cos^2\theta = 1$

(vi) $1 + \tan^2\theta = \sec^2\theta$ or, $\sec\theta - \tan\theta = \dfrac{1}{\sec\theta + \tan\theta}$

(vii) $1 + \cot^2\theta = \text{cosec}^2\theta$ or, $\text{cosec}\,\theta - \cot\theta = \dfrac{1}{\text{cosec}\,\theta + \cot\theta}$

2. (i) $\sin(-\theta) = -\sin\theta$ or, $\text{cosec}(-\theta) = -\text{cosec}\,\theta$

(ii) $\cos(-\theta) = \cos\theta$ or, $\sec(-\theta) = \sec\theta$

(iii) $\tan(-\theta) = -\tan\theta$ or, $\cot(-\theta) = -\cot\theta$

(iv) $\sin(90^\circ - \theta) = \cos\theta$, $\cos(90^\circ - \theta) = \sin\theta$
$\tan(90^\circ - \theta) = \cot\theta$, $\sec(90^\circ - \theta) = \text{cosec}\,\theta$
$\text{cosec}(90^\circ - \theta) = \sec\theta$, $\cot(90^\circ - \theta) = \tan\theta$

(v) $\sin(90^\circ + \theta) = \cos\theta$, $\cos(90^\circ + \theta) = -\sin\theta$
$\tan(90^\circ + \theta) = -\cot\theta$, $\cot(90^\circ + \theta) = -\tan\theta$
$\sec(90^\circ + \theta) = -\text{cosec}\,\theta$, $\text{cosec}(90^\circ + \theta) = \sec\theta$

(vi) $\sin(180^\circ - \theta) = \sin\theta$, $\cos(180^\circ - \theta) = -\cos\theta$
$\tan(180^\circ - \theta) = -\tan\theta$, $\cot(180^\circ - \theta) = -\cot\theta$
$\sec(180^\circ - \theta) = -\sec\theta$, $\text{cosec}(180^\circ - \theta) = \text{cosec}\,\theta$

(vii) $\sin(270^\circ - \theta) = -\cos\theta$, $\cos(270^\circ - \theta) = -\sin\theta$
$\tan(270^\circ - \theta) = \cot\theta$, $\cot(270^\circ - \theta) = \tan\theta$
$\text{cosec}(270^\circ - \theta) = -\sec\theta$, $\sec(270^\circ - \theta) = -\text{cosec}\,\theta$

(viii) $\sin(270^\circ + \theta) = -\cos\theta$, $\cos(270^\circ + \theta) = \sin\theta$
$\tan(270^\circ + \theta) = -\cot\theta$, $\cot(270^\circ + \theta) = -\tan\theta$
$\text{cosec}(270^\circ + \theta) = -\sec\theta$, $\sec(270^\circ + \theta) = \text{cosec}\,\theta$

(ix) $\sin(360^\circ - \theta) = -\sin\theta$, $\cos(360^\circ - \theta) = \cos\theta$
$\tan(360^\circ - \theta) = -\tan\theta$, $\text{cosec}(360^\circ - \theta) = -\text{cosec}\,\theta$
$\sec(360^\circ - \theta) = \sec\theta$, $\cot(360^\circ - \theta) = -\cot\theta$

(x) Sine and Cosine functions and their reciprocals i.e. Cosecant and Secant functions are periodic functions with period 2π. Tangent and Cotangent functions are periodic with period π.

(xi)

Odd functions	*Even functions*
sine, tangent	cosine, secant
cotangent, cosecant	

CHAPTER 6

GRAPHS OF TRIGONOMETRIC FUNCTIONS

6.1 INTRODUCTION

We have already studied in the previous chapters that all trigonometric functions are periodic. For example, sine and cosine functions are periodic with period 2π, while tangent and cotangent functions are periodic with period π. We know that if $f(x)$ is a periodic function with period T and $a > 0$, then $f(ax + b)$ is periodic with period T/a. Therefore, $\sin(ax + b)$ and $\cos ax$ are periodic functions with period $\frac{2\pi}{a}$. If the graph of a periodic function with period T is to be drawn in a given interval, then it is sufficient to draw its graph only in an interval of length T. Because, once it is drawn in one such interval, it can be easily drawn completely by repeating it over the intervals of lengths T. The amplitude of a function is defined as the greatest numerical value which it can attain.

Using the knowledge acquired in the above discussion let us now draw the graphs of various trigonometric functions.

6.2 GRAPHS OF TRIGONOMETRIC FUNCTIONS

6.2.1 GRAPH OF $y = \sin x$

Since $\sin x$ is a periodic function with period 2π. So, we will draw the graph of $y = \sin x$ in the interval $[0, 2\pi]$. In order to draw the graph of $y = \sin x$ in the interval $[0, 2\pi]$, we first draw it in $[0, \pi/2]$. The values of $\sin x$ for different values of x in the interval $[0, \pi/2]$ are given in the following table.

x	0°	30°	45°	60°	90°
$\sin x$	0	$1/2 = 0.5$	$1/\sqrt{2} = 0.707$	$\sqrt{3}/2 = 0.866$	1

Using the above table and the fact that $\sin x$ is an increasing function we obtain the graph of $y = \sin x$ in the interval $[0, \pi/2]$ as shown in Fig. 6.1. In the interval $[\pi/2, \pi]$, we draw the graph of $y = \sin x$ by using the fact that $\sin(\pi - x) = \sin x$. Finally, we draw it in the interval $[\pi, 2\pi]$, using the fact that $\sin(\pi + x) = -\sin x$ which means that the graph of $y = \sin x$ in $[\pi, 2\pi]$ is the mirror image of the graph of $y = \sin x$ in $[0, \pi]$. The graph is now sketched in Fig. 6.1.

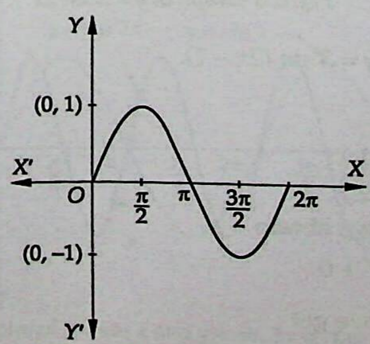


Fig. 6.1 Graph of $y = \sin x$

Since sin x is a periodic function with period 2π. Therefore, we use the following algorithm to draw the graphs of $y = c \sin ax$.

ALGORITHM

STEP I *Obtain the values of a and c.*

STEP II *Draw the graph of $y = \sin x$ and mark the points where it crosses x-axis.*

STEP III *Divide the x-coordinates of the points where $y = \sin x$ crosses x-axis by a and mark maximum and minimum values of $y = c \sin ax$ as c and $-c$ on y-axis.*

The graph so obtained is the graph of $y = c \sin ax$ as shown in Fig. 6.2.

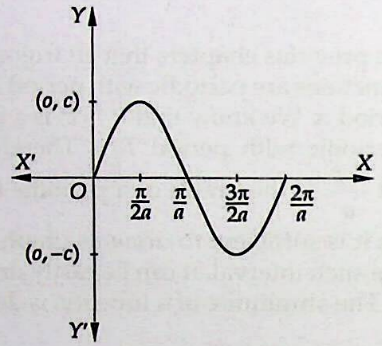


Fig. 6.2 Graph of $y = c \sin ax$

EXAMPLE 1 *Sketch the graph of $y = 3 \sin 2x$.*

SOLUTION To obtain the graph of $y = 3 \sin 2x$ we first draw the graph of $y = \sin x$ in the interval $[0, 2\pi]$ and then divide the x-coordinates of the points where it crosses x-axis by 2. The maximum and minimum values are 3 and -3 respectively as shown in Fig. 6.3.

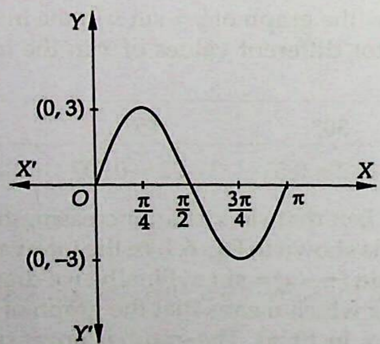


Fig. 6.3 Graph of $y = 3 \sin 2x$

EXAMPLE 2 *Sketch the graph of $y = 3 \sin (2x - 1)$.*

SOLUTION We have,

$$y = 3 \sin (2x - 1)$$

$$\Rightarrow \quad (y - 0) = 3 \sin 2\left(x - \frac{1}{2}\right) \qquad \text{...(i)}$$

Shifting the origin at $(1/2, 0)$, we obtain

$$x = X + \frac{1}{2} \text{ and } y = Y + 0$$

Substituting these values in (i), we get

$$Y = 3 \sin 2X$$

Thus, if we draw the graph of $Y = 3 \sin 2X$ and shift it by 1/2 unit to the right, we get the required graph as shown in Fig. 6.4.

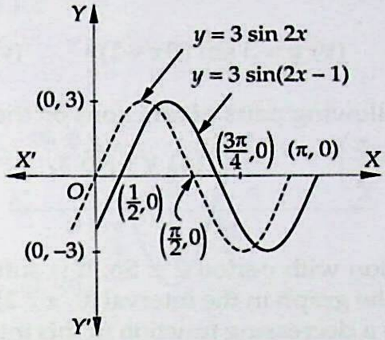


Fig. 6.4 Graph of $y = 3 \sin(2x - 1)$

EXAMPLE 3 *Sketch the graph of $y = \sin x$ and $y = \sin 2x$ on the same axes.*

SOLUTION Clearly $\sin 2x$ is a periodic function with period $2\pi/2 = \pi$ whereas $\sin x$ is periodic with period 2π. The graphs of $y = \sin x$ and $y = \sin 2x$ on different axes are shown in Figs. 6.5 and 6.6 respectively.

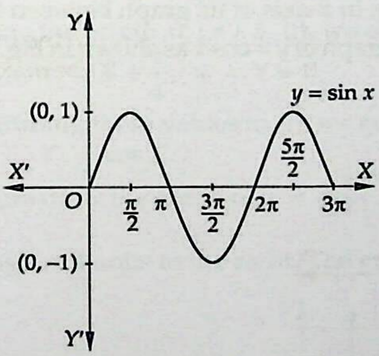


Fig. 6.5 Graph of $y = \sin x$

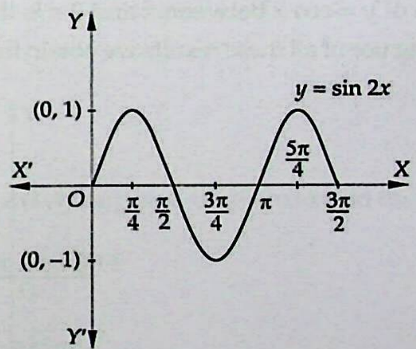


Fig. 6.6 Graph of $y = \sin 2x$

If these two graphs are drawn on the same axes, then the graphs are drawn in Fig. 6.7.

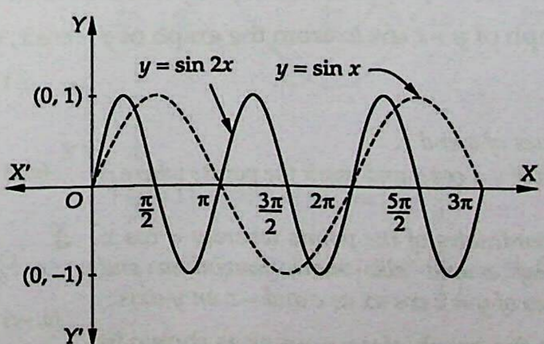


Fig. 6.7 Graphs of $y = \sin x$ and $y = \sin 2x$ on the same scale

EXERCISE 6.1

1. Sketch the following graphs:

   (i) $y=2\sin 2x$ (ii) $y=3\sin x$ (iii) $y=2\sin\left(x-\frac{\pi}{4}\right)$

   (iv) $y=2\sin(2x-1)$ (v) $y=3\sin(3x+1)$ (vi) $y=3\sin\left(2x-\frac{\pi}{4}\right)$

2. Sketch the graph of the following pairs of functions on the same axes :

   (i) $y=\sin x,\ y=\sin\left(x+\frac{\pi}{4}\right)$ (ii) $y=\sin x,\ y=\sin 3x$

6.2.2 GRAPH OF $y=\cos x$

Since $\cos x$ is a periodic function with period 2π. So, it is sufficient to draw the graph in the interval $[0, 2\pi]$. We first draw the graph in the interval $[0, \pi/2]$. For this we use the table given below and the fact that $\cos x$ is a decreasing function in this interval.

x	0°	30°	45°	60°	90°
$\cos x$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0

Now, to draw the graph of $y = \cos x$ in the interval $[\pi/2, \pi]$ we use the relation $\cos(\pi-x)=-\cos x$. If we give values of x between 0 and $\pi/2$, then this relation shows that the values of $\cos x$ between $\pi/2$ and π are negative of its values in the interval $[0, \pi/2]$. Thus, the graph of $y=\cos x$ between $\pi/2$ and π is below x-axis. Now, $\cos(\pi+x)=-\cos x$ shows that the graph of $y=\cos x$ between π and 2π is the mirror image in x-axis of its graph between 0 and π. Making use of all these results we obtain the sketch of the graph of $y=\cos x$ as shown in Fig. 6.8.

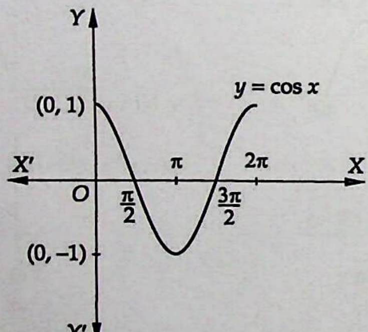


Fig. 6.8 Graph of $y=\cos x$

In order to draw the graph of $y=c\cos ax$ from the graph of $y=\cos x$, we may use the following algorithm.

ALGORITHM

STEP I *Obtain the values of a and c.*

STEP II *Draw the graph of $y=\cos x$ and mark the points where it crosses x-axis.*

STEP III *Divide the x-coordinates of the points where $y=\cos x$ meets x-axis by a and also mark maximum and minimum values of $y=c\cos ax$ as c and $-c$ on y-axis.*

The graph so obtained is the graph of $y=c\cos ax$ as shown in Fig. 6.9.

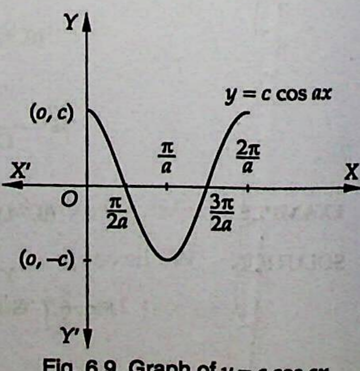


Fig. 6.9 Graph of $y=c\cos ax$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Draw the sketch of the graph of $y = 3\cos 2x$.*

SOLUTION Replacing c by 3 and a by 2 in the graph of $y = c\cos ax$, we obtain the graph of $y = 3\cos 2x$ as shown in Fig. 6.10.

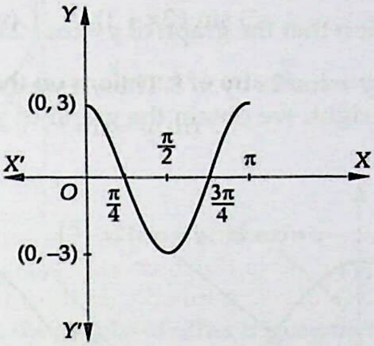


Fig. 6.10 Graph of $y = 3\cos 2x$

EXAMPLE 2 *Sketch the graph of $y = \cos\left(x - \frac{\pi}{4}\right)$.*

SOLUTION We have,

$$y = \cos\left(x - \frac{\pi}{4}\right) \Rightarrow y - 0 = \cos\left(x - \frac{\pi}{4}\right) \qquad \text{...(i)}$$

Shifting the origin at $(\pi/4, 0)$, we obtain

$$x = X + \frac{\pi}{4},\ y = Y + 0$$

Substituting these values in (i), we get

$$Y = \cos X.$$

Thus, to draw the graph of $y = \cos\left(x - \frac{\pi}{4}\right)$ we first draw the graph of $y = \cos x$ and then shift it through $\pi/4$ units to the right. The graph is drawn in Fig. 6.11.

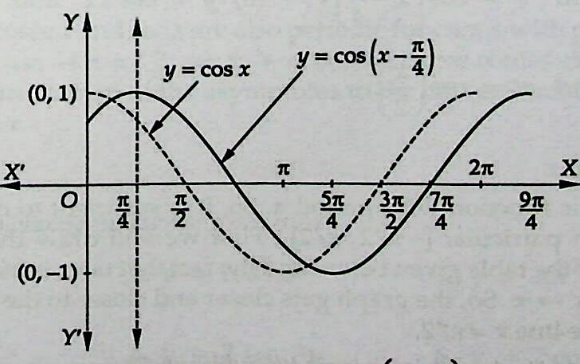


Fig. 6.11 Graph of $y = \cos\left(x - \frac{\pi}{4}\right)$

EXAMPLE 3 *Sketch the graphs of $y = \cos 2x$ and $y = \cos\left(2x - \frac{\pi}{4}\right)$ on the same scale.*

SOLUTION We have,

$$y = \cos\left(2x - \frac{\pi}{4}\right) = \cos 2\left(x - \frac{\pi}{8}\right) \Rightarrow y - 0 = \cos 2\left(x - \frac{\pi}{8}\right)$$

Shifting the origin at $(\pi/8, 0)$, we have

$$x = X + \frac{\pi}{8} \text{ and } y = Y + 0$$

Using these relations $y - 0 = \cos 2\left(x - \frac{\pi}{8}\right)$ reduces to $Y = \cos 2X$.

It follows from the above discussion that the graph of $y = \cos\left(2x - \frac{\pi}{4}\right)$ is similar to the graph of $y = \cos 2x$ but it lags the graph of $y = \cos 2x$ by $\pi/8$. Thus, if we draw the graph of $y = \cos 2x$ and shift it through $\pi/8$ units to the right, we obtain the graph of $y = \cos(2x - \pi/4)$ as shown in Fig. 6.12.

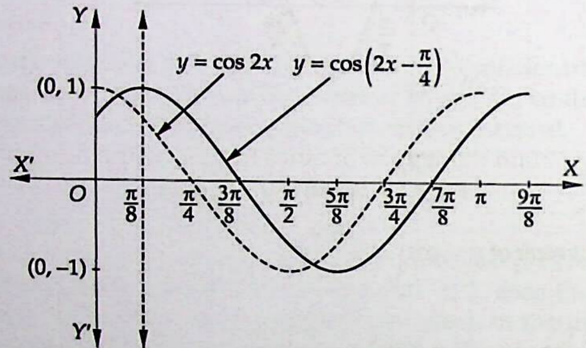


Fig. 6.12 Graph of $y = \cos 2x$ and $y = \cos\left(2x - \frac{\pi}{4}\right)$ on the same scale

EXERCISE 6.2

1. Sketch the following graphs:

   (i) $y = \cos\left(x + \frac{\pi}{4}\right)$ (ii) $y = \cos\left(x - \frac{\pi}{4}\right)$ (iii) $y = 3\cos(2x - 1)$ (iv) $y = 2\cos\left(x - \frac{\pi}{2}\right)$

2. Sketch the graphs of the following functions on the same scale.

   (i) $y = \cos x$ and $y = \cos\left(x - \frac{\pi}{4}\right)$ (ii) $y = \cos 2x$ and $y = \cos 2\left(x - \frac{\pi}{4}\right)$

   (iii) $y = \cos x$ and $y = \cos\left(\frac{x}{2}\right)$

6.2.3 GRAPH OF $y = \tan x$

Since $\tan x$ is a periodic function with period π. So, it is sufficient to draw the graph over an interval of length π, in particular $[-\pi/2, \pi/2]$. First we will draw the graph in the interval $[0, \pi/2]$. For this we use the table given below and the fact that $\tan x$ is increasing in this interval. Also, as $x \to \pi/2$, $\tan x \to \infty$. So, the graph gets closer and closer to the line $x = \pi/2$ as $x \to \pi/2$. But it never touches the line $x = \pi/2$.

x	0°	30°	45°	60°	90°
$\tan x$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞

Since $\tan(-x) = -\tan x$, therefore, if $(x, \tan x)$ is any point on the curve $y = \tan x$, then $(-x, -\tan x)$ will also be a point on it. This means that the graph is symmetric in opposite quadrants.

Using all the above points we obtain the sketch of the curve $y = \tan x$ as given in Fig. 6.13.

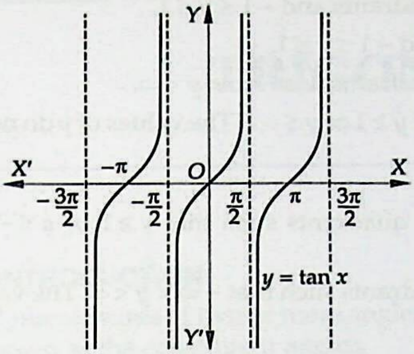


Fig. 6.13 Graph of $y = \tan x$

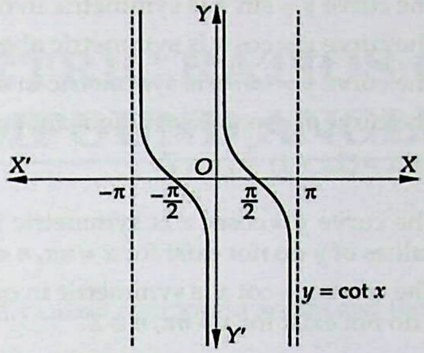


Fig. 6.14 Graph of $y = \cot x$

Proceeding as above we obtain the graphs of other trigonometric functions as shown in Figures 6.14, 6.15, and 6.16.

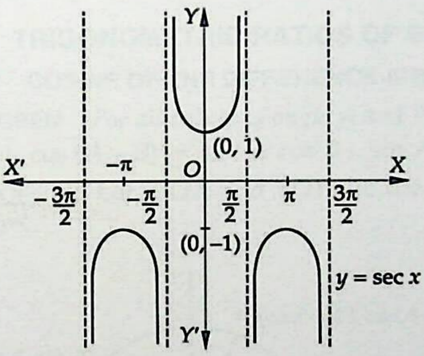


Fig. 6.15 Graph of $y = \sec x$

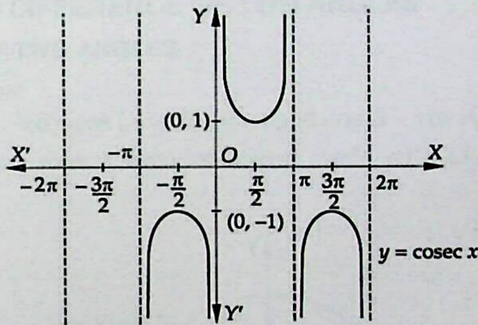


Fig. 6.16 Graph of $y = \text{cosec}\, x$

It should be noted that cosec x and sec x are also periodic functions with period 2π while cot x is periodic with period π. As $x \to \pm \pi/2$, sec $x \to \infty$. So, the curve comes closer and closer to ∞ as $x \to \pm \pi/2$. These lines are known as the asymptotes to the curve. Similarly, $x = 0$, $x = \pi$ etc. are asymptotes of $y = \text{cosec}\, x$.

EXERCISE 6.3

Sketch the graphs of the following functions:

1. $y = \sin^2 x$ **2.** $y = \cos^2 x$ **3.** $y = \sin^2\left(x - \frac{\pi}{4}\right)$

4. $y = \tan 2x$ **5.** $y = 2\tan 3x$ **6.** $y = 2\cot 2x$

Sketch the graphs of the following functions on the same scale:

7. $y = \cos 2x$, $y = \cos\left(2x - \frac{\pi}{3}\right)$ **8.** $y = \sin^2 x$, $y = \sin x$

9. $y = \tan x$, $y = \tan^2 x$ **10.** $y = \tan 2x$, $y = \tan x$

SUMMARY

1. The curve $y = \sin x$ is symmetric in opposite quadrants and $-1 \le y \le 1$.
2. The curve $y = \cos x$ is symmetric about y-axis and $-1 \le y \le 1$.
3. The curve $y = \tan x$ is symmetric in opposite quadrants and $-\infty < y < \infty$.
4. The curve $y = \sec x$ is symmetric about y-axis and $y \ge 1$ or $y \le -1$. The values of y do not exist for $x = (2x+1)\dfrac{\pi}{2}, n \in Z$.
5. The curve $y = \operatorname{cosec} x$ is symmetric in opposite quadrants such that $y \ge 1$ or $y \le -1$. The values of y do not exist for $x = n\pi, n \in Z$.
6. The curve $y = \cot x$ is symmetric in opposite quadrants such that $-\infty < y < \infty$. The values of y do not exist for $x = n\pi, n \in Z$.

CHAPTER 7

TRIGONOMETRIC RATIOS OF COMPOUND ANGLES

7.1 INTRODUCTION

The algebraic sums of two or more angles are generally called compound angles and the angles are known as the constituent angles.

For example, if A, B, C are three angles, then $A \pm B,\ A+B+C,\ A-B+C$ etc. are compound angles.

In this chapter, we shall derive formulae which will express the trigonometric ratios of compound angles in terms of trigonometric ratios of constituent angles.

7.2 TRIGONOMETRIC RATIOS OF SUM AND DIFFERENCE OF TWO ANGLES

7.2.1 COSINE OF THE DIFFERENCE AND SUM OF TWO ANGLES

THEOREM *For all values of angle A and B, prove that*

(i) $\cos(A-B) = \cos A \cos B + \sin A \sin B$ (ii) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

PROOF (i) Let $X'OX$ and YOY' be the coordinate axes. Consider a unit circle with O as the centre.

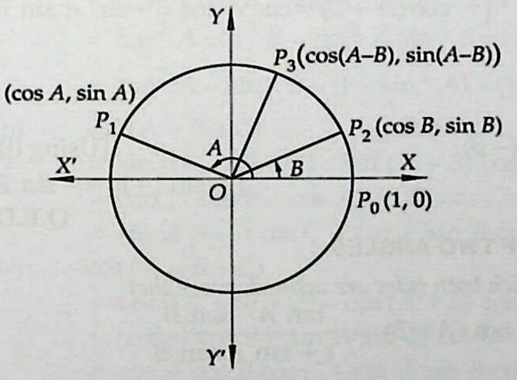


Fig. 7.1

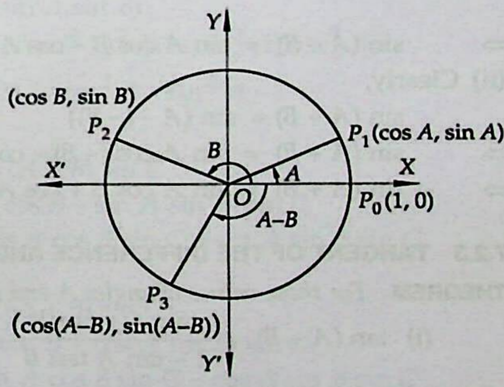


Fig. 7.2

Let P_1, P_2 and P_3 be three points on the circles such that $\angle XOP_1 = A,\ \angle XOP_2 = B$ and $\angle XOP_3 = A-B$. As we have seen in section 5.5 that the terminal side of any angle intersects the circle with centre at O and unit radius at a point whose coordinates are respectively the cosine and sine of the angle. Therefore, coordinates of P_1, P_2 and P_3 are $(\cos A, \sin A)$, $(\cos B, \sin B)$ and $(\cos(A-B), \sin(A-B))$ respectively.

We know that equal chords of a circle make equal angles at its centre and chords $P_0 P_3$ and $P_1 P_2$ subtend equal angles at O. Therefore,

$$\text{Chord } P_0 P_3 = \text{Chord } P_1 P_2$$

$$\Rightarrow \sqrt{\{\cos(A-B)-1\}^2 + \{\sin(A-B)-0\}^2} = \sqrt{(\cos B - \cos A)^2 + (\sin B - \sin A)^2}$$

$$\Rightarrow \quad \{\cos(A-B)-1\}^2+\sin^2(A-B) = (\cos B-\cos A)^2+(\sin B-\sin A)^2$$

$$\Rightarrow \quad \cos^2(A-B)-2\cos(A-B)+1+\sin^2(A-B) = \cos^2 B+\cos^2 A-2\cos A\cos B + \sin^2 B+\sin^2 A-2\sin A\sin B$$

$$\Rightarrow \quad 2-2\cos(A-B) = 2-2\cos A\cos B-2\sin A\sin B$$

$$\Rightarrow \quad \cos(A-B) = \cos A\cos B+\sin A\sin B$$

Hence, $\cos(A-B) = \cos A\cos B+\sin A\sin B$

(ii) Clearly,

$$\cos(A+B) = \cos(A-(-B))$$

$$\Rightarrow \quad \cos(A+B) = \cos A\cos(-B)+\sin A\sin(-B) \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \cos(A+B) = \cos A\cos B-\sin A\sin B \qquad [\because \cos(-B)=\cos B,\ \sin(-B)=-\sin B]$$

Hence, $\cos(A+B) = \cos A\cos B-\sin A\sin B$

Q.E.D.

<u>REMARK</u> *This method of proof of the above formula is true for all values of angles A and B whether positive, zero or negative.*

7.2.2 SINE OF THE DIFFERENCE AND SUM OF TWO ANGLES

THEOREM *For all values of angles A and B, prove that*

(i) $\sin(A-B) = \sin A\cos B-\cos A\sin B$ (ii) $\sin(A+B) = \sin A\cos B+\cos A\sin B$

<u>PROOF</u> (i) We have,

$$\sin(A-B) = \cos(90^\circ-(A-B)) \qquad [\because \cos(90^\circ-\theta)=\sin\theta]$$

$$\Rightarrow \quad \sin(A-B) = \cos((90^\circ-A)+B)$$

$$\Rightarrow \quad \sin(A-B) = \cos(90^\circ-A)\cos B-\sin(90^\circ-A)\sin B$$

$$[\because \cos(A+B)=\cos A\cos B-\sin A\sin B]$$

$$\Rightarrow \quad \sin(A-B) = \sin A\cos B-\cos A\sin B$$

(ii) Clearly,

$$\sin(A+B) = \sin(A-(-B))$$

$$\Rightarrow \quad \sin(A+B) = \sin A\cos(-B)-\cos A\sin(-B) \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \sin(A+B) = \sin A\cos B+\cos A\sin B \qquad [\because \sin(-B)=-\sin B]$$

Q.E.D.

7.2.3 TANGENT OF THE DIFFERENCE AND SUM OF TWO ANGLES

THEOREM *For those values of angles A and B for which both sides are defined, prove that*

(i) $\tan(A+B) = \dfrac{\tan A+\tan B}{1-\tan A\tan B}$ (ii) $\tan(A-B) = \dfrac{\tan A-\tan B}{1+\tan A\tan B}$

<u>PROOF</u> (i) We have,

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$\Rightarrow \quad \tan(A+B) = \frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

$$\Rightarrow \quad \tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B} \qquad \left[\begin{array}{l}\text{On dividing the numerator and denominator}\\ \text{by } \cos A\cos B\end{array}\right]$$

(ii) We have,

$$\tan(A-B) = \tan(A+(-B))$$

$$\Rightarrow \quad \tan(A-B) = \frac{\tan A+\tan(-B)}{1-\tan A\tan(-B)} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}$$ **Q.E.D.**

Similarly, it can be proved that

$$\cot(A+B) = \frac{\cot A \cot B - 1}{\cot B + \cot A} \text{ and, } \cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

7.3 MORE USEFUL RESULTS

THEOREM *Prove that:*

(i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(iii) $\sin(A+B+C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$

(iv) $\cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$

(v) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

PROOF (i) $\sin(A+B)\sin(A-B)$

$= (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B)$

$= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B$

$= \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B$

$= \sin^2 A - \sin^2 A \sin^2 B - \sin^2 B + \sin^2 A \sin^2 B$

$= \sin^2 A - \sin^2 B = (1-\cos^2 A) - (1-\cos^2 B) = \cos^2 B - \cos^2 A$

(ii) $\cos(A+B)\cos(A-B)$

$= (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B)$

$= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B = \cos^2 A(1-\sin^2 B) - (1-\cos^2 A)\sin^2 B$

$= \cos^2 A - \sin^2 B = (1-\sin^2 A) - (1-\cos^2 B) = \cos^2 B - \sin^2 A$

(iii) $\sin(A+B+C)$

$= \sin((A+B)+C) = \sin(A+B)\cos C + \cos(A+B)\sin C$

$= (\sin A \cos B + \cos A \sin B)\cos C + (\cos A \cos B - \sin A \sin B)\sin C$

$= \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$

(iv) $\cos(A+B+C)$

$= \cos((A+B)+C) = \cos(A+B)\cos C - \sin(A+B)\sin C$

$= (\cos A \cos B - \sin A \sin B)\cos C - (\sin A \cos B + \cos A \sin B)\sin C$

$= \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C$

(v) $\tan(A+B+C) = \tan((A+B)+C)$

$$\Rightarrow \quad \tan(A+B+C) = \frac{\tan(A+B) + \tan C}{1 - \tan(A+B)\tan C}$$

$$\Rightarrow \quad \tan(A+B+C) = \frac{\left(\dfrac{\tan A + \tan B}{1-\tan A \tan B}\right) + \tan C}{1 - \left(\dfrac{\tan A + \tan B}{1 - \tan A \tan B}\right)\tan C}$$

$$\Rightarrow \quad \tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

Q.E.D.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE VALUES OF $\sin(A \pm B)$, $\cos(A \pm B)$ AND $\tan(A \pm B)$ WHEN ONE OF THE TRIGONOMETRIC RATIOS OF EACH OF ANGLES A AND B is GIVEN**

EXAMPLE 1 *If* $\sin A = \frac{3}{5}$ *and* $\cos B = \frac{9}{41}$, $0 < A < \frac{\pi}{2}$, $0 < B < \frac{\pi}{2}$, *find the values of the following:*

(i) $\sin(A - B)$ (ii) $\sin(A + B)$ (iii) $\cos(A - B)$ (iv) $\cos(A + B)$

SOLUTION We have,

$$\sin A = \frac{3}{5} \text{ and, } \cos B = \frac{9}{41}$$

$$\therefore \quad \cos A = \sqrt{1 - \sin^2 A} \text{ and, } \sin B = \sqrt{1 - \cos^2 B}$$

$$\Rightarrow \quad \cos A = \sqrt{1 - \frac{9}{25}} = \frac{4}{5} \text{ and, } \sin B = \sqrt{1 - \frac{81}{1681}} = \frac{40}{41}$$

(i) $\sin(A - B) = \sin A \cos B - \cos A \sin B = \frac{3}{5} \times \frac{9}{41} - \frac{4}{5} \times \frac{40}{41} = -\frac{133}{205}$

(ii) $\sin(A + B) = \sin A \cos B + \cos A \sin B = \frac{3}{5} \times \frac{9}{41} + \frac{4}{5} \times \frac{40}{41} = \frac{187}{205}$

(iii) $\cos(A - B) = \cos A \cos B + \sin A \sin B = \frac{4}{5} \times \frac{9}{41} + \frac{3}{5} \times \frac{40}{41} = \frac{156}{205}$

(iv) $\cos(A + B) = \cos A \cos B - \sin A \sin B = \frac{4}{5} \times \frac{9}{41} - \frac{3}{5} \times \frac{40}{41} = -\frac{84}{205}$

EXAMPLE 2 *If* $\sin A = \frac{3}{5}$, $0 < A < \frac{\pi}{2}$ *and* $\cos B = \frac{-12}{13}$, $\pi < B < \frac{3\pi}{2}$, *find the following:*

(i) $\sin(A - B)$ (ii) $\cos(A + B)$ (iii) $\tan(A - B)$

SOLUTION We have, $\sin A = \frac{3}{5}$, where $0 < A < \frac{\pi}{2}$.

$$\therefore \quad \cos A = \pm\sqrt{1 - \sin^2 A} \Rightarrow \cos A = +\sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$$

In the first quadrant tangent function is positive. Therefore, $\tan A = \frac{\sin A}{\cos A} = \frac{3}{4}$

It is given that: $\cos B = -\frac{12}{13}$ and $\pi < B < \frac{3\pi}{2}$.

$$\therefore \quad \sin B = \pm\sqrt{1 - \cos^2 B}$$

$$\Rightarrow \quad \sin B = -\sqrt{1 - \cos^2 B} \qquad [\because \text{ Sine is negative in the third quadrant}]$$

$$\Rightarrow \quad \sin B = -\sqrt{1 - \left(\frac{-12}{13}\right)^2} = -\frac{5}{13}$$

In the III quadrant tangent function is positive. Therefore, $\tan B = \frac{\sin B}{\cos B} = \frac{5}{12}$.

Now,

(i) $\sin(A - B) = \sin A \cos B - \cos A \sin B = \frac{3}{5} \times \frac{-12}{13} - \frac{4}{5} \times \frac{-5}{13} = \frac{-16}{65}$

(ii) $\cos(A + B) = \cos A \cos B - \sin A \sin B = \frac{4}{5} \times \frac{-12}{13} - \frac{3}{5} \times \frac{-5}{13} = \frac{-33}{65}$

(iii) $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A \tan B} = \dfrac{\frac{3}{4}-\frac{5}{12}}{1+\frac{3}{4}\times\frac{5}{12}} = \dfrac{16}{63}$

EXAMPLE 3 *If* $\cos A = \frac{4}{5}$, $\cos B = \frac{12}{13}$, $\frac{3\pi}{2} < A,\ B < 2\pi$, *find the values of the following:*

(i) $\cos(A+B)$ (ii) $\sin(A-B)$

SOLUTION Since A and B both lie in the IV quadrant, it follows that $\sin A$ and $\sin B$ are negative. Therefore,

$$\sin A = -\sqrt{1-\cos^2 A} \Rightarrow \sin A = -\sqrt{1-\frac{16}{25}} = -\frac{3}{5}$$

and, $$\sin B = -\sqrt{1-\cos^2 B} \Rightarrow \sin B = -\sqrt{1-\frac{144}{169}} = -\frac{5}{13}$$

Now,

(i) $\cos(A+B) = \cos A\cos B - \sin A \sin B = \frac{4}{5}\times\frac{12}{13} - \left(\frac{-3}{5}\right)\left(\frac{-5}{13}\right) = \frac{33}{65}$

(ii) $\sin(A-B) = \sin A \cos B - \cos A \sin B = \frac{-3}{5}\times\frac{12}{13} - \frac{4}{5}\times\frac{-5}{13} = \frac{-16}{65}$

EXAMPLE 4 *If* $\cot\alpha = \frac{1}{2}$, $\sec\beta = -\frac{5}{3}$, *where* $\pi < \alpha < \frac{3\pi}{2}$ *and* $\frac{\pi}{2} < \beta < \pi$. *Find the value of* $\tan(\alpha+\beta)$. *State the quadrant in which* $\alpha+\beta$ *terminates.*

SOLUTION We have,

$$\cot\alpha = \frac{1}{2} \Rightarrow \tan\alpha = 2$$

Since β lies in the second quadrant. Therefore, $\tan\beta$ is negative. Consequently,

$$1+\tan^2\beta = \sec^2\beta \Rightarrow \tan\beta = -\sqrt{\sec^2\beta - 1} = -\sqrt{\frac{25}{9}-1} = -\frac{4}{3}$$

Hence, $$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{2-\frac{4}{3}}{1-2\times\frac{-4}{3}} = \frac{2}{11}$$

Now, $$\pi < \alpha < \frac{3\pi}{2} \text{ and } \frac{\pi}{2} < \beta < \pi \Rightarrow \frac{3\pi}{2} < \alpha+\beta < \frac{5\pi}{2}$$

We know that tangent function is positive in I and III quadrants.

$\therefore$ $\frac{3\pi}{2} < \alpha+\beta < \frac{5\pi}{2}$ and $\tan(\alpha+\beta) = \frac{2}{11} > 0 \Rightarrow \alpha+\beta$ lies in I quadrant.

Type II ON FINDING THE TRIGONOMETRIC RATIOS OF ANGLES WHICH ARE MULTIPLES OF 15°

EXAMPLE 5 *Find the values of the following:*

(i) $\sin 75°$ [NCERT] (ii) $\cos 75°$ [NCERT] (iii) $\sin 15°$ (iv) $\cos 15°$

SOLUTION (i) $\sin 75° = \sin(45°+30°) = \sin 45° \cos 30° + \cos 45° \sin 30°$

$$\Rightarrow \quad \sin 75° = \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(ii) $\cos 75° = \cos(45°+30°) = \cos 45° \cos 30° - \sin 45° \sin 30°$

$$\Rightarrow \quad \cos 75° = \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

(iii) $\sin 15° = \sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30°$

$\Rightarrow$ $\sin 15° = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(iv) $\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$

$\Rightarrow$ $\cos 15° = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

EXAMPLE 6 *Find the values of the following:*

(i) $\tan 15°$ [NCERT] (ii) $\tan 75°$ (iii) $\tan 105°$ (iv) $\tan \frac{13\pi}{12}$ [NCERT]

SOLUTION (i) $\tan 15° = \tan(45° - 30°) = \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

(ii) $\tan 75° = \tan(45° + 30°) = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

(iii) $\tan 105° = \tan(90° + 15°) = -\cot 15° = -\frac{1}{\tan 15°} = -\frac{\sqrt{3}+1}{\sqrt{3}-1}$ [Using (i)]

(iv) $\tan \frac{13\pi}{12} = \tan\left(\pi + \frac{\pi}{12}\right) = \tan \frac{\pi}{12} = \tan 15° = \frac{\sqrt{3}-1}{\sqrt{3}+1}$ [Using (i)]

EXAMPLE 7 *Prove that:* $\tan 75° + \cot 75° = 4$

SOLUTION We have,

$$\tan 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1} \quad \text{[See Ex. 6]}$$

$$\therefore \quad \cot 75° = \frac{1}{\tan 75°} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

Now, $\text{LHS} = \tan 75° + \cot 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1} + \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2}{(\sqrt{3}-1)(\sqrt{3}+1)}$

$$= \frac{(4+2\sqrt{3}) + (4-2\sqrt{3})}{3-1} = \frac{8}{2} = 4 = \text{RHS}$$

Type III **ON THE APPLICATIONS OF THE FOLLOWING FORMULAE:**

(i) $\sin A \cos B \pm \cos A \sin B = \sin(A \pm B)$ (ii) $\cos A \cos B \pm \sin A \sin B = \cos(A \mp B)$

EXAMPLE 8 *Evaluate the following:*

(i) $\sin \frac{7\pi}{12} \cos \frac{\pi}{4} - \cos \frac{7\pi}{12} \sin \frac{\pi}{4}$ (ii) $\sin \frac{\pi}{4} \cos \frac{\pi}{12} + \cos \frac{\pi}{4} \sin \frac{\pi}{12}$

(iii) $\cos \frac{2\pi}{3} \cos \frac{\pi}{4} - \sin \frac{2\pi}{3} \sin \frac{\pi}{4}$

SOLUTION (i) $\sin \frac{7\pi}{12} \cos \frac{\pi}{4} - \cos \frac{7\pi}{12} \sin \frac{\pi}{4} = \sin\left(\frac{7\pi}{12} - \frac{\pi}{4}\right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

(ii) $\sin \frac{\pi}{4} \cos \frac{\pi}{12} + \cos \frac{\pi}{4} \sin \frac{\pi}{12} = \sin\left(\frac{\pi}{4} + \frac{\pi}{12}\right) = \sin \frac{4\pi}{12} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

(iii) $\cos\frac{2\pi}{3}\cos\frac{\pi}{4}-\sin\frac{2\pi}{3}\sin\frac{\pi}{4}=\cos\left(\frac{2\pi}{3}+\frac{\pi}{4}\right)=\cos\frac{11\pi}{12}=\cos 165°$

$$=\cos(180°-15°)=-\cos 15°=-\frac{\sqrt{3}+1}{2\sqrt{2}}\quad \text{[See Ex. 5 (iv)]}$$

EXAMPLE 9 *Prove that:* $\cos\left(\frac{\pi}{4}-A\right)\cos\left(\frac{\pi}{4}-B\right)-\sin\left(\frac{\pi}{4}-A\right)\sin\left(\frac{\pi}{4}-B\right)=\sin(A+B)$ **[NCERT]**

SOLUTION We have,

$$\text{LHS}=\cos\left(\frac{\pi}{4}-A\right)\cos\left(\frac{\pi}{4}-B\right)-\sin\left(\frac{\pi}{4}-A\right)\sin\left(\frac{\pi}{4}-B\right)$$

$$=\cos\left\{\left(\frac{\pi}{4}-A\right)+\left(\frac{\pi}{4}-B\right)\right\}=\cos\left\{\frac{\pi}{2}-(A+B)\right\}=\sin(A+B)=\text{RHS}$$

EXAMPLE 10 *Prove that:* $\sin(n+1)A\sin(n+2)A+\cos(n+1)A\cos(n+2)A=\cos A$ **[NCERT]**

SOLUTION $\text{LHS}=\sin(n+1)A\sin(n+2)A+\cos(n+1)A\cos(n+2)A$

$$=\cos(n+2)A\cos(n+1)A+\sin(n+2)A\sin(n+1)A$$

$$=\cos\{(n+2)A-(n+1)A\}=\cos A=\text{RHS}$$

Type IV **ON APPLICATIONS OF THE FORMULAE:**

(i) $\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$ (ii) $\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$

(iii) $\tan(A\pm B)=\dfrac{\tan A\pm\tan B}{1\mp\tan A\tan B}$

EXAMPLE 11 *Prove that:*

(i) $\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)=\sqrt{2}\cos x$ **[NCERT]**

(ii) $\cos\left(\frac{3\pi}{4}+x\right)-\cos\left(\frac{3\pi}{4}-x\right)=-\sqrt{2}\sin x$ **[NCERT]**

SOLUTION (i) We have,

$$\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)$$

$$=\left(\cos\frac{\pi}{4}\cos x-\sin\frac{\pi}{4}\sin x\right)+\left(\cos\frac{\pi}{4}\cos x+\sin\frac{\pi}{4}\sin x\right)$$

$$=2\cos\frac{\pi}{4}\cos x=2\times\frac{1}{\sqrt{2}}\times\cos x=\sqrt{2}\cos x$$

(ii) We have,

$$\cos\left(\frac{3\pi}{4}+x\right)-\cos\left(\frac{3\pi}{4}-x\right)=-\sqrt{2}\sin x$$

$$=\left(\cos\frac{3\pi}{4}\cos x-\sin\frac{3\pi}{4}\sin x\right)+\left(\cos\frac{3\pi}{4}\cos x+\sin\frac{3\pi}{4}\sin x\right)$$

$$=\left(-\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x\right)-\left(-\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x\right)$$

$$=-\frac{1}{\sqrt{2}}\sin x-\frac{1}{\sqrt{2}}\sin x=-\frac{2}{\sqrt{2}}\sin x=-\sqrt{2}\sin x$$

EXAMPLE 12 *Prove that:* $\dfrac{\sin(x+y)}{\sin(x-y)}=\dfrac{\tan x+\tan y}{\tan x-\tan y}$ **[NCERT]**

SOLUTION We have,

$$\text{LHS} = \frac{\sin(x+y)}{\sin(x-y)} = \frac{\sin x \cos y + \cos x \sin y}{\sin x \cos y - \cos x \sin y}$$

$$= \frac{\dfrac{\sin x \cos y + \cos x \sin y}{\cos x \cos y}}{\dfrac{\sin x \cos y - \cos x \sin y}{\cos x \cos y}}$$ [Dividing the numerator and denominator by $\cos x \cos y$]

$$= \frac{\tan x + \tan y}{\tan x - \tan y} = \text{RHS}$$

EXAMPLE 13 *Prove that:* $\dfrac{\sin(B-C)}{\cos B \cos C} + \dfrac{\sin(C-A)}{\cos C \cos A} + \dfrac{\sin(A-B)}{\cos A \cos B} = 0$

SOLUTION We have,

$$\text{LHS} = \frac{\sin(B-C)}{\cos B \cos C} + \frac{\sin(C-A)}{\cos C \cos A} + \frac{\sin(A-B)}{\cos A \cos B}$$

$$= \frac{\sin B \cos C - \cos B \sin C}{\cos B \cos C} + \frac{\sin C \cos A - \cos C \sin A}{\cos C \cos A} + \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B}$$

$$= \frac{\sin B \cos C}{\cos B \cos C} - \frac{\cos B \sin C}{\cos B \cos C} + \frac{\sin C \cos A}{\cos C \cos A} - \frac{\cos C \sin A}{\cos C \cos A} + \frac{\sin A \cos B}{\cos A \cos B} - \frac{\cos A \sin B}{\cos A \cos B}$$

$$= \tan B - \tan C + \tan C - \tan A + \tan A - \tan B = 0 = \text{RHS}$$

EXAMPLE 14 *Prove that:* $\dfrac{\tan\left(\dfrac{\pi}{4}+x\right)}{\tan\left(\dfrac{\pi}{4}-x\right)} = \left(\dfrac{1+\tan x}{1-\tan x}\right)^2$ **[NCERT]**

SOLUTION $$\text{LHS} = \frac{\tan\left(\dfrac{\pi}{4}+x\right)}{\tan\left(\dfrac{\pi}{4}-x\right)} = \frac{\tan\dfrac{\pi}{4} + \tan x}{1 - \tan\dfrac{\pi}{4}\tan x} \times \frac{1 + \tan\dfrac{\pi}{4}\tan x}{\tan\dfrac{\pi}{4} - \tan x}$$

$$= \frac{1+\tan x}{1-\tan x} \times \frac{1+\tan x}{1-\tan x} = \left(\frac{1+\tan x}{1-\tan x}\right)^2 = \text{RHS}$$

EXAMPLE 15 *If* $\tan A - \tan B = x$ *and* $\cot B - \cot A = y$, *prove that* $\cot(A-B) = \dfrac{1}{x} + \dfrac{1}{y}$

[NCERT EXEMPAR]

SOLUTION We have, $\tan A - \tan B = x$ and $\cot B - \cot A = y$

Now,

$$\cot B - \cot A = y$$

$$\Rightarrow \quad \frac{1}{\tan B} - \frac{1}{\tan A} = y \Rightarrow \frac{\tan A - \tan B}{\tan A \tan B} = y \Rightarrow \frac{x}{\tan A \tan B} = y \Rightarrow \tan A \tan B = \frac{x}{y}$$

$$\therefore \quad \cot(A-B) = \frac{1}{\tan(A-B)} = \frac{1 + \tan A \tan B}{\tan A - \tan B} = \frac{1 + \dfrac{x}{y}}{x} = \frac{x+y}{xy} = \frac{1}{x} + \frac{1}{y}$$

EXAMPLE 16 *If* $\tan\alpha = \dfrac{1}{\sqrt{x(x^2+x+1)}}$, $\tan\beta = \dfrac{\sqrt{x}}{\sqrt{x^2+x+1}}$ *and* $\tan\gamma = \sqrt{x^{-3}+x^{-2}+x^{-1}}$, *prove that* $\alpha + \beta = \gamma$.

SOLUTION We have,

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \frac{\frac{1}{\sqrt{x(x^2+x+1)}}+\frac{\sqrt{x}}{\sqrt{x^2+x+1}}}{1-\frac{1}{\sqrt{x(x^2+x+1)}}\times\frac{\sqrt{x}}{\sqrt{x^2+x+1}}} = \frac{(x+1)\sqrt{x(x^2+x+1)}}{x(x^2+x+1)-x}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \frac{(x+1)\sqrt{x(x^2+x+1)}}{x^2(x+1)} = \frac{\sqrt{x(x^2+x+1)}}{x^2} = \sqrt{\frac{x(x^2+x+1)}{x^4}}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \sqrt{x^{-3}+x^{-2}+x^{-1}} = \tan\gamma$$

$$\Rightarrow \quad \alpha+\beta = \gamma.$$

EXAMPLE 17 *If α and β are acute angles such that* $\tan\alpha = \frac{m}{m+1}$ and $\tan\beta = \frac{1}{2m+1}$, *prove that* $\alpha+\beta = \frac{\pi}{4}$.

SOLUTION We have,

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \frac{\frac{m}{m+1}+\frac{1}{2m+1}}{1-\frac{m}{m+1}\times\frac{1}{2m+1}} = \frac{2m^2+m+m+1}{2m^2+3m+1-m} = \frac{2m^2+2m+1}{2m^2+2m+1} = 1 = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad \alpha+\beta = \frac{\pi}{4}$$

EXAMPLE 18 *If $A+B=\frac{\pi}{4}$, prove that:*

(i) $(1+\tan A)(1+\tan B)=2$ (ii) $(\cot A-1)(\cot B-1)=2$

SOLUTION (i) We have,

$$A+B = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan(A+B) = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{\tan A+\tan B}{1-\tan A\tan B} = 1$$

$$\Rightarrow \quad \tan A+\tan B = 1-\tan A\tan B$$

$$\Rightarrow \quad \tan A+\tan B+\tan A\tan B = 1$$

$$\Rightarrow \quad 1+\tan A+\tan B+\tan A\tan B = 2$$

$$\Rightarrow \quad (1+\tan A)+\tan B(1+\tan A) = 2 \Rightarrow (1+\tan A)(1+\tan B) = 2$$

(ii) We have,

$$A+B=\frac{\pi}{4}$$

$$\Rightarrow \quad \tan(A+B) = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{\tan A+\tan B}{1-\tan A\tan B} = 1$$

$$\tan A+\tan B = 1-\tan A\tan B$$

$$\Rightarrow \quad \tan A + \tan B + \tan A \tan B = 1$$

$$\Rightarrow \quad \frac{\tan A + \tan B + \tan A \tan B}{\tan A \tan B} = \frac{1}{\tan A \tan B}$$

$$\Rightarrow \quad \cot B + \cot A + 1 = \cot A \cot B$$

$$\Rightarrow \quad \cot A \cot B - \cot A - \cot B = 1$$

$$\Rightarrow \quad \cot A \cot B - \cot A - \cot B + 1 = 2$$

$$\Rightarrow \quad \cot A (\cot B - 1) - (\cot B - 1) = 2$$

$$\Rightarrow \quad (\cot A - 1)(\cot B - 1) = 2$$

EXAMPLE 19 *If* $\tan \beta = \dfrac{n \sin \alpha \cos \alpha}{1 - n \sin^2 \alpha}$, *show that* $\tan(\alpha - \beta) = (1 - n)\tan \alpha$.

SOLUTION We have,

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

$$\Rightarrow \quad \tan(\alpha - \beta) = \frac{\dfrac{\sin \alpha}{\cos \alpha} - \dfrac{n \sin \alpha \cos \alpha}{1 - n \sin^2 \alpha}}{1 + \dfrac{\sin \alpha}{\cos \alpha} \times \dfrac{n \sin \alpha \cos \alpha}{1 - n \sin^2 \alpha}} \qquad \left[\because \tan \beta = \frac{n \sin \alpha \cos \alpha}{1 - n \sin^2 \alpha}\right]$$

$$\Rightarrow \quad \tan(\alpha - \beta) = \frac{\sin \alpha - n \sin^3 \alpha - n \sin \alpha \cos^2 \alpha}{\cos \alpha (1 - n \sin^2 \alpha) + n \sin^2 \alpha \cos \alpha} \qquad \text{[On taking LCM]}$$

$$\Rightarrow \quad \tan(\alpha - \beta) = \frac{\sin \alpha - n \sin^3 \alpha - n \sin \alpha (1 - \sin^2 \alpha)}{\cos \alpha - n \sin^2 \alpha \cos \alpha + n \sin^2 \alpha \cos \alpha}$$

$$\Rightarrow \quad \tan(\alpha - \beta) = \frac{\sin \alpha - n \sin \alpha}{\cos \alpha} = \frac{\sin \alpha}{\cos \alpha}(1 - n) = (1 - n)\tan \alpha$$

EXAMPLE 20 *Prove that:*

(i) $\tan 3A \tan 2A \tan A = \tan 3A - \tan 2A - \tan A$ **[NCERT]**

(ii) $\cot A \cot 2A - \cot 2A \cot 3A - \cot 3A \cot A = 1$

SOLUTION (i) Clearly,

$$3A = 2A + A$$

$$\Rightarrow \quad \tan 3A = \tan(2A + A)$$

$$\Rightarrow \quad \tan 3A = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$$

$$\Rightarrow \quad \tan 3A (1 - \tan 2A \tan A) = \tan 2A + \tan A$$

$$\Rightarrow \quad \tan 3A - \tan 3A \tan 2A \tan A = \tan 2A + \tan A$$

$$\Rightarrow \quad \tan 3A - \tan 2A - \tan A = \tan 3A \tan 2A \tan A$$

(ii) Dividing both sides by $\tan A \tan 2A \tan 3A$, we get

$$\frac{\tan 3A \tan 2A \tan A}{\tan 3A \tan 2A \tan A} = \frac{\tan 3A - \tan 2A - \tan A}{\tan 3A \tan 2A \tan A}$$

$$\Rightarrow \quad 1 = \frac{1}{\tan 2A \tan A} - \frac{1}{\tan 3A \tan A} - \frac{1}{\tan 3A \tan 2A}$$

$$\Rightarrow \quad 1 = \cot A \cot 2A - \cot 3A \cot A - \cot 3A \cot 2A$$

Type V **ON THE APPLICATIONS OF THE FORMULAE:**
(i) $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B$ (ii) $\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B$

EXAMPLE 21 *Prove that:* $\dfrac{\tan(A+B)}{\cot(A-B)}=\dfrac{\sin^2 A-\sin^2 B}{\cos^2 A-\sin^2 B}$

SOLUTION $\text{LHS}=\dfrac{\tan(A+B)}{\cot(A-B)}=\dfrac{\sin(A+B)}{\cos(A+B)}\dfrac{\sin(A-B)}{\cos(A-B)}=\dfrac{\sin^2 A-\sin^2 B}{\cos^2 A-\sin^2 B}=\text{RHS}$

EXAMPLE 22 *Prove that :* $\sin^2 6x-\sin^2 4x=\sin 2x\sin 10x$ **[NCERT]**

SOLUTION We know that $\sin^2 A-\sin^2 B=\sin(A+B)\sin(A-B)$

$\therefore\quad \sin^2 6x-\sin^2 4x=\sin(6x+4x)\sin(6x-4x)=\sin 10x\sin 2x$

EXAMPLE 23 *Prove that :* $\cos^2 2x-\cos^2 6x=\sin 4x\sin 8x$ **[NCERT]**

SOLUTION We have,

$$\therefore\quad \text{LHS}=\cos^2 2x-\cos^2 6x$$
$$=(1-\sin^2 2x)-(1-\sin^2 6x)$$
$$=\sin^2 6x-\sin^2 2x=\sin(6x+2x)\sin(6x-2x)=\sin 8x\sin 4x=\text{RHS}$$

EXAMPLE 24 *Prove that* $\dfrac{\cos^2 33°-\cos^2 57°}{\sin^2\dfrac{21°}{2}-\sin^2\dfrac{69°}{2}}=-\sqrt{2}$

SOLUTION We have,

$$\text{LHS}=\frac{\cos^2 33°-\cos^2 57°}{\sin^2\frac{21°}{2}-\sin^2\frac{69°}{2}}$$
$$=\frac{(1-\sin^2 33°)-(1-\sin^2 57°)}{\sin^2\frac{21°}{2}-\sin^2\frac{69°}{2}}$$
$$=\frac{\sin^2 57°-\sin^2 33°}{\sin^2\frac{21°}{2}-\sin^2\frac{69°}{2}}$$
$$=\frac{\sin(57°+33°)\sin(57°-33°)}{\sin\left(\frac{21°}{2}+\frac{69°}{2}\right)\sin\left(\frac{21°}{2}-\frac{69°}{2}\right)}$$
$$=\frac{\sin 90°\sin 24°}{\sin 45°\sin(-24°)}=\frac{\sin 24°}{-\frac{1}{\sqrt{2}}\sin 24°}=-\sqrt{2}=\text{RHS}$$

EXAMPLE 25 *Prove that:* $\sin^2\left(\dfrac{\pi}{8}+\dfrac{A}{2}\right)-\sin^2\left(\dfrac{\pi}{8}-\dfrac{A}{2}\right)=\dfrac{1}{\sqrt{2}}\sin A$

SOLUTION Using $\sin^2 A-\sin^2 B=\sin(A+B)-\sin(A-B)$, we obtain

$$\text{LHS}=\sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right)-\sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right)$$
$$=\sin\left\{\left(\frac{\pi}{8}+\frac{A}{2}\right)+\left(\frac{\pi}{8}-\frac{A}{2}\right)\right\}\sin\left\{\left(\frac{\pi}{8}+\frac{A}{2}\right)-\left(\frac{\pi}{8}-\frac{A}{2}\right)\right\}$$
$$=\sin\frac{\pi}{4}\sin A=\frac{1}{\sqrt{2}}\sin A=\text{RHS}$$

EXAMPLE 26 *Prove that:* $\cos 2\alpha\cos 2\beta+\sin^2(\alpha-\beta)-\sin^2(\alpha+\beta)=\cos 2(\alpha+\beta)$

SOLUTION We have,

$$\text{LHS}=\cos 2\alpha\cos 2\beta+\sin^2(\alpha-\beta)-\sin^2(\alpha+\beta)$$

$$= \cos 2\alpha \cos 2\beta + \sin(\alpha-\beta+\alpha+\beta)\sin(\alpha-\beta-\alpha-\beta) \quad \left[\begin{array}{l} \because \ \sin^2 A - \sin^2 B \\ = \sin(A+B)\sin(A-B) \end{array}\right]$$

$$= \cos 2\alpha \cos 2\beta + \sin 2\alpha \sin(-2\beta)$$

$$= \cos 2\alpha \cos 2\beta - \sin 2\alpha \sin 2\beta = \cos(2\alpha + 2\beta)$$

EXAMPLE 27 *Prove that:* $\sin^2 A = \cos^2(A-B) + \cos^2 B - 2\cos(A-B)\cos A \cos B$.

SOLUTION We have,

$$\begin{aligned}
\text{RHS} &= \cos^2(A-B) + \cos^2 B - 2\cos(A-B)\cos A\cos B \\
&= \cos^2 B + \cos^2(A-B) - 2\cos(A-B)\cos A\cos B \\
&= \cos^2 B + \cos(A-B)\{\cos(A-B) - 2\cos A\cos B\} \\
&= \cos^2 B + \cos(A-B)\{\cos A\cos B + \sin A\sin B - 2\cos A\cos B\} \\
&= \cos^2 B + \cos(A-B)\{\sin A\sin B - \cos A\cos B\} \\
&= \cos^2 B - \cos(A-B)(\cos A\cos B - \sin A\sin B) \\
&= \cos^2 B - \cos(A-B)\cos(A+B) \\
&= \cos^2 B - (\cos^2 A - \sin^2 B) \\
&= \cos^2 B + \sin^2 B - \cos^2 A = 1 - \cos^2 A = \sin^2 A = \text{LHS}
\end{aligned}$$

LEVEL-2

EXAMPLE 28 *If* $3\tan A\tan B = 1$, *prove that* $2\cos(A+B) = \cos(A-B)$.

SOLUTION We have,

$$3\tan A\tan B = 1$$

$$\Rightarrow \quad \frac{3\sin A\sin B}{\cos A\cos B} = 1$$

$$\Rightarrow \quad \frac{\cos A\cos B}{\sin A\sin B} = \frac{3}{1}$$

$$\Rightarrow \quad \frac{\cos A\cos B + \sin A\sin B}{\cos A\cos B - \sin A\sin B} = \frac{3+1}{3-1} \quad \text{[Applying componendo-dividendo]}$$

$$\Rightarrow \quad \frac{\cos(A-B)}{\cos(A+B)} = 2 \Rightarrow 2\cos(A+B) = \cos(A-B)$$

EXAMPLE 29 *If* $\cos(\alpha-\beta) + \cos(\beta-\gamma) + \cos(\gamma-\alpha) = -\frac{3}{2}$, *prove that*

$$\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$$

SOLUTION We have,

$$\cos(\alpha-\beta) + \cos(\beta-\gamma) + \cos(\gamma-\alpha) = -\frac{3}{2}$$

$$\Rightarrow \quad 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta + 2\cos\beta\cos\gamma + 2\sin\beta\sin\gamma + 2\cos\gamma\cos\alpha + 2\sin\gamma\sin\alpha = -3$$

$$\Rightarrow \quad (2\cos\alpha\cos\beta + 2\cos\beta\cos\gamma + 2\cos\gamma\cos\alpha) + (2\sin\alpha\sin\beta + 2\sin\beta\sin\gamma + 2\sin\gamma\sin\alpha) + 3 = 0$$

$$\Rightarrow \quad (2\cos\alpha\cos\beta + 2\cos\beta\cos\gamma + 2\cos\gamma\cos\alpha) + (2\sin\alpha\sin\beta + 2\sin\beta\sin\gamma + 2\sin\gamma\sin\alpha + (\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) + (\cos^2\gamma + \sin^2\gamma) = 0$$

$$\Rightarrow \quad (\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta + 2\cos\beta\cos\gamma + 2\cos\gamma\cos\alpha) + (\sin^2\alpha + \sin^2\beta + \sin^2\gamma + 2\sin\alpha\sin\beta + 2\sin\beta\sin\gamma + 2\sin\gamma\sin\alpha) = 0$$

$\Rightarrow \quad (\cos\alpha + \cos\beta + \cos\gamma)^2 + (\sin\alpha + \sin\beta + \sin\gamma)^2 = 0$

$\Rightarrow \quad \cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$

EXAMPLE 30 *If* $\sin B = 3\sin(2A + B)$, *prove that* $2\tan A + \tan(A + B) = 0$.

SOLUTION We have,

$$\sin B = 3\sin(2A + B)$$

$$\Rightarrow \quad \frac{\sin(2A + B)}{\sin B} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{\sin\{(A + B) + A\}}{\sin\{(A + B) - A\}} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{\sin\{A + B) + A\} + \sin\{(A + B) - A\}}{\sin\{(A + B) + A\} - \sin\{(A + B) - A\}} = \frac{1 + 3}{1 - 3} \qquad \text{[Using componendo–dividendo]}$$

$$\Rightarrow \quad \frac{\{\sin(A + B)\cos A + \cos(A + B)\sin A\} + \{\sin(A + B)\cos A - \cos(A + B)\sin A\}}{\{\sin(A + B)\cos A + \cos(A + B)\sin A\} - \{\sin(A + B)\cos A - \cos(A + B)\sin A\}} = \frac{1 + 3}{1 - 3}$$

$$\Rightarrow \quad \frac{2\sin(A + B)\cos A}{2\cos(A + B)\sin A} = -2$$

$$\Rightarrow \quad \tan(A + B)\cot A = -2 \Rightarrow \tan(A + B) = -2\tan A \Rightarrow 2\tan A + \tan(A + B) = 0$$

EXAMPLE 31 *If* $2\tan\beta + \cot\beta = \tan\alpha$, *prove that* $\cot\beta = 2\tan(\alpha - \beta)$.

SOLUTION Clearly,

$$2\tan(\alpha - \beta) = 2\left\{\frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}\right\}$$

$$= 2\left\{\frac{2\tan\beta + \cot\beta - \tan\beta}{1 + (2\tan\beta + \cot\beta)\tan\beta}\right\} \qquad \text{[Using : } \tan\alpha = 2\tan\beta + \cot\beta\text{]}$$

$$= 2\left\{\frac{\tan\beta + \cot\beta}{1 + 2\tan^2\beta + 1}\right\}$$

$$= \frac{2(\tan\beta + \cot\beta)}{2 + 2\tan^2\beta} = \frac{2\left\{\tan\beta + \frac{1}{\tan\beta}\right\}}{2(1 + \tan^2\beta)} = \frac{1}{\tan\beta} = \cot\beta$$

Hence, $\cot\beta = 2\tan(\alpha - \beta)$.

EXAMPLE 32 *If* $\cos(\alpha + \beta)\sin(\gamma + \delta) = \cos(\alpha - \beta)\sin(\gamma - \delta)$, *prove that* $\cot\alpha\cot\beta\cot\gamma = \cot\delta$.

SOLUTION We have,

$$\cos(\alpha + \beta)\sin(\gamma + \delta) = \cos(\alpha - \beta)\sin(\gamma - \delta)$$

$$\Rightarrow \quad \frac{\cos(\alpha - \beta)}{\cos(\alpha + \beta)} = \frac{\sin(\gamma + \delta)}{\sin(\gamma - \delta)}$$

$$\Rightarrow \quad \frac{\cos(\alpha - \beta) + \cos(\alpha + \beta)}{\cos(\alpha - \beta) - \cos(\alpha + \beta)} = \frac{\sin(\gamma + \delta) + \sin(\gamma - \delta)}{\sin(\gamma + \delta) - \sin(\gamma - \delta)} \qquad \text{[Using componendo–dividendo]}$$

$$\Rightarrow \quad \frac{2\cos\alpha\cos\beta}{2\sin\alpha\sin\beta} = \frac{2\sin\gamma\cos\delta}{2\cos\gamma\sin\delta} \Rightarrow \cot\alpha\cot\beta = \tan\gamma\cot\delta \Rightarrow \cot\alpha\cot\beta\cot\gamma = \cot\delta$$

EXAMPLE 33 *Prove that:* $\dfrac{\sin(x + \theta)}{\sin(x + \phi)} = \cos(\theta - \phi) + \cot(x + \phi)\sin(\theta - \phi)$.

SOLUTION We have,

$$\frac{\sin(x + \theta)}{\sin(x + \phi)} = \frac{\sin\{(x + \phi) + (\theta - \phi)\}}{\sin(x + \phi)}$$

$$= \frac{\sin(x+\phi)\cos(\theta-\phi)+\cos(x+\phi)\sin(\theta-\phi)}{\sin(x+\phi)}$$

$$= \cos(\theta-\phi)+\cot(x+\phi)\sin(\theta-\phi)$$

EXAMPLE 34 *If* $\cos(\alpha+\beta)=\frac{4}{5}$, $\sin(\alpha-\beta)=\frac{5}{13}$ *and* α, β *lie between* 0 *and* $\frac{\pi}{4}$, *prove that* $\tan 2\alpha=\frac{56}{33}$ **[NCERT EXEMPLAR]**

SOLUTION It is given that α, β lie between 0 and $\pi/4$. Therefore, $-\pi/4<\alpha-\beta<\pi/4$ and $0<\alpha+\beta<\pi/2$.

So, $\cos(\alpha-\beta)$ and $\sin(\alpha+\beta)$ are positive.

Now, $\sin(\alpha+\beta)=\sqrt{1-\cos^2(\alpha+\beta)} \Rightarrow \sin(\alpha+\beta)=\sqrt{1-\frac{16}{25}}=\frac{3}{5}$

and, $\cos(\alpha-\beta)=\sqrt{1-\sin^2(\alpha-\beta)} \Rightarrow \cos(\alpha-\beta)=\sqrt{1-\frac{25}{169}}=\frac{12}{13}$

$\therefore$ $\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{3/5}{4/5}=\frac{3}{4}$ and, $\tan(\alpha-\beta)=\frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}=\frac{5}{12}$

Now,

$$\tan 2\alpha=\tan\{(\alpha+\beta)+(\alpha-\beta)\}=\frac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\tan(\alpha-\beta)}=\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\times\frac{5}{12}}=\frac{56}{33}$$

EXAMPLE 35 *Prove that:* $\tan 70°=\tan 20°+2\tan 50°$.

SOLUTION We have,

$$\tan A-\tan B=\tan(A-B)(1+\tan A\tan B)$$

$$\therefore\quad \tan 70°-\tan 20°=\tan(70°-20°)(1+\tan 70°\tan 20°)$$

$$=\tan 50°\ (1+\tan 70°\cot 70°)$$

$$=2\tan 50° \qquad [\because 1+\tan 70°\cot 70°=2]$$

EXAMPLE 36 *If* $\tan(\alpha+\theta)=n\tan(\alpha-\theta)$, *show that:* $(n+1)\sin 2\theta=(n-1)\sin 2\alpha$.

SOLUTION We have,

$$\tan(\alpha+\theta)=n\tan(\alpha-\theta)$$

$$\Rightarrow \frac{\tan(\alpha+\theta)}{\tan(\alpha-\theta)}=\frac{n}{1}$$

$$\Rightarrow \frac{\tan(\alpha+\theta)+\tan(\alpha-\theta)}{\tan(\alpha+\theta)-\tan(\alpha-\theta)}=\frac{n+1}{n-1} \qquad \text{[Applying componendo-dividendo]}$$

$$\Rightarrow \frac{\sin(\alpha+\theta)\cos(\alpha-\theta)+\cos(\alpha+\theta)\sin(\alpha-\theta)}{\sin(\alpha+\theta)\cos(\alpha-\theta)-\cos(\alpha+\theta)\sin(\alpha-\theta)}=\frac{n+1}{n-1}$$

$$\Rightarrow \frac{\sin(\alpha+\theta)+\alpha-\theta)}{\sin(\alpha+\theta)-(\alpha-\theta))}=\frac{n+1}{n-1}$$

$$\Rightarrow \frac{\sin 2\alpha}{\sin 2\theta}=\frac{n+1}{n-1}\Rightarrow (n+1)\sin 2\theta=(n-1)\sin 2\alpha$$

EXAMPLE 37 *Prove that:* $\cot\theta\cot 2\theta+\cot 2\theta\cot 3\theta+2=\cot\theta(\cot\theta-\cot 3\theta)$

SOLUTION

$$\therefore\quad \text{LHS}=\cot\theta\cot 2\theta+\cot 2\theta\cot 3\theta+2$$

$$=(\cot\theta\cot 2\theta+1)+(\cot 2\theta\cot 3\theta+1)$$

$$=\left(\frac{\cos\theta\cos 2\theta}{\sin\theta\sin 2\theta}\right)+\left(\frac{\cos 2\theta\cos 3\theta}{\sin 2\theta\sin 3\theta}+1\right)$$

$$=\left(\frac{\cos 2\theta \cos\theta+\sin 2\theta \sin\theta}{\sin\theta \sin 2\theta}\right)+\left(\frac{\cos 3\theta \cos 2\theta+\sin 3\theta \sin 2\theta}{\sin 2\theta \sin 3\theta}\right)$$

$$=\frac{\cos(2\theta-\theta)}{\sin\theta \sin 2\theta}+\frac{\cos(3\theta-2\theta)}{\sin 2\theta \sin 3\theta}$$

$$=\frac{\cos\theta}{\sin\theta \sin 2\theta}+\frac{\cos\theta}{\sin 2\theta \sin 3\theta}$$

$$=\cos\theta\left\{\frac{1}{\sin\theta \sin 2\theta}+\frac{1}{\sin 2\theta \sin 3\theta}\right\}$$

$$=\frac{\cos\theta}{\sin\theta}\left\{\frac{\sin\theta}{\sin\theta \sin 2\theta}+\frac{\sin\theta}{\sin 2\theta \sin 3\theta}\right\}$$

$$=\cot\theta\left\{\frac{\sin(2\theta-\theta)}{\sin\theta \sin 2\theta}+\frac{\sin(3\theta-2\theta)}{\sin 2\theta \sin 3\theta}\right\}$$

$$=\cot\theta\left\{\frac{\sin 2\theta \cos\theta-\cos 2\theta \sin\theta}{\sin\theta \sin 2\theta}+\frac{\sin 3\theta \cos 2\theta-\cos 3\theta \sin 2\theta}{\sin 2\theta \sin 3\theta}\right\}$$

$$=\cot\theta\,\{\cot\theta-\cot 2\theta+\cot 2\theta-\cot 3\theta\}=\cot\theta\,(\cot\theta-\cot 3\theta)=\text{RHS}$$

EXAMPLE 38 *If* $\tan(\pi\cos\theta)=\cot(\pi\sin\theta)$, *prove that* $\cos\left(\theta-\frac{\pi}{4}\right)=\pm\frac{1}{2\sqrt{2}}$.

SOLUTION We have,

$$\tan(\pi\cos\theta)=\cot(\pi\sin\theta)$$

$$\Rightarrow \frac{\sin(\pi\cos\theta)}{\cos(\pi\cos\theta)}=\frac{\cos(\pi\sin\theta)}{\sin(\pi\sin\theta)}$$

$$\Rightarrow \sin(\pi\cos\theta)\sin(\pi\sin\theta)=\cos(\pi\sin\theta)\cos(\pi\cos\theta)$$

$$\Rightarrow \cos(\pi\cos\theta)\cos(\pi\sin\theta)-\sin(\pi\cos\theta)\sin(\pi\sin\theta)=0$$

$$\Rightarrow \cos(\pi\cos\theta+\pi\sin\theta)=0$$

$$\Rightarrow \pi\cos\theta+\pi\sin\theta=\pm\frac{\pi}{2} \qquad \left[\because \cos\left(\pm\frac{\pi}{2}\right)=0\right]$$

$$\Rightarrow \cos\theta+\sin\theta=\pm\frac{1}{2}$$

$$\Rightarrow \frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta=\pm\frac{1}{2\sqrt{2}} \qquad [\text{Multiplying both sides by } 1/\sqrt{2}\,]$$

$$\Rightarrow \cos\theta\cos\frac{\pi}{4}+\sin\theta\sin\frac{\pi}{4}=\pm\frac{1}{2\sqrt{2}}$$

$$\Rightarrow \cos\left(\theta-\frac{\pi}{4}\right)=\pm\frac{1}{2\sqrt{2}}$$

EXAMPLE 39 *If* $a\tan\alpha+b\tan\beta=(a+b)\tan\left(\frac{\alpha+\beta}{2}\right)$, *where* $\alpha\neq\beta$, *prove that* $a\cos\beta=b\cos\alpha$.

SOLUTION We have,

$$a\tan\alpha+b\tan\beta=(a+b)\tan\left(\frac{\alpha+\beta}{2}\right)$$

$$\Rightarrow a\left\{\tan\alpha-\tan\left(\frac{\alpha+\beta}{2}\right)\right\}=b\left\{\tan\left(\frac{\alpha+\beta}{2}\right)-\tan\beta\right\}$$

$$\Rightarrow \quad \frac{a \sin\left(\alpha - \frac{\alpha+\beta}{2}\right)}{\cos\alpha \cos\left(\frac{\alpha+\beta}{2}\right)} = \frac{b \sin\left(\frac{\alpha+\beta}{2} - \beta\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)\cos\beta} \qquad \left[\because \tan A - \tan B = \frac{\sin(A-B)}{\cos A \cos B}\right]$$

$$\Rightarrow \quad \frac{a \sin\left(\frac{\alpha-\beta}{2}\right)}{\cos\alpha} = \frac{b \sin\left(\frac{\alpha-\beta}{2}\right)}{\cos\beta}$$

$$\Rightarrow \quad a\cos\beta = b\cos\alpha \qquad \left[\because \alpha \neq \beta \therefore \sin\left(\frac{\alpha-\beta}{2}\right) \neq 0\right]$$

EXAMPLE 40 *If* $\sin\alpha + \sin\beta = a$ *and* $\cos\alpha + \cos\beta = b$, *show that*

(i) $\cos(\alpha+\beta) = \dfrac{b^2 - a^2}{b^2 + a^2}$ (ii) $\sin(\alpha+\beta) = \dfrac{2ab}{a^2+b^2}$

SOLUTION (i) We have,

$$b^2 + a^2 = (\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2$$

$$\Rightarrow \quad b^2 + a^2 = (\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) + 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$

$$\Rightarrow \quad b^2 + a^2 = 1 + 1 + 2\cos(\alpha-\beta) = 2 + 2\cos(\alpha-\beta) \qquad \text{...(i)}$$

and, $$b^2 - a^2 = (\cos\alpha + \cos\beta)^2 - (\sin\alpha + \sin\beta)^2$$

$$\Rightarrow \quad b^2 - a^2 = \cos^2\alpha + \cos^2\beta - \sin^2\alpha - \sin^2\beta + 2(\cos\alpha\cos\beta - \sin\alpha\sin\beta)$$

$$\Rightarrow \quad b^2 - a^2 = (\cos^2\alpha - \sin^2\beta) + (\cos^2\beta - \sin^2\alpha) + 2\cos(\alpha+\beta)$$

$$\Rightarrow \quad b^2 - a^2 = \cos(\alpha+\beta)\cos(\alpha-\beta) + \cos(\beta+\alpha)\cos(\beta-\alpha) + 2\cos(\alpha+\beta)$$

$$\Rightarrow \quad b^2 - a^2 = 2\cos(\alpha+\beta)\cos(\alpha-\beta) + 2\cos(\alpha+\beta)$$

$$[\because \cos(\beta-\alpha) = \cos\{-(\alpha-\beta)\} = \cos(\alpha-\beta)]$$

$$\Rightarrow \quad b^2 - a^2 = \cos(\alpha+\beta)\{2\cos(\alpha-\beta) + 2\}$$

$$\Rightarrow \quad b^2 - a^2 = \cos(\alpha+\beta)(b^2 + a^2) \qquad [\text{Using (i)}]$$

Thus, $$b^2 - a^2 = (b^2 + a^2)\cos(\alpha+\beta) \Rightarrow \cos(\alpha+\beta) = \frac{b^2 - a^2}{b^2 + a^2}$$

(ii) $$\sin(\alpha+\beta) = \sqrt{1 - \cos^2(\alpha+\beta)}$$

$$\Rightarrow \quad \sin(\alpha+\beta) = \sqrt{1 - \left(\frac{b^2 - a^2}{b^2 + a^2}\right)^2} = \sqrt{\frac{4a^2b^2}{(a^2+b^2)^2}} = \frac{2ab}{b^2+a^2}$$

EXAMPLE 41 *If* α *and* β *are the solutions of the equation* $a\tan\theta + b\sec\theta = c$, *then show that* $\tan(\alpha+\beta) = \dfrac{2ac}{a^2 - c^2}$ **[NCERT EXEMPLAR]**

SOLUTION We have,

$$a\tan\theta + b\sec\theta = c \qquad \text{...(i)}$$

$$\Rightarrow \quad c - a\tan\theta = b\sec\theta$$

$$\Rightarrow \quad (c - a\tan\theta)^2 = b^2\sec^2\theta$$

$$\Rightarrow \quad c^2 + a^2\tan^2\theta - 2ac\tan\theta = b^2(1 + \tan^2\theta)$$

$$\Rightarrow \quad \tan^2\theta(a^2 - b^2) - 2ac\tan\theta + (c^2 - b^2) = 0 \qquad \text{...(ii)}$$

It is given that α & β are the solutions of (i). Therefore, $\tan\alpha$ and $\tan\beta$ are roots of equation (ii).

$$\therefore \quad \tan\alpha + \tan\beta = \frac{2ac}{a^2 - b^2} \text{ and } \tan\alpha \tan\beta = \frac{c^2 - b^2}{a^2 - b^2}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{\dfrac{2ac}{(a^2-b^2)}}{1 - \dfrac{c^2-b^2}{a^2-b^2}} = \frac{2ac}{a^2 - c^2}$$

EXAMPLE 42 *If* α *and* β *are the solutions of* $a\cos\theta + b\sin\theta = c$, *then show that*

(i) $\cos(\alpha+\beta) = \dfrac{a^2 - b^2}{a^2 + b^2}$ (ii) $\cos(\alpha - \beta) = \dfrac{2c^2 - (a^2 + b^2)}{a^2 + b^2}$ **[NCERT EXEMPLAR]**

SOLUTION We have,

$$a\cos\theta + b\sin\theta = c \qquad \text{...(i)}$$

$$\Rightarrow \quad a\cos\theta = c - b\sin\theta$$

$$\Rightarrow \quad a^2\cos^2\theta = (c - b\sin\theta)^2$$

$$\Rightarrow \quad a^2(1 - \sin^2\theta) = c^2 - 2bc\sin\theta + b^2\sin^2\theta$$

$$\Rightarrow \quad (a^2 + b^2)\sin^2\theta - 2bc\sin\theta + (c^2 - a^2) = 0 \qquad \text{...(ii)}$$

Since α, β are roots of equation (i). Therefore, $\sin\alpha$ and $\sin\beta$ are roots of equation (ii).

$$\therefore \quad \sin\alpha\sin\beta = \frac{c^2 - a^2}{a^2 + b^2} \qquad \text{...(iii)}$$

Again, $a\cos\theta + b\sin\theta = c$

$$\Rightarrow \quad b\sin\theta = c - a\cos\theta$$

$$\Rightarrow \quad b^2\sin^2\theta = (c - a\cos\theta)^2$$

$$\Rightarrow \quad b^2(1 - \cos^2\theta) = (c - a\cos\theta)^2$$

$$\Rightarrow \quad (a^2 + b^2)\cos^2\theta - 2ac\cos\theta + c^2 - b^2 = 0 \qquad \text{...(iv)}$$

It is given that α, β are the roots of equation (i). So, $\cos\alpha$, $\cos\beta$ are the roots of equation (iv).

$$\therefore \quad \cos\alpha\cos\beta = \frac{c^2 - b^2}{a^2 + b^2} \qquad \text{...(v)}$$

Now, $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$\Rightarrow \quad \cos(\alpha+\beta) = \frac{c^2 - b^2}{a^2 + b^2} - \frac{c^2 - a^2}{a^2 + b^2} = \frac{a^2 - b^2}{a^2 + b^2}$$

and, $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$\Rightarrow \quad \cos(\alpha-\beta) = \frac{c^2 - b^2}{a^2 + b^2} + \frac{c^2 - a^2}{a^2 + b^2} = \frac{2c^2 - (a^2 + b^2)}{a^2 + b^2}$$

EXAMPLE 43 *Prove that:*

$$\frac{\sin x}{\cos 3x} + \frac{\sin 3x}{\cos 9x} + \frac{\sin 9x}{\cos 27x} = \frac{1}{2}(\tan 27x - \tan x)$$

SOLUTION We have,

$$\frac{\sin x}{\cos 3x} + \frac{\sin 3x}{\cos 9x} + \frac{\sin 9x}{\cos 27x}$$

$$= \frac{1}{2}\left\{\frac{2\sin x\cos x}{\cos 3x\cos x}+\frac{2\sin 3x\cos 3x}{\cos 9x\cos 3x}+\frac{2\sin 9x\cos 9x}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin(x+x)}{\cos 3x\cos x}+\frac{\sin(3x+3x)}{\cos 9x\cos 3x}+\frac{\sin(9x+9x)}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin 2x}{\cos 3x\cos x}+\frac{\sin 6x}{\cos 9x\cos 3x}+\frac{\sin 18x}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin(3x-x)}{\cos 3x\cos x}+\frac{\sin(9x-3x)}{\cos 9x\cos 3x}+\frac{\sin(27x-9x)}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin 3x\cos x-\cos 3x\sin x}{\cos 3x\cos x}+\frac{\sin 9x\cos 3x-\cos 9x\sin 3x}{\cos 9x\cos 3x}+\frac{\sin 27x\cos 9x-\cos 27x\sin 9x}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin 3x\cos x}{\cos 3x\cos x}-\frac{\cos 3x\sin x}{\cos 3x\cos x}+\frac{\sin 9x\cos 3x}{\cos 9x\cos 3x}-\frac{\cos 9x\sin 3x}{\cos 9x\cos 3x}+\frac{\sin 27x\cos 9x}{\cos 27x\cos 9x}-\frac{\cos 27x\sin 9x}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\{(\tan 3x-\tan x)+(\tan 9x-\tan 3x)+(\tan 27x-\tan 9x)\} = \frac{1}{2}(\tan 27x-\tan x)$$

EXERCISE 7.1

LEVEL-1

1. If $\sin A = \frac{4}{5}$ and $\cos B = \frac{5}{13}$, where $0 < A,\ B < \frac{\pi}{2}$, find the values of the following:
   (i) $\sin(A+B)$ (ii) $\cos(A+B)$ (iii) $\sin(A-B)$ (iv) $\cos(A-B)$
2. (a) If $\sin A = \frac{12}{13}$ and $\sin B = \frac{4}{5}$, where $\frac{\pi}{2} < A < \pi$ and $0 < B < \frac{\pi}{2}$, find the following:
   (i) $\sin(A+B)$ (ii) $\cos(A+B)$
   (b) If $\sin A = \frac{3}{5}$, $\cos B = -\frac{12}{13}$, where A and B both lie in second quadrant, find the value of $\sin(A+B)$. **[NCERT]**
3. If $\cos A = -\frac{24}{25}$ and $\cos B = \frac{3}{5}$, where $\pi < A < \frac{3\pi}{2}$ and $\frac{3\pi}{2} < B < 2\pi$, find the following:
   (i) $\sin(A+B)$ (ii) $\cos(A+B)$
4. If $\tan A = \frac{3}{4}$, $\cos B = \frac{9}{41}$, where $\pi < A < \frac{3\pi}{2}$ and $0 < B < \frac{\pi}{2}$, find $\tan(A+B)$.
5. If $\sin A = \frac{1}{2}$, $\cos B = \frac{12}{13}$, where $\frac{\pi}{2} < A < \pi$ and $\frac{3\pi}{2} < B < 2\pi$, find $\tan(A-B)$.
6. If $\sin A = \frac{1}{2}$, $\cos B = \frac{\sqrt{3}}{2}$, where $\frac{\pi}{2} < A < \pi$ and $0 < B < \frac{\pi}{2}$, find the following:
   (i) $\tan(A+B)$ (ii) $\tan(A-B)$
7. Evaluate the following:
   (i) $\sin 78^\circ \cos 18^\circ - \cos 78^\circ \sin 18^\circ$ (ii) $\cos 47^\circ \cos 13^\circ - \sin 47^\circ \sin 13^\circ$
   (iii) $\sin 36^\circ \cos 9^\circ + \cos 36^\circ \sin 9^\circ$ (iv) $\cos 80^\circ \cos 20^\circ + \sin 80^\circ \sin 20^\circ$
8. If $\cos A = -\frac{12}{13}$ and $\cot B = \frac{24}{7}$, where A lies in the second quadrant and B in the third quadrant, find the values of the following:
   (i) $\sin(A+B)$ (ii) $\cos(A+B)$ (iii) $\tan(A+B)$
9. Prove that: $\cos 105^\circ + \cos 15^\circ = \sin 75^\circ - \sin 15^\circ$
10. Prove that: $\frac{\tan A+\tan B}{\tan A-\tan B} = \frac{\sin(A+B)}{\sin(A-B)}$

11. Prove that:

(i) $\dfrac{\cos 11° + \sin 11°}{\cos 11° - \sin 11°} = \tan 56°$ (ii) $\dfrac{\cos 9° + \sin 9°}{\cos 9° - \sin 9°} = \tan 54°$

(iii) $\dfrac{\cos 8° - \sin 8°}{\cos 8° + \sin 8°} = \tan 37°$

12. Prove that:

(i) $\sin(60° - \theta)\cos(30° + \theta) + \cos(60° - \theta)\sin(30° + \theta) = 1$

(ii) $\sin\left(\dfrac{4\pi}{9}+7\right)\cos\left(\dfrac{\pi}{9}+7\right) - \cos\left(\dfrac{4\pi}{9}+7\right)\sin\left(\dfrac{\pi}{9}+7\right) = \dfrac{\sqrt{3}}{2}$

(iii) $\sin\left(\dfrac{3\pi}{8}-5\right)\cos\left(\dfrac{\pi}{8}+5\right) + \cos\left(\dfrac{3\pi}{8}-5\right)\sin\left(\dfrac{\pi}{8}+5\right) = 1$

13. Prove that: $\dfrac{\tan 69° + \tan 66°}{1 - \tan 69° \tan 66°} = -1$

14. (i) If $\tan A = \dfrac{5}{6}$ and $\tan B = \dfrac{1}{11}$, prove that $A + B = \dfrac{\pi}{4}$

(ii) If $\tan A = \dfrac{m}{m-1}$ and $\tan B = \dfrac{1}{2m-1}$, then prove that $A - B = \dfrac{\pi}{4}$

15. Prove that:

(i) $\cos^2 45° - \sin^2 15° = \dfrac{\sqrt{3}}{4}$

(ii) $\sin^2 (n+1)A - \sin^2 nA = \sin(2n+1)A \sin A$

16. Prove that:

(i) $\dfrac{\sin(A+B) + \sin(A-B)}{\cos(A+B) + \cos(A-B)} = \tan A$

(ii) $\dfrac{\sin(A-B)}{\cos A \cos B} + \dfrac{\sin(B-C)}{\cos B \cos C} + \dfrac{\sin(C-A)}{\cos C \cos A} = 0$

(iii) $\dfrac{\sin(A-B)}{\sin A \sin B} + \dfrac{\sin(B-C)}{\sin B \sin C} + \dfrac{\sin(C-A)}{\sin C \sin A} = 0$

(iv) $\sin^2 B = \sin^2 A + \sin^2 (A-B) - 2\sin A \cos B \sin(A-B)$

(v) $\cos^2 A + \cos^2 B - 2\cos A \cos B \cos(A+B) = \sin^2 (A+B)$

(vi) $\dfrac{\tan(A+B)}{\cot(A-B)} = \dfrac{\tan^2 A - \tan^2 B}{1 - \tan^2 A \tan^2 B}$

17. Prove that:

(i) $\tan 8\theta - \tan 6\theta - \tan 2\theta = \tan 8\theta \tan 6\theta \tan 2\theta$

(ii) $\tan 15° + \tan 30° + \tan 15° \tan 30° = 1$

(iii) $\tan 36° + \tan 9° + \tan 36° \tan 9° = 1$

(iv) $\tan 13\theta - \tan 9\theta - \tan 4\theta = \tan 13\theta \tan 9\theta \tan 4\theta$

18. Prove that: $\dfrac{\tan^2 2\theta - \tan^2 \theta}{1 - \tan^2 2\theta \tan^2 \theta} = \tan 3\theta \tan \theta$

19. If $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{a+b}{a-b}$, show that $\dfrac{\tan x}{\tan y} = \dfrac{a}{b}$. [NCERT EXAMPLAR]

20. If $\tan A = x \tan B$, prove that $\dfrac{\sin(A-B)}{\sin(A+B)} = \dfrac{x-1}{x+1}$

21. If $\tan(A+B)=x$ and $\tan(A-B)=y$, find the values of $\tan 2A$ and $\tan 2B$.

22. If $\cos A+\sin B=m$ and $\sin A+\cos B=n$, prove that $2\sin(A+B)=m^2+n^2-2$.

LEVEL-2

23. If $\tan A+\tan B=a$ and $\cot A+\cot B=b$, prove that: $\cot(A+B)=\frac{1}{a}-\frac{1}{b}$.

24. If θ lies in the first quadrant and $\cos\theta=\frac{8}{17}$, then prove that

$$\cos\left(\frac{\pi}{6}+\theta\right)+\cos\left(\frac{\pi}{4}-\theta\right)+\cos\left(\frac{2\pi}{3}-\theta\right)=\left(\frac{\sqrt{3}-1}{2}+\frac{1}{\sqrt{2}}\right)\frac{23}{17}$$ [NCERT EXEMPLAR]

25. If $\tan x+\tan\left(x+\frac{\pi}{3}\right)+\tan\left(x+\frac{2\pi}{3}\right)=3$, then prove that $\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}=1$.

26. If $\sin(\alpha+\beta)=1$ and $\sin(\alpha-\beta)=\frac{1}{2}$, where $0\le\alpha,\ \beta\le\frac{\pi}{2}$, then find the values of $\tan(\alpha+2\beta)$ and $\tan(2\alpha+\beta)$.

27. If α,β are two different values of θ lying between 0 and 2π which satisfy the equation $6\cos\theta+8\sin\theta=9$, find the value of $\sin(\alpha+\beta)$.

28. If $\sin\alpha+\sin\beta=a$ and $\cos\alpha+\cos\beta=b$, show that

(i) $\sin(\alpha+\beta)=\frac{2ab}{a^2+b^2}$ (ii) $\cos(\alpha+\beta)=\frac{b^2-a^2}{b^2+a^2}$

29. Prove that :

(i) $\frac{1}{\sin(x-a)\sin(x-b)}=\frac{\cot(x-a)-\cot(x-b)}{\sin(a-b)}$

(ii) $\frac{1}{\sin(x-a)\cos(x-b)}=\frac{\cot(x-a)+\tan(x-b)}{\cos(a-b)}$

(iii) $\frac{1}{\cos(x-a)\cos(x-b)}=\frac{\tan(x-b)-\tan(x-a)}{\sin(a-b)}$

30. If $\sin\alpha\sin\beta-\cos\alpha\cos\beta+1=0$, prove that $1+\cot\alpha\tan\beta=0$.

31. If $\tan\alpha=x+1$, $\tan\beta=x-1$, show that $2\cot(\alpha-\beta)=x^2$.

32. If angle θ is divided into two parts such that the tangents of one part is λ times the tangent of other, and ϕ is their difference, then show that $\sin\theta=\frac{\lambda+1}{\lambda-1}\sin\phi$.

[NCERT EXEMPLAR]

33. If $\tan\theta=\frac{\sin\alpha-\cos\alpha}{\sin\alpha+\cos\alpha}$, then show that $\sin\alpha+\cos\alpha=\sqrt{2}\cos\theta$

[NCERT EXEMPLAR]

ANSWERS

1. (i) $\frac{56}{65}$ (ii) $\frac{-33}{65}$ (iii) $\frac{-16}{65}$ (iv) $\frac{63}{65}$

2. (a) (i) $\frac{16}{65}$ (ii) $-\frac{63}{65}$ (b) $\frac{-56}{65}$

3. (i) $\frac{3}{5}$ (ii) $-\frac{4}{5}$ 4. $-\frac{187}{84}$ 5. $\frac{5\sqrt{3}-12}{5+12\sqrt{3}}$

6. (i) 0 (ii) $-\sqrt{3}$

7. (i) $\frac{\sqrt{3}}{2}$ (ii) $\frac{1}{2}$ (iii) $\frac{1}{\sqrt{2}}$ (iv) $\frac{1}{2}$

8. (i) $\frac{-36}{325}$ (ii) $\frac{323}{325}$ (iii) $-\frac{36}{323}$ 18. $\frac{x+y}{1-xy}, \frac{x-y}{1+xy}$

26. $-\sqrt{3}, -\frac{1}{\sqrt{3}}$ 27. $\frac{24}{25}$

HINTS TO SELECTED PROBLEMS

9. LHS $= \cos(90° + 15°) + \cos(90° - 75°) = -\sin 15° + \sin 75° =$ RHS

11. $$\text{LHS} = \frac{\dfrac{\cos 11° + \sin 11°}{\cos 11°}}{\dfrac{\cos 11° - \sin 11°}{\cos 11°}} = \frac{1 + \tan 11°}{1 - \tan 11°} = \frac{\tan 45° + \tan 11°}{1 - \tan 45° \tan 11°} = \tan(45° + 11°) = \tan 56° = \text{RHS}$$

29. (i) We have,

$$\frac{1}{\sin(x-a)\sin(x-b)} = \frac{1}{\sin(a-b)}\left\{\frac{\sin(a-b)}{\sin(x-a)\sin(x-b)}\right\}$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\sin\{(x-b)-(x-a)\}}{\sin(x-a)\sin(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\sin(x-b)\cos(x-a)}{\sin(x-a)\sin(x-b)} - \frac{\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}\left[\cot(x-a) - \cot(x-b)\right]$$

Similarly, we can prove other two parts.

33. $$\tan\theta = \frac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}$$

$$\tan\theta = \frac{\dfrac{\sin\alpha - \cos\theta}{\cos\alpha}}{\dfrac{\sin\alpha + \cos\alpha}{\cos\alpha}}$$ [Dividing numerator and denominator by $\cos\alpha$]

$$\Rightarrow \quad \tan\theta = \frac{\tan\alpha - 1}{\tan\alpha + 1}$$

$$\Rightarrow \quad \tan\theta = \tan\left(\alpha - \frac{\pi}{4}\right) \Rightarrow \theta = \alpha - \frac{\pi}{4} \Rightarrow \alpha = \theta + \frac{\pi}{4}$$

$$\therefore \quad \sin\alpha + \cos\alpha = \sin\left(\theta + \frac{\pi}{4}\right) + \cos\left(\theta + \frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}(\sin\theta + \cos\theta) + \frac{1}{\sqrt{2}}(\cos\theta - \sin\theta) = \sqrt{2}\cos\theta$$

7.4 MAXIMUM AND MINIMUM VALUES OF TRIGONOMETRICAL EXPRESSIONS

We have learnt that for those values of θ for which trigonometrical functions are defined, we have

$$-1 \le \sin\theta \le 1,\ -1 \le \cos\theta \le 1,\ -\infty < \tan\theta < \infty\ \ \text{cosec}\,\theta \ge 1 \text{ or } \text{cosec}\,\theta \le -1,$$
$$\sec\theta \ge 1 \text{ or } \sec\theta \le -1 \text{ and, } -\infty < \cot\theta < \infty$$

In this section, we will find the maximum and minimum values of trigonometrical expressions of the form $a \cos\theta + b \sin\theta$ for varying values of θ.

Let $f(\theta) = a \cos\theta + b \sin\theta$ Further, let $a = r \sin\alpha$ and $b = r \cos\alpha$. Then,

$$a^2 + b^2 = r^2 \sin^2\alpha + r^2 \cos^2\alpha \text{ and, } \frac{a}{b} = \frac{r \sin\alpha}{r \cos\alpha}$$

$$\Rightarrow \quad a^2 + b^2 = r^2(\sin^2\alpha + \cos^2\alpha) \text{ and, } \frac{a}{b} = \tan\alpha$$

$$\Rightarrow \quad r = \sqrt{a^2 + b^2} \text{ and, } \tan\alpha = \frac{a}{b}$$

Substituting the values of a and b in $f(\theta)$, we obtain

$$f(\theta) = r \sin\alpha \cos\theta + r \cos\alpha \sin\theta = r(\sin\alpha \cos\theta + \cos\alpha \sin\theta) = r \sin(\alpha + \theta)$$

We know that

$$-1 \le \sin(\alpha + \theta) \le 1 \quad \text{for all } \theta$$

$$\Rightarrow \quad -r \le r \sin(\alpha + \theta) \le r \text{ for all } \theta \qquad \text{[Multiplying throughout by } r]$$

$$\Rightarrow \quad -\sqrt{a^2 + b^2} \le f(\theta) \le \sqrt{a^2 + b^2} \text{ for all } \theta$$

$$\Rightarrow \quad -\sqrt{a^2 + b^2} \le a \cos\theta + b \sin\theta \le \sqrt{a^2 + b^2} \text{ for all } \theta$$

Hence, maximum and minimum values of $a \cos\theta + b \sin\theta$ are $\sqrt{a^2 + b^2}$ and $-\sqrt{a^2 + b^2}$ respectively.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the maximum and minimum values of* $7 \cos\theta + 24 \sin\theta$.

SOLUTION We know that the maximum and minimum values of $a \cos\theta + b \sin\theta$ are $\sqrt{a^2 + b^2}$ and $-\sqrt{a^2 + b^2}$ respectively. Hence, the maximum and minimum values of $7 \cos\theta + 24 \sin\theta$ are $\sqrt{7^2 + 24^2} = 25$ and $-\sqrt{7^2 + 24^2} = -25$ respectively.

EXAMPLE 2 *Find the maximum and minimum values of the following expressions:*

(i) $3 \cos\theta + 5 \sin\left(\theta - \frac{\pi}{6}\right)$ (ii) $4 \sin\theta - 3 \cos\theta + 7$

SOLUTION (i) Let $f(\theta) = 3 \cos\theta + 5 \sin\left(\theta - \frac{\pi}{6}\right)$. Then,

$$f(\theta) = 3 \cos\theta + 5\left\{\sin\theta \cos\frac{\pi}{6} - \cos\theta \sin\frac{\pi}{6}\right\} = 3 \cos\theta + \frac{5\sqrt{3}}{2} \sin\theta - \frac{5}{2} \cos\theta$$

$$\Rightarrow \quad f(\theta) = \frac{1}{2} \cos\theta + \frac{5\sqrt{3}}{2} \sin\theta$$

$$\because \quad -\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{5\sqrt{3}}{2}\right)^2} \le \frac{1}{2} \cos\theta + \frac{5\sqrt{3}}{2} \sin\theta \le \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{5\sqrt{3}}{2}\right)^2} \text{ for all } \theta.$$

$$\therefore \quad -\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{5\sqrt{3}}{2}\right)^2} \le f(\theta) \le \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{5\sqrt{3}}{2}\right)^2} \text{ for all } \theta$$

$$\Rightarrow \quad -\sqrt{\frac{1}{4} + \frac{75}{4}} \le 3 \cos\theta + 5 \sin\left(\theta - \frac{\pi}{6}\right) \le \sqrt{\frac{1}{4} + \frac{75}{4}} \text{ for all } \theta$$

$$\Rightarrow \quad -\sqrt{19} \le 3 \cos\theta + 5 \sin\left(\theta - \frac{\pi}{6}\right) \le \sqrt{19} \quad \text{for all } \theta$$

Hence, $-\sqrt{19}$ and $\sqrt{19}$ are respectively the minimum and the maximum values of $3\cos\theta+5\sin\left(\theta-\frac{\pi}{6}\right)$.

(ii) Let $f(\theta) = 4\sin\theta-3\cos\theta+7$

We know that

$$-\sqrt{4^2+(-3)^2} \le 4\sin\theta-3\cos\theta \le \sqrt{4^2+(-3)^2} \text{ for all } \theta$$

$$\Rightarrow -5\le 4\sin\theta-3\cos\theta \le 5 \text{ for all } \theta$$

$$\Rightarrow -5+7\le 4\sin\theta-3\cos\theta+7 \le 5+7 \text{ for all } \theta$$

$$\Rightarrow 2\le f(\theta) \le 12 \text{ for all } \theta$$

Hence, minimum and maximum values of $4\sin\theta-3\cos\theta+7$ are 2 and 12 respectively.

EXAMPLE 3 *Prove that* $5\cos\theta+3\cos\left(\theta+\frac{\pi}{3}\right)+3$ *lies between* -4 *and* 10.

SOLUTION Let $f(\theta)=5\cos\theta+3\cos\left(\theta+\frac{\pi}{3}\right)+3$. Then,

$$f(\theta)=5\cos\theta+3\left(\cos\theta\cos\frac{\pi}{3}-\sin\theta\sin\frac{\pi}{3}\right)+3=5\cos\theta+\frac{3}{2}\cos\theta-\frac{3\sqrt{3}}{2}\sin\theta+3$$

$$\Rightarrow f(\theta)=\frac{13}{2}\cos\theta-\frac{3\sqrt{3}}{2}\sin\theta+3 \quad \text{...(i)}$$

$$\because -\sqrt{\left(\frac{13}{2}\right)^2+\left(\frac{3\sqrt{3}}{2}\right)^2}\le\frac{13}{2}\cos\theta-\frac{3\sqrt{3}}{2}\sin\theta \le\sqrt{\left(\frac{13}{2}\right)^2+\left(\frac{3\sqrt{3}}{2}\right)^2}$$

$$\Rightarrow -7\le\frac{13}{2}\cos\theta-\frac{3\sqrt{3}}{2}\sin\theta\le 7 \text{ for all } \theta$$

$$\Rightarrow -7+3\le\frac{13}{2}\cos\theta-\frac{3\sqrt{3}}{2}\sin\theta+3\le 7+3 \text{ for all } \theta$$

$$\Rightarrow -4\le 5\cos\theta+3\cos\left(\theta+\frac{\pi}{3}\right)+3\le 10 \text{ for all } \theta \quad \text{[Using (i)]}$$

EXAMPLE 4 *Find a and b such that the following inequality holds good for all* θ:

$$a\le 3\cos\theta+5\sin\left(\theta-\frac{\pi}{6}\right)\le b$$

SOLUTION Let $f(\theta)=3\cos\theta+5\sin\left(\theta-\frac{\pi}{6}\right)$. Then,

$$f(\theta)=3\cos\theta+5\left(\sin\theta\cos\frac{\pi}{6}-\cos\theta\sin\frac{\pi}{6}\right)=3\cos\theta+5\left(\frac{\sqrt{3}}{2}\sin\theta-\frac{1}{2}\cos\theta\right)$$

$$\Rightarrow f(\theta)=\frac{1}{2}\cos\theta+\frac{5\sqrt{3}}{2}\sin\theta$$

We have,

$$\because \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{5\sqrt{3}}{2}\right)^2}\le\frac{1}{2}\cos\theta+\frac{5\sqrt{3}}{2}\sin\theta\le\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{5\sqrt{3}}{2}\right)^2} \text{ for all } \theta$$

$$\Rightarrow -\sqrt{19}\le 3\cos\theta+5\sin\left(\theta-\frac{\pi}{6}\right)\le\sqrt{19} \text{ for all } \theta$$

Hence, $a=-\sqrt{19}$ and $b=\sqrt{19}$

7.5 TO EXPRESS $a\cos\theta+b\sin\theta$ IN THE FORM $r\sin(\theta\pm\alpha)$ OR $r\cos(\theta\pm\alpha)$

Sometimes we need to express trigonometrical expressions of the form $a\cos\theta+b\sin\theta$ in terms of sine or cosine of single term. We may use the following algorithm to do so.

ALGORITHM

STEP I Multiply and divide $f(\theta) = a\cos\theta + b\sin\theta$ by $\sqrt{a^2+b^2}$ to get

$$f(\theta) = \sqrt{a^2+b^2}\left\{\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta\right\}$$

STEP II In order to express $f(\theta)$ in terms of sine of some term, replace $\frac{a}{\sqrt{a^2+b^2}}$ i.e. coefficient of $\cos\theta$ by $\sin\alpha$ and $\frac{b}{\sqrt{a^2+b^2}}$ i.e. coefficient of $\sin\theta$ by $\cos\alpha$. This gives the following :

$$f(\theta) = \sqrt{a^2+b^2}\,\{\sin\alpha\cos\theta + \cos\alpha\sin\theta\} = \sqrt{a^2+b^2}\,\sin(\theta+\alpha)$$

To express $f(\theta)$ in terms of cosine of some term, replace coefficient of $\cos\theta$ i.e. $\frac{a}{\sqrt{a^2+b^2}}$ by $\cos\alpha$ and coefficient of $\sin\theta$ i.e. $\frac{b}{\sqrt{a^2+b^2}}$ by $\sin\alpha$. This gives the following:

$$f(\theta) = \sqrt{a^2+b^2}\,\{\cos\alpha\cos\theta + \sin\alpha\sin\theta\} = \sqrt{a^2+b^2}\,\cos(\theta-\alpha).$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Reduce* $\sqrt{3}\sin\theta + \cos\theta$ *as a single term consisting (i) sine only (ii) cosine only.*

SOLUTION Let $f(\theta) = \sqrt{3}\sin\theta + \cos\theta$. Then,

$$f(\theta) = \sqrt{3}\sin\theta + \cos\theta$$

$$\Rightarrow \quad f(\theta) = 2\left\{\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right\} \quad \left[\text{Multiplying and dividing by } \sqrt{(\sqrt{3})^2+1^2} \text{ i.e. by } 2\right]$$

$$\Rightarrow \quad f(\theta) = 2\left\{\sin\theta\cos\frac{\pi}{6} + \cos\theta\sin\frac{\pi}{6}\right\} \quad \left[\because \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} \text{ and } \frac{1}{2} = \sin\frac{\pi}{6}\right]$$

$$\Rightarrow \quad f(\theta) = 2\sin\left(\theta + \frac{\pi}{6}\right)$$

Again,

$$f(\theta) = 2\left\{\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right\} = 2\left\{\sin\frac{\pi}{3}\sin\theta + \cos\frac{\pi}{3}\cos\theta\right\}$$

$$\Rightarrow \quad f(\theta) = 2\left\{\cos\theta\cos\frac{\pi}{3} + \sin\theta\sin\frac{\pi}{3}\right\} = 2\cos\left(\theta - \frac{\pi}{3}\right).$$

EXAMPLE 2 *Express* $3\cos\theta - 4\sin\theta$ *as sines and cosines of a single expression.*

SOLUTION Let $f(\theta) = 3\cos\theta - 4\sin\theta$. Multiplying and dividing by $\sqrt{3^2+(-4)^2}$ i.e. by 5, we get

$$f(\theta) = \sqrt{3^2+(-4)^2}\left\{\frac{3}{\sqrt{3^2+(-4)^2}}\cos\theta - \frac{4}{\sqrt{3^2+(-4)^2}}\sin\theta\right\}$$

$$\Rightarrow \quad f(\theta) = 5\left(\frac{3}{5}\cos\theta - \frac{4}{5}\sin\theta\right)$$

$$\Rightarrow \quad f(\theta) = 5(\sin\alpha\cos\theta - \cos\alpha\sin\theta), \text{ where } \sin\alpha = \frac{3}{5} \text{ and } \cos\alpha = \frac{4}{5}$$

$$\Rightarrow \quad f(\theta) = 5\sin(\alpha - \theta), \text{ where } \tan\alpha = \frac{3}{4}$$

Again,

$$f(\theta) = 5\left(\frac{3}{5}\cos\theta - \frac{4}{5}\sin\theta\right)$$

$\Rightarrow$ $f(\theta) = 5(\cos\alpha\cos\theta - \sin\alpha\sin\theta)$, where $\cos\alpha = \frac{3}{5}$ and $\sin\alpha = \frac{4}{5}$

$\Rightarrow$ $f(\theta) = 5\cos(\alpha+\theta)$, where $\tan\alpha = \frac{4}{3}$

EXAMPLE 3 *Find the sign of the expression* $\sin 100° + \cos 100°$.

SOLUTION We have,

$$\sin 100° + \cos 100° = \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin 100° + \frac{1}{\sqrt{2}}\cos 100°\right)$$
$$= \sqrt{2}\,(\cos 45°\sin 100° + \sin 45°\cos 100°)$$
$$= \sqrt{2}\sin(100° + 45°)$$

$= \sqrt{2}\sin 145°$, which is a positive real number. [$\because$ $\sin 145°$ is positive]

EXERCISE 7.2

LEVEL-1

1. Find the maximum and minimum values of each of the following trigonometrical expressions:
   (i) $12\sin\theta - 5\cos\theta$ (ii) $12\cos\theta + 5\sin\theta + 4$
   (iii) $5\cos\theta + 3\sin\left(\frac{\pi}{6} - \theta\right) + 4$ (iv) $\sin\theta - \cos\theta + 1$
2. Reduce each of the following expressions to the sine and cosine of a single expression:
   (i) $\sqrt{3}\sin\theta - \cos\theta$ (ii) $\cos\theta - \sin\theta$ (iii) $24\cos\theta + 7\sin\theta$
3. Show that $\sin 100° - \sin 10°$ is positive.
4. Prove that $(2\sqrt{3}+3)\sin\theta + 2\sqrt{3}\cos\theta$ lies between $-(2\sqrt{3}+\sqrt{15})$ and $(2\sqrt{3}+\sqrt{15})$.

ANSWERS

1.

	Minimum	Maximum
(i)	-13	13
(ii)	-9	17
(iii)	-3	11
(iv)	$1-\sqrt{2}$	$1+\sqrt{2}$

2. (i) $2\sin\left(\theta - \frac{\pi}{6}\right), -2\cos\left(\frac{\pi}{3} + \theta\right)$ (ii) $\sqrt{2}\sin\left(\frac{\pi}{4} - \theta\right), \sqrt{2}\cos\left(\frac{\pi}{4} + \theta\right)$
   (iii) $25\sin(\alpha+\theta)$, where $\tan\alpha = \frac{24}{7}$, $25\cos(\theta-\alpha)$, where $\tan\alpha = \frac{7}{24}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If $\alpha+\beta-\gamma = \pi$, and $\sin^2\alpha + \sin^2\beta - \sin^2\gamma = \lambda\sin\alpha\sin\beta\cos\gamma$, then write the value of λ.
2. If $x\cos\theta = y\cos\left(\theta + \frac{2\pi}{3}\right) = z\cos\left(\theta + \frac{4\pi}{3}\right)$, then write the value of $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$.
3. Write the maximum and minimum values of $3\cos x + 4\sin x + 5$.
4. Write the maximum value of $12\sin\theta - 9\sin^2\theta$.
5. If $12\sin\theta - 9\sin^2\theta$ attains its maximum value at $\theta = \alpha$, then write the value of $\sin\alpha$.

6. Write the interval in which the values of $5 \cos \theta + 3 \cos\left(\theta + \frac{\pi}{3}\right) + 3$ lie.

7. If $\tan (A + B) = p$ and $\tan (A - B) = q$, then write the value of $\tan 2B$.

8. If $\frac{\cos (x - y)}{\cos (x + y)} = \frac{m}{n}$, then write the value of $\tan x \tan y$.

9. If $a = b \cos \frac{2\pi}{3} = c \cos \frac{4\pi}{3}$, then write the value of $ab + bc + ca$.

10. If $A + B = C$, then write the value of $\tan A \tan B \tan C$.

11. If $\sin \alpha - \sin \beta = a$ and $\cos \alpha + \cos \beta = b$, then write the value of $\cos (\alpha + \beta)$.

12. If $\tan \alpha = \frac{1}{1 + 2^{-x}}$ and $\tan \beta = \frac{1}{1 + 2^{x+1}}$, then write the value of $\alpha + \beta$ lying in the interval $(0, \pi/2)$.

ANSWERS

1. 2 2. 0 3. Maximum = 10, Minimum = 0 4. 4 5. $\frac{2}{3}$

6. $[-4, 10]$ 7. $\frac{p-q}{1+pq}$ 8. $\frac{m-n}{m+n}$ 9. 0 10. $\tan C - \tan A - \tan B$

11. $\frac{a^2 + b^2 - 2}{2}$ 12. $\frac{\pi}{4}$

MULTIPLE CHOICE QUATIONS (MCQs)

Mark the correct alternative in each of the following:

1. The value of $\sin^2 75° - \sin^2 15°$ is
   (a) $1/2$ (b) $\sqrt{3}/2$ (c) 1 (d) 0

2. If $A + B + C = 180°$, then $\sec A (\cos B \cos C - \sin B \sin C)$ is equal to
   (a) 0 (b) -1 (c) 1 (d) none of these

3. $\tan 20° + \tan 40° + \sqrt{3} \tan 20° \tan 40°$ is equal to
   (a) $\frac{\sqrt{3}}{4}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\sqrt{3}$ (d) 1

4. If $\tan A = \frac{a}{a+1}$ and $\tan B = \frac{1}{2a+1}$, then the value of $A + B$ is
   (a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

5. If $3 \sin \theta + 4 \cos \theta = 5$, then $4 \sin \theta - 3 \cos \theta =$
   (a) 0 (b) 5 (c) 1 (d) none of these

6. If in a ΔABC, $\tan A + \tan B + \tan C = 6$, then $\cot A \cot B \cot C =$
   (a) 6 (b) 1 (c) $1/6$ (d) none of these

7. $\tan 3A - \tan 2A - \tan A$ is equal to
   (a) $\tan 3A \tan 2A \tan A$ (b) $-\tan 3A \tan 2A \tan A$
   (c) $\tan A \tan 2A - \tan 2A \tan 3A - \tan 3A \tan A$
   (d) none of these.

8. If $A + B + C = 180°$, then $\frac{\tan A + \tan B + \tan C}{\tan A \ \tan B \ \tan C}$ is equal to
   (a) $\tan A \tan B \tan C$ (b) 0
   (c) 1 (d) none of these.

9. If $\cos P = \frac{1}{7}$ and $\cos Q = \frac{13}{14}$, where P and Q both are acute angles. Then, the value of $P - Q$ is
(a) 30° (b) 60° (c) 45° (d) 75°

10. If $\cot(\alpha + \beta) = 0$, then $\sin(\alpha + 2\beta)$ is equal to
(a) $\sin \alpha$ (b) $\cos 2\beta$ (c) $\cos \alpha$ (d) $\sin 2\alpha$

11. $\dfrac{\cos 10° + \sin 10°}{\cos 10° - \sin 10°}$ is equal to
(a) $\tan 55°$ (b) $\cot 55°$ (c) $-\tan 35°$ (d) $-\cot 35°$

12. The value of $\cos^2\left(\frac{\pi}{6} + \theta\right) - \sin^2\left(\frac{\pi}{6} - \theta\right)$ is
(a) $\frac{1}{2}\cos 2\theta$ (b) 0 (c) $-\frac{1}{2}\cos 2\theta$ (d) $\frac{1}{2}$

13. If $\tan \theta_1 \tan \theta_2 = k$, then $\dfrac{\cos(\theta_1 - \theta_2)}{\cos(\theta_1 + \theta_2)} =$
(a) $\frac{1+k}{1-k}$ (b) $\frac{1-k}{1+k}$ (c) $\frac{k+1}{k-1}$ (d) $\frac{k-1}{k+1}$

14. If $\sin(\pi \cos \theta) = \cos(\pi \sin \theta)$, then $\sin 2\theta =$
(a) $\pm\frac{3}{4}$ (b) $\pm\frac{4}{3}$ (c) $\pm\frac{1}{3}$ (d) none of these

15. If $\tan \theta = \frac{1}{2}$ and $\tan \phi = \frac{1}{3}$, then the value of $\theta + \phi$ is
(a) $\frac{\pi}{6}$ (b) π (c) 0 (d) $\frac{\pi}{4}$

16. The value of $\cos(36° - A)\cos(36° + A) + \cos(54° + A)\cos(54° - A)$ is
(a) $\sin 2A$ (b) $\cos 2A$ (c) $\cos 3A$ (d) $\sin 3A$

17. If $\tan(\pi/4 + \theta) + \tan(\pi/4 - \theta) = a$, then $\tan^2(\pi/4 + \theta) + \tan^2(\pi/4 - \theta) =$
(a) $a^2 + 1$ (b) $a^2 + 2$ (c) $a^2 - 2$ (d) none of these

18. If $\tan(A - B) = 1$, $\sec(A + B) = \frac{2}{\sqrt{3}}$, then the smallest positive value of B is
(a) $\frac{25\pi}{24}$ (b) $\frac{19\pi}{24}$ (c) $\frac{13\pi}{24}$ (d) $\frac{11\pi}{24}$

19. If $A - B = \pi/4$, then $(1 + \tan A)(1 - \tan B)$ is equal to
(a) 2 (b) 1 (c) 0 (d) 3

20. The maximum value of $\sin^2(120° + \theta) + \sin^2(120° - \theta)$ is
(a) 1/2 (b) 3/2 (c) 1/4 (d) 3/4

21. If $\cos(A - B) = \frac{3}{5}$ and $\tan A \tan B = 2$, then
(a) $\cos A \cos B = \frac{1}{5}$ (b) $\cos A \cos B = -\frac{1}{5}$
(c) $\sin A \sin B = -\frac{1}{5}$ (d) $\sin A \sin B = -\frac{1}{5}$

22. If $\tan 69° + \tan 66° - \tan 69° \tan 66° = 2k$, then $k =$
(a) -1 (b) 1/2 (c) $-1/2$ (d) none of these

23. If $\tan \alpha = \frac{x}{x+1}$ and $\tan \beta = \frac{1}{2x+1}$, then $\alpha + \beta$ is equal to
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{4}$

ANSWERS

1. (b)	2. (b)	3. (c)	4. (d)	5. (a)	6. (c)	7. (a)	8. (c)
9. (b)	10. (a)	11. (a)	12. (a)	13. (a)	14. (a)	15. (d)	16. (b)
17. (c)	18. (b)	19. (a)	20. (b)	21. (a)	22. (c)	23. (d)	

SUMMARY

1. (i) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

   (ii) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

   (iii) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

   (iv) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

   (v) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

   (vi) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

   (vii) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

   (viii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$

2. (i) $\sin(A+B+C) = \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C$
   $-\sin A\sin B\sin C$

   (ii) $\cos(A+B+C) = \cos A\cos B\cos C - \cos A\sin B\sin C - \sin A\cos B\sin C$
   $-\sin A\sin B\cos C$

   (iii) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$

3. (i) If $A + B = \pi$, then $\sin A = \sin B$, $\cos A = -\cos B$ and $\tan A = -\tan B$

   (ii) If $A + B = 2\pi$, then $\sin A = -\sin B$, $\cos A = \cos B$ and $\tan A = -\tan B$

CHAPTER 8

TRANSFORMATION FORMULAE

8.1 INTRODUCTION

In this chapter, we will establish two sets of transformation formulae: One to transform the products of two sines or two cosines or one sine and one cosine into the sum or difference of two sines or two cosines and the other to convert the sum or difference of two sines or two cosines in the product of two sines or two cosines or one sine and one cosine. These two sets of formulae are of fundamental importance and one should have thorough acquaintance with these formulae.

8.2 FORMULAE TO TRANSFORM THE PRODUCT INTO SUM OR DIFFERENCE

In the previous chapter we have derived the following formulae:

$$\sin A \cos B + \cos A \sin B = \sin (A + B) \quad \text{...(i)}$$
$$\sin A \cos B - \cos A \sin B = \sin (A - B) \quad \text{...(ii)}$$
$$\cos A \cos B - \sin A \sin B = \cos (A + B) \quad \text{...(iii)}$$
$$\cos A \cos B + \sin A \sin B = \cos (A - B) \quad \text{...(iv)}$$

Adding (i) and (ii), we obtain

$$2 \sin A \cos B = \sin (A + B) + \sin (A - B)$$

Subtracting (ii) from (i), we get

$$2 \cos A \sin B = \sin (A + B) - \sin (A - B)$$

Adding (iii) and (iv), we get

$$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$$

Subtracting (iii) from (iv), we get

$$2 \sin A \sin B = \cos (A - B) - \cos (A + B)$$

Thus, we obtain the following formulae :

(a) $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$ (b) $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$

(c) $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$ (d) $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$

These four formulae convert the product of two sines or two cosines or one sine and one cosine into the sum or difference of two sines or two cosines.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Convert each of the following products into the sum or difference of sines and cosines:*

(i) $2 \sin 5\theta \cos \theta$ (ii) $2 \cos 4\theta \cos 3\theta$ (iii) $2 \sin 3\theta \sin \theta$

(iv) $\sin 75° \cos 15°$ (v) $\cos 75° \cos 15°$

SOLUTION (i) Using $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$, we obtain

$$2 \sin 5\theta \cos \theta = \sin (5\theta + \theta) + \sin (5\theta - \theta) = \sin 6\theta + \sin 4\theta$$

(ii) Using $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$, we obtain

$$2 \cos 4\theta \cos 3\theta = \cos (4\theta + 3\theta) + \cos (4\theta - 3\theta) = \cos 7\theta + \cos \theta$$

(iii) Using $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$, we obtain

$$2 \sin 3\theta \sin \theta = \cos(3\theta - \theta) - \cos(3\theta + \theta) = \cos 2\theta - \cos 4\theta$$

(iv) $\sin 75° \cos 15° = \frac{1}{2}(2 \sin 75° \cos 15°)$

$$= \frac{1}{2}\{\sin(75° + 15°) + \sin(75° - 15°)\} = \frac{1}{2}(\sin 90° + \sin 60°)$$

(v) $\cos 75° \cos 15° = \frac{1}{2}(2 \cos 75° \cos 15°)$

$$= \frac{1}{2}\{\cos(75° + 15°) + \cos(75° - 15°)\} = \frac{1}{2}(\cos 90° + \cos 60°)$$

EXAMPLE 2 *Prove that:* $2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} = 0$ **[NCERT]**

SOLUTION We have,

$$\begin{aligned}
\text{LHS} &= 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} \\
&= \cos\left(\frac{9\pi}{13} + \frac{\pi}{13}\right) + \cos\left(\frac{9\pi}{13} - \frac{\pi}{13}\right) + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} \\
&= \cos \frac{10\pi}{13} + \cos \frac{8\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} \\
&= \cos\left(\pi - \frac{3\pi}{13}\right) + \cos\left(\pi - \frac{5\pi}{13}\right) + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} \\
&= -\cos \frac{3\pi}{13} - \cos \frac{5\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} = 0 = \text{RHS} \qquad [\because \cos(\pi - \theta) = -\cos\theta]
\end{aligned}$$

EXAMPLE 3 *Prove that:* $\cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16}$

SOLUTION We have,

$$\begin{aligned}
& \text{LHS} = \cos 20° \cos 40° \cos 60° \cos 80° \\
\Rightarrow\ & \text{LHS} = \cos 60° (\cos 20° \cos 40°) \cos 80° \\
\Rightarrow\ & \text{LHS} = \frac{1}{2} \times \frac{1}{2}(2 \cos 20° \cos 40°) \cos 80° \qquad \left[\because \cos 60° = \frac{1}{2}\right] \\
\Rightarrow\ & \text{LHS} = \frac{1}{4}\left[\{\cos(40° + 20°) + \cos(40° - 20°)\} \cos 80°\right] \\
& \qquad [\because 2 \cos A \cos B = \cos(A + B) + \cos(A - B)] \\
\Rightarrow\ & \text{LHS} = \frac{1}{4}\left\{(\cos 60° + \cos 20°) \cos 80°\right\} \\
\Rightarrow\ & \text{LHS} = \frac{1}{4}\left\{\left(\frac{1}{2} + \cos 20°\right) \cos 80°\right\} \\
\Rightarrow\ & \text{LHS} = \frac{1}{4}\left\{\frac{1}{2} \cos 80° + \cos 80° \cos 20°\right\} \\
\Rightarrow\ & \text{LHS} = \frac{1}{8}\left\{\cos 80° + 2 \cos 80° \cos 20°\right\} \\
\Rightarrow\ & \text{LHS} = \frac{1}{8}\left[\cos 80° + \left\{\cos(80° + 20°) + \cos(80° - 20°)\right\}\right] \\
& \qquad [\because 2 \cos A \cos B = \cos(A + B) + \cos(A - B)] \\
\Rightarrow\ & \text{LHS} = \frac{1}{8}\{\cos 80° + \cos 100° + \cos 60°\} \\
\Rightarrow\ & \text{LHS} = \frac{1}{8}\{\cos 80° + \cos(180° - 80°) + \cos 60°\}
\end{aligned}$$

$\Rightarrow \quad \text{LHS} = \frac{1}{8}\{\cos 80° - \cos 80° + \cos 60°\}$ $\left[\begin{array}{l}\because \cos(180° - \theta) = -\cos\theta \\ \therefore \cos(180° - 80°) = -\cos 80°\end{array}\right]$

$\Rightarrow \quad \text{LHS} = \frac{1}{8}\left\{\cos 80° - \cos 80° + \frac{1}{2}\right\} = \frac{1}{8} \times \frac{1}{2} = \frac{1}{16} = \text{RHS}$

EXAMPLE 4 *Prove that:* $\sin 10° \sin 30° \sin 50° \sin 70° = \frac{1}{16}$.

SOLUTION We have,

$\text{LHS} = \sin 30° (\sin 10° \sin 50°) \sin 70°$

$\Rightarrow \quad \text{LHS} = \frac{1}{2}(\sin 50° \sin 10°) \sin 70°$

$\Rightarrow \quad \text{LHS} = \frac{1}{2} \times \frac{1}{2}(2 \sin 50° \sin 10°) \sin 70°$

$\Rightarrow \quad \text{LHS} = \frac{1}{4}\left\{(2 \sin 50° \sin 10°) \sin 70°\right\}$

$\Rightarrow \quad \text{LHS} = \frac{1}{4}\left\{\{\cos(50° - 10°) - \cos(50° + 10°)\} \sin 70°\right\}$

$[\because 2 \sin A \sin B = \cos(A - B) - \cos(A + B)]$

$\Rightarrow \quad \text{LHS} = \frac{1}{4}\left[(\cos 40° - \cos 60°) \sin 70°\right]$

$\Rightarrow \quad \text{LHS} = \frac{1}{4}\left[\sin 70° \cos 40° - \sin 70° \cos 60°\right]$

$\Rightarrow \quad \text{LHS} = \frac{1}{4}\left[\sin 70° \cos 40° - \frac{1}{2}\sin 70°\right]$

$\Rightarrow \quad \text{LHS} = \frac{1}{8}\left[2 \sin 70° \cos 40° - \sin 70°\right]$

$\Rightarrow \quad \text{LHS} = \frac{1}{8}\left[\sin(70° + 40°) + \sin(70° - 40°) - \sin 70°\right]$

$[\because 2 \sin A \cos B = \sin(A + B) + \sin(A - B)]$

$\Rightarrow \quad \text{LHS} = \frac{1}{8}\left\{\sin 110° + \sin 30° - \sin 70°\right\}$

$\Rightarrow \quad \text{LHS} = \frac{1}{8}\left\{\sin(180° - 70°) + \sin 30° - \sin 70°\right\}$

$\Rightarrow \quad \text{LHS} = \frac{1}{8}\left\{\sin 70° + \frac{1}{2} - \sin 70°\right\}$ $\left[\because \sin(180 - \theta) = \sin\theta \therefore \sin(180° - 70°) = \sin 70°\right]$

$\Rightarrow \quad \text{LHS} = \frac{1}{8} \times \frac{1}{2} = \frac{1}{16} = \text{RHS}$

<u>ALITER</u> We have,

$\text{LHS} = \sin 10° \sin 30° \sin 50° \sin 70°$

$\Rightarrow \quad \text{LHS} = \sin(90° - 80°) \sin(90° - 60°) \sin(90° - 40°) \sin(90° - 20°)$

$\Rightarrow \quad \text{LHS} = \cos 80° \cos 60° \cos 40° \cos 20°$

$\Rightarrow \quad \text{LHS} = \cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16} = \text{RHS}$ [See Ex. 3]

EXAMPLE 5 *Prove that:* $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$

SOLUTION We have,

$\text{LHS} = \sin 20° \sin 40° \sin 60° \sin 80°$

$$\Rightarrow \quad LHS = \sin 60° (\sin 20° \sin 40°) \sin 80°$$

$$\Rightarrow \quad LHS = \frac{\sqrt{3}}{2} \times \frac{1}{2} (2 \sin 20° \sin 40°) \sin 80°$$

$$\Rightarrow \quad LHS = \frac{\sqrt{3}}{4} \left[\left\{ \cos (40° - 20°) - \cos (40° + 20°) \right\} \sin 80° \right]$$

$$[\because 2 \sin A \sin B = \cos (A - B) - \cos (A + B)]$$

$$\Rightarrow \quad LHS = \frac{\sqrt{3}}{4} \left[(\cos 20° - \cos 60°) \sin 80° \right]$$

$$\Rightarrow \quad LHS = \frac{\sqrt{3}}{4} \left[\left(\cos 20° - \frac{1}{2} \right) \sin 80° \right]$$

$$\Rightarrow \quad LHS = \frac{\sqrt{3}}{8} \left[2 \cos 20° \sin 80° - \sin 80° \right]$$

$$\Rightarrow \quad LHS = \frac{\sqrt{3}}{8} \left[\sin (80° + 20°) + \sin (80° - 20°) - \sin 80° \right]$$

$$[\because 2 \sin A \cos B = \sin (A + B) + \sin (A - B)]$$

$$\Rightarrow \quad LHS = \frac{\sqrt{3}}{8} \left\{ \sin 100° + \sin 60° - \sin 80° \right\}$$

$$\Rightarrow \quad LHS = \frac{\sqrt{3}}{8} \left\{ \sin (180° - 80°) + \frac{\sqrt{3}}{2} - \sin 80° \right\}$$

$$\Rightarrow \quad LHS = \frac{\sqrt{3}}{8} \left\{ \sin 80° + \frac{\sqrt{3}}{2} - \sin 80° \right\} \qquad [\because \sin (180° - 80°) = \sin 80°]$$

$$\Rightarrow \quad LHS = \frac{\sqrt{3}}{8} \times \frac{\sqrt{3}}{2} = \frac{3}{16} = RHS$$

EXAMPLE 6 *Prove that:* $4 \cos 12° \cos 48° \cos 72° = \cos 36°$

SOLUTION We have,

$$LHS = 4 \cos 12° \cos 48° \cos 72°$$

$$\Rightarrow \quad LHS = 2 (2 \cos 12° \cos 48°) \cos 72°$$

$$\Rightarrow \quad LHS = 2 (\cos 60° + \cos 36°) \cos 72°$$

$$\Rightarrow \quad LHS = 2 \cos 60° \cos 72° + 2 \cos 36° \cos 72°$$

$$\Rightarrow \quad LHS = \cos 72° + \cos 108° + \cos 36°$$

$$\Rightarrow \quad LHS = \cos 72° + \cos (180° - 72°) + \cos 36°$$

$$\Rightarrow \quad LHS = \cos 72° - \cos 72° + \cos 36°$$

$$\Rightarrow \quad LHS = \cos 36° = RHS$$

EXAMPLE 7 *Prove that:* $\tan 20° \tan 40° \tan 80° = \tan 60°$

SOLUTION We have,

$$LHS = \tan 20° \tan 40° \tan 80° = \frac{\sin 20° \sin 40° \sin 80°}{\cos 20° \cos 40° \cos 80°}$$

$$\Rightarrow \quad LHS = \frac{(2 \sin 20° \sin 40°) \sin 80°}{(2 \cos 20° \cos 40°) \cos 80°} = \frac{(\cos 20° - \cos 60°) \sin 80°}{(\cos 60° + \cos 20°) \cos 80°}$$

$$\Rightarrow \quad LHS = \frac{\sin 80° \cos 20° - (1/2) \sin 80°}{(1/2) \cos 80° + \cos 80° \cos 20°} = \frac{2 \sin 80° \cos 20° - \sin 80°}{\cos 80° + 2 \cos 80° \cos 20°}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin 100° + \sin 60° - \sin 80°}{\cos 80° + \cos 100° + \cos 60°} = \frac{\sin (180° - 80°) + \sin 60° - \sin 80°}{\cos 80° + \cos (180° - 80°) + \cos 60°}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin 80° + \sin 60° - \sin 80°}{\cos 80° - \cos 80° + \cos 60°} = \frac{\sin 60°}{\cos 60°} = \tan 60° = \text{RHS}$$

EXAMPLE 8 *Prove that:* $\sin A \sin (60° - A) \sin (60° + A) = \frac{1}{4} \sin 3A$

SOLUTION We have,

$$\begin{aligned}
\text{LHS} &= \sin A \sin (60° - A) \sin (60° + A) \\
&= \frac{1}{2} \sin A \left\{ 2 \sin (60° - A) \sin(60° + A) \right\} \\
&= \frac{1}{2} \sin A \left[\cos \left\{ (60° - A) - (60° + A) \right\} - \cos \left\{ (60° - A) + (60° + A) \right\} \right] \\
&= \frac{1}{2} \sin A \left\{ \cos (-2A) - \cos 120° \right\} \\
&= \frac{1}{2} \sin A \left\{ \cos 2A + \frac{1}{2} \right\} \\
&= \frac{1}{2} \sin A \cos 2A + \frac{1}{4} \sin A \\
&= \frac{1}{4} (2 \sin A \cos 2A) + \frac{1}{4} \sin A \\
&= \frac{1}{4} \left\{ \sin (A + 2A) + \sin (A - 2A) \right\} + \frac{1}{4} \sin A \\
&= \frac{1}{4} \left\{ \sin 3A + \sin (-A) \right\} + \frac{1}{4} \sin A \\
&= \frac{1}{4} \sin 3A - \frac{1}{4} \sin A + \frac{1}{4} \sin A = \frac{1}{4} \sin 3A = \text{RHS}
\end{aligned}$$

EXAMPLE 9 *Prove that:* $\cos A \cos (60° - A) \cos (60° + A) = \frac{1}{4} \cos 3A$.

SOLUTION We have,

$$\begin{aligned}
\text{LHS} &= \cos A \cos (60° - A) \cos (60° + A) \\
&= \frac{1}{2} \cos A \left\{ 2 \cos (60° - A) \cos (60° + A) \right\} \\
&= \frac{1}{2} \cos A \left[\cos \left\{ (60° - A) + (60° + A) \right\} + \cos \left\{ (60° - A) - (60° + A) \right\} \right] \\
&= \frac{1}{2} \cos A \left\{ \cos 120° + \cos (-2A) \right\} \\
&= \frac{1}{2} \cos A \left\{ -\frac{1}{2} + \cos 2A \right\} \\
&= -\frac{1}{4} \cos A + \frac{1}{2} \cos A \cos 2A \\
&= -\frac{1}{4} \cos A + \frac{1}{4} (2 \cos 2A \cos A) \\
&= -\frac{1}{4} \cos A + \frac{1}{4} \left\{ \cos (2A + A) + \cos (2A - A) \right\}
\end{aligned}$$

$$= -\frac{1}{4}\cos A + \frac{1}{4}(\cos 3A + \cos A)$$

$$= \frac{1}{4}\cos 3A = \text{RHS}.$$

LEVEL-2

EXAMPLE 10 *Prove that:* $4\sin\theta\sin\left(\frac{\pi}{3}+\theta\right)\sin\left(\frac{2\pi}{3}+\theta\right) = \sin 3\theta$

SOLUTION We have,

$$\text{LHS} = 4\sin\theta\sin\left(\frac{\pi}{3}+\theta\right)\sin\left(\frac{2\pi}{3}+\theta\right)$$

$$= 2\sin\theta\left\{2\sin\left(\frac{2\pi}{3}+\theta\right)\sin\left(\frac{\pi}{3}+\theta\right)\right\}$$

$$= 2\sin\theta\left[\cos\left\{\left(\frac{2\pi}{3}+\theta\right)-\left(\frac{\pi}{3}+\theta\right)\right\} - \cos\left\{\left(\frac{2\pi}{3}+\theta\right)+\left(\frac{\pi}{3}+\theta\right)\right\}\right]$$

$$= 2\sin\theta\left\{\cos\frac{\pi}{3} - \cos(\pi + 2\theta)\right\}$$

$$= 2\sin\theta\left\{\frac{1}{2} + \cos 2\theta\right\}$$

$$= \sin\theta + 2\sin\theta\cos 2\theta$$

$$= \sin\theta + \{\sin(\theta + 2\theta) + \sin(\theta - 2\theta)\}$$

$$= \sin\theta + \sin 3\theta + \sin(-\theta) = \sin\theta + \sin 3\theta - \sin\theta = \sin 3\theta = \text{RHS}$$

EXAMPLE 11 *Show that:* $\tan(60° + \theta)\tan(60° - \theta) = \frac{2\cos 2\theta + 1}{2\cos 2\theta - 1}$

SOLUTION We have

LH

$$\frac{60° - \theta)\}}{-(60° - \theta)\}}$$

$$= \frac{\cos 2\theta - \cos 120°}{\cos 120° + \cos 2\theta} = \frac{\cos 2\theta + \frac{1}{2}}{-\frac{1}{2} + \cos 2\theta} = \frac{2\cos 2\theta + 1}{2\cos 2\theta - 1} = \text{RHS}$$

EXAMPLE 12 *If* $\alpha + \beta = 90°$, *find the maximum and minimum values of* $\sin\alpha\sin\beta$.

SOLUTION Let $y = \sin\alpha\sin\beta$. Then,

$$y = \frac{1}{2}(2\sin\alpha\sin\beta)$$

$$\Rightarrow \quad y = \frac{1}{2}\{\cos(\alpha-\beta) - \cos(\alpha+\beta)\} = \frac{1}{2}\{\cos(\alpha-\beta) - \cos 90°\} = \frac{1}{2}\cos(\alpha-\beta)$$

We know that

$$-1 \le \cos(\alpha-\beta) \le 1$$

$$\Rightarrow \quad \frac{-1}{2} \le \frac{1}{2}\cos(\alpha-\beta) \le \frac{1}{2}$$

$$\Rightarrow \quad \frac{-1}{2} \le y \le \frac{1}{2}$$

$$\Rightarrow \quad \frac{-1}{2} \le \sin \alpha \sin \beta \le \frac{1}{2}$$

Hence, $-\frac{1}{2}$ and $\frac{1}{2}$ are respectively the minimum and maximum values of $\sin \alpha \sin \beta$.

EXERCISE 8.1

LEVEL-1

1. Express each of the following as the sum or difference of sines and cosines:
 (i) $2 \sin 3\theta \cos \theta$ (ii) $2 \cos 3\theta \sin 2\theta$
 (iii) $2 \sin 4\theta \sin 3\theta$ (iv) $2 \cos 7\theta \cos 3\theta$
2. Prove that:
 (i) $2 \sin \frac{5\pi}{12} \sin \frac{\pi}{12} = \frac{1}{2}$ (ii) $2 \cos \frac{5\pi}{12} \cos \frac{\pi}{12} = \frac{1}{2}$
 (iii) $2 \sin \frac{5\pi}{12} \cos \frac{\pi}{12} = \frac{\sqrt{3}+2}{2}$
3. Show that:
 (i) $\sin 50° \cos 85° = \frac{1-\sqrt{2} \sin 35°}{2\sqrt{2}}$ (ii) $\sin 25° \cos 115° = \frac{1}{2}(\sin 140° - 1)$
4. Prove that: $4 \cos \theta \cos\left(\frac{\pi}{3}+\theta\right) \cos\left(\frac{\pi}{3}-\theta\right) = \cos 3\theta$
5. Prove that :
 (i) $\cos 10° \cos 30° \cos 50° \cos 70° = \frac{3}{16}$ (ii) $\cos 40° \cos 80° \cos 160° = -\frac{1}{8}$
 (iii) $\sin 20° \sin 40° \sin 80° = \frac{\sqrt{3}}{8}$ (iv) $\cos 20° \cos 40° \cos 80° = \frac{1}{8}$
 (v) $\tan 20° \tan 40° \tan 60° \tan 80° = 3$ (vi) $\tan 20° \tan 30° \tan 40° \tan 80° = 1$
 (vii) $\sin 10° \sin 50° \sin 60° \sin 70° = \frac{\sqrt{3}}{16}$ (viii) $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$
6. Show that :
 (i) $\sin A \sin (B-C) + \sin B \sin (C-A) + \sin C \sin (A-B) = 0$
 (ii) $\sin (B-C) \cos (A-D) + \sin (C-A) \cos (B-D) + \sin (A-B) \cos (C-D) = 0$

LEVEL-2

7. Prove that : $\tan \theta \tan (60° - \theta) \tan (60° + \theta) = \tan 3\theta$.
8. If $\alpha + \beta = 90°$, show that the maximum value of $\cos \alpha \cos \beta$ is $\frac{1}{2}$.

ANSWERS

1. (i) $\sin 4\theta + \sin 2\theta$ (ii) $\sin 5\theta - \sin \theta$ (iii) $\cos \theta - \cos 7\theta$ (iv) $\cos 10\theta + \cos 4\theta$

HINTS TO SELECTED PROBLEM

2. (i) $\text{LHS} = \cos\left(\frac{5\pi}{12}-\frac{\pi}{12}\right) - \cos\left(\frac{5\pi}{12}+\frac{\pi}{12}\right) = \cos \frac{\pi}{3} - \cos \frac{\pi}{2} = \frac{1}{2} = \text{RHS}$
 (ii) $\text{LHS} = \cos\left(\frac{5\pi}{12}+\frac{\pi}{12}\right) + \cos\left(\frac{5\pi}{12}-\frac{\pi}{12}\right) = \cos \frac{\pi}{2} + \cos \frac{\pi}{3} = \frac{1}{2} = \text{RHS}$
 (iii) $\text{LHS} = \sin\left(\frac{5\pi}{12}+\frac{\pi}{12}\right) + \sin\left(\frac{5\pi}{12}-\frac{\pi}{12}\right) = \sin \frac{\pi}{2} + \sin \frac{\pi}{3} = 1 + \frac{\sqrt{3}}{2} = \text{RHS}$

8.3 FORMULAE TO TRANSFORM THE SUM OR DIFFERENCE INTO PRODUCT

In the previous section, we have used the following formulae:

$$\sin(A+B)+\sin(A-B) = 2\sin A\cos B, \qquad \sin(A+B)-\sin(A-B) = 2\cos A\sin B$$
$$\cos(A+B)+\cos(A-B) = 2\cos A\cos B \text{ and, } \cos(A-B)-\cos(A+B)=2\sin A\sin B.$$

Let $A+B=C$ and $A-B=D$. Then, $A=\dfrac{C+D}{2}$ and $B=\dfrac{C-D}{2}$.

Substituting the values of A, B, C and D in the above formulae, we get

$$\sin C+\sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \qquad ...(i)$$

$$\sin C-\sin D = 2\sin\left(\frac{C-D}{2}\right)\cos\left(\frac{C+D}{2}\right) \qquad ...(ii)$$

$$\cos C+\cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \qquad ...(iii)$$

$$\cos D-\cos C = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

or, $$\cos C-\cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \qquad ...(iv)$$

or, $$\cos C-\cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$$

These four formulae are used to convert the sum or difference of two sines or two cosines into the product of sines and cosines.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Express each of the following as a product:*

(i) $\sin 4\theta+\sin 2\theta$ (ii) $\sin 6\theta-\sin 2\theta$

(iii) $\cos 4\theta+\cos 8\theta$ (iv) $\cos 6\theta-\cos 8\theta$

SOLUTION (i) $\sin 4\theta+\sin 2\theta$

$$= 2\sin\left(\frac{4\theta+2\theta}{2}\right)\cos\left(\frac{4\theta-2\theta}{2}\right) \qquad \left[\because \sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\right]$$

$= 2\sin 3\theta\cos\theta$

(ii) $\sin 6\theta-\sin 2\theta$

$$= 2\sin\left(\frac{6\theta-2\theta}{2}\right)\cos\left(\frac{6\theta+2\theta}{2}\right) \qquad \left[\because \sin C-\sin D=2\sin\left(\frac{C-D}{2}\right)\cos\left(\frac{C+D}{2}\right)\right]$$

$= 2\sin 2\theta\cos 4\theta$

(iii) $\cos 4\theta+\cos 8\theta$

$$= 2\cos\left(\frac{8\theta+4\theta}{2}\right)\cos\left(\frac{8\theta-4\theta}{2}\right) \qquad \left[\because \cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$$

$= 2\cos 6\theta\cos 2\theta$

(iv) $\cos 6\theta-\cos 8\theta$

$$= 2\sin\left(\frac{6\theta+8\theta}{2}\right)\sin\left(\frac{8\theta-6\theta}{2}\right) \qquad \left[\because \cos C-\cos D=2\sin\frac{C+D}{2}\sin\frac{D-C}{2}\right]$$

$= 2\sin 7\theta\sin\theta$

EXAMPLE 2 *Prove that:* $\cos 18° - \sin 18° = \sqrt{2}\sin 27°$

SOLUTION We have,

$$\begin{aligned}\text{LHS} &= \cos 18° - \sin 18°\\ &= \cos 18° - \cos 72° \qquad [\because \sin 18° = \sin(90° - 72°) = \cos 72°]\\ &= 2\sin\left(\frac{18° + 72°}{2}\right)\sin\left(\frac{72° - 18°}{2}\right) = 2\sin 45°\sin 27° = \sqrt{2}\sin 27° = \text{RHS}\end{aligned}$$

EXAMPLE 3 *Prove that:*

(i) $\dfrac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A} = \tan A$ (ii) $\dfrac{\sin A + \sin 3A}{\cos A + \cos 3A} = \tan 2A$ **[NCERT]**

(iii) $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan\left(\dfrac{A+B}{2}\right)$ (iv) $\dfrac{\cos 7A + \cos 5A}{\sin 7A - \sin 5A} = \cot A$ **[NCERT]**

SOLUTION (i) LHS $= \dfrac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A}$

$$= \frac{2\sin\left(\frac{5A-3A}{2}\right)\cos\left(\frac{5A+3A}{2}\right)}{2\cos\left(\frac{5A+3A}{2}\right)\cos\left(\frac{5A-3A}{2}\right)} = \frac{2\sin A\cos 4A}{2\cos 4A\cos A} = \tan A = \text{RHS}$$

(ii) LHS $= \dfrac{\sin 3A + \sin A}{\cos 3A + \cos A}$

$$= \frac{2\sin\left(\frac{3A+A}{2}\right)\cos\left(\frac{3A-A}{2}\right)}{2\cos\left(\frac{3A+A}{2}\right)\cos\left(\frac{3A-A}{2}\right)} = \frac{\sin 2A\cos A}{\cos 2A\cos A} = \tan 2A = \text{RHS}$$

(iii) $$\text{LHS} = \frac{\sin A + \sin B}{\cos A + \cos B} = \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)} = \tan\left(\frac{A+B}{2}\right) = \text{RHS}$$

(iv) $$\text{LHS} = \frac{\cos 7A + \cos 5A}{\sin 7A - \sin 5A} = \frac{2\cos\left(\frac{7A+5A}{2}\right)\cos\left(\frac{7A-5A}{2}\right)}{2\sin\left(\frac{7A-5A}{2}\right)\cos\left(\frac{7A+5A}{2}\right)} = \frac{2\cos 6A\cos A}{2\sin A\cos 6A} = \cot A = \text{RHS}$$

EXAMPLE 4 *Prove that:*

(i) $\dfrac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = -\dfrac{\sin 2x}{\cos 10x}$ **[NCERT]** (ii) $\dfrac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x$ **[NCERT]**

(iii) $(\sin 3x + \sin x)\sin x + (\cos 3x - \cos x)\cos x = 0$ **[NCERT]**

SOLUTION (i) LHS $= \dfrac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x}$

$$= \frac{-2\sin\left(\frac{9x+5x}{2}\right)\sin\left(\frac{9x-5x}{2}\right)}{2\sin\left(\frac{17x-3x}{2}\right)\cos\left(\frac{17x+3x}{2}\right)} = \frac{-2\sin 7x\sin 2x}{2\sin 7x\cos 10x} = \frac{-\sin 2x}{\cos 10x} = \text{RHS}$$

(ii) LHS $= \dfrac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x}$

$$= \frac{2\sin\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)}{2\cos\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)} = \frac{2\sin 4x\cos x}{2\cos 4x\cos x} = \tan 4x = \text{RHS}$$

(iii) LHS $= (\sin 3x + \sin x)\sin x + (\cos 3x - \cos x)\cos x$

$$= \left\{2\sin\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right)\right\}\sin x + \left\{-2\sin\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)\right\}\cos x$$

$= 2\sin 2x\cos x\sin x - 2\sin 2x\sin x\cos x = 0 = \text{RHS}$

EXAMPLE 5 *Prove that:* $\cot 4x(\sin 5x + \sin 3x) = \cot x(\sin 5x - \sin 3x)$ **[NCERT]**

SOLUTION LHS $= \cot 4x(\sin 5x + \sin 3x)$

$$= \cot 4x \times 2\sin\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)$$

$$= \frac{\cos 4x}{\sin 4x} \times 2\sin 4x\cos x = 2\cos 4x\cos x \quad \text{...(i)}$$

RHS $= \cot x(\sin 5x - \sin 3x)$

$$= \cot x \times 2\sin\left(\frac{5x-3x}{2}\right)\cos\left(\frac{5x+3x}{2}\right)$$

$$= \frac{\cos x}{\sin x} \times 2\sin x\cos 4x = 2\cos 4x\cos x \quad \text{...(ii)}$$

From (i) and (ii), we obtain that LHS = RHS.

EXAMPLE 6 *Prove that:* $\sin x + \sin 3x + \sin 5x + \sin 7x = 4\cos x\cos 2x\sin 4x$ **[NCERT]**

SOLUTION LHS $= \sin x + \sin 3x + \sin 5x + \sin 7x$

$= (\sin 7x + \sin x) + (\sin 5x + \sin 3x)$

$$= 2\sin\left(\frac{7x+x}{2}\right)\cos\left(\frac{7x-x}{2}\right) + 2\sin\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)$$

$= 2\sin 4x\cos 3x + 2\sin 4x\cos x$

$= 2\sin 4x(\cos 3x + \cos x)$

$$= 2\sin 4x \times 2\cos\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right)$$

$= 2\sin 4x \times 2\cos 2x\cos x = 4\cos x\cos 2x\sin 4x = \text{RHS}$

EXAMPLE 7 *Prove that:* $1 + \cos 2x + \cos 4x + \cos 6x = 4\cos x\cos 2x\cos 3x$

SOLUTION LHS $= 1 + \cos 2x + \cos 4x + \cos 6x$

$= (\cos 0x + \cos 2x) + (\cos 4x + \cos 6x)$

$= 2\cos x\cos x + 2\cos 5x\cos x$

$= 2\cos x(\cos x + \cos 5x)$

$= 2\cos x(2\cos 3x\cos 2x) = 4\cos x\cos 2x\cos 3x = \text{RHS}$

EXAMPLE 8 *Prove that:*

(i) $(\sin 3A + \sin A)\sin A + (\cos 3A - \cos A)\cos A = 0$

(ii) $\cos 2\theta\cos\frac{\theta}{2} - \cos 3\theta\cos\frac{9\theta}{2} = \sin 5\theta\sin\frac{5\theta}{2}$ **[NCERT]**

(iii) $\sin\alpha + \sin\left(\alpha + \frac{2\pi}{3}\right) + \sin\left(\alpha + \frac{4\pi}{3}\right) = 0$

SOLUTION (i) We have,

LHS $= (\sin 3A + \sin A)\sin A + (\cos 3A - \cos A)\cos A$

$$= \left\{2\sin\left(\frac{3A+A}{2}\right)\cos\left(\frac{3A-A}{2}\right)\right\}\sin A + \left\{-2\sin\left(\frac{3A+A}{2}\right)\sin\left(\frac{3A-A}{2}\right)\right\}\cos A$$

$$= 2\sin 2A\cos A\sin A - 2\sin 2A\sin A\cos A = 0 = \text{RHS}$$

(ii) $\text{LHS} = \cos 2\theta\cos\frac{\theta}{2} - \cos 3\theta\cos\frac{9\theta}{2}$

$\Rightarrow \quad \text{LHS} = \frac{1}{2}\left\{2\cos 2\theta\cos\frac{\theta}{2} - 2\cos 3\theta\cos\frac{9\theta}{2}\right\}$

$\Rightarrow \quad \text{LHS} = \frac{1}{2}\left[\cos\left\{\left(2\theta+\frac{\theta}{2}\right) + \cos\left(2\theta-\frac{\theta}{2}\right)\right\} - \left\{\cos\left(3\theta+\frac{9\theta}{2}\right) + \cos\left(\frac{9\theta}{2}-3\theta\right)\right\}\right]$

[Using: $2\cos A\cos B = \cos(A+B) + \cos(A-B)$]

$\Rightarrow \quad \text{LHS} = \frac{1}{2}\left\{\cos\frac{5\theta}{2} + \cos\frac{3\theta}{2} - \cos\frac{15\theta}{2} - \cos\frac{3\theta}{2}\right\}$

$\Rightarrow \quad \text{LHS} = \frac{1}{2}\left\{\cos\frac{5\theta}{2} - \cos\frac{15\theta}{2}\right\}$

$\Rightarrow \quad \text{LHS} = \frac{1}{2}\left\{2\sin\left(\frac{\frac{5\theta}{2}+\frac{15\theta}{2}}{2}\right)\sin\left(\frac{\frac{15\theta}{2}-\frac{5\theta}{2}}{2}\right)\right\} \quad \left[\because \cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}\right]$

$\Rightarrow \quad \text{LHS} = \sin 5\theta\sin\frac{5\theta}{2} = \text{RHS}$

(iii) $\text{LHS} = \sin\alpha + \sin\left(\alpha+\frac{2\pi}{3}\right) + \sin\left(\alpha+\frac{4\pi}{3}\right)$

$\Rightarrow \quad \text{LHS} = \sin\alpha + \left[\sin\left(\alpha+\frac{2\pi}{3}\right) + \sin\left(\alpha+\frac{4\pi}{3}\right)\right]$

$\Rightarrow \quad \text{LHS} = \sin\alpha + \left[2\sin\left(\frac{\alpha+\frac{2\pi}{3}+\alpha+\frac{4\pi}{3}}{2}\right)\cos\left(\frac{\alpha+\frac{4\pi}{3}-\alpha-\frac{2\pi}{3}}{2}\right)\right]$

$\Rightarrow \quad \text{LHS} = \sin\alpha + 2\sin(\alpha+\pi)\cos\frac{\pi}{3} = \sin\alpha + 2(-\sin\alpha)\left(\frac{1}{2}\right) = \sin\alpha - \sin\alpha = 0 = \text{RHS}$

EXAMPLE 9 *Prove that:*

(i) $(\cos\alpha+\cos\beta)^2 + (\sin\alpha+\sin\beta)^2 = 4\cos^2\left(\frac{\alpha-\beta}{2}\right)$ **[NCERT]**

(ii) $(\cos\alpha-\cos\beta)^2 + (\sin\alpha-\sin\beta)^2 = 4\sin^2\left(\frac{\alpha-\beta}{2}\right)$ **[NCERT]**

(iii) $\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = 4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$

SOLUTION (i) $\text{LHS} = (\cos\alpha+\cos\beta)^2 + (\sin\alpha+\sin\beta)^2$

$\Rightarrow \quad \text{LHS} = \left\{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right\}^2 + \left\{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right\}^2$

$$\Rightarrow \quad \text{LHS} = 4\cos^2\left(\frac{\alpha+\beta}{2}\right)\cos^2\left(\frac{\alpha-\beta}{2}\right) + 4\sin^2\left(\frac{\alpha+\beta}{2}\right)\cos^2\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \quad \text{LHS} = 4\cos^2\left(\frac{\alpha-\beta}{2}\right)\left\{\cos^2\left(\frac{\alpha+\beta}{2}\right) + \sin^2\left(\frac{\alpha+\beta}{2}\right)\right\} = 4\cos^2\left(\frac{\alpha-\beta}{2}\right) = \text{RHS}$$

(ii) $\quad \text{LHS} = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$

$$\Rightarrow \quad \text{LHS} = \left\{-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right\}^2 + \left\{2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)\right\}^2$$

$$\Rightarrow \quad \text{LHS} = 4\sin^2\left(\frac{\alpha+\beta}{2}\right)\sin^2\left(\frac{\alpha-\beta}{2}\right) + 4\sin^2\left(\frac{\alpha-\beta}{2}\right)\cos^2\left(\frac{\alpha+\beta}{2}\right)$$

$$\Rightarrow \quad \text{LHS} = 4\sin^2\left(\frac{\alpha-\beta}{2}\right)\left\{\sin^2\left(\frac{\alpha+\beta}{2}\right) + \cos^2\left(\frac{\alpha+\beta}{2}\right)\right\}$$

$$\Rightarrow \quad \text{LHS} = 4\sin^2\left(\frac{\alpha-\beta}{2}\right) = \text{RHS} \qquad \left[\because \sin^2\left(\frac{\alpha+\beta}{2}\right) + \cos^2\left(\frac{\alpha+\beta}{2}\right) = 1\right]$$

(iii) $\quad \text{LHS} = \cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma)$

$\Rightarrow \quad \text{LHS} = (\cos\alpha + \cos\beta) + [\cos\gamma + \cos(\alpha+\beta+\gamma)]$

$$\Rightarrow \quad \text{LHS} = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + 2\cos\left(\frac{\alpha+\beta+\gamma+\gamma}{2}\right)\cos\left(\frac{\alpha+\beta+\gamma-\gamma}{2}\right)$$

$$\Rightarrow \quad \text{LHS} = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)$$

$$\Rightarrow \quad \text{LHS} = 2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)\right\}$$

$$\Rightarrow \quad \text{LHS} = 2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{2\cos\left(\frac{\frac{\alpha-\beta}{2}+\frac{\alpha+\beta+2\gamma}{2}}{2}\right)\cos\left(\frac{\frac{\alpha+\beta+2\gamma}{2}-\frac{\alpha-\beta}{2}}{2}\right)\right\}$$

$$\Rightarrow \quad \text{LHS} = 2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{2\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\right\}$$

$$\Rightarrow \quad \text{LHS} = 4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\gamma+\alpha}{2}\right) = \text{RHS}$$

EXAMPLE 10 *Prove that:* $\dfrac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x$ **[NCERT]**

SOLUTION $\text{LHS} = \dfrac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x}$

$$\Rightarrow \quad \text{LHS} = \frac{(\cos 4x + \cos 2x) + \cos 3x}{(\sin 4x + \sin 2x) + \sin 3x}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\cos\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right) + \cos 3x}{2\sin\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right) + \sin 3x} = \frac{2\cos 3x\cos x + \cos 3x}{2\sin 3x\cos x + \sin 3x}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos 3x(2\cos x + 1)}{\sin 3x(2\cos x + 1)} = \frac{\cos 3x}{\sin 3x} = \cot 3x = \text{RHS}$$

EXAMPLE 11 *Prove that:* $\dfrac{\sin A+\sin 3A+\sin 5A+\sin 7A}{\cos A+\cos 3A+\cos 5A+\cos 7A}=\tan 4A$

SOLUTION We have,

$$\text{LHS}=\frac{(\sin 7A+\sin A)+(\sin 5A+\sin 3A)}{(\cos 7A+\cos A)+(\cos 5A+\cos 3A)}$$

$$=\frac{2\sin\left(\frac{7A+A}{2}\right)\cos\left(\frac{7A-A}{2}\right)+2\sin\left(\frac{5A+3A}{2}\right)\cos\left(\frac{5A-3A}{2}\right)}{2\cos\left(\frac{7A+A}{2}\right)\cos\left(\frac{7A-A}{2}\right)+2\cos\left(\frac{5A+3A}{2}\right)\cos\left(\frac{5A-3A}{2}\right)}$$

$$=\frac{\sin 4A\cos 3A+\sin 4A\cos A}{\cos 4A\cos 3A+\cos 4A\cos A}=\frac{\sin 4A(\cos 3A+\cos A)}{\cos 4A(\cos 3A+\cos A)}=\tan 4A=\text{RHS}$$

EXAMPLE 12 *Prove that:* $\dfrac{\cos 8A\cos 5A-\cos 12A\cos 9A}{\sin 8A\cos 5A+\cos 12A\sin 9A}=\tan 4A$

SOLUTION We have,

$$\text{LHS}=\frac{2\cos 8A\cos 5A-2\cos 12A\cos 9A}{2\sin 8A\cos 5A+2\cos 12A\sin 9A}$$

$$\Rightarrow\quad \text{LHS}=\frac{\{\cos(8A+5A)+\cos(8A-5A)\}-\{\cos(12A+9A)+\cos(12A-9A)\}}{\{\sin(8A+5A)+\sin(8A-5A)\}+\{\sin(9A+12A)+\sin(9A-12A)\}}$$

$$\Rightarrow\quad \text{LHS}=\frac{\{\cos 13A+\cos 3A\}-\{\cos 21A+\cos 3A\}}{\{\sin 13A+\sin 3A\}+\{\sin 21A+\sin(-3A)\}}$$

$$\Rightarrow\quad \text{LHS}=\frac{(\cos 13A+\cos 3A)-(\cos 21A+\cos 3A)}{(\sin 13A+\sin 3A)+(\sin 21A-\sin 3A)}=\frac{\cos 13A-\cos 21A}{\sin 13A+\sin 21A}$$

$$\Rightarrow\quad \text{LHS}=\frac{2\sin\left(\frac{13A+21A}{2}\right)\sin\left(\frac{21A-13A}{2}\right)}{2\sin\left(\frac{3A+21A}{2}\right)\cos\left(\frac{21A-13A}{2}\right)}=\frac{\sin 17A\sin 4A}{\sin 17A\cos 4A}=\tan 4A=\text{RHS}$$

EXAMPLE 13 *Prove that:* $\dfrac{\cos 2A\cos 3A-\cos 2A\cos 7A+\cos A\cos 10A}{\sin 4A\sin 3A-\sin 2A\sin 5A+\sin 4A\sin 7A}=\cot 6A\cot 5A$

SOLUTION We have,

$$\text{LHS}=\frac{2\cos 3A\cos 2A-2\cos 7A\cos 2A+2\cos 10A\cos A}{2\sin 4A\sin 3A-2\sin 5A\sin 2A+2\sin 7A\sin 4A}$$

$$=\frac{(\cos 5A+\cos A)-(\cos 9A+\cos 5A)+(\cos 11A+\cos 9A)}{(\cos A-\cos 7A)-(\cos 3A-\cos 7A)+(\cos 3A-\cos 11A)}$$

$$=\frac{\cos A+\cos 11A}{\cos A-\cos 11A}=\frac{2\cos\left(\frac{11A+A}{2}\right)\cos\left(\frac{11A-A}{2}\right)}{2\sin\left(\frac{A+11A}{2}\right)\sin\left(\frac{11A-A}{2}\right)}$$

$$=\frac{\cos 6A\cos 5A}{\sin 6A\sin 5A}=\cot 6A\cot 5A=\text{RHS}$$

EXAMPLE 14 *Prove that:* $\dfrac{\sin(A-C)+2\sin A+\sin(A+C)}{\sin(B-C)+2\sin B+\sin(B+C)}=\dfrac{\sin A}{\sin B}$

SOLUTION We have,

$$\text{LHS} = \frac{\sin(A-C)+\sin(A+C)+2\sin A}{\sin(B-C)+\sin(B+C)+2\sin B}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\sin\left(\frac{A-C+A+C}{2}\right)\cos\left(\frac{A+C-A+C}{2}\right)+2\sin A}{2\sin\left(\frac{B+C+B-C}{2}\right)\cos\left(\frac{B+C-B+C}{2}\right)+2\sin B}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\sin A\cos C+2\sin A}{2\sin B\cos C+2\sin B} = \frac{2\sin A(\cos C+1)}{2\sin B(\cos C+1)} = \frac{\sin A}{\sin B} = \text{RHS}$$

LEVEL-2

EXAMPLE 15 *If* $\sin\theta = n\sin(\theta+2\alpha)$, *prove that* $\tan(\theta+\alpha) = \frac{1+n}{1-n}\tan\alpha$.

SOLUTION We have,

$$\sin\theta = n\sin(\theta+2\alpha)$$

$$\Rightarrow \quad \frac{\sin(\theta+2\alpha)}{\sin\theta} = \frac{1}{n}$$

$$\Rightarrow \quad \frac{\sin(\theta+2\alpha)+\sin\theta}{\sin(\theta+2\alpha)-\sin\theta} = \frac{1+n}{1-n} \quad \text{[Applying componendo-dividendo]}$$

$$\Rightarrow \quad \frac{2\sin(\theta+\alpha)\cos\alpha}{2\sin\alpha\cos(\theta+\alpha)} = \frac{1+n}{1-n}$$

$$\Rightarrow \quad \tan(\theta+\alpha) = \frac{1+n}{1-n}\tan\alpha$$

EXAMPLE 16 *Prove that:*

$$\left(\frac{\cos A+\cos B}{\sin A-\sin B}\right)^n + \left(\frac{\sin A+\sin B}{\cos A-\cos B}\right)^n = \begin{cases} 2\cot^n\left(\frac{A-B}{2}\right), & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd}\end{cases}$$

SOLUTION We have,

$$\text{LHS} = \left(\frac{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}{2\sin\frac{A-B}{2}\cos\frac{A+B}{2}}\right)^n + \left(\frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}}\right)^n$$

$$\Rightarrow \quad \text{LHS} = \left\{\cot\left(\frac{A-B}{2}\right)\right\}^n + \left\{-\cot\left(\frac{A-B}{2}\right)\right\}^n$$

$$\Rightarrow \quad \text{LHS} = \cot^n\left(\frac{A-B}{2}\right) + (-1)^n\cot^n\left(\frac{A-B}{2}\right)$$

$$\Rightarrow \quad \text{LHS} = \cot^n\left(\frac{A-B}{2}\right)^n\left\{1+(-1)^n\right\} = \begin{cases} 2\cot^n\left(\frac{A-B}{2}\right), & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd}\end{cases}$$

EXAMPLE 17 *If three angles A, B and C are in A.P., prove that:* $\cot B = \frac{\sin A - \sin C}{\cos C - \cos A}$.

SOLUTION We have,

$$\text{RHS} = \frac{2\sin\frac{A-C}{2}\cos\frac{A+C}{2}}{2\sin\frac{A+C}{2}\sin\frac{A-C}{2}} = \cot\left(\frac{A+C}{2}\right) = \cot B = \text{LHS} \qquad \left[\begin{array}{l}\because A, B, C \text{ are in A.P.}\\ \therefore 2B = A+C\end{array}\right]$$

EXAMPLE 18 *If* $\sin\theta + \sin\phi = \sqrt{3}(\cos\phi - \cos\theta)$, *prove that* $\sin 3\theta + \sin 3\phi = 0$ **[NCERT EXEMPLAR]**

SOLUTION We have,

$$\sin\theta + \sin\phi = \sqrt{3}(\cos\phi - \cos\theta)$$

$$\Rightarrow \quad 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} = 2\sqrt{3}\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$$

$$\Rightarrow \quad \left\{\cos\frac{\theta-\phi}{2} - \sqrt{3}\sin\frac{\theta-\phi}{2}\right\}\sin\left(\frac{\theta+\phi}{2}\right) = 0$$

$$\Rightarrow \quad \sin\left(\frac{\theta+\phi}{2}\right) = 0 \text{ or, } \cos\left(\frac{\theta-\phi}{2}\right) - \sqrt{3}\sin\left(\frac{\theta-\phi}{2}\right) = 0$$

$$\Rightarrow \quad \sin\left(\frac{\theta+\phi}{2}\right) = 0 \text{ or, } \tan\left(\frac{\theta-\phi}{2}\right) = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6}$$

$$\Rightarrow \quad \frac{\theta+\phi}{2} = 0 \quad \text{or,} \quad \frac{\theta-\phi}{2} = \frac{\pi}{6}$$

$$\Rightarrow \quad \theta = -\phi \quad \text{or,} \quad \theta - \phi = \frac{\pi}{3}$$

CASE I When $\theta = -\phi$: In this case, we have

$$\sin 3\theta + \sin 3\phi = \sin 3(-\phi) + \sin 3\phi = -\sin 3\phi + \sin 3\phi = 0$$

CASE II When $\theta - \phi = \frac{\pi}{3}$: In this case, we have

$$\theta - \phi = \frac{\pi}{3} \Rightarrow 3\theta - 3\phi = \pi \Rightarrow 3\theta = \pi + 3\phi$$

$$\therefore \quad \sin 3\theta + \sin 3\phi = \sin(\pi + 3\phi) + \sin 3\phi = -\sin 3\phi + \sin 3\phi = 0$$

EXAMPLE 19 *If* $\frac{\sin(\theta+\alpha)}{\cos(\theta-\alpha)} = \frac{1-m}{1+m}$, *prove that* $\tan\left(\frac{\pi}{4}-\theta\right)\tan\left(\frac{\pi}{4}-\alpha\right) = m$

SOLUTION We have,

$$\frac{\sin(\theta+\alpha)}{\cos(\theta-\alpha)} = \frac{1-m}{1+m}$$

$$\Rightarrow \quad \frac{\sin(\theta+\alpha)+\cos(\theta-\alpha)}{\sin(\theta+\alpha)-\cos(\theta-\alpha)} = \frac{2}{-2m} \qquad \text{[Using componendo-dividendo]}$$

$$\Rightarrow \quad \frac{\sin(\theta+\alpha) + \sin\left\{\frac{\pi}{2} - (\theta-\alpha)\right\}}{\sin(\theta+\alpha) - \sin\left\{\frac{\pi}{2} - (\theta-\alpha)\right\}} = -\frac{1}{m}$$

$$\Rightarrow \quad \frac{2\sin\left(\dfrac{\theta+\alpha+\frac{\pi}{2}-\theta+\alpha}{2}\right)\cos\left(\dfrac{\theta+\alpha-\frac{\pi}{2}+\theta-\alpha}{2}\right)}{2\sin\left(\dfrac{\theta+\alpha-\frac{\pi}{2}+\theta-\alpha}{2}\right)\cos\left(\dfrac{\theta+\alpha+\frac{\pi}{2}-\theta+\alpha}{2}\right)} = -\frac{1}{m}$$

$$\Rightarrow \quad \frac{\sin\left(\frac{\pi}{4}+\alpha\right)\cos\left(-\frac{\pi}{4}+\theta\right)}{\sin\left(-\frac{\pi}{4}+\theta\right)\cos\left(\frac{\pi}{4}+\alpha\right)} = -\frac{1}{m}$$

$$\Rightarrow \quad \frac{\sin\left(\frac{\pi}{4}+\alpha\right)}{\cos\left(\frac{\pi}{4}+\alpha\right)} \cdot \frac{\cos\left(\frac{\pi}{4}-\theta\right)}{\sin\left(\frac{\pi}{4}-\theta\right)} = \frac{1}{m}$$

$$\Rightarrow \quad \tan\left(\frac{\pi}{4}+\alpha\right)\cot\left(\frac{\pi}{4}-\theta\right) = \frac{1}{m}$$

$$\Rightarrow \quad m = \cot\left(\frac{\pi}{4}+\alpha\right)\tan\left(\frac{\pi}{4}-\theta\right)$$

$$\Rightarrow \quad m = \tan\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}+\alpha\right)\right\}\tan\left(\frac{\pi}{4}-\theta\right)$$

$$\Rightarrow \quad m = \tan\left(\frac{\pi}{4}-\alpha\right)\tan\left(\frac{\pi}{4}-\theta\right)$$

EXAMPLE 20 *If* $a\sin\theta = b\sin\left(\theta+\frac{2\pi}{3}\right) = c\sin\left(\theta+\frac{4\pi}{3}\right)$, *prove that* $ab+bc+ca=0$.

SOLUTION We have,

$$a\sin\theta = b\sin\left(\theta+\frac{2\pi}{3}\right) = c\sin\left(\theta+\frac{4\pi}{3}\right) = \lambda \text{ (say)}$$

$$\Rightarrow \quad \frac{\lambda}{a} = \sin\theta,\ \frac{\lambda}{b} = \sin\left(\theta+\frac{2\pi}{3}\right) \text{ and } \frac{\lambda}{c} = \sin\left(\theta+\frac{4\pi}{3}\right)$$

$$\Rightarrow \quad \frac{\lambda}{a}+\frac{\lambda}{b}+\frac{\lambda}{c} = \sin\theta+\sin\left(\theta+\frac{2\pi}{3}\right)+\sin\left(\theta+\frac{4\pi}{3}\right)$$

$$\Rightarrow \quad \frac{\lambda}{a}+\frac{\lambda}{b}+\frac{\lambda}{c} = \left\{\sin\left(\theta+\frac{4\pi}{3}\right)+\sin\theta\right\}+\sin\left(\theta+\frac{2\pi}{3}\right)$$

$$\Rightarrow \quad \frac{\lambda}{a}+\frac{\lambda}{b}+\frac{\lambda}{c} = 2\sin\left(\theta+\frac{2\pi}{3}\right)\cos\frac{2\pi}{3}+\sin\left(\theta+\frac{2\pi}{3}\right)$$

$$\Rightarrow \quad \frac{\lambda}{a}+\frac{\lambda}{b}+\frac{\lambda}{c} = -\sin\left(\theta+\frac{2\pi}{3}\right)+\sin\left(\theta+\frac{2\pi}{3}\right) = 0$$

$$\Rightarrow \quad \lambda\left(\frac{bc+ca+ab}{abc}\right) = 0 \Rightarrow ab+bc+ca = 0$$

EXAMPLE 21 *If* $\sin(y+z-x)$, $\sin(z+x-y)$, $\sin(x+y-z)$ *are in A.P., prove that* $\tan x$, $\tan y$, $\tan z$ *are also in A.P.*

SOLUTION It is given that $\sin(y+z-x)$, $\sin(z+x-y)$ and $\sin(x+y-z)$ are in A.P.

$\therefore \quad \sin(z+x-y)-\sin(y+z-x) = \sin(x+y-z)-\sin(z+x-y)$

$\Rightarrow \quad 2\sin(x-y)\cos z = 2\sin(y-z)\cos x$

$\Rightarrow \quad \sin(x-y)\cos z = \sin(y-z)\cos x$

$\Rightarrow \quad \sin x\cos y\cos z - \cos x\sin y\cos z = \sin y\cos z\cos x - \cos y\sin z\cos x$

$\Rightarrow \quad 2\sin y\cos x\cos z = \sin x\cos y\cos z + \cos x\cos y\sin z$

$\Rightarrow \quad 2\tan y = \tan x + \tan z$ [Dividing throughout by $\cos x\cos y\cos z$]

$\Rightarrow \quad \tan x$, $\tan y$, $\tan z$ are in A.P.

EXAMPLE 22 *If* $\dfrac{\tan(\theta+\alpha)}{a} = \dfrac{\tan(\theta+\beta)}{b} = \dfrac{\tan(\theta+\gamma)}{c}$, *prove that*

$$\frac{a+b}{a-b}\sin^2(\alpha-\beta) + \frac{b+c}{b-c}\sin^2(\beta-\gamma) + \frac{c+a}{c-a}\sin^2(\gamma-\alpha) = 0$$

SOLUTION We have,

$$\frac{\tan(\theta+\alpha)}{a} = \frac{\tan(\theta+\beta)}{b}$$

$$\Rightarrow \quad \frac{a}{b} = \frac{\tan(\theta+\alpha)}{\tan(\theta+\beta)}$$

$$\Rightarrow \quad \frac{a+b}{a-b} = \frac{\tan(\theta+\alpha)+\tan(\theta+\beta)}{\tan(\theta+\alpha)-\tan(\theta+\beta)} \qquad \text{[Applying Componendo-dividendo]}$$

$$\Rightarrow \quad \frac{a+b}{a-b} = \frac{\sin(2\theta+\alpha+\beta)}{\sin(\alpha-\beta)} \qquad \left[\because \frac{\tan A + \tan B}{\tan A - \tan B} = \frac{\sin(A+B)}{\sin(A-B)}\right]$$

$$\Rightarrow \quad \frac{a+b}{a-b}\sin^2(\alpha-\beta) = \sin(2\theta+\alpha+\beta)\sin(\alpha-\beta)$$

$$\Rightarrow \quad \frac{a+b}{a-b}\sin^2(\alpha-\beta) = \frac{1}{2}\left\{2\sin(2\theta+\alpha+\beta)\sin(\alpha-\beta)\right\}$$

$$\Rightarrow \quad \frac{a+b}{a-b}\sin^2(\alpha-\beta) = \frac{1}{2}\left\{\cos(2\theta+2\beta) - \cos(2\theta+2\alpha)\right\}$$

Similarly, we obtain

$$\frac{b+c}{b-c}\sin^2(\beta-\gamma) = \frac{1}{2}\left\{\cos(2\theta+2\gamma) - \cos(2\theta+2\beta)\right\}$$

and, $$\frac{c+a}{c-a}\sin^2(\gamma-\alpha) = \frac{1}{2}\left\{\cos(2\theta+2\alpha) - \cos(2\theta+2\gamma)\right\}$$

$$\therefore \quad \frac{a+b}{a-b}\sin^2(\alpha-\beta) + \frac{b+c}{b-c}\sin^2(\beta-\gamma) + \frac{c+a}{c-a}\sin^2(\gamma-\alpha)$$

$$= \frac{1}{2}\left\{\cos(2\theta+2\beta) - \cos(2\theta+2\alpha) + \cos(2\theta+2\gamma) - \cos(2\theta+2\beta) + \cos(2\theta+2\alpha) - \cos(2\theta+2\gamma)\right\}$$

$$= \frac{1}{2}\times 0 = 0$$

EXAMPLE 23 *Prove that :* $\dfrac{\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10}{\cos 5\theta + 5\cos 3\theta + 10\cos\theta} = 2\cos\theta$

SOLUTION We have,

$$\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$$

$$= (\cos 6\theta + \cos 4\theta) + (5\cos 4\theta + 5\cos 2\theta) + (10\cos 2\theta + 10)$$

$$= (\cos 6\theta + \cos 4\theta) + 5(\cos 4\theta + \cos 2\theta) + 10(\cos 2\theta + \cos 0\theta)$$

$$= 2\cos 5\theta\cos\theta + 5\times 2\cos 3\theta\cos\theta + 10\times 2\cos\theta\cos\theta$$

$$= 2\cos\theta\,(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$$

$$\therefore \quad \text{LHS} = \frac{\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10}{\cos 5\theta + 5\cos 3\theta + 10\cos\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\cos\theta\,(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)}{\cos 5\theta + 5\cos 3\theta + 10\cos\theta} = 2\cos\theta = \text{RHS}$$

EXERCISE 8.2

LEVEL-1

1. Express each of the following as the product of sines and cosines:
   (i) $\sin 12\theta + \sin 4\theta$ (ii) $\sin 5\theta - \sin \theta$ (iii) $\cos 12\theta + \cos 8\theta$
   (iv) $\cos 12\theta - \cos 4\theta$ (v) $\sin 2\theta + \cos 4\theta$
2. Prove that:
   (i) $\sin 38° + \sin 22° = \sin 82°$ (ii) $\cos 100° + \cos 20° = \cos 40°$
   (iii) $\sin 50° + \sin 10° = \cos 20°$ (iv) $\sin 23° + \sin 37° = \cos 7°$
   (v) $\sin 105° + \cos 105° = \cos 45°$ (vi) $\sin 40° + \sin 20° = \cos 10°$
3. Prove that:
   (i) $\cos 55° + \cos 65° + \cos 175° = 0$ (ii) $\sin 50° - \sin 70° + \sin 10° = 0$
   (iii) $\cos 80° + \cos 40° - \cos 20° = 0$ (iv) $\cos 20° + \cos 100° + \cos 140° = 0$
   (v) $\sin \frac{5\pi}{18} - \cos \frac{4\pi}{9} = \sqrt{3} \sin \frac{\pi}{9}$ (vi) $\cos \frac{\pi}{12} - \sin \frac{\pi}{12} = \frac{1}{\sqrt{2}}$
   (vii) $\sin 80° - \cos 70° = \cos 50°$ (viii) $\sin 51° + \cos 81° = \cos 21°$
4. Prove that:
   (i) $\cos\left(\frac{3\pi}{4} + x\right) - \cos\left(\frac{3\pi}{4} - x\right) = -\sqrt{2} \sin x$ [NCERT]
   (ii) $\cos\left(\frac{\pi}{4} + x\right) + \cos\left(\frac{\pi}{4} - x\right) = \sqrt{2} \cos x$ [NCERT]
5. Prove that :
   (i) $\sin 65° + \cos 65° = \sqrt{2} \cos 20°$ (ii) $\sin 47° + \cos 77° = \cos 17°$
6. Prove that :
   (i) $\cos 3A + \cos 5A + \cos 7A + \cos 15A = 4 \cos 4A \cos 5A \cos 6A$
   (ii) $\cos A + \cos 3A + \cos 5A + \cos 7A = 4 \cos A \cos 2A \cos 4A$
   (iii) $\sin A + \sin 2A + \sin 4A + \sin 5A = 4 \cos \frac{A}{2} \cos \frac{3A}{2} \sin 3A$
   (iv) $\sin 3A + \sin 2A - \sin A = 4 \sin A \cos \frac{A}{2} \cos \frac{3A}{2}$
   (v) $\cos 20° \cos 100° + \cos 100° \cos 140° - \cos 140° \cos 200° = -\frac{3}{4}$
   (vi) $\sin \frac{\theta}{2} \sin \frac{7\theta}{2} + \sin \frac{3\theta}{2} \sin \frac{11\theta}{2} = \sin 2\theta \sin 5\theta.$
   (vii) $\cos \theta \cos \frac{\theta}{2} - \cos 3\theta \cos \frac{9\theta}{2} = \sin 7\theta \sin 8\theta.$ [NCERT EXEMPLAR]
7. Prove that:
   (i) $\frac{\sin A + \sin 3A}{\cos A - \cos 3A} = \cot A$
   (ii) $\frac{\sin 9A - \sin 7A}{\cos 7A - \cos 9A} = \cot 8A$
   (iii) $\frac{\sin A - \sin B}{\cos A + \cos B} = \tan \frac{A - B}{2}$

(iv) $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \tan\left(\dfrac{A+B}{2}\right)\cot\left(\dfrac{A-B}{2}\right)$

(v) $\dfrac{\cos A + \cos B}{\cos B - \cos A} = \cot\left(\dfrac{A+B}{2}\right)\cot\left(\dfrac{A-B}{2}\right)$

8. Prove that:

(i) $\dfrac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} = \tan 3A$

(ii) $\dfrac{\cos 3A + 2\cos 5A + \cos 7A}{\cos A + 2\cos 3A + \cos 5A} = \dfrac{\cos 5A}{\cos 3A}$

(iii) $\dfrac{\cos 4A + \cos 3A + \cos 2A}{\sin 4A + \sin 3A + \sin 2A} = \cot 3A$ **[NCERT]**

(iv) $\dfrac{\sin 3A + \sin 5A + \sin 7A + \sin 9A}{\cos 3A + \cos 5A + \cos 7A + \cos 9A} = \tan 6A$

(v) $\dfrac{\sin 5A - \sin 7A + \sin 8A - \sin 4A}{\cos 4A + \cos 7A - \cos 5A - \cos 8A} = \cot 6A$

(vi) $\dfrac{\sin 5A \cos 2A - \sin 6A \cos A}{\sin A \sin 2A - \cos 2A \cos 3A} = \tan A$

(vii) $\dfrac{\sin 11A \sin A + \sin 7A \sin 3A}{\cos 11A \sin A + \cos 7A \sin 3A} = \tan 8A$

(viii) $\dfrac{\sin 3A \cos 4A - \sin A \cos 2A}{\sin 4A \sin A + \cos 6A \cos A} = \tan 2A$

(ix) $\dfrac{\sin A \sin 2A + \sin 3A \sin 6A}{\sin A \cos 2A + \sin 3A \cos 6A} = \tan 5A$

(x) $\dfrac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \dfrac{\sin 3A}{\sin 5A}$

(xi) $\dfrac{\sin(\theta+\phi) - 2\sin\theta + \sin(\theta-\phi)}{\cos(\theta+\phi) - 2\cos\theta + \cos(\theta-\phi)} = \tan\theta$

9. Prove that:

(i) $\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha+\beta+\gamma) = 4\sin\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\beta+\gamma}{2}\right)\sin\left(\dfrac{\gamma+\alpha}{2}\right)$

(ii) $\cos(A+B+C) + \cos(A-B+C) + \cos(A+B-C) + \cos(-A+B+C)$

$= 4\cos A\cos B\cos C$

LEVEL-2

10. If $\cos A + \cos B = \dfrac{1}{2}$ and $\sin A + \sin B = \dfrac{1}{4}$, prove that: $\tan\left(\dfrac{A+B}{2}\right) = \dfrac{1}{2}$

11. If $\operatorname{cosec} A + \sec A = \operatorname{cosec} B + \sec B$, prove that: $\tan A \tan B = \cot\dfrac{A+B}{2}$

12. If $\sin 2A = \lambda \sin 2B$, prove that: $\dfrac{\tan(A+B)}{\tan(A-B)} = \dfrac{\lambda+1}{\lambda-1}$

13. Prove that:

(i) $\dfrac{\cos(A+B+C)+\cos(-A+B+C)+\cos(A-B+C)+\cos(A+B-C)}{\sin(A+B+C)+\sin(-A+B+C)+\sin(A-B+C)-\sin(A+B-C)} = \cot C$

(ii) $\sin(B-C)\cos(A-D)+\sin(C-A)\cos(B-D)+\sin(A-B)\cos(C-D) = 0$

14. If $\dfrac{\cos(A-B)}{\cos(A+B)} + \dfrac{\cos(C+D)}{\cos(C-D)} = 0$, prove that $\tan A \tan B \tan C \tan D = -1$

15. If $\cos(\alpha+\beta)\sin(\gamma+\delta) = \cos(\alpha-\beta)\sin(\gamma-\delta)$, prove that $\cot\alpha\cot\beta\cot\gamma = \cot\delta$

16. If $y\sin\phi = x\sin(2\theta+\phi)$, prove that $(x+y)\cot(\theta+\phi) = (y-x)\cot\theta$

17. If $\cos(A+B)\sin(C-D) = \cos(A-B)\sin(C+D)$, prove that $\tan A\tan B\tan C + \tan D = 0$

18. If $x\cos\theta = y\cos\left(\theta+\dfrac{2\pi}{3}\right) = z\cos\left(\theta+\dfrac{4\pi}{3}\right)$, prove that $xy+yz+zx=0$.

[NCERT EXEMPLAR]

19. If $m\sin\theta = n\sin(\theta+2\alpha)$, prove that $\tan(\theta+\alpha)\cot\alpha = \dfrac{m+n}{m-n}$. [NCERT EXEMPLAR]

ANSWERS

1. (i) $2\sin 8\theta\cos 4\theta$ (ii) $2\sin 2\theta\cos 3\theta$ (iii) $2\cos 10\theta\cos 2\theta$

(iv) $-2\sin 8\theta\sin 4\theta$ (v) $2\cos\left(\dfrac{\pi}{4}+\theta\right)\cos\left(\dfrac{\pi}{4}-3\theta\right)$

HINTS TO NCERT & SELECTED PROBLEMS

4. (i) LHS $=\cos\left(\dfrac{3\pi}{4}+x\right)-\cos\left(\dfrac{3\pi}{4}-x\right)$

$$=-2\sin\left\{\frac{\left(\frac{3\pi}{4}+x\right)+\left(\frac{3\pi}{4}-x\right)}{2}\right\}\sin\left\{\frac{\left(\frac{3\pi}{4}+x\right)-\left(\frac{3\pi}{4}-x\right)}{2}\right\}$$

$$=-2\sin\frac{3\pi}{4}\sin x = -2\times\frac{1}{\sqrt{2}}\sin x = -\sqrt{2}\sin x = \text{RHS}$$

(ii) LHS $=\cos\left(\dfrac{\pi}{4}+x\right)+\cos\left(\dfrac{\pi}{4}-x\right)$

$$=2\cos\left\{\frac{\left(\frac{\pi}{4}+x\right)+\left(\frac{\pi}{4}-x\right)}{2}\right\}\cos\left\{\frac{\left(\frac{\pi}{4}+x\right)-\left(\frac{\pi}{4}-x\right)}{2}\right\}$$

$$=2\cos\frac{\pi}{4}\cos x = 2\times\frac{1}{\sqrt{2}}\cos x = \sqrt{2}\cos x = \text{RHS}$$

8. (iii) LHS $=\dfrac{\cos 4A+\cos 3A+\cos 2A}{\sin 4A+\sin 3A+\sin 2A}$

$$=\frac{(\cos 4A+\cos 2A)+\cos 3A}{(\sin 4A+\sin 2A)+\sin 3A}$$

$$=\frac{2\cos 3A\cos A+\cos 3A}{2\sin 3A\cos A+\sin 3A}=\frac{\cos 3A(2\cos A+1)}{\sin 3A(2\cos A+1)}=\cot 3A=\text{RHS}$$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = \lambda\cos^2\left(\frac{\alpha-\beta}{2}\right)$, write the value of λ.
2. Write the value of $\sin\frac{\pi}{12}\sin\frac{5\pi}{12}$.
3. If $\sin A + \sin B = \alpha$ and $\cos A + \cos B = \beta$, then write the value of $\tan\left(\frac{A+B}{2}\right)$.
4. If $\cos A = m\cos B$, then write the value of $\cot\frac{A+B}{2}\cot\frac{A-B}{2}$.
5. Write the value of the expression $\frac{1-4\sin 10^\circ\sin 70^\circ}{2\sin 10^\circ}$.
6. If $A + B = \frac{\pi}{3}$ and $\cos A + \cos B = 1$, then find the value of $\cos\frac{A-B}{2}$.
7. Write the value of $\sin 12^\circ\sin 48^\circ\sin 54^\circ$.
8. If $\sin 2A = \lambda\sin 2B$, then write the value of $\frac{\lambda+1}{\lambda-1}$.
9. Write the value of $\frac{\sin A + \sin 3A}{\cos A + \cos 3A}$.
10. If $\cos(A+B)\sin(C-D) = \cos(A-B)\sin(C+D)$, then write the value $\tan A\tan B\tan C$.

ANSWERS

1. 4 2. $\frac{1}{2}$ 3. $\frac{\alpha}{\beta}$ 4. $\frac{1+m}{1-m}$ 5. 1 6. $\frac{1}{\sqrt{3}}$

7. $\frac{1}{8}$ 8. $\frac{\tan(A+B)}{\tan(A-B)}$ 9. $\tan 2A$ 10. $-\tan D$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. $\cos 40^\circ + \cos 80^\circ + \cos 160^\circ + \cos 240^\circ =$

   (a) 0 (b) 1 (c) 1/2 (d) −1/2

2. $\sin 163^\circ\cos 347^\circ + \sin 73^\circ\sin 167^\circ =$

   (a) 0 (b) 1/2 (c) 1 (d) none of these

3. If $\sin 2\theta + \sin 2\phi = \frac{1}{2}$ and $\cos 2\theta + \cos 2\phi = \frac{3}{2}$, then $\cos^2(\theta-\phi) =$

   (a) 3/8 (b) 5/8 (c) 3/4 (d) 5/4

4. The value of $\cos 52^\circ + \cos 68^\circ + \cos 172^\circ$ is

   (a) 0 (b) 1 (c) 2 (d) 3/2

5. The value of $\sin 78^\circ - \sin 66^\circ - \sin 42^\circ + \sin 6^\circ$ is

   (a) 1/2 (b) −1/2 (c) −1 (d) none of these

6. If $\sin\alpha + \sin\beta = a$ and $\cos\alpha - \cos\beta = b$, then $\tan\frac{\alpha-\beta}{2} =$

(a) $-\frac{a}{b}$ (b) $-\frac{b}{a}$ (c) $\sqrt{a^2+b^2}$ (d) none of these

7. $\cos 35° + \cos 85° + \cos 155° =$

(a) 0 (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\cos 275°$

8. The value of $\sin 50° - \sin 70° + \sin 10°$ is equal to

(a) 1 (b) 0 (c) 1/2 (d) 2

9. $\sin 47° + \sin 61° - \sin 11° - \sin 25°$ is equal to

(a) $\sin 36°$ (b) $\cos 36°$ (c) $\sin 7°$ (d) $\cos 7°$

10. If $\cos A = m \cos B$, then $\cot \frac{A+B}{2} \cot \frac{B-A}{2} =$

(a) $\frac{m-1}{m+1}$ (b) $\frac{m+2}{m-2}$ (c) $\frac{m+1}{m-1}$ (d) none of these

11. If A, B, C are in A.P., then $\frac{\sin A - \sin C}{\cos C - \cos A} =$

(a) $\tan B$ (b) $\cot B$ (c) $\tan 2B$ (d) none of these

12. If $\sin (B + C - A)$, $\sin (C + A - B)$, $\sin (A + B - C)$ are in A.P., then $\cot A$, $\cot B$, $\cot C$ are in

(a) GP (b) HP (c) AP (d) none of these

13. If $\sin x + \sin y = \sqrt{3} (\cos y - \cos x)$, then $\sin 3x + \sin 3y =$

(a) $2 \sin 3x$ (b) 0 (c) 1 (d) none of these

14. If $\tan \alpha = \frac{x}{x+1}$ and $\tan \beta = \frac{1}{2x+1}$, then $\alpha + \beta$ is equal to

(a) $\pi/2$ (b) $\pi/3$ (c) $\pi/6$ (d) $\pi/4$

ANSWERS

1. (d) 2. (b) 3. (b) 4. (a) 5. (b) 6. (b) 7. (a) 8. (b)
9. (d) 10. (c) 11. (b) 12. (b) 13. (b) 14. (d)

SUMMARY

1. (i) $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$

(ii) $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$

(iii) $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$

(iv) $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$

2. (i) $\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$

(ii) $\sin C - \sin D = 2 \sin \frac{C-D}{2} \cos \frac{C+D}{2}$

(iii) $\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$

(iv) $\cos C - \cos D = -2 \sin \frac{C+D}{2} \sin \frac{C-D}{2}$

CHAPTER 9

TRIGONOMETRIC RATIOS OF MULTIPLE AND SUBMULTIPLE ANGLES

9.1 INTRODUCTION

In this chapter, we intend to express the trigonometric ratios of multiple angles $2A, 3A, 4A, \ldots$ etc. in terms of the trigonometric ratios of angle A and the trigonometric ratios of angle A in terms of the trigonometric ratios of sub-multiple angles $A/2, A/3, A/4 \ldots$ etc. These results will be used to find the trigonometric ratios of some important angles viz. $18°, 36°, 54°, 7\frac{1}{2}°, 11\frac{1}{4}°$ etc.

9.2 TRIGONOMETRIC RATIOS OF ANGLE 2A IN TERMS OF THAT OF ANGLE A

THEOREM 1 *For the values of angle A for which the two sides are meaningful prove that:*

(i) $\sin 2A = 2 \sin A \cos A$ (ii) $\cos 2A = \cos^2 A - \sin^2 A$

(iii) $\cos 2A = 2\cos^2 A - 1$ or, $1 + \cos 2A = 2\cos^2 A$

(iv) $\cos 2A = 1 - 2\sin^2 A$ or, $1 - \cos 2A = 2\sin^2 A$

(v) $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$ (vi) $\sin 2A = \dfrac{2\tan A}{1 + \tan^2 A}$

(vii) $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

PROOF (i) We know that

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\Rightarrow \sin 2A = \sin A \cos A + \cos A \sin A$ [Replacing B by A]

$\Rightarrow \sin 2A = 2 \sin A \cos A$

(ii) We know that

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\Rightarrow \cos 2A = \cos A \cos A - \sin A \sin A$ [Replacing B by A]

$\Rightarrow \cos 2A = \cos^2 A - \sin^2 A$

(iii) We have,

$\cos 2A = \cos^2 A - \sin^2 A$

$\Rightarrow \cos 2A = \cos^2 A - (1 - \cos^2 A)$

$\Rightarrow \cos 2A = 2\cos^2 A - 1$

Again, $\cos 2A = 2\cos^2 A - 1$

$\Rightarrow 1 + \cos 2A = 2\cos^2 A$

(iv) We have,

$\cos 2A = \cos^2 A - \sin^2 A$

$\Rightarrow \cos 2A = (1 - \sin^2 A) - \sin^2 A$

$$\Rightarrow \quad \cos 2A = 1 - 2\sin^2 A$$

Again, $\cos 2A = 1 - 2\sin^2 A$

$$\Rightarrow \quad 1 - \cos 2A = 2\sin^2 A$$

(v) We know that

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\Rightarrow \quad \tan 2A = \frac{\tan A + \tan A}{1 - \tan A \tan A} \qquad \text{[Replacing } B \text{ by } A]$$

$$\Rightarrow \quad \tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

(vi) We have,

$$\sin 2A = 2\sin A \cos A$$

$$\Rightarrow \quad \sin 2A = \frac{2\sin A \cos A}{1}$$

$$\Rightarrow \quad \sin 2A = \frac{2\sin A \cos A}{\sin^2 A + \cos^2 A} \qquad [\because \sin^2 A + \cos^2 A = 1]$$

$$\Rightarrow \quad \sin 2A = \frac{\frac{2\sin A \cos A}{\cos^2 A}}{\frac{\sin^2 A + \cos^2 A}{\cos^2 A}} \qquad \text{[Dividing Numerator and Denominator by } \cos^2 A]$$

$$\Rightarrow \quad \sin 2A = \frac{\frac{2\sin A}{\cos A}}{\frac{\sin^2 A}{\cos^2 A} + \frac{\cos^2 A}{\cos^2 A}}$$

$$\Rightarrow \quad \sin 2A = \frac{2\tan A}{1 + \tan^2 A}$$

(vii) We have,

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\Rightarrow \quad \cos 2A = \frac{\cos^2 A - \sin^2 A}{1}$$

$$\Rightarrow \quad \cos 2A = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} \qquad \text{[Replacing 1 by } \cos^2 A + \sin^2 A]$$

$$\Rightarrow \quad \cos 2A = \frac{\frac{\cos^2 A - \sin^2 A}{\cos^2 A}}{\frac{\cos^2 A + \sin^2 A}{\cos^2 A}} \qquad \text{[Dividing Numerator and Denominator by } \cos^2 A]$$

$$\Rightarrow \quad \cos 2A = \frac{\frac{\cos^2 A}{\cos^2 A} - \frac{\sin^2 A}{\cos^2 A}}{\frac{\cos^2 A}{\cos^2 A} + \frac{\sin^2 A}{\cos^2 A}}$$

$$\Rightarrow \quad \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

REMARK *In the above formulae it should be noted that the angle on the RHS is half of the angle on LHS.*

$$\therefore \quad \sin 60° = 2 \sin 30° \cos 30°, \cos 120° = \cos^2 60 - \sin^2 60°, \tan 60° = \frac{2 \tan 30°}{1 - \tan^2 30} \text{ etc.}$$

9.3 TRIGONOMETRIC RATIOS OF THE ANGLE A IN TERMS OF THAT OF ANGLE $\frac{A}{2}$

The relations in section 9.2 are true for all values of the angle A for which the two sides are meaningful. Replacing A by $A/2$ in the above relations, we obtain the following relations:

(i) $\sin A = 2 \sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)$ (ii) $\cos A = \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right)$

(iii) $\cos A = 2\cos^2\left(\frac{A}{2}\right) - 1$ or, $1 + \cos A = 2\cos^2\left(\frac{A}{2}\right)$

(iv) $\cos A = 1 - 2\sin^2\left(\frac{A}{2}\right)$ or, $1 - \cos A = 2\sin^2\left(\frac{A}{2}\right)$

(v) $\tan A = \dfrac{2\tan\left(\frac{A}{2}\right)}{1 - \tan^2\left(\frac{A}{2}\right)}$ (vi) $\sin A = \dfrac{2\tan\left(\frac{A}{2}\right)}{1 + \tan^2\left(\frac{A}{2}\right)}$

(vii) $\cos A = \dfrac{1 - \tan^2\left(\frac{A}{2}\right)}{1 + \tan^2\left(\frac{A}{2}\right)}$

9.4 TRIGONOMETRIC RATIOS OF THE ANGLE $A/2$ IN TERMS OF COS A

We have,

$$\cos A = 2\cos^2\frac{A}{2} - 1 \Rightarrow 2\cos^2\frac{A}{2} = 1 + \cos A \Rightarrow \cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$$

The sign on RHS depends upon the quadrant in which angle $\frac{A}{2}$ lies.

Also,

$$\cos A = 1 - 2\sin^2\frac{A}{2} \Rightarrow 2\sin^2\frac{A}{2} = 1 - \cos A \Rightarrow \sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$

The sign on RHS depends upon the quadrant in which angle $\frac{A}{2}$ lies.

Now, $$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \pm\frac{\sqrt{\frac{1-\cos A}{2}}}{\sqrt{\frac{1+\cos A}{2}}} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

The sign on RHS depends upon the quadrant in which angle $\frac{A}{2}$ lies.

REMARK *These relations are very useful to find the trigonometric ratios of the angles* $22\frac{1}{2}^\circ, 7\frac{1}{2}^\circ, 11\frac{1}{2}^\circ$ *etc.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE VALUES OF $\sin 2A, \cos 2A, \tan 2A$ ETC WHEN VALUES OF $\sin A$ OR $\cos A$ OR $\tan A$ ARE GIVEN**

EXAMPLE 1 *If* $\sin A = \frac{3}{5}$, *where* $0° < A < 90°$, *find the values of* $\sin 2A$, $\cos 2A$, $\tan 2A$ *and* $\sin 4A$.

SOLUTION We have, $\sin A = \frac{3}{5}$, where $0° < A < 90°$.

$$\therefore \quad \cos^2 A = 1 - \sin^2 A$$

$$\Rightarrow \quad \cos A = +\sqrt{1-\sin^2 A} = \sqrt{1-\frac{9}{25}} = \frac{4}{5} \qquad [\because \cos A > 0 \text{ for } 0 < A < 90°]$$

$$\therefore \quad \tan A = \frac{\sin A}{\cos A} = \frac{3}{4}$$

Now,

$$\sin 2A = 2\sin A\cos A = 2\times\frac{3}{5}\times\frac{4}{5} = \frac{24}{25}$$

$$\cos 2A = 1 - 2\sin^2 A = 1 - 2\times\left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2} = \frac{\frac{6}{4}}{1-\frac{9}{16}} = \frac{24}{7} \qquad \left[\because \tan A = \frac{3}{4}\right]$$

and, $$\sin 4A = 2\sin 2A\cos 2A = 2\times\frac{24}{25}\times\frac{7}{25} = \frac{336}{625} \qquad \left[\because \sin 2A = \frac{24}{25} \text{ and } \cos 2A = \frac{7}{25}\right]$$

EXAMPLE 2 *If* $\tan\alpha = \frac{1}{7}$, $\sin\beta = \frac{1}{\sqrt{10}}$. *Prove that* $\alpha + 2\beta = \frac{\pi}{4}$, *where* $0 < \alpha < \frac{\pi}{2}$ *and* $0 < \beta < \frac{\pi}{2}$.

SOLUTION In order to prove that $\alpha + 2\beta = \frac{\pi}{4}$, it is sufficient to prove that $\tan(\alpha+2\beta) = \tan\frac{\pi}{4} = 1$.

We have, $\sin\beta = \frac{1}{\sqrt{10}}$, where $0 < \beta < \frac{\pi}{2}$.

$$\therefore \quad \cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-\frac{1}{10}} = \frac{3}{\sqrt{10}} \text{ and, } \tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{\frac{1}{\sqrt{10}}}{\frac{3}{\sqrt{10}}} = \frac{1}{3}$$

In order to find the value of $\tan(\alpha+2\beta)$, we require the values of $\tan\alpha$ and $\tan 2\beta$. The values of $\tan\alpha$ is given. So, let us find $\tan 2\beta$.

Now, $$\tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta}$$

$$\Rightarrow \quad \tan 2\beta = \frac{2\times\frac{1}{3}}{1-\frac{1}{9}} = \frac{\frac{2}{3}}{\frac{8}{9}} = \frac{2\times 9}{3\times 8} = \frac{3}{4}$$

Thus, we have

$$\tan\alpha = \frac{1}{7} \text{ and } \tan 2\beta = \frac{3}{4}$$

$$\therefore \quad \tan(\alpha + 2\beta) = \frac{\tan\alpha + \tan 2\beta}{1 - \tan\alpha \tan 2\beta} = \frac{\frac{1}{7} + \frac{3}{4}}{1 - \frac{1}{7} \times \frac{3}{4}} = \frac{4 + 21}{28 - 3} = 1$$

$$\Rightarrow \quad \alpha + 2\beta = \frac{\pi}{4}$$

Type II **ON PROVING RESULTS AND IDENTITIES BASED UPON THE FOLLOWING FORMULAE:**

$$\sin 2\theta = 2\sin\theta\cos\theta,\ 1 + \cos 2\theta = 2\cos^2\theta,\ 1 - \cos 2\theta = 2\sin^2\theta$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}, 1 + \cos\theta = 2\cos^2\frac{\theta}{2}, 1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

EXAMPLE 3 *Prove that :*

(i) $\dfrac{\sin 2\theta}{1 + \cos 2\theta} = \tan\theta$ (ii) $\dfrac{\sin 2\theta}{1 - \cos 2\theta} = \cot\theta$

(iii) $\dfrac{1 + \sin 2\theta + \cos 2\theta}{1 + \sin 2\theta - \cos 2\theta} = \cot\theta$ (iv) $\dfrac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta} = \tan\dfrac{\theta}{2}$

(v) $\dfrac{\cos 2\theta}{1 + \sin 2\theta} = \tan\left(\dfrac{\pi}{4} - \theta\right)$ (vi) $\dfrac{\cos\theta}{1 + \sin\theta} = \tan\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)$

SOLUTION (i) We have, $\sin 2\theta = 2\sin\theta\cos\theta$ and $1 + \cos 2\theta = 2\cos^2\theta$.

$$\therefore \quad \text{LHS} = \frac{\sin 2\theta}{1 + \cos 2\theta} = \frac{2\sin\theta\cos\theta}{2\cos^2\theta} = \tan\theta = \text{RHS}$$

(ii) $$\text{LHS} = \frac{\sin 2\theta}{1 - \cos 2\theta} = \frac{2\sin\theta\cos\theta}{2\sin^2\theta} = \cot\theta = \text{RHS}$$

(iii) $$\text{LHS} = \frac{1 + \sin 2\theta + \cos 2\theta}{1 + \sin 2\theta - \cos 2\theta} = \frac{(1 + \cos 2\theta) + \sin 2\theta}{(1 - \cos 2\theta) + \sin 2\theta}$$

$$= \frac{2\cos^2\theta + 2\sin\theta\cos\theta}{2\sin^2\theta + 2\sin\theta\cos\theta} = \frac{2\cos\theta(\cos\theta + \sin\theta)}{2\sin\theta(\cos\theta + \sin\theta)} = \frac{\cos\theta}{\sin\theta} = \cot\theta = \text{RHS}$$

(iv) $$\text{LHS} = \frac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta} = \frac{(1 - \cos\theta) + \sin\theta}{(1 + \cos\theta) + \sin\theta}$$

$$= \frac{2\sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)} = \tan\frac{\theta}{2} = \text{RHS}$$

(v) $$\text{LHS} = \frac{\cos 2\theta}{1 + \sin 2\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin\left(\frac{\pi}{2} - 2\theta\right)}{1 + \cos\left(\frac{\pi}{2} - 2\theta\right)} \qquad \left[\because \cos A = \sin\left(\frac{\pi}{2} - A\right), \sin A = \cos\left(\frac{\pi}{2} - A\right)\right]$$

$$\Rightarrow \quad \text{LHS} = \frac{2\sin\left(\frac{\pi}{4} - \theta\right)\cos\left(\frac{\pi}{4} - \theta\right)}{2\cos^2\left(\frac{\pi}{4} - \theta\right)} \qquad \left[\because \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}\ \&\ 1 + \cos A = 2\cos^2\frac{A}{2}\right]$$

$\Rightarrow$ LHS $= \tan\left(\frac{\pi}{4}-\theta\right) =$ RHS

(vi) LHS $= \frac{\cos\theta}{1+\sin\theta} = \frac{\sin\left(\frac{\pi}{2}-\theta\right)}{1+\cos\left(\frac{\pi}{2}-\theta\right)} = \frac{2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)} = \tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right) =$ RHS

EXAMPLE 4 *Show that:* $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}} = 2\cos\theta,\ 0<\theta<\frac{\pi}{8}$

SOLUTION We have,

LHS $= \sqrt{2+\sqrt{2+\sqrt{2(1+\cos 8\theta)}}}$

$\Rightarrow$ LHS $= \sqrt{2+\sqrt{2+\sqrt{2(2\cos^2 4\theta)}}}$ $\left[\because 1+\cos 8\theta = 2\cos^2\frac{8\theta}{2} = 2\cos^2 4\theta\right]$

$\Rightarrow$ LHS $= \sqrt{2+\sqrt{2+\sqrt{(4\cos^2 4\theta)}}}$

$\Rightarrow$ LHS $= \sqrt{2+\sqrt{2+2\cos 4\theta}}$

$\Rightarrow$ LHS $= \sqrt{2+\sqrt{2(1+\cos 4\theta)}}$

$\Rightarrow$ LHS $= \sqrt{2+\sqrt{2(2\cos^2 2\theta)}}$ $[\because 1+\cos 4\theta = 2\cos^2 2\theta]$

$\Rightarrow$ LHS $= \sqrt{2+2\cos 2\theta} = \sqrt{2(1+\cos 2\theta)} = \sqrt{2(2\cos^2\theta)} = 2\cos\theta =$ RHS

EXAMPLE 5 *Prove that:* $\cos 4x = 1-8\sin^2 x\cos^2 x.$ **[NCERT]**

SOLUTION We have,

LHS $= \cos 4x = \cos 2(2x) = 1-2\sin^2 2x = 1-2(\sin 2x)^2$

$= 1-2(2\sin x\cos x)^2 = 1-8\sin^2 x\cos^2 x =$ RHS

EXAMPLE 6 *Prove that:* $(\cos A+\cos B)^2+(\sin A-\sin B)^2 = 4\cos^2\left(\frac{A+B}{2}\right).$

SOLUTION We have,

LHS $= (\cos A+\cos B)^2+(\sin A-\sin B)^2$

$\Rightarrow$ LHS $= (\cos^2 A+\cos^2 B+2\cos A\cos B)+(\sin^2 A+\sin^2 B-2\sin A\sin B)$

$\Rightarrow$ LHS $= (\cos^2 A+\sin^2 A)+(\cos^2 B+\sin^2 B)+2(\cos A\cos B-\sin A\sin B)$

$\Rightarrow$ LHS $= 1+1+2\cos(A+B)$

$\Rightarrow$ LHS $= 2+2\cos(A+B)$

$\Rightarrow$ LHS $= 2\{1+\cos(A+B)\}$

$\Rightarrow$ LHS $= 2\times 2\cos^2\left(\frac{A+B}{2}\right)$ $\left[\because 1+\cos\theta = 2\cos^2\frac{\theta}{2}\right]$

$\Rightarrow$ LHS $= 4\cos^2\left(\frac{A+B}{2}\right) =$ RHS.

EXAMPLE 7 *Prove that:* $\frac{\sec 8\theta-1}{\sec 4\theta-1} = \frac{\tan 8\theta}{\tan 2\theta}$ **[NCERT EXEMPLAR]**

SOLUTION We have,

$$\text{LHS} = \frac{\sec 8\theta - 1}{\sec 4\theta - 1}$$

$$\Rightarrow \text{LHS} = \frac{\frac{1}{\cos 8\theta} - 1}{\frac{1}{\cos 4\theta} - 1} = \frac{1 - \cos 8\theta}{\cos 8\theta} \times \frac{\cos 4\theta}{1 - \cos 4\theta}$$

$$\Rightarrow \text{LHS} = \frac{2\sin^2 4\theta}{\cos 8\theta} \times \frac{\cos 4\theta}{2\sin^2 2\theta} \qquad \left[\begin{array}{l} \because \quad 1 - \cos 8\theta = 2\sin^2 \frac{8\theta}{2} = 2\sin^2 4\theta \\ \text{and, } 1 - \cos 4\theta = 2\sin^2 \frac{4\theta}{2} = 2\sin^2 2\theta \end{array}\right]$$

$$\Rightarrow \text{LHS} = \frac{(2\sin 4\theta \cos 4\theta)}{\cos 8\theta} \times \frac{\sin 4\theta}{2\sin^2 2\theta}$$

$$\Rightarrow \text{LHS} = \left(\frac{2\sin 4\theta \cos 4\theta}{\cos 8\theta}\right) \times \left(\frac{2\sin 2\theta \cos 2\theta}{2\sin^2 2\theta}\right)$$

$$\Rightarrow \text{LHS} = \left(\frac{\sin 2(4\theta)}{\cos 8\theta}\right) \times \left(\frac{\cos 2\theta}{\sin 2\theta}\right) = \left(\frac{\sin 8\theta}{\cos 8\theta}\right) \times \left(\frac{\cos 2\theta}{\sin 2\theta}\right) = \tan 8\theta \cot 2\theta = \frac{\tan 8\theta}{\tan 2\theta} = \text{RHS}$$

EXAMPLE 8 *Prove that:* $\left(1 + \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)\left(1 + \cos\frac{5\pi}{8}\right)\left(1 + \cos\frac{7\pi}{8}\right) = \frac{1}{8}$

[NCERT EXEMPLAR]

SOLUTION We observe that

$$\cos\frac{7\pi}{8} = \cos\left(\pi - \frac{\pi}{8}\right) = -\cos\frac{\pi}{8} \quad \text{and,} \quad \cos\frac{5\pi}{8} = \cos\left(\pi - \frac{3\pi}{8}\right) = -\cos\frac{3\pi}{8}$$

$$\therefore \text{ LHS} = \left(1 + \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)\left(1 + \cos\frac{5\pi}{8}\right)\left(1 + \cos\frac{7\pi}{8}\right)$$

$$\Rightarrow \text{LHS} = \left(1 + \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)\left(1 - \cos\frac{3\pi}{8}\right)\left(1 - \cos\frac{\pi}{8}\right)$$

$$\Rightarrow \text{LHS} = \left\{\left(1 + \cos\frac{\pi}{8}\right)\left(1 - \cos\frac{\pi}{8}\right)\right\}\left\{\left(1 + \cos\frac{3\pi}{8}\right)\left(1 - \cos\frac{3\pi}{8}\right)\right\}$$

$$\Rightarrow \text{LHS} = \left(1 - \cos^2\frac{\pi}{8}\right)\left(1 - \cos^2\frac{3\pi}{8}\right)$$

$$\Rightarrow \text{LHS} = \sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}$$

$$\Rightarrow \text{LHS} = \frac{1}{4}\left(2\sin^2\frac{\pi}{8}\right)\left(2\sin^2\frac{3\pi}{8}\right)$$

$$\Rightarrow \text{LHS} = \frac{1}{4}\left\{\left(1 - \cos\frac{\pi}{4}\right)\left(1 - \cos\frac{3\pi}{4}\right)\right\} \qquad [\because 2\sin^2\theta = 1 - \cos 2\theta]$$

$$\Rightarrow \text{LHS} = \frac{1}{4}\left\{\left(1 - \frac{1}{\sqrt{2}}\right)\left(1 + \frac{1}{\sqrt{2}}\right)\right\} = \frac{1}{4}\left(1 - \frac{1}{2}\right) = \frac{1}{8} = \text{RHS}$$

EXAMPLE 9 *Prove that:*

(i) $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8} = \frac{3}{2}$ **[NCERT EXEMPLAR]**

(ii) $\sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{5\pi}{8} + \sin^4\frac{7\pi}{8} = \frac{3}{2}$

SOLUTION (i) We know that

$$\frac{7\pi}{8} = \pi - \frac{\pi}{8} \text{ and } \frac{5\pi}{8} = \pi - \frac{3\pi}{8}$$

$$\therefore \cos\frac{7\pi}{8} = -\cos\frac{\pi}{8} \text{ and } \cos\frac{5\pi}{8} = -\cos\frac{3\pi}{8} \qquad [\because \cos(\pi-\theta) = -\cos\theta]$$

$$\Rightarrow \cos^4\frac{7\pi}{8} = \cos^4\frac{\pi}{8} \text{ and } \cos^4\frac{5\pi}{8} = \cos^4\frac{3\pi}{8}$$

$$\therefore \text{LHS} = \cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8}$$

$$\Rightarrow \text{LHS} = \cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{\pi}{8}$$

$$\Rightarrow \text{LHS} = 2\cos^4\frac{\pi}{8} + 2\cos^4\frac{3\pi}{8}$$

$$\Rightarrow \text{LHS} = 2\left\{\left(\cos^2\frac{\pi}{8}\right)^2 + \left(\cos^2\frac{3\pi}{8}\right)^2\right\}$$

$$\Rightarrow \text{LHS} = 2\left[\left\{\frac{1+\cos\frac{\pi}{4}}{2}\right\}^2 + \left\{\frac{1+\cos\frac{3\pi}{4}}{2}\right\}^2\right] \qquad \left[\because \frac{1+\cos 2\theta}{2} = \cos^2\theta\right]$$

$$\Rightarrow \text{LHS} = \frac{2}{4}\left\{\left(1+\cos\frac{\pi}{4}\right)^2 + \left(1+\cos\frac{3\pi}{4}\right)^2\right\}$$

$$\Rightarrow \text{LHS} = \frac{2}{4}\left\{\left(1+\frac{1}{\sqrt{2}}\right)^2 + \left(1-\frac{1}{\sqrt{2}}\right)^2\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left\{\left(1+\frac{1}{\sqrt{2}}\right)^2 + \left(1-\frac{1}{\sqrt{2}}\right)^2\right\} = \frac{1}{2}\left\{\left(1+\frac{1}{2}+\sqrt{2}\right) + \left(1+\frac{1}{2}-\sqrt{2}\right)\right\} = \frac{3}{2} = \text{RHS}$$

(ii) We observe that

$$\sin^4\frac{7\pi}{8} = \sin^4\left(\pi - \frac{\pi}{8}\right) = \sin^4\frac{\pi}{8} \text{ and, } \sin^4\frac{5\pi}{8} = \sin^4\left(\pi - \frac{3\pi}{8}\right) = \sin^4\frac{3\pi}{8}$$

$$\therefore \text{LHS} = \sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{5\pi}{8} + \sin^4\frac{7\pi}{8}$$

$$\Rightarrow \text{LHS} = \sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{\pi}{8}$$

$$\Rightarrow \text{LHS} = 2\left\{\sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8}\right\}$$

$$\Rightarrow \text{LHS} = 2\left\{\left(\sin^2\frac{\pi}{8}\right)^2 + \left(\sin^2\frac{3\pi}{8}\right)^2\right\}$$

$$\Rightarrow \text{LHS} = 2\left[\left\{\frac{1-\cos\frac{\pi}{4}}{2}\right\}^2 + \left\{\frac{1-\cos\frac{3\pi}{4}}{2}\right\}^2\right] \qquad \left[\because \frac{1-\cos 2\theta}{2} = \sin^2\theta\right]$$

$$\Rightarrow \text{LHS} = \frac{2}{4}\left\{\left(1-\cos\frac{\pi}{4}\right)^2 + \left(1-\cos\frac{3\pi}{4}\right)^2\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left\{\left(1-\frac{1}{\sqrt{2}}\right)^2 + \left(1+\frac{1}{\sqrt{2}}\right)^2\right\} = \frac{1}{2}\left\{\left(1+\frac{1}{2}-\sqrt{2}\right) + \left(1+\frac{1}{2}+\sqrt{2}\right)\right\} = \frac{3}{2} = \text{RHS}$$

EXAMPLE 10 *Prove that:*

(i) $\cos^2 A + \cos^2\left(A + \frac{2\pi}{3}\right) + \cos^2\left(A - \frac{2\pi}{3}\right) = \frac{3}{2}$

(ii) $\cos^2 A + \cos^2\left(A + \frac{\pi}{3}\right) + \cos^2\left(A - \frac{\pi}{3}\right) = \frac{3}{2}$ **[NCERT EXEMPLAR]**

SOLUTION (i) We have,

$$\text{LHS} = \cos^2 A + \cos^2\left(A + \frac{2\pi}{3}\right) + \cos^2\left(A - \frac{2\pi}{3}\right)$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left\{2\cos^2 A + 2\cos^2\left(A + \frac{2\pi}{3}\right) + 2\cos^2\left(A - \frac{2\pi}{3}\right)\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left[1 + \cos 2A + \left\{1 + \cos 2\left(A + \frac{2\pi}{3}\right)\right\} + \left\{1 + \cos 2\left(A - \frac{2\pi}{3}\right)\right\}\right]$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left[1 + \cos 2A + 1 + \cos\left(2A + \frac{4\pi}{3}\right) + 1 + \cos\left(2A - \frac{4\pi}{3}\right)\right]$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left[3 + \cos 2A + \left\{\cos\left(2A + \frac{4\pi}{3}\right) + \cos\left(2A - \frac{4\pi}{3}\right)\right\}\right]$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left[3 + \cos 2A + 2\cos 2A \cos\frac{4\pi}{3}\right] \quad [\because \cos(A+B) + \cos(A-B) = 2\cos A\cos B]$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left[3 + \cos 2A + 2(\cos 2A)\left(-\frac{1}{2}\right)\right] = \frac{1}{2}(3 + \cos 2A - \cos 2A) = \frac{3}{2} = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \cos^2 A + \cos^2\left(A + \frac{\pi}{3}\right) + \cos^2\left(A - \frac{\pi}{3}\right)$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left\{2\cos^2 A + 2\cos^2\left(A + \frac{\pi}{3}\right) + 2\cos^2\left(A - \frac{\pi}{3}\right)\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left\{(1 + \cos 2A) + 1 + \cos\left(2A + \frac{2\pi}{3}\right) + 1 + \cos\left(2A - \frac{2\pi}{3}\right)\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left[3 + \cos 2A + \left\{\cos\left(2A + \frac{2\pi}{3}\right) + \cos\left(2A - \frac{2\pi}{3}\right)\right\}\right]$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left\{3 + \cos 2A + 2\cos 2A\cos\frac{2\pi}{3}\right\} \quad [\because \cos(A+B) + \cos(A-B) = 2\cos A\cos B]$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left\{3 + \cos 2A + 2(\cos 2A) \times -\frac{1}{2}\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\{3 + \cos 2A - \cos 2A\} = \frac{3}{2} = \text{RHS}$$

EXAMPLE 11 *Prove that:*

(i) $\dfrac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x} = \tan x$ **[NCERT]**

(ii) $\sin 2x + 2\sin 4x + \sin 6x = 4\cos^2 x \sin 4x$ **[NCERT]**

(iii) $\dfrac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = 2\sin x$ **[NCERT]**

SOLUTION (i) We have,

$$\text{LHS} = \frac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sin 5x + \sin x) - 2\sin 3x}{\cos 5x - \cos x}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\sin\left(\frac{5x+x}{2}\right)\cos\left(\frac{5x-x}{2}\right) - 2\sin 3x}{-2\sin\left(\frac{5x+x}{2}\right)\sin\left(\frac{5x-x}{2}\right)} = \frac{2\sin 3x\cos 2x - 2\sin 3x}{-2\sin 3x\sin 2x}$$

$$\Rightarrow \quad \text{LHS} = -\frac{2\sin 3x(1-\cos 2x)}{-2\sin 3x\sin 2x} = \frac{1-\cos 2x}{\sin 2x} = \frac{2\sin^2 x}{2\sin\cos x} = \tan x = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \sin 2x + 2\sin 4x + \sin 6x$$

$$\Rightarrow \quad \text{LHS} = (\sin 6x + \sin 2x) + 2\sin 4x$$

$$\Rightarrow \quad \text{LHS} = 2\sin\left(\frac{6x+2x}{2}\right)\cos\left(\frac{6x-2x}{2}\right) + 2\sin 4x$$

$$\Rightarrow \quad \text{LHS} = 2\sin 4x\cos 2x + 2\sin 4x$$

$$\Rightarrow \quad \text{LHS} = 2\sin 4x(\cos 2x + 1) = 2\sin 4x \times 2\cos^2 x = 4\cos^2 x\sin 4x = \text{RHS}$$

(iii) We have,

$$\text{LHS} = \frac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = \frac{2\sin\left(\frac{x-3x}{2}\right)\cos\left(\frac{x+3x}{2}\right)}{-(\cos^2 x - \sin^2 x)}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\sin(-x)\cos 2x}{-\cos 2x} = \frac{-2\sin x\cos 2x}{-\cos 2x} = 2\sin x = \text{RHS}$$

EXAMPLE 12 *Show that:* $2\sin^2\beta + 4\cos(\alpha+\beta)\sin\alpha\sin\beta + \cos 2(\alpha+\beta) = \cos 2\alpha$

[NCERT EXEMPLAR]

SOLUTION We have,

$$\text{LHS} = 2\sin^2\beta + 4\cos(\alpha+\beta)\sin\alpha\sin\beta + \cos 2(\alpha+\beta)$$

$$\Rightarrow \quad \text{LHS} = 2\sin^2\beta + 2\cos(\alpha+\beta)(2\sin\alpha\sin\beta) + \cos 2(\alpha+\beta)$$

$$\Rightarrow \quad \text{LHS} = 2\sin^2\beta + 2\cos(\alpha+\beta)\{\cos(\alpha-\beta) - \cos(\alpha+\beta)\} + \cos 2(\alpha+\beta)$$

$$\Rightarrow \quad \text{LHS} = 2\sin^2\beta + 2\cos(\alpha+\beta)\cos(\alpha-\beta) - 2\cos^2(\alpha+\beta) + \cos 2(\alpha+\beta)$$

$$\Rightarrow \quad \text{LHS} = 2\sin^2\beta + 2(\cos^2\alpha - \sin^2\beta) - 2\cos^2(\alpha+\beta) + 2\cos^2(\alpha+\beta) - 1$$

$$\Rightarrow \quad \text{LHS} = 2\sin^2\beta + 2\cos^2\alpha - 2\sin^2\beta - 2\cos^2(\alpha+\beta) + 2\cos^2(\alpha+\beta) - 1$$

$$\Rightarrow \quad \text{LHS} = 2\cos^2\alpha - 1 = \cos 2\alpha = \text{RHS}$$

EXAMPLE 13 *Show that* $\sqrt{3}\,\text{cosec}\,20^\circ - \sec 20^\circ = 4.$ **[NCERT EXEMPLAR]**

SOLUTION We have,

$$\text{LHS} = \sqrt{3}\,\text{cosec}\,20^\circ - \sec 20^\circ$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{3}}{\sin 20^\circ} - \frac{1}{\cos 20^\circ}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{3}\cos 20° - \sin 20°}{\sin 20° \cos 20°}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\left\{\frac{\sqrt{3}}{2}\cos 20° - \frac{1}{2}\sin 20°\right\}}{\sin 20° \cos 20°}$$

$$\Rightarrow \quad \text{LHS} = \frac{2(\sin 60° \cos 20° - \cos 60° \sin 20°)}{\sin 20° \cos 20°}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\sin(60° - 20°)}{\sin 20° \cos 20°} = \frac{2\sin 40°}{\sin 20° \cos 20°} = \frac{4\sin 40°}{2\sin 20° \cos 20°} = \frac{4\sin 40°}{\sin 40°} = 4 = \text{RHS}$$

EXAMPLE 14 *Prove that:* $\tan 4\theta = \dfrac{4\tan\theta(1-\tan^2\theta)}{1-6\tan^2\theta+\tan^4\theta}$ **[NCERT]**

SOLUTION We have,

$$\text{LHS} = \tan 4\theta = \tan\Big(2(2\theta)\Big)$$

$$\Rightarrow \quad \text{LHS} = \frac{2\tan 2\theta}{1-\tan^2 2\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right)}{1-\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right)^2}$$

$$\Rightarrow \quad \text{LHS} = \frac{4\tan\theta(1-\tan^2\theta)}{(1-\tan^2\theta)^2-4\tan^2\theta} = \frac{4\tan\theta(1-\tan^2\theta)}{1-6\tan^2\theta+\tan^4\theta} = \text{RHS}$$

Type III ON FINDING THE VALUES OF $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ AND $\tan\frac{x}{2}$ WHEN VALUES OF $\sin x$ OR $\cos x$ OR $\tan x$ ARE GIVEN

EXAMPLE 15 *If* $0 \leq x \leq 2\pi$, *find* $\sin\frac{x}{2}$, $\cos\frac{x}{2}$, *and* $\tan\frac{x}{2}$, *when:*

(i) $\tan x = -\frac{4}{3}$, *x lies in quadrant II* (ii) $\cos x = -\frac{1}{3}$, *x lies in quadrant III*

(iii) $\sin x = -\frac{1}{2}$, *x lies in quadrant IV.*

SOLUTION (i) It is given that x lies in IInd quadrant in which $\cos x$ is negative.

$$\therefore \quad \cos x = -\frac{1}{\sqrt{1+\tan^2 x}} = -\frac{1}{\sqrt{1+16/9}} = -\frac{3}{5}$$

It is given that x lies in IInd quadrant.

i.e. $\frac{\pi}{2} < x < \pi$

$\Rightarrow \quad \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$

$\Rightarrow \quad \frac{x}{2}$ lies in first quadrant $\Rightarrow \sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$ are positive

$\therefore \quad \cos\dfrac{x}{2} = \pm\sqrt{\dfrac{1+\cos x}{2}} \Rightarrow \cos\dfrac{x}{2} = \sqrt{\dfrac{1+\cos x}{2}} = \sqrt{\dfrac{1-3/5}{2}} = \dfrac{1}{\sqrt{5}}$

$\sin\dfrac{x}{2} = \pm\sqrt{\dfrac{1-\cos x}{2}} \Rightarrow \sin\dfrac{x}{2} = \sqrt{\dfrac{1-\cos x}{2}} = \sqrt{\dfrac{1+3/5}{2}} = \dfrac{2}{\sqrt{5}}$

and, $\tan\dfrac{x}{2} = \pm\sqrt{\dfrac{1-\cos x}{1+\cos x}} \Rightarrow \tan\dfrac{x}{2} = \sqrt{\dfrac{1-\cos x}{1+\cos x}} = \sqrt{\dfrac{1+3/5}{1-3/5}} = 2$

(ii) It is given that x lies in the III quadrant.

i.e. $\pi < x < \dfrac{3\pi}{2}$

$\Rightarrow \quad \dfrac{\pi}{2} < \dfrac{x}{2} < \dfrac{3\pi}{4}$

$\Rightarrow \quad \dfrac{x}{2}$ lies in IInd quadrant $\Rightarrow \cos\dfrac{x}{2} < 0$, $\sin\dfrac{x}{2} > 0$ and $\tan\dfrac{x}{2} < 0$

$\therefore \quad \cos\dfrac{x}{2} = \pm\sqrt{\dfrac{1+\cos x}{2}}$

$\Rightarrow \quad \cos\dfrac{x}{2} = -\sqrt{\dfrac{1+\cos x}{2}}$ $\quad \left[\because \cos\dfrac{x}{2} \text{ is} -\text{ve}\right]$

$\Rightarrow \quad \cos\dfrac{x}{2} = -\sqrt{\dfrac{1-1/3}{2}} = -\dfrac{1}{\sqrt{3}}$ $\quad \left[\because \cos x = -\dfrac{1}{3}\right]$

and, $\sin\dfrac{x}{2} = \pm\sqrt{\dfrac{1-\cos x}{2}}$

$\Rightarrow \quad \sin\dfrac{x}{2} = \sqrt{\dfrac{1-\cos x}{2}}$ $\quad \left[\because \sin\dfrac{x}{2} > 0\right]$

$\Rightarrow \quad \sin\dfrac{x}{2} = \sqrt{\dfrac{1+1/3}{2}} = \sqrt{\dfrac{2}{3}}$ $\quad \left[\because \cos x = \dfrac{1}{3}\right]$

and, $\tan\dfrac{x}{2} = \dfrac{\sin x/2}{\cos x/2} = \sqrt{\dfrac{2}{3}} \times -\sqrt{3} = -\sqrt{2}$

(iii) It is given that x lies in IVth quadrant in which $\cos x$ is positive.

$\therefore \quad \sin x = -\dfrac{1}{2} \Rightarrow \cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\dfrac{1}{4}} = \dfrac{\sqrt{3}}{2}$

Now,

x lies in IVth quadrant

$\Rightarrow \quad \dfrac{3\pi}{2} < x < 2\pi$

$\Rightarrow \quad \dfrac{3\pi}{4} < \dfrac{x}{2} < \pi$

$\Rightarrow \quad \dfrac{x}{2}$ lies in IInd quadrant $\Rightarrow \cos\dfrac{x}{2} < 0, \sin\dfrac{x}{2} > 0$ and $\tan\dfrac{x}{2} < 0$

$\therefore \quad \cos\dfrac{x}{2} = \pm\sqrt{\dfrac{1+\cos x}{2}}$

$\Rightarrow \quad \cos\dfrac{x}{2} = -\sqrt{\dfrac{1+\cos x}{2}}$ $\quad \left[\because \cos\dfrac{x}{2} < 0\right]$

$\Rightarrow \quad \cos\dfrac{x}{2} = -\sqrt{\dfrac{1+\sqrt{3}/2}{2}} = -\dfrac{\sqrt{2+\sqrt{3}}}{2}$ $\quad \left[\because \cos x = \dfrac{\sqrt{3}}{2}\right]$

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

$\Rightarrow \quad \sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}}$ $\qquad \left[\because \sin\frac{x}{2} > 0\right]$

$\Rightarrow \quad \sin\frac{x}{2} = \sqrt{\frac{1-\sqrt{3}/2}{2}} = \frac{\sqrt{2-\sqrt{3}}}{2}$ $\qquad \left[\because \cos x = \frac{\sqrt{3}}{2}\right]$

and, $\quad \tan\frac{x}{2} = \frac{\sin(x/2)}{\cos(x/2)} = \frac{\sqrt{2-\sqrt{3}}}{2} \times \frac{-2}{\sqrt{2+\sqrt{3}}} = -\sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}$

EXAMPLE 16 *If* $\tan x = \frac{3}{4}$, $\pi < x < \frac{3\pi}{2}$, *find the values of* $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$. **[NCERT]**

SOLUTION It is given that

$\tan x = \frac{3}{4}$ and, $\pi < x < \frac{3\pi}{2}$

$\therefore \quad \cos x = -\frac{1}{\sqrt{1+\tan^2 x}}$ $\qquad \left[\because \pi < x < \frac{3\pi}{2} \therefore \cos x \text{ is negative}\right]$

$\Rightarrow \quad \cos x = -\frac{1}{\sqrt{1+\frac{9}{16}}} = -\frac{4}{5}$

$\because \quad \pi < x < \frac{3\pi}{2} \Rightarrow \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4} \Rightarrow \cos x < 0 \text{ and } \sin x > 0$

$\therefore \quad \cos\frac{x}{2} = -\sqrt{\frac{1+\cos x}{2}}$ and, $\sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}}$

$\Rightarrow \quad \cos\frac{x}{2} = -\sqrt{\frac{1-4/5}{2}}$ and, $\sin\frac{x}{2} = \sqrt{\frac{1+\frac{4}{5}}{2}}$

$\Rightarrow \quad \cos\frac{x}{2} = -\frac{1}{\sqrt{10}}$ and, $\sin\frac{x}{2} = \frac{3}{\sqrt{10}}$

$\therefore \quad \tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{3/\sqrt{10}}{-1/\sqrt{10}} = -3$

Hence, $\cos\frac{x}{2} = -\frac{1}{\sqrt{10}}$, $\sin\frac{x}{2} = \frac{3}{\sqrt{10}}$ and $\tan\frac{x}{2} = -3$

Type IV **ON FINDING THE VALUES TRIGONOMETRICAL FUNCTIONS FOR** $\frac{\pi}{24}, \frac{\pi}{16}, \frac{\pi}{8}$

Formulae: $\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$, $\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$, $\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$

EXAMPLE 17 *Find the values of*

(i) $\cos\frac{\pi}{8}$ (ii) $\sin\frac{\pi}{8}$ (iii) $\tan\frac{\pi}{8}$ **[NCERT]** (iv) $\sin\frac{\pi}{24}$ (v) $\cos\frac{\pi}{24}$

SOLUTION (i) We know that $\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$

Putting $A = \frac{\pi}{4}$, we get

$$\cos\frac{\pi}{8} = \sqrt{\frac{1+\cos\pi/4}{2}} \qquad \left[\because \cos\frac{\pi}{8} \text{ is} + \text{ve}\right]$$

$$\Rightarrow \quad \cos\frac{\pi}{8} = \sqrt{\frac{1+1/\sqrt{2}}{2}} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

(ii) We have,

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$

Putting $A = \frac{\pi}{4}$, we get

$$\sin\frac{\pi}{8} = \sqrt{\frac{1-\cos\pi/4}{2}} \qquad \left[\because \sin\frac{\pi}{8} \text{ is} + \text{ve}\right]$$

$$\Rightarrow \quad \sin\frac{\pi}{8} = \sqrt{\frac{1-1/\sqrt{2}}{2}} = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$$

(iii) We have,

$$\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

Putting $A = \frac{\pi}{4}$, we get

$$\tan\frac{\pi}{8} = \sqrt{\frac{1-\cos\pi/4}{1+\cos\pi/4}} \qquad \left[\because \tan\frac{\pi}{8} \text{ is} + \text{ve}\right]$$

$$\Rightarrow \quad \tan\frac{\pi}{8} = \sqrt{\frac{1-1/\sqrt{2}}{1+1/\sqrt{2}}} = \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}} = \sqrt{\frac{(\sqrt{2}-1)^2}{(\sqrt{2}+1)(\sqrt{2}-1)}} = \sqrt{2}-1$$

(iv) We observe that

$$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Putting $A = \frac{\pi}{12}$ in $\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$, we get

$$\sin\frac{\pi}{24} = \sqrt{\frac{1-\cos\frac{\pi}{12}}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}+1}{2\sqrt{2}}}{2}} = \sqrt{\frac{2\sqrt{2}-\sqrt{3}-1}{4\sqrt{2}}} = \sqrt{\frac{4-\sqrt{6}-\sqrt{2}}{8}} = \frac{\sqrt{4-\sqrt{6}-\sqrt{2}}}{2\sqrt{2}}$$

(v) Putting $A = \frac{\pi}{12}$ in $\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$, we get

$$\cos\frac{\pi}{24} = \sqrt{\frac{1+\cos\frac{\pi}{12}}{2}}$$

$$\Rightarrow \quad \cos\frac{\pi}{24} = \sqrt{\frac{1+\frac{\sqrt{3}+1}{2\sqrt{2}}}{2}} \qquad \left[\because \cos\frac{\pi}{24} = \cos 7\frac{1}{2}^\circ \text{ is} + \text{ve}\right]$$

$$\Rightarrow \quad \cos\frac{\pi}{24} = \sqrt{\frac{2\sqrt{2}+\sqrt{3}+1}{4\sqrt{2}}} = \sqrt{\frac{4+\sqrt{6}+\sqrt{2}}{8}} = \frac{\sqrt{4+\sqrt{6}+\sqrt{2}}}{2\sqrt{2}}$$

EXAMPLE 18 *Prove that:*

(i) $\cot\dfrac{\pi}{24}=\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$ (ii) $\tan\dfrac{\pi}{16}=\sqrt{4+2\sqrt{2}}-(\sqrt{2}+1)$

(iii) $\tan 142\dfrac{1^\circ}{2}=2+\sqrt{2}-\sqrt{3}-\sqrt{6}$

SOLUTION (i) We have,

$$\text{LHS}=\cot\frac{\pi}{24}$$

$$\Rightarrow\quad \text{LHS}=\frac{\cos\frac{\pi}{24}}{\sin\frac{\pi}{24}}=\frac{2\cos\frac{\pi}{24}\cos\frac{\pi}{24}}{2\sin\frac{\pi}{24}\cos\frac{\pi}{24}}=\frac{2\cos^2\frac{\pi}{24}}{2\sin\frac{\pi}{24}\cos\frac{\pi}{24}}=\frac{1+\cos\frac{\pi}{12}}{\sin\frac{\pi}{12}}$$

$$\Rightarrow\quad \text{LHS}=\frac{1+\cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right)}=\frac{1+\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}=\frac{2\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1}=\frac{(2\sqrt{2}+\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$\Rightarrow\quad \text{LHS}=\frac{2\sqrt{6}+2\sqrt{2}+3+\sqrt{3}+\sqrt{3}+1}{3-1}=\frac{2\sqrt{6}+2\sqrt{2}+2\sqrt{3}+4}{2}$$

$$\Rightarrow\quad \text{LHS}=\sqrt{2}+\sqrt{3}+2+\sqrt{6}=\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}=\text{RHS}$$

(ii)
$$\text{LHS}=\tan\frac{\pi}{16}=\frac{\sin\frac{\pi}{16}}{\cos\frac{\pi}{16}}=\frac{\sin\frac{\pi}{16}}{\cos\frac{\pi}{16}}\times\frac{2\sin\frac{\pi}{16}}{2\sin\frac{\pi}{16}}=\frac{2\sin^2\frac{\pi}{16}}{2\sin\frac{\pi}{16}\cos\frac{\pi}{16}}=\frac{1-\cos\frac{\pi}{8}}{\sin\frac{\pi}{8}}$$

$$\Rightarrow\quad \text{LHS}=\frac{1-\sqrt{\dfrac{1+\cos\frac{\pi}{4}}{2}}}{\sqrt{\dfrac{1-\cos\frac{\pi}{4}}{2}}}\qquad\left[\because\ \cos\frac{\theta}{2}=\sqrt{\frac{1+\cos\theta}{2}}\ \text{and}\ \sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}\right]$$

$$\Rightarrow\quad \text{LHS}=\frac{\sqrt{2}-\sqrt{1+\cos\frac{\pi}{4}}}{\sqrt{1-\cos\frac{\pi}{4}}}=\frac{\sqrt{2}-\sqrt{1+\frac{1}{\sqrt{2}}}}{\sqrt{1-\frac{1}{\sqrt{2}}}}=\frac{\sqrt{2}-\sqrt{\frac{\sqrt{2}+1}{\sqrt{2}}}}{\sqrt{\frac{\sqrt{2}-1}{\sqrt{2}}}}$$

$$\Rightarrow\quad \text{LHS}=\frac{\sqrt{2\sqrt{2}}-\sqrt{\sqrt{2}+1}}{\sqrt{\sqrt{2}-1}}=\frac{\sqrt{2\sqrt{2}}-\sqrt{\sqrt{2}+1}}{\sqrt{\sqrt{2}-1}}\times\frac{\sqrt{\sqrt{2}+1}}{\sqrt{\sqrt{2}+1}}$$

$$\Rightarrow\quad \text{LHS}=\frac{\sqrt{2\sqrt{2}}\sqrt{\sqrt{2}+1}-\sqrt{(\sqrt{2}+1)^2}}{\sqrt{(\sqrt{2}+1)(\sqrt{2}-1)}}=\frac{\sqrt{2\sqrt{2}(\sqrt{2}+1)}-(\sqrt{2}+1)}{2-1}$$

$$\Rightarrow\quad \text{LHS}=\sqrt{4+2\sqrt{2}}-(\sqrt{2}+1)=\text{RHS}$$

(iii)
$$\text{LHS}=\tan 142\frac{1}{2}^\circ=\tan\left(180^\circ=37\frac{1}{2}^\circ\right)=\tan 37\frac{1}{2}^\circ=-\tan\frac{5\pi}{24}$$

$$\Rightarrow \quad \text{LHS} = -\frac{\sin\frac{5\pi}{24}}{\cos\frac{5\pi}{24}} = -\frac{2\sin^2\frac{5\pi}{24}}{2\sin\frac{5\pi}{24}\cos\frac{5\pi}{24}} = -\frac{1-\cos\frac{5\pi}{12}}{\sin\frac{5\pi}{12}}$$

$$\Rightarrow \quad \text{LHS} = -\frac{1-\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{4}+\frac{\pi}{6}\right)} = -\frac{1-\left(\cos\frac{\pi}{4}\cos\frac{\pi}{6}-\sin\frac{\pi}{4}\sin\frac{\pi}{6}\right)}{\sin\frac{\pi}{4}\cos\frac{\pi}{6}+\cos\frac{\pi}{4}\sin\frac{\pi}{6}}$$

$$\Rightarrow \quad \text{LHS} = \left(\frac{1-\frac{\sqrt{3}-1}{2\sqrt{2}}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}\right) = -\left(\frac{2\sqrt{2}-\sqrt{3}+1}{\sqrt{3}+1}\right) = -\frac{(2\sqrt{2}-\sqrt{3}+1)}{(\sqrt{3}+1)} \times \frac{(\sqrt{3}-1)}{\sqrt{3}-1}$$

$$\Rightarrow \quad \text{LHS} = -\left\{\frac{(2\sqrt{2}-\sqrt{3}+1)(\sqrt{3}-1)}{3-1}\right\} = -\left\{\frac{2\sqrt{2}(\sqrt{3}-1)-(\sqrt{3}-1)^2}{2}\right\}$$

$$\Rightarrow \quad \text{LHS} = -\left\{\frac{2\sqrt{2}(\sqrt{3}-1)-(3+1-2\sqrt{3})}{2}\right\} = -\left\{\sqrt{2}(\sqrt{3}-1) - (2-\sqrt{3})\right\}$$

$$\Rightarrow \quad \text{LHS} = -\sqrt{6}+\sqrt{2}+2-\sqrt{3} = 2+\sqrt{2}-\sqrt{3}-\sqrt{6} = \text{RHS}$$

ILLUSTRATIVE EXAMPLES

LEVEL-2

AN IMPORTANT RESULT *Prove that:* $\cos A \cos 2A \cos 2^2 A \cos 2^3 A \ldots \cos 2^{n-1} A = \dfrac{\sin 2^n A}{2^n \sin A}$

PROOF LHS $= \cos A \cos 2A \cos 2^2 A \cos 2^3 A \ldots \cos 2^{n-1} A$

$$= \frac{1}{2\sin A}\left\{(2\sin A\cos A)\cos 2A\cos 2^2 A\cos 2^3 A\ldots\cos 2^{n-1} A\right\}$$

$$= \frac{1}{2\sin A}\left\{\sin 2A\cos 2A\cos 2^2 A\cos 2^3 A\ldots\cos 2^{n-1} A\right\}$$

$$= \frac{1}{2^2\sin A}\left\{(2\sin 2A\cos 2A)\cos 2^2 A\cos 2^3 A\ldots\cos 2^{n-1} A\right\}$$

$$= \frac{1}{2^2\sin A}\left\{\sin 2(2A)\cos 2^2 A\cos 2^3 A\ldots\cos 2^{n-1} A\right\}$$

$$= \frac{1}{2^3\sin A}\left\{(2\sin 2^2 A\cos 2^2 A)\cos 2^3 A\ldots\cos 2^{n-1} A\right\}$$

$$= \frac{1}{2^3\sin A}\left\{\sin(2\times 2^2 A)\cos 2^3 A\ldots\cos 2^{n-1} A\right\}$$

$$= \frac{1}{2^3\sin A}\left\{\sin 2^3 A\cos 2^3 A\cos 2^4 A\ldots\cos 2^{n-1} A\right\}$$

$$= \ldots\ldots = \frac{1}{2^{n-1}\sin A}\left\{\sin 2^{n-1} A\cos 2^{n-1} A\right\} = \frac{1}{2^n\sin A}\left\{2\sin 2^{n-1} A\cos 2^{n-1} A\right\}$$

$$= \frac{1}{2^n\sin A}\sin(2\times 2^{n-1} A) = \frac{1}{2^n\sin A}\sin 2^n A = \text{RHS}$$

Type V PROBLEMS BASED ON THE FORMULA

$$\cos A \cos 2A \cos 2^2 A \cos 2^3 A \ldots \cos 2^{n-1} A = \frac{\sin 2^n A}{2^n \sin A}$$

EXAMPLE 19 *If* $\theta = \dfrac{\pi}{2^n + 1}$, *prove that:* $2^n \cos\theta \cos 2\theta \cos 2^2 \theta \ldots \cos 2^{n-1}\theta = 1$.

SOLUTION We have,

$$\theta = \frac{\pi}{2^n+1} \Rightarrow 2^n\theta + \theta = \pi \quad \Rightarrow 2^n \theta = \pi - \theta$$

$\therefore\ 2^n \cos\theta \cos 2\theta \cos 2^2 \theta \ldots \cos 2^{n-1}\theta$

$$= 2^n \left\{\frac{\sin 2^n \theta}{2^n \sin\theta}\right\} = \frac{\sin 2^n \theta}{\sin\theta} = \frac{\sin(\pi-\theta)}{\sin\theta} = 1 \qquad [\because\ 2^n\theta = \pi - \theta]$$

EXAMPLE 20 *Prove that:* $\cos 20° \cos 40° \cos 60° \cos 80° = \dfrac{1}{16}$

SOLUTION We have,

$$\text{LHS} = \cos 20° \cos 40° \cos 60° \cos 80°$$

$$\Rightarrow \text{LHS} = \frac{1}{2}(\cos 20° \cos 40° \cos 80°)$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left(\cos A \cos 2A \cos 2^2 A\right), \quad \text{where } A = 20°$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left(\frac{\sin 2^3 A}{2^3 \sin A}\right) = \frac{1}{2^4}\frac{\sin 8A}{\sin A}$$

$$\Rightarrow \text{LHS} = \frac{1}{2^4}\frac{\sin 160°}{\sin 20°} = \frac{1}{2^4}\frac{\sin(180° - 20°)}{\sin 20°} = \frac{1}{2^4}\frac{\sin 20°}{\sin 20°} = \frac{1}{2^4} = \frac{1}{16} = \text{RHS}$$

EXAMPLE 21 *Prove that:* $\cos\dfrac{\pi}{7}\cos\dfrac{2\pi}{7}\cos\dfrac{4\pi}{7} = -\dfrac{1}{8}$.

SOLUTION We have,

$$\text{LHS} = \cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$$

$$\Rightarrow \text{LHS} = \cos A \cos 2A \cos 2^2 A, \text{ where } A = \frac{\pi}{7}$$

$$\Rightarrow \text{LHS} = \frac{\sin 2^3 A}{2^3 \sin A}$$

$$\Rightarrow \text{LHS} = \frac{\sin 8\frac{\pi}{7}}{8\sin\frac{\pi}{7}}\ \frac{\sin\left(\pi + \frac{\pi}{7}\right)}{8\sin\frac{\pi}{7}} = -\frac{\sin\frac{\pi}{7}}{8\sin\frac{\pi}{7}} = -\frac{1}{8} = \text{RHS}$$

EXAMPLE 22 *Prove that:* $\cos\dfrac{\pi}{7}\cos\dfrac{2\pi}{7}\cos\dfrac{3\pi}{7}\cos\dfrac{3\pi}{7} = \dfrac{1}{8}$

SOLUTION We have,

$$\text{LHS} = \cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}$$

$$\Rightarrow \text{LHS} = \cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\left(\pi - \frac{4\pi}{7}\right)$$

$$\Rightarrow \text{LHS} = -\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} = -\left(-\frac{1}{8}\right)$$ [See Example 21]

$$\Rightarrow \text{LHS} = \frac{1}{8} = \text{RHS}$$

EXAMPLE 23 *Prove that:* $\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{14\pi}{15} = \frac{1}{16}$

SOLUTION We have,

$$\text{LHS} = \cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\left(\pi - \frac{\pi}{15}\right)$$

$$\Rightarrow \text{LHS} = \left(\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\right)\left(-\cos\frac{\pi}{15}\right) \qquad [\because \cos(\pi-\theta) = -\cos\theta]$$

$$\Rightarrow \text{LHS} = -\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}$$

$$\Rightarrow \text{LHS} = -\cos A\cos 2A\cos 2^2A\cos 2^3A, \text{ where } A = \frac{\pi}{15}$$

$$\Rightarrow \text{LHS} = -\frac{\sin 2^4 A}{2^4\sin A} = -\frac{\sin 16A}{2^4\sin A}$$

$$\text{LHS} = -\frac{\sin\frac{16\pi}{15}}{16\sin\frac{\pi}{15}} = -\frac{\sin\left(\pi+\frac{\pi}{15}\right)}{16\sin\frac{\pi}{15}} = \frac{\sin\frac{\pi}{15}}{16\sin\frac{\pi}{15}} = \frac{1}{16} = \text{RHS}$$

EXAMPLE 24 *Prove that:*

$$\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14} = \frac{1}{64}$$

SOLUTION We know that

$$A + B = \pi \Rightarrow \sin A = \sin(\pi - B) = \sin B$$

$$\because \frac{\pi}{14} + \frac{13\pi}{14} = \pi, \quad \frac{3\pi}{14} + \frac{11\pi}{14} = \pi, \quad \frac{5\pi}{14} + \frac{9\pi}{14} = \pi$$

$$\therefore \sin\frac{\pi}{14} = \sin\frac{13\pi}{14}, \quad \sin\frac{3\pi}{14} = \sin\frac{11\pi}{14}, \quad \sin\frac{5\pi}{14} = \sin\frac{9\pi}{14}$$

$$\therefore \text{LHS} = \sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$$

$$\Rightarrow \text{LHS LHS} = \sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{3\pi}{14}\sin\frac{\pi}{14}$$

$$\Rightarrow \text{LHS} = \left\{\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\right\}^2 \times \sin\frac{7\pi}{14}$$

$$\Rightarrow \text{LHS} = \left\{\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\right\}^2 \times 1$$

$$\Rightarrow \text{LHS} = \left\{\cos\left(\frac{\pi}{2}-\frac{\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{3\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{14}\right)\right\}^2$$

$$\Rightarrow \text{LHS} = \left\{\cos\frac{6\pi}{14}\cos\frac{4\pi}{14}\cos\frac{2\pi}{14}\right\}^2 = \left\{\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\right\}^2 = \left(\frac{1}{8}\right)^2 = \frac{1}{64}$$ [See Example 21]

EXAMPLE 25 *Find the value of* $\sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}$.

SOLUTION We have,

$$\sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}$$

$$=\cos\left(\frac{\pi}{2}-\frac{\pi}{18}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{18}\right)\cos\left(\frac{\pi}{2}-\frac{7\pi}{18}\right)$$

$$=\cos\frac{4\pi}{9}\cos\frac{2\pi}{9}\cos\frac{\pi}{9}$$

$$=\frac{\sin\left(2^3\times\frac{\pi}{9}\right)}{2^3\sin\frac{\pi}{9}}=\frac{\sin\frac{8\pi}{9}}{8\sin\frac{\pi}{9}}=\frac{\sin\left(\pi-\frac{\pi}{9}\right)}{8\sin\frac{\pi}{9}}=\frac{\sin\frac{\pi}{9}}{8\sin\frac{\pi}{9}}=\frac{1}{8}$$

EXAMPLE 26 *Prove that:*

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}=\frac{1}{128}$$

SOLUTION We have,

$$\text{LHS}=\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}$$

$$=\left\{\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15}\right\}\left\{\cos\frac{3\pi}{15}\cos\frac{6\pi}{15}\right\}\cos\frac{5\pi}{15}$$

$$=\frac{1}{2}\left\{\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15}\right\}\left\{\cos\frac{3\pi}{15}\cos\frac{6\pi}{15}\right\}\qquad\left[\because\cos\frac{5\pi}{15}=\cos\frac{\pi}{3}=\frac{1}{2}\right]$$

$$=-\frac{1}{2}\left\{\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\right\}\left\{\cos\frac{3\pi}{15}\cos\frac{6\pi}{15}\right\}\left[\because\cos\frac{7\pi}{15}=\cos\left(\pi-\frac{8\pi}{15}\right)=-\cos\frac{8\pi}{15}\right]$$

$$=-\frac{1}{2}\left\{\frac{\sin\frac{2^4\pi}{15}}{2^4\sin\frac{\pi}{15}}\right\}\times\left\{\frac{\sin\left(2^2\times\frac{3\pi}{15}\right)}{2^2\sin\frac{3\pi}{15}}\right\}$$

$$=-\frac{1}{2}\left\{\frac{\sin\frac{16\pi}{15}}{16\sin\frac{\pi}{15}}\right\}\times\left\{\frac{\sin\frac{12\pi}{15}}{4\sin\frac{3\pi}{15}}\right\}=-\frac{1}{2}\left\{\frac{\sin\left(\pi+\frac{\pi}{15}\right)}{16\sin\frac{\pi}{15}}\right\}\times\left\{\frac{\sin\left(\pi-\frac{3\pi}{15}\right)}{4\sin\frac{3\pi}{15}}\right\}$$

$$=-\frac{1}{2}\left\{\frac{-\sin\frac{\pi}{15}}{16\sin\frac{\pi}{15}}\right\}\times\left\{\frac{\sin\frac{3\pi}{15}}{4\sin\frac{3\pi}{15}}\right\}=-\frac{1}{2}\times-\frac{1}{16}\times\frac{1}{4}=\frac{1}{128}=\text{RHS}$$

EXAMPLE 27 *Prove that:* $(1+\sec 2\theta)(1+\sec 4\theta)(1+\sec 8\theta)\ldots(1+\sec 2^n\theta)=\tan 2^n\theta\cot\theta, n\in N.$

SOLUTION We have,

$$\text{LHS}=(1+\sec 2\theta)(1+\sec 4\theta)(1+\sec 8\theta)\ldots(1+\sec 2^n\theta)$$

$$\Rightarrow \text{LHS}=\frac{(1+\cos 2\theta)(1+\cos 4\theta)(1+\cos 8\theta)\ldots(1+\cos 2^n\theta)}{\cos 2\theta\cos 4\theta\cos 8\theta\ldots\cos 2^n\theta}$$

$$\Rightarrow \text{LHS}=\frac{(2\cos^2\theta)(2\cos^2 2\theta)(2\cos^2 2^2\theta)\ldots(2\cos^2 2^{n-1}\theta)}{\cos 2\theta\cos 2^2\theta\cos 2^3\theta\ldots\cos 2^n\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{2^n \cos\theta}{\cos 2^n\theta}\left\{\cos\theta\cos 2\theta\cos 2^2\theta\ldots\cos 2^{n-1}\theta\right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{2^n \cos\theta}{\cos 2^n\theta}\left\{\frac{\sin 2^n\theta}{2^n\sin\theta}\right\} = \tan 2^n\theta\cot\theta = \text{RHS}$$

Type VI MISCELLANEOUS PROBLEMS BASED UPON FOLLOWING FORMULAE

$\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = \cos^2\theta - \sin^2\theta$, $\cos 2\theta = 2\cos^2\theta - 1$,

$\cos 2\theta = 1 - 2\sin^2\theta$, $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$, $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$, $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

EXAMPLE 28 *If* $\tan^2\theta = 2\tan^2\phi + 1$, *prove that* $\cos 2\theta + \sin^2\phi = 0$.

SOLUTION We have,

$$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$\therefore \quad \cos 2\theta + \sin^2\phi = \frac{1-\tan^2\theta}{1+\tan^2\theta} + \sin^2\phi$$

$$\Rightarrow \quad \cos 2\theta + \sin^2\phi = \frac{1-(2\tan^2\phi+1)}{1+(2\tan^2\phi+1)} + \sin^2\phi \qquad [\because \tan^2\theta = 2\tan^2\phi + 1]$$

$$\Rightarrow \quad \cos 2\theta + \sin^2\phi = \frac{-2\tan^2\phi}{2+2\tan^2\phi} + \sin^2\phi = \frac{-\tan^2\phi}{\sec^2\phi} + \sin^2\phi = -\sin^2\phi + \sin^2\phi = 0$$

EXAMPLE 29 *Prove that:*

(i) $\dfrac{1+\cos 4x}{\cot x - \tan x} = \dfrac{1}{2}\sin 4x$ (ii) $\dfrac{\cos 5x + \cos 4x}{1 - 2\cos 3x} = -\cos 2x - \cos x$

(iii) $\dfrac{\cos 7x - \cos 8x}{1 + 2\cos 5x} = \cos 2x - \cos 3x$

SOLUTION (i) We have,

$$\text{LHS} = \frac{1+\cos 4x}{\cot x - \tan x} = \frac{2\cos^2 2x \times \cos x\sin x}{\cos^2 x - \sin^2 x} = \frac{2\cos^2 2x \times 2\sin x\cos x}{2\cos 2x}$$

$$= \cos 2x\sin 2x = \frac{1}{2}(2\sin 2x\cos 2x) = \frac{1}{2}\sin 4x = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin 3x(\cos 5x + \cos 4x)}{\sin 3x(1 - 2\cos 3x)} \qquad [\text{Multiplying and dividing by } \sin 3x]$$

$$\Rightarrow \quad \text{LHS} = \frac{\left\{2\sin\frac{3x}{2}\cos\frac{3x}{2}\right\}\left\{2\cos\frac{9x}{2}\cos\frac{x}{2}\right\}}{\sin 3x - 2\sin 3x\cos 3x}$$

$$\Rightarrow \quad \text{LHS} = \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{\sin 3x - \sin 6x} = \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{2\sin\left(\frac{3x-6x}{2}\right)\cos\left(\frac{3x+6x}{2}\right)}$$

$$\Rightarrow \text{LHS} = \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{2\sin\left(-\frac{3x}{2}\right)\cos\frac{9x}{2}}$$

$$\Rightarrow \text{LHS} = \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{-2\sin\frac{3x}{2}\cos\frac{9x}{2}} = -2\cos\frac{3x}{2}\cos\frac{x}{2} = -(\cos 2x + \cos x) = \text{RHS}$$

(iii) We have,

$$\text{LHS} = \frac{\cos 7x - \cos 8x}{1 + 2\cos 5x}$$

$$\Rightarrow \text{LHS} = \frac{2\sin\frac{5x}{2}(\cos 7x - \cos 8x)}{2\sin\frac{5x}{2}(1 + 2\cos 5x)} \quad \left[\text{Multiplying numerator and denominator by } 2\sin\frac{5x}{2}\right]$$

$$\Rightarrow \text{LHS} = \frac{2\sin\frac{5x}{2}\cos 7x - 2\sin\frac{5x}{2}\cos 8x}{2\sin\frac{5x}{2} + 4\sin\frac{5x}{2}\cos 5x} = \frac{2\sin\frac{5x}{2}\cos 7x - 2\sin\frac{5x}{2}\cos 8x}{2\left\{\sin\frac{5x}{2} + 2\sin\frac{5x}{2}\cos 5x\right\}}$$

$$\Rightarrow \text{LHS} = \frac{\left(\sin\frac{19x}{2} - \sin\frac{9x}{2}\right) - \left(\sin\frac{21x}{2} - \sin\frac{11x}{2}\right)}{2\left\{\sin\frac{5x}{2} + \sin\frac{15x}{2} - \sin\frac{5x}{2}\right\}}$$

$$\Rightarrow \text{LHS} = \frac{\left(\sin\frac{19x}{2} + \sin\frac{11x}{2}\right) - \left(\sin\frac{9x}{2} + \sin\frac{21x}{2}\right)}{2\sin\frac{15x}{2}}$$

$$\Rightarrow \text{LHS} = \frac{2\sin\left(\frac{\frac{19x}{2} + \frac{11x}{2}}{2}\right)\cos\left(\frac{\frac{19x}{2} - \frac{11x}{2}}{2}\right) - 2\sin\left(\frac{\frac{9x}{2} + \frac{21x}{2}}{2}\right)\cos\left(\frac{\frac{9x}{2} - \frac{21x}{2}}{2}\right)}{2\sin\frac{15x}{2}}$$

$$\Rightarrow \text{LHS} = \frac{2\sin\frac{15x}{2}\cos 2x - 2\sin\frac{15x}{2}\cos(-3x)}{2\sin\frac{15x}{2}} = \cos 2x - \cos 3x = \text{RHS}$$

EXAMPLE 30 *If* $\tan\alpha = \frac{p}{q}$, *where* $\alpha = 6\beta$, α *being an acute angle, prove that*

$$\frac{1}{2}\left\{p \operatorname{cosec} 2\beta - q \sec 2\beta\right\} = \sqrt{p^2 + q^2}$$

SOLUTION We have, $\tan\alpha = \frac{p}{q}$

$$\therefore \quad \sin\alpha = \frac{p}{\sqrt{p^2 + q^2}} \text{ and, } \cos\alpha = \frac{q}{\sqrt{p^2 + q^2}}$$

Now,

$$\text{LHS} = \frac{1}{2}\left\{p \operatorname{cosec} 2\beta - q \sec 2\beta\right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{p^2+q^2}}{2}\left\{\frac{p}{\sqrt{p^2+q^2}} \operatorname{cosec} 2\beta - \frac{q}{\sqrt{p^2+q^2}} \sec 2\beta\right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{p^2+q^2}}{2}\left\{\sin\alpha \operatorname{cosec} 2\beta - \cos\alpha \sec 2\beta\right\} = \frac{\sqrt{p^2+q^2}}{2}\left\{\frac{\sin\alpha}{\sin 2\beta} - \frac{\cos\alpha}{\cos 2\beta}\right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{p^2+q^2}}{2}\left\{\frac{\sin\alpha\cos 2\beta - \cos\alpha\sin 2\beta}{\sin 2\beta\cos 2\beta}\right\}$$

$$\Rightarrow \quad \text{LHS} = \sqrt{p^2+q^2}\left\{\frac{\sin(\alpha-2\beta)}{2\sin 2\beta\cos 2\beta}\right\} = \sqrt{p^2+q^2}\left\{\frac{\sin(6\beta-2\beta)}{\sin 4\beta}\right\} \quad [\because \alpha = 6\beta]$$

$$\Rightarrow \quad \text{LHS} = \sqrt{p^2+q^2} = \text{RHS}.$$

EXAMPLE 31 *Prove that:* $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = \cot\alpha$

SOLUTION We have,

$$\cot\theta - \tan\theta = \frac{1}{\tan\theta} - \tan\theta = \frac{1-\tan^2\theta}{\tan\theta} = 2\left\{\frac{1-\tan^2\theta}{2\tan\theta}\right\} = \frac{2}{\tan 2\theta}$$

$$\Rightarrow \quad \cot\theta - \tan\theta = 2\cot 2\theta \quad \text{...(i)}$$

Now,

$$\text{LHS} = \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha$$

$$\Rightarrow \quad \text{LHS} = \cot\alpha - \cot\alpha + \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha$$

$$\Rightarrow \quad \text{LHS} = \cot\alpha - \{\cot\alpha - \tan\alpha - 2\tan 2\alpha - 4\tan 4\alpha - 8\cot 8\alpha\}$$

$$\Rightarrow \quad \text{LHS} = \cot\alpha - \{(\cot\alpha - \tan\alpha) - 2\tan 2\alpha - 4\tan 4\alpha - 8\cot 8\alpha\}$$

$$\Rightarrow \quad \text{LHS} = \cot\alpha - \{2\cot 2\alpha - 2\tan 2\alpha - 4\tan 4\alpha - 8\cot 8\alpha\}$$

$$\Rightarrow \quad \text{LHS} = \cot\alpha - \{2(\cot 2\alpha - \tan 2\alpha) - 4\tan 4\alpha - 8\cot 8\alpha\} \quad \text{[Using (i)]}$$

$$\Rightarrow \quad \text{LHS} = \cot\alpha - \{2\times 2\cot 2(2\alpha) - 4\tan 4\alpha - 8\cot 8\alpha\} \quad \text{[Using (i)]}$$

$$\Rightarrow \quad \text{LHS} = \cot\alpha - \{4(\cot 4\alpha - \tan 4\alpha) - 8\cot 8\alpha\} \quad \text{[Using (i)]}$$

$$\Rightarrow \quad \text{LHS} = \cot\alpha - \{4\times 2\cot 2(4\alpha) - 8\cot 8\alpha\}$$

$$\Rightarrow \quad \text{LHS} = \cot\alpha - (8\cot 8\alpha - 8\cot 8\alpha) = \cot\alpha = \text{RHS}$$

EXAMPLE 32 *If* $\tan\beta = \dfrac{\tan\alpha + \tan\gamma}{1+\tan\alpha\tan\gamma}$, *prove that* $\sin 2\beta = \dfrac{\sin 2\alpha + \sin 2\gamma}{1+\sin 2\alpha\sin 2\gamma}$

SOLUTION It is given that

$$\tan\beta = \frac{\tan\alpha + \tan\gamma}{1+\tan\alpha\tan\gamma}$$

$$\Rightarrow \quad \tan\beta = \frac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\gamma}{\cos\gamma}}{1 + \dfrac{\sin\alpha}{\cos\alpha}\times\dfrac{\sin\gamma}{\cos\gamma}} = \frac{\sin\alpha\cos\gamma + \cos\alpha\sin\gamma}{\cos\alpha\cos\gamma + \sin\alpha\sin\gamma} = \frac{\sin(\alpha+\gamma)}{\cos(\alpha-\gamma)} \quad \text{...(i)}$$

Now, $\quad \sin 2\beta = \dfrac{2\tan\beta}{1+\tan^2\beta}$

$$\Rightarrow \sin 2\beta = \frac{\dfrac{2\sin(\alpha+\gamma)}{\cos(\alpha-\gamma)}}{1+\dfrac{\sin^2(\alpha+\gamma)}{\cos^2(\alpha-\gamma)}} \qquad \text{[Using (i)]}$$

$$\Rightarrow \sin 2\beta = \frac{2\sin(\alpha+\gamma)\cos(\alpha-\gamma)}{\cos^2(\alpha-\gamma)+\sin^2(\alpha+\gamma)}$$

$$\Rightarrow \sin 2\beta = \frac{2(\sin 2\alpha+\sin 2\gamma)}{2\cos^2(\alpha-\gamma)+2\sin^2(\alpha+\gamma)}$$

$$\Rightarrow \sin 2\beta = \frac{2(\sin 2\alpha+\sin 2\gamma)}{1+\cos(2\alpha-2\gamma)+1-\cos(2\alpha+2\gamma)} = \frac{2(\sin 2\alpha+\sin 2\gamma)}{2+\cos(2\alpha-2\gamma)-\cos(2\alpha+2\gamma)}$$

$$\Rightarrow \sin 2\beta = \frac{2(\sin 2\alpha+\sin 2\gamma)}{2+2\sin 2\alpha\sin 2\gamma} = \frac{\sin 2\alpha+\sin 2\gamma}{1+\sin 2\alpha\sin 2\gamma}$$

EXAMPLE 33 *If* $\sin(\theta+\alpha)=a$ *and* $\sin(\theta+\beta)=b$, *prove that* $\cos 2(\alpha-\beta)-4ab\cos(\alpha-\beta)=1-2a^2-2b^2$

[NCERT EXEMPLAR]

SOLUTION We have,

$\sin(\theta+\alpha)=a$ and $\sin(\theta+\beta)=b$

$\therefore\ \cos(\theta+\alpha)=\sqrt{1-\sin^2(\theta+\alpha)}=\sqrt{1-a^2}$ and $\cos(\theta+\beta)=\sqrt{1-\sin^2(\theta+\beta)}=\sqrt{1-b^2}$

Now,

$\cos(\alpha-\beta)=\cos\{(\theta+\alpha)-(\theta+\beta)\}$

$\Rightarrow \cos(\alpha-\beta)=\cos(\theta+\alpha)\cos(\theta+\beta)+\sin(\theta+\alpha)\sin(\theta+\beta)$

$\Rightarrow \cos(\alpha-\beta)=\sqrt{1-a^2}\,\sqrt{1-b^2}+ab$

$\Rightarrow \cos(\alpha-\beta)=ab+\sqrt{1-a^2-b^2+a^2b^2}$

$\therefore\ \cos 2(\alpha-\beta)-4ab\cos(\alpha-\beta)$

$=2\cos^2(\alpha-\beta)-1-4ab\cos(\alpha-\beta)$

$=2\left\{ab+\sqrt{1-a^2-b^2+a^2b^2}\right\}^2-1-4ab\left\{ab+\sqrt{1-a^2-b^2+a^2b^2}\right\}$

$=2\left\{a^2b^2+2ab\sqrt{1-a^2-b^2+a^2b^2}+1-a^2-b^2+a^2b^2\right\}-1-4a^2b^2-4ab\sqrt{1-a^2-b^2+a^2b^2}$

$=2-2a^2-2b^2-1=1-2a^2-2b^2.$

EXAMPLE 34 *If* $\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\tan^3\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$, *prove that* $\sin\theta=\dfrac{3\sin\alpha+\sin^3\alpha}{1+3\sin^2\alpha}$.

SOLUTION We have,

$$\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\tan^3\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

$$\Rightarrow \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}=\left\{\frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}\right\}^3$$

$$\Rightarrow \frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}=\left\{\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}\right\}^3$$

$$\Rightarrow \left\{\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right\}^2 = \left[\left\{\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}\right\}^2\right]^3 \quad \text{[On squaring both sides]}$$

$$\Rightarrow \frac{1+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \left\{\frac{1+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{1-2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right\}^3$$

$$\Rightarrow \frac{1+\sin\theta}{1-\sin\theta} = \left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)^3$$

$$\Rightarrow \frac{(1+\sin\theta)-(1-\sin\theta)}{(1+\sin\theta)+(1-\sin\theta)} = \frac{(1+\sin\alpha)^3-(1-\sin\alpha)^3}{(1+\sin\alpha)^3+(1-\sin\alpha)^3} \quad \left[\text{Applying componendo-dividendo}\right]$$

$$\Rightarrow \frac{2\sin\theta}{2} = \frac{6\sin\alpha+2\sin^3\alpha}{2+6\sin^2\alpha}$$

$$\Rightarrow \sin\theta = \frac{3\sin\alpha+\sin^3\alpha}{1+3\sin^2\alpha}$$

EXAMPLE 35 *If* $\tan\frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\phi}{2}$, *prove that* $\cos\phi = \frac{\cos\theta - e}{1-e\cos\theta}$.

SOLUTION We have,

$$\cos\phi = \frac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}$$

$$\Rightarrow \cos\phi = \frac{1-\frac{1+e}{1-e}\tan^2\frac{\theta}{2}}{1+\frac{1+e}{1-e}\tan^2\frac{\theta}{2}} \quad \left[\because \tan\frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\phi}{2} \Rightarrow \tan\frac{\phi}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{\theta}{2}\right]$$

$$\Rightarrow \cos\phi = \frac{(1-e)-(1+e)\tan^2\frac{\theta}{2}}{(1-e)+(1+e)\tan^2\frac{\theta}{2}}$$

$$\Rightarrow \cos\phi = \frac{(1-\tan^2\theta/2)-e(1+\tan^2\theta/2)}{(1+\tan^2\theta/2)-e(1-\tan^2\theta/2)}$$

$$\Rightarrow \cos\phi = \frac{\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}-e}{1-e\,\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}} \quad \left[\text{Dividing numerator and denominator by } 1+\tan^2\frac{\theta}{2}\right]$$

$$\Rightarrow \cos\phi = \frac{\cos\theta - e}{1-e\cos\theta} \quad \left[\because \cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right]$$

EXAMPLE 36 *Prove that:*

$$\frac{2\cos 2^n\theta + 1}{2\cos\theta + 1} = (2\cos\theta - 1)(2\cos 2\theta - 1)(2\cos 2^2\theta - 1)\ldots(2\cos 2^{n-1}\theta - 1)$$

SOLUTION RHS

$$= (2\cos\theta - 1)(2\cos 2\theta - 1)(2\cos 2^2\theta - 1)\ldots(2\cos 2^{n-1}\theta - 1)$$

$$= \frac{1}{(2\cos\theta + 1)}\left\{(2\cos\theta + 1)(2\cos\theta - 1)(2\cos 2\theta - 1)(2\cos 2^2\theta - 1)\ldots(2\cos 2^{n-1}\theta - 1\right\}$$

$$= \frac{1}{(2\cos\theta + 1)}\left\{(4\cos^2\theta - 1)(2\cos 2\theta - 1)(2\cos 2^2\theta - 1)\ldots(2\cos 2^{n-1}\theta - 1)\right\}$$

$$= \frac{1}{(2\cos\theta + 1)}\left\{\{2(1 + \cos 2\theta) - 1\}(2\cos 2\theta - 1)(2\cos 2^2\theta - 1)\ldots(2\cos 2^{n-1}\theta - 1)\right\}$$

$$= \frac{1}{(2\cos\theta + 1)}\left\{(2\cos 2\theta + 1)(2\cos 2\theta - 1)(2\cos 2^2\theta - 1)\ldots(2\cos 2^{n-1}\theta - 1)\right\}$$

$$= \frac{1}{(2\cos\theta + 1)}\left\{(4\cos^2 2\theta - 1)(2\cos 2^2\theta - 1)(2\cos 2^3\theta - 1)\ldots(2\cos 2^{n-1}\theta - 1)\right\}$$

$$= \frac{1}{(2\cos\theta + 1)}\left\{\left\{2(\cos 4\theta + 1) - 1\right\}(2\cos 2^2\theta - 1)(2\cos 2^3\theta - 1)\ldots(2\cos 2^{n-1}\theta - 1)\right\}$$

$$= \frac{1}{(2\cos\theta + 1)}\left\{(2\cos 2^2\theta + 1)(2\cos 2^2\theta - 1)(2\cos 2^3\theta - 1)\ldots\ldots(2\cos 2^{n-1}\theta - 1)\right\}$$

$$= \frac{1}{(2\cos\theta + 1)}\left\{(4\cos^2 2^2\theta - 1)(2\cos 2^3\theta - 1)\ldots\ldots(2\cos 2^{n-1}\theta - 1)\right\}$$

$$= \frac{1}{(2\cos q + 1)}\left\{\left\{2(1 + \cos 2^3\theta) - 1\right\}(2\cos 2^3\theta - 1)\ldots(2\cos 2^n - 1\theta - 1)\right\}$$

$$= \frac{1}{(2\cos\theta + 1)}\left\{(2\cos 2^3\theta + 1)(2\cos 2^3\theta - 1)\ldots(2\cos 2^{n-1}\theta - 1)\right\}$$

....

$$= \frac{1}{(2\cos\theta + 1)}(2\cos 2^{n-1}\theta + 1)(2\cos 2^{n-1}\theta - 1)$$

$$= \frac{1}{(2\cos\theta + 1)}(4\cos^2 2^{n-1}\theta - 1)$$

$$= \frac{1}{(2\cos\theta + 1)}\left\{2(\cos 2 . 2^{n-1}\theta + 1) - 1\right\}$$

$$= \frac{1}{(2\cos\theta + 1)}(2\cos 2^n\theta + 2 - 1) = \frac{2\cos 2^n\theta + 1}{2\cos\theta + 1} = \text{LHS}$$

Type VII ON CONDITIONAL IDENTITIES

EXAMPLE 37 *If* $\tan\frac{\theta}{2} = \sqrt{\frac{a-b}{a+b}}\tan\frac{\phi}{2}$, *prove that* $\cos\theta = \dfrac{a\cos\phi + b}{a + b\cos\phi}$.

SOLUTION We have,

$$\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$

$$\Rightarrow \quad \cos\theta = \frac{1-\frac{a-b}{a+b}\tan^2\frac{\phi}{2}}{1+\frac{a-b}{a+b}\tan^2\frac{\phi}{2}} \qquad \left[\text{Putting } \tan\frac{\theta}{2} = \sqrt{\frac{a-b}{a+b}}\tan\frac{\phi}{2}\right]$$

$$\Rightarrow \quad \cos\theta = \frac{(a+b)-(a-b)\tan^2\frac{\phi}{2}}{(a+b)+(a-b)\tan^2\frac{\phi}{2}}$$

$$\Rightarrow \quad \cos\theta = \frac{a\left(1-\tan^2\frac{\phi}{2}\right)+b\left(1+\tan^2\frac{\phi}{2}\right)}{a\left(1+\tan^2\frac{\phi}{2}\right)+b\left(1-\tan^2\frac{\phi}{2}\right)}$$

$$\Rightarrow \quad \cos\theta = \frac{a\left(\frac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}\right)+b}{a+b\left(\frac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}\right)} \qquad \left[\text{Dividing numerator and denominator } by\ 1+\tan^2\frac{\phi}{2}\right]$$

$$\Rightarrow \quad \cos\theta = \frac{a\cos\phi+b}{a+b\cos\phi}$$

EXAMPLE 38 *If* $\cos\theta = \cos\alpha\cos\beta$, *prove that* $\tan\frac{\theta+\alpha}{2}\tan\frac{\theta-\alpha}{2} = \tan^2\frac{\beta}{2}.$

SOLUTION We have,

$$\cos\theta = \cos\alpha\cos\beta$$

$$\Rightarrow \quad \cos\beta = \frac{\cos\theta}{\cos\alpha}$$

$$\Rightarrow \quad \frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}} = \frac{\cos\theta}{\cos\alpha}$$

$$\Rightarrow \quad \frac{\left(1-\tan^2\frac{\beta}{2}\right)+\left(1+\tan^2\frac{\beta}{2}\right)}{\left(1-\tan^2\frac{\beta}{2}\right)-\left(1+\tan^2\frac{\beta}{2}\right)} = \frac{\cos\theta+\cos\alpha}{\cos\theta-\cos\alpha} \qquad \text{[Applying componendo - dividendo]}$$

$$\Rightarrow \quad \frac{2}{-2\tan^2\frac{\beta}{2}} = \frac{2\cos\frac{\theta+\alpha}{2}\cos\frac{\theta-\alpha}{2}}{-2\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}}$$

$$\Rightarrow \quad \frac{1}{\tan^2\frac{\beta}{2}} = \frac{1}{\tan\frac{\theta+\alpha}{2}\tan\frac{\theta-\alpha}{2}} \Rightarrow \tan^2\frac{\beta}{2} = \tan\frac{\theta+\alpha}{2}\tan\frac{\theta-\alpha}{2}$$

EXAMPLE 39 *If* $\cos\theta = \dfrac{\cos\alpha - \cos\beta}{1 - \cos\alpha\cos\beta}$, *prove that* $\tan\dfrac{\theta}{2} = \pm\tan\dfrac{\alpha}{2}\cot\dfrac{\beta}{2}$.

SOLUTION We have,

$$\cos\theta = \frac{\cos\alpha - \cos\beta}{1 - \cos\alpha\cos\beta}$$

$$\Rightarrow \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}} = \frac{\cos\alpha - \cos\beta}{1 - \cos\alpha\cos\beta}$$

$$\Rightarrow \frac{\left(1 - \tan^2\frac{\theta}{2}\right) + \left(1 + \tan^2\frac{\theta}{2}\right)}{\left(1 - \tan^2\frac{\theta}{2}\right) - \left(1 + \tan^2\frac{\theta}{2}\right)} = \frac{(\cos\alpha - \cos\beta) + (1 - \cos\alpha\cos\beta)}{(\cos\alpha - \cos\beta) - (1 - \cos\alpha\cos\beta)}$$

$$\Rightarrow \frac{2}{-2\tan^2\frac{\theta}{2}} = \frac{1 + \cos\alpha - \cos\beta - \cos\alpha\cos\beta}{-\{1 - \cos\alpha + \cos\beta - \cos\alpha\cos\beta\}}$$

$$\Rightarrow \frac{1}{\tan^2\frac{\theta}{2}} = \frac{(1 + \cos\alpha)(1 - \cos\beta)}{(1 - \cos\alpha)(1 + \cos\beta)}$$

$$\Rightarrow \frac{1}{\tan^2\frac{\theta}{2}} = \frac{2\cos^2\frac{\alpha}{2} \times 2\sin^2\frac{\beta}{2}}{2\sin^2\frac{\alpha}{2} \times 2\cos^2\frac{\beta}{2}} \Rightarrow \tan^2\frac{\theta}{2} = \tan^2\frac{\alpha}{2}\cot^2\frac{\beta}{2} \Rightarrow \tan\frac{\theta}{2} = \pm\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$$

EXAMPLE 40 *If* $\cos\theta = \dfrac{\cos\alpha\cos\beta}{1 - \sin\alpha\sin\beta}$, *prove that one value of* $\tan\dfrac{\theta}{2} = \dfrac{\tan\frac{\alpha}{2} - \tan\frac{\beta}{2}}{1 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}}$.

SOLUTION We have, $\cos\theta = \dfrac{\cos\alpha\cos\beta}{1 - \sin\alpha\sin\beta}$

Now, $\tan^2\dfrac{\theta}{2} = \dfrac{1 - \cos\theta}{1 + \cos\theta}$

$$\Rightarrow \tan^2\frac{\theta}{2} = \frac{1 - \dfrac{\cos\alpha\cos\beta}{1 - \sin\alpha\sin\beta}}{1 + \dfrac{\cos\alpha\cos\beta}{1 - \sin\alpha\sin\beta}}$$

$$\Rightarrow \tan^2\frac{\theta}{2} = \frac{1 - \sin\alpha\sin\beta - \cos\alpha\cos\beta}{1 - \sin\alpha\sin\beta + \cos\alpha\cos\beta}$$

$$\Rightarrow \tan^2\frac{\theta}{2} = \frac{1 - (\cos\alpha\cos\beta + \sin\alpha\sin\beta)}{1 + (\cos\alpha\cos\beta - \sin\alpha\sin\beta)}$$

$$\Rightarrow \tan^2\frac{\theta}{2} = \frac{1 - \cos(\alpha - \beta)}{1 + \cos(\alpha + \beta)}$$

$$\Rightarrow \tan^2\frac{\theta}{2} = \frac{2\sin^2\left(\frac{\alpha - \beta}{2}\right)}{2\cos^2\left(\frac{\alpha + \beta}{2}\right)}$$

$$\Rightarrow \quad \tan\frac{\theta}{2} = \pm\frac{\sin\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}$$

$$\Rightarrow \quad \tan\frac{\theta}{2} = \pm\frac{\sin\frac{\alpha}{2}\cos\frac{\beta}{2}-\cos\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}$$

$$\Rightarrow \quad \tan\frac{\theta}{2} = \pm\frac{\tan\frac{\alpha}{2}-\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}} \quad \left[\text{Dividing numerator and denominator by } \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\right]$$

EXAMPLE 41 *If* $\sin\alpha+\sin\beta = a$ *and* $\cos\alpha+\cos\beta = b$, *prove that*

(i) $\cos(\alpha-\beta) = \dfrac{a^2+b^2-2}{2}$ (ii) $\tan\dfrac{\alpha-\beta}{2} = \pm\sqrt{\dfrac{4-a^2-b^2}{a^2+b^2}}$

SOLUTION (i) We have,

$$\sin\alpha+\sin\beta = a \text{ and } \cos\alpha+\cos\beta = b$$

$$\Rightarrow \quad (\sin\alpha+\sin\beta)^2+(\cos\alpha+\cos\beta)^2 = a^2+b^2$$

$$\Rightarrow \quad (\cos^2\alpha+\sin^2\alpha)+(\cos^2\beta+\sin^2\beta)+2(\cos\alpha\cos\beta+\sin\alpha\sin\beta) = a^2+b^2$$

$$\Rightarrow \quad 2+2\cos(\alpha-\beta) = a^2+b^2$$

$$\Rightarrow \quad \cos(\alpha-\beta) = \frac{a^2+b^2-2}{2}$$

(ii) Now, $\tan^2\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{1+\cos\theta}$

$$\Rightarrow \quad \tan^2\left(\frac{\alpha-\beta}{2}\right) = \frac{1-\cos(\alpha-\beta)}{1+\cos(\alpha-\beta)}$$

$$\Rightarrow \quad \tan^2\left(\frac{\alpha-\beta}{2}\right) = \frac{1-\frac{a^2+b^2-2}{2}}{1+\frac{a^2+b^2-2}{2}}$$

$$\Rightarrow \quad \tan^2\left(\frac{\alpha-\beta}{2}\right) = \frac{4-a^2-b^2}{a^2+b^2}$$

$$\Rightarrow \quad \tan\left(\frac{\alpha-\beta}{2}\right) = \pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$$

EXAMPLE 42 *If* α *and* β *are distinct roots of* $a\cos\theta+b\sin\theta = c$, *prove that* $\sin(\alpha+\beta) = \dfrac{2ab}{a^2+b^2}$.

SOLUTION It is given that α and β are distinct roots of $a\cos\theta+b\sin\theta = c$

$$\therefore \quad a\cos\alpha+b\sin\alpha = c \text{ and } a\cos\beta+b\sin\beta = c$$

$$\Rightarrow \quad (a\cos\alpha+b\sin\alpha)-(a\cos\beta+b\sin\beta) = c-c$$

$$\Rightarrow \quad a(\cos\alpha-\cos\beta)+(b\sin\alpha-\sin\beta) = 0$$

$\Rightarrow \quad -2a \sin \dfrac{\alpha+\beta}{2} \sin \dfrac{\alpha-\beta}{2} + 2b \sin \dfrac{\alpha-\beta}{2} \cos \dfrac{\alpha+\beta}{2} = 0$

$\Rightarrow \quad 2a \sin \dfrac{\alpha+\beta}{2} \sin \dfrac{\alpha-\beta}{2} = 2b \sin \dfrac{\alpha-\beta}{2} \cos \dfrac{\alpha+\beta}{2}$

$\Rightarrow \quad \tan \dfrac{\alpha+\beta}{2} = \dfrac{b}{a}$ $\qquad \left[\because \alpha \neq \beta \therefore \sin \dfrac{\alpha-\beta}{2} \neq 0\right]$

$\therefore \quad \sin(\alpha+\beta) = \dfrac{2\tan\dfrac{\alpha+\beta}{2}}{1+\tan^2\dfrac{\alpha+\beta}{2}} \Rightarrow \sin(\alpha+\beta) = \dfrac{\dfrac{2b}{a}}{1+\dfrac{b^2}{a^2}} = \dfrac{2ab}{a^2+b^2}$

ALITER We have,

$a\cos\theta + b\sin\theta = c$ ...(i)

$\Rightarrow \quad a\left(\dfrac{1-\tan^2\dfrac{\theta}{2}}{1+\tan^2\dfrac{\theta}{2}}\right) + b\left(\dfrac{2\tan\dfrac{\theta}{2}}{1+\tan^2\dfrac{\theta}{2}}\right) = c$

$\Rightarrow \quad a\left(1-\tan^2\dfrac{\theta}{2}\right) + 2b\tan\dfrac{\theta}{2} = c\left(1+\tan^2\dfrac{\theta}{2}\right)$

$\Rightarrow \quad (c+a)\tan^2\dfrac{\theta}{2} - 2b\tan\dfrac{\theta}{2} + (c-a) = 0$ ...(ii)

It is given that α and β are roots of the equation (i). Therefore, $\tan\dfrac{\alpha}{2}$ and $\tan\dfrac{\beta}{2}$ are roots of equation (ii).

$\therefore \quad \tan\dfrac{\alpha}{2} + \tan\dfrac{\beta}{2} = \dfrac{2b}{c+a}$ and, $\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2} = \dfrac{c-a}{c+a}$ ...(iii)

Now, $\tan\left(\dfrac{\alpha}{2}+\dfrac{\beta}{2}\right) = \dfrac{\tan\dfrac{\alpha}{2}+\tan\dfrac{\beta}{2}}{1-\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}}$

$\Rightarrow \quad \tan\left(\dfrac{\alpha+\beta}{2}\right) = \dfrac{2b/c+a}{1-\dfrac{c-a}{c+a}} = \dfrac{b}{a}$ [Using (iii)]

$\therefore \quad \sin(\alpha+\beta) = \dfrac{2\tan\left(\dfrac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\dfrac{\alpha+\beta}{2}\right)} = \dfrac{2\dfrac{b}{a}}{1+\dfrac{b^2}{a^2}} = \dfrac{2ab}{a^2+b^2}$

EXERCISE 9.1

LEVEL-1

Prove that: (1– 27)

1. $\sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}} = \tan\theta$

2. $\dfrac{\sin 2\theta}{1-\cos 2\theta} = \cot\theta$

3. $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$

4. $\sqrt{2+\sqrt{2+2\cos 4\theta}} = 2\cos\theta$

5. $\dfrac{1-\cos 2\theta+\sin 2\theta}{1+\cos 2\theta+\sin 2\theta} = \tan\theta$

6. $\dfrac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta} = \tan\theta$

7. $\dfrac{\cos 2\theta}{1+\sin 2\theta} = \tan\left(\dfrac{\pi}{4}-\theta\right)$

8. $\dfrac{\cos\theta}{1-\sin\theta} = \tan\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)$

9. $\cos^2\dfrac{\pi}{8}+\cos^2\dfrac{3\pi}{8}+\cos^2\dfrac{5\pi}{8}+\cos^2\dfrac{7\pi}{8} = 2$

10. $\sin^2\dfrac{\pi}{8}+\sin^2\dfrac{3\pi}{8}+\sin^2\dfrac{5\pi}{8}+\sin^2\dfrac{7\pi}{8} = 2$

11. $(\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2 = 4\cos^2\left(\dfrac{\alpha-\beta}{2}\right)$

12. $\sin^2\left(\dfrac{\pi}{8}+\dfrac{A}{2}\right)-\sin^2\left(\dfrac{\pi}{8}-\dfrac{A}{2}\right)=\dfrac{1}{\sqrt{2}}\sin A$

13. $1+\cos^2 2\theta = 2(\cos^4\theta+\sin^4\theta)$

14. $\cos^3 2\theta+3\cos 2\theta = 4(\cos^6\theta-\sin^6\theta)$

15. $(\sin 3A+\sin A)\sin A+(\cos 3A-\cos A)\cos A = 0$

16. $\cos^2\left(\dfrac{\pi}{4}-\theta\right)-\sin^2\left(\dfrac{\pi}{4}-\theta\right) = \sin 2\theta$

17. $\cos 4A = 1-8\cos^2 A+8\cos^4 A$

18. $\sin 4A = 4\sin A\cos^3 A-4\cos A\sin^3 A$ **[NCERT EXEMPLAR]**

19. $3(\sin x-\cos x)^4+6(\sin x+\cos x)^2+4(\sin^6 x+\cos^6 x) = 13$

20. $2(\sin^6 x+\cos^6 x)-3(\sin^4 x+\cos^4 x)+1 = 0$

21. $\cos^6 A-\sin^6 A = \cos 2A\left(1-\dfrac{1}{4}\sin^2 2A\right)$

22. $\tan\left(\dfrac{\pi}{4}+\theta\right)+\tan\left(\dfrac{\pi}{4}-\theta\right) = 2\sec 2\theta$

23. $\cot^2 A-\tan^2 A = 4\cot 2A\,\mathrm{cosec}\,2A$

24. $\cos 4\theta-\cos 4\alpha = 8(\cos\theta-\cos\alpha)(\cos\theta+\cos\alpha)(\cos\theta-\sin\alpha)(\cos\theta+\sin\alpha)$

25. $\sin 3x+\sin 2x-\sin x = 4\sin x\cos\dfrac{x}{2}\cos\dfrac{3x}{2}$ **[NCERT]**

26. $\tan 82\dfrac{1}{2}^\circ = (\sqrt{3}+\sqrt{2})(\sqrt{2}+1) = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$

27. $\cot 22\dfrac{1}{2}^\circ = \sqrt{2}+1$

28. (i) If $\cos x = -\frac{3}{5}$ and x lies in the IIIrd quadrant, find the values of $\cos\frac{x}{2}$, $\sin\frac{x}{2}$ and, $\sin 2x$.

(ii) If $\cos x = -\frac{3}{5}$ and x lies in IInd quadrant, find the values of $\sin 2x$ and $\sin\frac{x}{2}$.

29. If $\sin x = \frac{\sqrt{5}}{3}$ and x lies in IInd quadrant, find the values of $\cos\frac{x}{2}$, $\sin\frac{x}{2}$ and $\tan\frac{x}{2}$.

30. (i) If $0 \le x \le \pi$ and x lies in the IInd quadrant such that $\sin x = \frac{1}{4}$. Find the values of

$\cos\frac{x}{2}$, $\sin\frac{x}{2}$ and $\tan\frac{x}{2}$.

(ii) If $\cos\theta = \frac{4}{5}$ and θ is acute, find $\tan 2\theta$

(iii) If $\sin\theta = \frac{4}{5}$ and $0 < \theta < \frac{\pi}{2}$, find the value of $\sin 4\theta$

31. If $\tan x = \frac{b}{a}$, then find the value of $\sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}}$. **[NCERT]**

32. If $\tan A = \frac{1}{7}$ and $\tan B = \frac{1}{3}$, show that $\cos 2A = \sin 4B$.

LEVEL-2

33. Prove that: $\cos 7^\circ \cos 14^\circ \cos 28^\circ \cos 56^\circ = \frac{\sin 68^\circ}{16 \cos 83^\circ}$

34. Prove that: $\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15} = \frac{1}{16}$

35. Prove that: $\cos\frac{\pi}{5}\cos\frac{2\pi}{5}\cos\frac{4\pi}{5}\cos\frac{8\pi}{5} = \frac{-1}{16}$

36. Prove that: $\cos\frac{\pi}{65}\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65} = \frac{1}{64}$

37. If $2\tan\alpha = 3\tan\beta$, prove that $\tan(\alpha - \beta) = \frac{\sin 2\beta}{5 - \cos 2\beta}$.

38. If $\sin\alpha + \sin\beta = a$ and $\cos\alpha + \cos\beta = b$, prove that

(i) $\sin(\alpha+\beta) = \frac{2ab}{a^2+b^2}$ (ii) $\cos(\alpha-\beta) = \frac{a^2+b^2-2}{2}$

39. If $2\tan\frac{\alpha}{2} = \tan\frac{\beta}{2}$, prove that $\cos\alpha = \frac{3+5\cos\beta}{5+3\cos\beta}$

40. If $\cos\theta = \frac{\cos\alpha + \cos\beta}{1+\cos\alpha\cos\beta}$, prove that $\tan\frac{\theta}{2} = \pm\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$

41. If $\sec(\theta+\alpha) + \sec(\theta-\alpha) = 2\sec\theta$, prove that $\cos\theta = \pm\sqrt{2}\cos\frac{\alpha}{2}$

42. If $\cos\alpha + \cos\beta = \frac{1}{3}$ and $\sin\alpha + \sin\beta = \frac{1}{4}$, prove that $\cos\frac{\alpha-\beta}{2} = \pm\frac{5}{24}$.

43. If $\sin\alpha = \frac{4}{5}$ and $\cos\beta = \frac{5}{13}$, prove that $\cos\frac{\alpha-\beta}{2} = \frac{8}{\sqrt{65}}$.

44. If $a \cos 2\theta + b \sin 2\theta = c$ has α and β as its roots, then prove that

(i) $\tan \alpha + \tan \beta = \frac{2b}{a+c}$ [NCERT EXEMPLAR] (ii) $\tan \alpha \tan \beta = \frac{c-a}{c+a}$

(iii) $\tan (\alpha + \beta) = \frac{b}{a}$ [NCERT EXEMPLAR]

45. If $\cos \alpha + \cos \beta = 0 = \sin \alpha + \sin \beta$, then prove that $\cos 2\alpha + \cos 2\beta = -2 \cos (\alpha + \beta)$.

[NCERT EXEMPLAR]

ANSWERS

28. (i) $-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, \frac{24}{25}$ (ii) $-\frac{24}{25}, \frac{2}{\sqrt{5}}$ 29. $\frac{1}{\sqrt{6}}, \sqrt{\frac{5}{6}}, \sqrt{5}$

30. (i) $\sqrt{\frac{4-\sqrt{15}}{8}}, \sqrt{\frac{4+\sqrt{15}}{8}}, 4+\sqrt{15}$ (ii) $\frac{24}{7}$ (iii) $-\frac{336}{625}$ 31. $\frac{2 \cos x}{\sqrt{\cos 2x}}$

HINTS TO NCERT & SELECTED PROBLEMS

25. LHS $= \sin 3x + \sin 2x - \sin x$

$= (\sin 3x - \sin x) + \sin 2x$

$= 2 \sin x \cos 2x + \sin 2x$

$= 2 \sin x \cos 2x + 2 \sin x \cos x$

$= 2 \sin x (\cos 2x + \cos x) = 2 \sin x \left(2 \cos \frac{3x}{2} \cos \frac{x}{2}\right) = 4 \sin x \cos \frac{x}{2} \cos \frac{3x}{2} =$ RHS

32. Use: $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$, $\sin 4B = \frac{2 \tan 2B}{1+\tan^2 2B}$, where $\tan 2B = \frac{2 \tan B}{1-\tan^2 B}$

33. Let $A = 7°$. Then,

$\cos 7° \cos 14° \cos 28° \cos 56° = \cos A \cos 2A \cos 2^2 A \cos 2^3 A$

$$= \frac{\sin 2^4 A}{2^4 \sin A} = \frac{\sin 16A}{16 \sin A} = \frac{\sin 112°}{2 \sin 7°} = \frac{\sin 68°}{2 \cos 83°}$$

34. Let $A = \frac{2\pi}{15}$. Then,

$\cos \frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15} \cos \frac{16\pi}{15}$

$= \cos A \cos 2A \cos 2^2 A \cos 2^3 A$

$$= \frac{\sin 2^4 A}{2^4 \sin A} = \frac{\sin 16A}{16 \sin A} = \frac{\sin \frac{32\pi}{15}}{16 \sin \frac{2\pi}{15}} = \frac{\sin\left(2\pi + \frac{2\pi}{15}\right)}{16 \sin \frac{2\pi}{15}} = \frac{\sin \frac{2\pi}{15}}{16 \sin \frac{2\pi}{15}} = \frac{1}{16} = \text{RHS}$$

44. We have,

$a \cos 2\theta + b \sin 2\theta = c$ ... (i)

$\Rightarrow a\left(\frac{1-\tan^2 \theta}{1+\tan^2 \theta}\right) + \frac{2b \tan \theta}{1+\tan^2 \theta} = c$

$\Rightarrow (c+a) \tan^2 \theta - 2b \tan \theta + (c-a) = 0$ ... (ii)

It is given that α, β are roots of equation (i). Therefore, $\tan \alpha$, $\tan \beta$ are roots of equation (ii).

$\therefore \tan \alpha + \tan \beta = \frac{2b}{c+a}$ and, $\tan \alpha \tan \beta = \frac{c-a}{c+a}$

Now,

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \Rightarrow \tan(\alpha+\beta)=\frac{2b/c+a}{1-\dfrac{c-a}{c+a}}=\frac{b}{a}.$$

45. We have, $\cos\alpha+\cos\beta=0$ and $\sin\alpha+\sin\beta=0$

$\therefore \quad (\cos\alpha+\cos\beta)^2-(\sin\alpha+\sin\beta)^2=0^2-0^2$

$\Rightarrow \quad (\cos^2\alpha-\sin^2\alpha)+(\cos^2\beta-\sin^2\beta)+2(\cos\alpha\cos\beta-\sin\alpha\sin\beta)=0$

$\Rightarrow \quad \cos 2\alpha+\cos 2\beta+2\cos(\alpha+\beta)=0$

$\Rightarrow \quad \cos 2\alpha+\cos 2\beta=-2\cos(\alpha+\beta)$

9.5 TRIGONOMETRIC RATIOS OF ANGLE 3*A* IN TERMS OF ANGLE *A*

THEOREM *For the values of angle A, for which the two sides are meaningful prove that:*

(i) $\sin 3A = 3\sin A-4\sin^3 A$ (ii) $\cos 3A = 4\cos^3 A-3\cos A$

(iii) $\tan 3A = \dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}$

PROOF (i) We know that

$\sin(A+B) = \sin A\cos B+\cos A\sin B.$

$\Rightarrow \quad \sin(A+2A) = \sin A\cos 2A+\cos A\sin 2A$ [Replacing B by $2A$]

$\Rightarrow \quad \sin 3A = \sin A(1-2\sin^2 A)+\cos A(2\sin A\cos A)$

[$\because \cos 2A=1-2\sin^2 A$ & $\sin 2A=2\sin A\cos A$]

$\Rightarrow \quad \sin 3A = \sin A-2\sin^3 A+2\sin A(1-\sin^2 A)$

$\Rightarrow \quad \sin 3A = 3\sin A-4\sin^3 A$

Hence, $\sin 3A = 3\sin A-4\sin^3 A$

(ii) We know that

$\cos(A+B) = \cos A\cos B-\sin A\sin B$

$\Rightarrow \quad \cos(A+2A) = \cos A\cos 2A-\sin A\sin 2A$ [Replacing B by $2A$]

$\Rightarrow \quad \cos 3A = \cos A\cos 2A-\sin A(2\sin A\cos A)$ [$\because \sin 2A=2\sin A\cos A$]

$\Rightarrow \quad \cos 3A = \cos A(2\cos^2 A-1)-2\cos A(1-\cos^2 A)$ [$\because \cos 2A=2\cos^2 A-1$]

$\Rightarrow \quad \cos 3A = 4\cos^3 A-3\cos A$

Hence, $\cos 3A = 4\cos^3 A-3\cos A$

(iii) We know that

$$\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$\Rightarrow \quad \tan(A+2A) = \frac{\tan A+\tan 2A}{1-\tan A\tan 2A} \quad \text{[Replacing } B \text{ by } 2A]$$

$$\Rightarrow \quad \tan 3A = \frac{\tan A+\dfrac{2\tan A}{1-\tan^2 A}}{1-\tan A\times\dfrac{2\tan A}{1-\tan^2 A}} \quad \left[\because \tan 2A = \frac{2\tan A}{1-\tan^2 A}\right]$$

$$\Rightarrow \quad \tan 3A = \frac{\tan A(1-\tan^2 A)+2\tan A}{1-\tan^2 A-2\tan^2 A} = \frac{3\tan A-\tan^3 A}{1-3\tan^2 A}$$

$$\text{Hence,} \quad \tan 3A = \frac{3\tan A-\tan^3 A}{1-3\tan^2 A}$$

Q.E.D.

REMARK *It should be noted the angle on the RHS of these formulae is one third of the angle on LHS.*

$\therefore \quad \sin 60° = 3\sin 20°-4\sin^3 20°,\ \sin 30° = 3\sin 10°-4\sin^3 10°,$

$\cos 120° = 4\cos^3 40°-3\cos 40°$ etc.

9.6 TRIGONOMETRIC RATIOS OF ANGLE *A* IN TERMS OF ANGLE *A*/3

Replacing A by $A/3$ in the formulas in the above section, we obtain the following formulae:

(i) $\sin A = 3\sin\left(\frac{A}{3}\right) - 4\sin^3\left(\frac{A}{3}\right)$

(ii) $\cos A = 4\cos^3\left(\frac{A}{3}\right) - 3\cos\left(\frac{A}{3}\right)$

(iii) $\tan A = \dfrac{3\tan\left(\frac{A}{3}\right) - \tan^3\left(\frac{A}{3}\right)}{1 - 3\tan^2\left(\frac{A}{3}\right)}$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Prove that :* $8\cos^3\frac{\pi}{9} - 6\cos\frac{\pi}{9} = 1$

SOLUTION We have,

$$\text{LHS} = 2\left(4\cos^3\frac{\pi}{9} - 3\cos\frac{\pi}{9}\right) = 2\cos\left(3\times\frac{\pi}{9}\right) = 2\cos\frac{\pi}{3} = 1 = \text{RHS}$$

EXAMPLE 2 *Prove that:* $108\sin\frac{\pi}{18} - 144\sin^3\frac{\pi}{18} = 18$

SOLUTION We have,

$$\text{LHS} = 108\sin\frac{\pi}{18} - 144\sin^3\frac{\pi}{18} = 36\left(3\sin\frac{\pi}{18} - 4\sin^3\frac{\pi}{18}\right)$$

$$= 36\sin\left(3\times\frac{\pi}{18}\right) = 36\sin\frac{\pi}{6} = 36\times\frac{1}{2} = 18 = \text{RHS}$$

EXAMPLE 3 *Prove that:* $15\sin\frac{5\pi}{12} + 15\cos\frac{5\pi}{12} - 20\sin^3\frac{5\pi}{12} - 20\cos^3\frac{5\pi}{12} = 0$

SOLUTION We have,

$$\text{LHS} = 15\sin\frac{5\pi}{12} + 15\cos\frac{5\pi}{12} - 20\sin^3\frac{5\pi}{12} - 20\cos^3\frac{5\pi}{12}$$

$$= \left(15\sin\frac{5\pi}{12} - 20\sin^3\frac{5\pi}{12}\right) - \left(20\cos^3\frac{5\pi}{12} - 15\cos\frac{5\pi}{12}\right)$$

$$= 5\left(3\sin\frac{5\pi}{12} - 4\sin^3\frac{5\pi}{12}\right) - 5\left(4\cos^3\frac{5\pi}{12} - 3\cos\frac{5\pi}{12}\right)$$

$$= 5\sin\left(3\times\frac{5\pi}{12}\right) - 5\cos\left(3\times\frac{5\pi}{12}\right) = 5\sin\frac{5\pi}{4} - 5\cos\frac{5\pi}{4} = -5\sin\frac{\pi}{4} + 5\cos\frac{\pi}{4} = 0$$

EXAMPLE 4 *Prove that:* $\cos 6A = 32\cos^6 A - 48\cos^4 A + 18\cos^2 A - 1$ **[NCERT]**

SOLUTION We have,

$\text{LHS} = \cos 6A$

$\Rightarrow \text{LHS} = 2\cos^2 3A - 1$ $[\because \cos 2\theta = 2\cos^2\theta - 1]$

$\Rightarrow \text{LHS} = 2(4\cos^3 A - 3\cos A)^2 - 1$

$\Rightarrow \text{LHS} = 2(16\cos^6 A + 9\cos^2 A - 24\cos^4 A) - 1$

$\Rightarrow \text{LHS} = 32\cos^6 A - 48\cos^4 A + 18\cos^2 A - 1 = \text{RHS}$

EXAMPLE 5 *Prove that:* $\cos A \cos(60 - A)\cos(60 + A) = \frac{1}{4}\cos 3A$

SOLUTION We have,

$$\text{LHS} = \cos A \cos(60 - A)\cos(60 + A)$$

$$\Rightarrow \quad \text{LHS} = \cos A(\cos^2 60° - \sin^2 A) \qquad [\because \cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B]$$

$$\Rightarrow \quad \text{LHS} = \cos A\left(\frac{1}{4} - \sin^2 A\right) = \cos A\left\{\frac{1}{4} - (1 - \cos^2 A)\right\} = \cos A\left(-\frac{3}{4} + \cos^2 A\right)$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{4}\cos A(-3 + 4\cos^2 A) = \frac{1}{4}(4\cos^3 A - 3\cos A) = \frac{1}{4}\cos 3A = \text{RHS}$$

EXAMPLE 6 *Prove that:* $\sin A \sin(60 - A)\sin(60 + A) = \frac{1}{4}\sin 3A$

SOLUTION We have,

$$\text{LHS} = \sin A \sin(60 - A)\sin(60 + A)$$

$$\Rightarrow \quad \text{LHS} = \sin A(\sin^2 60° - \sin^2 A) \qquad [\because \sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B]$$

$$\Rightarrow \quad \text{LHS} = \sin A\left(\frac{3}{4} - \sin^2 A\right)$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{4}\sin A(3 - 4\sin^2 A)$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{4}(3\sin A - 4\sin^3 A) = \frac{1}{4}\sin 3A.$$

<u>NOTE</u> *Reader is advised to learn the results derived in the above two examples as standard results. The following example is an application of the above results.*

EXAMPLE 7 *Prove that:* $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$

SOLUTION We have,

$$\text{LHS} = \frac{\sqrt{3}}{2}\left\{\sin 20° \sin(60° - 20°)\sin(60 + 20°)\right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{3}}{2}\left\{(\sin A \sin(60° - A)\sin(60° + A)\right\}, \text{ where } A = 20°$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{3}}{2} \times \frac{1}{4}\sin 3A = \frac{\sqrt{3}}{8} \times \sin 60° = \frac{\sqrt{3}}{8} \times \frac{\sqrt{3}}{2} = \frac{3}{16} = \text{RHS}$$

EXAMPLE 8 *Prove that:*

(i) $\tan A + \tan(60° + A) - \tan(60° - A) = 3\tan 3A$

(ii) $\cot A + \cot(60° + A) - \cot(60° - A) = 3\cot 3A$

SOLUTION (i) We have,

$$\text{LHS} = \tan A + \tan(60° + A) - \tan(60° - A)$$

$$\Rightarrow \quad \text{LHS} = \tan A + \frac{\sqrt{3} + \tan A}{1 - \sqrt{3}\tan A} - \frac{\sqrt{3} - \tan A}{1 + \sqrt{3}\tan A}$$

$$\Rightarrow \quad \text{LHS} = \tan A + \frac{(\sqrt{3} + \tan A)(1 + \sqrt{3}\tan A) - (\sqrt{3} - \tan A)(1 - \sqrt{3}\tan A)}{(1 - \sqrt{3}\tan A)(1 + \sqrt{3}\tan A)}$$

$$\Rightarrow \quad \text{LHS} = \tan A + \frac{8\tan A}{1 - 3\tan^2 A}$$

$$\Rightarrow \quad \text{LHS} = \frac{9\tan A - 3\tan^3 A}{1 - 3\tan^2 A} = 3\left(\frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}\right) = 3\tan 3A = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \cot A + \cot(60° + A) - \cot(60° - A)$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{\tan A} + \frac{1}{\tan(60° + A)} - \frac{1}{\tan(60° - A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{\tan A} + \frac{1 - \sqrt{3}\tan A}{\sqrt{3} + \tan A} - \frac{1 + \sqrt{3}\tan A}{\sqrt{3} - \tan A}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{\tan A} + \frac{(1 - \sqrt{3}\tan A)(\sqrt{3} - \tan A) - (1 + \sqrt{3}\tan A)(\sqrt{3} + \tan A)}{(\sqrt{3} + \tan A)(\sqrt{3} - \tan A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{\tan A} - \frac{8\tan A}{3 - \tan^2 A}$$

$$\Rightarrow \quad \text{LHS} = \frac{3 - 9\tan^2 A}{3\tan A - \tan^3 A} = 3\left(\frac{1 - 3\tan^2 A}{3\tan A - \tan^3 A}\right) = \frac{3}{\tan 3A} = 3\cot 3A = \text{RHS}$$

LEVEL-2

EXAMPLE 9 *If* $\cos\alpha + \cos\beta + \cos\gamma = 0$, *then prove that*

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 12\cos\alpha\cos\beta\cos\gamma$$

SOLUTION $\cos 3\alpha + \cos 3\beta + \cos 3\gamma$

$$= (4\cos^3\alpha - 3\cos\alpha) + (4\cos^3\beta - 3\cos\beta) + (4\cos^3\gamma - 3\cos\gamma)$$

$$= 4(\cos^3\alpha + \cos^3\beta + \cos^3\gamma) - 3(\cos\alpha + \cos\beta + \cos\gamma)$$

$$= 4(\cos^3\alpha + \cos^3\beta + \cos^3\gamma) - 3 \times 0$$

$$= 4 \times 3\cos\alpha\cos\beta\cos\gamma \qquad [\because a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc]$$

$$= 12\cos\alpha\cos\beta\cos\gamma$$

EXAMPLE 10 *Prove that:* $\sin 3A\sin^3 A + \cos 3A\cos^3 A = \cos^3 2A$

SOLUTION We know that

$$\sin 3A = 3\sin A - 4\sin^3 A \Rightarrow \sin^3 A = \frac{3\sin A - \sin 3A}{4}$$

Similarly,

$$\cos 3A = 4\cos^3 A - 3\cos A \Rightarrow \cos^3 A = \frac{\cos 3A + 3\cos A}{4}$$

$$\therefore \quad \text{LHS} = \sin 3A\sin^3 A + \cos 3A\cos^3 A$$

$$\Rightarrow \quad \text{LHS} = \sin 3A\left\{\frac{3\sin A - \sin 3A}{4}\right\} + \cos 3A\left\{\frac{\cos 3A + 3\cos A}{4}\right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{4}\left\{3(\cos A\cos 3A + \sin A\sin 3A) + (\cos^2 3A - \sin^2 3A)\right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{4}\left\{3\cos(3A - A) + \cos 2(3A)\right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{4}\left\{3\cos 2A + \cos 3(2A)\right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{4}\left\{3\cos 2A + (4\cos^3 2A - 3\cos 2A)\right\} = \cos^3 2A = \text{RHS}$$

EXAMPLE 11 *Prove that:* $\cos^3 A + \cos^3(120° + A) + \cos^3(240° + A) = \dfrac{3}{4}\cos 3A$

SOLUTION We know that $\cos 3A = 4\cos^3 A - 3\cos A$

$$\therefore \cos^3 A = \frac{1}{4}(\cos 3A + 3\cos A)$$

Now,

$$\text{LHS} = \frac{1}{4}\left\{\cos 3A + 3\cos A\right\} + \frac{1}{4}\left\{\cos(360° + 3A) + 3\cos(120° + A)\right\}$$
$$+ \frac{1}{4}\left\{\cos(720° + 3A) + 3\cos(240° + A)\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{4}\left\{\cos 3A + 3\cos A\right\} + \frac{1}{4}\left\{\cos 3A + 3\cos 120° + A)\right\} + \frac{1}{4}\left\{\cos 3A + 3\cos(240° + A)\right\}$$

$$\Rightarrow \text{LHS} = \frac{3}{4}\cos 3A + \frac{3}{4}\left\{\cos A + \cos(120° + A) + \cos(240° + A)\right\}$$

$$\Rightarrow \text{LHS} = \frac{3}{4}\cos 3A + \frac{3}{4}\left\{\cos A + 2\cos(180° + A)\cos 60°\right\}$$

$$\Rightarrow \text{LHS} = \frac{3}{4}\cos 3A + \frac{3}{4}\left\{\cos A - 2\cos A \times \frac{1}{2}\right\} = \frac{3}{4}\cos 3A = \text{RHS}$$

<u>ALITER</u> We have,

$$\cos A + \cos(120° + A) + \cos(240° + A)$$
$$= \cos A + 2\cos\left(\frac{240° + A + 120° + A}{2}\right)\cos\left(\frac{240° + A - 120° - A}{2}\right)$$
$$= \cos A + 2\cos(180° + A)\cos 60° = \cos A - 2(\cos A) \times \frac{1}{2}$$
$$= \cos A - \cos A = 0$$

$$\therefore \cos^3 A + \cos^3(120° + A) + \cos^3(240° + A)$$
$$= 3\cos A\cos(120° + A)\cos(240° + A) \qquad [\because a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc]$$
$$= 3\cos A\cos(180° - 60° + A)\cos(180° + 60° + A)$$
$$= 3\cos A\cos\{180° - (60° - A)\}\cos\{180° + (60° + A)\}$$
$$= (3\cos A)\{-\cos(60° - A)\}\{-\cos(60° + A))\}$$
$$= 3\cos A\cos(60° - A)\cos(60° + A) = 3 \times \frac{1}{4}\cos 3A = \frac{3}{4}\cos 3A$$

EXAMPLE 12 *Prove that* $\dfrac{\tan 3x}{\tan x}$ *never lies between* $\dfrac{1}{3}$ *and 3.*

SOLUTION Let $y = \dfrac{\tan 3x}{\tan x}$. Then,

$$y = \frac{3\tan x - \tan^3 x}{\tan x(1 - 3\tan^2 x)}$$

$$\Rightarrow y = \frac{3 - \tan^2 x}{1 - 3\tan^2 x} \Rightarrow (3y - 1)\tan^2 x = y - 3 \Rightarrow \tan^2 x = \frac{y - 3}{3y - 1}$$

But, $\tan^2 x \geq 0$ for all x

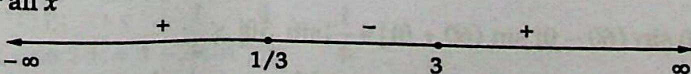


Fig. 9.1 Signs of $\dfrac{y-3}{3y-1}$ for different values of y

$\therefore \quad \dfrac{y-3}{3y-1} \geq 0$

$\Rightarrow \quad y < \dfrac{1}{3}$ or, $y \geq 3$

$\Rightarrow \quad y$ does not lie between 1/3 and 3.

Hence, $\dfrac{\tan 3x}{\tan x}$ never lies between $\dfrac{1}{3}$ and 3.

EXAMPLE 13 *Prove that:* $\cos 5A = 16 \cos^5 A - 20 \cos^3 A + 5 \cos A$

SOLUTION We have,

$$\cos 5A = \cos(3A + 2A) = \cos 3A \cos 2A - \sin 3A \sin 2A$$

$$\Rightarrow \quad \cos 5A = (4\cos^3 A - 3\cos A)(2\cos^2 A - 1) - (3\sin A - 4\sin^3 A)(2\sin A \cos A)$$

$$\Rightarrow \quad \cos 5A = (4\cos^3 A - 3\cos A)(2\cos^2 A - 1) - (3 - 4\sin^2 A)(2\sin^2 A \cos A)$$

$$\Rightarrow \quad \cos 5A = (4\cos^3 A - 3\cos A)(2\cos^2 A - 1) - \{3 - 4(1 - \cos^2 A)\} 2(1 - \cos^2 A)\cos A$$

$$\Rightarrow \quad \cos 5A = (8\cos^5 A - 10\cos^3 A + 3\cos A) - 2\cos A(1 - \cos^2 A)(4\cos^2 A - 1)$$

$$\Rightarrow \quad \cos 5A = (8\cos^5 A - 10\cos^3 A + 3\cos A) - 2\cos A(5\cos^2 A - 4\cos^4 A - 1)$$

$$\Rightarrow \quad \cos 5A = 16\cos^5 A - 20\cos^3 A + 5\cos A = \text{RHS}$$

EXERCISE 9.2

LEVEL-1

Prove the following identities (1–8)

1. $\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$
2. $4(\cos^3 10° + \sin^3 20°) = 3(\cos 10° + \sin 20°)$
3. $\cos^3\theta \sin 3\theta + \sin^3\theta \cos 3\theta = \dfrac{3}{4}\sin 4\theta$
4. $\sin 5A = 5\cos^4 A \sin A - 10\cos^2 A \sin^3 A + \sin^5 A$
5. $\tan\theta \tan(\theta + 60°) + \tan\theta \tan(\theta - 60°) + \tan(\theta + 60°)\tan(\theta - 60°) = -3$
6. $\tan A + \tan(60° + A) - \tan(60° - A) = 3\tan 3A$
7. $\cot A + \cot(60° + A) - \cot(60° - A) = 3\cot 3A$
8. $\cot A + \cot(60° + A) + \cot(120° + A) = 3\cot 3A$

LEVEL-2

9. Prove that: $\sin^3 A + \sin^3\left(\dfrac{2\pi}{3} + A\right) + \sin^3\left(\dfrac{4\pi}{3} + A\right) = -\dfrac{3}{4}\sin 3A.$
10. Prove that: $|\sin\theta \sin(60 - \theta)\sin(60 + \theta)| \leq \dfrac{1}{4}$ for all values of θ.
11. Prove that: $|\cos\theta \cos(60° - \theta)\cos(60° + \theta)| \leq \dfrac{1}{4}$ for all values of θ.

HINTS TO SELECTED PROBLEMS

10. We have, $|\sin\theta \sin(60 - \theta)\sin(60 + \theta)| = \dfrac{1}{4}|\sin 3\theta| \leq \dfrac{1}{4}$ $\quad [\because |\sin 3\theta| \leq 1]$
11. We have, $|\cos\theta \cos(60° - \theta)\cos(60° + \theta)| = \left|\dfrac{1}{4}\cos 3\theta\right| \leq \dfrac{1}{4}$

9.7 TRIGONOMETRICAL RATIOS OF SOME IMPORTANT ANGLES

By using the formulae introduced in the previous sections we can now find the trigonometrical ratios of some important angles of degree measures 18°, 36°, 54° etc.

THEOREM 1 *Prove that:* $\sin 18° = \dfrac{\sqrt{5}-1}{4}$.

PROOF Let $\theta = 18°$. Then,

$$5\theta = 90°$$
$$\Rightarrow \quad 2\theta + 3\theta = 90°$$
$$\Rightarrow \quad 2\theta = 90° - 3\theta$$
$$\Rightarrow \quad \sin 2\theta = \sin(90° - 3\theta)$$
$$\Rightarrow \quad \sin 2\theta = \cos 3\theta$$
$$\Rightarrow \quad 2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$$
$$\Rightarrow \quad \cos\theta(2\sin\theta - 4\cos^2\theta + 3) = 0$$
$$\Rightarrow \quad 2\sin\theta - 4\cos^2\theta + 3 = 0 \qquad [\because \cos\theta = \cos 18° \neq 0]$$
$$\Rightarrow \quad 2\sin\theta - 4(1-\sin^2\theta) + 3 = 0$$
$$\Rightarrow \quad 4\sin^2\theta + 2\sin\theta - 1 = 0$$
$$\Rightarrow \quad \sin\theta = \frac{-2 \pm \sqrt{4+16}}{8}$$
$$\Rightarrow \quad \sin\theta = \frac{-1 \pm \sqrt{5}}{4}$$
$$\Rightarrow \quad \sin\theta = \frac{-1+\sqrt{5}}{4} = \frac{\sqrt{5}-1}{4} \qquad [\because \theta \text{ lies in Ist quadrant } \therefore \sin\theta > 0]$$

Hence, $\sin 18° = \dfrac{\sqrt{5}-1}{4}$

Q.E.D.

THEOREM 2 *Prove that:* $\cos 18° = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$.

PROOF Putting $\theta = 18°$ in $\cos\theta = \sqrt{1-\sin^2\theta}$, we get

$$\cos 18° = \sqrt{1-\sin^2 18°} = \sqrt{1-\left(\frac{\sqrt{5}-1}{4}\right)^2} = \sqrt{\frac{16-(5+1-2\sqrt{5})}{16}} = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

Hence, $\cos 18° = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$

Q.E.D.

REMARK *The complement of* 18° *is* 72°.

$$\therefore \quad \sin 72° = \sin(90° - 18°) = \cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

and, $\quad \cos 72° = \cos(90° - 18°) = \sin 18° = \dfrac{\sqrt{5}-1}{4}$

The remaining trigonometrical ratios of **18°** ***may be obtained from the above values.***

THEOREM 3 Prove that: $\cos 36° = \dfrac{\sqrt{5}+1}{4}$.

PROOF We have, $\cos 2\theta = 1 - 2\sin^2\theta$

$$\therefore \quad \cos 36° = 1 - 2\sin^2 18° \qquad [\text{Putting } \theta = 18°]$$

$$\Rightarrow \quad \cos 36^\circ = 1-2\left(\frac{\sqrt{5}-1}{4}\right)^2 = 1-2\left(\frac{6-2\sqrt{5}}{16}\right) = 1-\left(\frac{3-\sqrt{5}}{4}\right) = \frac{\sqrt{5}+1}{4}$$

Hence, $\cos 36^\circ = \dfrac{\sqrt{5}+1}{4}$

Q.E.D.

THEOREM 4 *Prove that:* $\sin 36^\circ = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$.

PROOF Putting $\theta = 36^\circ$ in $\sin\theta = \sqrt{1-\cos^2\theta}$, we obtain

$$\sin 36^\circ = \sqrt{1-\cos^2 36^\circ} = \sqrt{1-\left(\frac{\sqrt{5}+1}{4}\right)^2} = \sqrt{\frac{16-(6+2\sqrt{5})}{16}} = \frac{\sqrt{10-2\sqrt{5}}}{4}$$

Hence, $\sin 36^\circ = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$

Q.E.D.

REMARK *The complement of 36° is 54°.*

$\therefore \ \sin 54^\circ = \sin(90^\circ - 36^\circ) = \cos 36^\circ = \dfrac{\sqrt{5}+1}{4}$ *and,* $\cos 54^\circ = \cos(90^\circ-36^\circ) = \sin 36^\circ = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$

The other trigonometrical ratios of 36° may be obtained from the above values.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Prove that:*

(i) $\sin^2 72^\circ - \sin^2 60^\circ = \dfrac{\sqrt{5}-1}{8}$ (ii) $\cos^2 48^\circ - \sin^2 12^\circ = \dfrac{\sqrt{5}+1}{8}$

(iii) $\sin\dfrac{\pi}{10} + \sin\dfrac{13\pi}{10} = -\dfrac{1}{2}$ (iv) $\sin\dfrac{\pi}{10}\sin\dfrac{13\pi}{10} = -\dfrac{1}{4}$

(v) $\sin^2 24^\circ - \sin^2 6^\circ = \dfrac{\sqrt{5}-1}{8}$

SOLUTION (i) We have,

$$\text{LHS} = \sin^2 72^\circ - \sin^2 60^\circ$$

$$\Rightarrow \quad \text{LHS} = \cos^2 18^\circ - \sin^2 60^\circ \qquad [\because \sin 72^\circ = \cos 18^\circ]$$

$$\Rightarrow \quad \text{LHS} = \left\{\frac{\sqrt{10+2\sqrt{5}}}{4}\right\}^2 - \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{10+2\sqrt{5}}{16} - \frac{3}{4} = \frac{2\sqrt{5}-2}{16} = \frac{\sqrt{5}-1}{8} = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \cos^2 48^\circ - \sin^2 12^\circ$$

$$\Rightarrow \quad \text{LHS} = \cos(48^\circ+12^\circ)\cos(48^\circ-12^\circ) \qquad [\because \cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B)]$$

$$\Rightarrow \quad \text{LHS} = \cos 60^\circ \cos 36^\circ = \frac{1}{2}\times\frac{\sqrt{5}+1}{4} = \frac{\sqrt{5}+1}{8} = \text{RHS}$$

(iii) We have,

$$\text{LHS} = \sin\frac{\pi}{10} + \sin\frac{13\pi}{10}$$

$$\Rightarrow \quad \text{LHS} = \sin 18^\circ + \sin 234^\circ = \sin 18^\circ + \sin(270^\circ - 36^\circ)$$

$$\Rightarrow \quad \text{LHS} = \sin 18^\circ - \cos 36^\circ = \frac{\sqrt{5}-1}{4} - \frac{\sqrt{5}+1}{4} = -\frac{1}{2}$$

(iv) We have,

$$\Rightarrow \quad \text{LHS} = \sin\frac{\pi}{10}\sin\frac{13\pi}{10}$$

$$\Rightarrow \quad \text{LHS} = \sin 18^\circ \sin 234^\circ = -\sin 18^\circ \cos 36^\circ = -\frac{\sqrt{5}-1}{4}\times\frac{\sqrt{5}+1}{4} = -\left(\frac{5-1}{16}\right) = -\frac{1}{4} = \text{RHS}$$

(v) We have,

$$\text{LHS} = \sin^2 24^\circ - \sin^2 6^\circ$$

$$\Rightarrow \quad \text{LHS} = \sin(24^\circ + 6^\circ)\sin(24^\circ - 6^\circ) \qquad [\because \sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B]$$

$$\Rightarrow \quad \text{LHS} = \sin 30^\circ \sin 18^\circ = \frac{1}{2}\times\frac{\sqrt{5}-1}{4} = \frac{\sqrt{5}-1}{8} = \text{RHS}$$

EXAMPLE 2 *Prove that:* $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}$

SOLUTION If $A + B = \pi$, then $A = \pi - B \Rightarrow \sin A = \sin(\pi - B) \Rightarrow \sin A = \sin B$

$$\therefore \quad \frac{\pi}{5}+\frac{4\pi}{5} = \pi \Rightarrow \sin\frac{\pi}{5} = \sin\frac{4\pi}{5} \text{ and } \frac{2\pi}{5}+\frac{3\pi}{5} = \pi \Rightarrow \sin\frac{2\pi}{5} = \sin\frac{3\pi}{5}$$

Using these values, we obtain

$$\text{LHS} = \sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5}$$

$$\Rightarrow \quad \text{LHS} = \sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{2\pi}{5}\sin\frac{\pi}{5}$$

$$\Rightarrow \quad \text{LHS} = \left(\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\right)^2 = (\sin 36^\circ \sin 72^\circ)^2 = (\sin 36^\circ \cos 18^\circ)^2$$

$$\Rightarrow \quad \text{LHS} = \left\{\frac{\sqrt{10-2\sqrt{5}}}{4}\times\frac{\sqrt{10+2\sqrt{5}}}{4}\right\}^2 = \frac{10-2\sqrt{5}}{16}\times\frac{10+2\sqrt{5}}{16} = \frac{100-20}{256} = \frac{80}{256} = \frac{5}{16} = \text{RHS}$$

EXAMPLE 3 *Prove that:* $16\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{14\pi}{15} = 1$

SOLUTION We have,

$$\text{LHS} = 16\cos 24^\circ \cos 48^\circ \cos 96^\circ \cos 168^\circ$$

$$\Rightarrow \quad \text{LHS} = 4(2\cos 24^\circ \cos 96^\circ)(2\cos 48^\circ \cos 168^\circ)$$

$$\Rightarrow \quad \text{LHS} = 4(\cos 120^\circ + \cos 72^\circ)(\cos 216^\circ + \cos 120^\circ)$$

$$\Rightarrow \quad \text{LHS} = 4(-\sin 30^\circ + \sin 18^\circ)(-\cos 36^\circ - \sin 30^\circ)$$

$$\Rightarrow \quad \text{LHS} = 4\left(-\frac{1}{2}+\frac{\sqrt{5}-1}{4}\right)\left(-\frac{\sqrt{5}+1}{4}-\frac{1}{2}\right)$$

$$\Rightarrow \quad \text{LHS} = 4\left(\frac{\sqrt{5}-3}{4}\right)\left(\frac{-\sqrt{5}-3}{4}\right) = 4\left(\frac{3-\sqrt{5}}{4}\right)\left(\frac{3+\sqrt{5}}{4}\right) = \left(\frac{9-5}{4}\right) = 1 = \text{RHS}$$

EXAMPLE 4 *Prove that:* $\sin 12^\circ \sin 48^\circ \sin 54^\circ = \frac{1}{8}$.

SOLUTION We have,

$$\text{LHS} = \frac{1}{2}(2\sin 48^\circ \sin 12^\circ)\sin 54^\circ$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{2}(\cos 36^\circ - \cos 60^\circ)\cos 36^\circ \qquad [\because 2\sin A \sin B = \cos(A-B) - \cos(A+B)]$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{2}\left(\frac{\sqrt{5}+1}{4}-\frac{1}{2}\right)\left(\frac{\sqrt{5}+1}{4}\right) = \frac{1}{2}\left(\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4}\right) = \frac{1}{8} = \text{RHS}$$

EXAMPLE 5 *Prove that:* $\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right)=\frac{1}{16}$.

SOLUTION If $A+B=\pi$, then $\cos A=\cos(\pi-B)=-\cos B$

$$\therefore \quad \frac{\pi}{10}+\frac{9\pi}{10}=\pi \Rightarrow \cos\frac{9\pi}{10}=-\cos\frac{\pi}{10} \text{ and, } \frac{3\pi}{10}+\frac{7\pi}{10}=\pi \Rightarrow \cos\frac{7\pi}{10}=-\cos\frac{3\pi}{10}$$

Using these values, we obtain

$$\text{LHS}=\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1-\cos\frac{3\pi}{10}\right)\left(1-\cos\frac{\pi}{10}\right)$$

$$\Rightarrow \quad \text{LHS}=\left(1-\cos^2\frac{\pi}{10}\right)\left(1-\cos^2\frac{3\pi}{10}\right)$$

$$\Rightarrow \quad \text{LHS}=(1-\cos^2 18°)(1-\cos^2 54°)=\sin^2 18°\sin^2 54=\sin^2 18°\cos^2 36°$$

$$\Rightarrow \quad \text{LHS}=(\sin 18°\cos 36°)^2=\left(\frac{\sqrt{5}-1}{4}\times\frac{\sqrt{5}+1}{4}\right)^2=\left(\frac{1}{4}\right)^2=\frac{1}{16}=\text{RHS}$$

EXAMPLE 6 *Prove that:* $\tan 6°\tan 42°\tan 66°\tan 78°=1$.

SOLUTION We have,

$$\text{LHS}=\frac{\sin 6°\sin 42°\sin 66°\sin 78°}{\cos 6°\cos 42°\cos 66°\cos 78°}=\frac{(2\sin 66°\sin 6°)(2\sin 78°\sin 42°)}{(2\cos 66°\cos 6°)(2\cos 78°\cos 42°)}$$

$$\Rightarrow \quad \text{LHS}=\frac{(\cos 60°-\cos 72°)(\cos 36°-\cos 120°)}{(\cos 60°+\cos 72°)(\cos 36°+\cos 120°)}$$

$$\Rightarrow \quad \text{LHS}=\frac{(\cos 60°-\sin 18°)(\cos 36°+\sin 30°)}{(\cos 60°+\sin 18°)(\cos 36°-\sin 30°)}$$

$$\Rightarrow \quad \text{LHS}=\frac{\left(\frac{1}{2}-\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4}+\frac{1}{2}\right)}{\left(\frac{1}{2}+\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}-1}{4}-\frac{1}{2}\right)}=\frac{(3-\sqrt{5})(3+\sqrt{5})}{(\sqrt{5}+1)(\sqrt{5}-1)}=\frac{9-5}{5-1}=1=\text{RHS}$$

LEVEL-2

EXAMPLE 7 *Prove that :* $4\sin 27°=\sqrt{(5+\sqrt{5})}-\sqrt{(3-\sqrt{5})}$.

SOLUTION We have,

$$16\sin^2 27°=8(2\sin^2 27°)=8(1-\cos 54°)=8(1-\sin 36°)$$

$$=8\left\{1-\frac{\sqrt{10-2\sqrt{5}}}{4}\right\}=2\left\{4-\sqrt{10-2\sqrt{5}}\right\}=8-2\sqrt{10-2\sqrt{5}}$$

$$=(5+\sqrt{5})+(3-\sqrt{5})-2\sqrt{(5+\sqrt{5})(3-\sqrt{5})}$$

$$=\left\{\sqrt{5+\sqrt{5}}\right\}^2+\left\{\sqrt{3-\sqrt{5}}\right\}^2-2\sqrt{(5+\sqrt{5})(3-\sqrt{5})}$$

$$=\left\{\sqrt{5+\sqrt{5}}-\sqrt{3-\sqrt{5}}\right\}^2$$

Taking square roots of both sides, we obtain

$$\therefore \quad 4\sin 27°=\sqrt{5+\sqrt{5}}-\sqrt{3-\sqrt{5}} \qquad [\because \sin 27° \text{ is positive}]$$

EXAMPLE 8 *Find the value of* $\tan 9° - \tan 27° - \tan 63° + \tan 81°$.

SOLUTION $\tan 9° - \tan 27° - \tan 63° + \tan 81°$

$= (\tan 9° + \tan 81°) - (\tan 27° + \tan 63°)$

$= (\tan 9° + \cot 9°) - (\tan 27° + \cot 27°)$ $\left[\because \begin{array}{l} \tan 81° = \tan(90° - 9°) = \cot 9° \\ \tan 63° = \tan(90° - 27°) = \cot 27° \end{array}\right]$

$= \dfrac{1}{\sin 9° \cos 9°} - \dfrac{1}{\sin 27° \cos 27°}$ $\left[\because \tan\theta + \cot\theta = \dfrac{1}{\sin\theta\cos\theta}\right]$

$= \dfrac{2}{\sin 18°} - \dfrac{2}{\sin 54°} = \dfrac{2}{\sin 18°} - \dfrac{2}{\cos 36°} = \dfrac{8}{\sqrt{5}-1} - \dfrac{8}{\sqrt{5}+1} = \dfrac{8 \times 2}{5-1} = 4$

EXERCISE 9.3

LEVEL-1

Prove that:

1. $\sin^2 72° - \sin^2 60° = \dfrac{\sqrt{5}-1}{8}$
2. $\sin^2 24° - \sin^2 6° = \dfrac{\sqrt{5}-1}{8}$
3. $\sin^2 42° - \cos^2 78° = \dfrac{\sqrt{5}+1}{8}$
4. $\cos 78° \cos 42° \cos 36° = \dfrac{1}{8}$
5. $\cos \dfrac{\pi}{15} \cos \dfrac{2\pi}{15} \cos \dfrac{4\pi}{15} \cos \dfrac{7\pi}{15} = \dfrac{1}{16}$
6. $\cos \dfrac{\pi}{15} \cos \dfrac{2\pi}{15} \cos \dfrac{3\pi}{15} \cos \dfrac{4\pi}{15} \cos \dfrac{5\pi}{15} \cos \dfrac{6\pi}{15} \cos \dfrac{7\pi}{15} = \dfrac{1}{128}$
7. $\cos 6° \cos 42° \cos 66° \cos 78° = \dfrac{1}{16}$
8. $\sin 6° \sin 42° \sin 66° \sin 78° = \dfrac{1}{16}$
9. $\cos 36° \cos 42° \cos 60° \cos 78° = \dfrac{1}{16}$
10. $\sin 36° \sin 72° \sin 108° \sin 144° = \dfrac{5}{16}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If $\cos 4x = 1 + k \sin^2 x \cos^2 x$, then write the value of k.
2. If $\tan \dfrac{x}{2} = \dfrac{m}{n}$, then write the value of $m \sin x + n \cos x$.
3. If $\dfrac{\pi}{2} < \theta < \dfrac{3\pi}{2}$, then write the value of $\sqrt{\dfrac{1+\cos 2\theta}{2}}$.
4. If $\dfrac{\pi}{2} < \theta < \pi$, then write the value of $\sqrt{2 + \sqrt{2 + 2\cos 2\theta}}$ in the simplest form.
5. If $\dfrac{\pi}{2} < \theta < \pi$, then write the value of $\sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}}$.

6. If $\pi < \theta < \frac{3\pi}{2}$, then write the value of $\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}$.

7. In a right angled triangle ABC, write the value of $\sin^2 A + \sin^2 B + \sin^2 C$.

8. Write the value of $\cos^2 76° + \cos^2 16° - \cos 76° \cos 16°$.

9. If $\frac{\pi}{4} < \theta < \frac{\pi}{2}$, then write the value of $\sqrt{1-\sin 2\theta}$.

10. Write the value of $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$.

11. If $\tan A = \frac{1-\cos B}{\sin B}$, then find the value of $\tan 2A$.

12. If $\sin x + \cos x = a$, find the value of $\sin^6 x + \cos^6 x$.

13. If $\sin x + \cos x = a$, find the value of $|\sin x - \cos x|$.

ANSWERS

1. -8 2. n 3. $-\cos\theta$ 4. $2\sin\frac{\theta}{2}$ 5. $-\tan\theta$ 6. $\tan\theta$ 7. 2

8. $\frac{3}{4}$ 9. $\sin\theta - \cos\theta$ 10. $-\frac{1}{8}$ 11. $\tan B$ 12. $\frac{1}{4}\{4-3(a^2-1)^2\}$ 13. $\sqrt{2-a^2}$.

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. $8\sin\frac{x}{8}\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}$ is equal to
   (a) $8\cos x$ (b) $\cos x$ (c) $8\sin x$ (d) $\sin x$

2. $\frac{\sec 8A - 1}{\sec 4A - 1}$ is equal to
   (a) $\frac{\tan 2A}{\tan 8A}$ (b) $\frac{\tan 8A}{\tan 2A}$ (c) $\frac{\cot 8A}{\cot 2A}$ (d) none of these

3. The value of $\cos\frac{\pi}{65}\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}$ is
   (a) $\frac{1}{8}$ (b) $\frac{1}{16}$ (c) $\frac{1}{32}$ (d) none of these

4. If $\cos 2x + 2\cos x = 1$ then, $(2-\cos^2 x)\sin^2 x$ is equal to
   (a) 1 (b) -1 (c) $-\sqrt{5}$ (d) $\sqrt{5}$

5. For all real values of x, $\cot x - 2\cot 2x$ is equal to
   (a) $\tan 2x$ (b) $\tan x$ (c) $-\cot 3x$ (d) none of these

6. The value of $2\tan\frac{\pi}{10} + 3\sec\frac{\pi}{10} - 4\cos\frac{\pi}{10}$ is
   (a) 0 (b) $\sqrt{5}$ (c) 1 (d) none of these

7. If in a ΔABC, $\tan A + \tan B + \tan C = 0$, then $\cot A \cot B \cot C =$
   (a) 6 (b) 1 (c) $\frac{1}{6}$ (d) none of these

8. If $\cos\theta = \frac{1}{2}\left(a+\frac{1}{a}\right)$, and $\cos 3\theta = \lambda\left(a^3+\frac{1}{a^3}\right)$, then $\lambda =$

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 1 (d) none of these

9. If $2 \tan \alpha = 3 \tan \beta$, then $\tan (\alpha - \beta) =$

(a) $\frac{\sin 2\beta}{5 - \cos 2\beta}$ (b) $\frac{\cos 2\beta}{5 - \cos 2\beta}$ (c) $\frac{\sin 2\beta}{5 + \cos 2\beta}$ (d) none of these

10. If $\tan \alpha = \frac{1 - \cos \beta}{\sin \beta}$, then

(a) $\tan 3\alpha = \tan 2\beta$ (b) $\tan 2\alpha = \tan \beta$
(c) $\tan 2\beta = \tan \alpha$ (d) none of these

11. If $\sin \alpha + \sin \beta = a$ and $\cos \alpha - \cos \beta = b$, then $\tan \frac{\alpha - \beta}{2} =$

(a) $-\frac{a}{b}$ (b) $-\frac{b}{a}$ (c) $\sqrt{a^2 + b^2}$ (d) none of these

12. The value of $\left(\cot \frac{x}{2} - \tan \frac{x}{2}\right)^2 (1 - 2 \tan x \cot 2x)$ is

(a) 1 (b) 2 (c) 3 (d) 4

13. The value of $\tan \theta \sin \left(\frac{\pi}{2} + \theta\right) \cos \left(\frac{\pi}{2} - \theta\right)$ is

(a) 1 (b) -1 (c) $\frac{1}{2} \sin 2\theta$ (d) none of these

14. The value of $\sin^2 \left(\frac{\pi}{18}\right) + \sin^2 \left(\frac{\pi}{9}\right) + \sin^2 \left(\frac{7\pi}{18}\right) + \sin^2 \left(\frac{4\pi}{9}\right)$ is

(a) 1 (b) 2 (c) 4 (d) none of these

15. If $5 \sin \alpha = 3 \sin (\alpha + 2\beta) \neq 0$, then $\tan (\alpha + \beta)$ is equal to

(a) $2 \tan \beta$ (b) $3 \tan \beta$ (c) $4 \tan \beta$ (d) $6 \tan \beta$

16. The value of $2 \cos \theta - \cos 3\theta - \cos 5\theta - 16 \cos^3 \theta \sin^2 \theta$ is

(a) 2 (b) 1 (c) 0 (d) -1

17. If $A = 2 \sin^2 \theta - \cos 2\theta$, then A lies in the interval

(a) $[-1, 3]$ (b) $[1, 2]$ (c) $[-2, 4]$ (d) none of these

18. The value of $\frac{\cos 3\theta}{2 \cos 2\theta - 1}$ is equal to

(a) $\cos \theta$ (b) $\sin \theta$ (c) $\tan \theta$ (d) none of these

19. If $\tan (\pi/4 + \theta) + \tan (\pi/4 - \theta) = \lambda \sec 2\theta$, them

(a) 3 (b) 4 (c) 1 (d) 2

20. The value of $\cos^2 \left(\frac{\pi}{6} + \theta\right) - \sin^2 \left(\frac{\pi}{6} - \theta\right)$ is

(a) $\frac{1}{2} \cos 2\theta$ (b) 0 (c) $-\frac{1}{2} \cos 2\theta$ (d) $\frac{1}{2}$

21. $\frac{\sin 3\theta}{1 + 2 \cos 2\theta}$ is equal to

(a) $\cos \theta$ (b) $\sin \theta$ (c) $-\cos \theta$ (d) $\sin \theta$

22. The value of $2 \sin^2 B + 4 \cos (A + B) \sin A \sin B + \cos 2 (A + B)$ is

(a) 0 (b) $\cos 3A$ (c) $\cos 2A$ (d) none of these

CHAPTER 10

SINE AND COSINE FORMULAE AND THEIR APPLICATIONS

10.1 INTRODUCTION

In any triangle the three sides and the three angles are generally called the elements of the triangle. A triangle which does not contain a right angle is called an *oblique triangle.*

In any triangle ABC, the measures of the angles $\angle BAC$, $\angle CBA$ and $\angle ACB$ are denoted by the letters A, B and C respectively. The sides BC, CA and AB opposite to the angles A, B and C respectively are denoted by a, b and c. These six elements of a triangle are not independent and are connected by the relations: (i) $A + B + C = \pi$ (ii) $a + b > c;\ b + c > a;\ c + a > b$. In addition to these relations, the elements of a triangle are connected by some trigonometric relations. We intend to discuss those relations in the sections to follow of this chapter.

10.2 THE LAW OF SINES OR SINE RULE

THEOREM *The sides of a triangle are proportional to the sines of the angles opposite to them i.e. in a* ΔABC, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

PROOF The following cases arise:

CASE I *When ΔABC is an acute angled triangle*:

Draw AD perpendicular from A to the opposite side BC meeting it in the point D.

In the triangle ABD, we have

$$\sin B = \frac{AD}{AB} \Rightarrow \sin B = \frac{AD}{c} \Rightarrow AD = c \sin B \quad \text{...(i)}$$

In the triangle ACD, we have

$$\sin C = \frac{AD}{AC} \Rightarrow \sin C = \frac{AD}{b} \Rightarrow AD = b \sin C \quad \text{...(ii)}$$

From (i) and (ii), we get

$$c \sin B = b \sin C \Rightarrow \frac{b}{\sin B} = \frac{c}{\sin C}$$

In a similar manner, by drawing a perpendicular from B on AC, we obtain

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Hence, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

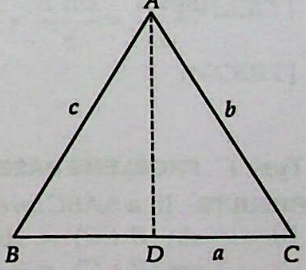


Fig. 10.1

CASE II *When ΔABC is an obtuse angled triangle.*

Draw AD perpendicular from A on CB produced meeting it in D.

In ΔADB, we have

$$\sin \angle ABD = \frac{AD}{AB} \Rightarrow \sin (180 - B) = \frac{AD}{c} \Rightarrow \sin B = \frac{AD}{c} \Rightarrow AD = c \sin B \quad \text{...(i)}$$

In ΔACD, we have

$$\sin C = \frac{AD}{AC} \Rightarrow \sin C = \frac{AD}{b} \Rightarrow AD = b \sin C \quad \text{...(ii)}$$

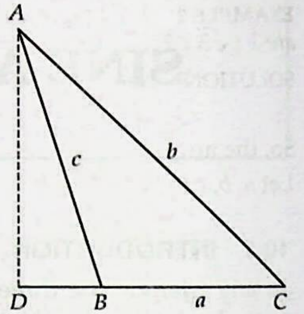


Fig. 10.2

From (i) and (ii), we obtain

$$c \sin B = b \sin C \Rightarrow \frac{b}{\sin B} = \frac{c}{\sin C}$$

Similarly, by drawing perpendicular from B on AC, we obtain

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Hence, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

<u>CASE III</u> *When ΔABC is a right angled triangle:*

In ΔABC, we have

$$\sin C = \sin \frac{\pi}{2} = 1, \ \sin A = \frac{BC}{AB} = \frac{a}{c} \text{ and, } \sin B = \frac{AC}{AB} = \frac{b}{c}$$

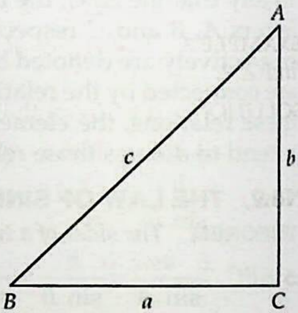


Fig. 10.3

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = c$$

$$\Rightarrow \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{1}$$

$$\Rightarrow \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \left[\because \sin C = \sin \frac{\pi}{2} = 1\right]$$

Hence, in all the cases, we obtain

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Q.E.D.

<u>REMARK 1</u> *The above rule may also be expressed as* $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

<u>REMARK 2</u> *The sine rule is a very useful tool to express sides of a triangle in terms of the sines of angles and vice-versa in the following manner.*

Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$ (say) Then, $a = k \sin A$, $b = k \sin B$, $c = k \sin C$.

Similarly,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \lambda \text{ (say)} \Rightarrow \sin A = \lambda a, \ \sin B = \lambda b \text{ and } \sin C = \lambda c.$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **PROBLEMS BASED ON SINE RULE**

RESULTS In a ΔABC, we have

$$\sin(B+C) = \sin A, \ \sin(C+A) = \sin B, \ \sin(A+B) = \sin C$$
$$\cos(B+C) = -\cos A, \ \cos(C+A) = -\cos B, \ \cos(A+B) = -\cos C$$
$$\tan(B+C) = -\tan A, \ \tan(C+A) = -\tan B, \ \tan(A+B) = -\tan C$$

EXAMPLE 1 *In a ΔABC, if $a = 2$, $b = 3$ and $\sin A = \dfrac{2}{3}$, find $\angle B$.*

SOLUTION We have,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$\Rightarrow \quad \dfrac{2}{(2/3)} = \dfrac{3}{\sin B} \Rightarrow 3 = \dfrac{3}{\sin B} \Rightarrow \sin B = 1 \Rightarrow \angle B = 90°$

EXAMPLE 2 *If in any triangle the angles be to one another as 1 : 2 : 3, prove that the corresponding sides are $1 : \sqrt{3} : 2$.*

SOLUTION Let the measures of the angles be x, $2x$ and $3x$. Then,

$$x + 2x + 3x = 180° \Rightarrow 6x = 180° \Rightarrow x = 30°$$

So, the angles are 30°, 60° and 90°

Let a, b, c be the lengths of the sides apposite to these angles. Then,

$$\frac{a}{\sin 30°} = \frac{b}{\sin 60°} = \frac{c}{\sin 90°}$$

$\Rightarrow \quad a : b : c = \sin 30° : \sin 60° : \sin 90°$

$\Rightarrow \quad a : b : c : = \dfrac{1}{2} : \dfrac{\sqrt{3}}{2} : 1 \Rightarrow a : b : c = 1 : \sqrt{3} : 2$

EXAMPLE 3 *The angles of a triangle ABC are in A. P. and it is being given that $b : c = \sqrt{3} : \sqrt{2}$, find $\angle A$.*

SOLUTION It is given that the angles $\angle A, \angle B, \angle C$ are in A.P.

$\therefore \quad 2\angle B = \angle A + \angle C \Rightarrow 3\angle B = \angle A + \angle B + \angle C \Rightarrow 3\angle B = 180° \Rightarrow \angle B = 60°$

Now, $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$\Rightarrow \quad \dfrac{b}{c} = \dfrac{\sin B}{\sin C}$

$\Rightarrow \quad \dfrac{\sqrt{3}}{\sqrt{2}} = \dfrac{\sin 60°}{\sin C} \qquad [\because b : c = \sqrt{3} : \sqrt{2}]$

$\Rightarrow \quad \dfrac{\sqrt{3}}{\sqrt{2}} = \dfrac{\sqrt{3}/2}{\sin C}$

$\Rightarrow \quad \sin C = \dfrac{1}{\sqrt{2}} \Rightarrow \angle C = 45°$

$\therefore \quad \angle A = 180° - (\angle B + \angle C) = 180° - (60° + 45°) = 75°$

EXAMPLE 4 *In any triangle ABC, prove that:*

(i) $\dfrac{\sin(B - C)}{\sin(B + C)} = \dfrac{b^2 - c^2}{a^2}$ **[NCERT]**

(ii) $a \sin(B - C) + b \sin(C - A) + c \sin(A - B) = 0$ **[NCERT]**

(iii) $a^3 \sin(B - C) + b^3 \sin(C - A) + c^3 \sin(A - B) = 0$

SOLUTION Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$. Then,

$$a = k \sin A,\ b = k \sin B \text{ and } c = k \sin C. \qquad \text{...(i)}$$

(i) $\quad \text{RHS} = \dfrac{b^2 - c^2}{a^2} = \dfrac{k^2 \sin^2 B - k^2 \sin^2 C}{k^2 \sin^2 A} \qquad$ [Using (i)]

$\Rightarrow \quad \text{RHS} = \dfrac{\sin^2 B - \sin^2 C}{\sin^2 A} = \dfrac{\sin(B + C)\sin(B - C)}{\sin^2 A}$

$\Rightarrow \quad \text{RHS} = \dfrac{\sin(\pi - A)\sin(B - C)}{\sin^2 A} \qquad [\because A + B + C = \pi \Rightarrow B + C = \pi - A]$

$$\Rightarrow \quad \text{RHS} = \frac{\sin A \sin (B-C)}{\sin^2 A} = \frac{\sin (B-C)}{\sin A} = \frac{\sin (B-C)}{\sin (B+C)} = \text{LHS}$$

(ii) $\text{LHS} = a \sin (B-C) + b \sin (C-A) + c \sin (A-B)$

$= k \sin A \sin (B-C) + k \sin B \sin (C-A) + k \sin C \sin (A-B)$ [Using (i)]

$$= k \left\{ \sin (B+C) \sin (B-C) + \sin (C+A) \sin (C-A) + \sin (A+B) \sin (A-B) \right\}$$

$$= k \left\{ \sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A + \sin^2 A - \sin^2 B \right\} = k\,(0) = \text{RHS}$$

(iii) $\text{LHS} = k^3 \sin^3 A \sin (B-C) + k^3 \sin^3 B \sin (C-A) + k^3 \sin^3 C \sin (A-B)$

$$= k^3 \left\{ \sin^2 A \sin A \sin (B-C) + \sin^2 B \sin B \sin (C-A) + \sin^2 C \sin C \sin (A-B) \right\}$$

$$= k^3 \left\{ \sin^2 A \sin (B+C) \sin (B-C) + \sin^2 B \sin (C+A) \sin (C-A) + \sin^2 C \sin (A+B) \sin (A-B) \right\}$$

$$= k^3 \left\{ \sin^2 A (\sin^2 B - \sin^2 C) + \sin^2 B (\sin^2 C - \sin^2 A) + \sin^2 C (\sin^2 A - \sin^2 B) \right\}$$

$$= k^3 \left\{ \sin^2 A \sin^2 B - \sin^2 A \sin^2 C + \sin^2 B \sin^2 C - \sin^2 B \sin^2 A + \sin^2 C \sin^2 A - \sin^2 C \sin^2 B \right\}$$

$$= k^3 \times 0 = 0 = \text{RHS}$$

EXAMPLE 5 *In any triangle ABC, prove that:*

$$\frac{a^2 \sin (B-C)}{\sin B + \sin C} + \frac{b^2 \sin (C-A)}{\sin C + \sin A} + \frac{c^2 \sin (A-B)}{\sin A + \sin B} = 0$$

SOLUTION Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$. Then, $a = k \sin A,\ b = k \sin B,\ c = k \sin C$

Now,

$$\text{LHS} = \frac{a^2 \sin (B-C)}{\sin B + \sin C} + \frac{b^2 \sin (C-A)}{\sin C + \sin A} + \frac{c^2 \sin (A-B)}{\sin A + \sin B}$$

$$= \frac{k^2 \sin^2 A \sin (B-C)}{\sin B + \sin C} + \frac{k^2 \sin^2 B \sin (C-A)}{\sin C + \sin A} + \frac{k^2 \sin^2 C \sin (A-B)}{\sin A + \sin B}$$

$$= k^2 \left\{ \frac{\sin A \sin (B+C) \sin (B-C)}{\sin B + \sin C} + \frac{\sin B \sin (C+A) \sin (C-A)}{\sin C + \sin A} + \frac{\sin C \sin (A+B) \sin (A-B)}{\sin A + \sin B} \right\}$$

$$= k^2 \left\{ \frac{\sin A (\sin^2 B - \sin^2 C)}{\sin B + \sin C} + \frac{\sin B (\sin^2 C - \sin^2 A)}{\sin C + \sin A} + \frac{\sin C (\sin^2 A - \sin^2 B)}{\sin A + \sin B} \right\}$$

$$= k^2 \{ \sin A (\sin B - \sin C) + \sin B (\sin C - \sin A) + \sin C (\sin A - \sin B) \}$$

$$= k^2 \times 0 = 0 = \text{RHS}$$

EXAMPLE 6 *In any triangle ABC, prove that:*

(i) $\dfrac{a \sin (B-C)}{b^2 - c^2} = \dfrac{b \sin (C-A)}{c^2 - a^2} = \dfrac{c \sin (A-B)}{a^2 - b^2}$

(ii) $\dfrac{b^2 - c^2}{\cos B + \cos C} + \dfrac{c^2 - a^2}{\cos C + \cos A} + \dfrac{a^2 - b^2}{\cos A + \cos B} = 0$

SOLUTION Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$. Then, $a = k \sin A, b = k \sin B, c = k \sin C$

(i) $$\frac{a \sin (B-C)}{b^2 - c^2} = \frac{k \sin A \sin (B-C)}{k^2 \sin^2 B - k^2 \sin^2 C} = \frac{k \sin (B+C) \sin (B-C)}{k^2 (\sin^2 B - \sin^2 C)}$$

$$= \frac{k (\sin^2 B - \sin^2 C)}{k^2 (\sin^2 B - \sin^2 C)} = \frac{1}{k}$$

and, $$\frac{b \sin (C-A)}{c^2 - a^2} = \frac{k \sin B \sin (C-A)}{k^2 \sin^2 C - k^2 \sin^2 A} = \frac{k \sin (C+A) \sin (C-A)}{k^2 (\sin^2 C - \sin^2 A)}$$

$$= \frac{k (\sin^2 C - \sin^2 A)}{k^2 (\sin^2 C - \sin^2 A)} = \frac{1}{k}$$

Similarly, it can be shown that $\dfrac{c \sin (A-B)}{a^2 - b^2} = \dfrac{1}{k}$

Hence, $$\frac{a \sin (B-C)}{b^2 - c^2} = \frac{b \sin (C-A)}{c^2 - a^2} = \frac{c \sin (A-B)}{a^2 - b^2}$$

(ii) $$\text{LHS} = \frac{k^2 (\sin^2 B - \sin^2 C)}{\cos B + \cos C} + \frac{k^2 (\sin^2 C - \sin^2 A)}{\cos C + \cos A} + \frac{k^2 (\sin^2 A - \sin^2 B)}{\cos A + \cos B} \quad \text{[Using (i)]}$$

$$= \frac{k^2 \{(1 - \cos^2 B) - (1 - \cos^2 C)\}}{\cos B + \cos C} + \frac{k^2 \{(1 - \cos^2 C) - (1 - \cos^2 A)\}}{\cos C + \cos A} + \frac{k^2 \{(1 - \cos^2 A) - (1 - \cos^2 B)\}}{\cos A + \cos B}$$

$$= k^2 \left\{ \frac{(\cos^2 C - \cos^2 B)}{\cos B + \cos C} + \frac{(\cos^2 A - \cos^2 C)}{\cos C + \cos A} + \frac{(\cos^2 B - \cos^2 A)}{\cos A + \cos B} \right\}$$

$$= k^2 \{(\cos C - \cos B) + (\cos A - \cos C) + (\cos B - \cos A)\} = k \times 0 = 0 = \text{RHS}$$

EXAMPLE 7 *In any triangle ABC, prove that:*

(i) $\dfrac{1 + \cos (A-B) \cos C}{1 + \cos (A-C) \cos B} = \dfrac{a^2 + b^2}{a^2 + c^2}$

(ii) $a \cos A + b \cos B + c \cos C = 2a \sin B \sin C$ **[NCERT]**

SOLUTION Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$. Then, $a = k \sin A, b = k \sin B, c = k \sin C$

(i) $$\text{LHS} = \frac{1 + \cos (A-B) \cos C}{1 + \cos (A-C) \cos B} = \frac{1 - \cos (A-B) \cos (A+B)}{1 - \cos (A-C) \cos (A+C)} = \frac{1 - (\cos^2 A - \sin^2 B)}{1 - (\cos^2 A - \sin^2 C)}$$

$$= \frac{1 - \cos^2 A + \sin^2 B}{1 - \cos^2 A + \sin^2 C} = \frac{\sin^2 A + \sin^2 B}{\sin^2 A + \sin^2 C} = \frac{a^2/k^2 + b^2/k^2}{a^2/k^2 + c^2/k^2} = \frac{a^2 + b^2}{a^2 + c^2} = \text{RHS}$$

(ii) $$\begin{aligned}\text{LHS} &= a\cos A + b\cos B + c\cos C\\ &= k\sin A\cos A + k\sin B\cos B + k\sin C\cos C\\ &= \frac{k}{2}\{\sin 2A + \sin 2B + \sin 2C\} = \frac{k}{2}(4\sin A\sin B\sin C)\\ &= 2k\sin A\sin B\sin C = 2a\sin B\sin C = \text{RHS}\qquad [\because k\sin A = a]\end{aligned}$$

EXAMPLE 8 *In any triangle ABC, prove that:*

(i) $\sin\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{a}\right)\cos\frac{A}{2}$ [NCERT] (ii) $a\cos\left(\frac{B-C}{2}\right) = (b+c)\sin\frac{A}{2}$ [NCERT]

(iii) $\frac{b-c}{b+c} = \frac{\tan\left(\frac{B-C}{2}\right)}{\tan\left(\frac{B+C}{2}\right)}$

SOLUTION Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$. Then, $a = k\sin A, b = k\sin B, c = k\sin C$

(i) $$\begin{aligned}\text{RHS} &= \left(\frac{b-c}{a}\right)\cos\frac{A}{2} = \left\{\frac{k\sin B - k\sin C}{k\sin A}\right\}\cos\frac{A}{2} = \left\{\frac{\sin B - \sin C}{\sin A}\right\}\cos\frac{A}{2}\\ &= \frac{2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)}{2\sin\frac{A}{2}\cos\frac{A}{2}}\cos\frac{A}{2} = \frac{\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{\pi-A}{2}\right)}{\sin\frac{A}{2}}\\ &= \frac{\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)}{\sin\frac{A}{2}} = \frac{\sin\left(\frac{B-C}{2}\right)\sin\frac{A}{2}}{\sin\frac{A}{2}} = \sin\left(\frac{B-C}{2}\right) = \text{LHS}\end{aligned}$$

(ii) $$\begin{aligned}\text{RHS} &= (b+c)\sin\frac{A}{2} = (k\sin B + k\sin C)\sin\frac{A}{2} = k(\sin B + \sin C)\sin\frac{A}{2}\\ &= k\times 2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)\sin\frac{A}{2} = 2k\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)\sin\frac{A}{2}\\ &= 2k\cos\frac{A}{2}\cos\left(\frac{B-C}{2}\right)\sin\frac{A}{2} = k\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)\\ &= k\sin A\cos\left(\frac{B-C}{2}\right) = a\cos\left(\frac{B-C}{2}\right) = \text{LHS}\qquad [\because k\sin A = a]\end{aligned}$$

(iii) $$\begin{aligned}\text{LHS} &= \frac{b-c}{b+c} = \frac{k\sin B - k\sin C}{k\sin B + k\sin C} = \frac{\sin B - \sin C}{\sin B + \sin C}\\ &= \frac{2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)} = \frac{\tan\left(\frac{B-C}{2}\right)}{\tan\left(\frac{B+C}{2}\right)} = \text{RHS}\end{aligned}$$

EXAMPLE 9 *In a triangle ABC, if $a\cos A = b\cos B$, show that the triangle is either isosceles or right angled.*

SOLUTION Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$. Then, $a = k\sin A,\ b = k\sin B,\ c = k\sin C$

Now, $a\cos A = b\cos B$

$$\Rightarrow \quad k \sin A \cos A = k \sin B \cos B$$
$$\Rightarrow \quad 2 \sin A \cos A = 2 \sin B \cos B$$
$$\Rightarrow \quad \sin 2A = \sin 2B$$
$$\Rightarrow \quad 2A = 2B \text{ or, } 2A = \pi - 2B$$
$$\Rightarrow \quad A = B \text{ or, } A + B = \pi/2$$
$$\Rightarrow \quad A = B \text{ or, } C = \frac{\pi}{2}$$
$$\Rightarrow \quad BC = CA \text{ or, } C = \frac{\pi}{2}$$

$\Rightarrow$ ΔABC is either isosceles or right angled.

EXAMPLE 10 *If in a ΔABC, $\dfrac{\sin A}{\sin C} = \dfrac{\sin (A-B)}{\sin (B-C)}$, prove that a^2, b^2, c^2 are in A.P.*

SOLUTION Let $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = k$. Then, $\sin A = ak$, $\sin B = bk$, $\sin C = ck$

Now, $\dfrac{\sin A}{\sin C} = \dfrac{\sin (A-B)}{\sin (B-C)}$

$$\Rightarrow \quad \frac{\sin (B+C)}{\sin (A+B)} = \frac{\sin (A-B)}{\sin (B-C)} \qquad [\because \sin A = \sin (B+C) \text{ and } \sin C = \sin (A+B)]$$
$$\Rightarrow \quad \sin (B+C) \sin (B-C) = \sin (A+B) \sin (A-B)$$
$$\Rightarrow \quad \sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$$
$$\Rightarrow \quad k^2 b^2 - k^2 c^2 = k^2 a^2 - k^2 b^2$$
$$\Rightarrow \quad b^2 - c^2 = a^2 - b^2 \Rightarrow 2b^2 = a^2 + c^2 \Rightarrow a^2, b^2, c^2 \text{ are in A.P.}$$

Type II APPLICATIONS OF SINE FORMULA IN PROBLEMS ON HEIGHTS AND DISTANCES

EXAMPLE 11 *A tree stands vertically on a hill side which makes an angle of 15° with the horizontal. From a point on the ground 35 m down the hill from the base of the tree, the angle of elevation of the top of the tree is 60°. Find the height of the tree.* **[NCERT]**

SOLUTION Let PQ be the tree on the hill which makes an angel of 15° with the horizontal AR, where A is a point on the ground 35 m down the hill from the base P of the tree.

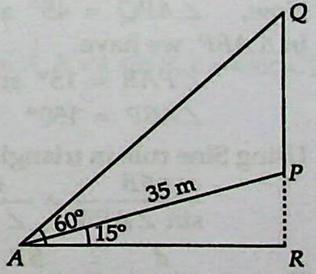


Fig. 10.4

In ΔARQ, we have

$$\angle RAQ = 60° \text{ and } \angle ARQ = 90°$$
$$\therefore \quad \angle AQP = 30°$$

In ΔAPQ, we have

$$\angle PAQ = 45° \text{ and } \angle AQP = 30°$$

Using Sine rule in ΔAPQ, we get

$$\frac{AP}{\sin \angle AQP} = \frac{PQ}{\sin \angle PAQ}$$

$$\Rightarrow \quad \frac{35}{\sin 30°} = \frac{PQ}{\sin 45°} \Rightarrow \frac{35}{1/2} = \frac{PQ}{1/\sqrt{2}} \Rightarrow PQ = \frac{70}{\sqrt{2}} \Rightarrow PQ = 35\sqrt{2} \text{ m}.$$

Hence, height of the tree is $35\sqrt{2}$ m.

EXAMPLE 12 *A person, standing on the bank of a river, observes that the angle subtended by a tree on the opposite bank is 60°, when he retreats 20 m from the bank, he finds the angle to be 30°. Find the height of the tree and the breadth of the river.*

SOLUTION Let AB be the tree at one bank of the river and let the P be the position of the person at the other bank of the river. After retreating 20 m from the bank, let the man be at Q. It is given that $\angle APB = 60°$ and $\angle AQB = 30°$.

Now, $\angle APB = 60° \Rightarrow \angle BPQ = 120°$ and $\angle PBA = 30°$

In ΔBPQ, we have

$$\angle PQB = 30°, \angle BPQ = 120° \text{ and, } \angle PBQ = 30°$$

Using Sine rule in triangles BPQ and PAB, we get

$$\frac{BP}{\sin \angle PQB} = \frac{PQ}{\sin \angle PBQ} \text{ and } \frac{BP}{\sin \angle PAB} = \frac{AP}{\sin \angle ABP} = \frac{AB}{\sin \angle APB}$$

$$\Rightarrow \quad \frac{BP}{\sin 30°} = \frac{PQ}{\sin 30°} \text{ and } \frac{BP}{\sin 90°} = \frac{AP}{\sin 30°} = \frac{AB}{\sin 60°}$$

$$\Rightarrow \quad \frac{BP}{1/2} = \frac{20}{1/2} \text{ and } BP = 2AP = \frac{2AB}{\sqrt{3}}$$

$$\Rightarrow \quad BP = 20 \text{ and } BP = 2AP = \frac{2AB}{\sqrt{3}} \quad [\because PQ = 20 \text{ m}]$$

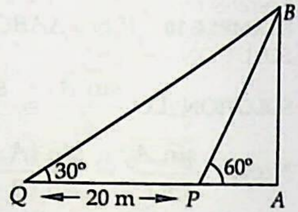


Fig. 10.5

$$\Rightarrow \quad 20 = 2AP = \frac{2AB}{\sqrt{3}}$$

$$\Rightarrow \quad AP = 10 \text{ and } AB = 10\sqrt{3}$$

Hence, the breadth of the river is 10 m and height of the tree is $10\sqrt{3}$ m.

EXAMPLE 13 *The angle of elevation of the top point P of the vertical tower PQ of heigth h from a point A is 45° and from a point B, the angle of elevation is 60°, where B is a point at a distance d from the point A measured along the line AB which makes an angle 30° with AQ. Prove that $d = (\sqrt{3} - 1)\, h$.* **[NCERT]**

SOLUTION It is given that $\angle PAQ = 45°$ and $\angle BAQ = 30°$. Therefore, $\angle BAP = 15°$.

In ΔAQP, we have

$$\angle PAQ = 45° \text{ and } \angle PQA = 90°$$

$$\therefore \quad \angle APQ = 45°$$

In ΔBRP, we have

$$\angle PBR = 60° \text{ and } \angle PRB = 90°$$

$$\therefore \quad \angle BPR = 30°$$

Now, $\angle APQ = 45°$ and $\angle BPR = 30° \Rightarrow \angle BPA = 15°$

In ΔABP, we have

$$\angle PAB = 15° \text{ and } \angle BPA = 15°.$$

$$\therefore \quad \angle ABP = 150°$$

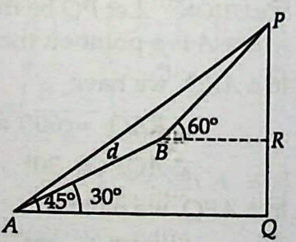


Fig. 10.6

Using Sine rule in triangle ABP, we get

$$\frac{AB}{\sin \angle APB} = \frac{BP}{\angle PAB} = \frac{AP}{\sin \angle ABP}$$

$$\Rightarrow \quad \frac{d}{\sin 15°} = \frac{BP}{\sin 15°} = \frac{AP}{\sin 150°}$$

$$\Rightarrow \quad \frac{d}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{AP}{\frac{1}{2}} \Rightarrow AP = \frac{\sqrt{2}\, d}{\sqrt{3}-1} \quad \text{...(i)}$$

Using Sine rule in ΔAQP, we get

$$\frac{AP}{\sin \angle AQP} = \frac{AQ}{\sin \angle APQ} = \frac{PQ}{\sin \angle PAQ}$$

$$\Rightarrow \quad \frac{AP}{\sin 90°} = \frac{PQ}{\sin 45°} \Rightarrow AP = \sqrt{2}\ PQ \Rightarrow AP = \sqrt{2}\ h \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$\sqrt{2}h = \frac{\sqrt{2}\ d}{\sqrt{3}-1} \Rightarrow d = (\sqrt{3}-1)\ h$$

LEVEL-2

EXAMPLE 14 *If in a* $\Delta\, ABC$, $\dfrac{a^2-b^2}{a^2+b^2} = \dfrac{\sin(A-B)}{\sin(A+B)}$, *prove that it is either a right angled or an isosceles triangle.*

SOLUTION Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$, (say). Then, $a = k \sin A, b = k\sin B, c = k\sin C$

$$\therefore \quad \frac{a^2-b^2}{a^2+b^2} = \frac{\sin(A-B)}{\sin(A+B)}$$

$$\Rightarrow \quad \frac{k^2\sin^2 A - k^2 \sin^2 B}{k^2 \sin^2 A + k^2 \sin^2 B} = \frac{\sin(A-B)}{\sin(A+B)}$$

$$\Rightarrow \quad \frac{\sin(A-B)}{\sin(\pi - C)} = \frac{\sin(A+B)\sin(A-B)}{\sin^2 A + \sin^2 B}$$

$$\Rightarrow \quad \frac{\sin(A-B)}{\sin C} = \frac{\sin(\pi - C)\sin(A-B)}{\sin^2 A + \sin^2 B}$$

$$\Rightarrow \quad \frac{\sin(A-B)}{\sin C} = \frac{\sin C\sin(A-B)}{\sin^2 A + \sin^2 B}$$

$$\Rightarrow \quad \sin(A-B)\left\{\frac{1}{\sin C} - \frac{\sin C}{\sin^2 A + \sin^2 B}\right\} = 0$$

$$\Rightarrow \quad \text{either } \sin(A-B) = 0 \quad \text{or,} \quad \frac{1}{\sin C} - \frac{\sin C}{\sin^2 A + \sin^2 B} = 0$$

$$\Rightarrow \quad \text{either } A - B = 0 \quad \text{or,} \quad \sin^2 A + \sin^2 B - \sin^2 C = 0$$

$$\Rightarrow \quad \text{either } A = B \quad \text{or,} \quad \frac{a^2}{k^2} + \frac{b^2}{k^2} - \frac{c^2}{k^2} = 0 \qquad [\because\ a = k\sin A, b = k\sin B, c = k\sin C]$$

$$\Rightarrow \quad \text{either } A = B \quad \text{or,} \quad a^2 + b^2 = c^2$$

$\Rightarrow$ either the triangle is isosceles or it is right angled.

EXAMPLE 15 *In any triangle ABC, prove that:*

$$(b-c)\cot\frac{A}{2} + (c-a)\cot\frac{B}{2} + (a-b)\cot\frac{C}{2} = 0$$

SOLUTION We have,

$$\text{LHS} = (b-c)\cot\frac{A}{2} + (c-a)\cot\frac{B}{2} + (a-b)\cot\frac{C}{2}$$

$$= k(\sin B - \sin C)\cot\frac{A}{2} + k(\sin C - \sin A)\cot\frac{B}{2} + k(\sin A - \sin B)\cot\frac{C}{2}$$

$$= k\left[2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)\cot\frac{A}{2} + 2\sin\left(\frac{C-A}{2}\right)\cos\left(\frac{C+A}{2}\right)\cot\frac{B}{2}\right.$$

$$\left. + 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)\cot\frac{C}{2}\right]$$

$$= k\left[2\sin\left(\frac{B-C}{2}\right)\sin\frac{A}{2}\cot\frac{A}{2} + 2\sin\left(\frac{C-A}{2}\right)\sin\frac{B}{2}\cot\frac{B}{2} + 2\sin\left(\frac{A-B}{2}\right)\sin\frac{C}{2}\cot\frac{C}{2}\right]$$

$$= k\left[2\cos\frac{A}{2}\sin\left(\frac{B-C}{2}\right) + 2\cos\frac{B}{2}\sin\left(\frac{C-A}{2}\right) + 2\cos\frac{C}{2}\sin\left(\frac{A-B}{2}\right)\right]$$

$$= 2k\left[\sin\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right) + \sin\left(\frac{C+A}{2}\right)\sin\left(\frac{C-A}{2}\right) + \sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right]$$

$$\left[\because \cos\frac{A}{2} = \sin\left(\frac{B+C}{2}\right) \text{ etc.}\right]$$

$$= 2k\left\{\sin^2\frac{B}{2} - \sin^2\frac{C}{2} + \sin^2\frac{C}{2} - \sin^2\frac{A}{2} + \sin^2\frac{A}{2} - \sin^2\frac{B}{2}\right\} = 2k \times 0 = 0 = \text{RHS}$$

EXAMPLE 16 *Let O be a point inside a triangle ABC such that* $\angle OAB = \angle OBC = \angle OCA = \omega$, *then show that:*

(i) $\cot\omega = \cot A + \cot B + \cot C$ (ii) $\text{cosec}^2\,\omega = \text{cosec}^2 A + \text{cosec}^2 B + \text{cosec}^2 C$

SOLUTION (i) In ΔOBC,

$\angle OCB = \angle C - \omega$ and, $\angle BOC = 180 - \omega - (C - \omega) = 180° - C$

Similarly, we obtain $\angle AOB = 180° - B$

Applying sine rule in ΔOAB, we obtain

$$\frac{OB}{\sin\angle OAB} = \frac{AB}{\sin\angle AOB}$$

$$\frac{OB}{\sin\omega} = \frac{AB}{\sin(180° - B)}$$

$$\Rightarrow \quad \frac{OB}{\sin\omega} = \frac{c}{\sin B}$$

$$\Rightarrow \quad OB = \frac{c\sin\omega}{\sin B} \quad \text{...(i)}$$

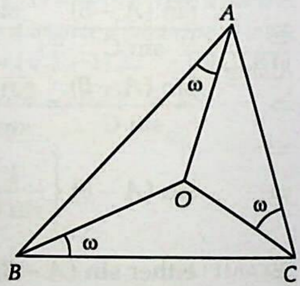


Fig. 10.7

Applying sine rule in ΔOBC, we get

$$\frac{OB}{\sin\angle BCO} = \frac{BC}{\sin\angle BOC}$$

$$\Rightarrow \quad \frac{OB}{\sin(C-\omega)} = \frac{BC}{\sin(180° - C)} \Rightarrow \frac{OB}{\sin(C-\omega)} = \frac{a}{\sin C} \Rightarrow OB = \frac{a\sin(C-\omega)}{\sin C} \quad \text{...(ii)}$$

From (i) and (ii), we get

$$\frac{c\sin\omega}{\sin B} = \frac{a\sin(C-\omega)}{\sin C}$$

$$\Rightarrow \quad \frac{k\sin C\sin\omega}{\sin B} = \frac{k\sin A\sin(C-\omega)}{\sin C} \quad \text{[Using sine rule]}$$

$$\Rightarrow \quad \sin^2 C\sin\omega = \sin A\sin B\sin(C-\omega)$$

$$\Rightarrow \quad \sin C\sin(A+B)\sin\omega = \sin A\sin B\sin(C-\omega) \quad [\because \sin C = \sin(\pi - (A+B)) = \sin(A+B)]$$

$$\Rightarrow \quad \frac{\sin(A+B)}{\sin A\sin B} = \frac{\sin(C-\omega)}{\sin C\sin\omega}$$

$$\Rightarrow \quad \frac{\sin A\cos B + \cos A\sin B}{\sin A\sin B} = \frac{\sin C\cos\omega - \cos C\sin\omega}{\sin C\sin\omega}$$

$$\Rightarrow \quad \cot B + \cot A = \cot\omega - \cot C \Rightarrow \cot\omega = \cot A + \cot B + \cot C$$

(ii) From (i), we have

$$\cot \omega = \cot A + \cot B + \cot C$$

$$\Rightarrow \quad \cot^2 \omega = \cot^2 A + \cot^2 B + \cot^2 C + 2(\cot A \cot B + \cot B \cot C + \cot C \cot A)$$

$$\Rightarrow \quad \cot^2 \omega = \cot^2 A + \cot^2 B + \cot^2 C + 2 \quad [\because \cot A \cot B + \cot B \cot C + \cot C \cot A = 1]$$

$$\Rightarrow \quad \operatorname{cosec}^2 \omega - 1 = (\operatorname{cosec}^2 A - 1) + (\operatorname{cosec}^2 B - 1) + (\operatorname{cosec}^2 C - 1) + 2$$

$$\Rightarrow \quad \operatorname{cosec}^2 \omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$$

EXAMPLE 17 *The angle of elevation of the top of a tower from a point A due South of the tower is α and from B due East of the tower is β. If AB = d, show that the height of the tower is* $\dfrac{d}{\sqrt{\cot^2 \alpha + \cot^2 \beta}}$.

SOLUTION Let OP be the tower and let A and B be two points due South and East respectively of the tower such that $\angle OAP = \alpha$ and $\angle OBP = \beta$. Then,

$$\angle OPA = \frac{\pi}{2} - \alpha \text{ and } \angle OPB = \frac{\pi}{2} - \beta.$$

Using sine rule in Δ's OAP and OBP, we have

$$\frac{OA}{\sin\left(\frac{\pi}{2} - \alpha\right)} = \frac{OP}{\sin \alpha} \text{ and, } \frac{OB}{\sin\left(\frac{\pi}{2} - \beta\right)} = \frac{OP}{\sin \beta}$$

$$\Rightarrow \quad \frac{OA}{\cos \alpha} = \frac{OP}{\sin \alpha} \text{ and } \frac{OB}{\cos \beta} = \frac{OP}{\sin \beta}$$

$$\Rightarrow \quad OA = OP \cot \alpha \text{ and } OB = OP \cot \beta$$

Using Pythagoras theorem in $\Delta\, AOB$, we get

$$AB^2 = OA^2 + OB^2$$

$$\Rightarrow \quad d^2 = OP^2 \cot^2 \alpha + OP^2 \cot^2 \beta$$

$$\Rightarrow \quad OP = \frac{d}{\sqrt{\cot^2 \alpha + \cot^2 \beta}}$$

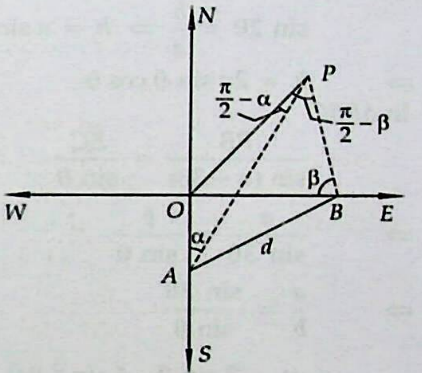


Fig. 10.8

EXAMPLE 18 *The elevation of a tower at a station A due North of it is α and at a station B due West of A is β. Prove that the height of the tower is* $\dfrac{AB \sin \alpha \sin \beta}{\sqrt{\sin^2 \alpha - \sin^2 \beta}}$.

SOLUTION Let OP be the tower and let A be a point due north of the tower OP and let B be the point due west of A such that $\angle OAP = \alpha$ and $\angle OBP = \beta$.

Clearly, triangles AOP and BOP are right triangles right angled at O.

$$\therefore \quad \angle OPA = \frac{\pi}{2} - \alpha \text{ and } \angle OPB = \frac{\pi}{2} - \beta$$

Using sine rule in triangles AOP and BOP, we get

$$\frac{OA}{\sin\left(\frac{\pi}{2} - \alpha\right)} = \frac{OP}{\sin \alpha} \text{ and } \frac{OB}{\sin\left(\frac{\pi}{2} - \beta\right)} = \frac{OP}{\sin \beta}$$

$$\Rightarrow \quad OA = OP \cot \alpha \text{ and } OB = OP \cot \beta$$

Applying Pythagoras theorem in ΔOAB, we get

$$OB^2 = OA^2 + AB^2$$

$$\Rightarrow \quad OB^2 - OA^2 = AB^2$$

$$\Rightarrow \quad OP^2 \cot^2 \beta - OP^2 \cot \alpha^2 = AB^2$$

$$\Rightarrow \quad OP^2 (\cot^2 \beta - \cot^2 \alpha) = AB^2$$

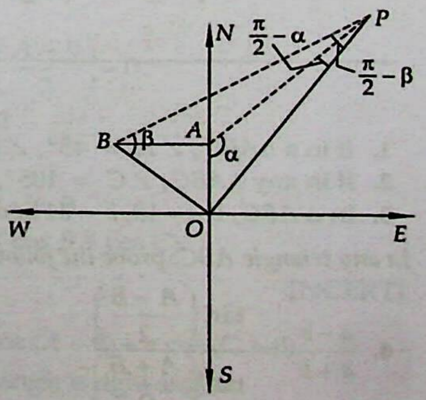


Fig. 10.9

$$\Rightarrow \quad OP^2 (\operatorname{cosec}^2 \beta - \operatorname{cosec}^2 \alpha) = AB^2$$

$$\Rightarrow \quad OP^2 \frac{(\sin^2 \alpha - \sin^2 \beta)}{\sin^2 \alpha \sin^2 \beta} = AB^2 \Rightarrow OP = \frac{AB \sin \alpha \sin \beta}{\sqrt{\sin^2 \alpha - \sin^2 \beta}}$$

EXAMPLE 19 *An object is observed from three points A, B, C in the same horizontal line passing through the base of the object. The angle of elevation at B is twice and at C thrice that at A. If AB = a, BC = b prove that the height of the object is* $\frac{a}{2b}\sqrt{(a+b)(3b-a)}$.

SOLUTION Let the object be at P at a height h from OA. Let the object when observed from A, B and C the angles of elevation are θ, 2θ and 3θ respectively.

In $\Delta\, PAB$, we have

$$2\theta = \theta + \angle APB \Rightarrow \angle APB = \theta$$

$$\therefore \quad \angle PAB = \angle APB = \theta \Rightarrow AB = BP = a$$

Similarly, in triangle BPC, $\angle\, BPC = \theta$.

In ΔOPB

$$\sin 2\theta = \frac{h}{a} \Rightarrow h = a \sin 2\theta$$

$$\Rightarrow \quad h = 2a \sin\theta \cos\theta \qquad \text{...(i)}$$

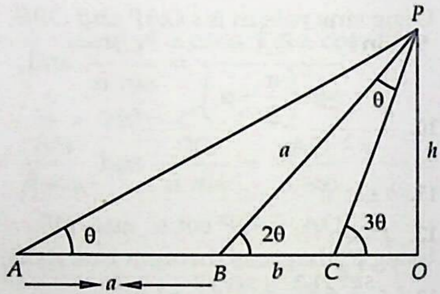


Fig. 10.10

In ΔPBC

$$\frac{PB}{\sin(\pi - 3\theta)} = \frac{BC}{\sin\theta} \qquad \text{[Using sine rule]}$$

$$\Rightarrow \quad \frac{a}{\sin 3\theta} = \frac{b}{\sin\theta}$$

$$\Rightarrow \quad \frac{a}{b} = \frac{\sin 3\theta}{\sin\theta}$$

$$\Rightarrow \quad \frac{a}{b} = \frac{3\sin\theta - 4\sin^3\theta}{\sin\theta}$$

$$\Rightarrow \quad \frac{a}{b} = 3 - 4\sin^2\theta \Rightarrow 4\sin^2\theta = 3 - \frac{a}{b} \Rightarrow \sin^2\theta = \frac{3b-a}{4b} \Rightarrow \sin\theta = \sqrt{\frac{3b-a}{4b}}$$

$$\therefore \quad \cos^2\theta = 1 - \sin^2\theta \Rightarrow \cos^2\theta = 1 - \frac{3b-a}{4b} = \frac{a+b}{4b} \Rightarrow \cos\theta = \sqrt{\frac{a+b}{4b}}$$

Substituting the values of $\sin\theta$ and $\cos\theta$ in (i), we get

$$h = 2a\sqrt{\frac{3b-a}{4b}} \times \sqrt{\frac{a+b}{4b}} = \frac{a}{2b}\sqrt{(a+b)(3b-a)}$$

EXERCISE 10.1

LEVEL-1

1. If in a ΔABC, $\angle A = 45°$, $\angle B = 60°$, and $\angle C = 75°$; find the ratio of its sides.
2. If in any ΔABC, $\angle C = 105°$, $\angle B = 45°$, $a = 2$, then find b.
3. In $\Delta\, ABC$, if $a = 18$, $b = 24$ and C = 30 and $\angle C = 90°$, find $\sin A$, $\sin B$ and $\sin C$.

In any triangle ABC, prove the following: (4-24)

4. $\dfrac{a-b}{a+b} = \dfrac{\tan\left(\dfrac{A-B}{2}\right)}{\tan\left(\dfrac{A+B}{2}\right)}$

5. $(a-b)\cos\frac{C}{2} = c\sin\left(\frac{A-B}{2}\right)$ **[NCERT]**

6. $\frac{c}{a-b} = \frac{\tan\left(\frac{A}{2}\right)+\tan\left(\frac{B}{2}\right)}{\tan\left(\frac{A}{2}\right)-\tan\left(\frac{B}{2}\right)}$

7. $\frac{c}{a+b} = \frac{1-\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)}{1+\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)}$

8. $\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$ **[NCERT]**

9. $\sin\left(\frac{B-C}{2}\right) = \frac{b-c}{a}\cos\frac{A}{2}$

10. $\frac{a^2-c^2}{b^2} = \frac{\sin(A-C)}{\sin(A+C)}$

11. $b\sin B - c\sin C = a\sin(B-C)$

12. $a^2\sin(B-C) = (b^2-c^2)\sin A$

13. $\frac{\sqrt{\sin A}-\sqrt{\sin B}}{\sqrt{\sin A}+\sqrt{\sin B}} = \frac{a+b-2\sqrt{ab}}{a-b}$

14. $a(\sin B-\sin C)+b(\sin C-\sin A)+c(\sin A-\sin B) = 0$

15. $\frac{a^2\sin(B-C)}{\sin A}+\frac{b^2\sin(C-A)}{\sin B}+\frac{c^2\sin(A-B)}{\sin C} = 0$

16. $a^2(\cos^2 B-\cos^2 C)+b^2(\cos^2 C-\cos^2 A)+c^2(\cos^2 A-\cos^2 B) = 0$

17. $b\cos B+c\cos C = a\cos(B-C)$

18. $\frac{\cos 2A}{a^2}-\frac{\cos 2B}{b^2} = \frac{1}{a^2}-\frac{1}{b^2}$

19. $\frac{\cos^2 B-\cos^2 C}{b+c}+\frac{\cos^2 C-\cos^2 A}{c+a}+\frac{\cos^2 A-\cos^2 B}{a+b} = 0$

20. $a\sin\frac{A}{2}\sin\left(\frac{B-C}{2}\right)+b\sin\frac{B}{2}\sin\left(\frac{C-A}{2}\right)+c\sin\frac{C}{2}\sin\left(\frac{A-B}{2}\right)=0.$

21. $\frac{b\sec B+c\sec C}{\tan B+\tan C} = \frac{c\sec C+a\sec A}{\tan C+\tan A} = \frac{a\sec A+b\sec B}{\tan A+\tan B}.$

22. $a\cos A+b\cos B+c\cos C = 2b\sin A\sin C = 2c\sin A\sin B$

23. $a(\cos B\cos C+\cos A)=b(\cos C\cos A+\cos B)=c(\cos A\cos B+\cos C).$

24. $a(\cos C-\cos B)=2(b-c)\cos^2\frac{A}{2}.$ **[NCERT]**

25. In ΔABC prove that, if θ be any angle, then $b\cos\theta = c\cos(A-\theta)+a\cos(C+\theta)$.

26. In a ΔABC, if $\sin^2 A+\sin^2 B = \sin^2 C$, show that the triangle is right angled.

LEVEL-2

27. In any ΔABC, if a^2, b^2, c^2 are in A.P., prove that $\cot A$, $\cot B$ and $\cot C$ are also in A.P.
28. The upper part of a tree broken over by the wind makes an angle of 30° with the ground and the distance from the root to the point where the top of the tree touches the ground is 15 m. Using sine rule, find the height of the tree.
29. At the foot of a mountain the elevation of its summit is 45°; after ascending 1000 m towards the mountain up a slope of 30° inclination, the elevation is found to be 60°. Find the height of the mountain.
30. A person observes the angle of elevation of the peak of a hill from a station to be α. He walks c metres along a slope inclined at the angle β and finds the angle of elevation of the peak of the hill to be γ. Show that the height of the peak above the ground is $\dfrac{c \sin \alpha \sin (\gamma-\beta)}{(\sin \gamma-\alpha)}$.
31. If the sides a, b, c of a ΔABC are in H.P., prove that $\sin^2 \dfrac{A}{2}, \sin^2 \dfrac{B}{2}, \sin^2 \dfrac{C}{2}$ are in H.P.

ANSWERS

1. $2 : \sqrt{6} : \sqrt{3}+1$ 2. $2\sqrt{2}$ 3. $\sin A = \dfrac{3}{5}$, $\sin B = \dfrac{4}{5}$

28. $15\sqrt{3}$ m 29. $500(\sqrt{3}+1)$ metres

10.3 THE LAW OF COSINES

THEOREM *In any ΔABC, we have:*

(i) $a^2 = b^2 + c^2 - 2bc \cos A$ or, $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$

(ii) $b^2 = c^2 + a^2 - 2ac \cos B$ or, $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$

(iii) $c^2 = a^2 + b^2 - 2ab \cos C$ or, $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

PROOF The following cases may arise:

CASE I *When ΔABC is an acute angled triangle:*

Draw perpendicular AD from A on BC.

In ΔABD, we have

$$\cos B = \frac{BD}{c} \Rightarrow BD = c \cos B \qquad \text{...(i)}$$

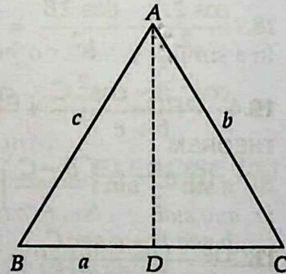


Fig. 10.11

In ΔACD, we have

$$\cos C = \frac{CD}{b} \Rightarrow CD = b \cos C$$

In ΔACD, using Pythagoras theorem, we have

$$AC^2 = AD^2 + CD^2$$

$$\Rightarrow \quad AC^2 = AD^2 + (BC - BD)^2$$

$$\Rightarrow \quad AC^2 = AD^2 + BC^2 + BD^2 - 2\,BC.BD$$

$$\Rightarrow \quad AC^2 = BC^2 + (AD^2 + BD^2) - 2BC.BD$$

$$\Rightarrow \quad AC^2 = BC^2 + AB^2 - 2\,BC.BD \qquad [\because AB^2 = BD^2 + AD^2]$$

$$\Rightarrow \quad b^2 = a^2 + c^2 - 2ac \cos B \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad b^2 = c^2 + a^2 - 2ca \cos B$$

$$\Rightarrow \quad \cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

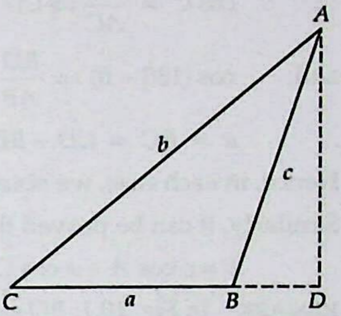


Fig. 10.12

CASE II *When ΔABC is an obtuse angled triangle:*
Draw perpendicular AD from A on CB produced.
In ΔABD, we have

$$\cos(180 - B) = \frac{BD}{AB} \Rightarrow BD = -AB \cos B = -c \cos B \quad \text{...(i)}$$

Using Pythagoras theorem in ΔACD, we have

$$AC^2 = AD^2 + CD^2$$

$$\Rightarrow \quad AC^2 = AD^2 + (CB + BD)^2$$

$$\Rightarrow \quad AC^2 = AD^2 + CB^2 + BD^2 + 2CB \,.\, BD$$

$$\Rightarrow \quad AC^2 = BC^2 + (BD^2 + AD^2) + 2BC \,.\, BD$$

$$\Rightarrow \quad AC^2 = BC^2 + AB^2 + 2BC \,.\, BD \qquad [\text{In } \Delta ABD,\ AB^2 = AD^2 + BD^2]$$

$$\Rightarrow \quad b^2 = a^2 + c^2 + 2a(-c \cos B) \qquad [\text{Using (i)}]$$

$$\Rightarrow \quad b^2 = c^2 + a^2 - 2ac \cos B$$

$$\Rightarrow \quad \cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

CASE III When ΔABC is a right angled triangle:
Let ΔABC be a right angled triangle with right angle at B. Then, by Pythagoras theorem, we obtain

$$b^2 = a^2 + c^2$$

$$\Rightarrow \quad b^2 = a^2 + c^2 - 2ac \cos B \qquad \left[\because\ B = \frac{\pi}{2} \ \therefore \cos B = 0\right]$$

$$\Rightarrow \quad \cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

Hence, in all the cases, we have

$$b^2 = c^2 + a^2 - 2ac \cos B \Rightarrow \cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

In a similar manner other results can be proved.

10.4 PROJECTION FORMULAE

THEOREM *In any ΔABC, we have*

(i) $a = b \cos C + c \cos B$ (ii) $b = c \cos A + a \cos C$ (iii) $c = a \cos B + b \cos A$

i.e. any side of a triangle is equal to the sum of the projections of other two sides on it.

PROOF The following cases arise:

CASE I When ΔABC is an acute angled triangle:
In Fig. 10.1, we have

$$\cos B = \frac{BD}{AB} \Rightarrow BD = AB \cos B \Rightarrow BD = c \cos B$$

and, $$\cos C = \frac{CD}{AC} \Rightarrow CD = AC \cos C \Rightarrow CD = b \cos C$$

Hence, $a = BC = BD + CD \Rightarrow a = c \cos B + b \cos C$

CASE II *When ΔABC is an obtuse angled triangle*

In Fig. 10.2, we have

$$\cos C = \frac{CD}{AC} \Rightarrow CD = AC \cos C \Rightarrow CD = b \cos C$$

and, $$\cos(180 - B) = \frac{BD}{AB} \Rightarrow BD = AB \cos(180 - B) \Rightarrow BD = -c \cos B$$

$\therefore$ $$a = BC = CD - BD \Rightarrow a = b\cos C + c \cos B$$

Hence, in each case, we obtain $a = b \cos C + c \cos B$

Similarly, it can be proved that

$$b = c\cos A + a\cos C \text{ and } c = a\cos B + b\cos A$$ Q.E.D.

REMARK *In Fig. 10.1, BD and CD are the projections of AB and AC respectively on BC.*

10.5 NAPIER'S ANALOGY (LAW OF TANGENTS)

THEOREM *In any ΔABC, we have*

(i) $\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\frac{A}{2}$ (ii) $\tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$

(iii) $\tan\left(\frac{C-A}{2}\right) = \left(\frac{c-a}{c+a}\right)\cot\frac{B}{2}$

PROOF Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$. Then, $a = k\sin A, b = k\sin B, c = k\sin C$ ...(i)

(i) $$\text{RHS} = \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{k\sin B - k\sin C}{k\sin B + k\sin C}\cot\frac{A}{2}$$ [Using (i)]

$$= \left(\frac{\sin B - \sin C}{\sin B + \sin C}\right)\cot\frac{A}{2} = \left\{\frac{2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}\right\}$$

$$= \tan\left(\frac{B-C}{2}\right)\cot\left(\frac{B+C}{2}\right)\cot\frac{A}{2} = \tan\left(\frac{B-C}{2}\right)\cot\left(\frac{\pi}{2}-\frac{A}{2}\right)\cot\frac{A}{2}$$

$$= \tan\left(\frac{B-C}{2}\right)\tan\frac{A}{2}\cot\frac{A}{2} = \tan\left(\frac{B-C}{2}\right) = \text{LHS}$$

Similarly, (ii) and (iii) can be proved.

10.6 AREA OF A TRIANGLE

THEOREM *Prove that the area of ΔABC is given by*

$$\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$$

PROOF Let *ABC* be a triangle. Then the following cases arise :

CASE I *When ΔABC is an acute angled triangle:*

In Fig. 10.1, we have

$$\sin B = \frac{AD}{AB} \Rightarrow AD = AB\sin B = c\sin B$$

$\therefore \quad \Delta = \text{Area of } \Delta ABC = \frac{1}{2} BC \times AD = \frac{1}{2} a\, c \sin B$

CASE II *When ΔABC is an obtuse angled triangle:*

In Fig. 10.2, we have

$$\sin(180 - B) = \frac{AD}{AB} \Rightarrow AD = AB \sin B = c \sin B$$

$\therefore \quad \Delta = \text{Area of } \Delta ABC = \frac{1}{2} BC \times AD = \frac{1}{2} a\, c \sin B$

Thus, in each case, we have $\Delta = \frac{1}{2} ac \sin B$

Similarly, it can be proved that $\Delta = \frac{1}{2} ab \sin C$ and $\Delta = \frac{1}{2} bc \sin A$

Q.E.D.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **PROBLEMS ON APPLICATIONS OF COSINE FORMULA AND SINE RULE**

EXAMPLE 1 *In a ΔABC, if $a = 3$, $b = 5$ and $c = 7$, find $\cos A$, $\cos B$ and $\cos C$.*

SOLUTION We have, $a = 3$, $b = 5$ and $c = 7$

$$\therefore \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{25 + 49 - 9}{2 \times 5 \times 7} = \frac{65}{70} = \frac{13}{14}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2bc} = \frac{49 + 9 - 25}{2 \times 3 \times 7} = \frac{33}{42} = \frac{11}{14}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{9 + 25 - 49}{2 \times 3 \times 5} = -\frac{15}{30} = -\frac{1}{2}$$

EXAMPLE 2 *If the sides of a Δ ABC are $a = 4$, $b = 6$ and $c = 8$, show that $4 \cos B + 3 \cos C = 2$.*

SOLUTION We have, $a = 4$, $b = 6$ and, $c = 8$

$$\therefore \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{16 + 64 - 36}{2 \times 4 \times 8} = \frac{44}{64} = \frac{11}{16}$$

$$\text{and,} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{16 + 36 - 64}{2 \times 4 \times 6} = -\frac{12}{48} = -\frac{1}{4}$$

$$\therefore \quad 4 \cos B + 3 \cos C = 4 \times \frac{11}{16} - \frac{3}{4} = \frac{11}{4} - \frac{3}{4} = 2$$

EXAMPLE 3 *In any ΔABC, prove that:*

(i) $a(b \cos C - c \cos B) = b^2 - c^2$ **[NCERT]**

(ii) $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$ **[NCERT]**

(iii) $2(bc \cos A + ca \cos B + ab \cos C) = a^2 + b^2 + c^2$

SOLUTION (i) LHS $= a(b \cos C - c \cos B) = ab \cos C - ac \cos B$

$$= ab\left(\frac{a^2 + b^2 - c^2}{2ab}\right) - ac\left(\frac{a^2 + c^2 - b^2}{2ac}\right)$$

$$= \frac{1}{2}\left\{(a^2 + b^2 - c^2) - (a^2 + c^2 - b^2)\right\} = b^2 - c^2 = \text{RHS}$$

(ii) $$\text{LHS} = \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$$

$$= \frac{b^2 + c^2 - a^2}{2abc} + \frac{c^2 + a^2 - b^2}{2acb} + \frac{a^2 + b^2 - c^2}{2abc} = \frac{a^2 + b^2 + c^2}{2abc} = \text{RHS}$$

(iii) $$\text{LHS} = 2bc \cos A + 2ca \cos B + 2ab \cos C$$

$$= 2bc\left(\frac{b^2 + c^2 - a^2}{2bc}\right) + 2ca\left(\frac{c^2 + a^2 - b^2}{2ac}\right) + 2ab\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$$

$$= (b^2 + c^2 - a^2) + (c^2 + a^2 - b^2) + (a^2 + b^2 - c^2) = a^2 + b^2 + c^2 = \text{RHS}$$

EXAMPLE 4 *In any ΔABC, prove that:* $(a - b)^2 \cos^2 \frac{C}{2} + (a + b)^2 \sin^2 \frac{C}{2} = c^2$

<u>PROOF</u> We have,

$$\text{LHS} = (a - b)^2 \cos^2 \frac{C}{2} + (a + b)^2 \sin^2 \frac{C}{2}$$

$$= a^2\left(\cos^2 \frac{C}{2} + \sin^2 \frac{C}{2}\right) + b^2\left(\cos^2 \frac{C}{2} + \sin^2 \frac{C}{2}\right) - 2ab\left(\cos^2 \frac{C}{2} - \sin^2 \frac{C}{2}\right)$$

$$= a^2 + b^2 - 2ab \cos C = c^2 = \text{RHS}$$

EXAMPLE 5 *In a ΔABC, prove that:*

(i) $(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$ **[NCERT]**

(ii) $\left(\frac{b^2 - c^2}{a^2}\right) \sin 2A + \left(\frac{c^2 - a^2}{b^2}\right) \sin 2B + \left(\frac{a^2 - b^2}{c^2}\right) \sin 2C = 0$ **[NCERT]**

SOLUTION Let $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$. Then, $\sin A = ak$, $\sin B = bk$ and $\sin C = ck$

(i) $$\text{LHS} = (b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C$$

$$= (b^2 - c^2) \frac{\cos A}{\sin A} + (c^2 - a^2) \frac{\cos B}{\sin B} + (a^2 - b^2) \frac{\cos C}{\sin C}$$

$$= \frac{(b^2 - c^2)}{ka}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) + \frac{(c^2 - a^2)}{kb}\left(\frac{a^2 + c^2 - b^2}{2ac}\right) + \frac{(a^2 - b^2)}{kc}\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$$

$$= \frac{1}{2kabc}\left\{(b^2 - c^2)(b^2 + c^2 - a^2) + (c^2 - a^2)(a^2 + c^2 - b^2) + (a^2 - b^2)(a^2 + b^2 - c^2)\right\}$$

$$= \frac{1}{2kabc}\left\{(b^2 - c^2)(b^2 + c^2) - a^2(b^2 - c^2) + (c^2 - a^2)(c^2 + a^2) - b^2(c^2 - a^2)\right.$$
$$\left. + (a^2 - b^2)(a^2 + b^2) - c^2(a^2 - b^2)\right\}$$

$$= \frac{1}{2kabc}\left\{(b^2 - c^2)(b^2 + c^2) + (c^2 - a^2)(c^2 + a^2) + (a^2 - b^2)(a^2 + b^2)\right.$$
$$\left. - a^2(b^2 - c^2) - b^2(c^2 - a^2) - c^2(a^2 - b^2)\right\}$$

$$= \frac{1}{2kabc}\left\{(b^4 - c^4) + (c^4 - a^4) + (a^4 - b^4) - (a^2b^2 - a^2c^2) - (b^2c^2 - b^2a^2) - (c^2a^2 - c^2b^2)\right\}$$

$$= \frac{1}{2kabc} \times 0 = 0 = \text{RHS}$$

(ii) LHS $= \left(\frac{b^2-c^2}{a^2}\right)\sin 2A + \left(\frac{c^2-a^2}{b^2}\right)\sin 2B + \left(\frac{a^2-b^2}{c^2}\right)\sin 2C$

$= \left(\frac{b^2-c^2}{a^2}\right) 2\sin A\cos A + \left(\frac{c^2-a^2}{b^2}\right) 2\sin B\cos B + \left(\frac{a^2-b^2}{c^2}\right) 2\sin C\cos C$

$= \left(\frac{b^2-c^2}{a^2}\right) 2ka\left(\frac{b^2+c^2-a^2}{2bc}\right) + \left(\frac{c^2-a^2}{b^2}\right) 2kb\left(\frac{a^2+c^2-b^2}{2ac}\right) + \left(\frac{a^2-b^2}{c^2}\right) 2kc\left(\frac{a^2+b^2-c^2}{2ab}\right)$

$= \frac{k}{abc}\left\{(b^2-c^2)(b^2+c^2-a^2) + (c^2-a^2)(c^2+a^2-b^2) + (a^2-b^2)(a^2+b^2-c^2)\right\}$

$= \frac{k}{abc} \times 0 = 0 = \text{RHS}$

Type II PROBLEMS BASED ON COSINE, SINE AND PROJECTION FORMULAE

EXAMPLE 6 *In any ΔABC, prove that:*

(i) $\frac{\sin B}{\sin C} = \frac{c-a\cos B}{b-a\cos C}$

(ii) $2\left\{a\sin^2\frac{C}{2} + c\sin^2\frac{A}{2}\right\} = a+c-b$

(iii) $2\left\{b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2}\right\} = a+b+c$

(iv) $(b+c)\cos A + (c+a)\cos B + (a+b)\cos C = a+b+c$

SOLUTION Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$. Then, $a = k\sin A, b = k\sin B, c = k\sin C$

(i) RHS $= \frac{c-a\cos B}{b-a\cos C} = \frac{(a\cos B + b\cos A) - a\cos B}{(a\cos C + c\cos A) - a\cos C} = \frac{b\cos A}{c\cos A} = \frac{b}{c} = \frac{k\sin B}{k\sin C} = \frac{\sin B}{\sin C} = \text{LHS}$

(ii) LHS $= 2\left(a\sin^2\frac{C}{2} + c\sin^2\frac{A}{2}\right) = \left\{a(1-\cos C) + c(1-\cos A)\right\}$

$= a + c - (a\cos C + c\cos A) = a+c-b = \text{RHS}$

(iii) LHS $= 2\left(b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2}\right) = \left\{b(1+\cos C) + c(1+\cos B)\right\}$

$= (b + c + b\cos C + c\cos B) = b+c+a = a+b+c = \text{RHS}$

(iv) LHS $= (b+c)\cos A + (c+a)\cos B + (a+b)\cos C$

$= (b\cos A + a\cos B) + (c\cos A + a\cos C) + (b\cos C + c\cos B)$

$= c+b+a = a+b+c = \text{RHS}$

EXAMPLE 7 *In any ΔABC, prove that:*

$$\frac{\cos A}{b\cos C + c\cos B} + \frac{\cos B}{c\cos A + a\cos C} + \frac{\cos C}{a\cos B + b\cos A} = \frac{a^2+b^2+c^2}{2abc}$$

SOLUTION We have,

LHS $= \frac{\cos A}{b\cos C + c\cos B} + \frac{\cos B}{c\cos A + a\cos C} + \frac{\cos C}{a\cos B + b\cos A}$

$= \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$

$= \frac{b^2+c^2-a^2}{2abc} + \frac{c^2+a-b^2}{2abc} + \frac{a^2+b^2-c^2}{2abc} = \frac{a^2+b^2+c^2}{2abc} = \text{RHS}$

EXAMPLE 8 *In a triangle ABC, if* $\cos A = \dfrac{\sin B}{2 \sin C}$, *show that the triangle is isosceles.*

SOLUTION Let $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = k$. Then, $\sin A = ka$, $\sin B = kb$, $\sin C = kc$.

Now, $\cos A = \dfrac{\sin B}{2 \sin C}$

$\Rightarrow \quad 2 \cos A \sin C = \sin B$

$\Rightarrow \quad 2\left(\dfrac{b^2 + c^2 - a^2}{2bc}\right) kc = kb$

$\Rightarrow \quad b^2 + c^2 - a^2 = b^2$

$\Rightarrow \quad c^2 = a^2 \Rightarrow c = a$

$\Rightarrow \quad \Delta ABC$ is isosceles.

EXAMPLE 9 *If in a triangle ABC,* $\dfrac{2 \cos A}{a} + \dfrac{\cos B}{b} + \dfrac{2 \cos C}{c} = \dfrac{a}{bc} + \dfrac{b}{ca}$, *then prove that the triangle is right angled.*

SOLUTION We have,

$$\frac{2 \cos A}{a} + \frac{\cos B}{b} + \frac{2 \cos C}{c} = \frac{a}{bc} + \frac{b}{ca}$$

$$\Rightarrow \quad 2\left(\frac{b^2 + c^2 - a^2}{2abc}\right) + \left(\frac{c^2 + a^2 - b^2}{2abc}\right) + 2\left(\frac{a^2 + b^2 - c^2}{2abc}\right) = \frac{a}{bc} + \frac{b}{ca}$$

$$\Rightarrow \quad 2(b^2 + c^2 - a^2) + (c^2 + a^2 - b^2) + 2(a^2 + b^2 - c^2) = 2a^2 + 2b^2$$

$$\Rightarrow \quad b^2 + c^2 = a^2$$

$\Rightarrow \quad \Delta ABC$ is a right angled triangle

Type III ON FINDING THE AREA OF A TRIANGLE WHEN ITS PARTS ARE GIVEN

EXAMPLE 10 *Find the area of a triangle ABC in which* $\angle A = 60°$, $b = 4$ *cm and* $c = \sqrt{3}$ *cm.*

SOLUTION The area Δ of triangle ABC is given by

$$\Delta = \frac{1}{2} bc \sin A = \frac{1}{2} \times 4\sqrt{3} \times \sin 60° = 2\sqrt{3} \times \frac{\sqrt{3}}{2} = 3 \text{ sq. cm.}$$

EXAMPLE 11 *In any triangle ABC, prove that:* $\Delta = \dfrac{b^2 + c^2 - a^2}{4 \cot A}$.

SOLUTION We have,

$$\text{RHS} = \frac{b^2 + c^2 - a^2}{4 \cot A} = \frac{b^2 + c^2 - a^2}{4 \cos A} \sin A = \frac{b^2 + c^2 - a^2}{4(b^2 + c^2 - a^2)} \times 2bc \sin A$$

$$= \frac{1}{2} bc \sin A = \Delta = \text{LHS}$$

EXAMPLE 12 *In any* ΔABC, *prove that:* $\Delta = \dfrac{a^2 - b^2}{2} \cdot \dfrac{\sin A \sin B}{\sin (A - B)}$

SOLUTION Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$. Then, $a = k \sin A$, $b = k \sin B$, $c = k \sin C$

$$\text{RHS} = \frac{a^2 - b^2}{2} \times \frac{\sin A \sin B}{\sin (A - B)} = \frac{k^2 \sin^2 A - k^2 \sin^2 B}{2} \times \frac{\sin A \sin B}{\sin (A - B)}$$

$$= \frac{k^2}{2}(\sin^2 A - \sin^2 B) \times \frac{\sin A \sin B}{\sin (A - B)} = \frac{k^2}{2}\sin (A + B) \sin (A - B) \frac{\sin A \sin B}{\sin (A - B)}$$

$$= \frac{1}{2}k^2 \sin (A + B) \sin A \sin B = \frac{1}{2}(k \sin A)(k \sin B) \sin (\pi - C)$$

$$= \frac{1}{2}ab \sin C = \Delta = \text{LHS}.$$

EXAMPLE 13 *In any ΔABC, prove that: $a \cos A + b \cos B + c \cos C = \frac{8\Delta^2}{abc}$.* **[NCERT]**

SOLUTION Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$. Then, $a = k \sin A,\ b = k \sin B,\ c = k \sin C$

Now, $a \cos A + b \cos B + c \cos C$

$$= k \sin A \cos A + k \sin B \cos B + k \sin C \cos C$$

$$= \frac{k}{2}(\sin 2A + \sin 2B + \sin 2C)$$

$$= \frac{k}{2}(4 \sin A \sin B \sin C) = 2k \sin A \sin B \sin C = 2a \sin B \sin C$$

$$= 2a \times \frac{2\Delta}{ac} \times \frac{2\Delta}{ab} \qquad \left[\because \Delta = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B \therefore \sin B = \frac{2\Delta}{ac},\ \sin C = \frac{2\Delta}{ab}\right]$$

$$= \frac{8\Delta^2}{abc} = \text{RHS}$$

EXAMPLE 14 *Two ships leave a port at the same time. One goes 24 km per hour in the direction N 45° E and other travels 32 km per hour in the direction S 75° E. Find the distance between the ships at the end of 3 hours.* **[NCERT]**

SOLUTION Let P and Q be the positions of two ships at the end of 3 hours. Then,

$$OP = 3 \times 24 = 72 \text{ km and } OQ = 3 \times 32 = 96 \text{ km}$$

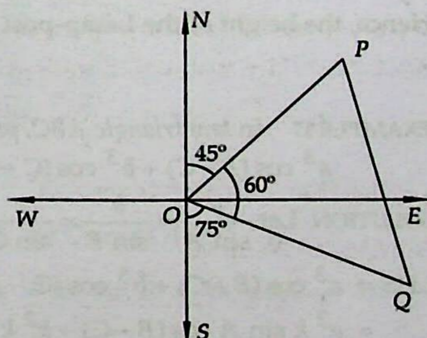


Fig. 10.13

Using cosine formula in ΔOPQ, we get

$$PQ^2 = OP^2 + OQ^2 - 2\, OP \times OQ \cos 60°$$

$$\Rightarrow \quad PQ^2 = 72^2 + 96^2 - 2 \times 72 \times 96 \times \frac{1}{2}$$

$$\Rightarrow \quad PQ^2 = 5184 + 9216 - 6912 = 7488$$

$$\Rightarrow \quad PQ = \sqrt{7488} \text{ km} = 86{\cdot}533 \text{ km}$$

EXAMPLE 15 *Two boats leave a place at the same time. One travels 56 km in the direction N 50° E, while other travels 48 km in the direction S 80° E. What is the distance between the two positions of the boats?*

SOLUTION Let A and B be the position of the boats such that $AB = x$.

Clearly, $\angle AOB = 180° - (50° + 80°) = 50°$

Using cosine formula, we have

$$AB^2 = OA^2 + OB^2 - 2\,OA\,OB \cos \angle AOB$$

$\Rightarrow \quad x^2 = (56)^2 + (48)^2 - 2 \times 56 \times 48 \cos 50°$

$\Rightarrow \quad x^2 = 3136 + 2304 - 2 \times 56 \times 48 \times 0.6428$

$\Rightarrow \quad x^2 = 5440 - 3455.69 = 1984.31$

$\Rightarrow \quad x = \sqrt{1984.31} = 44.54 \text{ m}$

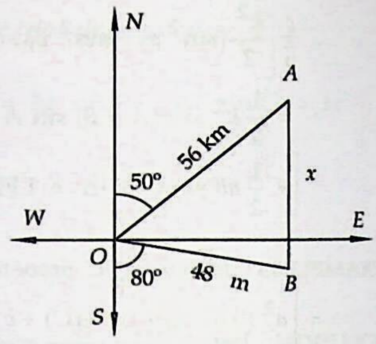


Fig. 10.14

Hence, the distance between the boats is 44.54 km.

EXAMPLE 16 *A lamp-post is situated at the middle point M of the side AC of a triangular plot ABC with BC = 7 m, CA = 8 m and AB = 9 m. Lamp-post subtends an angle of 15° at the point B. Determine the height of the lamp-post.* **[NCERT]**

SOLUTION Using cosine formula in $\Delta\, ABC$, we get

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{49 + 64 - 81}{2 \times 7 \times 8} = \frac{32}{112} = \frac{2}{7}$$

Using cosine formula in $\Delta\, BMC$, we get

$$BM^2 = BC^2 + CM^2 - 2BC\,CM \cos C$$

$\Rightarrow \quad BM^2 = 49 + 16 - 2 \times 7 \times 4 \times \frac{2}{7} \quad \left[\because CM = \frac{1}{2} AC = 4\right]$

$\Rightarrow \quad BM^2 = 49 \Rightarrow BM = 7$

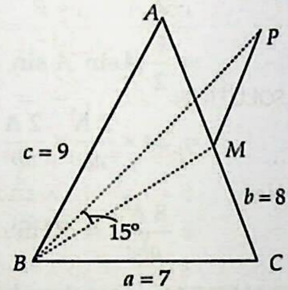


Fig. 10.15

In right triangle BMP, we have

$$\tan 15° = \frac{PM}{BM} \Rightarrow \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{PM}{7} \Rightarrow PM = 7\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) = 7\,(2 - \sqrt{3}) \text{ m}$$

Hence, the height of the Lamp-post is $7\,(2 - \sqrt{3})$ m.

LEVEL-2

EXAMPLE 17 *In any triangle ABC, prove that:*

$$a^3 \cos (B - C) + b^3 \cos (C - A) + c^3 \cos (A - B) = 3abc$$

SOLUTION Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$. Then, $a = k \sin A, b = k \sin B, c = k \sin C$

$$\text{LHS} = a^3 \cos (B - C) + b^3 \cos (C - A) + c^3 \cos (A - B)$$

$$= a^2 k \sin A \cos (B - C) + b^2 k \sin B \cos (C - A) + c^2 k \sin C \cos (A - B)$$

$$= \frac{k}{2}\left[a^2\left\{2 \sin A \cos (B - C)\right\} + b^2\left\{2 \sin B \cos (C - A)\right\} + c^2\left\{2 \sin C \cos (A - B)\right\}\right]$$

$$= \frac{k}{2}\left[a^2\left\{2 \sin (B + C) \cos (B - C)\right\} + b^2\left\{2 \sin (C + A) \cos (C - A)\right\} + c^2\left\{2 \sin (A + B) \cos (A - B)\right\}\right]$$

$$= \frac{k}{2}\left[a^2 (\sin 2B + \sin 2C) + b^2 (\sin 2C + \sin 2A) + c^2 (\sin 2A + \sin 2B)\right]$$

$$= \frac{k}{2}\left[2a^2 (\sin B \cos B + \sin C \cos C) + 2b^2 (\sin C \cos C + \sin A \cos A) + 2c^2 (\sin A \cos A + \sin B \cos B)\right]$$

$$= \left[a^2 (k \sin B \cos B + k \sin C \cos C) + b^2 (k \sin C \cos C + k \sin A \cos A) + c^2 (k \sin A \cos A + k \sin B \cos B)\right]$$

$$= \left[a^2 (b \cos B + c \cos C) + b^2 (c \cos C + a \cos A) + c^2 (a \cos A + b \cos B)\right]$$

$$[\because k \sin A = a, k \sin B = b, k \sin C = c]$$

$$= ab (a \cos B + b \cos A) + bc (b \cos C + c \cos B) + ca (a \cos C + c \cos A)$$

$$= abc + bca + cab = 3abc$$

EXAMPLE 18 *With usual notations, if in a triangle ABC* $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$, *then prove that:*

$$\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$$

SOLUTION Let $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13} = \lambda$ (say) Then, $b+c = 11\lambda$, $c+a = 12\lambda$, $a+b = 13\lambda$

$\therefore \quad (b+c+c+a+a+b) = 11\lambda + 12\lambda + 13\lambda \Rightarrow 2(a+b+c) = 36\lambda \Rightarrow a+b+c = 18\lambda$

Now, $b+c = 11\lambda$ and $a+b+c = 18\lambda \Rightarrow a = 7\lambda$

$c+a = 12\lambda$ and $a+b+c = 18\lambda \Rightarrow b = 6\lambda$

$a+b = 13\lambda$ and $a+b+c = 18\lambda \Rightarrow c = 5\lambda$

$$\therefore \quad \cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{36\lambda^2 + 25\lambda^2 - 49\lambda^2}{60\lambda^2} = \frac{12}{60} = \frac{1}{5}$$

$$\cos B = \frac{c^2+a^2-b^2}{2ac} = \frac{25\lambda^2 + 49\lambda^2 - 36\lambda^2}{70\lambda^2} = \frac{38}{70} = \frac{19}{35}$$

and, $$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{49\lambda^2 + 36\lambda^2 - 25\lambda^2}{84\lambda^2} = \frac{60}{84} = \frac{5}{7}$$

$$\therefore \quad \cos A : \cos B : \cos C = \frac{1}{5} : \frac{19}{35} : \frac{5}{7} = 7 : 19 : 25$$

EXAMPLE 19 *If* a^2, b^2, c^2 *are in A.P., prove that* $\cot A, \cot B, \cot C$ *are in A.P.*

SOLUTION Let $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$. Then, $\sin A = ka$, $\sin B = kb$, $\sin C = kc$.

Now, $\cot A, \cot B, \cot C$ will be in A.P.

$\Leftrightarrow \quad 2 \cot B = \cot A + \cot C$

$$\Leftrightarrow \quad \frac{2 \cos B}{\sin B} = \frac{\cos A}{\sin A} + \frac{\cos C}{\sin C}$$

$$\Leftrightarrow \quad \frac{2 \cos B}{kb} = \frac{\cos A}{ka} + \frac{\cos C}{kc}$$

$$\Leftrightarrow \quad 2\left(\frac{a^2+c^2-b^2}{2abc}\right) = \left(\frac{b^2+c^2-a^2}{2abc}\right) + \left(\frac{a^2+b^2-c^2}{2abc}\right)$$

$$\Leftrightarrow \quad 2(a^2+c^2-b^2) = (b^2+c^2-a^2) + (a^2+b^2-c^2) \Leftrightarrow a^2+c^2 = 2b^2 \Leftrightarrow a^2, b^2, c^2 \text{ are in A.P.}$$

ALITER It is given that a^2, b^2, c^2 are in A.P.

$\therefore\quad -2a^2, -2b^2, -2c^2$ are in A.P.

$\Rightarrow\quad (a^2+b^2+c^2)-2a^2, (a^2+b^2+c^2)-2b^2, (a^2+b^2+c^2)-2c^2$ are in A.P.

$\Rightarrow\quad b^2+c^2-a^2,\ c^2+a^2-b^2,\ a^2+b^2-c^2$ are in A.P.

$\Rightarrow\quad \dfrac{b^2+c^2-a^2}{2abc}, \dfrac{c^2+a^2-b^2}{2abc}, \dfrac{a^2+b^2-c^2}{2abc}$ are in A.P.

$\Rightarrow\quad \dfrac{1}{a}\left(\dfrac{b^2+c^2-a^2}{2bc}\right), \dfrac{1}{b}\left(\dfrac{c^2+a^2-b^2}{2ac}\right), \dfrac{1}{c}\left(\dfrac{a^2+b^2+c^2}{2ab}\right)$ are in A.P.

$\Rightarrow\quad \dfrac{\cos A}{k\sin A}, \dfrac{\cos B}{k\sin B}, \dfrac{\cos C}{k\sin C}$ are in A.P. $\qquad [\because\ a = k\sin A, b = k\sin B, c = k\sin C]$

$\Rightarrow\quad \cot A, \cot B, \cot C$ are in A.P.

EXAMPLE 20 *If in a triangle ABC, $\cos A + 2\cos B + \cos C = 2$ prove that the sides of the triangle are in A.P.*

SOLUTION We have,

$$\cos A + 2\cos B + \cos C = 2$$

$\Rightarrow\quad \cos A + \cos C = 2 - 2\cos B$

$\Rightarrow\quad \cos A + \cos C = 2(1-\cos B)$

$\Rightarrow\quad 2\cos\left(\dfrac{A+C}{2}\right)\cos\left(\dfrac{A-C}{2}\right) = 2\left(2\sin^2\dfrac{B}{2}\right)$

$\Rightarrow\quad 2\sin\dfrac{B}{2}\cos\left(\dfrac{A-C}{2}\right) = 4\sin^2\dfrac{B}{2} \qquad \left[\because\ \cos\dfrac{A+C}{2} = \cos\left(\dfrac{\pi}{2}-\dfrac{B}{2}\right) = \sin\dfrac{B}{2}\right]$

$\Rightarrow\quad \cos\left(\dfrac{A-C}{2}\right) = 2\sin\dfrac{B}{2} \qquad \left[\because\ 2\sin\dfrac{B}{2} \neq 0\right]$

$\Rightarrow\quad 2\cos\dfrac{B}{2}\cos\left(\dfrac{A-C}{2}\right) = 4\sin\dfrac{B}{2}\cos\dfrac{B}{2} \qquad$ [Multiplying both sides by $2\cos\dfrac{B}{2}$]

$\Rightarrow\quad 2\sin\dfrac{A+C}{2}\cos\dfrac{A-C}{2} = 2\left(2\sin\dfrac{B}{2}\cos\dfrac{B}{2}\right) \qquad \left[\because\ \cos\dfrac{B}{2} = \sin\dfrac{A+C}{2}\right]$

$\Rightarrow\quad \sin A + \sin C = 2\sin B$

$\Rightarrow\quad ka + kc = 2kb \qquad \left[\because\ \dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = k\right]$

$\Rightarrow\quad a + c = 2b \Rightarrow a, b, c$ are in A.P.

EXAMPLE 21 *In a triangle ABC, $\angle C = 60°$, then prove that :* $\dfrac{1}{a+c} + \dfrac{1}{b+c} = \dfrac{3}{a+b+c}$.

SOLUTION We have, $\angle C = 60°$

$\Rightarrow\quad \cos C = \dfrac{1}{2} \Rightarrow \dfrac{a^2+b^2-c^2}{2ab} = \dfrac{1}{2} \Rightarrow a^2+b^2-c^2 = ab \Rightarrow a^2+b^2-ab = c^2 \qquad$ ...(i)

Now, $\qquad \dfrac{1}{a+c} + \dfrac{1}{b+c} = \dfrac{3}{a+b+c}$

if $\qquad \dfrac{a+b+2c}{(a+c)(b+c)} = \dfrac{3}{a+b+c}$

i.e. if $\qquad (a+b+2c)(a+b+c) = 3(a+c)(b+c)$

i.e. if $\qquad (a+b)^2 + 2c^2 + 3c(a+b) = 3(ab+ac+bc+c^2)$

i.e. if $\qquad a^2+b^2+2ab+2c^2+3ac+3bc = 3ab+3ac+3bc+3c^2$

i.e. if $\qquad a^2+b^2-ab = c^2$, which is given $\qquad$ [see (i)]

EXAMPLE 22 *Two trees, A and B are on the same side of a river. From a point C in the river the distance of trees A and B are 250 m and 300 m respectively. If the angle C is 45°, find the distance between the trees (Use $\sqrt{2} = 1.44$).*

SOLUTION Using cosine formula in ΔABC, we have

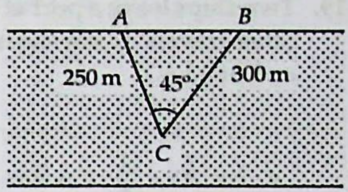


Fig. 10.16

$$AB^2 = AC^2 + BC^2 - 2AC \cdot BC \cos\frac{\pi}{4}$$

$$\Rightarrow \quad AB = \sqrt{(250)^2 + (300)^2 - 2 \times 250 \times 300 \times \frac{1}{\sqrt{2}}}$$

$$\Rightarrow \quad AB = \sqrt{62500 + 90000 - 75000\sqrt{2}}$$

$$\Rightarrow \quad AB = \sqrt{152500 - 75000 \times 1.44}$$

$$\Rightarrow \quad AB = \sqrt{152500 - 108000} = \sqrt{44500} = 210.95 \text{ m}$$

EXERCISE 10.2

LEVEL-1

In any ΔABC, prove the following : (1-13)

1. In a ΔABC, if $a = 5$, $b = 6$ and $C = 60°$, show that its area is $\frac{15\sqrt{3}}{2}$ sq. units.
2. In a ΔABC, if $a = \sqrt{2}$, $b = \sqrt{3}$ and $c = \sqrt{5}$, show that its area is $\frac{1}{2}\sqrt{6}$ sq. units.
3. The sides of a triangle are $a = 5$, $b = 6$ and $c = 8$, show that: $8\cos A + 16\cos B + 4\cos C = 17$.
4. In a ΔABC, if $a = 18$, $b = 24$, $c = 30$, find $\cos A$, $\cos B$ and $\cos C$.
5. $b(c\cos A - a\cos C) = c^2 - a^2$
6. $c(a\cos B - b\cos A) = a^2 - b^2$
7. $2(bc\cos A + ca\cos B + ab\cos C) = a^2 + b^2 + c^2$
8. $(c^2 - a^2 + b^2)\tan A = (a^2 - b^2 + c^2)\tan B = (b^2 - c^2 + a^2)\tan C$
9. $\dfrac{c - b\cos A}{b - c\cos A} = \dfrac{\cos B}{\cos C}$
10. $a(\cos B + \cos C - 1) + b(\cos C + \cos A - 1) + c(\cos A + \cos B - 1) = 0$
11. $a\cos A + b\cos B + c\cos C = 2b\sin A\sin C$
12. $a^2 = (b + c)^2 - 4bc\cos^2\frac{A}{2}$
13. $4\left(bc\cos^2\frac{A}{2} + ca\cos^2\frac{B}{2} + ab\cos^2\frac{C}{2}\right) = (a + b + c)^2$

LEVEL-2

14. In a ΔABC, prove that
$$\sin^3 A\cos(B - C) + \sin^3 B\cos(C - A) + \sin^3 C\cos(A - B) = 3\sin A\sin B\sin C$$
15. In in any ΔABC, $\dfrac{b+c}{12} = \dfrac{c+a}{13} = \dfrac{a+b}{15}$, then prove that $\dfrac{\cos A}{2} = \dfrac{\cos B}{7} = \dfrac{\cos C}{11}$.
16. In a ΔABC, if $\angle B = 60°$, prove that $(a + b + c)(a - b + c) = 3ca$
17. If in a ΔABC, $\cos^2 A + \cos^2 B + \cos^2 C = 1$, prove that the triangle is right angled.

18. In a $\Delta\, ABC$, if $\cos C = \dfrac{\sin A}{2 \sin B}$, prove that the triangle is isosceles.

19. Two ships leave a port at the same time. One goes 24 km/hr in the direction $N\,38°\,E$ and other travels 32 km/hr in the direction $S\,52°\,E$. Find the distance between the ships at the end of 3 hrs.

ANSWERS

4. $\cos A = \dfrac{4}{5}, \cos B = \dfrac{3}{5}, \cos C = 0$ 19. 120 km

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Find the area of the triangle ΔABC in which $a = 1, b = 2$ and $\angle c = 60°$.
2. In a ΔABC, if $b = \sqrt{3}, c = 1$ and $\angle A = 30°$, find a.
3. In a ΔABC, if $\cos A = \dfrac{\sin B}{2 \sin C}$, then show that $c = a$.
4. In a ΔABC, if $b = 20, c = 21$ and $\sin A = \dfrac{3}{5}$, find a.
5. In a ΔABC, if $\sin A$ and $\sin B$ are the roots of the equation $c^2x^2 - c\,(a+b)\,x + ab = 0$, then find $\angle C$.
6. In ΔABC, if $a = 8, b = 10, c = 12$ and $C = \lambda A$, find the value of λ.
7. If the sides of a triangle are proportional to $2, \sqrt{6}$ and $\sqrt{3} - 1$, find the measure of its greatest angle.
8. If in a ΔABC, $\dfrac{\cos A}{a} = \dfrac{\cos B}{b} = \dfrac{\cos C}{c}$, then find the measures of angles A, B, C.
9. In any triangle ABC, find the value of $a \sin (B - C) + b \sin (C - A) + c \sin (A - B)$.
10. In any ΔABC, find the value of $\Sigma\, a\,(\sin B - \sin C)$

ANSWERS

1. $\sqrt{3}$ sq. units 2. 1 4. 13 5. 90° 6. 2 7. 120° 8. $A = B = C = 60°$ 9. 0 10. 0

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. In any ΔABC, $\sum a^2\,(\sin B - \sin C) =$

   (a) $a^2 + b^2 + c^2$ (b) a^2 (c) b^2 (d) 0

2. In a ΔABC, if $a = 2$, $\angle B = 60°$ and $\angle C = 75°$, then $b =$

   (a) $\sqrt{3}$ (b) $\sqrt{6}$ (c) $\sqrt{9}$ (d) $1 + \sqrt{2}$

3. In the sides of a triangle are in the ratio $1 : \sqrt{3} : 2$, then the measure of its greatest angle is

   (a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{3}$ (c) $\dfrac{\pi}{2}$ (d) $\dfrac{2\pi}{3}$

4. In any $\Delta\, ABC$, $2\,(bc \cos A + ca \cos B + ab \cos C) =$

   (a) abc (b) $a + b + c$ (c) $a^2 + b^2 + c^2$ (d) $\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}$

5. In a triangle ABC, $a = 4$, $b = 3$, $\angle A = 60°$ then c is a root of the equation

(a) $c^2 - 3c - 7 = 0$ (b) $c^2 + 3c + 7 = 0$ (c) $c^2 - 3c + 7 = 0$ (d) $c^2 + 3c - 7 = 0$

6. In a ΔABC, if $(c + a + b)(a + b - c) = ab$, then the measure of angle C is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{2}$

7. In any ΔABC, the value of $2ac \sin\left(\frac{A - B + C}{2}\right)$ is

(a) $a^2 + b^2 - c^2$ (b) $c^2 + a^2 - b^2$ (c) $b^2 - c^2 - a^2$ (d) $c^2 - a^2 - b^2$

8. In any ΔABC, $a(b \cos C - c \cos B) =$

(a) a^2 (b) $b^2 - c^2$ (c) 0 (d) $b^2 + c^2$

ANSWERS

1. (d) 2. (b) 3. (c) 4. (c) 5. (a) 6. (a) 7. (c) 8. (c)

CHAPTER 11

TRIGONOMETRIC EQUATIONS

11.1 SOME DEFINITIONS

TRIGONOMETRIC EQUATIONS *The equations containing trigonometric functions of unknown angles are known as trigonometric equations.*

$\cos\theta = \frac{1}{2}$, $\sin\theta = 0$, $\tan\theta = \sqrt{3}$ etc. are trigonometric equations.

SOLUTION OF A TRIGONOMETRIC EQUATION *A solution of a trigonometric equation is the value of the unknown angle that satisfies the equation.*

Consider the equation $\sin\theta = \frac{1}{2}$. This equation is clearly satisfied by $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ etc. So, these are its solutions.

Solving an equation means to find the set of all values of the unknown angle which satisfy the given equation.

Consider the equation $2\cos\theta + 1 = 0$ or $\cos\theta = -1/2$. This equation is clearly satisfied by $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ etc.

Since the trigonometric functions are periodic. Therefore, if a trigonometric equation has a solution, it will have infinitely many solutions. For example, $\theta = \frac{2\pi}{3}$, $2\pi \pm \frac{2\pi}{3}$, $4\pi \pm \frac{2\pi}{3}$, ... are solutions of $2\cos\theta + 1 = 0$. These solutions can be put together in compact form as $2n\pi \pm \frac{2\pi}{3}$, where n is an integer. This solution is known as the general solution.

Thus, a solution generalised by means of periodicity is known as the general solution.

It also follows from the above discussion that solving an equation means to find its general solution.

11.2 GENERAL SOLUTIONS OF TRIGONOMETRIC EQUATIONS

In this section, we shall obtain the general solutions of the trigonometric equations $\sin\theta = 0$, $\cos\theta = 0$, $\tan\theta = 0$ and $\cot\theta = 0$.

THEOREM 1 *Prove that the general solution of* $\sin\theta = 0$ *is given by* $\theta = n\pi, n \in Z$.

PROOF In $\Delta\, OMP$, we obtain

$\sin\theta = \frac{PM}{OP}$

$\therefore \quad \sin\theta = 0$

$\Rightarrow \quad \frac{PM}{OP} = 0$

$\Rightarrow \quad PM = 0$

$\Rightarrow \quad OP$ coincides with OX or, OX'

$\Rightarrow \quad \theta = 0, \pi, 2\pi, \ldots, -\pi, -2\pi, -3\pi, \ldots$

$\Rightarrow \quad \theta = n\pi, n \in Z$.

Hence, $\theta = n\pi, n \in Z$ is the general solution of $\sin\theta = 0$.

Q.E.D.

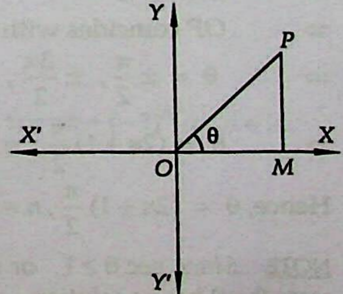


Fig. 11.1

THEOREM 2 *Prove that the general solution of* $\tan\theta = 0$ *is* $\theta = n\pi,\ n \in Z$.

<u>PROOF</u> By definition,

$$\tan\theta = \frac{PM}{OM} \qquad \text{[See Fig. 11.1]}$$

$$\therefore \quad \tan\theta = 0$$

$$\Rightarrow \quad \frac{PM}{OM} = 0$$

X or, OX′

π, − 2π, ...

ral solution of $\sin\theta = 0$. Q.E.D.

ral solution of $\cos\theta = 0$ is $\theta = (2n+1)\frac{\pi}{2},\ n \in Z$.

[See Fig. 11.1]

OY′

.....

2

, $n \in Z$.

2

Hence, the general solution of $\cos\theta = 0$ is $\theta = (2n+1)\frac{\pi}{2},\ n \in Z$. Q.E.D.

THEOREM 4 *Prove that the general solution of* $\cot\theta = 0$ *is* $\theta = (2n+1)\frac{\pi}{2},\ n \in Z$.

<u>PROOF</u> By definition,

$$\cot\theta = \frac{OM}{PM} \qquad \text{[See Fig. 11.1]}$$

$$\therefore \quad \cot\theta = 0$$

$$\Rightarrow \quad \frac{OM}{PM} = 0$$

$$\Rightarrow \quad OM = 0$$

$\Rightarrow$ OP coincides with OY or, OY'

$$\Rightarrow \quad \theta = \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2},\ \ldots$$

$$\Rightarrow \quad \theta = (2n+1)\frac{\pi}{2},\ n \in Z.$$

Hence, $\theta = (2n+1)\frac{\pi}{2},\ n \in Z$ is the general solution of $\cot\theta = 0$. Q.E.D.

<u>NOTE</u> *Since* $\sec\theta \geq 1$, or $\sec\theta \leq -1$, *therefore* $\sec\theta = 0$ *does not have any solution. Similarly,* $\operatorname{cosec}\theta = 0$ *has no solution.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the general solutions of the following equations:*

(i) $\sin 2\theta = 0$ (ii) $\sin \frac{3\theta}{2} = 0$ (iii) $\sin^2 2\theta = 0$

SOLUTION (i) We have,

$$\sin 2\theta = 0$$

$\Rightarrow \quad 2\theta = n\pi, \quad \text{where } n \in Z$ $\quad [\because \sin\theta = 0 \Rightarrow \theta = n\pi]$

$\Rightarrow \quad \theta = \frac{n\pi}{2}, \quad n \in Z.$

(ii) $\sin \frac{3\theta}{2} = 0$

$\Rightarrow \quad \frac{3\theta}{2} = n\pi, \; n \in Z$ $\quad [\sin\theta = 0 \Rightarrow \theta = n\pi]$

$\Rightarrow \quad \theta = \frac{2n\pi}{3}, n \in Z.$

(iii) $\sin^2 2\theta = 0 \Rightarrow \sin 2\theta = 0 \Rightarrow 2\theta = n\pi, \; n \in Z \Rightarrow \theta = \frac{n\pi}{2}, \; n \in Z.$

EXAMPLE 2 *Find the general solutions of the following equations:*

(i) $\cos 3\theta = 0$ (ii) $\cos \frac{3\theta}{2} = 0$ (iii) $\cos^2 3\theta = 0$

SOLUTION We know that the general solution of the equation $\cos\theta = 0$ is $\theta = (2n+1)\frac{\pi}{2}, n \in Z.$

Therefore,

(i) $\cos 3\theta = 0 \Rightarrow 3\theta = (2n+1)\frac{\pi}{2}, n \in Z \Rightarrow \theta = (2n+1)\frac{\pi}{6}, \; n \in Z$

(ii) $\cos \frac{3\theta}{2} = 0 \Rightarrow \frac{3\theta}{2} = (2n+1)\frac{\pi}{2}, n \in Z \Rightarrow \theta = (2n+1)\frac{\pi}{3}, \; n \in Z.$

(iii) $\cos^2 3\theta = 0 \Rightarrow \cos 3\theta = 0 \Rightarrow 3\theta = (2n+1)\frac{\pi}{2}, \; n \in Z \Rightarrow \theta = (2n+1)\frac{\pi}{6}, \; n \in Z.$

EXAMPLE 3 *Find the general solutions of the following equations:*

(i) $\tan 2\theta = 0$ (ii) $\tan \frac{\theta}{2} = 0$ (iii) $\tan \frac{3\theta}{4} = 0$

SOLUTION We know that the general solution of the equation $\tan\theta = 0$ is $\theta = n\pi, n \in Z.$
Therefore,

(i) $\tan 2\theta = 0 \Rightarrow 2\theta = n\pi, n \in Z \Rightarrow \theta = \frac{n\pi}{2}, \; n \in Z$

(ii) $\tan \frac{\theta}{2} = 0 \Rightarrow \frac{\theta}{2} = n\pi, n \in Z \Rightarrow \theta = 2n\pi, \; n \in Z$

(iii) $\tan \frac{3\theta}{4} = 0 \Rightarrow \frac{3\theta}{4} = n\pi, n \in Z \Rightarrow \theta = \frac{4n\pi}{3}, n \in Z$

THEOREM 5 *Prove that the general solution of* $\sin\theta = \sin\alpha$ *is given by:* $\theta = n\pi + (-1)^n \alpha, \; n \in Z.$

PROOF We have,

$$\sin\theta = \sin\alpha$$

$\Leftrightarrow \quad \sin\theta - \sin\alpha = 0$

$\Leftrightarrow \quad 2\sin\left(\frac{\theta-\alpha}{2}\right)\cos\left(\frac{\theta+\alpha}{2}\right) = 0$

$$\Leftrightarrow \quad \sin\left(\frac{\theta-\alpha}{2}\right)=0 \text{ or, } \cos\left(\frac{\theta+\alpha}{2}\right)=0$$

$$\Leftrightarrow \quad \frac{\theta-\alpha}{2} = m\pi, \text{ or, } \frac{\theta+\alpha}{2} = (2m+1)\frac{\pi}{2}, m \in Z$$

$$\Leftrightarrow \quad \theta = 2m\pi + \alpha, \in Z \text{ or, } \theta = (2m+1)\pi - \alpha, m \in Z$$

$$\Leftrightarrow \quad \theta = (\text{Any even multiple of } \pi) + \alpha \quad \text{or,} \quad \theta = (\text{Any odd multiple of } \pi) - \alpha$$

$$\Leftrightarrow \quad \theta = n\pi + (-1)^n \alpha, \quad \text{where } n \in Z.$$

Q.E.D.

REMARK 1 *The equation* $\operatorname{cosec}\theta = \operatorname{cosec}\alpha$ *is equivalent to* $\sin\theta = \sin\alpha$. *Thus,* $\operatorname{cosec}\theta = \operatorname{cosec}\alpha$ *and* $\sin\theta = \sin\alpha$ *have the same general solution.*

THEOREM 6 *Prove that the general solution of* $\cos\theta = \cos\alpha$ *is given by:* $\theta = 2n\pi \pm \alpha$, where $n \in Z$.

PROOF We have,

$$\cos\theta = \cos\alpha$$

$$\Leftrightarrow \quad \cos\theta - \cos\alpha = 0$$

$$\Leftrightarrow \quad -2\sin\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$$\Leftrightarrow \quad \sin\left(\frac{\theta+\alpha}{2}\right) = 0 \text{ or, } \sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$$\Leftrightarrow \quad \frac{\theta+\alpha}{2} = n\pi, \text{ or } \frac{\theta-\alpha}{2} = n\pi, n \in Z$$

$$\Leftrightarrow \quad \theta = 2n\pi - \alpha \text{ or, } \theta = 2n\pi + \alpha, n \in Z$$

$$\Leftrightarrow \quad \theta = 2n\pi \pm \alpha, n \in Z.$$

Q.E.D.

REMARK 2 *Since* $\sec\theta = \sec\alpha \Leftrightarrow \cos\theta = \cos\alpha$. *So, the general solutions of* $\cos\theta = \cos\alpha$ *and* $\sec\theta = \sec\alpha$ *are same.*

THEOREM 7 *Prove that the general solution of* $\tan\theta = \tan\alpha$ *is given by:* $\theta = n\pi + \alpha, n \in Z$.

PROOF We have,

$$\tan\theta = \tan\alpha$$

$$\Leftrightarrow \quad \frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$$

$$\Leftrightarrow \quad \sin\theta\cos\alpha - \cos\theta\sin\alpha = 0$$

$$\Leftrightarrow \quad \sin(\theta-\alpha) = 0$$

$$\Leftrightarrow \quad \theta - \alpha = n\pi, n \in Z$$

$$\Leftrightarrow \quad \theta = n\pi + \alpha, n \in Z$$

Q.E.D.

REMARK 3 *Since* $\tan\theta = \tan\alpha \Leftrightarrow \cot\theta = \cot\alpha$. *So, general solutions of* $\cot\theta = \cot\alpha$ *and* $\tan\theta = \tan\alpha$ *are same.*

In order to find the general solutions of trigonometrical equations of the formsin $\sin\theta = \sin\alpha$, $\cos\theta = \cos\alpha$ and $\tan\theta = \tan\alpha$, we may use the following algorithm.

ALGORITHM

STEP I *Find a value of* θ, *preferably between 0 and* 2π or between $-\pi$ and π, *satisfying the given equation and call it* α.

STEP II *If the equation is* $\sin\theta = \sin\alpha$, write $\theta = n\pi + (-1)^n \alpha, n \in Z$ *as the general solution.*

For the equation $\cos\theta = \cos\alpha$, write $\theta = 2n\pi \pm \alpha, n \in Z$ *as the general solution.*

For the equation $\tan\theta = \tan\alpha$, write $\theta = n\pi + \alpha, n \in Z$ *as the general solution.*

Following examples illustrate the algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE GENERAL SOLUTIONS OF THE EQUATIONS OF THE FORM** $\sin\theta = \sin\alpha, \cos\theta = \cos\alpha, \tan\theta = \tan\alpha$

EXAMPLE 1 *Find the general solutions of the following equations:*

(i) $\sin\theta = \frac{\sqrt{3}}{2}$ (ii) $2\sin\theta + 1 = 0$ (iii) $\operatorname{cosec}\theta = 2$

SOLUTION (i) A value of θ satisfying $\sin\theta = \frac{\sqrt{3}}{2}$ is $\frac{\pi}{3}$.

$\therefore \quad \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \sin\theta = \sin\frac{\pi}{3} \Rightarrow \theta = n\pi + (-1)^n \frac{\pi}{3}, n \in Z$

(ii) We have,

$2\sin\theta + 1 = 0 \Rightarrow \sin\theta = -\frac{1}{2}$

A value of θ satisfying this equation is $-\pi/6$.

$\therefore \quad \sin\theta = -\frac{1}{2}$

$\Rightarrow \quad \sin\theta = \sin\left(-\frac{\pi}{6}\right)$

$\Rightarrow \quad \theta = n\pi + (-1)^n\left(-\frac{\pi}{6}\right), n \in Z$

$\Rightarrow \quad \theta = n\pi + (-1)^{n+1}\frac{\pi}{6}, n \in Z.$

(iii) We have,

$\operatorname{cosec}\theta = 2 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \sin\theta = \sin\frac{\pi}{6} \Rightarrow \theta = n\pi + (-1)^n\frac{\pi}{6}, n \in Z.$

EXAMPLE 2 *Find the general solutions of the following equations:*

(i) $\cos\theta = \frac{1}{2}$ (ii) $\cos 3\theta = -\frac{1}{2}$ (iii) $\sqrt{3}\sec 2\theta = 2$

SOLUTION (i) $\cos\theta = \frac{1}{2} \Rightarrow \cos\theta = \cos\frac{\pi}{3} \Rightarrow \theta = 2n\pi \pm \frac{\pi}{3}, n \in Z$

(ii) $\quad \cos 3\theta = -\frac{1}{2}$

$\Rightarrow \quad \cos 3\theta = \cos\frac{2\pi}{3}$

$\Rightarrow \quad 3\theta = 2n\pi \pm \frac{2\pi}{3}, n \in Z$

$\Rightarrow \quad \theta = \frac{2n\pi}{3} \pm \frac{2\pi}{9}, n \in Z$

(iii) $\quad \sqrt{3}\sec 2\theta = 2$

$\Rightarrow \quad \cos 2\theta = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad \cos 2\theta = \cos\frac{\pi}{6}$

$\Rightarrow \quad 2\theta = 2n\pi \pm \frac{\pi}{6}, n \in Z$

$\Rightarrow \quad \theta = n\pi \pm \frac{\pi}{12}, n \in Z$

EXAMPLE 3 *Solve the following trigonometric equations:*

(i) $\tan\theta = \frac{1}{\sqrt{3}}$ (ii) $\tan 2\theta = \sqrt{3}$ (iii) $\tan 3\theta = -1$

SOLUTION (i) $\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \tan\theta = \tan\frac{\pi}{6} \Rightarrow \theta = n\pi + \frac{\pi}{6}, n \in Z$

(ii) $\tan 2\theta = \sqrt{3}$

$\Rightarrow \tan 2\theta = \tan\frac{\pi}{3}$

$\Rightarrow 2\theta = n\pi + \frac{\pi}{3}, n \in Z$

$\Rightarrow \theta = \frac{n\pi}{2} + \frac{\pi}{6}, n \in Z$

(iii) $\tan 3\theta = -1$

$\Rightarrow \tan 3\theta = \tan\left(-\frac{\pi}{4}\right)$

$\Rightarrow 3\theta = n\pi + \left(-\frac{\pi}{4}\right), n \in Z$

$\Rightarrow \theta = \frac{n\pi}{3} - \frac{\pi}{12}, n \in Z.$

EXAMPLE 4 *Solve the following trigonometric equations:*

(i) $\sin\frac{\theta}{2} = -1$ (ii) $\cos\frac{3\theta}{2} = \frac{1}{2}$ (iii) $\tan\left(\frac{2}{3}\theta\right) = \sqrt{3}$

SOLUTION (i) $\sin\frac{\theta}{2} = -1$

$\Rightarrow \sin\frac{\theta}{2} = \sin\left(-\frac{\pi}{2}\right)$

$\Rightarrow \frac{\theta}{2} = n\pi + (-1)^n\left(-\frac{\pi}{2}\right), n \in Z$

$\Rightarrow \theta = 2n\pi + (-1)^{n+1}\pi, n \in Z$

(ii) $\cos\frac{3\theta}{2} = \frac{1}{2}$

$\Rightarrow \cos\frac{3\theta}{2} = \cos\frac{\pi}{3}$

$\Rightarrow \frac{3\theta}{2} = 2n\pi \pm \frac{\pi}{3}, n \in Z$

$\Rightarrow \theta = \frac{4n\pi}{3} \pm \frac{2\pi}{9}, n \in Z$

(iii) $\tan\left(\frac{2\theta}{3}\right) = \sqrt{3}$

$\Rightarrow \tan\left(\frac{2\theta}{3}\right) = \tan\frac{\pi}{3}$

$\Rightarrow \frac{2\theta}{3} = n\pi + \frac{\pi}{3}, n \in Z$

$\Rightarrow \theta = \frac{3n\pi}{2} + \frac{\pi}{2}, n \in Z.$

Type II **ON FINDING THE GENERAL SOLUTION OF THE EQUATIONS REDUCIBLE TO THE FORMS** $\sin\theta = \sin\alpha$, $\cos\theta = \cos\alpha$, $\tan\theta = \tan\alpha$.

EXAMPLE 5 *Solve the equation:* $\sin\theta + \sin 3\theta + \sin 5\theta = 0$. **[NCERT EXEMPLAR]**

SOLUTION We have,

$$\sin\theta + \sin 3\theta + \sin 5\theta = 0$$
$$\Rightarrow \quad (\sin 5\theta + \sin\theta) + \sin 3\theta = 0$$
$$\Rightarrow \quad 2\sin 3\theta\cos 2\theta + \sin 3\theta = 0$$
$$\Rightarrow \quad \sin 3\theta(2\cos 2\theta + 1) = 0$$
$$\Rightarrow \quad \sin 3\theta = 0 \text{ or } 2\cos 2\theta + 1 = 0$$
$$\Rightarrow \quad \sin 3\theta = 0 \text{ or, } \cos 2\theta = -\frac{1}{2}$$

Now, $\sin 3\theta = 0 \Rightarrow 3\theta = n\pi, n \in Z \Rightarrow \theta = \frac{n\pi}{3}, n \in Z$

And, $\cos 2\theta = -\frac{1}{2}$

$$\Rightarrow \quad \cos 2\theta = \cos\frac{2\pi}{3}$$
$$\Rightarrow \quad 2\theta = 2m\pi \pm \frac{2\pi}{3}, m \in Z$$
$$\Rightarrow \quad \theta = m\pi \pm \frac{\pi}{3}, m \in Z.$$

Hence, the general solution of the given equation is: $\theta = \frac{n\pi}{3}$ or, $\theta = m\pi \pm \frac{\pi}{3}$, where $m, n \in Z$.

EXAMPLE 6 *Solve the equation:* $\cos\theta + \cos 3\theta - 2\cos 2\theta = 0$

SOLUTION We have,

$$\cos\theta + \cos 3\theta - 2\cos 2\theta = 0$$
$$\Leftrightarrow \quad 2\cos 2\theta\cos\theta - 2\cos 2\theta = 0$$
$$\Leftrightarrow \quad 2\cos 2\theta(\cos\theta - 1) = 0$$
$$\Rightarrow \quad \cos 2\theta = 0 \text{ or, } \cos\theta - 1 = 0$$

Now, $\cos 2\theta = 0 \Rightarrow 2\theta = (2n+1)\frac{\pi}{2}, n \in Z \quad \Rightarrow \theta = (2n+1)\frac{\pi}{4}, n \in Z$

And, $\cos\theta - 1 = 0$

$$\Rightarrow \quad \cos\theta = 1$$
$$\Rightarrow \quad \cos\theta = \cos 0$$
$$\Rightarrow \quad \theta = 2m\pi \pm 0, m \in Z$$
$$\Rightarrow \quad \theta = 2m\pi, m \in Z$$

Hence, $\theta = (2n+1)\frac{\pi}{4}$ or, $\theta = 2m\pi$, where $m, n \in Z$.

EXAMPLE 7 *Solve the equation:* $\sin m\theta + \sin n\theta = 0$.

SOLUTION We have,

$$\sin m\theta + \sin n\theta = 0$$
$$\Rightarrow \quad 2\sin\left(\frac{m+n}{2}\right)\theta\cos\left(\frac{m-n}{2}\right)\theta = 0$$
$$\Rightarrow \quad \sin\left(\frac{m+n}{2}\right)\theta = 0 \text{ or, } \cos\left(\frac{m-n}{2}\right)\theta = 0$$

Now, $\sin\left(\frac{m+n}{2}\right)\theta = 0$

$\Rightarrow \quad \left(\frac{m+n}{2}\right)\theta = r\pi, r \in Z$

$\Rightarrow \quad \theta = \frac{2r\pi}{m+n}, \; r \in Z$

And, $\cos\left(\frac{m-n}{2}\right)\theta = 0$

$\Rightarrow \quad \left(\frac{m-n}{2}\right)\theta = (2s+1)\frac{\pi}{2}, s \in Z$

$\Rightarrow \quad \theta = \frac{(2s+1)\pi}{m-n}, s \in Z$

Hence, $\theta = \frac{2r\pi}{m+n}$ or, $\theta = \frac{(2s+1)\pi}{m-n}$, where $r, \; s \in Z$.

EXAMPLE 8 *Solve the following equations:*

(i) $\sin 2\theta + \cos\theta = 0$ [NCERT] (ii) $\sin 3\theta + \cos 2\theta = 0$

(iii) $\sin 2\theta + \sin 4\theta + \sin 6\theta = 0$

SOLUTION (i) $\sin 2\theta + \cos\theta = 0$

$\Rightarrow \quad \cos\theta = -\sin 2\theta$

$\Rightarrow \quad \cos\theta = \cos\left(\frac{\pi}{2} + 2\theta\right)$

$\Rightarrow \quad \theta = 2n\pi \pm \left(\frac{\pi}{2} + 2\theta\right), \; n \in Z$

Taking positive sign, we have

$\theta = 2n\pi + \frac{\pi}{2} + 2\theta$

$\Rightarrow \quad -\theta = 2n\pi + \frac{\pi}{2}, n \in Z$

$\Rightarrow \quad \theta = -2n\pi - \frac{\pi}{2}, \; n \in Z$

$\Rightarrow \quad \theta = 2m\pi - \frac{\pi}{2}$, where $m = -n \in Z$.

Taking negative sign, we have

$\theta = 2n\pi - \left(\frac{\pi}{2} + 2\theta\right)$

$\Rightarrow \quad 3\theta = 2n\pi - \frac{\pi}{2} \Rightarrow \theta = \frac{2n\pi}{3} - \frac{\pi}{6}, n \in Z.$

Hence, $\theta = 2m\pi - \frac{\pi}{2}$, or, $\theta = \frac{2n\pi}{3} - \frac{\pi}{6}$, where $m, \; n \in Z$.

(ii) $\sin 3\theta + \cos 2\theta = 0$

$\Rightarrow \quad \cos 2\theta = -\sin 3\theta$

$\Rightarrow \quad \cos 2\theta = \cos\left(\frac{\pi}{2} + 3\theta\right)$

$\Rightarrow \quad 2\theta = 2n\pi \pm \left(\frac{\pi}{2} + 3\theta\right), \; n \in Z$

Taking positive sign, we have

$$2\theta = 2n\pi + \frac{\pi}{2} + 3\theta$$

$$\Rightarrow \quad -\theta = 2n\pi + \frac{\pi}{2}$$

$$\Rightarrow \quad \theta = -2n\pi - \frac{\pi}{2}$$

$$\Rightarrow \quad \theta = 2m\pi - \frac{\pi}{2}, \text{ where } -n = m.$$

Taking negative sign, we have

$$2\theta = 2n\pi - \frac{\pi}{2} - 3\theta \Rightarrow 5\theta = 2n\pi - \frac{\pi}{2} \Rightarrow \theta = \frac{2n\pi}{5} - \frac{\pi}{10}, \ n \in Z$$

Hence, $\theta = \frac{2n\pi}{5} - \frac{\pi}{10}$ or, $\theta = 2m\pi - \frac{\pi}{2}$, where $m, n \in Z$.

(iii) We have,

$$\sin 2\theta + \sin 4\theta + \sin 6\theta = 0$$

$$\Rightarrow \quad \sin 4\theta + (\sin 2\theta + \sin 6\theta) = 0$$

$$\Rightarrow \quad \sin 4\theta + 2\sin 4\theta \cos 2\theta = 0$$

$$\Rightarrow \quad \sin 4\theta (1 + 2\cos 2\theta) = 0$$

$$\Rightarrow \quad \sin 4\theta = 0 \text{ or, } 1 + 2\cos 2\theta = 0 \Rightarrow \sin 4\theta = 0 \text{ or, } \cos 2\theta = -\frac{1}{2}$$

Now, $\sin 4\theta = 0 \Rightarrow 4\theta = n\pi, n \in Z \Rightarrow \theta = \frac{n\pi}{4}, \ n \in Z$

And, $\cos 2\theta = -\frac{1}{2}$

$$\Rightarrow \quad \cos 2\theta = \cos \frac{2\pi}{3}$$

$$\Rightarrow \quad 2\theta = 2m\pi \pm \frac{2\pi}{3}, \ m \in Z$$

$$\Rightarrow \quad \theta = m\pi \pm \frac{\pi}{3}, \ m \in Z$$

Hence, $\theta = \frac{n\pi}{4}$ or, $\theta = m\pi \pm \frac{\pi}{3}$, where $m, n \in Z$.

EXAMPLE 9 *Solve the following equations:*

(i) $2\cos^2\theta + 3\sin\theta = 0$ **[NCERT]** (ii) $\cot^2\theta + \frac{3}{\sin\theta} + 3 = 0$

(iii) $2\tan\theta - \cot\theta = -1$ (iv) $4\cos\theta - 3\sec\theta = \tan\theta$

(v) $\tan^2\theta + (1-\sqrt{3})\tan\theta - \sqrt{3} = 0$ (vi) $\sec^2 2x = 1 - \tan 2x$ **[NCERT]**

SOLUTION (i) $2\cos^2\theta + 3\sin\theta = 0$

$$\Rightarrow \quad 2(1-\sin^2\theta) + 3\sin\theta = 0$$

$$\Rightarrow \quad 2\sin^2\theta - 3\sin\theta - 2 = 0$$

$$\Rightarrow \quad 2\sin^2\theta - 4\sin\theta + \sin\theta - 2 = 0$$

$$\Rightarrow \quad 2\sin\theta(\sin\theta - 2) + 1(\sin\theta - 2) = 0$$

$$\Rightarrow \quad (\sin\theta - 2)(2\sin\theta + 1) = 0$$

$$\Rightarrow \quad 2\sin\theta + 1 = 0 \qquad [\because \sin\theta \neq 2 \ \therefore \sin\theta - 2 \neq 0]$$

$\Rightarrow \quad \sin\theta = -\dfrac{1}{2}$

$\Rightarrow \quad \sin\theta = \sin\left(-\dfrac{\pi}{6}\right)$

$\Rightarrow \quad \theta = n\pi + (-1)^n\left(-\dfrac{\pi}{6}\right),\ n \in Z$

$\Rightarrow \quad \theta = n\pi + (-1)^{n+1}\,\dfrac{\pi}{6},\ \ n \in Z.$

(ii) $\quad \cot^2\theta + \dfrac{3}{\sin\theta} + 3$

$\Rightarrow \quad \mathrm{cosec}^2$

$\Rightarrow \quad$ cc

$\Rightarrow$

$\Rightarrow$

Now,

$\Rightarrow$

$\Rightarrow \quad$ sin

$\Rightarrow \quad \sin\theta = \sin\left(-\dfrac{\pi}{6}\right)$

$\Rightarrow \quad \theta = n\pi + (-1)^n\left(-\dfrac{\pi}{6}\right),\ n \in Z$

$\Rightarrow \quad \theta = n\pi + (-1)^{n+1}\,\dfrac{\pi}{6},\ n \in Z$

And, $\quad \mathrm{cosec}\,\theta + 1 = 0$

$\Rightarrow \quad \dfrac{1}{\sin\theta} + 1 = 0$

$\Rightarrow \quad \sin\theta = -1$

$\Rightarrow \quad \sin\theta = \sin\left(-\dfrac{\pi}{2}\right)$

$\Rightarrow \quad \theta = m\pi + (-1)^m\left(-\dfrac{\pi}{2}\right),\ m \in Z$

$\Rightarrow \quad \theta = m\pi + (-1)^{m+1}\,\dfrac{\pi}{2},\ m \in Z$

Hence, $\theta = n\pi + (-1)^{n+1}\dfrac{\pi}{6}$ or, $\theta = m\pi + (-1)^{m+1}\dfrac{\pi}{2}, m, n \in Z$

(iii) $\quad 2\tan\theta - \cot\theta = -1$

$\Rightarrow \quad 2\tan\theta - \dfrac{1}{\tan\theta} = -1$

$\Rightarrow \quad 2\tan^2\theta + \tan\theta - 1 = 0$

$\Rightarrow \quad 2\tan^2\theta + 2\tan\theta - \tan\theta - 1 = 0$

$\Rightarrow \quad 2\tan\theta(\tan\theta + 1) - (\tan\theta + 1) = 0$

$\Rightarrow \quad (\tan\theta + 1)(2\tan\theta - 1) = 0$

$\Rightarrow \quad \tan\theta = -1$ or, $\tan\theta = \dfrac{1}{2}$

Now, $\quad \tan\theta = -1$

$\Rightarrow \quad \tan\theta = \tan\left(-\dfrac{\pi}{4}\right)$

$\Rightarrow \quad \theta = n\pi + \left(-\dfrac{\pi}{4}\right),\ n \in Z$

$\Rightarrow \quad \theta = n\pi - \dfrac{\pi}{4},\ n \in Z$

And, $\quad \tan\theta = \dfrac{1}{2}$

$\Rightarrow \quad \tan\theta = \tan\alpha$, where $\tan\alpha = \dfrac{1}{2}$

$\Rightarrow \quad \theta = m\pi + \alpha$, where $\tan\alpha = \dfrac{1}{2}$ and $m \in Z$

Hence, $\theta = n\pi - \dfrac{\pi}{4}$ or, $\theta = m\pi + \alpha$, where $m,\ n \in Z$ and $\tan\alpha = \dfrac{1}{2}$

(iv) $\quad 4\cos\theta - 3\sec\theta = \tan\theta$

$\Rightarrow \quad 4\cos\theta - \dfrac{3}{\cos\theta} = \dfrac{\sin\theta}{\cos\theta}$

$\Rightarrow \quad 4\cos^2\theta - 3 = \sin\theta$

$\Rightarrow \quad 4(1-\sin^2\theta) - 3 = \sin\theta$

$\Rightarrow \quad 4\sin^2\theta + \sin\theta - 1 = 0$

$\Rightarrow \quad \sin\theta = \dfrac{-1 \pm \sqrt{1+16}}{8}$

$\Rightarrow \quad \sin\theta = \dfrac{-1 \pm \sqrt{17}}{8}$

$\Rightarrow \quad \sin\theta = \dfrac{-1 + \sqrt{17}}{8}$ or, $\sin\theta = \dfrac{-1 - \sqrt{17}}{8}$

Now, $\quad \sin\theta = \dfrac{-1 + \sqrt{17}}{8}$

$\Rightarrow \quad \sin\theta = \sin\alpha$, where $\sin\alpha = \dfrac{-1 + \sqrt{17}}{8}$

$\Rightarrow \quad \theta = n\pi + (-1)^n\alpha$, where $\sin\alpha = \dfrac{-1 + \sqrt{17}}{8}$ and $n \in Z$

And, $\quad \sin\theta = \dfrac{-1 - \sqrt{17}}{8}$

$\Rightarrow \quad \sin\theta = \sin\beta$, where $\sin\beta = \dfrac{-1 - \sqrt{17}}{8}$

$\Rightarrow \quad \theta = n\pi + (-1)^n\beta$, where $\sin\beta = \dfrac{-1 - \sqrt{17}}{8}$

(v) $\quad \tan^2\theta + (1-\sqrt{3})\tan\theta - \sqrt{3} = 0$

$\Rightarrow \quad \tan^2\theta + \tan\theta - \sqrt{3}\tan\theta - \sqrt{3} = 0$

$\Rightarrow \quad \tan(\tan\theta + 1) - \sqrt{3}(\tan\theta + 1) = 0$

$\Rightarrow \quad (\tan\theta + 1)(\tan\theta - \sqrt{3}) = 0$

$\Rightarrow \quad \tan\theta + 1 = 0$ or $\tan\theta - \sqrt{3} = 0$

$\Rightarrow \quad \tan\theta = -1$ or, $\tan\theta = \sqrt{3}$

Now, $\quad \tan\theta = -1 \Rightarrow \tan\theta = \tan\left(-\frac{\pi}{4}\right) \Rightarrow \theta = n\pi - \frac{\pi}{4},\ n \in Z$

And, $\quad \tan\theta = \sqrt{3} \Rightarrow \tan\theta = \tan\frac{\pi}{3} \Rightarrow \theta = m\pi + \frac{\pi}{3},\ m \in Z$

Hence, $\theta = n\pi - \frac{\pi}{4}$ or, $\theta = m\pi + \frac{\pi}{3}$, where $m, n \in Z$.

(vi) $\quad \sec^2 2x = 1 - \tan 2x$

$\Rightarrow \quad 1 + \tan^2 2x = 1 - \tan 2x$

$\Rightarrow \quad \tan^2 2x + \tan 2x = 0$

$\Rightarrow \quad \tan 2x(\tan 2x + 1) = 0$

$\Rightarrow \quad \tan 2x = 0$ or, $\tan 2x + 1 = 0$

$\Rightarrow \quad \tan 2x = 0$ or, $\tan 2x = -1$

$\Rightarrow \quad \tan 2x = 0$ or, $\tan 2x = \tan\frac{3\pi}{4}$

$\Rightarrow \quad 2x = n\pi$ or, $2x = n\pi + \frac{3\pi}{4},\ n \in Z$

$\Rightarrow \quad x = \frac{n\pi}{2}$ or, $x = \frac{n\pi}{2} + \frac{3\pi}{8},\ n \in Z$

EXAMPLE 10 *Solve the following equations:*

(i) $\tan\theta + \tan 2\theta + \tan\theta\tan 2\theta = 1$ (ii) $\tan\theta + \tan 2\theta + \tan 3\theta = \tan\theta\tan 2\theta\tan 3\theta$

(iii) $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$ (iv) $\tan\theta + \tan\left(\theta + \frac{\pi}{3}\right) + \tan\left(\theta + \frac{2\pi}{3}\right) = 3$

SOLUTION (i) $\tan\theta + \tan 2\theta + \tan\theta\tan 2\theta = 1$

$\Rightarrow \quad \tan\theta + \tan 2\theta = 1 - \tan\theta\tan 2\theta$

$\Rightarrow \quad \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} = 1$

$\Rightarrow \quad \tan 3\theta = 1$

$\Rightarrow \quad \tan 3\theta = \tan\frac{\pi}{4}$

$\Rightarrow \quad 3\theta = n\pi + \frac{\pi}{4},\ n \in Z$

$\Rightarrow \quad \theta = \frac{n\pi}{3} + \frac{\pi}{12},\ n \in Z$

(ii) $\quad \tan\theta + \tan 2\theta + \tan 3\theta = \tan\theta\tan 2\theta\tan 3\theta$

$\Rightarrow \quad \tan\theta + \tan 2\theta = -\tan 3\theta + \tan\theta\tan 2\theta\tan 3\theta$

$\Rightarrow \quad \tan\theta + \tan 2\theta = -\tan 3\theta(1 - \tan\theta\tan 2\theta)$

$\Rightarrow \quad \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} = -\tan 3\theta$

$\Rightarrow \quad \tan(\theta + 2\theta) = -\tan 3\theta$

$\Rightarrow \quad \tan 3\theta = -\tan 3\theta$

$\Rightarrow \quad 2\tan 3\theta = \theta$

$\Rightarrow \quad \tan 3\theta = 0$

$\Rightarrow \quad 3\theta = n\pi,\ n \in Z \Rightarrow \theta = \frac{n\pi}{3},\ n \in Z.$

(iii) $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$

$\Rightarrow \quad \tan\theta + \tan 2\theta = \sqrt{3}(1 - \tan\theta\tan 2\theta)$

$\Rightarrow \quad \dfrac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} = \sqrt{3}$

$\Rightarrow \quad \tan 3\theta = \sqrt{3}$

$\Rightarrow \quad \tan 3\theta = \tan\dfrac{\pi}{3}$

$\Rightarrow \quad 3\theta = n\pi + \dfrac{\pi}{3}, n \in Z$

$\Rightarrow \quad \theta = \dfrac{n\pi}{3} + \dfrac{\pi}{9}, \ n \in Z$

(iv) $\tan\theta + \tan\left(\theta + \dfrac{\pi}{3}\right) + \tan\left(\theta + \dfrac{2\pi}{3}\right) = 3$

$\Rightarrow \quad \tan\theta + \dfrac{\tan\theta + \tan\frac{\pi}{3}}{1 - \tan\theta\tan\frac{\pi}{3}} + \dfrac{\tan\theta + \tan\frac{2\pi}{3}}{1 - \tan\theta\tan\frac{2\pi}{3}} = 3$

$\Rightarrow \quad \tan\theta + \dfrac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta} + \dfrac{\tan\theta - \sqrt{3}}{1 + \sqrt{3}\tan\theta} = 3$

$\Rightarrow \quad \tan\theta + \dfrac{8\tan\theta}{1 - 3\tan^2\theta} = 3$

$\Rightarrow \quad \dfrac{\tan\theta - 3\tan^3\theta + 8\tan\theta}{1 - 3\tan^2\theta} = 3$

$\Rightarrow \quad \dfrac{3(3\tan\theta - \tan^3\theta)}{1 - 3\tan^2\theta} = 3$

$\Rightarrow \quad 3\tan 3\theta = 3$

$\Rightarrow \quad \tan 3\theta = 1$

$\Rightarrow \quad \tan 3\theta = \tan\dfrac{\pi}{4}$

$\Rightarrow \quad 3\theta = n\pi + \dfrac{\pi}{4}, n \in Z$

$\Rightarrow \quad \theta = \dfrac{n\pi}{3} + \dfrac{\pi}{12}, \ n \in Z.$

11.3 GENERAL SOLUTIONS OF TRIGONOMETRICAL EQUATIONS OF THE FORM $\sin^2\theta = \sin^2\alpha$, $\cos^2\theta = \cos^2\alpha$, $\tan^2\theta = \tan^2\alpha$

THEOREM *Prove that:*

(i) $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in Z$ (ii) $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in Z$

(iii) $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in Z$

PROOF (i) $\sin^2\theta = \sin^2\alpha$

$\Rightarrow \quad 2\sin^2\theta = 2\sin^2\alpha$

$\Rightarrow \quad 1 - \cos 2\theta = 1 - \cos 2\alpha$

$\Rightarrow \quad \cos 2\theta = \cos 2\alpha$

$\Rightarrow \quad 2\theta = 2n\pi \pm 2\alpha, \ n \in Z$

$\Rightarrow \quad \theta = n\pi \pm \alpha, \ n \in Z$

(ii) $\cos^2\theta = \cos^2\alpha$

$\Rightarrow \quad 2\cos^2\theta = 2\cos^2\alpha$

$\Rightarrow \quad 1+\cos 2\theta = 1+\cos 2\alpha$

$\Rightarrow \quad \cos 2\theta = \cos 2\alpha$

$\Rightarrow \quad 2\theta = 2n\pi \pm 2\alpha, n \in Z$

$\Rightarrow \quad \theta = n\pi \pm \alpha, n \in Z$

(iii) $\tan^2\theta = \tan^2\alpha$

$$\Rightarrow \quad \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$$

$\Rightarrow \quad \cos 2\theta = \cos 2\alpha$

$\Rightarrow \quad 2\theta = 2n\pi \pm 2\alpha, n \in Z$

$\Rightarrow \quad \theta = n\pi \pm \alpha, \; n \in Z$ Q.E.D.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Solve:* $7\cos^2\theta + 3\sin^2\theta = 4$

SOLUTION We have,

$7\cos^2\theta + 3\sin^2\theta = 4$

$\Rightarrow \quad 7(1-\sin^2\theta) + 3\sin^2\theta = 4$

$\Rightarrow \quad 4\sin^2\theta = 3$

$\Rightarrow \quad 4\sin^2\theta = 3$

$$\Rightarrow \quad \sin^2\theta = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2$$

$$\Rightarrow \quad \sin^2\theta = \sin^2\frac{\pi}{3} \Rightarrow \theta = n\pi \pm \frac{\pi}{3}, \; n \in Z$$

EXAMPLE 2 *Solve:* $2\sin^2 x + \sin^2 2x = 2$

SOLUTION We have,

$2\sin^2 x + \sin^2 2x = 2$

$\Rightarrow \quad 2\sin^2 x + (2\sin x\cos x)^2 = 2$

$\Rightarrow \quad 4\sin^2 x\cos^2 x + 2\sin^2 x = 2$

$\Rightarrow \quad 2\sin^2 x\cos^2 x + \sin^2 x = 1$

$\Rightarrow \quad 2\sin^2 x\cos^2 x - (1-\sin^2 x) = 0$

$\Rightarrow \quad 2\sin^2 x\cos^2 x - \cos^2 x = 0$

$\Rightarrow \quad \cos^2 x(2\sin^2 x - 1) = 0 \Rightarrow \cos^2 x = 0$ or $2\sin^2 x - 1 = 0$

$$\Rightarrow \quad \cos^2 x = 0 \text{ or, } \sin^2 x = \frac{1}{2}$$

Now, $\cos^2 x = 0 \Rightarrow \cos^2 x = \cos^2\frac{\pi}{2} \Rightarrow x = n\pi \pm \frac{\pi}{2}, \; n \in Z$

And, $\sin^2 x = \frac{1}{2} \Rightarrow \sin^2 x = \sin^2\frac{\pi}{4} \Rightarrow x = m\pi \pm \frac{\pi}{4}, \; m \in Z$

Hence, $x = n\pi \pm \frac{\pi}{2}$ or $x = m\pi \pm \frac{\pi}{4}$, where $m, n \in Z$

LEVEL-2

EXAMPLE 3 *Solve:* $\sin 3\alpha = 4\sin\alpha \sin(x+\alpha)\sin(x-\alpha)$, where $\alpha \neq n\pi, n \in Z$

SOLUTION We have,

$$\sin 3\alpha = 4\sin\alpha\sin(x+\alpha)\sin(x-\alpha)$$

$$\Rightarrow \quad \sin 3\alpha = 4\sin\alpha(\sin^2 x - \sin^2\alpha)$$

$$\Rightarrow \quad 3\sin\alpha - 4\sin^3\alpha = 4\sin^2 x\sin\alpha - 4\sin^3\alpha$$

$$\Rightarrow \quad 3\sin\alpha = 4\sin^2 x\sin\alpha$$

$$\Rightarrow \quad \sin^2 x = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2$$

$$\Rightarrow \quad \sin^2 x = \sin^2\frac{\pi}{3} \Rightarrow x = n\pi \pm \frac{\pi}{3},\ n \in Z$$

EXAMPLE 4 *Solve:* $4\sin x\sin 2x\sin 4x = \sin 3x$

SOLUTION We have,

$$4\sin x\sin 2x\sin 4x = \sin 3x$$

$$\Rightarrow \quad 4\sin x\sin(3x-x)\cdot\sin(3x+x) = \sin 3x$$

$$\Rightarrow \quad 4[\sin x(\sin^2 3x - \sin^2 x)] = 3\sin x - 4\sin^3 x$$

$$\Rightarrow \quad 4\sin x\sin^2 3x - 4\sin^3 x = 3\sin x - 4\sin^3 x$$

$$\Rightarrow \quad 4\sin x\sin^2 3x = 3\sin x$$

$$\Rightarrow \quad \sin x(4\sin^2 3x - 3) = 0$$

$$\Rightarrow \quad \sin x = 0 \text{ or, } 4\sin^2 3x - 3 = 0$$

$$\Rightarrow \quad \sin x = 0 \text{ or, } \sin^2 3x = \frac{3}{4}$$

Now, $\sin x = 0 \Rightarrow x = n\pi,\ n \in Z$

And, $\sin^2 3x = \dfrac{3}{4}$

$$\Rightarrow \quad \sin^2 3x = \left(\frac{\sqrt{3}}{2}\right)^2$$

$$\Rightarrow \quad \sin^2 3x = \sin^2\frac{\pi}{3}$$

$$\Rightarrow \quad 3x = m\pi \pm \frac{\pi}{3},\ m \in Z$$

$$\Rightarrow \quad x = \frac{m\pi}{3} \pm \frac{\pi}{9}$$

Hence, $x = n\pi$ or, $x = m\pi \pm \dfrac{\pi}{3}$, where $m, n \in Z$

11.4 TRIGONOMETRIC EQUATIONS OF THE FORM

$a\cos\theta + b\sin\theta = c$, where $a, b, c \in R$ such that $|c| \leq \sqrt{a^2+b^2}$

To solve this type of equations, we first reduce them in the form $\cos\theta = \cos\alpha$, or $\sin\theta = \sin\alpha$.

The following algorithm provides the method of solution.

ALGORITHM

STEP I *Obtain the equation* $a\cos\theta + b\sin\theta = c$.

STEP II *Put $a = r\cos\alpha$ and $b = r\sin\alpha$, where $r = \sqrt{a^2+b^2}$ and $\tan\alpha = b/a$* i.e. $\alpha = \tan^{-1}(b/a)$.

STEP III *Using the substitution in step II, the equation reduces to*

$$r\cos(\theta-\alpha) = c \Rightarrow \cos(\theta-\alpha) = \frac{c}{r} = \cos\beta \text{ (say)}.$$

STEP IV *Solve the equation obtained in step III by using the formulas discussed earlier.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Solve:* $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$ **[NCERT EXEMPLAR]**

SOLUTION We have,

$$\sqrt{3}\cos\theta + \sin\theta = \sqrt{2} \quad \text{...(i)}$$

This is of the form $a\cos\theta + b\sin\theta = c$, where $a = \sqrt{3}, b = 1$ and $c = \sqrt{2}$.

Let $a = r\cos\alpha$ and $b = r\sin\alpha$. Then,

$\sqrt{3} = r\cos\alpha$ and $1 = r\sin\alpha$.

$\Rightarrow$ $r = \sqrt{a^2+b^2} = \sqrt{(\sqrt{3})^2+1^2} = 2$ and $\tan\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}$

Substituting $a = \sqrt{3} = r\cos\alpha$ and $b = 1 = r\sin\alpha$ in the equation (i) it reduces to

$$r\cos\alpha\cos\theta + r\sin\alpha\sin\theta = \sqrt{2}$$

$\Rightarrow$ $r\cos(\theta-\alpha) = \sqrt{2}$

$\Rightarrow$ $2\cos\left(\theta-\frac{\pi}{6}\right) = \sqrt{2}$

$\Rightarrow$ $\cos\left(\theta-\frac{\pi}{6}\right) = \frac{1}{\sqrt{2}}$

$\Rightarrow$ $\cos\left(\theta-\frac{\pi}{6}\right) = \cos\frac{\pi}{4}$

$\Rightarrow$ $\theta-\frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4}, n \in Z$

$\Rightarrow$ $\theta = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{6}, n \in Z$

$\Rightarrow$ $\theta = 2n\pi + \frac{\pi}{4} + \frac{\pi}{6}$ or, $\theta = 2n\pi - \frac{\pi}{4} + \frac{\pi}{6}$

$\Rightarrow$ $\theta = 2n\pi + \frac{5\pi}{12}$ or, $\theta = 2n\pi - \frac{\pi}{12}$

Hence, $\theta = 2n\pi + \frac{5\pi}{12}$ or, $\theta = 2n\pi - \frac{\pi}{12}$, where $n \in Z$

EXAMPLE 2 *Solve:* $\sqrt{2}\sec\theta + \tan\theta = 1$

SOLUTION We have,

$$\sqrt{2}\sec\theta + \tan\theta = 1$$

$\Rightarrow$ $\frac{\sqrt{2}}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = 1$

$\Rightarrow$ $\sqrt{2} + \sin\theta = \cos\theta$

$\Rightarrow$ $\cos\theta - \sin\theta = \sqrt{2}$ ...(i)

This is of the form, $a\cos\theta - b\sin\theta = c$, where $a = 1$, $b = 1$ and $c = \sqrt{2}$

Let $a = r\cos\alpha$, and $b = r\sin\alpha$.

$\Rightarrow$ $1 = r\cos\alpha$ and $1 = r\sin\alpha$.

$\Rightarrow \quad r = \sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \sqrt{2}$ and, $\tan\alpha = \dfrac{r\sin\alpha}{r\cos\alpha} = 1$

$\Rightarrow \quad r = \sqrt{2}$ and, $\alpha = \dfrac{\pi}{4}$

Substituting $a = 1 = r\cos\alpha$ and $b = 1 = r\sin\alpha$ in (i), we get

$$r\cos\theta\cos\alpha - r\sin\theta\sin\alpha = \sqrt{2}$$

$$\Rightarrow \quad r\cos(\theta+\alpha) = \sqrt{2}$$

$$\Rightarrow \quad \cos\left(\theta+\frac{\pi}{4}\right) = 1$$

$$\Rightarrow \quad \cos\left(\theta+\frac{\pi}{4}\right) = \cos 0^\circ$$

$$\Rightarrow \quad \theta+\frac{\pi}{4} = 2n\pi \pm 0,\ n\in Z \quad \Rightarrow \theta = 2n\pi - \frac{\pi}{4},\ n\in Z$$

EXAMPLE 3 *Solve:* $\cot\theta + \operatorname{cosec}\theta = \sqrt{3}$

SOLUTION We have,

$$\cot\theta + \operatorname{cosec}\theta = \sqrt{3}$$

$$\Rightarrow \quad \frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta} = \sqrt{3}$$

$$\Rightarrow \quad \cos\theta + 1 = \sqrt{3}\sin\theta$$

$$\Rightarrow \quad \sqrt{3}\sin\theta - \cos\theta = 1 \qquad \text{...(i)}$$

This is of the form $a\sin\theta + b\cos\theta = c$, where $a = \sqrt{3}, b = -1$ and $c = 1$.

$\therefore \quad \sqrt{3} = r\sin\alpha$ and $1 = r\cos\alpha$

$\Rightarrow \quad r = \sqrt{a^2+b^2} = \sqrt{3+1} = 2$ and, $\tan\alpha = \dfrac{\sqrt{3}}{1} = \sqrt{3} \Rightarrow r = 2$ and $\alpha = \pi/3$

Substituting $a = \sqrt{3} = r\sin\alpha$ and $b = 1 = r\cos\alpha$ in (i), we get

$$r\sin\alpha\sin\theta - r\cos\alpha\cos\theta = 1$$

$$\Rightarrow \quad -r\cos(\theta+\alpha) = 1$$

$$\Rightarrow \quad -2\cos\left(\theta+\frac{\pi}{3}\right) = 1$$

$$\Rightarrow \quad \cos\left(\theta+\frac{\pi}{3}\right) = -\frac{1}{2}$$

$$\Rightarrow \quad \cos\left(\theta+\frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$$

$$\Rightarrow \quad \theta+\frac{\pi}{3} = 2n\pi \pm \frac{2\pi}{3},\ n\in Z$$

$$\Rightarrow \quad \theta = 2n\pi \pm \frac{2\pi}{3} - \frac{\pi}{3}, n\in Z$$

$$\Rightarrow \quad \theta = 2n\pi + \frac{\pi}{3},\ n\in Z \quad \text{or,} \quad \theta = 2n\pi - \pi = (2n-1)\pi,\ n\in Z$$

But, θ cannot be equal to $(2n-1)\pi$ as it makes $\sin\theta = 0$.

Hence, $\theta = 2n\pi + \dfrac{\pi}{3}, n\in Z$

EXERCISE 11.1

LEVEL-1

1. Find the general solutions of the following equations:

(i) $\sin\theta = \frac{1}{2}$ (ii) $\cos\theta = -\frac{\sqrt{3}}{2}$ (iii) $\operatorname{cosec}\theta = -\sqrt{2}$

(iv) $\sec\theta = \sqrt{2}$ (v) $\tan\theta = -\frac{1}{\sqrt{3}}$ (vi) $\sqrt{3}\sec\theta = 2$

2. Find the general solutions of the following equations:

(i) $\sin 2\theta = \frac{\sqrt{3}}{2}$ (ii) $\cos 3\theta = \frac{1}{2}$ (iii) $\sin 9\theta = \sin\theta$

(iv) $\sin 2\theta = \cos 3\theta$ (v) $\tan\theta + \cot 2\theta = 0$ (vi) $\tan 3\theta = \cot\theta$

(vii) $\tan 2\theta \tan\theta = 1$ (viii) $\tan m\theta + \cot n\theta = 0$ (ix) $\tan p\theta = \cot q\theta$

(x) $\sin 2\theta + \cos\theta = 0$ (xi) $\sin\theta = \tan\theta$ (xii) $\sin 3\theta + \cos 2\theta = 0$

3. Solve the following equations:

(i) $\sin^2\theta - \cos\theta = \frac{1}{4}$ (ii) $2\cos^2\theta - 5\cos\theta + 2 = 0$

(iii) $2\sin^2 x + \sqrt{3}\cos x + 1 = 0$ (iv) $4\sin^2\theta - 8\cos\theta + 1 = 0$

(v) $\tan^2 x + (1-\sqrt{3})\tan x - \sqrt{3} = 0$

(vi) $3\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta - 3\sin^2\theta = 0$ (vii) $\cos 4\theta = \cos 2\theta$

4. Solve the following equations:

(i) $\cos\theta + \cos 2\theta + \cos 3\theta = 0$ (ii) $\cos\theta + \cos 3\theta - \cos 2\theta = 0$ [NCERT]

(iii) $\sin\theta + \sin 5\theta = \sin 3\theta$ (iv) $\cos\theta\cos 2\theta\cos 3\theta = \frac{1}{4}$

(v) $\cos\theta + \sin\theta = \cos 2\theta + \sin 2\theta$ (vi) $\sin\theta + \sin 2\theta + \sin 3\theta = 0$

(vii) $\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0$

(viii) $\sin 3\theta - \sin\theta = 4\cos^2\theta - 2$ (ix) $\sin 2\theta - \sin 4\theta + \sin 6\theta = 0$ [NCERT]

5. Solve the following equations:

(i) $\tan\theta + \tan 2\theta + \tan 3\theta = 0$ (ii) $\tan\theta + \tan 2\theta = \tan 3\theta$

(iii) $\tan 3\theta + \tan\theta = 2\tan 2\theta$

6. Solve the following equations:

(i) $\sin\theta + \cos\theta = \sqrt{2}$ (ii) $\sqrt{3}\cos\theta + \sin\theta = 1$

(iii) $\sin\theta + \cos\theta = 1$ (iv) $\operatorname{cosec}\theta = 1 + \cot\theta$

(v) $(\sqrt{3}-1)\cos\theta + (\sqrt{3}+1)\sin\theta = 2$ [NCERT EXEMPLAR]

7. Solve the following equations:

(i) $\cot\theta + \tan\theta = 2$ [NCERT EXEMPLAR]

(ii) $2\sin^2\theta = 3\cos\theta,\ 0 \le \theta \le 2\pi$ [NCERT EXEMPLAR]

(iii) $\sec\theta\cos 5\theta + 1 = 0,\ 0 < \theta < \frac{\pi}{2}$ [NCERT EXEMPLAR]

(iv) $5\cos^2\theta + 7\sin^2\theta - 6 = 0$ [NCERT EXEMPLAR]

(v) $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$ [NCERT EXEMPLAR]

ANSWERS

1. (i) $\theta = n\pi + (-1)^n \frac{\pi}{6},\ n \in Z$ (ii) $\theta = 2n\pi \pm \frac{7\pi}{6},\ n \in Z$

(iii) $\theta = n\pi + (-1)^{n+1}\frac{\pi}{4},\ n \in Z$ (iv) $\theta = 2n\pi \pm \frac{\pi}{4},\ n \in Z$

(v) $\theta = n\pi - \frac{\pi}{6},\ n \in Z$ (vi) $\theta = 2n\pi \pm \frac{\pi}{6},\ n \in Z$

2. (i) $\theta = \frac{n\pi}{2} + (-1)^n \frac{\pi}{6},\ n \in Z$ (ii) $\theta = \frac{2n\pi}{3} \pm \frac{\pi}{9},\ n \in Z$

(iii) $\theta = \frac{r\pi}{4}$ or $\theta = (2r+1)\frac{\pi}{10}$, where $r \in Z$

(iv) $\theta = (4n+1)\frac{\pi}{10}$ or $\theta = (4n-1)\frac{\pi}{2}$, where $n \in Z$

(v) $\theta = n\pi - \frac{\pi}{2},\ n \in Z$ (vi) $\theta = \frac{n\pi}{4} + \frac{\pi}{8},\ n \in Z$

(vii) $\theta = \frac{n\pi}{3} + \frac{\pi}{6},\ n \in Z$ (viii) $\theta = \frac{(2r+1)\pi}{m-n},\ r \in Z$

(ix) $\theta = \left(\frac{2n+1}{p+q}\right)\frac{\pi}{2},\ n \in Z$

(x) $\theta = (4n-1)\frac{\pi}{2}$ or $\theta = (4m-1)\frac{\pi}{6}$, where $m,\ n \in Z$

(xi) $\theta = m\pi$ or $\theta = 2n\pi$, where $m,\ n \in Z$

(xii) $\theta = (4n-1)\frac{\pi}{10}$ or $\theta = (4m-1)\frac{\pi}{2},\ m,\ n \in Z$

3. (i) $\theta = 2n\pi \pm \frac{\pi}{3},\ n \in Z$ (ii) $\theta = 2n\pi \pm \frac{\pi}{3},\ n \in Z$

(iii) $\theta = 2n\pi \pm \frac{5\pi}{6},\ n \in Z$ (iv) $\theta = 2n\pi \pm \frac{\pi}{3},\ n \in Z$

(v) $\theta = n\pi - \frac{\pi}{4}$ or $\theta = m\pi + \frac{\pi}{3}$, where $m,\ n \in Z$

(vi) $\theta = n\pi - \frac{\pi}{3}$ or $\theta = m\pi + \frac{\pi}{6}$, where $m,\ n \in Z$

(vii) $x = n\pi,\ x = \frac{n\pi}{3},\ n \in Z$

4. (i) $\theta = (2n+1)\frac{\pi}{4}$ or $\theta = 2m\pi \pm \frac{2\pi}{3}$, where $m, n \in Z$

(ii) $\theta = (2n+1)\frac{\pi}{4}$ or $\theta = 2m\pi \pm \frac{\pi}{3}$ where $m, n \in Z$

(iii) $\theta = \frac{n\pi}{3}$ or $\theta = m\pi \pm \frac{\pi}{6}$, where $m, n \in Z$

(iv) $\theta = 2n+1\frac{\pi}{8}$ or $\theta = m\pi \pm \frac{\pi}{3}$, where $m, n \in Z$

(v) $\theta = \frac{(2n\pi)}{3} + \frac{\pi}{6}$ or $\theta = 2m\pi$, where $m, n \in Z$

(vi) $\theta = \frac{n\pi}{2}$ or $\theta = 2n\pi \pm \frac{2\pi}{3}$, where $m, n \in Z$

(vii) $\theta = n\pi + \frac{\pi}{2},\ \theta = (2m+1)\pi,\ \theta = \frac{2r\pi}{5}$, where $m, n\, r \in Z$

(viii) $\theta = n\pi + (-1)^n\frac{\pi}{2}$ or $\theta = (2m+1)\frac{\pi}{4}$, where $m, n \in Z$

(ix) $\theta = \frac{n\pi}{4},\ \theta = n\pi \pm \frac{\pi}{6},\ n \in Z$

5. (i) $\theta = \frac{m\pi}{3}$ or $\theta = n\pi \pm \alpha$, where $\alpha = \tan^{-1}\frac{1}{\sqrt{2}}$ and $m, n \in Z$

(ii) $\theta = m\pi$ or $\theta = \frac{n\pi}{3}$, where $m, n \in Z$

(iii) $\theta = n\pi$, where $n \in Z$

6. (i) $\theta = (8n +)\dfrac{\pi}{4}$, $n \in Z$

(ii) $\theta = (4n+1)\dfrac{\pi}{2}$ or $\theta = (12m-1)\dfrac{\pi}{6}$, where $m, n \in Z$

(iii) $\theta = 2n\pi$ or $\theta = 2m\pi + \dfrac{\pi}{2}$, where $m, n \in Z$

(iv) $\theta = 2m\pi + \dfrac{\pi}{2}$, where $m, n \in Z$ (v) $\theta = 2n\pi + \dfrac{\pi}{3}$ or, $\theta = 2n\pi - \dfrac{\pi}{6}$, $n \in Z$

7. (i) $\theta = 2n\pi \pm \dfrac{\pi}{3}, n \in Z$ (ii) $\theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$ (iii) $\theta = \dfrac{\pi}{6}, \dfrac{\pi}{4}, \dfrac{\pi}{2}$

(iv) $\theta = n\pi \pm \dfrac{\pi}{4}, n \in Z$ (v) $x = \dfrac{n\pi}{2} \pm \dfrac{\pi}{8}, n \in Z$

HINTS TO NCERT & SELECTED PROBLEMS

4. (ii) W

1/2

$2n+1)\dfrac{\pi}{2}, n \in Z \Rightarrow \theta = (2n+1)\dfrac{\pi}{4}, n \in Z$

$= \cos\dfrac{\pi}{3} \Rightarrow \theta = 2m\pi \pm \dfrac{\pi}{3}, m \in Z$

(iv) $3\theta = \dfrac{1}{4}$

$\Rightarrow$ 2 (2 ... .θ) cos 3θ = 1

$\Rightarrow 2(\cos 3\theta + \cos\theta)\cos 3\theta = 1$

$\Rightarrow 2\cos^2 3\theta + 2\cos 3\theta \cos\theta = 1$

$\Rightarrow 2\cos^2 3\theta + \cos 4\theta + \cos 2\theta = 1$

$\Rightarrow (2\cos^2 3\theta - 1) + \cos 4\theta + \cos 2\theta = 0$

$\Rightarrow \cos 6\theta + \cos 2\theta + \cos 4\theta = 0$

$\Rightarrow 2\cos 4\theta \cos 2\theta + \cos 4\theta = 0$

$\Rightarrow \cos 4\theta(2\cos 2\theta + 1) = 0$

$\Rightarrow \cos 4\theta = 0, 2\cos 2\theta + 1 = 0$

$\Rightarrow \cos 4\theta = 0, \cos 2\theta = \cos\dfrac{2\pi}{3}$

$\Rightarrow 4\theta = (2n+1)\dfrac{\pi}{2}, 2\theta = 2n\pi \pm \dfrac{2\pi}{3}, n \in Z$

$\Rightarrow \theta = (2n+1)\dfrac{\pi}{8}, \theta = n\pi \pm \dfrac{\pi}{3}, n \in Z$

(ix) We have,

$\sin 2\theta - \sin 4\theta + \sin 6\theta = 0$

$\Rightarrow \sin 6\theta + \sin 2\theta - \sin 4\theta = 0$

$\Rightarrow 2\sin 4\theta \cos 2\theta - \sin 4\theta = 0$

$\Rightarrow \quad \sin 4\theta\,(2\cos 2\theta - 1) = 0$

$\Rightarrow \quad \sin 4\theta = 0$ or, $2\cos 2\theta - 1 = 0$

Now, $\sin 4\theta = 0 \Rightarrow 4\theta = n\pi, n \in Z \Rightarrow \theta = \dfrac{n\pi}{4}, n \in Z$

and, $2\cos 2\theta - 1 = 0$

$\Rightarrow \quad \cos 2\theta = \dfrac{1}{2} \Rightarrow \cos 2\theta = \cos\dfrac{\pi}{3} \Rightarrow 2\theta = 2n\pi \pm \dfrac{\pi}{3}, n \in Z \Rightarrow \theta = n\pi \pm \dfrac{\pi}{6}, n \in Z$

5. (i) $\tan\theta + \tan 2\theta + \tan 3\theta = 0$

$\Rightarrow \quad \tan\theta + \tan 2\theta + \tan(\theta + 2\theta) = 0$

$\Rightarrow \quad \tan\theta + \tan 2\theta + \dfrac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} = 0$

$\Rightarrow \quad (\tan\theta + \tan 2\theta)\,(2 - \tan\theta\tan 2\theta) = 0$

$\Rightarrow \quad \tan\theta + \tan 2\theta = 0$ or $\tan\theta\tan 2\theta = 2$

Now, $\tan\theta + \tan 2\theta = 0$

$\Rightarrow \quad \tan\theta + \dfrac{2\tan\theta}{1 - \tan^2\theta} = 0$

$\Rightarrow \quad \tan\theta\,(3 - \tan^2\theta) = 0$

$\Rightarrow \quad \tan\theta = 0$ or, $\tan^2\theta = 3$

$\Rightarrow \quad \tan\theta = 0$ or $\tan^2\theta = \tan^2\dfrac{\pi}{3}$

$\Rightarrow \quad \theta = n\pi,\ \theta = n\pi \pm \dfrac{\pi}{3}, n \in Z$

and, $\tan\theta\tan 2\theta = 2$

$\Rightarrow \quad \tan\theta \times \dfrac{2\tan\theta}{1 - \tan^2\theta} = 2$

$\Rightarrow \quad \tan^2\theta = 1 - \tan^2\theta$

$\Rightarrow \quad 2\tan^2\theta = 1$

$\Rightarrow \quad \tan^2\theta = \left(\dfrac{1}{\sqrt{2}}\right)^2 = \tan^2\alpha$ (say), where $\tan\alpha = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \quad \theta = n\pi \pm \alpha$, where $\tan\alpha = \dfrac{1}{\sqrt{2}}$

(iii) $\tan 3\theta - \tan 2\theta = \tan 2\theta - \tan\theta$

$\Rightarrow \quad \dfrac{\sin(3\theta - 2\theta)}{\cos 3\theta\cos 2\theta} = \dfrac{\sin(2\theta - \theta)}{\cos 2\theta\cos\theta}$

$\Rightarrow \quad \dfrac{\sin\theta}{\cos 3\theta\cos 2\theta} = \dfrac{\sin\theta}{\cos 2\theta\cos\theta}$

$\Rightarrow \quad \sin\theta\cos 2\theta\,(\cos 3\theta - \cos\theta) = 0$

$\Rightarrow \quad -2\sin\theta\cos 2\theta\sin 2\theta\sin\theta = 0$

$\Rightarrow \quad 2\sin^2\theta\sin 2\theta = 0$ $\quad [\because \cos 2\theta \neq 0]$

$\Rightarrow \quad \sin\theta = 0$ or, $\sin 2\theta = 0 \Rightarrow \theta = n\pi$ or, $2\theta = m\pi \Rightarrow \theta = n\pi$ or, $\theta = \dfrac{m\pi}{2}$, where $n, m \in Z$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the number of solutions of the equation $\tan x + \sec x = 2\cos x$ in the interval $[0, 2\pi]$.
2. Write the number of solutions of the equation $4\sin x - 3\cos x = 7$.
3. Write the general solution of $\tan^2 2x = 1$.
4. Write the set of values of a for which the equation $\sqrt{3}\sin x - \cos x = a$ has no solution.
5. If $\cos x = k$ has exactly one solution in $[0, 2\pi]$, then write the value(s) of k.
6. Write the number of points of intersection of the curves $2y = 1$ and $y = \cos x$, $0 \le x \le 2\pi$.
7. Write the values of x in $[0, \pi]$ for which $\sin 2x$, $\frac{1}{2}$ and $\cos 2x$ are in A.P.
8. Write the number of points of intersection of the curves $2y = -1$ and $y = \operatorname{cosec} x$.
9. Write the solution set of the equation $(2\cos\theta + 1)(4\cos\theta + 5) = 0$ in the interval $[0, 2\pi]$.
10. Write the number of values of θ in $[0, 2\pi]$ that satisfy the equation $\sin^2\theta - \cos\theta = \frac{1}{4}$.
11. If $3\tan(\theta - 15°) = \tan(\theta + 15°)$, $0 < \theta < 90°$, find θ.
12. If $2\sin^2\theta = 3\cos\theta$, where $0 \le \theta \le 2\pi$, then find the value of θ.
13. If $\sec x \cos 5x + 1 = 0$, where $0 < x \le \frac{\pi}{2}$, find the value of x.

ANSWERS

1. 2	2. 0	3. $\frac{n\pi}{2} + \frac{\pi}{8}, n \in Z$	4. $a \in (-\infty, -2) \cup (2, \infty)$
5. -1	6. 2	7. $0, \frac{\pi}{4}, \pi$	8. 0
9. $\frac{2\pi}{3}, \frac{4\pi}{3}$	10. 2	11. $\frac{\pi}{4}$	12. $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$
13. $\theta = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}$			

MULTIPLE CHOICES QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The smallest value of θ satisfying the equation $\sqrt{3}(\cot\theta + \tan\theta) = 4$ is
   (a) $2\pi/3$ (b) $\pi/3$ (c) $\pi/6$ (d) $\pi/12$
2. If $\cos\theta + \sqrt{3}\sin\theta = 2$, then $\theta =$
   (a) $\pi/3$ (b) $2\pi/3$ (c) $4\pi/3$ (d) $5\pi/3$
3. If $\tan p\theta - \tan q\theta = 0$, then the values of θ form a series in
   (a) AP (b) GP (c) HP (d) none of these
4. If a is any real number, the number of roots of $\cot x - \tan x = a$ in the first quadrant is (are).
   (a) 2 (b) 0 (c) 1 (d) none of these
5. The general solution of the equation $7\cos^2\theta + 3\sin^2\theta = 4$ is
   (a) $\theta = 2n\pi \pm \frac{\pi}{6}, n \in Z$ (b) $\theta = 2n\pi \pm \frac{2\pi}{3}, n \in Z$

(c) $\theta = n\pi \pm \frac{\pi}{3}, n \in Z$ (d) none of these

6. A solution of the equation $\cos^2\theta + \sin\theta + 1 = 0$, lies in the interval
   (a) $(-\pi/4, \pi/4)$ (b) $(\pi/4, 3\pi/4)$
   (c) $(3\pi/4, 5\pi/4)$ (d) $(5\pi/4, 7\pi/4)$
7. The number of solution in $[0, \pi/2]$ of the equation $\cos 3x \tan 5x = \sin 7x$ is
   (a) 5 (b) 7 (c) 6 (d) none of these
8. The general value of x satisfying the equation $\sqrt{3}\sin x + \cos x = \sqrt{3}$ is given by
   (a) $x = n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{3}, n \in Z$ (b) $x = n\pi + (-1)^n \frac{\pi}{3} - \frac{\pi}{6}, n \in Z$
   (c) $x = n\pi \pm \frac{\pi}{6}, n \in Z$ (d) $x = n\pi \pm \frac{\pi}{3}, n \in Z$
9. The smallest positive angle which satisfies the equation
   $2\sin^2\theta + \sqrt{3}\cos\theta + 1 = 0$ is
   (a) $\frac{5\pi}{6}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$
10. If $4\sin^2\theta = 1$, then the values of θ are
    (a) $2n\pi \pm \frac{\pi}{3}, n \in Z$ (b) $n\pi \pm \frac{\pi}{3}, n \in Z$ (c) $n\pi \pm \frac{\pi}{6}, n \in Z$ (d) $2n\pi \pm \frac{\pi}{6}, n \in Z$
11. If $\cot\theta - \tan\theta = \sec\theta$, then, θ is equal to
    (a) $2n\pi + \frac{3\pi}{2}, n \in Z$ (b) $n\pi + (-1)^n \frac{\pi}{6}, n \in Z$
    (c) $n\pi + \frac{\pi}{2}, n \in Z$ (d) none of these.
12. A value of θ satisfying $\cos\theta + \sqrt{3}\sin\theta = 2$ is
    (a) $\frac{5\pi}{3}$ (b) $\frac{4\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{3}$
13. In $(0, \pi)$, the number of solutions of the equation
    $\tan\theta + \tan 2\theta + \tan 3\theta = \tan\theta \tan 2\theta \tan 3\theta$ is
    (a) 7 (b) 5 (c) 4 (d) 2.
14. The number of values of θ in $[0, 2\pi]$ that satisfy the equation $\sin^2\theta - \cos\theta = \frac{1}{4}$
    (a) 1 (b) 2 (c) 3 (d) 4
15. If $e^{\sin x} - e^{-\sin x} - 4 = 0$, then $x =$
    (a) 0 (b) $\sin^{-1}\{\log_e(2 - \sqrt{5})\}$
    (c) 1 (d) none of these
16. The equation $3\cos x + 4\sin x = 6$ has .... solution.
    (a) finite (b) infinite (c) one (d) no
17. If $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$, then general value of θ is
    (a) $n\pi + (-1)^n \frac{\pi}{4}, n \in Z$ (b) $(-1)^n \frac{\pi}{4} - \frac{\pi}{3}, n \in Z$

(c) $n\pi + \frac{\pi}{4} - \frac{\pi}{3}, n \in Z$ (d) $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{3}, n \in Z$

18. General solution of $\tan 5\theta = \cot 2\theta$ is

(a) $\frac{n\pi}{7} + \frac{\pi}{2}, n \in Z$ (b) $\theta = \frac{n\pi}{7} + \frac{\pi}{3}, n \in Z$

(c) $\theta = \frac{n\pi}{7} + \frac{\pi}{14}, n \in Z$ (d) $\theta = \frac{n\pi}{7} - \frac{\pi}{14}, n \in Z$

19. The solution of the equation $\cos^2\theta + \sin\theta + 1 = 0$ lies in the interval

(a) $(-\pi/4, \pi/4)$ (b) $(\pi/4, 3\pi/4)$ (c) $(3\pi/4, 5\pi/4)$ (d) $(5\pi/4, 7\pi/4)$

20. If $\cos\theta = -\frac{1}{2}$ and $0 < \theta < 360°$, then the solutions are

(a) $\theta = 60°, 240°$ (b) $\theta = 120°, 240°$

(c) $\theta = 120°, 210°$ (d) $\theta = 120°, 300°$

21. The number of values of x in the interval $[0, 5\pi]$ satisfying the equation $3\sin^2 x - 7\sin x + 2 = 0$ is

(a) 0 (b) 5 (c) 6 (d) 10

ANSWERS

1. (c)	2. (a)	3. (a)	4. (c)	5. (a)	6. (d)	7. (c)	8. (b)
9. (a)	10. (c)	11. (b)	12. (d)	13. (d)	14. (b)	15. (d)	16. (d)
17. (d)	18. (c)	19. (d)	20. (b)	21. (c)			

SUMMARY

1. An equation containing trigonometric functions of unknown angles is known as a trigonometric equation.
2. A solution of a trigonometric equation is the value of the unknown angle that satisfies the equation.
3. Following are the general solutions of trigonometric equations in standard forms:

	Trigonometric equation	*General solution*
(i)	$\sin\theta = 0$	$\theta = n\pi, n \in Z$
(ii)	$\cos\theta = 0$	$\theta = (2n+1)\frac{\pi}{2}, n \in Z$
(iii)	$\tan\theta = 0$	$\theta = n\pi, n \in Z$
(iv)	$\sin\theta = \sin\alpha$	$\theta = n\pi + (-1)^n \alpha, n \in Z$
(v)	$\cos\theta = \cos\alpha$	$\theta = 2n\pi \pm \alpha, n \in Z$
(vi)	$\tan\theta = \tan\alpha$	$\theta = n\pi + \alpha, n \in Z$
(vii)	$\left.\begin{array}{l}\sin^2\theta = \sin^2\alpha \\ \cos^2\theta = \cos^2\alpha \\ \tan^2\theta = \tan^2\alpha\end{array}\right\}$	$\theta = n\pi \pm \alpha, n \in Z$

4. The equation $a\cos\theta + b\sin\theta = c$ is solvable for $|c| \le \sqrt{a^2 + b^2}$.

CONTENTS

Volume I

MATHEMATICS - XII

CHAPTER **4**

INVERSE TRIGONOMETRIC FUNCTIONS

4.1 INTRODUCTION

In chapter 3, we have learnt about functions, types of functions, composition of functions and inverse of a function. In this chapter, we shall use these concepts to define the inverses of all trigonometric functions and to study their properties. Let us first recall the definition of inverse of a function.

4.2 INVERSE OF A FUNCTION

We know that corresponding to every bijection (one-one onto function) $f : A \to B$ there exists a bijection $g : B \to A$ defined by

$$g(y) = x \text{ if and only } f(x) = y.$$

The function $g : B \to A$ is called the inverse of function $f : A \to B$ and is denoted by f^{-1}.

Thus, we have

$$f(x) = y \Leftrightarrow f^{-1}(y) = x.$$

We have also learnt that

$$(f^{-1} of)(x) = f^{-1}(f(x)) = f^{-1}(y) = x, \text{ for all } x \in A.$$

and, $\quad (fo f^{-1})(y) = f(f^{-1}(y)) = f(x) = y, \text{ for all } y \in B.$

We know that trigonometric functions are periodic functions, and hence, in general, all trigonometric functions are not bijections. Consequently, their inverses do not exist. However, if we restrict their domains and co-domains, they can be made bijections and we can obtain their inverses. In the following sections, we shall do all these things to obtain the inverses of trigonometric functions.

4.3 INVERSE OF SINE FUNCTION

Consider the function $f : R \to R$ given by $f(x) = \sin x$. The graph of this function is shown in Fig. 4.1. Clearly, it is a many-one into function as it attains same value at infinitely many points

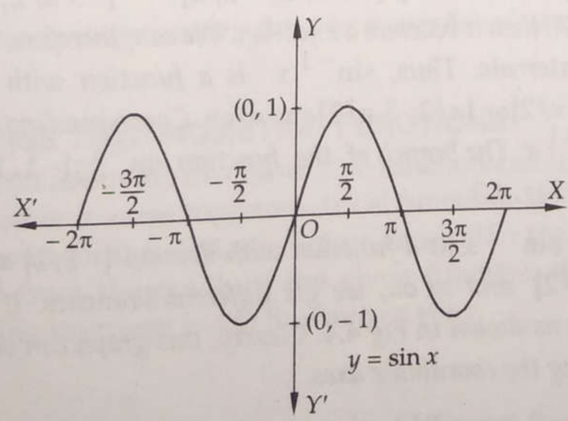


Fig. 4.1

and its range $[-1, 1]$ is not same as its co-domain. We know that any function can be made an onto function, if we replace its co-domain by its range. Therefore, $f : R \rightarrow [-1, 1]$ is a many-one onto function.

In order to make f a one-one function, we will have to restrict its domain in such a way that in that domain there is no turn in the graph of the function and the function takes every value between -1 and 1. It is evident from the graph of $f(x) = \sin x$ that if we take the domain as $[-\pi/2, \pi/2]$, then $f(x)$ becomes one-one. Thus, $f : [-\pi/2, \pi/2] \rightarrow [-1, 1]$ given by $f(\theta) = \sin \theta$ is a bijection and hence invertible.

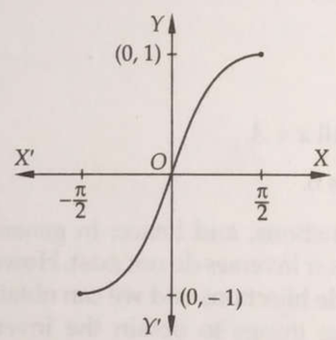
The inverse of the sine function is denoted by $\sin^{-1}$. Thus, $\sin^{-1}$ is a function with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$ such that

$$\sin^{-1} x = \theta \Leftrightarrow \sin \theta = x.$$

Also, $\sin^{-1}(\sin \theta) = \theta$ for all $\theta \in [-\pi/2, \pi/2]$ $\quad [\because f^{-1} of(x) = f^{-1}(f(x)) = x]$

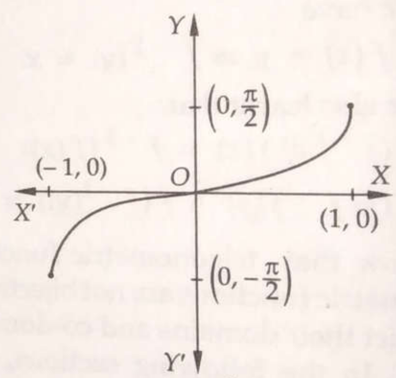
and, $\sin(\sin^{-1} x) = x$ for all $x \in [-1, 1]$ $\quad [\because fof^{-1}(y) = f(f^{-1}(y)) = y]$

The graph of the function $f : [-\pi/2, \pi/2] \rightarrow [-1, 1]$ given by $f(x) = \sin x$ is shown in Fig. 4.2. In order to obtain the graph of $\sin^{-1} : [-1, 1] \rightarrow [-\pi/2, \pi/2]$ we interchange x and y axes as shown in Fig. 4.3.



$y = \sin x$

Fig. 4.2

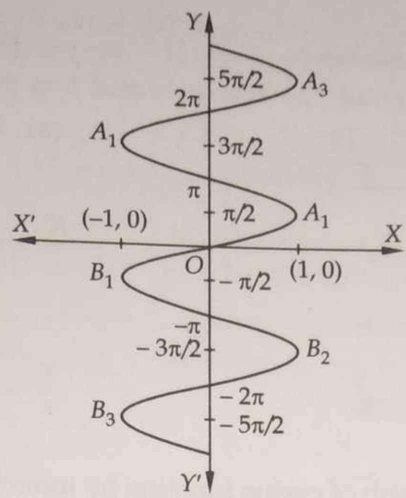


$y = \sin^{-1} x$

Fig. 4.3

<u>REMARK 1</u> *In the above discussion, we have restricted the domain of sine function to the interval* $[-\pi/2, \pi/2]$ *to make it a bijection. In fact, if we restrict its domain to any one of the intervals* $[-\pi/2, \pi/2]$, $[\pi/2, 3\pi/2]$, $[3\pi/2, 5\pi/2]$, $[-3\pi/2, -\pi/2]$, $[-5\pi/2, -3\pi/2]$ *in general* $[n/\pi - \pi/2, n\pi + \pi/2]$, $n \in Z$, *then it becomes a bijection. We can, therefore, define the inverse of the sine function in each of these intervals. Thus,* $\sin^{-1} x$ *is a function with domain* $[-1, 1]$ *and range* $[-\pi/2, \pi/2]$ *or* $[-3\pi/2, -\pi/2]$ *or* $[\pi/2, 3\pi/2]$ *and so on. Corresponding to each such interval, we get a branch of the function* $\sin^{-1} x$. *The branch of the function* $\sin^{-1} : [-1, 1] \rightarrow [-\pi/2, \pi/2]$ *called the principal branch as shown in Fig. 4.3.*

<u>REMARK 2</u> *By considering* $\sin^{-1} x$ *as a function with domain* $[-1, 1]$ *and range* $[-\pi/2, \pi/2]$ *or* $[\pi/2, 3\pi/2]$ *or* $[3\pi/2, 5\pi/2]$ *and so on, we get different branches. If all these branches are put together, we obtain the graph as shown in Fig 4.4. Clearly, this graph can be obtained from the graph of sine function by interchanging the coordinate axes.*



Graph of $y = \sin^{-1} x$

Fig. 4.4

NOTE 1 $\sin^{-1} x$ *is not equal to* $(\sin x)^{-1}$, or $\dfrac{1}{\sin x}$.

4.4 INVERSES OF OTHER TRIGONOMETRIC FUNCTIONS

In the above section, we have discussed about the inverse of sine function and its graph. Similarly, we can define the inverses of other five trigonometrical functions and their principal branches. The following table gives the domains, ranges and the principal value branches of all inverse trigonometric functions:

Function	*Domain*	*Range*	*Principal value branch*
$\sin^{-1}$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$-\pi/2 \le y \le \pi/2$, where $y = \sin^{-1} x$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$	$0 \le y \le \pi$, where $y = \cos^{-1} x$
$\tan^{-1}$	R	$(-\pi/2, \pi/2)$	$-\pi/2 < y < \pi/2$, where $y = \tan^{-1} x$
cosec^{-1}	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, \pi/2] - \{0\}$	$-\pi/2 \le y \le \pi/2$, where $y = \text{cosec}^{-1} x$, $y \ne 0$
$\sec^{-1}$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \{\pi/2\}$	$0 \le y \le \pi$, where $y = \sec^{-1} x$, $y \ne \pi/2$
$\cot^{-1}$	R	$(0, \pi)$	$0 < y < \pi$, where $y = \cot^{-1} x$

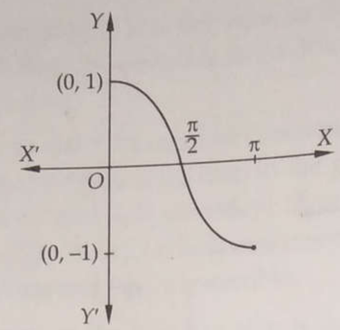
NOTE 1 *If no branch of an inverse trigonometric function is mentioned, then it means the principal value branch of that function.*

4.5 GRAPHS OF INVERSE TRIGONOMETRIC FUNCTIONS

In section 4.3, we have learnt about the inverse of sine function and its graph. In this section , we shall draw the graphs of other inverse trigonometrical functions with the help of the graphs of the corresponding trigonometrical functions. Let us recall that the graph of the inverse of a function can be obtained from the graph of the given function either by interchanging the coordinate axes or by taking its image in the line mirror $y = x$.

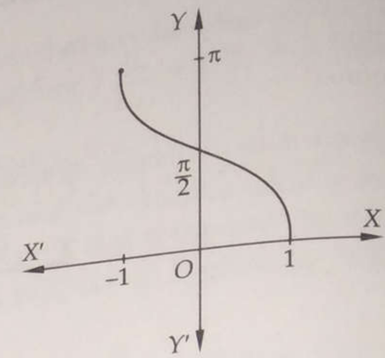
GRAPH OF $\cos^{-1} x$

We know that the function $f : [0, \pi] \to [-1, 1]$ given by $f(\theta) = \cos\theta$ is a bijection and its graph is shown in Fig. 4.5. Therefore, $\cos^{-1}$ is a function with domain $[-1, 1]$ and range $[0, \pi]$. The graph



$y = \cos x$

Fig. 4.5



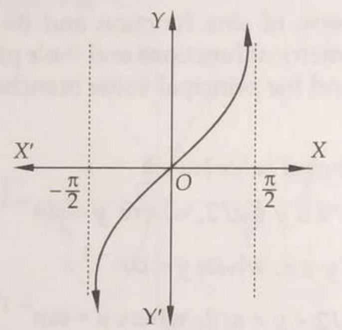
$y = \cos^{-1} x$

Fig. 4.6

of $\cos^{-1}$ is obtained from the graph of cosine function by interchanging x and y-axes as shown in Fig. 4.6.

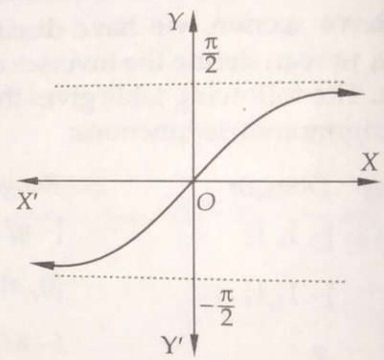
GRAPH OF $\tan^{-1} x$

We have seen in earlier sections that the function $f : (-\pi/2, \pi/2) \to R$ given by $f(\theta) = \tan\theta$ is a bijection and hence invertible. Therefore, $\tan^{-1}$ is a function with domain R and range $(-\pi/2, \pi/2)$. The graphs of $y = \tan x$ and $y = \tan^{-1} x$ are shown in Figs. 4.7 and 4.8 respectively.



$y = \tan x$

Fig. 4.7

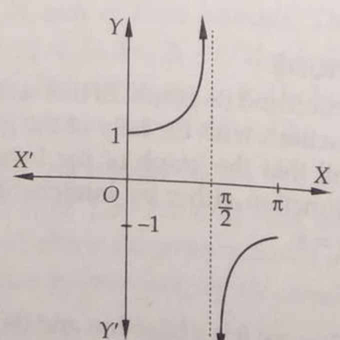


$y = \tan^{-1} x$

Fig. 4.8

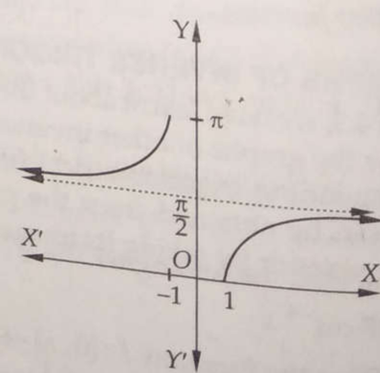
GRAPH OF $\sec^{-1} x$

The function $f : [0, \pi] - \{\pi/2\} \to (-\infty, -1] \cup [1, \infty)$ given by $f(\theta) = \sec\theta$ is a bijection and hence invertible. Therefore, $\sec^{-1}$ is a function with domain $(-\infty, -1] \cup [1, \infty)$ and range $[0, \pi] - \{\pi/2\}$. The graph of $y = \sec x$ and $y = \sec^{-1} x$ are shown in Figs. 4.9 and 4.10 respectively.



$y = \sec x$

Fig. 4.9

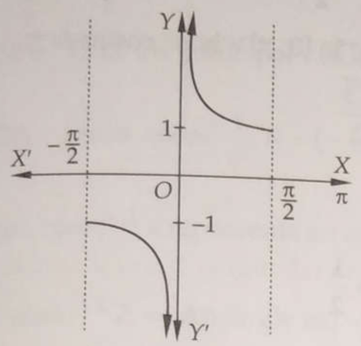


$y = \sec^{-1} x$

Fig. 4.10

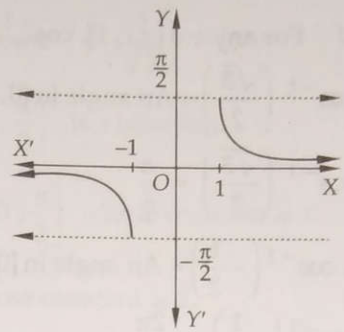
GRAPH OF $\text{cosec}^{-1} x$

The function $f:[-\pi/2, \pi/2]-\{0\} \to (-\infty, -1] \cup [-1, \infty)$ defined by $f(\theta) = \text{cosec}\,\theta$ is a bijection and is defined by $f(\theta) = \text{cosec}\,\theta$ and hence invertible. Therefore, cosec^{-1} is a function with domain $(-\infty, -1] \cup [1, \infty)$ and range $[-\pi/2, \pi/2]-\{0\}$. The graphs of $y = \text{cosec}\, x$ and $y = \text{cosec}^{-1} x$ are shown in Figs. 4.11 and 4.12 respectively.



$y = \text{cosec}\, x$

Fig. 4.11

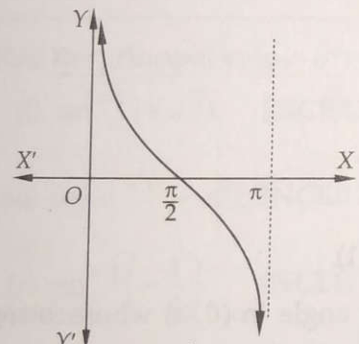


$y = \text{cosec}^{-1} x$

Fig. 4.12

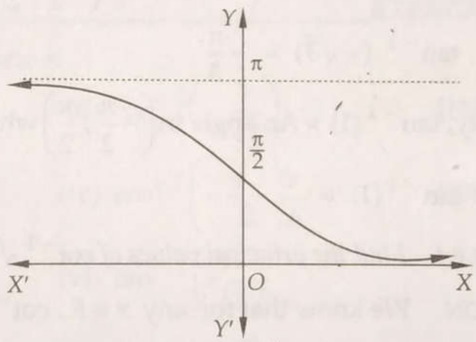
GRAPH OF $\cot^{-1} x$

We know that the function $f:(0, \pi) \to R$ given by $f(\theta) = \cot\theta$ is a bijection and hence invertible. Therefore, $\cot^{-1}: R \to (0, \pi)$ exists. The graphs of $y = \cot x$ and $y = \cot^{-1} x$ are shown in Figs. 4.13 and 4.14 respectively.



$y = \cot x$

Fig. 4.13



$y = \cot^{-1} x$

Fig. 4.14

<u>REMARK</u> *The graphs of the principal values of inverse trigonometric functions are generally known as their principal value branches. In case no branch of an inverse trigonometric function is mentioned, it will mean the principal value branch of that function.*

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the principal values of* $\sin^{-1}\left(\frac{1}{2}\right)$ *and* $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$.

SOLUTION We know that $\sin^{-1} x$ denotes an angle in the interval $[-\pi/2, \pi/2]$ whose sine is x for $x \in [-1, 1]$.

$\therefore$ $\sin^{-1}\left(\frac{1}{2}\right) =$ An angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $\frac{1}{2}$

$\Rightarrow$ $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

Similarly, $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ = An angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $-\frac{1}{\sqrt{2}}$

$$\Rightarrow \quad \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$$

EXAMPLE 2 *Find the principal values of* $\cos^{-1}\frac{\sqrt{3}}{2}$ *and* $\cos^{-1}\left(-\frac{1}{2}\right)$. **[NCERT]**

SOLUTION For any $x \in [-1, 1]$, $\cos^{-1} x$ represents an angle in $[0, \pi]$ whose cosine is x.

$\therefore \quad \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ = An angle in $[0, \pi]$ whose cosine is $\frac{\sqrt{3}}{2}$

$$\Rightarrow \quad \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

Similarly, $\cos^{-1}\left(-\frac{1}{2}\right)$ = An angle in $[0, \pi]$ whose cosine is $-\frac{1}{2}$

$$\Rightarrow \quad \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}.$$

EXAMPLE 3 *Find the principal values of* $\tan^{-1}(-\sqrt{3})$ *and* $\tan^{-1}(1)$. **[NCERT]**

SOLUTION We know that for any $x \in R$, $\tan^{-1} x$ represents an angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is x.

$\therefore \quad \tan^{-1}(-\sqrt{3})$ = An angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is $(-\sqrt{3})$.

$$\Rightarrow \quad \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$$

Similarly, $\tan^{-1}(1)$ = An angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is 1

$$\Rightarrow \quad \tan^{-1}(1) = \frac{\pi}{4}.$$

EXAMPLE 4 *Find the principal values of* $\cot^{-1}\sqrt{3}$ *and* $\cot^{-1}(-1)$.

SOLUTION We know that for any $x \in R$, $\cot^{-1} x$ denotes an angle in $(0, \pi)$ whose cotangent is x.

$\therefore \quad \cot^{-1}\sqrt{3}$ = An angle in $(0, \pi)$ whose cotangent is $\sqrt{3}$

$$\Rightarrow \quad \cot^{-1}(\sqrt{3}) = \frac{\pi}{6}$$

Similarly, $\cot^{-1}(-1)$ = An angle in $(0, \pi)$ whose cotangent is (-1)

$$\Rightarrow \quad \cot^{-1}(-1) = \frac{3\pi}{4}.$$

EXAMPLE 5 *Find the principal values of* $\sec^{-1}\frac{2}{\sqrt{3}}$ *and* $\sec^{-1}(-2)$.

SOLUTION Since $\sec^{-1} : R - (-1, 1) \to [0, \pi] - \left\{\frac{\pi}{2}\right\}$ is a bijection. Therefore, $\sec^{-1} x$ represents an angle in $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ whose secant is x.

$\therefore \quad \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ = An angle in $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ whose secant is $\frac{2}{\sqrt{3}}$

$\Rightarrow \quad \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{6}$

Similarly, $\sec^{-1}(-2)$ = An angle in $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ whose secant is (-2)

$\Rightarrow \quad \sec^{-1}(-2) = \frac{2\pi}{3}$

EXAMPLE 6 Find the principal values of $\text{cosec}^{-1} 2$ and $\text{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right)$.

SOLUTION Since $\text{cosec}^{-1}: R-(-1,1) \to \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ is a bijection.

Therefore, $\text{cosec}^{-1} x$ represents an angle in $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ whose cosecant is x.

$\therefore \quad \text{cosec}^{-1} 2$ = An angle in $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ whose cosecant is 2.

$\Rightarrow \quad \text{cosec}^{-1} 2 = \frac{\pi}{6}$

Similarly, $\text{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right)$ = An angle in $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ whose cosecant is $\left(-\frac{2}{\sqrt{3}}\right)$

$\Rightarrow \quad \text{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right) = -\frac{\pi}{3}$.

EXERCISE 4.1

1. Find the principal values of each of the following:

(i) $\tan^{-1}(-\sqrt{3})$ **[NCERT, CBSE 2011]**

(ii) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ **[NCERT]**

(iii) $\text{cosec}^{-1}(-\sqrt{2})$ **[NCERT]**

(iv) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **[NCERT]**

(v) $\sin^{-1}\left(-\frac{1}{2}\right)$ **[NCERT, CBSE 2011]**

(vi) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(vii) $\sec^{-1}(-\sqrt{2})$

(viii) $\cot^{-1}(-\sqrt{3})$

(ix) $\sec^{-1}(2)$

(x) $\text{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right)$

2. Evaluate each of the following:

(i) $\cos^{-1}\frac{1}{2} + 2\sin^{-1}\left(\frac{1}{2}\right)$ **[NCERT, CBSE 2012]**

(ii) $\tan^{-1}\left\{2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right\}$ **[NCERT]**

(iii) $\tan^{-1} 1 + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$ **[NCERT]**

(iv) $\tan^{-1}\sqrt{3} - \sec^{-1}(-2) + \text{cosec}^{-1}\frac{2}{\sqrt{3}}$

(v) $\cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(-\frac{1}{2}\right)$ **[CBSE 2012]**

3. For the principal values, evaluate the following:

(i) $\sin^{-1}\frac{1}{2} - 2\sin^{-1}\frac{1}{\sqrt{2}}$

(ii) $\sin^{-1}\left(-\frac{1}{2}\right) + 2\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

(iii) $\tan^{-1}(-1) + \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ (iv) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

(v) $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ **[CBSE 2012]**

ANSWERS

1. (i) $-\frac{\pi}{3}$ (ii) $\frac{3\pi}{4}$ (iii) $-\frac{\pi}{4}$ (iv) $\frac{5\pi}{6}$ (v) $-\frac{\pi}{6}$ (vi) $\frac{\pi}{6}$ (vii) $\frac{3\pi}{4}$ (viii) $\frac{5\pi}{6}$ (ix) $\frac{\pi}{3}$

(x) $\frac{\pi}{3}$ **2.** (i) $\frac{2\pi}{3}$ (ii) $\frac{\pi}{4}$ (iii) $\frac{3\pi}{4}$ (iv) 0

3. (i) $\frac{-\pi}{3}$ (ii) $\frac{3\pi}{2}$ (iii) $\frac{\pi}{2}$ (iv) $-\frac{\pi}{6}$ (v) $-\frac{\pi}{2}$

4.6 PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

PROPERTY I *Prove that:*

(i) $\sin^{-1}(\sin\theta) = \theta$, for all $\theta \in [-\pi/2, \pi/2]$

(ii) $\cos^{-1}(\cos\theta) = \theta$, for all $\theta \in [0, \pi]$

(iii) $\tan^{-1}(\tan\theta) = \theta$, for all $\theta \in (-\pi/2, \pi/2)$

(iv) $\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta$, for all $\theta \in [-\pi/2, \pi/2]$, $\theta \neq 0$

(v) $\sec^{-1}(\sec\theta) = \theta$, for all $\theta \in [0, \pi]$, $\theta \neq \pi/2$

(vi) $\cot^{-1}(\cot\theta) = \theta$, for all $\theta \in (0, \pi)$.

PROOF We know that, if $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ exists such that

$$f^{-1}of(x) = f^{-1}(f(x)) = x \text{ for all } x \in A.$$

Clearly, all these results are direct consequences of this property.

ALITER For any $\theta \in [-\pi/2, \pi/2]$, let $\sin\theta = x$. Then,

$$\theta = \sin^{-1} x.$$

$$\Rightarrow \quad \theta = \sin^{-1}(\sin\theta) \qquad [\because x = \sin\theta]$$

Hence, $\sin^{-1}(\sin\theta) = \theta$ for all $\theta \in [-\pi/2, \pi/2]$.

Similarly, we can prove other results.

ILLUSTRATION 1 *Evaluate each of the following:*

(i) $\sin^{-1}\left(\sin\frac{\pi}{3}\right)$ (ii) $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)$ (iii) $\tan^{-1}\left(\tan\frac{\pi}{4}\right)$

(iv) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ (v) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ **[CBSE 2009]** (vi) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

SOLUTION Recall that $\sin^{-1}(\sin\theta) = \theta$, if $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $\cos^{-1}(\cos\theta) = \theta$, if $0 \leq \theta \leq \pi$ and $\tan^{-1}(\tan\theta) = \theta$ if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Therefore,

(i) $$\sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$$

(ii) $$\cos^{-1}\left(\cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}$$

(iii) $$\tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$$

(iv) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) \neq \frac{2\pi}{3}$ as $\frac{2\pi}{3}$ does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Now, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left\{\sin\left(\pi-\frac{\pi}{3}\right)\right\}$ $\left[\because \sin\frac{2\pi}{3} = \sin\left(\pi-\frac{\pi}{3}\right)\right]$

$= \sin^{-1}\left(\sin\frac{\pi}{3}\right)$ $[\because \sin(\pi-\theta) = \sin\theta]$

$= \frac{\pi}{3}$

(v) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) \neq \frac{7\pi}{6}$, because $\frac{7\pi}{6}$ does not lie between 0 and π.

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left\{\cos\left(2\pi-\frac{5\pi}{6}\right)\right\}$ $\left[\because \frac{7\pi}{6} = 2\pi-\frac{5\pi}{6}\right]$

$= \cos^{-1}\left(\cos\frac{5\pi}{6}\right)$ $[\because \cos(2\pi-\theta) = \cos\theta]$

$= \frac{5\pi}{6}$

(vi) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) \neq \frac{3\pi}{4}$, because $\frac{3\pi}{4}$ does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Now, $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left\{\tan\left(\pi-\frac{\pi}{4}\right)\right\}$ $\left[\because \frac{3\pi}{4} = \pi-\frac{\pi}{4}\right]$

$= \tan^{-1}\left(-\tan\frac{\pi}{4}\right)$ $[\because \tan(\pi-\theta) = -\tan\theta]$

$= \tan^{-1}\left\{\tan\left(-\frac{\pi}{4}\right)\right\} = -\frac{\pi}{4}$

REMARK 1 *It should be noted that* $\sin^{-1}(\sin\theta) \neq \theta$, *if,* $\theta \notin [-\pi/2, \pi/2]$.

In fact, we have

$$\sin^{-1}(\sin\theta) = \begin{cases} -\pi-\theta & , \text{ if } \theta \in [-3\pi/2, -\pi/2] \\ \theta & , \text{ if } \theta \in [-\pi/2, \pi/2] \\ \pi-\theta & , \text{ if } \theta \in [\pi/2, 3\pi/2] \\ -2\pi+\theta & , \text{ if } \theta \in [3\pi/2, 5\pi/2] \end{cases} \quad \textit{and so on.}$$

Similarly, we have

$$\cos^{-1}(\cos\theta) = \begin{cases} -\theta & , \text{ if } \theta \in [-\pi, 0] \\ \theta & , \text{ if } \theta \in [0, \pi] \\ 2\pi-\theta & , \text{ if } \theta \in [\pi, 2\pi] \\ -2\pi+\theta & , \text{ if } \theta \in [2\pi, 3\pi] \end{cases} \quad \textit{and so on.}$$

$$\tan^{-1}(\tan\theta) = \begin{cases} \pi+\theta & , \text{ if } \theta \in (-3\pi/2, -\pi/2) \\ \theta & , \text{ if } \theta \in (-\pi/2, \pi/2) \\ \theta-\pi & , \text{ if } \theta \in (\pi/2, 3\pi/2) \\ \theta-2\pi & , \text{ if } \theta \in (3\pi/2, 5\pi/2) \end{cases} \quad \textit{and so on.}$$

ILLUSTRATION 2 *Express each of the following in the simplest form:* [NCERT]

(i) $\tan^{-1}\left\{\sqrt{\frac{1-\cos x}{1+\cos x}}\right\}, -\pi < x < \pi$ [CBSE 2012]

(ii) $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right), -\frac{\pi}{2} < x < \frac{\pi}{2}$ [NCERT]

(iii) $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right), -\frac{\pi}{2} < x < \frac{\pi}{2}$ [NCERT]

(iv) $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right), -\frac{\pi}{4} < x < \frac{\pi}{4}$

SOLUTION (i) We have,

$$\tan^{-1}\left\{\sqrt{\frac{1-\cos x}{1+\cos x}}\right\}$$

$$= \tan^{-1}\left\{\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right\} = \tan^{-1}\left\{\sqrt{\tan^2\frac{x}{2}}\right\} = \tan^{-1}\left(\left|\tan\frac{x}{2}\right|\right)$$

$$= \begin{cases} \tan^{-1}\left(-\tan\frac{x}{2}\right), \text{ if } -\pi < x < 0 \\ \tan^{-1}\left(\tan\frac{x}{2}\right), \text{ if } 0 \le x < \pi \end{cases}$$

$$= \begin{cases} \tan^{-1}\left\{\tan\left(\frac{-x}{2}\right)\right\} = -\frac{x}{2}, \text{ if } -\pi < x < 0 \\ \tan^{-1}\left\{\tan\frac{x}{2}\right\} = \frac{x}{2}, \text{ if } 0 < x < \pi \end{cases}$$

(ii) We have,

$$\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \tan^{-1}\left\{\frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}\right\}$$

$$= \tan^{-1}\left\{\frac{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}\right\} = \tan^{-1}\left\{\frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}}\right\}$$

$$= \tan^{-1}\left\{\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right\} = \tan^{-1}\left\{\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right\}$$

$$= \frac{\pi}{4} - \frac{x}{2}$$

$$\left[\because -\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow -\frac{\pi}{4} < -\frac{x}{2} < \frac{\pi}{4} \Rightarrow 0 < \frac{\pi}{4} - \frac{x}{2} < \frac{\pi}{2}\right]$$

ALITER We have,

$$\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \tan^{-1}\left\{\frac{\sin\left(\frac{\pi}{2}+x\right)}{1-\cos\left(\frac{\pi}{2}+x\right)}\right\}$$

$$= \tan^{-1}\left\{\frac{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right\} = \tan^{-1}\left\{\cot\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$$

$$= \tan^{-1}\left\{\tan\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}\right\} = \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\} = \frac{\pi}{4}-\frac{x}{2}$$

(iii) We have,

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \tan^{-1}\left\{\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}\right\}$$

$$= \tan^{-1}\left\{\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}\right\}$$

$$= \tan^{-1}\left\{\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right\}$$

$$= \tan^{-1}\left\{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$$

$$= \frac{\pi}{4}+\frac{x}{2} \qquad \left[\because -\frac{\pi}{2}<x<\frac{\pi}{2} \Rightarrow -\frac{\pi}{4}<\frac{x}{2}<\frac{\pi}{4} \Rightarrow 0<\frac{\pi}{4}+\frac{x}{2}<\frac{\pi}{2}\right]$$

ALITER We have,

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \tan^{-1}\left\{\frac{\sin\left(\frac{\pi}{2}-x\right)}{1-\cos\left(\frac{\pi}{2}-x\right)}\right\}$$

$$= \tan^{-1}\left\{\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right\}$$

$$= \tan^{-1}\left\{\cot\left\{\frac{\pi}{4}-\frac{x}{2}\right)\right\}$$

$$= \tan^{-1}\left[\tan\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}\right] = \tan^{-1}\left\{\tan^{-1}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\} = \frac{\pi}{4}+\frac{x}{2}$$

(iv) We have,

$$\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) = \tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-x\right)\right\} \qquad \left[\because -\frac{\pi}{4} < x < \frac{\pi}{4} \Rightarrow 0 < \frac{\pi}{4} - x < \frac{\pi}{2}\right]$$

$$= \frac{\pi}{4} - x$$

<u>REMARK</u> *In order to simplify trigonometrical expressions involving inverse trigonometrical functions, following substitutions are very helpful:*

Expression	*Substitution*
$a^2 + x^2$	$x = a\tan\theta$ or, $x = a\cot\theta$
$a^2 - x^2$	$x = a\sin\theta$ or, $x = a\cos\theta$
$x^2 - a^2$	$x = a\sec\theta$ or, $x = a\,\text{cosec}\,\theta$
$\sqrt{\frac{a-x}{a+x}}$ or, $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ or, $\sqrt{\frac{a^2+x^2}{a^2-x^2}}$	$x^2 = a^2\cos 2\theta$

ILLUSTRATION 3 *Write the following functions in the simplest form:*

(i) $\tan^{-1}\left\{\frac{x}{\sqrt{a^2-x^2}}\right\}, -a < x < a$ **[NCERT]** (ii) $\tan^{-1}\left\{\sqrt{\frac{a-x}{a+x}}\right\}, -a < x < a$

(iii) $\sin^{-1}\left\{\frac{x}{\sqrt{x^2+a^2}}\right\}$ (iv) $\cos^{-1}\left\{\frac{x}{\sqrt{x^2+a^2}}\right\}$

SOLUTION (i) Putting $x = a\sin\theta$, we have

$$\tan^{-1}\left\{\frac{x}{\sqrt{a^2-x^2}}\right\}$$

$$= \tan^{-1}\left\{\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{a\sin\theta}{a\cos\theta}\right\} = \tan^{-1}(\tan\theta) = \theta = \sin^{-1}\frac{x}{a} \qquad \left[\because x = a\sin\theta \Rightarrow \sin\theta = \frac{x}{a} \Rightarrow \theta = \sin^{-1}\frac{x}{a}\right]$$

(ii) Putting $x = a\cos\theta$, we have

$$\tan^{-1}\sqrt{\frac{a-x}{a+x}}$$

$$= \tan^{-1}\sqrt{\frac{a - a\cos\theta}{a + a\cos\theta}}$$

$$= \tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

$$= \tan^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}} = \tan^{-1}\left(\left|\tan\frac{\theta}{2}\right|\right)$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) \qquad \left[\because -a < x < a \Rightarrow 0 < \theta < \pi \Rightarrow 0 < \frac{\theta}{2} < \frac{\pi}{2}\right]$$

$$= \frac{\theta}{2} = \frac{1}{2}\cos^{-1}\frac{x}{a} \qquad \left[\because x = a\cos\theta \Rightarrow \cos\theta = \frac{x}{a} \Rightarrow \theta = \cos^{-1}\frac{x}{a}\right]$$

(iii) Putting $x = a\tan\theta$, we have

$$\sin^{-1}\left\{\frac{x}{\sqrt{x^2+a^2}}\right\}$$

$$= \sin^{-1}\left\{\frac{a\tan\theta}{\sqrt{a^2\tan^2\theta + a^2}}\right\}$$

$$= \sin^{-1}\left\{\frac{a\tan\theta}{a\sec\theta}\right\}$$

$$= \sin^{-1}(\sin\theta)$$

$$= \theta = \tan^{-1}\frac{x}{a} \qquad \left[\because x = a\tan\theta \Rightarrow \tan\theta = \frac{x}{a} \Rightarrow \theta = \tan^{-1}\frac{x}{a}\right]$$

(iv) Putting $x = a\cot\theta$, we have

$$\cos^{-1}\left\{\frac{x}{\sqrt{x^2+a^2}}\right\}$$

$$= \cos^{-1}\left\{\frac{a\cot\theta}{\sqrt{a^2\cot^2\theta + a^2}}\right\}$$

$$= \cos^{-1}\left\{\frac{a\cot\theta}{a\,\mathrm{cosec}\,\theta}\right\}$$

$$= \cos^{-1}(\cos\theta) = \theta = \cot^{-1}\frac{x}{a} \qquad \left[\because x = a\cot\theta \Rightarrow \cot\theta = \frac{x}{a} \Rightarrow \cot^{-1}\frac{x}{a} = \theta\right]$$

PROPERTY II *Prove that:*

(i) $\sin(\sin^{-1}x) = x$ for all $x \in [-1, 1]$

(ii) $\cos(\cos^{-1}x) = x$, for all $x \in [-1, 1]$

(iii) $\tan(\tan^{-1}x) = x$ for all $x \in R$

(iv) $\mathrm{cosec}(\mathrm{cosec}^{-1}x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(v) $\sec(\sec^{-1}x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\cot(\cot^{-1}x) = x$, for all $x \in R$.

PROOF We know that, if $f: A \to B$ is a bijection, then $f^{-1}: B \to A$ exists such that

$$fof^{-1}(y) = f(f^{-1}(y)) = y \text{ for all } y \in B.$$

Clearly, all these results are direct consequences of this property.

ALITER Let $\theta \in [-\pi/2, \pi/2]$ and $x \in [-1, 1]$ such that $\sin\theta = x$. Then, $\theta = \sin^{-1} x$.

$\therefore \quad x = \sin\theta = \sin(\sin^{-1} x)$

Hence, $\sin(\sin^{-1} x) = x$ for all $x \in [-1, 1]$.

Similarly, we can prove other results.

ILLUSTRATION 1 *Evaluate each of the following:*

(i) $\sin\left(\sin^{-1}\frac{5}{13}\right)$ (ii) $\sin\left(\cos^{-1}\frac{4}{5}\right)$ (iii) $\sin\left(\tan^{-1}\frac{15}{8}\right)$

(iv) $\sin\left(\cot^{-1}\frac{4}{3}\right)$ (v) $\sin\left(\sec^{-1}\frac{17}{15}\right)$ (vi) $\sin\left(\operatorname{cosec}^{-1}\frac{17}{8}\right)$

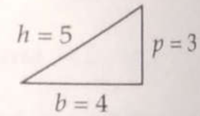
SOLUTION We know that $\sin(\sin^{-1} x) = x$ for all $x \in [-1, 1]$. So, will convert each expression in the form $\sin(\sin^{-1} x)$ by using

$$\cos^{-1}\frac{b}{h} = \sin^{-1}\frac{p}{h},\ \tan^{-1}\frac{p}{b} = \sin^{-1}\frac{p}{h},\ \cot^{-1}\frac{p}{b} = \sin^{-1}\frac{b}{h} \text{ etc.}$$

where b, p and h denote the base, perpendicular and hypotenuse respectively of a right triangle.

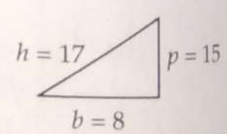
(i) $\sin\left(\sin^{-1}\frac{5}{13}\right) = \frac{5}{13}$

(ii) $\sin\left(\cos^{-1}\frac{4}{5}\right) = \sin\left(\sin^{-1}\frac{3}{5}\right) = \frac{3}{5}$



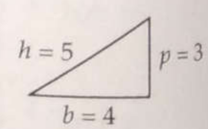
(iii) $\sin\left(\tan^{-1}\frac{15}{8}\right)$

$= \sin\left(\sin^{-1}\frac{15}{17}\right) = \frac{15}{17}$



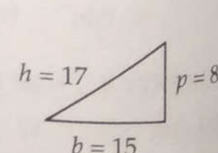
(iv) $\sin\left(\cot^{-1}\frac{4}{3}\right) = \sin\left(\cos^{-1}\frac{4}{5}\right)$

$= \sin\left(\sin^{-1}\frac{3}{5}\right) = \frac{3}{5}$



(v) $\sin\left(\sec^{-1}\frac{17}{15}\right)$

$= \sin\left(\sin^{-1}\frac{8}{17}\right) = \frac{8}{17}$



(vi) $\sin\left(\operatorname{cosec}^{-1}\frac{17}{8}\right)$

$= \sin\left(\sin^{-1}\frac{8}{17}\right) = \frac{8}{17}$

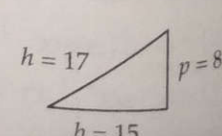


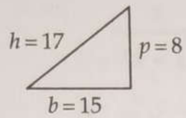
ILLUSTRATION 2 *Evaluate each of the following:*

(i) $\cos\left(\cos^{-1}\dfrac{5}{13}\right)$ (ii) $\cos\left(\sin^{-1}\dfrac{8}{17}\right)$ (iii) $\cos\left(\tan^{-1}\dfrac{3}{4}\right)$

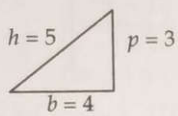
(iv) $\cos\left(\cot^{-1}\dfrac{15}{8}\right)$ (v) $\cos\left(\sec^{-1}\dfrac{5}{3}\right)$ (vi) $\cos\left(\operatorname{cosec}^{-1}\dfrac{13}{12}\right)$

SOLUTION (i) $\cos\left(\cos^{-1}\dfrac{5}{13}\right) = \dfrac{5}{13}$

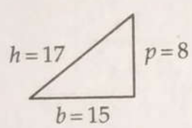
(ii) $\cos\left(\sin^{-1}\dfrac{8}{17}\right) = \cos\left(\cos^{-1}\dfrac{15}{17}\right) = \dfrac{15}{17}$ $\left[\because \sin^{-1}\dfrac{8}{17} = \cos^{-1}\dfrac{15}{17}\right]$



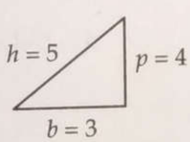
(iii) $\cos\left(\tan^{-1}\dfrac{3}{4}\right) = \cos\left(\cos^{-1}\dfrac{4}{5}\right) = \dfrac{4}{5}$ $\left[\because \tan^{-1}\dfrac{3}{4} = \cos^{-1}\dfrac{4}{5}\right]$



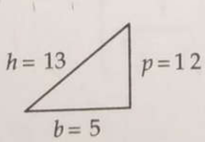
(iv) $\cos\left(\cot^{-1}\dfrac{15}{8}\right) = \cos\left(\cos^{-1}\dfrac{15}{17}\right) = \dfrac{15}{17}$ $\left[\because \cot^{-1}\dfrac{15}{8} = \cos^{-1}\dfrac{15}{17}\right]$



(v) $\cos\left(\sec^{-1}\dfrac{5}{3}\right) = \cos\left(\cos^{-1}\dfrac{3}{5}\right) = \dfrac{3}{5}$ $\left[\because \sec^{-1}\dfrac{5}{3} = \cos^{-1}\dfrac{3}{5}\right]$



(vi) $\cos\left(\operatorname{cosec}^{-1}\dfrac{13}{12}\right) = \cos\left(\cos^{-1}\dfrac{5}{13}\right) = \dfrac{5}{13}$ $\left[\because \operatorname{cosec}^{-1}\dfrac{13}{12} = \cos^{-1}\dfrac{5}{13}\right]$



REMARK *It follows from the above property that if b denotes the base, p denotes the perpendicular and h the hypotenuse of a right triangle, then*

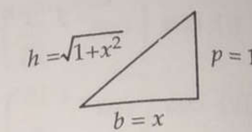
$$\sin^{-1}\left(\frac{p}{h}\right) = \cos^{-1}\left(\frac{b}{h}\right) = \tan^{-1}\left(\frac{p}{b}\right) = \sec^{-1}\left(\frac{h}{b}\right) = \operatorname{cosec}^{-1}\left(\frac{h}{p}\right)$$

For example,

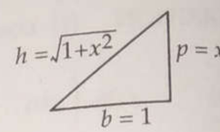
$$\sin^{-1}\left(\frac{3}{5}\right) = \cos^{-1}\left(\frac{4}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right) = \sec^{-1}\left(\frac{5}{4}\right) = \operatorname{cosec}^{-1}\left(\frac{5}{3}\right)$$

ILLUSTRATION 3 *Evaluate:*

(i) $\sin\,(\cot^{-1} x)$ (ii) $\cos\,(\tan^{-1} x)$



SOLUTION (i) $\sin(\cot^{-1} x) = \sin\left(\sin^{-1}\dfrac{1}{\sqrt{1+x^2}}\right) = \dfrac{1}{\sqrt{1+x^2}}$



(ii) $\cos\,(\tan^{-1} x) = \cos\left(\cos^{-1}\dfrac{1}{\sqrt{1+x^2}}\right) = \dfrac{1}{\sqrt{1+x^2}}$

PROPERTY III *Prove that:*

(i) $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$

$= \cot^{-1}\dfrac{\sqrt{1-x^2}}{x} = \sec^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right) = \operatorname{cosec}^{-1}\left(\dfrac{1}{x}\right)$

(ii) $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$

$= \cot^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\dfrac{1}{x} = \operatorname{cosec}^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right)$

(iii) $\tan^{-1} x = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)$

$= \cot^{-1}\left(\dfrac{1}{x}\right) = \sec^{-1}\sqrt{1+x^2} = \operatorname{cosec}^{-1}\left(\dfrac{\sqrt{1+x^2}}{x}\right)$

PROOF (i) Let $\sin^{-1} x = \theta$. Then, $x = \sin\theta$.

Now, $\cos\theta = \sqrt{1-\sin^2\theta}$

$\Rightarrow \quad \cos\theta = \sqrt{1-x^2}$

$\Rightarrow \quad \theta = \cos^{-1}\sqrt{1-x^2}$

$\Rightarrow \quad \sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \sec^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right)$ $[\because\ \theta = \sin^{-1} x]$

Again, $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\sin\theta}{\sqrt{1-\sin^2\theta}}$

$\Rightarrow \quad \tan\theta = \dfrac{x}{\sqrt{1-x^2}}$

$\Rightarrow \quad \theta = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$

$\Rightarrow \quad \sin^{-1} x = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$ $[\because\ \theta = \sin^{-1} x]$

$$\Rightarrow \quad \sin^{-1} x = \tan^{-1} \frac{x}{\sqrt{1-x^2}} = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) \qquad \left[\because \tan^{-1} x = \cot^{-1}\left(\frac{1}{x}\right)\right]$$

Hence, $\sin^{-1} x = \cos^{-1} \sqrt{1-x^2} = \sec^{-1} \frac{1}{\sqrt{1-x^2}}$

$$= \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

Similarly, other results can be proved.

PROPERTY IV *Prove that:*

(i) $\sin^{-1}(-x) = -\sin^{-1} x$, for all $x \in [-1, 1]$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1} x$, for all $x \in [-1, 1]$

(iii) $\tan^{-1}(-x) = -\tan^{-1} x$, for all $x \in R$

(iv) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$, for all $x \in \beta(-\infty, -1] \cup [1, \infty)$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1} x$, for all $x \in R$

PROOF (i) Clearly, $-x \in [-1, 1]$ for all $x \in [-1, 1]$

Let $\sin^{-1}(-x) = \theta$ ...(i)

Then, $-x = \sin \theta$

$\Rightarrow \quad x = -\sin \theta$

$\Rightarrow \quad x = \sin(-\theta)$

$\Rightarrow \quad -\theta = \sin^{-1} x \qquad [\because x\ [-1, 1]$ and $-\theta \in [-\pi/2, \pi/2]$ for all $\theta \in [-\pi/2, \pi/2]]$

$\Rightarrow \quad \theta = -\sin^{-1} x$ ...(ii)

From (i) and (ii), we get

$$\sin^{-1}(-x) = -\sin^{-1} x$$

(ii) Clearly, $-x \in [-1, 1]$ for all $x \in [-1, 1]$.

Let $\cos^{-1}(-x) = \theta$ ...(i)

Then, $-x = \cos \theta$

$\Rightarrow \quad x = -\cos \theta$

$\Rightarrow \quad x = \cos(\pi - \theta)$

$\Rightarrow \quad \cos^{-1} x = \pi - \theta \qquad [\because x \in [-1, 1]$ and $\pi - \theta \in [0, \pi]$ for all $\theta \in [0, \pi]]$

$\Rightarrow \quad \theta = \pi - \cos^{-1} x$ ...(ii)

From (i) and (ii), we get

$$\cos^{-1}(-x) = \pi - \cos^{-1} x$$

Similarly, we can prove other results.

PROPERTY V *Prove that:*

(i) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x$, *for all* $x \in (-\infty, -1] \cup [1, \infty)$

(ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$, *for all* $x \in (-\infty, -1] \cup [1, \infty)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x & , \textit{for } x > 0 \\ -\pi + \cot^{-1} x, & \textit{for } x < 0 \end{cases}$ ...(i)

PROOF (i) Let $\operatorname{cosec}^{-1} x = \theta$

Then,

$$x = \operatorname{cosec}\theta$$

$$\Rightarrow \quad \frac{1}{x} = \sin\theta$$

$$\Rightarrow \quad \theta = \sin^{-1}\frac{1}{x} \quad \text{...(ii)} \quad \left[\begin{array}{l}\because x \in (-\infty, -1] \cup [1, \infty) \Rightarrow 1/x \; [-1, 1] - \{0\} \\ \operatorname{cosec}^{-1} x = \theta \Rightarrow \theta \in [-\pi/2, \pi/2] - \{0\}\end{array}\right]$$

From (i) and (ii), we get

$$\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x$$

(ii) Let $\sec^{-1} x = \theta$ ...(i)

Then, $x \in (-\infty, -1] \cup [1, \infty)$ and $\theta \in [0, \pi] - \{\pi/2\}$.

Now, $\sec^{-1} x = \theta$

$$\Rightarrow \quad x = \sec\theta$$

$$\Rightarrow \quad \frac{1}{x} = \cos\theta$$

$$\Rightarrow \quad \theta = \cos^{-1}\frac{1}{x} \quad \left[\because x \in (-\infty, -1] \cup [1, \infty) \Rightarrow \frac{1}{x} \in [-1, 1] - \{0\} \text{ and } \theta \in [0, \pi]\right] \quad \text{...(ii)}$$

From (i) and (ii), we get

$$\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$$

(iii) Let $\cot^{-1} x = \theta$. Then, $x \in R$, $x \neq 0$ and $\theta \in (0, \pi)$ ...(i)

Now, two cases arise:

CASE I *When* $x > 0$

In this case, $\theta \in (0, \pi/2)$

$$\therefore \quad \cot^{-1} x = \theta$$

$$\Rightarrow \quad x = \cot\theta$$

$$\Rightarrow \quad \frac{1}{x} = \tan\theta$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{1}{x}\right) \quad [\because \theta \in (0, \pi/2)] \quad \text{...(ii)}$$

From (i) and (ii), we get

$$\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x \text{, for all } x > 0.$$

CASE II *When* $x < 0$

In this case, $\theta \in (\pi/2, \pi)$

Now, $\frac{\pi}{2} < \theta < \pi$ $\quad [\because x = \cot\theta < 0]$

$$\Rightarrow \quad -\frac{\pi}{2} < \theta - \pi < 0$$

$$\Rightarrow \quad \theta - \pi \in (-\pi/2, 0)$$

$$\therefore \quad \cot^{-1} x = \theta$$

$\Rightarrow \quad x = \cot\theta$

$\Rightarrow \quad \frac{1}{x} = \tan\theta$

$\Rightarrow \quad \frac{1}{x} = -\tan(\pi-\theta)$ $\quad$ [$\because \tan(\pi-\theta) = -\tan\theta$]

$\Rightarrow \quad \frac{1}{x} = \tan(\theta-\pi)$

$\Rightarrow \quad \theta-\pi = \tan^{-1}\left(\frac{1}{x}\right)$ $\quad$ [$\because \theta-\pi \in (-\pi/2, 0)$]

$\Rightarrow \quad \tan^{-1}\left(\frac{1}{x}\right) = -\pi+\theta$ $\quad$ ...(iii)

From (i) and (iii), we get

$$\tan^{-1}\left(\frac{1}{x}\right) = -\pi+\cot^{-1}x, \text{ if } x<0.$$

Hence, $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}x & \text{, for } x>0 \\ -\pi+\cot^{-1}x & \text{, for } x<0 \end{cases}$

PROPERTY VI *Prove that:*

(i) $\sin^{-1}x+\cos^{-1}x = \frac{\pi}{2}$, *for all* $x \in [-1, 1]$

(ii) $\tan^{-1}x+\cot^{-1}x = \frac{\pi}{2}$, *for all* $x \in R$

(iii) $\sec^{-1}x+\operatorname{cosec}^{-1}x = \frac{\pi}{2}$, *for all* $x \in (-\infty, -1] \cup [1, \infty)$.

PROOF (i) Let $\sin^{-1}x = \theta$ $\quad$ ...(i)

Then, $\theta \in [-\pi/2, \pi/2]$ $\quad$ [$\because x \in [-1, 1]$]

$\Rightarrow \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \Rightarrow -\frac{\pi}{2} \le -\theta \le \frac{\pi}{2} \Rightarrow 0 \le \frac{\pi}{2}-\theta \le \pi \Rightarrow \frac{\pi}{2}-\theta \in [0, \pi]$

Now, $\sin^{-1}x = \theta$

$\Rightarrow \quad x = \sin\theta$

$\Rightarrow \quad x = \cos\left(\frac{\pi}{2}-\theta\right)$

$\Rightarrow \quad \cos^{-1}x = \frac{\pi}{2}-\theta$ $\quad$ [$\because x \; [-1, 1]$ and $(\pi/2-\theta) \; [0, \pi]$]

$\Rightarrow \quad \theta+\cos^{-1}x = \frac{\pi}{2}$ $\quad$ ...(ii)

From (i) and (ii), we get

$$\sin^{-1}x+\cos^{-1}x = \frac{\pi}{2}$$

(ii) Let $\tan^{-1}x = \theta$ $\quad$ ...(i)

Then, $\theta \in (\pi/2, \pi/2)$ $\quad$ [$\because x \in R$]

$\Rightarrow \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\Rightarrow \quad -\frac{\pi}{2} < -\theta < \frac{\pi}{2}$

$\Rightarrow \quad 0 < \frac{\pi}{2}-\theta < \pi$

$$\Rightarrow \quad \left(\frac{\pi}{2}-\theta\right) \in (0, \pi)$$

Now, $\tan^{-1} x = \theta$

$\Rightarrow \quad x = \tan\theta$

$\Rightarrow \quad x = \cot\left(\frac{\pi}{2}-\theta\right)$ $\left[\because \frac{\pi}{2}-\theta \in (0, \pi)\right]$

$\Rightarrow \quad \cot^{-1} x = \frac{\pi}{2}-\theta$

$\Rightarrow \quad \theta + \cot^{-1} x = \frac{\pi}{2}$ ...(ii)

From (i) and (ii), we get

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}.$$

(iii) Let $\sec^{-1} x = \theta$ ...(i)

Then, $\theta \in [0, \pi] - \{\pi/2\}$ $[\because x \in (-\infty, -1] \cup [1, \infty)]$

$\Rightarrow \quad 0 \le \theta \le \pi, \theta \ne \frac{\pi}{2}$

$\Rightarrow \quad -\pi \le -\theta \le 0, \theta \ne \frac{\pi}{2}$

$\Rightarrow \quad -\frac{\pi}{2} \le \frac{\pi}{2}-\theta \le \frac{\pi}{2}, \frac{\pi}{2}-\theta \ne 0$

$\Rightarrow \quad \left(\frac{\pi}{2}-\theta\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\frac{\pi}{2}-\theta \ne 0.$

Now, $\sec^{-1} x = \theta$

$\Rightarrow \quad x = \sec\theta$

$\Rightarrow \quad x = \text{cosec}\left(\frac{\pi}{2}-\theta\right)$

$\Rightarrow \quad \text{cosec}^{-1} x = \frac{\pi}{2}-\theta$ $\left[\because \left(\frac{\pi}{2}-\theta\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ and, } \frac{\pi}{2}-\theta \ne 0\right]$

$\Rightarrow \quad \theta + \text{cosec}^{-1} x = \frac{\pi}{2}$ ...(ii)

From (i) and (ii), we get

$$\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2}$$

ILLUSTRATION 1 *Find the value of* $\cot(\tan^{-1} a + \cot^{-1} a)$ **[CBSE 2012, NCERT]**

SOLUTION $\cot(\tan^{-1} a + \cot^{-1} a) = \cot\frac{\pi}{2} = 0$

ILLUSTRATION 2 *Prove that* $\tan^{-1} x + \tan^{-1}\frac{1}{x} = \begin{cases} \pi/2, & \text{if } x > 0 \\ -\pi/2, & \text{if } x < 0 \end{cases}$

SOLUTION We have,

$$\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x, & \text{for } x > 0 \\ -\pi + \cot^{-1} x, & \text{for } x < 0 \end{cases}$$

$$\therefore \quad \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \tan^{-1} x + \cot^{-1} x = \pi/2, & \text{if } x > 0 \\ \tan^{-1} x + \cot^{-1} x - \pi = \pi/2 - \pi = -\pi/2, & \text{if } x < 0 \end{cases}$$

PROPERTY VII *Prove that:*

(i)
$$\tan^{-1}x+\tan^{-1}y=\begin{cases}\tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & \text{, if } xy<1\\ \pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & \text{, if } x>0,\ y>0 \text{ and } xy>1\\ -\pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } x<0,\ y<0 \text{ and } xy>1\end{cases}$$

(ii)
$$\tan^{-1}x-\tan^{-1}y=\begin{cases}\tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & \text{, if } xy>-1\\ \pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & \text{, if } x>0,\ y<0 \text{ and } xy<-1\\ \pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & \text{, if } x<0,\ y>0 \text{ and } xy<-1\end{cases}$$

PROOF (i) Let $\tan^{-1}x=A$ and $\tan^{-1}y=B$. Then,

$x=\tan A$ and $y=\tan B$ and $A, B\in(-\pi/2, \pi/2)$.

$$\therefore \quad \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}=\frac{x+y}{1-xy} \qquad \text{...(i)}$$

Now, the following cases arise.

CASE I *When $x>0$, $y>0$ and $xy<1$*

In this case, we have

$x>0, y>0$ and $xy<1$

$\Rightarrow \quad \dfrac{x+y}{1-xy}>0$

$\Rightarrow \quad \tan(A+B)>0$ [Using (i)]

$\Rightarrow \quad A+B$ lies in I quadrant or in III quadrant

$\Rightarrow \quad 0<A+B<\dfrac{\pi}{2}$ $\left[\begin{array}{l}\because x>0<1\ \ 0<A<\dfrac{\pi}{2}\\ \quad y>0\Rightarrow 0<B<\pi/2\end{array}\right\}\Rightarrow 0<A+B<\pi\right]$

$\therefore \quad \tan(A+B)=\dfrac{x+y}{1-xy}$ [From (i)]

$\Rightarrow \quad A+B=\tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$ $\left[\because\ 0<A+B<\dfrac{\pi}{2}\right]$

$\Rightarrow \quad \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

CASE II *When $x<0$, $y<0$ and $xy<1$*

In this case, we have

$x<0, y<0$ and $xy<1$

$\Rightarrow \quad \dfrac{x+y}{1-xy}<0$

$\Rightarrow \quad \tan(A+B)<0$

$\Rightarrow \quad A+B$ lies in II quadrant or in IV quadrant. [From (i)]

$$\left[\begin{array}{l} \because x < 0 \Rightarrow -\pi/2 < A < 0 \\ y < 0 \Rightarrow -\pi/2 < B < 0 \end{array}\right\} \Rightarrow -\pi < A + B < 0\right]$$

$\Rightarrow$ $A + B$ lies in IV quadrant [From (i)]

$\Rightarrow$ $-\frac{\pi}{2} < A + B < 0$

$\therefore$ $\tan(A + B) = \frac{x + y}{1 - xy}$

$\Rightarrow$ $A + B = \tan^{-1}\left(\frac{x + y}{1 - xy}\right)$

$\Rightarrow$ $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right)$

CASE III *When $x > 0$ and $y < 0$ or $x < 0$ and $y > 0$*

In this case, we have

$x > 0$ and $y < 0$

$\Rightarrow$ $A \in (0, \pi/2)$ and $B \in (-\pi/2, 0)$

$\Rightarrow$ $A + B \in (-\pi/2, \pi/2)$ [From (i)]

$\therefore$ $\tan(A + B) = \frac{x + y}{1 - xy}$

$\Rightarrow$ $A + B = \tan^{-1}\left(\frac{x + y}{1 - xy}\right)$

$\Rightarrow$ $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right)$

Similarly, if $x < 0$ and $y > 0$, we have

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right)$$

It follows from the above three cases that

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right), \text{ if } xy < 1.$$

CASE IV *When $x > 0$, $y > 0$ and $xy > 1$*

In this case, we have

$x > 0, y > 0$ and $xy > 1$

$\Rightarrow$ $\frac{x + y}{1 - xy} < 0$

$\Rightarrow$ $\tan(A + B) < 0$

$\Rightarrow$ $A + B$ lies either in II quadrant or in IV quadrant $\left[\text{From (i), } \tan(A + B) = \frac{x + y}{1 - xy}\right]$

$\Rightarrow$ $A + B$ lies in II quadrant $[\because x > 0, y > 0 \Rightarrow A, B \in (-\pi/2, 0) \Rightarrow A + B \in (-0, \pi)]$

$\Rightarrow$ $\frac{\pi}{2} < A + B < \pi$

$\Rightarrow$ $\frac{\pi}{2} - \pi < (A + B) - \pi < 0$

$\Rightarrow$ $-\frac{\pi}{2} < (A + B) - \pi < 0$

$$\therefore \quad \tan(A+B) = \frac{x+y}{1-xy} \qquad \text{[From (i)]}$$

$$\Rightarrow \quad -\tan\{\pi-(A+B)\} = \frac{x+y}{1-xy} \qquad [\because \tan\{\pi-(A+B)\} = -\tan(A+B)]$$

$$\Rightarrow \quad \tan\{(A+B)-\pi\} = \frac{x+y}{1-xy}$$

$$\Rightarrow \quad A+B-\pi = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \quad A+B = \pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \quad \tan^{-1}x+\tan^{-1}y = \pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right).$$

CASE V *When $x<0$, $y<0$ and $xy>1$:*

In this case, we have

$$x<0,\ y<0 \text{ and } xy>1$$

$$\Rightarrow \quad \frac{x+y}{1-xy} > 0$$

$$\Rightarrow \quad \tan(A+B) > 0 \qquad \left[\text{From (i), } \tan(A+B) = \frac{x+y}{1-xy}\right]$$

$\Rightarrow$ $A+B$ lies either in I quadrant or III quadrant

$\Rightarrow$ $A+B$ lies in III quadrant $\quad [\because x<0, y<0 \Rightarrow A, B \in (-\pi/2, 0) \Rightarrow A+B \in (-\pi, 0)]$

$$\Rightarrow \quad -\pi < A+B < -\frac{\pi}{2}$$

$$\Rightarrow \quad \pi-\pi < \pi+(A+B) < \pi-\frac{\pi}{2}$$

$$\Rightarrow \quad 0 < \pi+(A+B) < \frac{\pi}{2}$$

$$\text{Now,} \quad \tan(A+B) = \frac{x+y}{1-xy} \qquad \text{[From (i)]}$$

$$\Rightarrow \quad \tan(\pi+A+B) = \frac{x+y}{1-xy} \qquad [\because \tan(\pi+\theta) = \tan\theta]$$

$$\Rightarrow \quad \pi+A+B = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \quad A+B = -\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \quad \tan^{-1}x+\tan^{-1}y = -\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

(ii) Let $\tan^{-1}x = A$ and $\tan^{-1}y = B$. Then,

$$\Rightarrow \quad x=\tan A,\ y=\tan B \text{ and } A, B \in (-\pi/2, \pi/2)$$

$$\therefore \quad \tan(A-B) = \frac{\tan A-\tan B}{1+\tan A\tan B}$$

$$\Rightarrow \quad \tan(A-B) = \frac{x-y}{1+xy} \qquad \text{...(i)}$$

CASE I *When* $xy > -1$

If $x > 0$ and $y > 0$, then

$$A \in (0, \pi/2), B \in (0, \pi/2)$$

$$\Rightarrow \quad A - B \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \qquad \text{[From (i)]}$$

$$\therefore \quad \tan(A - B) = \frac{x-y}{1+xy}$$

$$\Rightarrow \quad A - B = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

$$\Rightarrow \quad \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

$$\Rightarrow \quad \tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right) \text{ for all } x, y \text{ with } xy > -1.$$

CASE II *When* $x > 0, y < 0$ *and* $xy < -1$:

In this case, we have

$$x > 0, y < 0$$

$$\Rightarrow \quad A \in (0, \pi/2), B \in (-\pi/2, 0)$$

$$\Rightarrow \quad A \in (0, \pi/2), -B \in (0, \pi/2)$$

$$\Rightarrow \quad A - B \in (0, \pi)$$

Again, $x > 0, y < 0$ and $xy < -1$

$$\Rightarrow \quad x > 0, -y > 0 \text{ and } 1 + xy < 0$$

$$\Rightarrow \quad x - y > 0 \quad \text{and } 1 + xy < 0$$

$$\Rightarrow \quad \frac{x-y}{1+xy} < 0$$

$$\Rightarrow \quad \tan(A - B) < 0$$

$$\Rightarrow \quad A - B \in (\pi/2, \pi) \qquad [\because A - B \in (0, \pi)]$$

$$\Rightarrow \quad \frac{\pi}{2} < A - B < \pi$$

$$\Rightarrow \quad -\frac{\pi}{2} < (A - B) - \pi < 0$$

$$\therefore \quad \tan(A - B) = \frac{x-y}{1+xy} \qquad \text{[From (i)]}$$

$$\Rightarrow \quad -\tan\{\pi - (A - B)\} = \frac{x-y}{1+xy}$$

$$\Rightarrow \quad \tan\{(A - B) - \pi\} = \frac{x-y}{1+xy}$$

$$\Rightarrow \quad (A - B) - \pi = \tan^{-1} \frac{x-y}{1+xy}$$

$$\Rightarrow \quad A - B = \pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

$$\Rightarrow \quad \tan^{-1} x - \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

CASE III *When* $x < 0, y > 0$ *and* $xy < -1$

In this case, we have

$$x<0, y>0 \text{ and } xy<-1$$

$$\Rightarrow \quad x-y<0 \text{ and } 1+xy<0$$

$$\Rightarrow \quad \frac{x-y}{1+xy}>0$$

$$\Rightarrow \quad \tan(A-B)>0 \qquad \text{[From (i)]}$$

$\Rightarrow$ $(A-B)$ lies either in I quadrant or in III quadrant

$$\Rightarrow \quad -\pi<A-B<-\frac{\pi}{2} \qquad [\because x<0, y>0 \Rightarrow A\in(-\pi/2, 0), B\in(0,\pi/2) \Rightarrow -\pi<A-B<0]$$

$$\Rightarrow \quad 0<\pi+(A-B)<\frac{\pi}{2}$$

$$\therefore \quad \tan(A-B)=\frac{x-y}{1+xy}$$

$$\Rightarrow \quad \tan\{\pi+(A-B)\}=\frac{x-y}{1+xy}$$

$$\Rightarrow \quad \pi+A-B=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

$$\Rightarrow \quad A-B=-\pi+\tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

$$\Rightarrow \quad \tan^{-1}x-\tan^{-1}y=-\pi+\tan^{-1}\left(\frac{x-y}{1+xy}\right).$$

<u>REMARK</u> *If $x_1, x_2, x_3, \ldots, x_n \in R$, then*

$$\tan^{-1}x_1+\tan^{-1}x_2+\ldots+\tan^{-1}x_n=\tan^{-1}\left(\frac{S_1-S_3+S_5-S_7+\ldots}{1-S_2+S_4-S_6+\ldots}\right)$$

where S_k denotes the sum of the products of $x_1, x_2, \ldots, x_n$ taken k at a time.

ILLUSTRATION 1 *Prove that :* $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\frac{1}{2}$

SOLUTION We have,

$$\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}$$

$$=\tan^{-1}\left\{\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}\times\frac{7}{24}}\right\} \qquad \left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{If } xy<1\right]$$

$$=\tan^{-1}\left\{\frac{48+77}{264-14}\right\}=\tan^{-1}\left(\frac{125}{250}\right)=\tan^{-1}\left(\frac{1}{2}\right)$$

ILLUSTRATION 2 *Prove that:* $\tan^{-1}2+\tan^{-1}3=\frac{3\pi}{4}$

SOLUTION We have,

$$\tan^{-1}2+\tan^{-1}3$$

$$=\pi+\tan^{-1}\left\{\frac{2+3}{1-2\times3}\right\} \qquad \left[\because \tan^{-1}x+\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{if } xy>1\right]$$

$$=\pi+\tan^{-1}(-1)=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$$

ILLUSTRATION 3 *Prove that:* $\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\pi$ [CBSE 2010]

SOLUTION $\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\tan^{-1}1+(\tan^{-1}2+\tan^{-1}3)$ [See illustration 2]

$$=\frac{\pi}{4}+\frac{3\pi}{4}$$
$$=\pi$$

ILLUSTRATION 4 *Prove that:* $\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}=\pi$

SOLUTION We have,

$$\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}$$

$$=\tan^{-1}\frac{12}{5}+\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{63}{16} \quad \left[\because \sin^{-1}\frac{12}{13}=\tan^{-1}\frac{12}{5} \text{ and } \cos^{-1}\frac{4}{5}=\tan^{-1}\frac{3}{4}\right]$$

$$=\pi+\tan^{-1}\left\{\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\times\frac{3}{4}}\right\}+\tan^{-1}\frac{63}{16} \quad \left[\because \tan^{-1}x+\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ if } xy>1\right]$$

$$=\pi+\tan^{-1}\left(\frac{63}{-16}\right)+\tan^{-1}\left(\frac{63}{16}\right)$$

$$=\pi-\tan^{-1}\frac{63}{16}+\tan^{-1}\frac{63}{16} \quad [\because \tan^{-1}(-x)=-\tan^{-1}x]$$

$$=\pi$$

PROPERTY VIII *Prove that:*

$$\text{(i) } \sin^{-1}x+\sin^{-1}y=\begin{cases}\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\} & \text{, if } -1\le x, y\le 1 \text{ and } x^2+y^2\le 1 \\ & \text{or} \\ & \text{if } xy<0 \text{ and } x^2+y^2>1 \\ \pi-\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\} & \text{, if } 0<x, y\le 1 \text{ and } x^2+y^2>1 \\ -\pi-\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\} & \text{, if } -1\le x, y<0 \text{ and } x^2+y^2>1\end{cases}$$

$$\text{(iii) } \sin^{-1}x-\sin^{-1}y=\begin{cases}\sin^{-1}\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\} & \text{, if } -1\le x, y\le 1 \text{ and } x^2+y^2\le 1 \\ & \text{or} \\ & \text{if } xy>0 \text{ and } x^2+y^2>1 \\ \pi-\sin^{-1}\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\} & \text{, if } 0<x\le 1, -1\le y\le 0 \text{ and } x^2+y^2>1 \\ -\pi-\sin^{-1}\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\} & \text{, if } -1\le x<0, 0<y\le 1 \text{ and } x^2+y^2>1\end{cases}$$

PROOF *Let* $\sin^{-1}x=A$ *and* $\sin^{-1}y=B$. *Then,*

$$x=\sin A, y=\sin B \text{ and } A, B\in[-\pi/2, \pi/2]$$

$\Rightarrow$ $\cos A=\sqrt{1-x^2}$, $\cos B=\sqrt{1-y^2}$ $\quad [\because A, B\in[-\pi/2, \pi/2] \therefore \cos A, \cos B\le 0]$

$\therefore$ $\sin(A+B)=\sin A\cos B+\cos A\sin B$

$\Rightarrow$ $\sin(A+B)=x\sqrt{1-y^2}+y\sqrt{1-x^2}$, ...(i)

$$\sin(A-B) = x\sqrt{1-y^2} - y\sqrt{1-x^2}, \quad \text{...(ii)}$$

$$\cos(A+B) = \sqrt{1-x^2}\sqrt{1-y^2} - xy \quad \text{...(iii)}$$

and, $$\cos(A-B) = \sqrt{1-x^2}\sqrt{1-y^2} + xy \quad \text{...(iv)}$$

CASE I *When* $-1 \le x, y \le 1$ *and* $x^2 + y^2 \le 1$

In this case, we have

$x^2 + y^2 \le 1$

$\Rightarrow \quad 1 - x^2 \ge y^2$ and $1 - y^2 \ge x^2$

$\Rightarrow \quad (1-x^2)(1-y^2) \ge x^2 y^2$

$\Rightarrow \quad \sqrt{1-x^2}\sqrt{1-y^2} \ge xy$

$\Rightarrow \quad \sqrt{1-x^2}\sqrt{1-y^2} - xy \ge 0$

$\Rightarrow \quad \cos(A+B) \ge 0$ [Using (iii)]

$\Rightarrow \quad A + B$ lies either in I quadrant or in IV quadrant

$\Rightarrow \quad A + B \in [-\pi/2, \pi/2]$ $[\because A, B \in [-\pi/2, \pi/2] \Rightarrow -\pi \le A + B \le \pi]$

$\therefore \quad \sin(A+B) = x\sqrt{1-y^2} + y\sqrt{1-x^2}$ [From (i)]

$\Rightarrow \quad A + B = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$ $\left[\because -\frac{\pi}{2} \le A + B \le \frac{\pi}{2}\right]$

$\Rightarrow \quad \sin^{-1}x + \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$

CASE II *When* $xy < 0$ *and* $x^2 + y^2 > 1$:

In this case, we have

$xy < 0$

$\Rightarrow \quad x > 0$ and $y < 0$ or $x < 0$ and $y > 0$

$\Rightarrow \quad \{A \in (0, \pi/2]$ and $B \in [-\pi/2, 0)\}$ or $\left\{A \in \left[-\frac{\pi}{2}, 0\right)\right.$ and $\left.B \in \left(0, \frac{\pi}{2}\right]\right\}$

$$\Rightarrow \quad -\frac{\pi}{2} \le A + B \le \frac{\pi}{2} \quad \text{...(v)}$$

and, $x^2 + y^2 > 1$

$\Rightarrow \quad 1 - x^2 < y^2$ and $1 - y^2 < x^2$

$\Rightarrow \quad (1-x^2)(1-y^2) < x^2 y^2$

$\Rightarrow \quad (\sqrt{1-x^2}\sqrt{1-y^2})^2 < (|xy|)^2$ $[\because xy < 0]$

$\Rightarrow \quad -|xy| < \sqrt{1-x^2}\sqrt{1-y^2} < |xy|$

$\Rightarrow \quad xy < \sqrt{1-x^2}\sqrt{1-y^2} < -xy$ $[\because xy < 0 \therefore |xy| = -xy]$

$\Rightarrow \quad \sqrt{1-x^2}\sqrt{1-y^2} - xy > 0$

$\Rightarrow \quad \cos(A+B) > 0$

$\Rightarrow \quad A + B$ lies either in I quadrant or in IV quadrant

$\Rightarrow \quad A + B \in [-\pi/2, \pi/2]$ [Using (v)]

$\therefore \quad \sin(A+B) = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$\Rightarrow \quad A + B = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$ $[\because A + B \in [-\pi/2, \pi/2]]$

$$\Rightarrow \quad \sin^{-1} x + \sin^{-1} y = \sin^{-1} \{(x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$$

CASE III *When* $0 < x, y \le 1$ *and* $x^2 + y^2 > 1$

In this case, we have

$$0 < x, y \le 1 \qquad \text{...(vi)}$$

$\Rightarrow \quad A \in (0, \pi/2]$ and $B \in (0, \pi/2]$

$\Rightarrow \quad A + B \in (0, \pi]$

and, $\quad x^2 + y^2 > 1$

$\Rightarrow \quad 1 - x^2 < y^2$ and $1 - y^2 < x^2$

$\Rightarrow \quad (1-x^2)(1-y^2) < x^2 y^2 \qquad [\because xy > 0]$

$\Rightarrow \quad \sqrt{1-x^2}\sqrt{1-y^2} < xy$

$\Rightarrow \quad \sqrt{1-x^2}\sqrt{1-y^2} - xy < 0 \qquad$ [Using (iii)]

$\Rightarrow \quad \cos(A+B) < 0$

$\Rightarrow \quad A + B$ lies either in II quadrant or in III quadrant $\qquad [\because A + B \ (0, \pi$, from (vi)]

$\Rightarrow \quad \frac{\pi}{2} \le A + B \le \pi$

$\Rightarrow \quad -\pi \le -(A+B) \le -\frac{\pi}{2}$

$\Rightarrow \quad 0 \le \pi - (A+B) \le \frac{\pi}{2}$

$\therefore \quad \sin(A+B) = x\sqrt{1-y^2} + y\sqrt{1-x^2} \qquad$ [From (i)]

$\Rightarrow \quad \sin\{\pi - (A+B)\} = x\sqrt{1-y^2} + y\sqrt{1-x^2} \qquad [\because \sin(\pi - \theta) = \sin\theta]$

$\Rightarrow \quad \pi - (A+B) = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$

$\Rightarrow \quad A + B = \pi - \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$

$\Rightarrow \quad \sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$.

CASE IV *When* $-1 \le x, y < 0$ *and* $x^2 + y^2 > 1$:

In this case, we have

$$-1 \le x, y < 0$$

$\Rightarrow \quad A \in [-\pi/2, 0)$ and $B \in [-\pi/2, 0)$

$\Rightarrow \quad A + B \in [-\pi, 0) \qquad \text{...(vii)}$

and, $\quad x^2 + y^2 > 1$

$\Rightarrow \quad 1 - x^2 < y^2$ and $1 - y^2 < x^2$

$\Rightarrow \quad (1-x^2)(1-y^2) < x^2 y^2 \qquad [\because xy > 0]$

$\Rightarrow \quad \sqrt{1-x^2}\sqrt{1-y^2} < xy$

$\Rightarrow \quad \sqrt{1-x^2}\sqrt{1-y^2} - xy < 0$

$\Rightarrow \quad \cos(A+B) < 0 \qquad$ [Using (iii)]

$\Rightarrow \quad A + B$ lies either in II quadrant or in III quadrant $\qquad$ [Using (vii)]

$\Rightarrow \quad -\pi \le A + B \le \frac{\pi}{2}$

$\Rightarrow \quad \frac{\pi}{2} \le -(A+B) \le \pi$

$\Rightarrow \quad -\frac{\pi}{2} \le -\pi-(A+B) \le 0$

$\therefore \quad \sin(A+B) = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$\Rightarrow \quad -\sin\{\pi+(A+B)\} = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$\Rightarrow \quad \sin\{-\pi-(A+B)\} = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$\Rightarrow \quad -\pi-(A+B) = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$

$\Rightarrow \quad A+B = -\pi - \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$

$\Rightarrow \quad \sin^{-1}x + \sin^{-1}y = -\pi - \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}.$

(ii) Do yourself.

ILLUSTRATION 1 *Prove that:* $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{77}{85} = \tan^{-1}\left(\frac{77}{36}\right)$

[CBSE 2012, NCERT]

SOLUTION We have,

$$\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5}$$

$$= \sin^{-1}\left\{\frac{8}{17}\sqrt{1-\left(\frac{3}{5}\right)^2} + \frac{3}{5}\sqrt{1-\left(\frac{8}{17}\right)^2}\right\}$$

$$= \sin^{-1}\left\{\frac{8}{17}\times\frac{4}{5} + \frac{3}{5}\times\frac{15}{17}\right\} = \sin^{-1}\left(\frac{77}{85}\right) = \tan^{-1}\left(\frac{77}{36}\right)$$

ILLUSTRATION 2 *Prove that:* $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$ **[NCERT, CBSE 2010, 2012]**

SOLUTION We have,

$$\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5}$$

$$= \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{3}{5} \qquad \left[\because \cos^{-1}\frac{12}{13} = \sin^{-1}\frac{5}{13}\right]$$

$$= \sin^{-1}\left\{\frac{5}{13}\times\sqrt{1-\left(\frac{3}{5}\right)^2} + \frac{3}{5}\times\sqrt{1-\left(\frac{5}{13}\right)^2}\right\}$$

$$= \sin^{-1}\left\{\frac{5}{13}\times\frac{4}{5} + \frac{3}{5}\times\frac{12}{13}\right\} = \sin^{-1}\frac{56}{65}$$

PROPERTY IX *Prove that:*

(i) $$\cos^{-1}x + \cos^{-1}y = \begin{cases} \cos^{-1}\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\}, & \text{if } -1 \le x, y \le 1 \text{ and } x+y \ge 0 \\ 2\pi - \cos^{-1}\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\}, & \text{if } -1 \le x, y \le 1 \text{ and } x+y \le 0 \end{cases}$$

(ii) $$\cos^{-1}x - \cos^{-1}y = \begin{cases} \cos^{-1}\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\}, & \text{if } -1 \le x, y \le 1 \text{ and } x \le y \\ -\cos^{-1}\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\}, & \text{if } -1 \le y \le 0,\ 0 < x \le 1 \text{ and } x \ge y \end{cases}$$

PROOF Let $\cos^{-1}x = A$ and $\cos^{-1}y = B$. Then,

$$x = \cos A\,,\ y = \cos B \text{ and } A, B \in [0\,,\pi] \qquad [\because \sin A\,,\ \sin B \geq 0 \text{ for } A, B \in [0\,,\pi]]$$

$\Rightarrow \quad \sin A = \sqrt{1-x^2}$ and $\sin B = \sqrt{1-y^2}$ ...(i)

$\therefore \quad \cos(A+B) = xy - \sqrt{1-x^2}\,\sqrt{1-y^2}$ ...(ii)

$\cos(A-B) = xy + \sqrt{1-x^2}\,\sqrt{1-y^2}$

<u>CASE I</u> *When* $-1 \leq x, y \leq 1$ *and* $x+y \geq 0$:

In this case, we have

$-1 \leq x, y \leq 1$

$\Rightarrow \quad A, B \in [0\,,\pi]$

$\Rightarrow \quad 0 \leq A+B \leq 2\pi$ ...(iii)

and, $\quad x+y \geq 0$

$\Rightarrow \quad \cos A + \cos B \geq 0$

$\Rightarrow \quad \cos A \geq -\cos B$

$\Rightarrow \quad \cos A \geq \cos(\pi - B)$

$\Rightarrow \quad A \leq \pi - B \qquad [\because \cos\theta$ is decreasing on $[0, \pi]]$

$\Rightarrow \quad A+B \leq \pi$ ...(iv)

From (iii) and (iv), we get

$0 \leq A+B \leq \pi$

$\therefore \quad \cos(A+B) = xy - \sqrt{1-x^2}\,\sqrt{1-y^2}$

$\Rightarrow \quad A+B = \cos^{-1}\{xy - \sqrt{1-x^2}\,\sqrt{1-y^2}\}$

$\Rightarrow \quad \cos^{-1}x + \cos^{-1}y = \cos^{-1}\{xy - \sqrt{1-x^2}\,\sqrt{1-y^2}\}$

<u>CASE II</u> *When* $-1 \leq x, y < 0$ *and* $x+y \leq 0$:

In this case, we have

$-1 \leq x, y < 1$

$\Rightarrow \quad A, B \in [0, \pi]$

$\Rightarrow \quad 0 \leq A+B \leq 2\pi$ ...(v)

and, $\quad x+y \leq 0$

$\Rightarrow \quad \cos A + \cos B \leq 0$

$\Rightarrow \quad \cos A \leq -\cos B$

$\Rightarrow \quad \cos A \leq \cos(\pi - B)$

$\Rightarrow \quad A \geq \pi - B \qquad [\because \cos\theta$ is decreasing on $[0\,,\pi]]$

$\Rightarrow \quad A+B \geq \pi$ ...(vi)

From (v) and (vi), we get

$\pi \leq A+B \leq 2\pi$

$\Rightarrow \quad -\pi \geq -(A+B) \geq -2\pi$

$\Rightarrow \quad \pi \geq 2\pi - (A+B) \geq 0$

$\Rightarrow \quad 0 \leq 2\pi - (A+B) \leq \pi$

$\therefore \quad \cos(A+B) = xy - \sqrt{1-x^2}\,\sqrt{1-y^2}$

$\Rightarrow \quad \cos\{2\pi - (A+B)\} = xy - \sqrt{1-x^2}\,\sqrt{1-y^2}$

$\Rightarrow \quad 2\pi - (A+B) = \cos^{-1}\{xy - \sqrt{1-x^2}\,\sqrt{1-y^2}\}$

$\Rightarrow \quad A+B = 2\pi - \cos^{-1}\{xy - \sqrt{1-x^2}\,\sqrt{1-y^2}\}$

$\Rightarrow \quad \cos^{-1}x + \cos^{-1}y = 2\pi - \cos^{-1}(xy - \sqrt{1-x^2}\,\sqrt{1-y^2})$.

(ii) Do yourself.

ILLUSTRATION 1 *Prove that:* $\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{33}{65}$ [NCERT, CBSE 2010, 2012]

SOLUTION We have,

$$\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}$$

$$=\cos^{-1}\left\{\frac{4}{5}\times\frac{12}{13}-\sqrt{1-\left(\frac{4}{5}\right)^2}\sqrt{1-\left(\frac{12}{13}\right)^2}\right\}$$

$$=\cos^{-1}\left\{\frac{48}{65}-\frac{3}{5}\times\frac{5}{13}\right\}=\cos^{-1}\left\{\frac{48}{65}-\frac{15}{65}\right\}=\cos^{-1}\frac{33}{65}$$

ILLUSTRATION 2 *Prove that:* $\sin^{-1}\frac{3}{5}-\sin^{-1}\frac{8}{17}=\cos^{-1}\frac{84}{85}$ [NCERT]

SOLUTION We have,

$$\sin^{-1}\frac{3}{5}-\sin^{-1}\frac{8}{17}$$

$$=\cos^{-1}\frac{4}{5}-\cos^{-1}\frac{15}{17} \qquad \left[\because \sin^{-1}\frac{3}{5}=\cos^{-1}\frac{4}{5},\ \sin^{-1}\frac{8}{17}=\cos^{-1}\frac{15}{17}\right]$$

$$=\cos^{-1}\left\{\frac{4}{5}\times\frac{15}{17}+\sqrt{1-\left(\frac{4}{5}\right)^2}\times\sqrt{1-\left(\frac{15}{17}\right)^2}\right\}$$

$$=\cos^{-1}\left\{\frac{4}{5}\times\frac{15}{17}+\frac{3}{5}\times\frac{8}{17}\right\}=\cos^{-1}\left\{\frac{60}{85}+\frac{24}{85}\right\}=\cos^{-1}\frac{84}{85}$$

PROPERTY X *Prove that:*

(i) $$2\sin^{-1}x=\begin{cases}\sin^{-1}(2x\sqrt{1-x^2}) & \text{, if } -\frac{1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}\\ \pi-\sin^{-1}(2x\sqrt{1-x^2}) & \text{, if } \frac{1}{\sqrt{2}}\le x\le 1\\ -\pi-\sin^{-1}(2x\sqrt{1-x^2}) & \text{, if } -1\le x\le-\frac{1}{\sqrt{2}}\end{cases}$$ [NCERT]

(ii) $$3\sin^{-1}x=\begin{cases}\sin^{-1}(3x-4x^3) & \text{, if } -\frac{1}{2}\le x\le\frac{1}{2}\\ \pi-\sin^{-1}(3x-4x^3) & \text{, if } \frac{1}{2}< x\le 1\\ -\pi-\sin^{-1}(3x-4x^3) & \text{, if } -1\le x<-\frac{1}{2}\end{cases}$$

<u>PROOF</u> (i) Let $\sin^{-1}x=\theta$. Then,

$$x=\sin\theta,$$

$$\Rightarrow \quad \cos\theta=\sqrt{1-x^2} \qquad [\because \cos\theta>0 \text{ for } \theta\in[-\pi/2,\pi/2]]$$

$$\therefore \quad \sin 2\theta=2\sin\theta\cos\theta$$

$$\Rightarrow \quad \sin 2\theta=2x\sqrt{1-x^2} \qquad \text{...(i)}$$

<u>CASE I</u> *When* $-\frac{1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}$

We have,

$$-\frac{1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}\Rightarrow -\frac{\pi}{4}\le\theta\le\frac{\pi}{4}\Rightarrow -\frac{\pi}{2}\le 2\theta\le\frac{\pi}{2}$$

Also, $-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \Rightarrow -1 \le 2x\sqrt{1-x^2} \le 1$ [From (i)]

$\therefore \quad \sin 2\theta = 2x\sqrt{1-x^2}$

$\Rightarrow \quad 2\theta = \sin^{-1}(2x\sqrt{1-x^2})$

$\Rightarrow \quad 2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$

CASE II *When* $\frac{1}{\sqrt{2}} \le x \le 1$:

We have,

$\frac{1}{\sqrt{2}} \le x \le 1 \Rightarrow \frac{1}{\sqrt{2}} \le \sin\theta \le 1 \Rightarrow \frac{\pi}{4} \le \theta \le \frac{\pi}{2} \Rightarrow \frac{\pi}{2} \le 2\theta \le \pi \Rightarrow -\pi \le -2\theta \le -\frac{\pi}{2} \Rightarrow 0 \le \pi - 2\theta \le \frac{\pi}{2}$

Also, $\frac{1}{\sqrt{2}} \le x \le 1 \Rightarrow 0 \le 2x\sqrt{1-x^2} < 1$

[From (i)]

$\therefore \quad \sin 2\theta = 2x\sqrt{1-x^2}$

$\Rightarrow \quad \sin(\pi - 2\theta) = 2x\sqrt{1-x^2}$

$\Rightarrow \quad \pi - 2\theta = \sin^{-1}(2x\sqrt{1-x^2})$

$\Rightarrow \quad \pi - 2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$

$\Rightarrow \quad 2\sin^{-1}x = \pi - \sin^{-1}(2x\sqrt{1-x^2})$.

CASE III *When* $-1 \le x < -\frac{1}{\sqrt{2}}$

We have,

$-1 \le x < -\frac{1}{\sqrt{2}} \Rightarrow -1 \le \sin\theta \le -\frac{1}{\sqrt{2}} \Rightarrow -\frac{\pi}{2} \le \theta \le -\frac{\pi}{4} \Rightarrow -\pi \le 2\theta \le -\frac{\pi}{2} \Rightarrow 0 \le \pi + 2\theta \le \frac{\pi}{2}$

Also, $-1 \le x \le -\frac{1}{\sqrt{2}} \Rightarrow -1 \le 2x\sqrt{1-x^2} \le 0$

$\therefore \quad \sin 2\theta = 2x\sqrt{1-x^2}$

$\Rightarrow \quad -\sin(\pi + 2\theta) = 2x\sqrt{1-x^2}$ [From (i)]

$\Rightarrow \quad \sin(-\pi - 2\theta) = 2x\sqrt{1-x^2}$

$\Rightarrow \quad -\pi - 2\theta = \sin^{-1}(2x\sqrt{1-x^2})$

$\Rightarrow \quad 2\theta = -\pi - \sin^{-1}(2x\sqrt{1-x^2})$

$\Rightarrow \quad 2\sin^{-1}x = -\pi - \sin^{-1}(2x\sqrt{1-x^2})$

(ii) Let $\sin^{-1}x = \theta$. Then, $x = \sin\theta$

$\therefore \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\Rightarrow \quad \sin 3\theta = 3x - 4x^3$

CASE I *When* $-\frac{1}{2} \le x \le \frac{1}{2}$

We have,

$$-\frac{1}{2} \le x \le \frac{1}{2} \Rightarrow -\frac{1}{2} \le \sin\theta \le \frac{1}{2} \Rightarrow -\frac{\pi}{6} \le \theta \le \frac{\pi}{6} \Rightarrow -\frac{\pi}{2} \le 3\theta \le \frac{\pi}{2}$$

Also, $-\frac{1}{2} \le x \le \frac{1}{2} \Rightarrow -1 \le 3x - 4x^3 \le 1$

$\therefore\ \sin 3\theta = 3x - 4x^3$

$\Rightarrow 3\theta = \sin^{-1}(3x - 4x^3)$

$\Rightarrow 3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$

CASE II *When* $1/2 < x \le 1$:

We have,

$$\frac{1}{2} < x \le 1 \Rightarrow \frac{1}{2} < \sin\theta \le 1 \Rightarrow \frac{\pi}{6} < \theta \le \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < 3\theta \le \frac{3\pi}{2} \Rightarrow -\frac{3\pi}{2} \le -3\theta < -\frac{\pi}{2} \Rightarrow -\frac{\pi}{2} \le \pi - 3\theta < \frac{\pi}{2}$$

Also, $\frac{1}{2} < x \le 1 \Rightarrow -1 \le 3x - 4x^3 \le 1$

$\therefore\ \sin 3\theta = 3x - 4x^3$

$\Rightarrow \sin(\pi - 3\theta) = (3x - 4x^3)$

$\Rightarrow \pi - 3\theta = \sin^{-1}(3x - 4x^3)$

$\Rightarrow \pi - 3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$

$\Rightarrow 3\sin^{-1}x = \pi - \sin^{-1}(3x - 4x^3)$.

CASE III *When* $-1 \le x < -\frac{1}{2}$

We have,

$-1 \le x < -\frac{1}{2}$

$$\Rightarrow -1 \le \sin\theta < -\frac{1}{2} \Rightarrow -\frac{\pi}{2} \le \theta < -\frac{\pi}{6} \Rightarrow -\frac{3\pi}{2} \le 3\theta \le -\frac{\pi}{2} \Rightarrow -\frac{\pi}{2} \le \pi + 3\theta \le \theta \Rightarrow 0 \le -\pi - \theta \le \frac{\pi}{2}$$

Also, $-\frac{1}{2} \le x < -\frac{1}{2} \Rightarrow -1 \le 3x - 4x^3 \le 1$

$\therefore\ \sin 3\theta = 3x - 4x^3$

$\Rightarrow -\sin(\pi + 3\theta) = 3x - 4x^3$ $\quad$ $[\sin(\pi + 3\theta) = -\sin 3\theta]$

$\Rightarrow \sin(-\pi - 3\theta) = 3x - 4x^3$

$\Rightarrow -\pi - 3\theta = \sin^{-1}(3x - 4x^3)$

$\Rightarrow -\pi - 3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$

$\Rightarrow 3\sin^{-1}x = -\pi - \sin^{-1}(3x - 4x^3)$

ILLUSTRATION 1 *Evaluate:*

(i) $\sin(2\sin^{-1} 0.6)$ (ii) $\sin(2\sin^{-1} 0.8)$

SOLUTION (i) $\sin(2\sin^{-1} 0.6)$

$$= \sin\left[\sin^{-1}\left\{2 \times 0.6 \times \sqrt{1-(0.6)^2}\right\}\right] \quad \left[\because 2\sin^{-1} x = \sin^{-1}\{2x\sqrt{1-x^2}\}, \text{ if } -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}\right]$$

$$= \sin(\sin^{-1} 0.96) = 0.96$$

(ii) We have,

$$\sin(2\sin^{-1} 0.8)$$

$$= \sin\left[\pi - \sin^{-1}\{2 \times 0.8 \times \sqrt{1-(0.8)^2}\}\right] \quad \left[\because 2\sin^{-1} x = \pi - \sin^{-1}(2x\sqrt{1-x^2}), \text{ if, } \frac{1}{\sqrt{2}} \le x < 1\right]$$

$$= \sin(\pi - \sin^{-1} 0.96)$$

$$= \sin\{\sin^{-1}(0.96)\} \quad [\because \sin(\pi - \theta) = \sin\theta]$$

$$= 0.96$$

ILLUSTRATION 2 *Evaluate:* $\sin(3\sin^{-1} 0.4)$

SOLUTION $\sin(3\sin^{-1} 0.4)$

$$= \sin[\sin^{-1}\{3 \times 0.4 - 4 \times (0.4)^3\}] \quad [\because 3\sin^{-1} x = \sin^{-1}(3x - 4x^3)]$$

$$= \sin\{\sin^{-1}(1.2 - 0.256)\} = \sin\{\sin^{-1}(0.944)\} = 0.944$$

PROPERTY XI *Prove that*

(i) $$2\cos^{-1} x = \begin{cases} \cos^{-1}(2x^2-1) & \text{, if } 0 \le x \le 1 \\ 2\pi - \cos^{-1}(2x^2-1) & \text{, if } -1 \le x \le 0 \end{cases}.$$

(ii) $$3\cos^{-1} x = \begin{cases} \cos^{-1}(4x^3-3x) & \text{, if } \frac{1}{2} \le x \le 1 \\ 2\pi - \cos^{-1}(4x^3-3x) & \text{, if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 2\pi + \cos^{-1}(4x^3-3x) & \text{, if } -1 \le x \le -\frac{1}{2} \end{cases}$$

PROOF (i) Let $\cos^{-1} x = \theta$. Then, $x = \cos\theta$

$$\therefore \quad \cos 2\theta = 2\cos^2\theta - 1 \Rightarrow \cos 2\theta = 2x^2 - 1$$

CASE I *When* $0 \le x \le 1$

We have,

$$0 \le x \le 1 \Rightarrow 0 \le \cos\theta \le 1 \Rightarrow 0 \le \theta \le \frac{\pi}{2} \Rightarrow 0 \le 2\theta \le \pi$$

Also, $0 \le x \le 1 \Rightarrow -1 \le 2x^2 - 1 \le 1$

$$\therefore \quad \cos 2\theta = 2x^2 - 1$$

$$\Rightarrow \quad 2\theta = \cos^{-1}(2x^2 - 1)$$

$$\Rightarrow \quad 2\cos^{-1} x = \cos^{-1}(2x^2 - 1).$$

CASE II *When* $-1 \le x \le 0$

We have,

$-1 \le x \le 0$

$\Rightarrow -1 \le \cos\theta \le 0 \Rightarrow \frac{\pi}{2} \le \theta \le \pi \Rightarrow \pi \le 2\theta \le 2\pi \Rightarrow -2\pi \le -2\theta \le -\pi \Rightarrow 0 \le 2\pi - 2\theta \le \pi$

Also, $-1 \le x \le 0 \Rightarrow -1 \le 2x^2 - 1 \le 1$

$\therefore \cos 2\theta = (2x^2 - 1)$

$\Rightarrow \cos(2\pi - 2\theta) = (2x^2 - 1)$

$\Rightarrow 2\pi - 2\theta = \cos^{-1}(2x^2 - 1)$

$\Rightarrow 2\theta = 2\pi - \cos^{-1}(2x^2 - 1)$

$\Rightarrow 2\cos^{-1} x = 2\pi - \cos^{-1}(2x^2 - 1).$

(ii) Let $\cos^{-1} x = \theta$. Then, $x = \cos\theta$

$\therefore \cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\Rightarrow \cos 3\theta = 4x^3 - 3x$

<u>CASE I</u> *When* $\frac{1}{2} \le x \le 1$

We have,

$\frac{1}{2} \le x \le 1 \Rightarrow \frac{1}{2} \le \cos\theta \le 1 \Rightarrow 0 \le \theta \le \frac{\pi}{3} \Rightarrow 0 \le 3\theta \le \pi$

Also, $\frac{1}{2} \le x \le 1 \Rightarrow -1 \le 4x^3 - 3x \le 1$

$\therefore \cos 3\theta = 4x^3 - 3x$

$\Rightarrow 3\theta = \cos^{-1}(4x^3 - 3x)$

$\Rightarrow 3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$

<u>CASE II</u> *When* $-\frac{1}{2} \le x \le \frac{1}{2}$

We have,

$-\frac{1}{2} \le x \le \frac{1}{2}$

$\Rightarrow -\frac{1}{2} \le \cos\theta \le \frac{1}{2} \Rightarrow \frac{\pi}{3} \le \theta \le \frac{2\pi}{3} \Rightarrow \pi \le 3\theta \le 2\pi \Rightarrow -2\pi \le -3\theta \le -\pi \Rightarrow 0 \le 2\pi - 3\theta \le \pi$

$\therefore \cos 3\theta = 4x^3 - 3x$

$\Rightarrow \cos(2\pi - 3\theta) = 4x^3 - 3x$

$\Rightarrow 2\pi - 3\theta = \cos^{-1}(4x^3 - 3x)$

$\Rightarrow 3\theta = 2\pi - \cos^{-1}(4x^3 - 3x)$

$\Rightarrow 3\cos^{-1} x = 2\pi - \cos^{-1}(4x^3 - 3x)$

<u>CASE III</u> *When* $-1 \le x \le -\frac{1}{2}$

We have,

$-1 \le x \le -\frac{1}{2}$

$\Rightarrow -1 \le \cos\theta \le -\frac{1}{2}$

$\Rightarrow \frac{2\pi}{3} \le \theta \le \pi \Rightarrow 2\pi \le 3\theta \le 3\pi \Rightarrow -3\pi \le -3\theta \le -2\pi \Rightarrow -\pi \le 2\pi - 3\theta \le 0 \Rightarrow 0 \le 3\theta - 2\pi \le \pi$

$\therefore \cos 3\theta = 4x^3 - 3x$

$\Rightarrow \cos(2\pi - 3\theta) = 4x^3 - 3x$

$\Rightarrow \cos(3\theta - 2\pi) = 4x^3 - 3x$

$\Rightarrow 3\theta - 2\pi = \cos^{-1}(4x^3 - 3x)$

$\Rightarrow 3\theta = 2\pi + \cos^{-1}(4x^3 - 3x)$

$\Rightarrow 3\cos^{-1}x = 2\pi + \cos^{-1}(4x^3 - 3x).$

PROPERTY XII *Prove that:*

$$\text{(i)}\quad 2\tan^{-1}x = \begin{cases} \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) & \text{, if } -1 < x < 1 \\ \pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) & \text{, if } x > 1 \\ -\pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) & \text{, if } x < -1 \end{cases}$$

$$\text{(ii)}\quad 3\tan^{-1}x = \begin{cases} \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right) & \text{, if } -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right) & \text{, if } x > \dfrac{1}{\sqrt{3}} \\ -\pi + \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right) & \text{, if } x < -\dfrac{1}{\sqrt{3}} \end{cases}$$

PROOF (i) Let $\tan^{-1}x = \theta$. Then, $x = \tan\theta$.

$\therefore \tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow \tan 2\theta = \dfrac{2x}{1-x^2}$

CASE I *When* $-1 < x < 1$

We have,

$-1 < x -1 \Rightarrow -1 < \tan\theta < 1 \Rightarrow -\frac{\pi}{2} < \theta < \frac{\pi}{4} \Rightarrow -\frac{\pi}{2} < 2\theta < \frac{\pi}{2}$

$\therefore \tan 2\theta = \dfrac{2x}{1-x^2}$

$\Rightarrow 2\theta = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

$$\Rightarrow \quad 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

CASE II *When $x > 1$*

We have,

$$x > 1$$

$$\Rightarrow \quad \tan\theta > 1$$

$$\Rightarrow \quad \frac{\pi}{2} > \theta > \frac{\pi}{4} \Rightarrow \pi > 2\theta > \frac{\pi}{2} \Rightarrow -\pi < -2\theta < -\frac{\pi}{2} \Rightarrow 0 < \pi - 2\theta < \frac{\pi}{2} \Rightarrow -\frac{\pi}{2} < -\pi + 2\theta < 0$$

$$\therefore \quad \tan 2\theta = \frac{2x}{1-x^2}$$

$$\Rightarrow \quad -\tan(\pi - 2\theta) = \frac{2x}{1-x^2}$$

$$\Rightarrow \quad \tan(-\pi + 2\theta) = \frac{2x}{1-x^2}$$

$$\Rightarrow \quad -\pi + 2\theta = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$\Rightarrow \quad 2\theta = \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$\Rightarrow \quad 2\tan^{-1} x = \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

CASE III *When $x < -1$*

We have,

$$x < -1$$

$$\Rightarrow \quad \tan\theta < -1 \Rightarrow -\frac{\pi}{2} < \theta < -\frac{\pi}{4} \Rightarrow -\pi < 2\theta < -\frac{\pi}{2} \Rightarrow 0 < \pi + 2\theta < \frac{\pi}{2}$$

$$\therefore \quad \tan 2\theta = \frac{2x}{1-x^2}$$

$$\Rightarrow \quad \tan(\pi + 2\theta) = \frac{2x}{1-x^2} \qquad [\because \tan(\pi + \alpha) = \tan\alpha]$$

$$\Rightarrow \quad \pi + 2\theta = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$\Rightarrow \quad \pi + 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$\Rightarrow \quad 2\tan^{-1} x = -\pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

(ii) Let $\tan^{-1} x = \theta$. Then, $x = \tan\theta$.

$$\therefore \quad \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$$\Rightarrow \quad \tan 3\theta = \frac{3x - x^3}{1 - 3x^2}$$

CASE I *When* $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

We have,

$$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \Rightarrow -\frac{1}{\sqrt{3}} < \tan\theta < \frac{1}{\sqrt{3}} \Rightarrow -\frac{\pi}{6} < \theta < \frac{\pi}{6} \Rightarrow -\frac{\pi}{2} < 3\theta < \frac{\pi}{2}$$

$$\therefore \quad \tan 3\theta = \frac{3x - x^3}{1 - 3x^2}$$

$$\Rightarrow \quad 3\theta = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

$$\Rightarrow \quad 3\tan^{-1} x = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

CASE II *When* $x > \frac{1}{\sqrt{3}}$

We have,

$$x > \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \tan\theta > \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \frac{\pi}{2} > \theta > \frac{\pi}{6}$$

$$\Rightarrow \quad \frac{\pi}{2} < 3\theta < \frac{3\pi}{2} \Rightarrow -\frac{3\pi}{2} < -3\theta < -\frac{\pi}{2} \Rightarrow -\frac{\pi}{2} < \pi - 3\theta < \frac{\pi}{2} \Rightarrow -\frac{\pi}{2} < 3\theta - \pi < \frac{\pi}{2}$$

$$\therefore \quad \tan 3\theta = \frac{3x - x^3}{1 - 3x^2}$$

$$\Rightarrow \quad -\tan(\pi - 3\theta) = \frac{3x - x^3}{1 - 3x^2} \qquad [\because \tan(\pi - 3\theta) = -\tan 3\theta)]$$

$$\Rightarrow \quad \tan(3\theta - \pi) = \frac{3x - x^3}{1 - 3x^2}$$

$$\Rightarrow \quad 3\theta - \pi = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

$$\Rightarrow \quad 3\tan^{-1} x - \pi = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

$$\Rightarrow \quad 3\tan^{-1} x = \pi + \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

CASE III *When* $x < -\frac{1}{\sqrt{3}}$

We have,

$$x < -\frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \tan\theta < -\frac{1}{\sqrt{3}} \Rightarrow -\frac{\pi}{2} < \theta < -\frac{\pi}{6} \Rightarrow -\frac{3\pi}{2} < 3\theta < -\frac{\pi}{2} \Rightarrow -\frac{\pi}{2} < \pi + 3\theta < \frac{\pi}{2}$$

$$\therefore \quad \tan 3\theta = \frac{3x - x^3}{1 - 3x^2}$$

$$\Rightarrow \quad \tan(\pi + 3\theta) = \frac{3x - x^3}{1 - 3x^2} \qquad [\because \tan(\pi + x) = \tan x]$$

$$\Rightarrow \quad \pi + 3\theta = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

$$\Rightarrow \quad \pi + 3\tan^{-1} x = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

$$\Rightarrow \quad 3\tan^{-1} x = -\pi + \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

ILLUSTRATION 1 *Prove that:*

$$\tan^{-1} x + \tan^{-1}\left(\frac{2x}{1 - x^2}\right) = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right), |x| < \frac{1}{\sqrt{3}}$$ **[NCERT, CBSE 2010]**

SOLUTION We have,

$$\tan^{-1} x + \tan^{-1}\left(\frac{2x}{1 - x^2}\right)$$

$$= \tan^{-1}\left\{\frac{x + \frac{2x}{1 - x^2}}{1 - \frac{2x^2}{1 - x^2}}\right\} \qquad \left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right), \text{ if } xy < 1\right]$$

$$= \tan^{-1}\left(\frac{x - x^3 + 2x}{1 - x^2 - 2x^2}\right) = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

ILLUSTRATION 2 *Prove that:* $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

SOLUTION We have,

$$2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left\{\frac{2 \times \frac{1}{2}}{1 - \left(\frac{1}{2}\right)^2}\right\} + \tan^{-1}\frac{1}{7} \qquad \left[\because 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1 - x^2}\right), \text{ if } -1 < x < 1\right]$$

$$= \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{17} = \tan^{-1}\left\{\frac{\frac{4}{3} + \frac{1}{7}}{1 - \frac{4}{3} \times \frac{1}{7}}\right\} = \tan^{-1}\frac{31}{17}$$

ILLUSTRATION 3 *Evaluate:* $\tan\left(2\tan^{-1}\frac{1}{5}\right)$ **[CBSE 2013]**

SOLUTION We have,

$$\tan\left(2\tan^{-1}\frac{1}{5}\right) = \tan\left\{\tan^{-1}\left(\frac{2 \times \frac{1}{5}}{1 - \frac{1}{25}}\right)\right\} = \tan\left(\tan^{-1}\frac{5}{12}\right) = \frac{5}{12}$$

PROPERTY XIII *Prove that:*

$$\text{(i) } 2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) & , \text{ if } -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) & , \text{ if } x > 1 \\ -\pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) & , \text{ if } x < -1 \end{cases}$$

$$\text{(ii) } 2\tan^{-1}x = \begin{cases} \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) & , \text{ if } 0 \le x < \infty \\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) & , \text{ if } -\infty < x \le 0 \end{cases}$$

PROOF (i) Let $\tan^{-1}x = \theta$. Then, $x = \tan\theta$

$$\therefore \quad \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} \Rightarrow \sin 2\theta = \frac{2x}{1+x^2}$$

CASE I *When* $-1 \le x \le 1$

We have,

$$-1 \le x \le 1 \Rightarrow -1 \le \tan\theta \le 1 \Rightarrow -\frac{\pi}{4} \le \theta \le \frac{\pi}{4} \Rightarrow -\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2}$$

$$\therefore \quad \sin 2\theta = \frac{2x}{1+x^2}$$

$$\Rightarrow \quad 2\theta = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \quad 2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

CASE II *When* $x > 1$

We have,

$$x > 1 \Rightarrow \tan\theta > 1 \Rightarrow \frac{\pi}{4} < \theta < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < 2\theta < \pi \Rightarrow -\pi < -2\theta < -\frac{\pi}{2} \Rightarrow 0 < \pi - 2\theta < \frac{\pi}{2}$$

$$\therefore \quad \sin 2\theta = \frac{2x}{1+x^2}$$

$$\Rightarrow \quad \sin(\pi - 2\theta) = \frac{2x}{1+x^2}$$

$$\Rightarrow \quad \pi - 2\theta = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \quad \pi - 2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \quad 2\tan^{-1}x = \pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

CASE III When $x < -1$

We have,

$$x < -1$$

$$\Rightarrow \quad \tan\theta < -1$$

$$\Rightarrow \quad -\frac{\pi}{2} < \theta < -\frac{\pi}{4} \Rightarrow -\pi < 2\theta < -\frac{\pi}{2} \Rightarrow 0 < \pi + 2\theta < \frac{\pi}{2} \Rightarrow -\frac{\pi}{2} < -\pi - 2\theta < 0$$

$$\therefore \quad \sin 2\theta = \frac{2x}{1+x^2}$$

$$\Rightarrow \quad -\sin(\pi + 2\theta) = \frac{2x}{1+x^2}$$

$$\Rightarrow \quad \sin(-\pi - 2\theta) = \frac{2x}{1+x^2}$$

$$\Rightarrow \quad -\pi - 2\theta = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \quad -\pi - 2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \quad 2\tan^{-1}x = -\pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

(ii) Let $\tan^{-1}x = \theta$. Then, $x = \tan\theta$.

$$\therefore \quad \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} \Rightarrow \cos 2\theta = \frac{1-x^2}{1+x^2}$$

CASE I *When* $0 \le x < \infty$

We have,

$$0 \le x < \infty \Rightarrow 0 \le \tan\theta < \infty \Rightarrow 0 \le \theta < \frac{\pi}{2} \Rightarrow 0 \le 2\theta < \pi$$

$$\therefore \quad \cos 2\theta = \frac{1-x^2}{1+x^2}$$

$$\Rightarrow \quad 2\theta = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \quad 2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right).$$

CASE II *When* $-\infty < x \le 0$

We have,

$$-\infty < x \le 0 \Rightarrow -\infty < \tan\theta \le 0 \Rightarrow -\frac{\pi}{2} < \theta \le 0 \Rightarrow -\pi < 2\theta \le 0 \Rightarrow 0 \le -2\theta < \pi$$

$$\therefore \quad \cos 2\theta = \frac{1-x^2}{1+x^2}$$

$$\Rightarrow \quad \cos(-2\theta) = \frac{1-x^2}{1+x^2}$$

$$\Rightarrow \quad -2\theta = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \quad -2\tan^{-1} x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \quad 2\tan^{-1} x = -\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

ILLUSTRATION 1 *Prove that:*

$$\tan\frac{1}{2}\left\{\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right\}=\frac{x+y}{1-xy}, \text{ if } |x|<1,\ y>0 \text{ and } xy<1.$$

[NCERT, CBSE 2013]

SOLUTION We have,

$$\tan\frac{1}{2}\left\{\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right\}$$

$$=\tan\frac{1}{2}\{2\tan^{-1}x+2\tan^{-1}y\} \quad \left[\because \sin^{-1}\frac{2x}{1+x^2}=2\tan^{-1}x \text{ for all } x\in[-1,1] \text{ and, } \cos^{-1}\frac{1-y^2}{1+y^2}=2\tan^{-1}y \text{ for all } y\geq 0\right]$$

$$=\tan(\tan^{-1}x+\tan^{-1}y)$$

$$=\tan\left\{\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right\} \quad [\because xy<1]$$

$$=\frac{x+y}{1-xy}$$

ILLUSTRATION 2 *Prove that:*

$$\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x\in[0,1]$$

[NCERT, CBSE 2010]

SOLUTION We have,

$$\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\cos^{-1}\left\{\frac{1-(\sqrt{x})^2}{1+(\sqrt{x})^2}\right\}=\frac{1}{2}\times 2\tan^{-1}\sqrt{x}=\tan^{-1}\sqrt{x}.$$

ALITER Putting $x=\tan^2\theta$, we have

$$\text{RHS}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\frac{1}{2}\cos^{-1}(\cos 2\theta)=\theta=\tan^{-1}\sqrt{x}=\text{LHS}$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If* $x, y, z\in[-1,1]$ *such that* $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$, *find the value of*

$$x^{2014}+y^{2015}+z^{2016}-\frac{9}{x^{2014}+y^{2015}+z^{2016}}$$

SOLUTION We know that: $-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$ for all $x \in [-1, 1]$. Therefore, the maximum and minimum values of $\sin^{-1} x$ are $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ respectively.

Now,

$$\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$$

$$\Rightarrow \quad \sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2}$$

$$\Rightarrow \quad \sin^{-1} x = \frac{\pi}{2}, \sin^{-1} y = \frac{\pi}{2}, \sin^{-1} z = \frac{\pi}{2}$$

$$\Rightarrow \quad x = 1, y = 1, z = 1$$

$$\therefore \quad x^{2014} + y^{2015} + z^{2016} - \frac{9}{x^{2014} + y^{2015} + z^{2016}} = 1 + 1 + 1 - \frac{9}{1 + 1 + 1} = 3 - 3 = 0$$

EXAMPLE 2 *Evaluate the following:*

(i) $\sin^{-1}(\sin 10)$ (ii) $\sin^{-1}(\sin 5)$ (iii) $\cos^{-1}(\cos 10)$ (iv) $\tan^{-1}\{\tan(-6)\}$

SOLUTION (i) We know that $\sin^{-1}(\sin\theta) = \theta$, if $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Here, $\theta = 10$ radians which does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. But, $3\pi - \theta$ i.e. $3\pi - 10$ lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Also, $\sin(3\pi - 10) = \sin 10$.

$$\therefore \quad \sin^{-1}(\sin 10) = \sin^{-1}(\sin(3\pi - 10)) = 3\pi - 10.$$

(ii) Here, $\theta = 5$ radians. Clearly, it does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. But, $2\pi - 5$ and $5 - 2\pi$ both lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ such that

$$\sin(5 - 2\pi) = \sin(-(2\pi - 5)) = -\sin(2\pi - 5) = -(-\sin 5) = \sin 5$$

$$\therefore \quad \sin^{-1}(\sin 5) = \sin^{-1}(\sin(5 - 2\pi)) = 5 - 2\pi.$$

(iii) We know that $\cos^{-1}(\cos\theta) = \theta$, if $0 \le \theta \le \pi$. Here, $\theta = 10$ radians.

Clearly, it does not lie between 0 and π. However, $(4\pi - 10)$ lies between 0 and π such that $\cos(4\pi - 10) = \cos 10$.

$$\therefore \cos^{-1}(\cos 10) = \cos^{-1}(\cos(4\pi - 10)) = 4\pi - 10$$

(iv) We know that $\tan^{-1}(\tan\theta) = \theta$, if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Here, $\theta = -6$ radians which does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. We find that $2\pi - 6$ lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ such that

$$\tan(2\pi - 6) = -\tan 6 = \tan(-6)$$

$$\tan^{-1}\{\tan(-6)\} = \tan^{-1}\{\tan(2\pi - 6)\} = 2\pi - 6$$

EXAMPLE 3 *Prove that:*

(i) $\tan^{-1}\left\{\dfrac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right\}=\dfrac{\pi}{4}+\dfrac{x}{2}, 0<x<\dfrac{\pi}{2}$ **[NCERT, CBSE 2009]**

(ii) $\cot^{-1}\left\{\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}=\dfrac{x}{2}, 0<x<\dfrac{\pi}{2}$

SOLUTION (i) We have,

$$\tan^{-1}\left\{\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right\}$$

$$=\tan^{-1}\left\{\frac{\sqrt{2\cos^2\frac{x}{2}}+\sqrt{2\sin^2\frac{x}{2}}}{\sqrt{2\cos^2\frac{x}{2}}-\sqrt{2\sin^2\frac{x}{2}}}\right\}$$

$$=\tan^{-1}\left\{\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right\} \qquad \left[\because 0<\frac{x}{2}<\frac{\pi}{4} \therefore \cos\frac{x}{2}>0, \sin\frac{x}{2}>0\right]$$

$$=\tan^{-1}\left\{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right\}=\tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}=\frac{\pi}{4}+\frac{x}{2} \quad \left[\because 0<x<\frac{\pi}{2} \therefore \frac{\pi}{4}<\frac{\pi}{4}+\frac{x}{2}<\frac{\pi}{2}\right]$$

(ii) We know that

$$1\pm\sin x=\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\pm 2\sin\frac{x}{2}\cos\frac{x}{2}=\left(\cos\frac{x}{2}\pm\sin\frac{x}{2}\right)^2$$

$$\therefore \quad \cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}$$

$$=\cot^{-1}\left\{\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right\}$$

$$=\cot^{-1}\left\{\frac{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|+\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|-\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}\right\} \qquad \left[\because \sqrt{x^2}=|x|\right]$$

$$=\cot^{-1}\left\{\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right\} \qquad \left[\because 0<\frac{x}{2}<\frac{\pi}{4} \therefore \cos\frac{x}{2}>\sin\frac{x}{2}\right]$$

$$=\cot^{-1}\left(\cot\frac{x}{2}\right)=\frac{x}{2} \qquad \left[\because 0<\frac{x}{2}<\frac{\pi}{4}\right]$$

EXAMPLE 4 *Prove that:*

(i) $\tan^{-1}\left\{\dfrac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right\}=\dfrac{\pi}{4}-\dfrac{x}{2}$, if $\pi < x < \dfrac{3\pi}{2}$

(ii) $\cot^{-1}\left\{\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}=\dfrac{\pi}{2}-\dfrac{x}{2}$, if $\dfrac{\pi}{2} < x < \pi$ **[CBSE 2011]**

SOLUTION (i) We have,

$$\tan^{-1}\left\{\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right\}$$

$$=\tan^{-1}\left\{\frac{\sqrt{2\cos^2\frac{x}{2}}+\sqrt{2\sin^2\frac{x}{2}}}{\sqrt{2\cos^2\frac{x}{2}}-\sqrt{2\sin^2\frac{x}{2}}}\right\}$$

$$=\tan^{-1}\left\{\frac{\sqrt{2}\left|\cos\frac{x}{2}\right|+\sqrt{2}\left|\sin\frac{x}{2}\right|}{\sqrt{2}\left|\cos\frac{x}{2}\right|-\sqrt{2}\left|\sin\frac{x}{2}\right|}\right\}$$

$$=\tan^{-1}\left\{\frac{-\cos\frac{x}{2}+\sin\frac{x}{2}}{-\cos\frac{x}{2}-\sin\frac{x}{2}}\right\} \qquad \left[\begin{array}{l}\because\ \pi < x < \frac{3\pi}{2}\ \therefore \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}\\ \therefore\ \left|\cos\frac{x}{2}\right| = -\cos\frac{x}{2},\ \left|\sin\frac{x}{2}\right| = \sin\frac{x}{2}\end{array}\right]$$

$$=\tan^{-1}\left\{\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\right\}$$

$$=\tan^{-1}\left\{\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right\}=\tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}=\frac{\pi}{4}-\frac{x}{2} \qquad \left[\begin{array}{l}\because\ \pi < x < \frac{3\pi}{2}\\ \therefore\ -\frac{\pi}{2} < \frac{\pi}{4}-\frac{x}{2} < -\frac{\pi}{4}\end{array}\right]$$

(ii) We have,

$$\cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}$$

$$=\cot^{-1}\left\{\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right\}$$

$$=\cot^{-1}\left\{\frac{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|+\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|-\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}\right\}$$

$$= \cot^{-1}\left\{\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right\} \quad \left[\because \frac{\pi}{2}<x<\pi \Rightarrow \frac{\pi}{4}<\frac{x}{2}<\frac{\pi}{2} \Rightarrow \cos\frac{x}{2}<\sin\frac{x}{2}\right]$$

$$= \cot^{-1}\left(\tan\frac{x}{2}\right) = \cot^{-1}\left\{\cot\left(\frac{\pi}{2}-\frac{x}{2}\right)\right\} = \frac{\pi}{2}-\frac{x}{2} \quad \left[\because \frac{\pi}{2}<x<\pi \Rightarrow 0<\frac{\pi}{2}-\frac{x}{2}<\frac{\pi}{4}\right]$$

EXAMPLE 5 *Prove that:*

(i) $\tan^{-1}\left\{\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right\} = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x,\ 0<x<1$ **[NCERT, CBSE 2010, CBSE 2011]**

(ii) $\tan^{-1}\left\{\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right\} = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2,\ -1<x<1$

SOLUTION (i) Putting $x=\cos 2\theta$, we have

$$\tan^{-1}\left\{\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right\} \quad \left[\because 0<x<1 \Rightarrow 0<\cos 2\theta<1 \Rightarrow 0<2\theta<\frac{\pi}{2} \Rightarrow 0<\theta<\frac{\pi}{4}\right]$$

$$= \tan^{-1}\left\{\frac{1-\tan\theta}{1+\tan\theta}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\theta\right)\right\}$$

$$= \frac{\pi}{4}-\theta \quad \left[\because 0<\theta<\frac{\pi}{4} \Rightarrow 0<\frac{\pi}{4}-\theta<\frac{\pi}{4}\right]$$

$$= \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x \quad \left[\because \cos 2\theta = x \therefore 2\theta=\cos^{-1}x \Rightarrow \theta=\frac{1}{2}\cos^{-1}x\right]$$

(ii) Putting $x^2=\cos 2\theta$, we have

$$\tan^{-1}\left\{\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{1+\tan\theta}{1-\tan\theta}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\theta\right)\right\}$$

$$= \frac{\pi}{4}+\theta \qquad \left[\because -1 < x < 1 \Rightarrow 0 < x^2 < 1 \Rightarrow 0 < 2\theta < \frac{\pi}{2} \Rightarrow 0 < \theta < \frac{\pi}{4}\right]$$

$$= \frac{\pi}{4}+\frac{1}{2}\cos^{-1} x^2 \qquad [\because x^2 = \cos 2\theta \Rightarrow 2\theta = \cos^{-1} x^2]$$

EXAMPLE 6 *Simplify each of the following:*

(i) $\cos^{-1}\left(\frac{3}{5}\cos x+\frac{4}{5}\sin x\right)$ (ii) $\sin^{-1}\left(\frac{5}{13}\cos x+\frac{12}{13}\sin x\right)$

SOLUTION (i) In order to simplify $\cos^{-1}\left(\frac{3}{5}\cos x+\frac{4}{5}\sin x\right)$, we will have to express $\frac{3}{5}\cos x+\frac{4}{5}\sin x$ in the form of cosine of some expression. For this, let

$$\frac{3}{5}=r\cos\theta \text{ and } \frac{4}{5}=r\sin\theta$$

$$\Rightarrow \quad r=\sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}=1 \text{ and } \tan\theta=\frac{r\sin\theta}{r\cos\theta}=\frac{4}{3}$$

$$\Rightarrow \quad r=1 \text{ and } \theta=\tan^{-1}\frac{4}{3}$$

$$\Rightarrow \quad \frac{3}{5}=\cos\theta \text{ and } \frac{4}{5}=\sin\theta, \text{ where } \theta=\tan^{-1}\frac{4}{3}$$

$$\therefore \quad \cos^{-1}\left(\frac{3}{5}\cos x+\frac{4}{5}\sin x\right)$$

$$= \cos^{-1}(\cos\theta\cos x+\sin\theta\sin x) = \cos^{-1}\{\cos(x-\theta)\} = x-\theta = x-\tan^{-1}\frac{4}{3}.$$

(ii) In order to simplify $\sin^{-1}\left(\frac{5}{13}\cos x+\frac{12}{13}\sin x\right)$, we will have to express $\frac{5}{13}\cos x+\frac{12}{13}\sin x$ in the form of sine of some expression. For this, let

$$\frac{5}{13}=r\sin\theta \text{ and } \frac{12}{13}=r\cos\theta$$

$$\Rightarrow \quad r=\sqrt{\left(\frac{5}{13}\right)^2+\left(\frac{12}{13}\right)^2}=1 \text{ and } \tan\theta=\frac{r\sin\theta}{r\cos\theta}=\frac{5}{12}$$

$$\Rightarrow \quad r=1 \text{ and } \theta=\tan^{-1}\frac{5}{12}$$

$$\Rightarrow \quad \frac{5}{13}=\sin\theta \text{ and } \frac{12}{13}=\cos\theta, \text{ where } \theta=\tan^{-1}\frac{5}{12}$$

$$\therefore \quad \sin^{-1}\left(\frac{5}{13}\cos x+\frac{12}{13}\sin x\right)$$

$$= \sin^{-1}(\sin\theta\cos x+\cos\theta\sin x) = \sin^{-1}\{\sin(x+\theta)\} = x+\theta = x+\tan^{-1}\frac{5}{12}$$

EXAMPLE 7 *Simplify each of the following:*

(i) $\sin^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right), -\frac{\pi}{4}<x<\frac{\pi}{4}$ (ii) $\cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right), \frac{\pi}{4}<x<\frac{5\pi}{4}$

SOLUTION (i) $\sin^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right)$

$$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)$$

$$= \sin^{-1}\left(\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}\right)$$

$$= \sin^{-1}\left\{\sin\left(x + \frac{\pi}{4}\right)\right\} = x + \frac{\pi}{4} \qquad \left[\because -\frac{\pi}{4} < x < \frac{\pi}{4} \Rightarrow 0 < x + \frac{\pi}{4} < \frac{\pi}{2}\right]$$

(ii) $\cos^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right)$

$$= \cos^{-1}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)$$

$$= \cos^{-1}\left(\sin x \sin\frac{\pi}{4} + \cos x \cos\frac{\pi}{4}\right)$$

$$= \cos^{-1}\left\{\cos\left(x - \frac{\pi}{4}\right)\right\} = x - \frac{\pi}{4} \qquad \left[\because \frac{\pi}{4} < x < \frac{5\pi}{4} \Rightarrow 0 < x - \frac{\pi}{4} < \pi\right]$$

REMARK *This example can be solved by using the procedure given in the earlier example.*

EXAMPLE 8 *Prove that:* $\sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3) = 15$

SOLUTION We have,

$$\sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3)$$

$$= \{\sec(\tan^{-1} 2)\}^2 + \{\operatorname{cosec}(\cot^{-1} 3)\}^2$$

$$= \left\{\sec\left(\tan^{-1}\frac{2}{1}\right)\right\}^2 + \left\{\operatorname{cosec}\left(\cot^{-1}\frac{3}{1}\right)\right\}^2$$

$$= \{\sec(\sec^{-1}\sqrt{5})\}^2 + \{\operatorname{cosec}(\operatorname{cosec}^{-1}\sqrt{10})\}^2 = (\sqrt{5})^2 + (\sqrt{10})^2 = 15$$

EXAMPLE 9 *Prove that:*

(i) $\sin[\cot^{-1}\{\cos(\tan^{-1} x)\}] = \sqrt{\frac{x^2+1}{x^2+2}}$

(ii) $\cos[\tan^{-1}\{\sin(\cot^{-1} x)\}] = \sqrt{\frac{x^2+1}{x^2+2}}$ **[CBSE 2010]**

SOLUTION (i) We have,

$$\cos(\tan^{-1} x) = \cos\left\{\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right\} = \frac{1}{\sqrt{1+x^2}} \qquad \left[\because \tan^{-1} x = \cos^{-1}\frac{1}{\sqrt{1+x^2}}\right]$$

$\therefore$ $\sin[\cot^{-1}\{\cos(\tan^{-1} x)\}]$

$$= \sin\left\{\cot^{-1}\frac{1}{\sqrt{1+x^2}}\right\} = \sin\left\{\sin^{-1}\frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}\right\} = \frac{\sqrt{1+x^2}}{\sqrt{2+x^2}} = \sqrt{\frac{x^2+1}{x^2+2}}$$

(ii) We have,

$$\sin(\cot^{-1} x) = \sin\left\{\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right\} = \frac{1}{\sqrt{1+x^2}}$$

$\cos[\tan^{-1}\{\sin(\cot^{-1}x)\}]$

$$\therefore \quad = \cos\left\{\tan^{-1}\frac{1}{\sqrt{1+x^2}}\right\} = \cos\left\{\cos^{-1}\frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}\right\} = \sqrt{\frac{1+x^2}{2+x^2}} = \sqrt{\frac{x^2+1}{x^2+2}}$$

EXAMPLE 10 *If* $x = \operatorname{cosec}\left[\tan^{-1}\left\{\cos\left(\cot^{-1}\left(\sec\left(\sin^{-1}a\right)\right)\right)\right\}\right]$

and, $y = \sec\left[\cot^{-1}\left\{\sin\left(\tan^{-1}\left(\operatorname{cosec}\left(\cos^{-1}a\right)\right)\right)\right\}\right]$

where $a \in [0, 1]$*. Find the relationship between x and y in terms of a.*

SOLUTION We have,

$$x = \operatorname{cosec}\left[\tan^{-1}\left\{\cos\left(\cot^{-1}\left(\sec\left(\sin^{-1}a\right)\right)\right)\right\}\right]$$

$$= \operatorname{cosec}\left[\tan^{-1}\left\{\cos\left(\cot^{-1}\left(\sec\left(\sec^{-1}\frac{1}{\sqrt{1-a^2}}\right)\right)\right)\right\}\right] \quad \left[\because \sin^{-1}a = \sec^{-1}\frac{1}{\sqrt{1-a^2}}\right]$$

$$= \operatorname{cosec}\left[\tan^{-1}\left\{\cos\left(\cot^{-1}\frac{1}{\sqrt{1-a^2}}\right)\right\}\right]$$

$$= \operatorname{cosec}\left[\tan^{-1}\left\{\cos\left(\cos^{-1}\frac{1}{\sqrt{2-a^2}}\right)\right\}\right] \quad \left[\because \cot^{-1}\frac{1}{\sqrt{1-a^2}} = \cos^{-1}\frac{1}{\sqrt{2-a^2}}\right]$$

$$= \operatorname{cosec}\left(\tan^{-1}\frac{1}{\sqrt{2-a^2}}\right) = \operatorname{cosec}\left(\operatorname{cosec}^{-1}\sqrt{3-a^2}\right) = \sqrt{3-a^2}$$

and, $$y = \sec\left[\cot^{-1}\left\{\sin\left(\tan^{-1}\left(\operatorname{cosec}\left(\cos^{-1}a\right)\right)\right)\right\}\right]$$

$$= \sec\left[\cot^{-1}\left\{\sin\left(\tan^{-1}\left(\operatorname{cosec}\left(\operatorname{cosec}^{-1}\frac{1}{\sqrt{1-a^2}}\right)\right)\right)\right\}\right] \quad \left[\because \cos^{-1}a = \operatorname{cosec}^{-1}\frac{1}{\sqrt{1-a^2}}\right]$$

$$= \sec\left[\cot^{-1}\left\{\sin\left(\tan^{-1}\frac{1}{\sqrt{1-a^2}}\right)\right\}\right]$$

$$= \sec\left[\cot^{-1}\left\{\sin\left(\sin^{-1}\frac{1}{\sqrt{2-a^2}}\right)\right\}\right] \quad \left[\because \tan^{-1}\frac{1}{\sqrt{1-a^2}} = \sin^{-1}\frac{1}{\sqrt{2-a^2}}\right]$$

$$= \sec\left(\cot^{-1}\frac{1}{\sqrt{2-a^2}}\right) = \sec\left(\sec^{-1}\sqrt{3-a^2}\right) = \sqrt{3-a^2}$$

$$\therefore \quad x^2 = y^2 = 3 - a^2.$$

EXAMPLE 11 *Simplify each of the following:*

(i) $\tan^{-1}\left(\frac{a+bx}{b-ax}\right), x < \frac{b}{a}$ **[NCERT]**

(ii) $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right), -\frac{\pi}{2} < x < \frac{\pi}{2}, \frac{a}{b}\tan x > -1$ **[NCERT]**

(iii) $\tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right), -\frac{1}{\sqrt{3}} < \frac{x}{a} < \frac{1}{\sqrt{3}}$

SOLUTION (i) $\tan^{-1}\left(\frac{a+bx}{b-ax}\right) = \tan^{-1}\left(\frac{\frac{a}{b}+x}{1-\frac{a}{b}x}\right) = \tan^{-1}\frac{a}{b} + \tan^{-1}x$

(ii) $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right) = \tan^{-1}\left(\frac{\frac{a}{b} - \tan x}{1 + \frac{a}{b}\tan x}\right) = \tan^{-1}\frac{a}{b} - \tan^{-1}(\tan x) = \tan^{-1}\frac{a}{b} - x$

(iii) $\tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right)$

$$= \tan^{-1}\left\{\frac{3\left(\frac{x}{a}\right) - \left(\frac{x}{a}\right)^3}{1 - 3\left(\frac{x}{a}\right)^2}\right\} = 3\tan^{-1}\frac{x}{a} \qquad \left[\because \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) = 3\tan^{-1}x\right]$$

EXAMPLE 12 *Prove that:*

(i) $\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} = \tan^{-1}\frac{2}{9}$

(ii) $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$ **[CBSE 2011, 2013]**

(iii) $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{8}{19} = \frac{\pi}{4}$ **[CBSE 2012]**

(iv) $\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$ **[NCERT, CBSE 2008, 2010]**

SOLUTION (i) LHS $= \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13}$

$$= \tan^{-1}\left\{\frac{\frac{1}{7}+\frac{1}{13}}{1 - \frac{1}{7}\times\frac{1}{13}}\right\} \qquad \left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}, \text{ if } xy < 1\right]$$

$$= \tan^{-1}\frac{20}{90} = \tan^{-1}\frac{2}{9} = \text{R.H.S.}$$

(ii) LHS $= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}$

$$= \left\{\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5}\right\} + \tan^{-1}\frac{1}{8}$$

$$= \tan^{-1}\left\{\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\times\frac{1}{5}}\right\} + \tan^{-1}\frac{1}{8} \qquad \left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ if } xy<1\right]$$

$$= \tan^{-1}\frac{7}{9} + \tan^{-1}\frac{1}{8} = \tan^{-1}\left\{\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}\times\frac{1}{8}}\right\} = \tan^{-1}\left(\frac{65}{65}\right) = \tan^{-1}1 = \frac{\pi}{4} = \text{R.H.S.}$$

(iii) $$\text{LHS} = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{8}{19}$$

$$\Rightarrow \quad \text{LHS} = \left\{\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5}\right\} - \tan^{-1}\frac{8}{19}$$

$$\Rightarrow \quad \text{LHS} = \tan^{-1}\left\{\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\times\frac{3}{5}}\right\} - \tan^{-1}\frac{8}{19}$$

$$\Rightarrow \quad \text{LHS} = \tan^{-1}\frac{27}{11} - \tan^{-1}\frac{8}{19}$$

$$\Rightarrow \quad \text{LHS} = \tan^{-1}\left\{\frac{\frac{27}{11}-\frac{8}{19}}{1+\frac{27}{11}\times\frac{8}{19}}\right\} = \tan^{-1}\frac{425}{425} = \tan^{-1}1 = \frac{\pi}{4} = \text{R.H.S.}$$

(iv) $$\text{LHS} = \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8}$$

$$\Rightarrow \quad \text{LHS} = \left(\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7}\right) + \left(\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8}\right)$$

$$\Rightarrow \quad \text{LHS} = \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right)$$

$$\Rightarrow \quad \text{LHS} = \tan^{-1}\frac{6}{17} + \tan^{-1}\frac{11}{23}$$

$$\Rightarrow \quad \text{LHS} = \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right) = \tan^{-1}\left(\frac{325}{325}\right) = \tan^{-1}1 = \frac{\pi}{4} = \text{R.H.S.}$$

EXAMPLE 13 *Prove that:*

(i) $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17} = \sin^{-1}\frac{77}{85}$ **[CBSE 2009]**

(ii) $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = \frac{\pi}{2}$ **[CBSE 2010]**

(iii) $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{36}{85}$ **[CBSE 2010]**

(iv) $\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65} = \sin^{-1}\frac{56}{65}$

SOLUTION Using $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right\}$, we obtain

(i) $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17}$

$$= \sin^{-1}\left\{\frac{3}{5}\sqrt{1-\left(\frac{8}{17}\right)^2} + \frac{8}{17}\sqrt{1-\left(\frac{3}{5}\right)^2}\right\} = \sin^{-1}\left\{\frac{3}{5}\times\frac{15}{17} + \frac{8}{17}\times\frac{4}{5}\right\} = \sin^{-1}\frac{77}{85}$$

(ii) $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65}$

$$= \left\{\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13}\right\} + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left\{\frac{4}{5}\sqrt{1-\left(\frac{5}{13}\right)^2} + \frac{5}{13}\sqrt{1-\left(\frac{4}{5}\right)^2}\right\} + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left\{\frac{4}{5}\times\frac{12}{13} + \frac{5}{13}\times\frac{3}{5}\right\} + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\frac{63}{65} + \sin^{-1}\frac{16}{65}$$

$$= \cos^{-1}\frac{16}{65} + \sin^{-1}\frac{16}{65} \qquad \left[\because \sin^{-1}\frac{63}{65} = \cos^{-1}\sqrt{1-\left(\frac{63}{65}\right)^2} = \cos^{-1}\frac{16}{65}\right]$$

$$= \frac{\pi}{2} \qquad \left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\right]$$

(iii) $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17}$

$$= \sin^{-1}\left\{\frac{3}{5}\sqrt{1-\left(\frac{8}{17}\right)^2} + \frac{8}{17}\sqrt{1-\left(\frac{3}{5}\right)^2}\right\} = \sin^{-1}\left\{\frac{3}{5}\times\frac{15}{17} + \frac{8}{17}\times\frac{4}{5}\right\}$$

$$= \sin^{-1}\frac{77}{85} = \cos^{-1}\sqrt{1-\left(\frac{77}{85}\right)^2} = \cos^{-1}\frac{36}{85} \qquad \left[\because \sin^{-1} x = \cos^{-1}\sqrt{1-x^2}\right]$$

(iv) We have,

$$\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{12}{13}$$

$$= \sin^{-1}\frac{3}{5} + \sin^{-1}\frac{5}{13} \qquad \left[\because \cos^{-1} x = \sin^{-1}\sqrt{1-x^2}\right]$$

$$= \sin^{-1}\left\{\frac{3}{5}\times\sqrt{1-\left(\frac{5}{13}\right)^2} + \frac{5}{13}\times\sqrt{1-\left(\frac{3}{5}\right)^2}\right\}$$

$$= \sin^{-1}\left\{\frac{3}{5}\times\frac{12}{13} + \frac{5}{13}\times\frac{4}{5}\right\} = \sin^{-1}\frac{56}{65} = \cos^{-1}\sqrt{1-\left(\frac{56}{65}\right)^2} = \cos^{-1}\frac{33}{65}$$

EXAMPLE 14 *Prove the following:*

(i) $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} = \frac{\pi}{4}$ (ii) $2\tan^{-1}\frac{1}{5} + \sec^{-1}\frac{5\sqrt{2}}{7} + 2\tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

SOLUTION (i) We have,

$$4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$$

$$= 2\left\{2\tan^{-1}\frac{1}{5}\right\} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$$

$$= 2\left\{\tan^{-1}\frac{2\times 1/5}{1-(1/5)^2}\right\} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} \qquad \left[\because\ 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2},\ \text{if } |x|<1\right]$$

$$= 2\tan^{-1}\frac{5}{12} - \left\{\tan^{-1}\frac{1}{70} - \tan^{-1}\frac{1}{99}\right\}$$

$$= \tan^{-1}\left\{\frac{2\times 5/12}{1-(5/12)^2}\right\} - \tan^{-1}\left\{\frac{\frac{1}{70}-\frac{1}{99}}{1+\frac{1}{70}\times\frac{1}{99}}\right\}$$

$$= \tan^{-1}\frac{120}{119} - \tan^{-1}\frac{29}{6931}$$

$$= \tan^{-1}\frac{120}{119} - \tan^{-1}\frac{1}{239} = \tan^{-1}\left\{\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\times\frac{1}{239}}\right\} = \tan^{-1}1 = \frac{\pi}{4}$$

(ii) $$2\tan^{-1}\frac{1}{5} + \sec^{-1}\frac{5\sqrt{2}}{7} + 2\tan^{-1}\frac{1}{8}$$

$$= 2\left\{\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}\right\} + \sec^{-1}\frac{5\sqrt{2}}{7}$$

$$= 2\tan^{-1}\left\{\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\right\} + \tan^{-1}\sqrt{\left(\frac{5\sqrt{2}}{7}\right)^2 - 1} \qquad [\because\ \sec^{-1}x = \tan^{-1}\sqrt{x^2-1}]$$

$$= 2\tan^{-1}\frac{13}{39} + \tan^{-1}\frac{1}{7}$$

$$= 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left\{\frac{2\times 1/3}{1-(1/3)^2}\right\} + \tan^{-1}\frac{1}{7} \qquad \left[\because\ 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2},\ \text{if } |x|<1\right]$$

$$= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7} = \tan^{-1}\left\{\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right\} = \tan^{-1}1 = \frac{\pi}{4}$$

EXAMPLE 15 *If $a>b>c>0$, prove that*

$$\cot^{-1}\left(\frac{ab+1}{a-b}\right) + \cot^{-1}\left(\frac{bc+1}{b-c}\right) + \cot^{-1}\left(\frac{ca+1}{c-a}\right) = \pi$$

SOLUTION We know that

$$\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases}\cot^{-1}x & \text{, for } x>0\\ -\pi + \cot^{-1}x, & \text{for } x<0\end{cases}$$

$$\Rightarrow \quad \cot^{-1}x = \begin{cases}\tan^{-1}\frac{1}{x}, & \text{for } x>0\\ \pi + \tan^{-1}\frac{1}{x}, & \text{for } x<0\end{cases}$$

$$\therefore \quad \cot^{-1}\left(\frac{ab+1}{a-b}\right)+\cot^{-1}\left(\frac{bc+1}{b-c}\right)+\cot^{-1}\left(\frac{ca+1}{c-a}\right)$$

$$= \tan^{-1}\left(\frac{a-b}{1+ab}\right)+\tan^{-1}\left(\frac{b-c}{1+bc}\right)+\pi+\tan^{-1}\left(\frac{c-a}{1+ca}\right)$$

$$= \tan^{-1}a-\tan^{-1}b+\tan^{-1}b-\tan^{-1}c+\pi+\tan^{-1}c-\tan^{-1}a = \pi.$$

EXAMPLE 16 *Prove that*

$$\cos^{-1}x = 2\sin^{-1}\sqrt{\frac{1-x}{2}} = 2\cos^{-1}\sqrt{\frac{1+x}{2}}$$

SOLUTION Let $\cos^{-1}x = y$. Then, $x = \cos y$.

$$\therefore \quad 2\sin^{-1}\sqrt{\frac{1-x}{2}} = 2\sin^{-1}\left\{\sqrt{\frac{1-\cos y}{2}}\right\} = 2\sin^{-1}\left\{\sin\frac{y}{2}\right\} = 2\left(\frac{y}{2}\right) = y$$

and, $$2\cos^{-1}\sqrt{\frac{1+x}{2}} = 2\cos^{-1}\left\{\sqrt{\frac{1+\cos y}{2}}\right\} = 2\cos^{-1}\left\{\cos\frac{y}{2}\right\} = 2\left(\frac{y}{2}\right) = y$$

Hence, $\cos^{-1}x = 2\sin^{-1}\sqrt{\frac{1-x}{2}} = 2\cos^{-1}\sqrt{\frac{1+x}{2}}$.

EXAMPLE 17 *If* $y = \cot^{-1}(\sqrt{\cos x}) - \tan^{-1}(\sqrt{\cos x})$, *prove that* $\sin y = \tan^2\frac{x}{2}$.

SOLUTION We have,

$$y = \cot^{-1}(\sqrt{\cos x}) - \tan^{-1}(\sqrt{\cos x})$$

$$\Rightarrow \quad y = \frac{\pi}{2} - \tan^{-1}(\sqrt{\cos x}) - \tan^{-1}(\sqrt{\cos x})$$

$$\Rightarrow \quad y = \frac{\pi}{2} - 2\tan^{-1}(\sqrt{\cos x})$$

$$\Rightarrow \quad y = \frac{\pi}{2} - \cos^{-1}\left\{\frac{1-(\sqrt{\cos x})^2}{1+(\sqrt{\cos x})^2}\right\} \qquad \left[\because 2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$$

$$\Rightarrow \quad y = \frac{\pi}{2} - \cos^{-1}\left(\frac{1-\cos x}{1+\cos x}\right)$$

$$\Rightarrow \quad y = \frac{\pi}{2} - \cos^{-1}\left(\tan^2\frac{x}{2}\right)$$

$$\Rightarrow \quad \cos^{-1}\left(\tan^2\frac{x}{2}\right) = \frac{\pi}{2} - y$$

$$\Rightarrow \quad \tan^2\frac{x}{2} = \cos\left(\frac{\pi}{2} - y\right)$$

$$\Rightarrow \quad \tan^2\frac{x}{2} = \sin y.$$

EXAMPLE 18 *Prove that* $\tan^{-1}\frac{yz}{xr} + \tan^{-1}\frac{zx}{yr} + \tan^{-1}\frac{xy}{zr} = \frac{\pi}{2}$, where $x^2+y^2+z^2 = r^2$

SOLUTION We know that

$$\tan^{-1}x_1 + \tan^{-1}x_2 + \tan^{-1}x_3 + \dots + \tan^{-1}x_n = \tan^{-1}\left\{\frac{S_1 - S_3 + \dots}{1 - S_2 + S_4 + \dots}\right\},$$

where, S_k denotes the sum of the product of $x_1, x_2, \dots, x_n$ taking k at a time. Therefore,

$$\text{LHS} = \tan^{-1}\frac{yz}{xr} + \tan^{-1}\frac{xz}{yr} + \tan^{-1}\frac{xy}{zr}$$

$$\Rightarrow \quad \text{LHS} = \tan^{-1}\left\{\frac{\frac{yz}{xr} + \frac{xz}{yr} + \frac{xy}{zr} - \frac{xyz}{r^3}}{1 - \left(\frac{x^2+y^2+z^2}{r^2}\right)}\right\} = \tan^{-1}\infty = \frac{\pi}{2} = \text{R.H.S}$$

EXAMPLE 19 (i) *If* $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \alpha$, *prove that* $\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha$.

(ii) *If* $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, *prove that* $x^2+y^2+z^2+2xyz = 1$

(iii) *If* $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$, *prove that*

(a) $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$

(b) $x^4+y^4+z^4+4x^2y^2z^2 = 2(x^2y^2+y^2z^2+z^2x^2)$

SOLUTION (i) We have,

$$\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \alpha$$

$$\Rightarrow \quad \cos^{-1}\left\{\frac{xy}{ab} - \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right\} = \alpha$$

$$\Rightarrow \quad \frac{xy}{ab} - \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}} = \cos\alpha$$

$$\Rightarrow \quad \frac{xy}{ab} - \cos\alpha = \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}$$

$$\Rightarrow \quad \left(\frac{xy}{ab} - \cos\alpha\right)^2 = \left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)$$

$$\Rightarrow \quad \frac{x^2y^2}{a^2b^2} - \frac{2xy}{ab}\cos\alpha + \cos^2\alpha = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{x^2y^2}{a^2b^2}$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\alpha = 1 - \cos^2\alpha$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\alpha = \sin^2\alpha$$

(ii) We have,

$$\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$$

$$\Rightarrow \quad \cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z$$

$$\Rightarrow \quad \cos^{-1}x + \cos^{-1}y = \cos^{-1}(-z) \qquad [\because \cos^{-1}(-z) = \pi - \cos^{-1}z]$$

$$\Rightarrow \quad \cos^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\} = \cos^{-1}(-z)$$

$$\Rightarrow \quad xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$$

$$\Rightarrow \quad (xy+z)^2 = (1-x^2)(1-y^2)$$

$$\Rightarrow \quad x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2$$

$\Rightarrow \quad x^2+y^2+z^2+2xyz = 1$

(iii) (a) Let $\sin^{-1} x = A$, $\sin^{-1} y = B$ and $\sin^{-1} z = C$. Then, $x = \sin A$, $y = \sin B$ and $z = \sin C$

We have,

$$\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$$

$$\Rightarrow \quad A+B+C = \pi$$

$$\Rightarrow \quad \sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$$

$$\Rightarrow \quad 2 \sin A \cos A + 2 \sin B \cos B + 2 \sin C \cos C = 4 \sin A \sin B \sin C$$

$$\Rightarrow \quad \sin A \sqrt{1-\sin^2 A} + \sin B \sqrt{1-\sin^2 B} + \sin C \sqrt{1-\sin^2 C} = 2 \sin A \sin B \sin C$$

$$\Rightarrow \quad x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$$

(b) We have,

$$\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$$

$$\Rightarrow \quad \sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1} z$$

$$\Rightarrow \quad \cos(\sin^{-1} x + \sin^{-1} y) = \cos(\pi - \sin^{-1} z)$$

$$\Rightarrow \quad \cos(\sin^{-1} x)\cos(\sin^{-1} y) - \sin(\sin^{-1} x)\sin(\sin^{-1} y) = -\cos(\sin^{-1} z)$$

$$\Rightarrow \quad \sqrt{1-x^2}\sqrt{1-y^2} - xy = -\sqrt{1-z^2} \quad \left[\because \cos(\sin^{-1} x) = \cos\left(\cos^{-1}\sqrt{1-x^2}\right) = \sqrt{1-x^2}\right]$$

$$\Rightarrow \quad \sqrt{(1-x^2)(1-y^2)} = xy - \sqrt{1-z^2}$$

$$\Rightarrow \quad 1-x^2-y^2+x^2y^2 = x^2y^2+1-z^2-2xy\sqrt{1-z^2} \quad \text{[On squaring both sides]}$$

$$\Rightarrow \quad x^2+y^2-z^2 = 2xy\sqrt{1-z^2}$$

$$\Rightarrow \quad (x^2+y^2-z^2)^2 = 4x^2y^2(1-z^2)$$

$$\Rightarrow \quad x^4+y^4+z^4-2x^2z^2-2y^2z^2+2x^2y^2 = 4x^2y^2-4x^2y^2z^2$$

$$\Rightarrow \quad x^4+y^4+z^4+4x^2y^2z^2 = 2(x^2y^2+y^2z^2+z^2x^2)$$

EXAMPLE 20 *Evaluate:*

(i) $\tan\left\{2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right\}$ (ii) $\tan\left\{\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right\}$

SOLUTION (i) $\tan\left\{2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right\}$

$$= \tan\left\{\tan^{-1}\left(\frac{2\times\frac{1}{5}}{1-\frac{1}{25}}\right) - \tan^{-1} 1\right\} \quad \left[\because 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right), \text{ if } |x|<1\right]$$

$$= \tan\left\{\tan^{-1}\frac{5}{12} - \tan^{-1} 1\right\} = \tan\left\{\tan^{-1}\left(\frac{\frac{5}{12}-1}{1+\frac{5}{12}}\right)\right\} = \tan\left\{\tan^{-1}\left(\frac{-7}{17}\right)\right\} = \frac{-7}{17}$$

(ii) Let $\cos^{-1}\frac{\sqrt{5}}{3} = \theta$. Then, $\cos\theta = \frac{\sqrt{5}}{3}$.

$$\therefore \quad \tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right) = \tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{1-\frac{\sqrt{5}}{3}}{1+\frac{\sqrt{5}}{3}}}$$

$$= \sqrt{\frac{3-\sqrt{5}}{3+\sqrt{5}}} = \sqrt{\frac{(3-\sqrt{5})^2}{(3+\sqrt{5})(3-\sqrt{5})}} = \sqrt{\frac{(3-\sqrt{5})^2}{9-5}} = \frac{3-\sqrt{5}}{2}$$

EXAMPLE 21 *If* $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\} = \alpha$, *then prove that* $x^2 = \sin 2\alpha$.

SOLUTION We have,

$$\tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\} = \alpha$$

$$\Rightarrow \quad \frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}} = \tan\alpha$$

$$\Rightarrow \quad -\frac{\sqrt{1+x^2}}{\sqrt{1-x^2}} = \frac{\tan\alpha+1}{\tan\alpha-1}$$

$$\Rightarrow \quad \frac{\sqrt{1-x^2}}{\sqrt{1+x^2}} = \frac{1-\tan\alpha}{1+\tan\alpha}$$

$$\Rightarrow \quad \sqrt{\frac{1-x^2}{1+x^2}} = \frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}$$

$$\Rightarrow \quad \frac{1-x^2}{1+x^2} = \left(\frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}\right)^2 \Rightarrow \frac{1-x^2}{1+x^2} = \frac{1-\sin 2\alpha}{1+\sin 2\alpha} \Rightarrow x^2 = \sin 2\alpha$$

EXAMPLE 22 *Prove that:* $\cos^{-1}\left(\frac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}\right) = 2\tan^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)$.

SOLUTION We have,

$$\text{RHS} = 2\tan^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)$$

$$\Rightarrow \quad \text{RHS} = \cos^{-1}\left\{\frac{1-\tan^2\alpha/2\tan^2\beta/2}{1+\tan^2\alpha/2\tan^2\beta/2}\right\} \qquad \left[\because\ 2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$$

$$\Rightarrow \quad \text{RHS} = \cos^{-1}\left\{\frac{\cos^2\alpha/2\cos^2\beta/2-\sin^2\alpha/2\sin^2\beta/2}{\cos^2\alpha/2\cos^2\beta/2+\sin^2\alpha/2\sin^2\beta/2}\right\}$$

$$\Rightarrow \quad \text{RHS} = \cos^{-1}\left\{\frac{(2\cos^2\alpha/2)\,(2\cos^2\beta/2)-(2\sin^2\alpha/2)\,(2\sin^2\beta/2)}{(2\cos^2\alpha/2)\,(2\cos^2\beta/2)+(2\sin^2\alpha/2)\,(2\sin^2\beta/2)}\right\}$$

$$\Rightarrow \quad \text{RHS} = \cos^{-1}\left\{\frac{(1+\cos\alpha)(1+\cos\beta)-(1-\cos\alpha)(1-\cos\beta)}{(1+\cos\alpha)(1+\cos\beta)+(1-\cos\alpha)(1-\cos\beta)}\right\}$$

$$\Rightarrow \quad \text{RHS} = \cos^{-1}\left(\frac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}\right) = \text{LHS}.$$

EXAMPLE 23 *Evaluate:* $\cos(2\cos^{-1}x+\sin^{-1}x)$ *at* $x = \frac{1}{5}$, *where* $0 \le \cos^{-1}x \le \pi$ *and* $-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$.

SOLUTION We have,

$$\cos(2\cos^{-1}x + \sin^{-1}x)$$
$$= \cos(\cos^{-1}x + \cos^{-1}x + \sin^{-1}x) \qquad \left[\because \cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}\right]$$
$$= \cos\left(\cos^{-1}x + \frac{\pi}{2}\right)$$
$$= -\sin(\cos^{-1}x) = -\sin\left(\sin^{-1}\sqrt{1-x^2}\right) = -\sqrt{1-x^2} = -\sqrt{1-\frac{1}{25}} = -\sqrt{\frac{24}{25}}$$

EXAMPLE 24 *Prove that:* $\dfrac{\alpha^3}{2}\operatorname{cosec}^2\left(\dfrac{1}{2}\tan^{-1}\dfrac{\alpha}{\beta}\right) + \dfrac{\beta^3}{2}\sec^2\left(\dfrac{1}{2}\tan^{-1}\dfrac{\beta}{\alpha}\right) = (\alpha+\beta)(\alpha^2+\beta^2)$

SOLUTION We have,

$$\frac{\alpha^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right) + \frac{\beta^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right)$$
$$= \frac{\alpha^3}{2\sin^2\theta} + \frac{\beta^3}{2\cos^2\phi}, \text{ where } \theta = \frac{1}{2}\tan^{-1}\frac{\alpha}{\beta} \text{ and } \phi = \frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}$$
$$= \frac{\alpha^3}{1-\cos 2\theta} + \frac{\beta^3}{1+\cos 2\phi}$$
$$= \frac{\alpha^3}{1-\cos(\tan^{-1}\alpha/\beta)} + \frac{\beta^3}{1+\cos(\tan^{-1}\beta/\alpha)}$$
$$= \frac{\alpha^3}{1-\cos\left(\cos^{-1}\dfrac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)} + \frac{\beta^3}{1+\cos\left(\cos^{-1}\dfrac{\alpha}{\sqrt{\alpha^2+\beta^2}}\right)}$$
$$= \frac{\alpha^3}{1-\dfrac{\beta}{\sqrt{\alpha^2+\beta^2}}} + \frac{\beta^3}{1+\dfrac{\alpha}{\sqrt{\alpha^2+\beta^2}}}$$
$$= \left\{\frac{\alpha^3}{\sqrt{\alpha^2+\beta^2}-\beta} + \frac{\beta^3}{\sqrt{\alpha^2+\beta^2}+\alpha}\right\}\sqrt{\alpha^2+\beta^2}$$
$$= \left[\frac{\alpha^3\left\{\sqrt{\alpha^2+\beta^2}+\beta\right\}}{\alpha^2+\beta^2-\beta^2} + \frac{\beta^3\left\{\sqrt{\alpha^2+\beta^2}-\alpha\right\}}{\alpha^2+\beta^2-\alpha^2}\right]\sqrt{\alpha^2+\beta^2}$$
$$= \left\{\alpha\left(\sqrt{\alpha^2+\beta^2}+\beta\right) + \beta\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)\right\}\sqrt{\alpha^2+\beta^2}$$
$$= \alpha(\alpha^2+\beta^2) + \beta(\alpha^2+\beta^2) = (\alpha+\beta)(\alpha^2+\beta^2)$$

EXAMPLE 25 *Prove that:* $\tan^{-1}\dfrac{1-x}{1+x} - \tan^{-1}\dfrac{1-y}{1+y} = \sin^{-1}\dfrac{y-x}{\sqrt{1+x^2}\sqrt{1+y^2}}$.

SOLUTION We have,

$$\text{LHS} = \tan^{-1}\frac{1-x}{1+x} - \tan^{-1}\frac{1-y}{1+y}$$
$$\Rightarrow \quad \text{LHS} = (\tan^{-1}1 - \tan^{-1}x) - (\tan^{-1}1 - \tan^{-1}y)$$

$\Rightarrow$ $\text{LHS} = \tan^{-1} y - \tan^{-1} x$

$\Rightarrow$ $\text{LHS} = \tan^{-1}\left(\dfrac{y-x}{1+yx}\right)$

$\Rightarrow$ $\text{LHS} = \sin^{-1} \dfrac{y-x}{\sqrt{(1+yx)^2+(y-x)^2}} = \sin^{-1}\left\{\dfrac{y-x}{\sqrt{(1+x^2)(1+y^2)}}\right\} = \text{RHS}.$

EXAMPLE 26 *Evaluate:*

$$\tan^{-1}\left(\frac{3\sin 2\alpha}{5+3\cos 2\alpha}\right) + \tan^{-1}\left(\frac{1}{4}\tan\alpha\right), \textit{ where } -\frac{\pi}{2} < \alpha < \frac{\pi}{2}.$$

SOLUTION We have,

$$\tan^{-1}\left(\frac{3\sin 2\alpha}{5+3\cos 2\alpha}\right) + \tan^{-1}\left(\frac{1}{4}\tan\alpha\right)$$

$$= \tan^{-1}\left(\frac{6\tan\alpha}{8+2\tan^2\alpha}\right) + \tan^{-1}\left(\frac{1}{4}\tan\alpha\right) \qquad \left[\begin{array}{l}\text{Using: } \cos 2\alpha = \dfrac{1-\tan^2\alpha}{1+\tan^2\alpha} \\ \text{and, } \sin 2\alpha = \dfrac{2\tan\alpha}{1+\tan^2\alpha}\end{array}\right]$$

$$= \tan^{-1}\left(\frac{3\tan\alpha}{4+\tan^2\alpha}\right) + \tan^{-1}\left(\frac{1}{4}\tan\alpha\right)$$

$$= \tan^{-1}\left\{\frac{\dfrac{3\tan\alpha}{4+\tan^2\alpha}+\dfrac{1}{4}\tan\alpha}{1-\dfrac{3\tan^2\alpha}{16+4\tan^2\alpha}}\right\}$$

$$= \tan^{-1}\left\{\frac{(16+\tan^2\alpha)\tan\alpha}{16+\tan^2\alpha}\right\} = \tan^{-1}(\tan\alpha) = \alpha \qquad [\because -\pi/2 < \alpha < \pi/2]$$

EXAMPLE 27 *Prove that:*

$$\tan\left\{\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right\} + \tan\left\{\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right\} = \frac{2b}{a}$$

SOLUTION Let $\cos^{-1}\left(\dfrac{a}{b}\right) = \theta$. Then, $\cos\theta = \dfrac{a}{b}$

$\therefore$ $\text{LHS} = \tan\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right) + \tan\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)$

$\Rightarrow$ $\text{LHS} = \dfrac{1+\tan\theta/2}{1-\tan\theta/2} + \dfrac{1-\tan\theta/2}{1+\tan\theta/2}$

$\Rightarrow$ $\text{LHS} = \dfrac{(1+\tan\theta/2)^2+(1-\tan\theta/2)^2}{1-\tan^2\theta/2} = 2\left(\dfrac{1+\tan^2\theta/2}{1-\tan^2\theta/2}\right) = \dfrac{2}{\cos\theta} = \dfrac{2b}{a} = \text{RHS}.$

EXAMPLE 28 *Prove that:*

$$\tan^{-1}\left\{\frac{\cos 2\alpha\sec 2\beta+\cos 2\beta\sec 2\alpha}{2}\right\} = \tan^{-1}\{\tan^2(\alpha+\beta)\tan^2(\alpha-\beta)\} + \tan^{-1} 1$$

SOLUTION We have,

$\text{RHS} = \tan^{-1}\{\tan^2(\alpha+\beta)\ \tan^2(\alpha-\beta)\} + \tan^{-1} 1$

$$\Rightarrow \quad \text{RHS} = \tan^{-1}\left\{\frac{\tan^2(\alpha+\beta)\tan^2(\alpha-\beta)+1}{1-\tan^2(\alpha+\beta)\tan^2(\alpha-\beta)}\right\}$$

$$\Rightarrow \quad \text{RHS} = \tan^{-1}\left\{\frac{\sin^2(\alpha+\beta)\sin^2(\alpha-\beta)+\cos^2(\alpha+\beta)\cos^2(\alpha-\beta)}{\cos^2(\alpha+\beta)\cos^2(\alpha-\beta)-\sin^2(\alpha+\beta)\sin^2(\alpha-\beta)}\right\}$$

$$\Rightarrow \quad \text{RHS} = \tan^{-1}\left\{\frac{\{2\sin(\alpha+\beta)\sin(\alpha-\beta)\}^2+\{2\cos(\alpha+\beta)\cos(\alpha-\beta)\}^2}{\{2\cos(\alpha+\beta)\cos(\alpha-\beta)\}^2-\{2\sin(\alpha+\beta)\sin(\alpha-\beta)\}^2}\right\}$$

$$\Rightarrow \quad \text{RHS} = \tan^{-1}\left\{\frac{(\cos 2\beta-\cos 2\alpha)^2+(\cos 2\alpha+\cos 2\beta)^2}{(\cos 2\alpha+\cos 2\beta)^2-(\cos 2\beta-\cos 2\alpha)^2}\right\}$$

$$\Rightarrow \quad \text{RHS} = \tan^{-1}\left\{\frac{\cos^2 2\alpha+\cos^2 2\beta}{2\cos 2\alpha\cos 2\beta}\right\} = \tan^{-1}\left\{\frac{\cos 2\alpha\sec 2\beta+\cos 2\beta\sec 2\alpha}{2}\right\} = \text{LHS.}$$

EXAMPLE 29 *Prove that:*

$$\tan^{-1}\left(\frac{1}{2}\tan 2A\right)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A) = \begin{cases} 0, \text{ if } \dfrac{\pi}{4} < A < \dfrac{\pi}{2} \\ \pi, \text{ if } 0 < A < \pi/4 \end{cases}$$

SOLUTION We know that

$$\tan^{-1}x+\tan^{-1}y = \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), \text{ if } xy<1 \\ \pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right), \text{ if } xy>1 \end{cases}$$

Also, $\cot A > 1$, if $0 < A < \dfrac{\pi}{4}$ and, $0 < \cot A < 1$, if $\dfrac{\pi}{4} < A < \dfrac{\pi}{2}$

$\therefore$ $\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)$

$$= \begin{cases} \tan^{-1}\left(\dfrac{\cot A+\cot^3 A}{1-\cot^4 A}\right), \text{ if } \dfrac{\pi}{4} < A < \dfrac{\pi}{2} \\ \pi+\tan^{-1}\left(\dfrac{\cot A+\cot^3 A}{1-\cot^4 A}\right), \text{ if } 0 < A < \dfrac{\pi}{4} \end{cases}$$

$$= \begin{cases} \tan^{-1}\left(\dfrac{\cot A}{1-\cot^2 A}\right), \text{ if } \dfrac{\pi}{4} < A < \dfrac{\pi}{2} \\ \pi+\tan^{-1}\left(\dfrac{\cot A}{1-\cot^2 A}\right), \text{ if } 0 < A < \dfrac{\pi}{4} \end{cases}$$

$$= \begin{cases} \tan^{-1}\left(-\dfrac{1}{2}\tan 2A\right), \text{ if } \dfrac{\pi}{4} < A < \dfrac{\pi}{2} \\ \pi+\tan^{-1}\left(-\dfrac{1}{2}\tan 2A\right), \text{ if } 0 < A < \dfrac{\pi}{4} \end{cases}$$

$$= \begin{cases} -\tan^{-1}\left(\dfrac{1}{2}\tan 2A\right), \text{ if } \dfrac{\pi}{4} < A < \dfrac{\pi}{4} \\ \pi-\tan^{-1}\left(\dfrac{1}{2}\tan 2A\right), \text{ if } 0 < A < \dfrac{\pi}{4} \end{cases}$$

Adding $\tan^{-1}\left(\frac{1}{2}\tan 2A\right)$ on both sides, we get

$$\tan^{-1}\left(\frac{1}{2}\tan 2A\right)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)=\begin{cases}0, \text{ if } \pi/4<A<\pi/2\\ \pi, \text{ if } 0<A<\pi/4\end{cases}$$

EXAMPLE 30 *If* $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, *then find the value of* x. **[NCERT]**

SOLUTION We have,

$$\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$$

$$\Rightarrow \sin^{-1}\frac{1}{5}+\cos^{-1}x=\sin^{-1}1$$

$$\Rightarrow \sin^{-1}\frac{1}{5}+\cos^{-1}x=\frac{\pi}{2}$$

$$\Rightarrow \cos^{-1}x=\frac{\pi}{2}-\sin^{-1}\frac{1}{5}$$

$$\Rightarrow \cos^{-1}x=\cos^{-1}\frac{1}{5} \qquad \left[\because \sin^{-1}\frac{1}{5}+\cos^{-1}\frac{1}{5}=\frac{\pi}{2}\right]$$

$$\Rightarrow x=\frac{1}{5}$$

EXAMPLE 31 *Solve the following equations:*

(i) $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$ **[NCERT, CBSE 2010]**

(ii) $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$ **[NCERT, CBSE 2009, 2012]**

(iii) $\tan^{-1}\frac{x-1}{x+1}+\tan^{-1}\frac{2x-1}{2x+1}=\tan^{-1}\frac{23}{36}$

(iv) $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$ **[NCERT, CBSE 2009, 2010 C]**

SOLUTION (i) We have,

$$\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\tan^{-1}1$$

$$\Rightarrow \tan^{-1}\frac{x-1}{x-2}=\tan^{-1}1-\tan^{-1}\frac{x+1}{x+2}$$

$$\Rightarrow \tan^{-1}\frac{x-1}{x-2}=\tan^{-1}\left(\frac{1-\frac{x+1}{x+2}}{1+\frac{x+1}{x+2}}\right)$$

$$\tan^{-1}\frac{x-1}{x-2}=\tan^{-1}\frac{x+2-x-1}{x+2+x+1}$$

$$\Rightarrow \tan^{-1}\frac{x-1}{x-2}=\tan^{-1}\frac{1}{2x+3}$$

$$\Rightarrow \frac{x-1}{x-2}=\frac{1}{2x+3}$$

$$\Rightarrow (2x+3)(x-1)=x-2 \Rightarrow 2x^2+x-3=x-2 \Rightarrow 2x^2-1=0 \Rightarrow x=\pm\frac{1}{\sqrt{2}}$$

(ii) We have,

$$\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left\{\frac{2x+3x}{1-2x\times 3x}\right\} = \tan^{-1}1, \text{ if } 6x^2<1$$

$$\Rightarrow \quad \frac{5x}{1-6x^2} = 1, \text{ if } 6x^2<1$$

$$\Rightarrow \quad 6x^2+5x-1 = 0 \text{ and } x^2<\frac{1}{6}$$

$$\Rightarrow \quad (6x-1)(x+1) = 0 \text{ and } -\frac{1}{\sqrt{6}}<x<\frac{1}{\sqrt{6}}$$

$$\Rightarrow \quad x = -1, \frac{1}{6} \text{ and } -\frac{1}{\sqrt{6}}<x<\frac{1}{\sqrt{6}}$$

$$\Rightarrow \quad x = \frac{1}{6}$$

(iii) We have,

$$\tan^{-1}\frac{x-1}{x+1}+\tan^{-1}\frac{2x-1}{2x+1} = \tan^{-1}\frac{23}{36}$$

$$\Rightarrow \quad \tan^{-1}\left\{\frac{\frac{x-1}{x+1}+\frac{2x-1}{2x+1}}{1-\frac{x-1}{x+1}\cdot\frac{2x-1}{2x+1}}\right\} = \tan^{-1}\frac{23}{36}, \text{ if } \frac{(x-1)(2x-1)}{(x+1)(2x+1)}<1$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2x^2-1}{3x}\right) = \tan^{-1}\frac{23}{36} \text{ and } \frac{(x-1)(2x-1)}{(x+1)(2x+1)}-1<0$$

$$\Rightarrow \quad \frac{2x^2-1}{3x} = \frac{23}{36} \quad \text{and} \quad \frac{-6x}{(x+1)(2x+1)}<0$$

$$\Rightarrow \quad 24x^2-23x-12 = 0 \text{ and } \frac{x}{(x+1)(2x+1)}>0$$

$$\Rightarrow \quad (3x-4)(8x+3) = 0 \text{ and } x\in(-1,-1/2)\cup(0,\infty)$$

$$\Rightarrow \quad x = \frac{4}{3}$$

(iv) We have,

$$2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\operatorname{cosec} x)$$

$$\Rightarrow \quad \frac{2\cos x}{\sin^2 x} = 2\operatorname{cosec} x \Rightarrow \cos x = \sin x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$$

EXAMPLE 32 *Solve the following equations:*

(i) $\sin^{-1}\frac{3x}{5}+\sin^{-1}\frac{4x}{5} = \sin^{-1}x$

(ii) $\sin^{-1}(1-x)-2\sin^{-1}x = \frac{\pi}{2}$

(iii) $\sin[2\cos^{-1}\{\cot(2\tan^{-1}x)\}] = 0$

SOLUTION (i) We have,

$$\sin^{-1}\frac{3x}{5}+\sin^{-1}\frac{4x}{5} = \sin^{-1}x$$

$$\Rightarrow \quad \sin^{-1}\left\{\frac{3x}{5}\sqrt{1-\frac{16x^2}{25}}+\frac{4x}{5}\sqrt{1-\frac{9x^2}{25}}\right\} = \sin^{-1}x$$

$$\Rightarrow \quad \frac{3x}{5}\sqrt{1-\frac{16x^2}{25}}+\frac{4x}{5}\sqrt{1-\frac{9x^2}{25}} = x$$

$$\Rightarrow \quad 3x\sqrt{25-16x^2} + 4x\sqrt{25-9x^2} = 25x$$

$$\Rightarrow \quad x = 0 \quad \text{or,} \quad 3\sqrt{25-16x^2} + 4\sqrt{25-9x^2} = 25$$

Now, $3\sqrt{25-16x^2} + 4\sqrt{25-9x^2} = 25$

$$\Rightarrow \quad 4\sqrt{25-9x^2} = 25 - 3\sqrt{25-16x^2}$$

$$\Rightarrow \quad 16(25-9x^2) = 625 + 9(25-16x^2) - 150\sqrt{25-16x^2}$$

$$\Rightarrow \quad 150\sqrt{25-16x^2} = 450$$

$$\Rightarrow \quad 25-16x^2 = 9 \Rightarrow x = \pm 1$$

Hence, $x = 0, 1, -1$ are roots of the given equation.

(ii) We have,

$$\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow \quad \sin^{-1}(1-x) = \frac{\pi}{2} + 2\sin^{-1}x$$

$$\Rightarrow \quad 1-x = \sin(\pi/2 + 2\sin^{-1}x)$$

$$\Rightarrow \quad 1-x = \cos(2\sin^{-1}x)$$

$$\Rightarrow \quad 1-x = \cos\{\cos^{-1}(1-2x^2)\} \qquad [\because 2\sin^{-1}x = \cos^{-1}(1-2x^2)]$$

$$\Rightarrow \quad 1-x = (1-2x^2) \Rightarrow x = 2x^2 \Rightarrow x(2x-1) = 0 \Rightarrow x = 0, \frac{1}{2}$$

For, $x = \frac{1}{2}$, we have

$$\text{LHS} = \sin^{-1}(1-x) - 2\sin^{-1}x = \sin^{-1}\frac{1}{2} - 2\sin^{-1}\frac{1}{2} = -\sin^{-1}\frac{1}{2} = -\frac{\pi}{6} \neq \text{R.H.S.}$$

So, $x = 1/2$ is not a root of the given equation.

Clearly, $x = 0$ satisfies the equation. Hence, $x = 0$ is a root of the given equation.

(iii) We have,

$$\sin[2\cos^{-1}\{\cot(2\tan^{-1}x)\}] = 0$$

$$\Rightarrow \quad \sin\left[2\cos^{-1}\left\{\cot\left(\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right)\right\}\right] = 0 \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$\Rightarrow \quad \sin\left[2\cos^{-1}\left\{\cot\left(\cot^{-1}\left(\frac{1-x^2}{2x}\right)\right)\right\}\right] = 0 \qquad \left[\because \cot^{-1}x = \tan^{-1}\frac{1}{x}\right]$$

$$\Rightarrow \quad \sin\left[2\cos^{-1}\left(\frac{1-x^2}{2x}\right)\right] = 0$$

$$\Rightarrow \quad \sin\left[\sin^{-1}\left\{2\left(\frac{1-x^2}{2x}\right)\sqrt{1-\left(\frac{1-x^2}{2x}\right)^2}\right\}\right] = 0 \qquad [\because 2\cos^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})]$$

$$\Rightarrow \quad \left(\frac{1-x^2}{x}\right)\sqrt{1-\left(\frac{1-x^2}{2x}\right)^2} = 0$$

$$\Rightarrow \quad \frac{1-x^2}{x}=0 \text{ or, } \sqrt{1-\left(\frac{1-x^2}{2x}\right)^2}=0$$

$$\Rightarrow \quad 1-x^2=0 \text{ or, } \left(\frac{1-x^2}{2x}\right)^2=1$$

$$\Rightarrow \quad x=\pm 1 \text{ or, } (1-x^2)^2=4x^2$$

Now, $(1-x^2)^2=4x^2$

$$\Rightarrow \quad (1-x^2)^2-(2x)^2=0$$

$$\Rightarrow \quad (1-x^2-2x)(1-x^2+2x)=0$$

$$\Rightarrow \quad 1-x^2-2x=0 \text{ or, } 1-x^2+2x=0$$

$$\Rightarrow \quad x^2+2x-1=0 \text{ or, } x^2-2x-1=0$$

$$\Rightarrow \quad x=-1\pm\sqrt{2} \text{ or, } x=1\pm\sqrt{2}$$

Hence, $x=\pm 1, -1\pm\sqrt{2}, 1\pm\sqrt{2}$ are the roots of the given equation.

EXAMPLE 33 *Solve the equation*

$$\tan^{-1}\sqrt{x^2+x}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}$$

SOLUTION This equation holds, if $x^2+x\geq 0$ and $0\leq x^2+x+1\leq 1$

Now, $x^2+x\geq 0$ and $0\leq x^2+x+1\leq 1$

$\Rightarrow \quad x^2+x\geq 0$ and $x^2+x+1\leq 1$ $\qquad [\because\ x^2+x+1>0$ for all $x]$

$\Rightarrow \quad x^2+x\geq 0$ and $x^2+x\leq 0$

$\Rightarrow \quad x^2+x=0 \Rightarrow x=0,-1$

Clearly, these two values satisfy the given equation. Hence, $x=0,-1$ are the solutions of the given equation.

EXAMPLE 34 *Solve for x:*

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{3}, -1<x<1$$

[CBSE 2011]

SOLUTION We know that

$$\tan^{-1}\left(\frac{1}{x}\right)=\begin{cases}\cot^{-1}x & ,\text{ if } x>0\\ -\pi+\cot^{-1}x & ,\text{ if } x<0\end{cases} \quad \text{i.e. } \cot^{-1}x=\begin{cases}\tan^{-1}\left(\frac{1}{x}\right) & ,\text{ if } x>0\\ \pi+\tan^{-1}\left(\frac{1}{x}\right) & ,\text{ if } x<0\end{cases}$$

So, following cases arise:

CASE I *When* $0<x<1$

In this case, we have

$$\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\tan^{-1}\left(\frac{2x}{1-x^2}\right) \quad \text{...(i)}$$

Given equation is

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{3}$$

$$\Rightarrow \quad 2\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{3} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad 4\tan^{-1}x=\frac{\pi}{3} \Rightarrow \tan^{-1}x=\frac{\pi}{12} \Rightarrow x=\tan\frac{\pi}{12}=\tan 15^\circ=\frac{\sqrt{3}-1}{\sqrt{3}+1}$$

CASE II *When* $-1 < x < 0$

In this case, we have

$$\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right) \qquad \ldots\text{(ii)}$$

Given equation is

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{3}$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2x}{1-x^2}\right)+\pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{3} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad 2\tan^{-1}\left(\frac{2x}{1-x^2}\right)=-\frac{2\pi}{3}$$

$$\Rightarrow \quad 4\tan^{-1}x=-\frac{2\pi}{3} \Rightarrow \tan^{-1}x=-\frac{\pi}{6} \Rightarrow x=\tan\left(\frac{-\pi}{6}\right)=-\frac{1}{\sqrt{3}}$$

CASE III *When* $x = 0$

In this case, we have

$$\text{LHS}=\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\tan^{-1}(0)+\cot^{-1}(\infty)=\frac{\pi}{2} \text{ and, RHS } =\frac{\pi}{3}$$

So, $x = 0$ is not a solution of the given equation.

Hence, $x=\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ and $x=\dfrac{1}{\sqrt{3}}$ are solutions of the given equation.

EXERCISE 4.2

1. Evaluate the following:

(i) $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$ (ii) $\cos^{-1}\left\{\cos\left(-\frac{\pi}{4}\right)\right\}$ (iii) $\tan^{-1}\left\{\tan\frac{3\pi}{4}\right\}$

(iv) $\sin^{-1}(\sin 2)$ (v) $\sin\left\{\frac{\pi}{3}-\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right\}$ (vi) $\cos\left\{\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)+\frac{\pi}{4}\right\}$

(vii) $\cos\left(\tan^{-1}\frac{3}{4}\right)$ (viii) $\cos^{-1}\left\{\cos\frac{5\pi}{4}\right\}$ (ix) $\cos^{-1}\left\{\cos\left(\frac{4\pi}{3}\right)\right\}$

(x) $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$ (xi) $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ (xii) $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

2. Evaluate the following:

(i) $\cos\left(\sin^{-1}\frac{3}{5}\right)$ (ii) $\sin\left(\cos^{-1}\frac{4}{5}\right)$ (iii) $\cos\left(\sin^{-1}-\frac{3}{5}\right)$

(iv) $\tan\left(\cos^{-1}\frac{8}{17}\right)$ (v) $\operatorname{cosec}\left\{\cos^{-1}\left(-\frac{12}{13}\right)\right\}$ (vi) $\tan\left\{2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right\}$

(vii) $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)$ [CBSE 2013] (viii) $\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right)$

(ix) $\cos\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}\right)$ (x) $\sin(\tan^{-1}x+\cot^{-1}x)$

3. Prove the following results:

(i) $\cos^{-1}\left(\frac{5}{13}\right)=\tan^{-1}\left(\frac{12}{5}\right)$ (ii) $\sin^{-1}\left(-\frac{4}{5}\right)=\tan^{-1}\left(-\frac{4}{3}\right)=\cos^{-1}\left(-\frac{3}{5}\right)-\pi$

(iii) $\tan\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right)=\frac{17}{6}$ (iv) $2\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{24}{7}$ [NCERT]

(v) $\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}=\tan^{-1}\frac{63}{16}$ [CBSE 2012]

(vi) $\cos\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)=\frac{6}{5\sqrt{13}}$

4. Prove the following results:

(i) $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}=\tan^{-1}\frac{2}{9}$

(ii) $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\cos^{-1}\frac{3}{5}=\frac{1}{2}\sin^{-1}\left(\frac{4}{5}\right)$ [CBSE 2010 C]

(iii) $\tan^{-1}\frac{2}{3}=\frac{1}{2}\tan^{-1}\frac{12}{5}$

(iv) $\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{3}=\frac{\pi}{4}$ [CBSE 2010]

(v) $\sin^{-1}\frac{4}{5}+2\tan^{-1}\frac{1}{3}=\frac{\pi}{2}$

(vi) $\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}=\pi$

(vii) $2\sin^{-1}\frac{3}{5}-\tan^{-1}\frac{17}{31}=\frac{\pi}{4}$

(viii) $\cot^{-1}7+\cot^{-1}8+\cot^{-1}18=\cot^{-1}3$

(ix) $2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\tan^{-1}\frac{4}{7}$ [NCERT]

(x) $2\tan^{-1}\frac{3}{4}-\tan^{-1}\frac{17}{31}=\frac{\pi}{4}$ [CBSE 2011]

(xi) $2\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\tan^{-1}\left(\frac{31}{17}\right)$ [CBSE 2011]

(xii) $\sin^{-1}\left(\frac{63}{65}\right)=\sin^{-1}\left(\frac{5}{13}\right)+\cos^{-1}\left(\frac{3}{5}\right)$ [CBSE 2012]

5. If $\sin^{-1}\frac{2a}{1+a^2}-\cos^{-1}\frac{1-b^2}{1+b^2}=\tan^{-1}\frac{2x}{1-x^2}$, then prove that $x=\frac{a-b}{1+ab}$

6. If $\cos^{-1}\frac{x}{2}+\cos^{-1}\frac{y}{3}=\alpha$, then prove that $9x^2-12xy\cos\alpha+4y^2=36\sin^2\alpha$.

7. Prove that

(i) $\tan^{-1}\left(\frac{1-x^2}{2x}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{2}$ (ii) $\sin\left\{\tan^{-1}\frac{1-x^2}{2x}+\cos^{-1}\frac{1-x^2}{1+x^2}\right\}=1$

(iii) $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}=\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$ **[NCERT]**

8. Prove that $2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right)=\cos^{-1}\left(\frac{a\cos\theta+b}{a+b\cos\theta}\right)$

9. If $\sin^{-1}\frac{2a}{1+a^2}+\sin^{-1}\frac{2b}{1+b^2}=2\tan^{-1}x$, prove that $x=\frac{a+b}{1-ab}$.

10. Show that $2\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}$ is constant for $x\geq 1$, find that constant.

11. Prove that:

$$\tan^{-1}\frac{2ab}{a^2-b^2}+\tan^{-1}\frac{2xy}{x^2-y^2}=\tan^{-1}\frac{2\alpha\beta}{\alpha^2-\beta^2}, \text{ where } \alpha=ax-by \text{ and } \beta=ay+bx.$$

12. Write each of the following in the simplest form:

(i) $\sin^{-1}\{x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\}$ (ii) $\tan^{-1}\{x+\sqrt{1+x^2}\}$, $x\in R$

(iii) $\tan^{-1}\left\{\sqrt{1+x^2}-x\right\}$, $x\in R$ (iv) $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-1}{x}\right\}$, $x\neq 0$

(v) $\tan^{-1}\left\{\frac{\sqrt{1+x^2}+1}{x}\right\}$, $x\neq 0$ (vi) $\tan^{-1}\sqrt{\frac{a-x}{a+x}}$, $-a<x<a$

(vii) $\tan^{-1}\left\{\frac{x}{a+\sqrt{a^2-x^2}}\right\}$, $-a<x<a$ (viii) $\sin^{-1}\left\{\frac{x+\sqrt{1-x^2}}{\sqrt{2}}\right\}$, $-\frac{1}{2}<x<\frac{1}{\sqrt{2}}$

(ix) $\sin^{-1}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right\}$, $0<x<1$ (x) $\sin\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}$

(xi) $\cot^{-1}\frac{a}{\sqrt{x^2-a^2}}$, $|x|>a$

13. Find the values of each of the following:

(i) $\tan^{-1}\left\{2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right\}$ (ii) $\cot(\tan^{-1}a+\cot^{-1}a)$ **[CBSE 2012]**

(iii) $\cos(\sec^{-1}x+\operatorname{cosec}^{-1}x)$, $|x|\geq 1$

14. For any $a, b, x, y>0$, prove that :

$$\frac{2}{3}\tan^{-1}\left(\frac{3ab^2-a^3}{b^3-3a^2b}\right)+\frac{2}{3}\tan^{-1}\left(\frac{3xy^2-x^3}{y^3-3x^2y}\right)=\tan^{-1}\frac{2\alpha\beta}{\alpha^2-\beta^2}$$

where $\alpha=-ax+by$, $\beta=bx+ay$

15. Solve the following equations for x :

(i) $\tan^{-1}2x+\tan^{-1}3x=n\pi+\frac{3\pi}{4}$

(ii) $\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31}$

(iii) $\tan^{-1}\frac{1}{4} + 2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{6} + \tan^{-1}\frac{1}{x} = \frac{\pi}{4}$

(iv) $\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}$

(v) $3\sin^{-1}\frac{2x}{1+x^2} - 4\cos^{-1}\frac{1-x^2}{1+x^2} + 2\tan^{-1}\frac{2x}{1-x^2} = \frac{\pi}{3}$ [CBSE 2010]

(vi) $\cos^{-1} x + \sin^{-1}\frac{x}{2} = \frac{\pi}{6}$

(vii) $\tan^{-1}(x-1) + \tan^{-1} x + \tan^{-1}(x+1) = \tan^{-1} 3x$

(viii) $\tan(\cos^{-1} x) = \sin\left(\cot^{-1}\frac{1}{2}\right)$ [NCERT, CBSE 2010]

(ix) $\tan^{-1}\left(\frac{1-x}{1+x}\right) - \frac{1}{2}\tan^{-1} x = 0$, where $x > 0$

(x) $\cot^{-1} x - \cot^{-1}(x+2) = \frac{\pi}{12}$, where $x > 0$ [CBSE 2010]

(xi) $\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cot^{-1}\left(\frac{1-x^2}{2x}\right) = \frac{2\pi}{3}, x > 0$ [CBSE 2010]

(xii) $\tan^{-1}(x+2) + \tan^{-1}(x-2) = \tan^{-1}\left(\frac{8}{79}\right), x > 0$ [CBSE 2010]

(xiii) $\tan^{-1}\frac{x}{2} + \tan^{-1}\frac{x}{3} = \frac{\pi}{4}, 0 < x < \sqrt{6}$ [CBSE 2010C]

(xiv) $2\tan^{-1}(\sin x) = \tan^{-1}(2\sec x), x \neq \frac{\pi}{2}$ [CBSE 2012]

(xv) $\cos\left(\tan^{-1} x\right) = \sin\left(\cot^{-1}\frac{3}{4}\right)$ [CBSE 2013]

16. Find the value of $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$ [CBSE 2011]

ANSWERS

1. (i) $\frac{\pi}{6}$ (ii) $\frac{\pi}{4}$ (iii) $-\frac{\pi}{4}$ (iv) $\pi - 2$ (v) $\frac{\sqrt{3}}{2}$ (vi) $-\frac{\sqrt{3}+1}{2\sqrt{2}}$
(vii) $\frac{4}{5}$ (viii) $\frac{3\pi}{4}$ (ix) $\frac{2\pi}{3}$ (x) $\frac{-\pi}{3}$ (xi) $\frac{\pi}{6}$ (xii) $\frac{\pi}{6}$

2. (i) $\frac{4}{5}$ (ii) $\frac{3}{5}$ (iii) $\frac{4}{5}$ (iv) $\frac{15}{8}$ (v) $\frac{13}{5}$ (vi) $-\frac{7}{17}$
(vii) $\frac{4-\sqrt{7}}{3}$ (viii) $\frac{1}{\sqrt{10}}$ (ix) $\frac{33}{65}$ (x) 1 (xi) $\frac{1}{\sqrt{3}}$ (xii) $\frac{1}{4}$ (xiii) 1 **10.** π

12. (i) $\sin^{-1} x - \sin^{-1}\sqrt{x}$ (ii) $\frac{\pi}{2} - \frac{1}{2}\cot^{-1} x$ (iii) $\frac{1}{2}\cot^{-1} x$
(iv) $\frac{1}{2}\tan^{-1} x$ (v) $\frac{\pi}{2} - \frac{1}{2}\tan^{-1} x$ (vi) $\frac{1}{2}\cos^{-1}\frac{x}{a}$
(vii) $\frac{1}{2}\sin^{-1}\frac{x}{a}$ (viii) $\frac{\pi}{4} + \sin^{-1} x$ (ix) $\frac{\pi}{4} + \frac{1}{2}\cos^{-1} x$
(x) $\sqrt{1-x^2}$ (xi) $\sec^{-1}\frac{x}{a}$

13. (i) $\frac{\pi}{4}$ (ii) 0 (iii) 0

15. (i) $-\frac{1}{6}$ (ii) $\frac{1}{4}$ (iii) $-\frac{461}{9}$ (iv) $\frac{1}{2}\sqrt{\frac{3}{7}}$ (v) $\frac{1}{\sqrt{3}}$ (vi) 1

(vii) $0, \pm\frac{1}{2}$ (viii) $\pm\frac{\sqrt{5}}{3}$ (ix) $\frac{1}{\sqrt{3}}$ (x) $\sqrt{3}$ (xi) $\frac{1}{\sqrt{3}}$ (xii) $\frac{1}{4}$

(xiii) 1 (xiv) $x = n\pi + \pi/4,\ n \in Z$ (xv) $\pm\frac{3}{4}$ **16.** $\frac{\pi}{4}$

HINTS TO NCERT & SELECTED PROBLEMS

3. $\sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5} = \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{4}{3} = \tan^{-1}\left(\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\times\frac{4}{3}}\right) = \tan^{-1}\left(\frac{63}{16}\right)$

7. (iii) LHS $= \frac{9}{4}\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right) = \frac{9}{4}\times\cos^{-1}\frac{1}{3} = \frac{9}{4}\sin^{-1}\sqrt{1-\frac{1}{9}} = \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

8. Using: $2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, we get

$$2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right) = \cos^{-1}\left\{\frac{1-\frac{a-b}{a+b}\tan^2\frac{\theta}{2}}{1+\frac{a-b}{a+b}\tan^2\frac{\theta}{2}}\right\}$$

$$\Rightarrow\quad 2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right) = \cos^{-1}\left\{\frac{a(1-\tan^2\theta/2)+b(1+\tan^2\theta/2)}{a(1+\tan^2\theta/2)+b(1-\tan^2\theta/2)}\right\}$$

$$\Rightarrow\quad 2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right) = \cos^{-1}\left\{\frac{a\left(\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)+b}{a+b\left(\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)}\right\} = \cos^{-1}\left(\frac{a\cos\theta+b}{a+b\cos\theta}\right)$$

12. (i) $\sin^{-1}\left\{x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right\} = \sin^{-1}\left\{x\sqrt{1-(\sqrt{x})^2} - \sqrt{x}\sqrt{1-x^2}\right\}$

$$= \sin^{-1}x - \sin^{-1}\sqrt{x}.$$

(ii) Putting $x = \cot\theta$, we obtain

$$\tan^{-1}\left\{x+\sqrt{1+x^2}\right\} = \tan^{-1}\{\cot\theta + \operatorname{cosec}\theta\} = \tan^{-1}\left(\frac{1+\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left\{\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right\} = \tan^{-1}\left(\cot\frac{\theta}{2}\right) = \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\}$$

$$= \frac{\pi}{2} - \frac{\theta}{2} = \frac{\pi}{2} - \frac{1}{2}\cot^{-1}x$$

(ix) Putting $x = \sin\theta$, we obtain

$$\sin^{-1}\left\{\frac{x+\sqrt{1-x^2}}{2}\right\} = \sin^{-1}\left\{\frac{\sin\theta+\cos\theta}{\sqrt{2}}\right\} = \sin^{-1}\left\{\sin\left(\frac{\pi}{4}+\theta\right)\right\}$$

$$= \frac{\pi}{4}+\theta = \frac{\pi}{4}+\sin^{-1}x$$

15. (iv) We have, $\sin^{-1}x + \sin^{-1}2x = \dfrac{\pi}{3}$

$$\Rightarrow \quad \sin^{-1}x - \sin^{-1}\frac{\sqrt{3}}{2} = -\sin^{-1}2x$$

$$\Rightarrow \quad \sin^{-1}\left\{x\sqrt{1-\frac{3}{4}}-\frac{\sqrt{3}}{2}\sqrt{1-x^2}\right\} = \sin^{-1}(-2x)$$

$$\Rightarrow \quad \frac{x}{2}-\frac{\sqrt{3}}{2}\sqrt{1-x^2} = -2x \Rightarrow 5x = \sqrt{3}\sqrt{1-x^2} \Rightarrow 25x^2 = 3-3x^2 \Rightarrow 28x^2 = 3 \Rightarrow x = \frac{\sqrt{3}}{28}$$

(v) Given equation reduces to:

$$3(2\tan^{-1}x) - 4(2\tan^{-1}x) + 2(2\tan^{-1}x) = \pi/3$$

$$\Rightarrow \quad 2\tan^{-1}x = \frac{\pi}{3} \Rightarrow \tan^{-1}x = \frac{\pi}{6} \Rightarrow x = \frac{1}{\sqrt{3}}$$

(vi) We have,

$$\cos^{-1}x + \sin^{-1}\frac{x}{2} = \frac{\pi}{6}$$

$$\Rightarrow \quad \cos^{-1}x + \frac{\pi}{2} - \cos^{-1}\frac{x}{2} = \frac{\pi}{6}$$

$$\Rightarrow \quad \cos^{-1}x - \cos^{-1}\frac{x}{2} = -\frac{\pi}{3}$$

$$\Rightarrow \quad \cos^{-1}x = \cos^{-1}\frac{x}{2} - \cos^{-1}\frac{1}{2}$$

$$\Rightarrow \quad \cos^{-1}x = \cos^{-1}\left\{\frac{x}{4}+\frac{\sqrt{3}}{2}\sqrt{1-\frac{x^2}{4}}\right\}$$

$$\Rightarrow \quad x = \frac{x}{4}+\frac{\sqrt{3}}{4}\sqrt{4-x^2} \Rightarrow \frac{3x}{4} = \frac{\sqrt{3}}{4}\sqrt{4-x^2} \Rightarrow 9x^2 = 3(4-x^2) \Rightarrow 4x^2 = 4 \Rightarrow x = 1$$

(viii) We have,

$$\tan(\cos^{-1}x) = \sin\left(\cot^{-1}\frac{1}{2}\right)$$

$$\Rightarrow \quad \tan\left\{\tan^{-1}\frac{\sqrt{1-x^2}}{x}\right\} = \sin\left(\sin^{-1}\frac{2}{\sqrt{5}}\right) \Rightarrow \frac{\sqrt{1-x^2}}{x} = \frac{2}{\sqrt{5}} \Rightarrow x = \pm\frac{\sqrt{5}}{3}$$

(ix) We have,

$$\tan^{-1}\left(\frac{1-x}{1+x}\right) - \frac{1}{2}\tan^{-1}x = 0,\ x > 0$$

$$\Rightarrow \tan^{-1}1 - \tan^{-1}x - \frac{1}{2}\tan^{-1}x = 0 \Rightarrow \frac{\pi}{4}-\frac{3}{2}\tan^{-1}x = 0 \Rightarrow \tan^{-1}x = \frac{\pi}{6} \Rightarrow x = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

(x) We have,

$$\tan^{-1}\frac{1}{x} - \tan^{-1}\frac{1}{x+2} = \frac{\pi}{12}$$

$$\Rightarrow \quad \tan^{-1}\left\{\frac{\frac{1}{x} - \frac{1}{x+2}}{1 + \frac{1}{x(x+2)}}\right\} = \frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2}{x^2 + 2x + 1}\right) = \frac{\pi}{12}$$

$$\Rightarrow \quad \frac{2}{x^2 + 2x + 1} = \tan\frac{\pi}{12}$$

$$\Rightarrow \quad \frac{2}{x^2 + 2x + 1} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$

$$\Rightarrow \quad \frac{2}{x^2 + 2x + 1} = \frac{2}{(\sqrt{3} + 1)^2} \Rightarrow (x+1)^2 = (\sqrt{3} + 1)^2 \Rightarrow x = \sqrt{3}$$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the value of $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) + \cos^{-1}\left(\frac{-1}{2}\right)$.
2. Write the difference between maximum and minimum values of $\sin^{-1} x$ for $x \in [-1, 1]$.
3. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$, then write the value of $x + y + z$.
4. If $x > 1$, then write the value of $\sin^{-1}\left(\frac{2x}{1 + x^2}\right)$ in terms of $\tan^{-1} x$.
5. If $x < 0$, then write the value of $\cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$ in terms of $\tan^{-1} x$.
6. Write the value of $\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$ for $x > 0$.
7. Write the value of $\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$ for $x < 0$.
8. What is the value of $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right)$?
9. If $-1 < x < 0$, then write the value of $\sin^{-1}\left(\frac{2x}{1 + x^2}\right) + \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$.
10. Write the value of $\sin(\cot^{-1} x)$.
11. Write the value of $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$.
12. Write the range of $\tan^{-1} x$.
13. Write the value of $\cos^{-1}(\cos 1540°)$.

14. Write the value of $\sin^{-1}(\sin(-600°))$.
15. Write the value of $\cos\left(2\sin^{-1}\frac{1}{3}\right)$.
16. Write the value of $\sin^{-1}(\sin 1550°)$.
17. Evaluate: $\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right)$.
18. Evaluate: $\sin\left(\tan^{-1}\frac{3}{4}\right)$.
19. Write the value of $\cos^{-1}\left(\tan\frac{3\pi}{4}\right)$.
20. Write the value of $\cos\left(2\sin^{-1}\frac{1}{2}\right)$.
21. Write the value of $\cos^{-1}(\cos 350°) - \sin^{-1}(\sin 350°)$
22. Write the value of $\cos^2\left(\frac{1}{2}\cos^{-1}\frac{3}{5}\right)$.
23. If $\tan^{-1}x + \tan^{-1}y = \frac{\pi}{4}$, then write the value of $x + y + xy$.
24. Write the value of $\cos^{-1}(\cos 6)$.
25. Write the value of $\sin^{-1}\left(\cos\frac{\pi}{9}\right)$.
26. Write the value of $\sin\left\{\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right\}$. **[CBSE 2011]**
27. Write the value of $\tan^{-1}\left\{\tan\left(\frac{15\pi}{4}\right)\right\}$.
28. Write the value of $\sin^{-1}\frac{1}{2} + \cos^{-1}\left(-\frac{1}{2}\right)$.
29. Write the value of $\tan^{-1}\frac{a}{b} - \tan^{-1}\left(\frac{a-b}{a+b}\right)$.
30. Write the value of $\cos^{-1}\left(\cos\frac{5\pi}{4}\right)$.
31. Show that $\sin^{-1}(2x\sqrt{1-x^2}) = 2\sin^{-1}x$
32. Evaluate: $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$. **[CBSE 2009]**
33. If $\tan^{-1}(\sqrt{3}) + \cot^{-1}x = \frac{\pi}{2}$, find x. **[CBSE 2010]**
34. If $\sin^{-1}\left(\frac{1}{3}\right) + \cos^{-1}x = \frac{\pi}{2}$, then find x. **[CBSE 2010]**
35. Write the value of $\sin^{-1}\left(\frac{1}{3}\right) - \cos^{-1}\left(-\frac{1}{3}\right)$.
36. If $4\sin^{-1}x + \cos^{-1}x = \pi$, then what is the value of x?
37. If $x < 0$, $y < 0$ such that $xy = 1$, then write the value of $\tan^{-1}x + \tan^{-1}y$.

38. What is the principal value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$? [CBSE 2010]

39. Write the principal value of $\sin^{-1}\left(\frac{1}{2}\right)$. [CBSE 2011]

40. Write the principal value of $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)+\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$.

41. Write the value of $\tan\left(2\tan^{-1}\frac{1}{5}\right)$. [CBSE 2013]

42. Write the principal value of $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)$. [CBSE 2013]

43. Write the value of $\tan^{-1}\left\{2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right\}$. [CBSE 2013]

44. Write the principal value of $\tan^{-1}\sqrt{3}-\cot^{-1}\sqrt{3}$. [CBSE 21013]

ANSWERS

1. $\frac{\pi}{3}$ 2. π 3. 3 4. $\pi-2\tan^{-1}x$ 5. $-2\tan^{-1}x$

6. $\frac{\pi}{2}$ 7. $-\frac{\pi}{2}$ 8. π 9. 0 10. $\frac{1}{\sqrt{1+x^2}}$ 11. $\frac{2\pi}{3}$

12. $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ 13. 100° 14. 60° 15. $\frac{7}{9}$ 16. 70 17. $\frac{1}{\sqrt{10}}$

18. $\frac{3}{5}$ 19. π 20. $\frac{1}{2}$ 21. 20° 22. $\frac{4}{5}$ 23. 1

24. $2\pi-6$ 25. $\frac{7\pi}{18}$ 26. 1 27. $-\frac{\pi}{4}$ 28. $\frac{5\pi}{6}$ 29. $\frac{\pi}{4}$

30. $\frac{3\pi}{4}$ 32. $\frac{2\pi}{5}$ 33. $\sqrt{3}$ 34. $\frac{1}{3}$ 35. $-\frac{\pi}{2}$ 36. $\frac{1}{2}$

37. $-\frac{\pi}{2}$ 38. $-\frac{\pi}{3}$ 39. $\frac{-\pi}{6}$ 40. π 41. $\frac{5}{12}$ 42. $\frac{11\pi}{12}$

43. $\frac{\pi}{3}$ 44. $\frac{\pi}{2}$

MULTIPLE CHOICE QUESTIONS (MCQs)

1. If $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}=\alpha$, then $x^2=$

   (a) $\sin 2\alpha$ (b) $\sin\alpha$ (c) $\cos 2\alpha$ (d) $\cos\alpha$

2. The value of $\tan\left\{\cos^{-1}\frac{1}{5\sqrt{2}}-\sin^{-1}\frac{4}{\sqrt{17}}\right\}$ is

   (a) $\frac{\sqrt{29}}{3}$ (b) $\frac{29}{3}$ (c) $\frac{\sqrt{3}}{29}$ (d) $\frac{3}{29}$

3. $2\tan^{-1}\{\text{cosec}(\tan^{-1}x)-\tan(\cot^{-1}x)\}$ is equal to

   (a) $\cot^{-1}x$ (b) $\cot^{-1}\frac{1}{x}$ (c) $\tan^{-1}x$ (d) none of these

4. If $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\alpha$, then $\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=$

   (a) $\sin^2\alpha$ (b) $\cos^2\alpha$ (c) $\tan^2\alpha$ (d) $\cot^2\alpha$

5. The positive integral solution of the equation $\tan^{-1}x+\cos^{-1}\frac{y}{\sqrt{1+y^2}}=\sin^{-1}\frac{3}{\sqrt{10}}$ is

   (a) $x=1, y=2$ (b) $x=2, y=1$ (c) $x=3, y=2$ (d) $x=-2, y=-1$

6. If $\sin^{-1} x - \cos^{-1} x = \frac{\pi}{6}$, then $x =$

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $-\frac{1}{2}$ (d) none of these

7. $\sin\left[\cot^{-1}\left\{\tan\left(\cos^{-1} x\right)\right\}\right]$ is equal to

(a) x (b) $\sqrt{1-x^2}$ (c) $\frac{1}{x}$ (d) none of these

8. The number of solutions of the equation $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$ is

(a) 2 (b) 3 (c) 1 (d) none of these

9. If $\alpha = \tan^{-1}\left(\tan \frac{5\pi}{4}\right)$ and $\beta = \tan^{-1}\left(-\tan \frac{2\pi}{3}\right)$, then

(a) $4\alpha = 3\beta$ (b) $3\alpha = 4\beta$ (c) $\alpha - \beta = \frac{7\pi}{12}$ (d) none of these

10. The number of real solutions of the equation $\sqrt{1 + \cos 2x} = \sqrt{2} \sin^{-1}(\sin x), -\pi \le x \le \pi$ is

(a) 0 (b) 1 (c) 2 (d) infinite

11. If $x < 0, y < 0$ such that $xy = 1$, then $\tan^{-1} x + \tan^{-1} y$ equals

(a) $\frac{\pi}{2}$ (b) $-\frac{\pi}{2}$ (c) $-\pi$ (d) none of these

12. If $u = \cot^{-1}\{\sqrt{\tan \theta}\} - \tan^{-1}\{\sqrt{\tan \theta}\}$ then, $\tan\left(\frac{\pi}{4} - \frac{u}{2}\right) =$

(a) $\sqrt{\tan \theta}$ (b) $\sqrt{\cot \theta}$ (c) $\tan \theta$ (d) $\cot \theta$

13. If $\cos^{-1}\frac{x}{3} + \cos^{-1}\frac{y}{2} = \frac{\theta}{2}$, then $4x^2 - 12xy \cos\frac{\theta}{2} + 9y^2 =$

(a) 36 (b) $36 - 36\cos\theta$ (c) $18 - 18\cos\theta$ (d) $18 + 18\cos\theta$

14. If $\alpha = \tan^{-1}\left(\frac{\sqrt{3}\,x}{2y - x}\right)$, $\beta = \tan^{-1}\left(\frac{2x - y}{\sqrt{3}\,y}\right)$, then $\alpha - \beta =$

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $-\frac{\pi}{3}$

15. Let $f(x) = e^{\cos^{-1}\{\sin(x + \pi 3)\}}$. Then, $f(8\pi/9) =$

(a) $e^{5\pi/18}$ (b) $e^{13\pi/18}$ (c) $e^{-2\pi/18}$ (d) none of these

16. $\tan^{-1}\frac{1}{11} + \tan^{-1}\frac{2}{11}$ is equal to

(a) 0 (b) 1/2 (c) -1 (d) none of these

17. If $\cos^{-1}\frac{x}{2} + \cos^{-1}\frac{y}{3} = \theta$, then $9x^2 - 12xy\cos\theta + 4y^2$ is equal to

(a) 36 (b) $-36\sin^2\theta$ (c) $36\sin^2\theta$ (d) $36\cos^2\theta$

18. If $\tan^{-1} 3 + \tan^{-1} x = \tan^{-1} 8$, then $x =$

(a) 5 (b) 1/5 (c) 5/14 (d) 14/5

19. The value of $\sin^{-1}\left(\cos\frac{33\pi}{5}\right)$ is

(a) $\frac{3\pi}{5}$ (b) $-\frac{\pi}{10}$ (c) $\frac{\pi}{10}$ (d) $\frac{7\pi}{5}$

20. The value of $\cos^{-1}\left(\cos\frac{5\pi}{3}\right)+\sin^{-1}\left(\sin\frac{5\pi}{3}\right)$ is

(a) $\frac{\pi}{2}$ (b) $\frac{5\pi}{3}$ (c) $\frac{10\pi}{3}$ (d) 0

21. $\sin\left\{2\cos^{-1}\left(\frac{-3}{5}\right)\right\}$ is equal to

(a) $\frac{6}{25}$ (b) $\frac{24}{25}$ (c) $\frac{4}{5}$ (d) $-\frac{24}{25}$

22. If $\theta=\sin^{-1}\{\sin(-600°)\}$, then one of the possible values of θ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $-\frac{2\pi}{3}$

23. If $3\sin^{-1}\left(\frac{2x}{1+x^2}\right)-4\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)+2\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{3}$ is equal to

(a) $\frac{1}{\sqrt{3}}$ (b) $-\frac{1}{\sqrt{3}}$ (c) $\sqrt{3}$ (d) $-\frac{\sqrt{3}}{4}$

24. If $4\cos^{-1}x+\sin^{-1}x=\pi$, then the value of x is

(a) $\frac{3}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{2}{\sqrt{3}}$

25. If $\tan^{-1}\frac{x+1}{x-1}+\tan^{-1}\frac{x-1}{x}=\tan^{-1}(-7)$, then the value of x is

(a) 0 (b) -2 (c) 1 (d) 2

26. If $\sin^{-1}x-\cos^{-1}x=\pi/6$, then $x=$

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $-\frac{1}{2}$ (d) $-\frac{\sqrt{3}}{2}$

27. In a ΔABC, if C is a right angle, then $\tan^{-1}\left(\frac{a}{b+c}\right)+\tan^{-1}\left(\frac{b}{c+a}\right)=$

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{5\pi}{2}$ (d) $\frac{\pi}{6}$

28. The value of $\sin\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)$ is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{2\sqrt{2}}$ (d) $\frac{1}{3\sqrt{3}}$

29. $\cot\left(\frac{\pi}{4}-2\cot^{-1}3\right)=$

(a) 7 (b) 6 (c) 5 (d) none of these

30. If $\tan^{-1}(\cot\theta)=2\theta$, then $\theta=$

(a) $\pm\frac{\pi}{3}$ (b) $\pm\frac{\pi}{4}$ (c) $\pm\frac{\pi}{6}$ (d) none of these

ANSWERS

1. (a)	2. (d)	3. (c)	4. (a)	5. (a)	6. (b)	7. (a)	8. (a)	9. (a)
10. (c)	11. (b)	12. (a)	13. (c)	14. (a)	15. (b)	16. (d)	17. (c)	18. (b)
19. (b)	20. (d)	21. (d)	22. (a)	23. (a)	24. (c)	25. (d)	26. (b)	27. (b)
28. (c)	29. (a)	30. (c)						

SUMMARY

1. (i) $\sin^{-1}(\sin\theta) = \theta,$ for all $\theta \in [-\pi/2, \pi/2]$
   (ii) $\cos^{-1}(\cos\theta) = \theta,$ for all $\theta \in [0, \pi]$
   (iii) $\tan^{-1}(\tan\theta) = \theta,$ for all $\theta \in (-\pi/2, \pi/2)$
   (iv) $\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta,$ for all $\theta \in [-\pi/2, \pi/2],\ \theta \neq 0$
   (v) $\sec^{-1}(\sec\theta) = \theta,$ for all $\theta \in [0, \pi],\ \theta \neq \pi/2$
   (vi) $\cot^{-1}(\cot\theta) = \theta,$ for all $\theta \in (0, \pi).$
2. (i) $\sin(\sin^{-1}x) = x,$ for all $x \in [-1, 1]$
   (ii) $\cos(\cos^{-1}x) = x,$ for all $x \in [-1, 1]$
   (iii) $\tan(\tan^{-1}x) = x,$ for all $x \in R$
   (iv) $\text{cosec}(\text{cosec}^{-1}x) = x,$ for all $x \in (-\infty, -1] \cup [1, \infty)$
   (v) $\sec(\sec^{-1}x) = x,$ for all $x \in (-\infty, -1] \cup [1, \infty)$
   (vi) $\cot(\cot^{-1}x) = x,$ for all $x \in R.$

<u>REMARK</u> *It should be noted that* $\sin^{-1}(\sin\theta) \neq \theta$, *if* $\theta \in [-\pi/2, \pi/2].$

In fact, we have

$$\sin^{-1}(\sin\theta) = \begin{cases} -\pi-\theta & , \text{ if } \theta\ [-3\pi/2, -\pi/2] \\ \theta & , \text{ if } \theta\ [-\pi/2, \pi/2] \\ \pi-\theta & , \text{ if } \theta\ [\pi/2, 3\pi/2] \\ -2\pi+\theta & , \text{ if } \theta\ [3\pi/2, 5\pi/2] \end{cases} \quad \text{and so on.}$$

Similarly, we have

$$\cos^{-1}(\cos\theta) = \begin{cases} -\theta & , \text{ if } \theta \in [-\pi, 0] \\ \theta & , \text{ if } \theta \in [0, \pi] \\ 2\pi-\theta & , \text{ if } \theta \in [\pi, 2/\pi] \\ -2\pi+\theta & , \text{ if } \theta \in [2\pi, 3/\pi] \end{cases} \quad \text{and so on.}$$

$$\tan^{-1}(\tan\theta) = \begin{cases} \pi-\theta & , \text{ if } \theta \in (-3\pi/2, -\pi/2) \\ \theta & , \text{ if } \theta \in (-\pi/2, \pi/2) \\ \theta-\pi & , \text{ if } \theta \in (\pi/2, 3\pi/2) \\ \theta-2\pi & , \text{ if } \theta \in (3\pi/2, 5\pi/2) \end{cases} \quad \text{and so on.}$$

3. (i) $\sin^{-1}(-x) = -\sin^{-1}x,$ for all $x \in [-1, 1]$
   (ii) $\cos^{-1}(-x) = \pi - \cos^{-1}x,$ for all $x \in [-1, 1]$
   (iii) $\tan^{-1}(-x) = -\tan^{-1}x,$ for all $x \in R$
   (iv) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x,$ for all $x \in (-\infty, -1] \cup [1, \infty)$
   (v) $\sec^{-1}(-x) = \pi - \sec^{-1}x,$ for all $x \in (-\infty, -1] \cup [1, \infty)$
   (vi) $\cot^{-1}(-x) = \pi - \cot^{-1}x,$ for all $x \in R$
4. (i) $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1}x,$ for all $x \in (-\infty, -1] \cup [1, \infty)$
   (ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x,$ for all $x \in (-\infty, -1] \cup [1, \infty)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x & , \text{ for } x > 0 \\ -\pi + \cot^{-1} x, & \text{for } x < 0 \end{cases}$

5. (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, for all $x \in [-1, 1]$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$, for all $x \in R$

(iii) $\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}$, for all $x \in (-\infty, -1] \cup [1, \infty)$.

6. (i) $$\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right) & , \text{ if } xy < 1 \\ \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } x > 0, y > 0 \text{ and } xy > 1 \\ -\pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } x < 0, y < 0 \text{ and } xy > 1 \end{cases}$$

(ii) $$\tan^{-1} x - \tan^{-1} y = \begin{cases} \tan^{-1}\left(\frac{x-y}{1+xy}\right) & , \text{ if } xy > -1 \\ \pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right) & , \text{ if } x > 0, y < 0 \text{ and } xy < -1 \\ \pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right) & , \text{ if } x < 0, y > 0 \text{ and } xy < -1 \end{cases}$$

<u>REMARK</u> *If $x_1, x_2, x_3, \ldots, x_n \in R$, then*

$$\tan^{-1} x_1 + \tan^{-1} x_2 + \ldots + \tan^{-1} x_n = \tan^{-1}\left(\frac{S_1 - S_3 + S_5 - S_7 + \ldots}{1 - S_2 + S_4 - S_6 + \ldots}\right),$$

where S_k denotes the sum of the products of $x_1, x_2, \ldots, x_n$ taken k at a time.

7. (i) $\sin^{-1} x + \sin^{-1} y$

$$= \begin{cases} \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\} & , \text{ if } -1 \le x, y \le 1 \text{ and } x^2 + y^2 \le 1 \\ & \text{or} \\ & \text{if } xy < 0 \text{ and } x^2 + y^2 > 1 \\ \pi - \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\} & , \text{ if } 0 < x, y \le 1 \text{ and } x^2 + y^2 > 1 \\ -\pi - \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}, & \text{if } -1 \le x, y < 0 \text{ and } x^2 + y^2 > 1 \end{cases}$$

(ii) $\sin^{-1} x - \sin^{-1} y$

$$= \begin{cases} \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}, & \text{if } -1 \le x, y \le 1 \text{ and } x^2 + y^2 \le 1 \\ & \text{or} \\ & \text{if } xy > 0 \text{ and } x^2 + y^2 > 1 \\ \pi - \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}, & \text{if } 0 < x \le 1, -1 \le y \le 0 \text{ and } x^2 + y^2 > 1 \\ -\pi - \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}, & \text{if } -1 \le x < 0, 0 < y \le 1 \text{ and } x^2 + y^2 \end{cases}$$

8. (i) $\cos^{-1} x + \cos^{-1} y$

$$= \begin{cases} \cos^{-1}\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\} & , \text{ if } -1 \le x, y \le 1 \text{ and } x + y \ge 0 \\ 2\pi - \cos^{-1}\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\} & , \text{ if } -1 \le x, y \le 1 \text{ and } x + y \le 0 \end{cases}$$

(ii) $\cos^{-1} x - \cos^{-1} y$

$$= \begin{cases} \cos^{-1}\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\} & , \text{ if } -1 \le x, y \le 1 \text{ and } x \le y \\ -\cos^{-1}\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\} & , \text{ if } -1 \le y \le 0, 0 < x \le 1 \text{ and } x \ge y \end{cases}$$

9. (i) $$2\sin^{-1} x = \begin{cases} \sin^{-1}(2x\sqrt{1-x^2}) & , \text{ if } -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - \sin^{-1}(2x\sqrt{1-x^2}) & , \text{ if } \frac{1}{\sqrt{2}} \le x \le 1 \\ -\pi - \sin(2x\sqrt{1-x^2}) & , \text{ if } -1 \le x \le -\frac{1}{\sqrt{2}} \end{cases}$$

(ii) $$3\sin^{-1} x = \begin{cases} \sin^{-1}(3x - 4x^3) & , \text{ if } -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - \sin^{-1}(3x - 4x^3) & , \text{ if } \frac{1}{2} < x \le 1 \\ -\pi - \sin^{-1}(3x - 4x^3) & , \text{ if } -1 \le x < -\frac{1}{2} \end{cases}$$

10. (i) $$2\cos^{-1} x = \begin{cases} \cos^{-1}(2x^2 - 1) & , \text{ if } 0 \le x \le 1 \\ 2\pi - \cos^{-1}(2x^2 - 1) & , \text{ if } -1 \le x \le 0 \end{cases}$$

(ii) $$3\cos^{-1} x = \begin{cases} \cos^{-1}(4x^3 - 3x) & , \text{ if } \frac{1}{2} \le x \le 1 \\ 2\pi - \cos^{-1}(4x^3 - 3x) & , \text{ if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 2\pi + \cos^{-1}(4x^3 - 3x) & , \text{ if } -1 \le x \le -\frac{1}{2} \end{cases}$$

11. (i) $$2\tan^{-1} x = \begin{cases} \tan^{-1}\left(\frac{2x}{1-x^2}\right) & , \text{ if } -1 < x < 1 \\ \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right) & , \text{ if } x > 1 \\ -\pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right) & , \text{ if } x < -1 \end{cases}$$

(ii) $3\tan^{-1}x = \begin{cases} \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right) & \text{, if } -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right) & \text{, if } x > \dfrac{1}{\sqrt{3}} \\ -\pi + \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right) & \text{, if } x < -\dfrac{1}{\sqrt{3}} \end{cases}$

12. (i) $2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) & \text{, if } -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) & \text{, if } x > 1 \\ -\pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) & \text{, if } x < -1 \end{cases}$

(ii) $2\tan^{-1}x = \begin{cases} \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) & \text{, if } 0 \le x < \infty \\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) & \text{, if } -\infty < x \le 0 \end{cases}$

13. (i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$

$$= \cot^{-1}\frac{\sqrt{1-x^2}}{x} = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

(ii) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$

$$= \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

(iii) $\tan^{-1}x = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)$

$$= \cot^{-1}\left(\frac{1}{x}\right) = \sec^{-1}\sqrt{1+x^2} = \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$$

14. If $x_1, x_2, \ldots, x_n \in R$, then

$$\tan^{-1}x_1 + \tan^{-1}x_2 + \ldots + \tan^{-1}x_n = \tan^{-1}\left(\frac{S_1 - S_3 + S_5 - S_7 + \ldots}{1 - S_2 + S_4 - S_6 + \ldots}\right),$$

where S_k = Sum of the products of $x_1, x_2, \ldots, x_n$ taken k at a time.

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TRIGONOMETRIC RATIOS AND IDENTITIES

NTRODUCTION

ITION *Measure of an angle is the amount of rotation of a ing line with respect to a fixed line.*

otation is in clock-wise sense, the measure of the angle tive and it is positive if the rotation is in anti-clockwise

re three systems of measuring an angle viz.
exagesimal system or English system
ircular system
ench system

o of these three systems are commonly used. In mal system, a right angle is divided into 90 equal parts grees. Further, each degree is divided into sixty equal ed *minutes* and each minute is divided into sixty equal ed *seconds*.

$$1 \text{ right angle} = 90 \text{ degrees } (90^\circ)$$
$$1^\circ = 60 \text{ minutes } (60')$$
$$1' = 60 \text{ seconds } (60'')$$

system the unit of measurement is *radian*. One radian made by an arc of length equal to radius of a given centre.

ween degree and radian : If D is the degree measure and R is its measure in radians, then

$$\frac{D}{90} = \frac{2R}{\pi}$$

$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees}$$
$$= 57^\circ\, 17'\, 45'' \text{ (approximately)}$$

$$1 \text{ degree} = \frac{\pi}{180} \text{ radian}$$

ASIC FORMULAE

$\cos^2 A = 1$

$^2 A = \sec^2 A$ or, $\sec^2 A - \tan^2 A = 1$

$+ \tan A = \dfrac{1}{\sec A - \tan A}$, where $A \neq n\pi + \dfrac{\pi}{2}$

$A = \mathrm{cosec}^2 A$ or, $\mathrm{cosec}^2 A - \cot^2 A = 1$

$A + \cot A = \dfrac{1}{\quad}$

3. DOMAIN AND RANGE OF TRIGONOMETRICAL FUNCTIONS

	Domain	*Range*
$\sin A$	R	$[-1, 1]$
$\cos A$	R	$[-1, 1]$
$\tan A$	$R - \{(2n+1)\pi/2 : n \in Z\}$	$(-\infty, \infty) = R$
$\mathrm{cosec}\, A$	$R - \{n\pi : n \in Z\}$	$(-\infty, -1] \cup [1, \infty)$
$\sec A$	$R - \{(2n+1)\pi/2 : n \in Z\}$	$(-\infty, -1] \cup [1, \infty)$
$\cot A$	$R - \{n\pi : n \in Z\}$	$(-\infty, \infty) = R$

Thus, $|\sin A| \le 1$, $|\cos A| \le 1$, $\sec A \ge 1$ or, $\sec A \le -1$ and, $\mathrm{cosec}\, A \ge 1$ or, $\mathrm{cosec}\, A \le -1$.

4. SUM AND DIFFERENCE FORMULAE

1. $\sin(A + B) = \sin A \cos B + \cos A \sin B$
2. $\sin(A - B) = \sin A \cos B - \cos A$ si
3. $\cos(A + B) = \cos A \cos B -$ sin
4. $\cos(A - B) = \cos A \cos B +$
5. $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan}$ ... $n\pi +$ ... $/2$
6. $\tan(A - B) = \dfrac{\tan A - \tan}{1 + \tan A}$ ... $B \neq$
7. $\cot(A + B) = \dfrac{\cot A\, c}{\cot}$ ... re

   $\cot(A - B) = \dfrac{\cot}{}$
8. $\sin(A + B)$
9. $\cos(A +$
10. $\sin 2$
11. c

$\Rightarrow \quad \dfrac{S_1 - S_3}{1 - S_2} = 0$ and $\dfrac{1 - S_2}{S_1 - S_3} = 0$

$\Rightarrow \quad S_1 = S_3$ and $S_2 = 1$

$\Rightarrow \quad \tan A + \tan B + \tan C = \tan A \tan B \tan C \quad$ ...(i)

and, $\tan\dfrac{A}{2}\tan\dfrac{B}{2} + \tan\dfrac{B}{2}\tan\dfrac{C}{2} + \tan\dfrac{C}{2}\tan\dfrac{A}{2} = 1 \quad$ ...(ii)

Using A.M. $\geq$ G.M., we have

$$\frac{\tan A + \tan B + \tan C}{3} \geq (\tan A \tan B \tan C)^{1/3}$$

$$\Rightarrow \quad \frac{\tan A \tan B \tan C}{3} \geq (\tan A \tan B \tan C)^{1/3} \quad \text{[Using (i)]}$$

$\Rightarrow \quad \tan A \tan B \tan C \geq 3\sqrt{3}$

So, statement–2 is true.

From (ii), we have

$$xy + yz + zx = 1, \text{ where } x = \tan\frac{A}{2}, y = \tan\frac{B}{2} \text{ and } z = \tan\frac{C}{2}$$

$$\therefore \quad x^2 + y^2 + z^2 - 1$$
$$= x^2 + y^2 + z^2 - (xy + yz + zx)$$
$$= \frac{1}{2}\left\{(x-y)^2 + (y-z)^2 + (z-x)^2\right\} \geq 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 \geq 1 \Rightarrow \tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \geq 1$$

Hence, both the statements are true.

EXAMPLE 11 *Statement–1: If A, B, C are the angles of a triangle such that angle A is obtuse, then* $\tan B \tan C > 1$.

Statement–2: In any ΔABC, *we have*

$$\tan A = \frac{\tan B + \tan C}{\tan B \tan C - 1}$$

(a) 1 (b) 2 (c) 3 (d) 4

Ans. (d)

SOLUTION In any ΔABC, we have

$$A + B + C = \pi$$

$\Rightarrow \quad A = \pi - (B + C)$

$\Rightarrow \quad \tan A = -\tan(B + C)$

$$\Rightarrow \quad \tan A = \frac{\tan B + \tan C}{\tan B \tan C - 1}$$

So, statement–2 is true.

If angle A is obtuse, then B and C are acute angles.

$\therefore \quad \tan A < 0, \ \tan B > 0$ and $\tan C > 0$

$$\Rightarrow \quad \frac{\tan B + \tan C}{\tan B \tan C - 1} < 0 \Rightarrow \tan B \tan C < 1$$

So, statement–1 is not correct.

EXAMPLE 12 *Statement–1: The numbers* $\sin 18^\circ$ *and* $-\sin 54^\circ$ *are the roots of the quadratic equation with integer coefficients.*

Statement–2: If $x = 18^\circ, \cos 3x = \sin 2x$ *and if* $y = -54^\circ$; $\sin 2y = \cos 3y$.

(a) 1 (b) 2 (c) 3 (d) 4

Ans. (a)

SOLUTION Clearly, statement–2 is true.

Using statement–2, we have

$$\cos 3x = \sin 2x$$

$\Rightarrow \quad 4\cos^3 x - 3\cos x = 2\sin x \cos x$

$\Rightarrow \quad 4(1 - \sin^2 x) - 3 = 2\sin x$

$\Rightarrow \quad 4\sin^2 x + 2\sin x - 1 = 0$

$\Rightarrow \quad \sin x = \sin 18^\circ$ is a root of a quadratic equation with integer coefficients.

Similarly, $\sin 2y = \cos 3y$

$\Rightarrow \quad \sin y = \sin(-54^\circ) = -\sin 54^\circ$ is a root of the equation $4\sin^2 y + 2\sin y - 1 = 0$

Hence, $\sin 18^\circ$ and $-\sin 54^\circ$ are the roots of a quadratic equation with integer coefficients.

EXAMPLE 13 *Statement–1: If* $2\sin\dfrac{\theta}{2} = \sqrt{1 + \sin\theta} + \sqrt{1 - \sin\theta}$, *then* $\theta \in \left((8n+1)\dfrac{\pi}{2}, (8n+3)\dfrac{\pi}{2}\right)$

Statement–2: If $\dfrac{\pi}{4} \leq \theta \leq \dfrac{3\pi}{4}$, *then* $\sin\dfrac{\theta}{2} > 0$.

(a) 1 (b) 2 (c) 3 (d) 4

Ans. (b)

SOLUTION We have,

$$2\sin\frac{\theta}{2} = \sqrt{1 + \sin\theta} + \sqrt{1 - \sin\theta}$$

$$\Rightarrow \quad 2\sin\frac{\theta}{2} = \sqrt{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2} + \sqrt{\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)^2}$$

$$\Rightarrow \quad 2\sin\frac{\theta}{2} = \left|\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right| + \left|\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right|$$

$$\Rightarrow \quad \cos\frac{\theta}{2} + \sin\frac{\theta}{2} > 0 \text{ and } \cos\frac{\theta}{2} - \sin\frac{\theta}{2} < 0$$

$$\Rightarrow \quad \sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right) > 0 \text{ and } \cos\left(\frac{\theta}{2} + \frac{\pi}{4}\right) < 0$$

$$\Rightarrow \quad 2n\pi + \frac{\pi}{2} < \frac{\theta}{2} + \frac{\pi}{4} < 2n\pi + \pi$$

$$\Rightarrow \quad (8n+1)\frac{\pi}{2} < \theta < (8n+3)\frac{\pi}{2}$$

So, statement–1 is true.

Statement–2 is also true, but it is not a correct explanation for statement–1.

EXERCISE

This exercise contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) for its answer, out of which only one is correct.

1. The value of $\cos 10^\circ - \sin 10^\circ$ is

(a) positive (b) negative (c) 0 (d) 1

2. The value of $\cos 1^\circ \cos 2^\circ \cos 3^\circ \ldots \cos 179^\circ$ is

(a) $\dfrac{1}{\sqrt{2}}$ (b) 0 (c) 1 (d) none of these

3. The value of $\tan 1^\circ \tan 2^\circ \tan 3^\circ \ldots \tan 89^\circ$ is
(a) 1 (b) 0 (c) ∞ (d) 1/2

4. The maximum value of $\cos^2\left(\frac{\pi}{3}-x\right)-\cos^2\left(\frac{\pi}{3}+x\right)$, is
(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$ (c) $-\frac{\sqrt{3}}{2}$ (d) $\frac{3}{2}$

5. Which of the following is correct
(a) $\sin 1^\circ > \sin 1$ (b) $\sin 1^\circ < \sin 1$
(c) $\sin 1^\circ = \sin 1$ (d) $\sin 1^\circ = \frac{\pi}{180}\sin 1$

6. Given $A = \sin^2\theta + \cos^4\theta$, then for all real θ
(a) $1 \le A \le 2$ (b) $\frac{3}{4} \le A \le 1$
(c) $\frac{13}{16} \le A \le 1$ (d) $\frac{3}{4} \le A \le \frac{13}{16}$

7. The expression $\tan^2\alpha + \cot^2\alpha$, is
(a) ≥ 2 (b) ≤ 2 (c) ≥ -2 (d) none of these

8. If $\tan\theta = -4/3$, then $\sin\theta$ is
(a) $-4/5$ but not $4/5$ (b) $-4/5$ or $4/5$
(c) $4/5$ but not $-4/5$ (d) none of these
[JEE (Orissa) 2002]

9. If $\sin\theta + \cos\theta = \sqrt{2}\cos\theta$ then $\cos\theta - \sin\theta$ is equal to
(a) $\sqrt{2}\cos\theta$ (b) $\sqrt{2}\sin\theta$
(c) $\sqrt{2}(\cos\theta + \sin\theta)$ (d) none of these

10. In a right angled triangle, the hypotenuse is four times as long as the perpendicular drawn to it from the opposite vertex. One of the acute angle is
(a) 15° (b) 30° (c) 45° (d) none of these

11. If $\cos\theta = \frac{8}{17}$ and θ lies in the first quadrant, then the value of $\cos(30+\theta)+\cos(45-\theta)+\cos(120-\theta)$, is
(a) $\frac{23}{17}\left(\frac{\sqrt{3}-1}{2}+\frac{1}{\sqrt{2}}\right)$ (b) $\frac{23}{17}\left(\frac{\sqrt{3}+1}{2}+\frac{1}{\sqrt{2}}\right)$
(c) $\frac{23}{17}\left(\frac{\sqrt{3}-1}{2}-\frac{1}{\sqrt{2}}\right)$ (d) $\frac{23}{17}\left(\frac{\sqrt{3}+1}{2}-\frac{1}{\sqrt{2}}\right)$

12. The value of $\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}$, is
(a) 0 (b) 1/2 (c) 3/2 (d) 1

13. If $\tan(A+B)=p$ and $\tan(A-B)=q$, then the value of $\tan 2A$ is
(a) $\frac{p+q}{p-q}$ (b) $\frac{p-q}{1+pq}$ (c) $\frac{1+pq}{1-p}$ (d) $\frac{p+q}{1-pq}$
[PET (MP) 2002]

4. In a triangle ABC, $\sin A - \cos B = \cos C$, then angle B is
(a) $\pi/2$ (b) $\pi/3$ (c) $\pi/4$ (d) $\pi/6$

. If θ lies in the first quadrant which of the following is not true
(a) $\frac{\theta}{2} < \tan\left(\frac{\theta}{2}\right)$ (b) $\frac{\theta}{2} < \sin\frac{\theta}{2}$
(c) $\theta\cos^2\left(\frac{\theta}{2}\right) < \sin\theta$ (d) $\theta\sin\frac{\theta}{2} < 2\sin\frac{\theta}{2}$

16. $\cos 2\theta + 2\cos\theta$ is always
(a) greater than $-\frac{3}{2}$ (b) less than or equal to $\frac{3}{2}$
(c) greater than or equal to $-\frac{3}{2}$ (d) none of these

17. If the interior angles of a polygon are in A.P. with common difference 5° and the smallest angle 120°, then the number of sides of the polygon is
(a) 9 or 16 (b) 9 (c) 13 (d) 16

18. The maximum value of $5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right)+3$ is
(a) 5 (b) 10 (c) 11 (d) -11

19. The value of $16\sin 144^\circ \sin 108^\circ \sin 72^\circ \sin 36^\circ$ is equal to
(a) 5 (b) 4 (c) 3 (d) 1
[JEE (WB) 2007]

20. If $A = \tan 6^\circ \tan 42^\circ$ and $B = \cot 66^\circ \cot 78^\circ$, then
(a) $A = 2B$ (b) $A = 1/3$ (c) $A = B$ (d) $3A = 2B$

21. If $\sin x + \operatorname{cosec} x = 2$, then $\sin^n x + \operatorname{cosec}^n x$ is equal to
(a) 2 (b) $2n$ (c) $2n-1$ (d) $2n-2$
[JEE (WB) 2006]

22. If $\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1$, $\frac{x}{a}\cos\beta + \frac{y}{b}\sin\beta = 1$ and

$\frac{\cos\alpha\cos\beta}{a^2}+\frac{\sin\alpha\sin\beta}{b^2}=0$, then

(a) $\tan\alpha\tan\beta = \frac{b^2(x^2-a^2)}{a^2(y^2-b^2)}$ and $x^2+y^2=a^2-b^2$
(b) $\tan\alpha\tan\beta = \frac{a^2}{b^2}$
(c) $x^2+y^2=a^2-b^2$ (d) none of these

23. The values of θ lying between 0 and $\pi/2$ and satisfying the equation
$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+\sin^4\theta \end{vmatrix} = 0 \text{ are}$$
(a) $\frac{7\pi}{24}$ and $\frac{11\pi}{24}$ (b) $\frac{7\pi}{24}$ and $\frac{5\pi}{24}$
(c) $\frac{5\pi}{24}$ and $\frac{\pi}{24}$ (d) none of these

24. The value of $\sqrt{3}\cot 20^\circ - 4\cos 20^\circ$ is
(a) 1 (b) -1 (c) 0 (d) none of these

25. The value of $\sqrt{3}\operatorname{cosec} 20^\circ - \sec 20^\circ$ is equal to
(a) 2 (b) 1 (c) 4 (d) -4
[EAMCET 2008

26. The equation $\sin^2\theta = \frac{x^2+y^2}{2xy}$, is possible if
(a) $x=y$ (b) $x=-y$ (c) $2x=y$ (d) none of the

27. The value of $\sin(\pi+\theta)\sin(\pi-\theta)\operatorname{cosec}^2\theta$ is equal to
(a) -1 (b) 0 (c) $\sin\theta$ (d) none of the

28. If $\frac{\sin(x+y)}{\sin(x-y)}=\frac{a+b}{a-b}$, then $\frac{\tan x}{\tan y}$ is equal to
(a) b/a (b) a/b (c) ab (d) none of these

29. If $\sin x+\sin^2 x=1$, then value of $\cos^2 x+\cos^4 x$ is
(a) 1 (b) 2 (c) 1.5 (d) none of these

30. If $\tan\frac{x}{2}$ = cosec $x-\sin x$, then the value of $\tan^2\frac{x}{2}$, is
(a) $2-\sqrt{5}$ (b) $2+\sqrt{5}$ (c) $-2-\sqrt{5}$ (d) $-2+\sqrt{5}$

31. If $\cos A=\frac{3}{4}$, then $32\sin\left(\frac{A}{2}\right)\sin\left(\frac{5A}{2}\right)=$
(a) 7 (b) 8 (c) 11 (d) none of these

32. The value of
$\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$ is
(a) $\frac{1}{2}$ (b) $\cos\frac{\pi}{8}$ (c) $\frac{1}{8}$ (d) $\frac{1+\sqrt{2}}{2\sqrt{2}}$

33. If $\tan^2\theta=2\tan^2\phi+1$, then $\cos 2\theta+\sin^2\phi$ equals
(a) -1 (b) 0 (c) 1 (d) none of these

34. If $\sin 2\theta=\cos 3\theta$ and θ is an acute angle, then $\sin\theta$ equals
(a) $\frac{\sqrt{5}-1}{4}$ (b) $-\left(\frac{\sqrt{5}-1}{4}\right)$
(c) $\frac{\sqrt{5}+1}{4}$ (d) $\frac{-\sqrt{5}-1}{4}$

35. If $y=\sec^2\theta+\cos^2\theta, \theta\neq 0$, then
(a) $y=0$ (b) $y\leq 2$ (c) $y\geq -2$ (d) $y\neq 2$
[JEE (WB) 2006]

36. The value of
$\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$ is
(a) 1/16 (b) 1/64 (c) 1/128 (d) 1/32

37. The value of $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}$ is
(a) 1/16 (b) 1/8 (c) 1/2 (d) 1/4

38. If $\sin(\alpha+\beta)=1$, $\sin(\alpha-\beta)=1/2$; α, β . $[0, \pi/2]$, then $\tan(\alpha+2\beta)\tan(2\alpha+\beta)$ is equal to
(a) 1 (b) -1 (c) 0 (d) 1/2

39. If $\cos(\theta-\alpha)=a$, $\cos(\theta-\beta)=b$,
then $\sin^2(\alpha-\beta)+2ab\cos(\alpha-\beta)=$
(a) a^2+b^2 (b) a^2-b^2 (c) b^2-a^2 (d) $-a^2-b^2$
[JEE (WB) 2007]

40. The value of $\sin\left(\frac{\pi}{18}\right)\sin\left(\frac{5\pi}{18}\right)\sin\left(\frac{7\pi}{18}\right)$, is
(a) 1/2 (b) 1/4 (c) 1/8 (d) 1/16

41. The value of $\log\tan 1^\circ+\log\tan 2^\circ+\ldots+\log\tan 89^\circ$, is
(a) 0 (b) -1 (c) 1 (d) ∞

42. If $1+\sin x+\sin^2 x+\sin^3 x+\ldots+\ldots\infty$ is equal to $4+2\sqrt{3}, 0<x<\pi$, then $x=$
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ or $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$ or $\frac{2\pi}{3}$
[I.P. (Delhi) 2003]

43. If $x\cos\alpha+y\sin\alpha=2a$, $x\cos\beta+y\sin\beta=2a$ and $2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}=1$, then
(a) $\cos\alpha+\cos\beta=\frac{2ax}{x^2+y^2}$ (b) $\cos\alpha\cos\beta=\frac{2a^2-y^2}{x^2+y^2}$
(c) $y^2=4a(\alpha-x)$ (d) $\cos\alpha+\cos\beta=2\cos\alpha\cos\beta$.

44. If $\tan x=\frac{2b}{a-c}$, $a\neq c$;
and $y=a\cos^2 x+2b\sin x\cos x+c\sin^2 x$
$z=a\sin^2 x-2b\sin x\cos x+c\cos^2 x$, then
(a) $y=z$ (b) $y+z=a-c$
(c) $y-z=a-c$ (d) $(y-z)=(a-c)^2+4b^2$

45. If $\alpha+\beta+\gamma=2\pi$, then
(a) $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$
(b) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=1$
(c) $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$
(d) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=0$.

46. If $\sin\theta-\cos\theta<0$, then θ lies between
(a) $n\pi-\frac{3\pi}{4}$ and $n\pi+\frac{\pi}{4}$, n . Z
(b) $n\pi-\frac{\pi}{4}$ and $n\pi+\frac{3\pi}{4}$ n . Z
(c) $2n\pi-\frac{3\pi}{4}$ and $2n\pi-\frac{\pi}{4}$, n . Z
(d) $2n\pi-\frac{3\pi}{4}$ and $2n\pi+\frac{\pi}{4}$, n . Z

47. If $2\sin\frac{A}{2}=\sqrt{1+\sin A}+\sqrt{1-\sin A}$, then $\frac{A}{2}$ lies between
(a) $2n\pi+\frac{\pi}{4}$ and $2n\pi+\frac{3\pi}{4}$, n . Z
(b) $2n\pi-\frac{\pi}{4}$ and $2n\pi+\frac{\pi}{4}$, n . Z
(c) $2n\pi-\frac{3\pi}{4}$ and $2n\pi-\frac{\pi}{4}$, n . Z
(d) $-\infty$ and $+\infty$

48. If $2\cos\frac{A}{2}=\sqrt{1+\sin A}+\sqrt{1-\sin A}$, then $\frac{A}{2}$ lies between, (n
(a) $2n\pi+\frac{\pi}{4}$ and $2n\pi+\frac{3\pi}{4}$ (b) $2n\pi-\frac{\pi}{4}$ and $2n\pi+\frac{\pi}{4}$
(c) $2n\pi-\frac{3\pi}{4}$ and $2n\pi-\frac{\pi}{4}$ (d) $-\infty$ and $+\infty$

49. The angle θ whose cosine equals to its tangent is given by
(a) $\cos\theta=2\cos 18^\circ$ (b) cos theta $=2\sin 18^\circ$
(c) $\sin\theta=2\sin 18^\circ$ (d) $\sin\theta=2\cos 18^\circ$

50. The value of $\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{14\pi}{15}$ is

(a) 1 (b) 1/2 (c) 1/4 (d) 1/16

51. The value of

$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}$ is

(a) $\frac{1}{2^6}$ (b) $\frac{1}{2^7}$ (c) $\frac{1}{2^8}$ (d) none of these

52. The value of $\tan 5\theta$ is

(a) $\frac{5\tan\theta-10\tan^3\theta+\tan^5\theta}{1-10\tan^2\theta+5\tan^4\theta}$

(b) $\frac{5\tan\theta+10\tan^3\theta-\tan^5\theta}{1+10\tan^2\theta-5\tan^4\theta}$

(c) $\frac{5\tan^5\theta-10\tan^3\theta+\tan\theta}{1-10\tan^2\theta+5\tan^4\theta}$

(d) none of these

53. If $\cos\theta = \cos\alpha\cos\beta$, then $\tan\left(\frac{\theta+\alpha}{2}\right)\tan\left(\frac{\theta-\alpha}{2}\right)$ is equal to

(a) $\tan^2\frac{\alpha}{2}$ (b) $\tan^2\frac{\beta}{2}$ (c) $\tan^2\frac{\theta}{2}$ (d) $\cot^2\frac{\beta}{2}$

54. If $\left|\cos\theta\left\{\sin\theta+\sqrt{\sin^2\theta+\sin^2\alpha}\right\}\right|\le k$, then the value of k is

(a) $\sqrt{1+\cos^2\alpha}$ (b) $\sqrt{1+\sin^2\alpha}$

(c) $\sqrt{2+\sin^2\alpha}$ (d) $\sqrt{2+\cos^2\alpha}$

55. The value of $\sin 10^\circ+\sin 20^\circ+\sin 30^\circ+\ldots+\sin 360^\circ$ is

(a) 1 (b) 0 (c) −1 (d) 1/2

56. The expression

$3\left\{\sin^4\left(\frac{3\pi}{2}-\alpha\right)+\sin^4(3\pi-\alpha)\right\}$

$-2\left\{\sin^6\left(\frac{\pi}{2}+\alpha\right)+\sin^6(5\pi-\alpha)\right\}$ is equal to

(a) 0 (b) 1 (c) 3 (d) $\sin 4\alpha+\cos 6\alpha$

57. If $A+B=\frac{\pi}{4}$, then $(\tan A+1)(\tan B+1)$ is equal to

(a) 1 (b) 2 (c) $\sqrt{3}$ (d) −1

58. If $\sin A+\sin B=a$ and $\cos A+\cos B=b$, then $\cos(A+B)$

(a) $\frac{a^2+b^2}{b^2-a^2}$ (b) $\frac{2ab}{a^2+b^2}$ (c) $\frac{b^2-a^2}{a^2+b^2}$ (d) $\frac{a^2-b^2}{a^2+b^2}$

59. If an angle α is divided into two parts A and B such that $A-B=x$ and $\tan A:\tan B=k:1$, then the value of $\sin x$ is

(a) $\frac{k+1}{k-1}\sin\alpha$ (b) $\frac{k}{k+1}\sin\alpha$

(c) $\frac{k-1}{k+1}\sin\alpha$ (d) none of these

60. The value of the expression

$3(\sin\theta-\cos\theta)^4+6(\sin\theta+\cos\theta)^2+4(\sin^6\theta+\cos^6\theta)$ is

(a) 1 (b) −1 (c) 13 (d) 0

61. If $\tan\left(\frac{\theta}{2}\right)=\frac{5}{2}$ and $\tan\left(\frac{\phi}{2}\right)=\frac{3}{4}$, the value of $\cos(\theta+\phi)$ is

(a) $-\frac{364}{725}$ (b) $-\frac{627}{725}$ (c) $-\frac{240}{339}$ (d) $-\frac{339}{725}$

62. If $\alpha,\beta,\gamma\in(0,\pi/2)$, then the value of $\frac{\sin(\alpha+\beta+\gamma)}{\sin\alpha+\sin\beta+\sin\gamma}$, is

(a) <1 (b) >1 (c) $=1$ (d) $=-1$

63. If $\sin x+\sin y=3(\cos y-\cos x)$, then the value of $\frac{\sin 3x}{\sin 3y}$ is

(a) 1 (b) −1 (c) 0 (d) ±1

64. If $\cos x=\tan y$, $\cos y=\tan z$, $\cos z=\tan x$, then the value of $\sin x$ is

(a) $2\cos 18^\circ$ (b) $\cos 18^\circ$ (c) $\sin 18^\circ$ (d) $2\sin 18^\circ$

65. If $k=\sin^6 x+\cos^6 x$, then k belongs to the interval

(a) [7/8, 5/4] (b) [1/2, 5/8]

(c) [1/4, 1] (d) none of these

66. The value of $\tan 9^\circ-\tan 27^\circ-\tan 63^\circ+\tan 81^\circ$ is

(a) 2 (b) 3 (c) 4 (d) 1

67. If $\tan^2\alpha\tan^2\beta+\tan^2\beta\tan^2\gamma+\tan^2\gamma\tan^2\alpha+2\tan^2\alpha$ $\tan^2\beta\tan^2\gamma=1$, then the value of $\sin^2\alpha+\sin^2\beta+\sin^2\gamma$ is

(a) 0 (b) −1 (c) 1 (d) ±1

68. The value of

$e^{\log_{10}\tan 1^\circ+\log_{10}\tan 2^\circ+\log_{10}\tan 3^\circ+\ldots+\log_{10}\tan 89^\circ}$, is

(a) 0 (b) e (c) $1/e$ (d) 1

69. For what and only what values of α lying between 0 and π is the inequality $\sin\alpha\cos^3\alpha>\sin^3\alpha\cos\alpha$ valid ?

(a) $\alpha\in(0,\pi/4)$ (b) $\alpha\in(0,\pi/2)$

(c) $\alpha\in(\pi/4,\pi/2)$ (d) none of these

70. If $(\sec A-\tan A)(\sec B-\tan B)(\sec C-\tan C)$ $=(\sec A+\tan A)(\sec B+\tan B)(\sec C+\tan C)$ then each side is equal to

(a) 0 (b) 1 (c) −1 (d) ±1

71. If $\pi<\alpha<\frac{3\pi}{2}$, then the expression

$\sqrt{4\sin^4\alpha+\sin^2 2\alpha}+4\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$ is equal to

(a) $2+4\sin\alpha$ (b) $2-4\sin\alpha$ (c) 2 (d) none of t

72. If α is an acute angle and $\sin\frac{\alpha}{2}=\sqrt{\frac{x-1}{2x}}$, then $\tan\alpha$

(a) $\sqrt{\frac{x-1}{x+1}}$ (b) $\frac{\sqrt{x-1}}{x+1}$ (c) $\sqrt{x^2-1}$ (d) $\sqrt{}$

73. The value of $\tan 82\frac{1}{2}^\circ$, is

(a) $\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$ (b) $(\sqrt{3}+\sqrt{2})(\sqrt{2}-$

(c) $-(\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$ (d) none of these

74. The value of $\tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ$ is
(a) 1 (b) 1/2 (c) 1/4 (d) 1/8

75. The value of $\cot 36^\circ \cot 72^\circ$ is
(a) 1/5 (b) $1/\sqrt{5}$ (c) 1 (d) 1/3

76. The value of
$$\cos\frac{\pi}{7}+\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{5\pi}{7}+\cos\frac{6\pi}{7}+\cos\frac{7\pi}{7},$$
is
(a) 1 (b) −1 (c) 0 (d) −2

77. The value of $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}$, is
(a) 1 (b) −1 (c) 1/2 (d) −1/2

78. The value of $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}$, is
(a) 1/8 (b) −1/8 (c) 1 (d) 0

79. The value of $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{3\pi}{9}\cos\frac{4\pi}{9}$ is
(a) $\frac{1}{8}$ (b) $\frac{1}{16}$ (c) $\frac{1}{64}$ (d) $\frac{1}{4}$

80. The value of $\operatorname{cosec}^2\frac{\pi}{7}+\operatorname{cosec}^2\frac{2\pi}{7}+\operatorname{cosec}^2\frac{3\pi}{7}$ is
(a) 2^0 (b) 2 (c) 2^2 (d) 2^3

81. The value of $\sin 12^\circ \sin 48^\circ \sin 54^\circ$ is
(a) 1/4 (b) 1/8 (c) 1/16 (d) 1/64

82. The value of $\sin\frac{\pi}{7}+\sin\frac{2\pi}{7}+\sin\frac{3\pi}{7}$ is
(a) $\cot\frac{\pi}{14}$ (b) $\frac{1}{2}\cot\frac{\pi}{14}$ (c) $\tan\frac{\pi}{14}$ (d) $\frac{1}{2}\tan\frac{\pi}{14}$

83. $\tan^6\frac{\pi}{9}-33\tan^4\frac{\pi}{9}+27\tan^2\frac{\pi}{9}=$
(a) 0 (b) $\sqrt{3}$ (c) 3 (d) 9

84. $\frac{\sin^2 3A}{\sin^2 A}-\frac{\cos^2 3A}{\cos^2 A}=$
(a) $\cos 2A$ (b) $8\cos 2A$
(c) $1/8\cos 2A$ (d) none of these

85. If $\sin A=\frac{336}{625}$ where $450^\circ<A<540^\circ$, then $\sin\frac{A}{4}=$
(a) 3/5 (b) −3/5 (c) 4/5 (d) −4/5
[IP (Delhi) 2003]

86. If $y=\frac{\tan x}{\tan 3x}$, then
(a) $y\in[1/3,3]$ (b) $y\notin[1/3,3]$
(c) $y\in[-3,-1/3]$ (d) $y\notin[-3,-1/3]$

87. The value of $\cot^2\frac{\pi}{9}+\cot^2\frac{2\pi}{9}+\cot^2\frac{4\pi}{9}$, is
(a) 0 (b) 3 (c) 9 (d) 1/3
[IP (Delhi) 2003]

88. The value of $\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}$, is
(a) 1/8 (b) $\sqrt{7}/8$ (c) $\sqrt{7}/2$ (d) $\sqrt{7}/16$

89. The value of $\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}$, is
(a) $\sqrt{7}/8$ (b) 1/8 (c) $\sqrt{7}/2$ (d) $-\sqrt{7}/2$

90. The value of $\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15}$, is
(a) 0 (b) 1 (c) −1 (d) 1/8

91. If $\sin A+\cos A=m$ and $\sin^3 A+\cos^3 A=n$, then
(a) $m^3-3m+n=0$ (b) $n^3-3n+2m=0$
(c) $m^3-3m+2n=0$ (d) $m^3+3m+2n=0$

92. If $\cos A+\cos B=m$ and $\sin A+\sin B=n$ where $m,n\neq 0$, then $\sin(A+B)$ is equal to
(a) $\frac{mn}{m^2+n^2}$ (b) $\frac{2mn}{m^2+n^2}$ (c) $\frac{m^2+n^2}{2mn}$ (d) $\frac{mn}{m+n}$

93. If $0<A<\frac{\pi}{6}$ and $\sin A+\cos A=\frac{\sqrt{7}}{2}$, then $\tan\frac{A}{2}=$
(a) $\frac{\sqrt{7}-2}{3}$ (b) $\frac{\sqrt{7}+2}{3}$ (c) $\frac{\sqrt{7}}{3}$ (d) none of these

94. The value of $\cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11}$, is
(a) 0 (b) −1/2 (c) 1/2 (d) 1

95. If $4n\alpha=\pi$, then the value of
$\tan\alpha\tan 2\alpha\tan 3\alpha\tan 4\alpha\ldots\tan(2n-2)\alpha\ \tan(2n-1)\alpha$, is
(a) 0 (b) 1 (c) −1 (d) none of these

96. $\tan 9^\circ-\tan 27^\circ-\tan 63^\circ+\tan 81^\circ$ is equal to
(a) 0 (b) 1 (c) −1 (d) 4

97. For $x\in R$,
$$\tan x+\frac{1}{2}\tan\frac{x}{2}+\frac{1}{2^2}\tan\frac{x}{2^2}+\ldots+\frac{1}{2^{n-1}}\tan\left(\frac{x}{2^{n-1}}\right)$$
is equal to
(a) $2\cot 2x-\frac{1}{2^{n-1}}\cot\left(\frac{x}{2^{n-1}}\right)$
(b) $\frac{1}{2^{n-1}}\cot\left(\frac{x}{2^{n-1}}\right)-2\cot 2x$
(c) $\cot\left(\frac{x}{2^{n-1}}\right)-\cot 2x$
(d) none of these

98. If $\frac{\tan 3A}{\tan A}=k$, then $\frac{\sin 3A}{\sin A}$ is equal to
(a) $\frac{2k}{k-1}$, $k\in R$ (b) $\frac{2k}{k-1}$, $k\in[1/3,3]$
(c) $\frac{2k}{k-1}$, $k\notin[1/3,3]$ (d) $\frac{k-1}{2k}$, $k\notin[1/3,3]$
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99. If $y=\frac{\sec^2\theta-\tan\theta}{\sec^2\theta+\tan\theta}$, then
(a) $\frac{1}{3}<y<3$ (b) $y\notin[1/3,3]$
(c) $-3<y<-\frac{1}{3}$ (d) none of these

100. If $\cos A=\tan B$, $\cos B=\tan C$, $\cos C=\tan A$, then sin
equal to

(a) $\sin 18^\circ$ (b) $2 \sin 18^\circ$
(c) $2 \cos 18^\circ$ (d) $2 \cos 36^\circ$

101. If $A_1 A_2 A_3 A_4 A_5$ be a regular pentagon inscribed in a unit circle. Then, $(A_1 A_2)(A_1 A_3)$ is equal to
(a) 1 (b) 3 (c) 4 (d) $\sqrt{5}$

102. If $\tan \alpha$ equals the integral solution of the inequality $4x^2 - 16x + 15 < 0$ and $\cos \beta$ equals to the slope of the bisector of the first quadrant, then $\sin(\alpha+\beta)\sin(\alpha-\beta)$ is equal to
(a) $\frac{3}{5}$ (b) $-\frac{3}{5}$ (c) $\frac{4}{5}$ (d) $-\frac{4}{5}$

103. If $\frac{x}{\cos\theta} = \frac{y}{\cos\left(\theta - \frac{2\pi}{3}\right)} = \frac{z}{\cos\left(\theta + \frac{2\pi}{3}\right)}$,
then $x + y + z =$
(a) 1 (b) 0 (c) -1 (d) 2

104. If $\cos A = \frac{3}{4}$, then the value of $\sin\frac{A}{2}\sin\frac{5A}{2}$ is
(a) $\frac{1}{32}$ (b) $\frac{11}{8}$ (c) $\frac{11}{32}$ (d) $\frac{11}{16}$

105. The minimum value of $9\tan^2\theta + 4\cot^2\theta$ is
(a) 13 (b) 9 (c) 6 (d) 12

106. If $x_1, x_2, x_3, \ldots, x_n$ are in A.P. whose common difference is α, then the value of
$\sin\alpha\,(\sec x_1 \sec x_2 + \sec x_2 \sec x_3 + \ldots + \sec x_{n-1} \sec x_n)$ is
(a) $\frac{\sin(n-1)\alpha}{\cos x_1 \cos x_n}$ (b) $\frac{\sin n\alpha}{\cos x_1 \cos x_n}$
(c) $\sin(n-1)\alpha \cos x_1 \cos x_n$ (d) $\sin n\alpha \cos x_1 \cos x_n$

107. If $a\sin^2 x + b\cos^2 x = c$, $b\sin^2 y + a\cos^2 y = d$ and $a\tan x = b\tan y$, then $\frac{a^2}{b^2}$ is equal to
(a) $\frac{(b-c)(d-b)}{(a-d)(c-a)}$ (b) $\frac{(a-d)(c-a)}{(b-c)(d-b)}$
(c) $\frac{(d-a)(c-a)}{(b-c)(d-b)}$ (d) $\frac{(b-c)(b-d)}{(a-c)(a-d)}$

108. If $a_{n+1} = \sqrt{\frac{1}{2}(1 + a_n)}$, then $\cos\left(\frac{\sqrt{1 - a_0^{\,2}}}{a_1 a_2 a_3 \ldots \text{to } \infty}\right) =$
(a) 1 (b) -1 (c) a_0 (d) $1/a_0$

109. If $\alpha, \beta, \gamma, \delta$ are the smallest positive angles in ascending order of magnitude which have their sines equal to the positive quantity k, then the value of $4\sin\frac{\alpha}{2} + 3\sin\frac{\beta}{2} + 2\sin\frac{\gamma}{2} + \sin\frac{\delta}{2}$ is equal to
(a) $2\sqrt{1-k}$ (b) $2\sqrt{1+k}$ (c) $\frac{\sqrt{1+k}}{2}$ (d) $\frac{\sqrt{1-k}}{2}$

10. The value of
$\cos y \cos\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - y\right)\cos x + \sin y \cos\left(\frac{\pi}{2} - x\right) + \cos x \sin\left(\frac{\pi}{2} - y\right)$ is zero, if
(a) $x = 0$ (b) $y = 0$
(c) $x = y + \frac{\pi}{4}$ (d) $x = \frac{3\pi}{4} + y$

111. If $\cos x - \sin\alpha \cot\beta \sin x = \cos\alpha$, then $\tan\frac{x}{2}$ is equal to
(a) $\cot\frac{\alpha}{2}\tan\frac{\beta}{2}$ (b) $-\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$
(c) $-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$ (d) $\cot\frac{\alpha}{2}\cot\frac{\beta}{2}$

112. The expression
$\operatorname{cosec}^2 A \cot^2 A - \sec^2 A \tan^2 A - (\cot^2 A - \tan^2 A)$
$(\sec^2 A \operatorname{cosec}^2 A - 1)$ is equal to
(a) 1 (b) -1 (c) 0 (d) 2

113. If $\sin\alpha + \cos\alpha = m$, then $\sin^6\alpha + \cos^6\alpha$ is equal to
(a) $\frac{4 - 3(m^2-1)^2}{4}$ (b) $\frac{4 + 3(m^2-1)^2}{4}$
(c) $\frac{3 + 4(m^2-1)^2}{4}$ (d) none of these

114. If $0 \le x \le \pi$ and $81^{\sin^2 x} + 81^{\cos^2 x} = 30$, then x is equal to
(a) $\frac{\pi}{6}, \frac{\pi}{3}$ (b) $\frac{\pi}{3}, \frac{\pi}{2}$ (c) $\frac{5\pi}{6}, \frac{\pi}{3}$ (d) $\frac{2\pi}{3}, \frac{\pi}{3}$

115. If $\cos(A - B) = \frac{3}{5}$ and $\tan A \tan B = 2$, then
(a) $\cos A \cos B = \frac{1}{5}$ (b) $\sin A \sin B = -\frac{2}{5}$
(c) $\cos(A + B) = -\frac{1}{5}$ (d) none of these

116. The value of $\frac{(3 + \cot 76^\circ \cot 16^\circ)}{\cot 76^\circ + \cot 16^\circ}$ is
(a) $\cot 44^\circ$ (b) $\tan 44^\circ$ (c) $\tan 2^\circ$ (d) $\cot 46^\circ$

117. If $\sin x + \sin^2 x = 1$, then $\cos^8 x + 2\cos^6 x + \cos^4 x =$
(a) 0 (b) -1 (c) 2 (d) 1

118. If $x = y\cos\frac{2\pi}{3} = z\cos\frac{4\pi}{3}$, then $xy + yz + zx =$
(a) -1 (b) 0 (c) 1 (d) 2

119. If $\sin\alpha = \sin\beta$ and $\cos\alpha = \cos\beta$, then
(a) $\sin\frac{\alpha+\beta}{2} = 0$ (b) $\cos\frac{\alpha+\beta}{2} = 0$
(c) $\sin\frac{\alpha-\beta}{2} = 0$ (d) $\cos\left(\frac{\alpha-\beta}{2}\right) = 0$

120. If
$\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 3$, then $\cos\theta_1 + \cos\theta_2 + \cos\theta_3 =$
(a) 3 (b) 2 (c) 1 (d) 0

121. If A lies in the third quadrant and $3\tan A - 4 = 0$, then $5\sin 2A + 3\sin A + 4\cos A =$
(a) 0 (b) $\frac{-24}{5}$ (c) $\frac{24}{5}$ (d) $\frac{48}{5}$

122. $\tan 5x \tan 3x \tan 2x =$

(a) $\tan 5x - \tan 3x - \tan 2x$ (b) $\frac{\sin 5x - \sin 3x - \sin 2x}{\cos 5x - \cos 3x - \cos 2x}$

(c) 0 (d) none of these

123. The value of $\sin 12^\circ \sin 24^\circ \sin 48^\circ \sin 84^\circ$, is

(a) $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ$

(b) $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ$

(c) 3/15

(d) none of these

124. If $A + B + C = \frac{3\pi}{2}$, then $\cos 2A + \cos 2B + \cos 2C =$

(a) $1 - 4\cos A \cos B \cos C$ (b) $4 \sin A \sin B \sin C$

(c) $1 + 2\cos A \cos B \cos C$ (d) $1 - 4\sin A \sin B \sin C$

125. If $A + C = B$, then $\tan A \tan B \tan C =$

(a) $\tan A \tan B + \tan C$ (b) $\tan B - \tan C - \tan A$

(c) $\tan A + \tan C - \tan B$ (d) $-(\tan A \tan B + \tan C)$

126. If $\tan(\cot x) = \cot(\tan x)$, then $\sin 2x$ is equal to

(a) $\frac{2}{(2n+1)\pi}$ (b) $\frac{4}{(2n+1)\pi}$

(c) $\frac{2}{n(n+1)\pi}$ (d) $\frac{4}{n(n+1)\pi}$

127. If $\tan\theta = \frac{a}{b}$, then $b\cos 2\theta + a\sin 2\theta =$

(a) a (b) b (c) b/a (d) a/b

128. If $A = 130°$ and $x = \sin A + \cos A$, then

(a) $x > 0$ (b) $x < 0$ (c) $x = 0$ (d) $x \geq 0$

129. The value of the expression $\sin^6\theta + \cos^6\theta + 3\sin^2\theta\cos^2\theta$ equals

(a) 0 (b) 2 (c) 3 (d) 1

30. If $A = \cos^2\theta + \sin^4\theta$, then for all values of θ,

(a) $1 \leq A \leq 2$ (b) $\frac{13}{16} \leq A \leq 1$

(c) $\frac{3}{4} \leq A \leq \frac{13}{16}$ (d) $\frac{3}{4} \leq A \leq 1$

31. The minimum value of the expression $\sin\alpha + \sin\beta + \sin\gamma$, where α, β, γ are real numbers satisfying $\alpha + \beta + \gamma = \pi$, is

(a) positive (b) zero (c) negative (d) -3

32. Which of the following statement is incorrect

(a) $\sin\theta = -1/5$ (b) $\cos\theta = 1$

(c) $\sec\theta = 1/2$ (d) $\tan\theta = 20$

3. The value of $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}$, is

(a) 1 (b) 1/4 (c) 1/8 (d) $\sqrt{2}/7$

4. If $\sin\theta + \text{cosec}\,\theta = 2$, then $\sin^2\theta + \text{cosec}^2\theta$ is equal to

(a) 1 (b) 4 (c) 2 (d) none of these

135. If $\tan\theta = \frac{1}{2}$ and $\tan\phi = \frac{1}{3}$, then the value of $\theta + \phi$, is

(a) $\pi/6$ (b) π (c) zero (d) $\pi/4$

136. If $\sin x + \sin^2 x = 1$, then the value of $\cos^{12} x + 3\cos^{10} x + 3\cos^8 x + \cos^6 x + 2\cos^4 x + \cos^2 x - 2$, is equal to

(a) 0 (b) 1 (c) 2 (d) $\sin^2 x$.

137. The maximum value of $12\sin\theta - 9\sin^2\theta$, is

(a) 3 (b) 4 (c) 5 (d) 2

138. If $f(x) = \cos^2 x + \sec^2 x$, its value always is

(a) $f(x) < 1$ (b) $f(x) = 1$

(c) $2 > f(x) > 1$ (d) $f(x) \geq 2$

139. The maximum value of $3\cos x + 4\sin x + 5$, is

(a) 5 (b) 9 (c) 7 (d) none of these

140. Let A and B denote the statements:

$$A: \cos\alpha + \cos\beta + \cos\gamma = 0$$

$$B: \sin\alpha + \sin\beta + \sin\gamma = 0$$

If $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -\frac{3}{2}$, then

(a) A is true and B is false (b) A is false and B is true

(c) both A and B are true (d) both A and B are false

[AIEEE 2009]

141. The numberof ordered pairs (α, β), where $\alpha, \beta \in (-\pi, \pi)$ satisfying $\cos(\alpha - \beta) = 1$ and $\cos(\alpha + \beta) = \frac{1}{e}$, is

(a) 0 (b) 1 (c) 2 (d) 4

[IIT (S) 2005]

142. The maximum value of $\sin(x + \pi/6) + \cos(x + \pi/6)$ in the interval $(0, \pi/2)$ is attained at

(a) $\pi/12$ (b) $\pi/6$ (c) $\pi/3$ (d) $\pi/2$

143. If $A + B + C = \pi$ $(A, B, C > 0)$ and the angle C is obtuse, then

(a) $\tan A \tan B > 1$ (b) $\tan A \tan B < 1$

(c) $\tan A \tan B = 1$ (d) none of these

144. If $0 < \theta < 2\pi$, then the intervals of values of θ for which $2\sin^2\theta - 5\sin\theta + 2 > 0$, is

(a) $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$ (b) $\left(\frac{\pi}{8}, \frac{5\pi}{6}\right)$

(c) $\left(0, \frac{\pi}{8}\right) \cup \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ (d) $\left(\frac{41\pi}{48}, \pi\right)$

[IIT 2006]

145. If $\tan\theta = x - \frac{1}{4x}$, then $\sec\theta - \tan\theta$ is equal to

(a) $-2x, \frac{1}{2x}$ (b) $-\frac{1}{2x}, 2x$ (c) $2x$ (d) $2x, \frac{1}{2x}$

146. If $\sec\theta = x + \frac{1}{4x}$, then $\sec\theta + \tan\theta =$

(a) $x, \frac{1}{x}$ (b) $2x, \frac{1}{2x}$ (c) $-2x, \frac{1}{2x}$ (d) $-\frac{1}{x},$

147. If $\pi < \theta < 2\pi$, then $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$ is equal to

(a) $\operatorname{cosec}\theta + \cot\theta$ (b) $\operatorname{cosec}\theta - \cot\theta$

(c) $-\operatorname{cosec}\theta + \cot\theta$ (d) $-\operatorname{cosec}\theta - \cot\theta$

148. If $\frac{\pi}{2} < \theta < \pi$, then $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} + \sqrt{\frac{1+\sin\theta}{1-\sin\theta}}$ is equal to

(a) $2\sec\theta$ (b) $-2\sec\theta$ (c) $\sec\theta$ (d) $-\sec\theta$

149. $\sin^2\theta = \frac{(x+y)^2}{4xy}$, where $x, y \in R$, gives real θ if and only if

(a) $x + y = 0$ (b) $x = y$

(c) $|x| = |y| \neq 0$ (d) none of these

150. $\sec\theta = \frac{a^2+b^2}{a^2-b^2}$, where $a, b, \in R$, gives real values of θ if and only if

(a) $a = b \neq 0$ (b) $|a| \neq |b| \neq 0$

(c) $a + b = 0, a \neq 0$ (d) none of these

151. If $0 < \theta < \pi$, then

$$\sqrt{2+\sqrt{2+\sqrt{2+ \dots + \sqrt{2+2\cos\theta}}}}$$

there being n number of 2's, is equal to

(a) $2\cos\frac{\theta}{2^n}$ (b) $2\cos\frac{\theta}{2^{n-1}}$

(c) $2\cos\frac{\theta}{2^{n+1}}$ (d) none of these

152. $\sin 65° + \sin 43° - \sin 29° - \sin 7°$ is equal to

(a) $\cos 36°$ (b) $\cos 18°$

(c) $\cos 9°$ (d) none of these

153. If $\sec\alpha$ and $\operatorname{cosec}\alpha$ are the roots of the equation $x^2 - ax + b = 0$, then

(a) $a^2 = b(b-2)$ (b) $a^2 = b(b+2)$

(c) $a^2 + b^2 = 2b$ (d) none of these

154. The value of the expression

$3(\sin x - \cos x)^4 + 4(\sin^6 x + \cos^6 x) + 6(\sin x + \cos x)^2$ is

(a) 10 (b) 12 (c) 13 (d) none of these

155. If $\cos(\alpha+\beta)\sin(\gamma+\delta) = \cos(\alpha-\beta)\sin(\gamma-\delta)$, then $\cot\alpha\cot\beta\cot\gamma$ is equal to

(a) $\cot\delta$ (b) $-\cot\delta$ (c) $\tan\delta$ (d) $-\tan\delta$

156. The value of

$\sin^2 5° + \sin^2 10° + \sin^2 15° + \dots + \sin^2 85° + \sin^2 90°$, is

(a) $7\frac{1}{2}$ (b) $8\frac{1}{2}$ (c) $9\frac{1}{2}$ (d) none of these

157. If $\sin x + \sin^2 x = 1$, then the value of

$\cos^{12} x + 3\cos^{10} x + 3\cos^8 x + \cos^6 x - 1$ is equal to

(a) 2 (b) 1 (c) 0 (d) -1

[JEE (WB) 2006]

158. In a cyclic quadrilateral $ABCD$, the value of $\cos A + \cos B + \cos C + \cos D$, is

(a) 1 (b) 0

(c) -1 (d) none of these

[IP (Delhi) 2003]

159. If $ABCD$ is a cyclic quadrilateral such that $12\tan A - 5 = 0$ and $5\cos B + 3 = 0$, then the quadratic equation whose roots are $\cos C$ and $\tan D$, is

(a) $39x^2 - 16x - 48 = 0$ (b) $39x^2 + 88x + 48 = 0$

(c) $39x^2 - 88x + 48 = 0$ (d) none of these

160. If $\sin(\pi\cot\theta) = \cos(\pi\tan\theta)$, then

(a) $\cot 2\theta = \pm\frac{1}{4}, -\frac{3}{4}$ (b) $\cot 2\theta = 4, \frac{4}{3}$

(c) $\cot 2\theta = -\frac{3}{4}, -\frac{1}{4}$ (d) none of these

161. The value of

$\cos x\cos y\sin(x-y) + \cos y\cos z\sin(y-z)$
$+ \cos z\cos x\sin(z-x) + \sin(x-y)\sin(y-z)\sin(z-x)$, is

(a) 0 (b) 1 (c) 2 (d) -1

162. If $ABCD$ is a convex quadrilateral such that $4\sec A + 5 = 0$ then the quadratic equation whose roots are $\tan A$ and $\operatorname{cosec} A$ is

(a) $12x^2 - 29x + 15 = 0$ (b) $12x^2 - 11x - 15 = 0$

(c) $12x^2 + 11x - 15 = 0$ (d) none of these

163. If $\frac{\tan\alpha + \tan\beta}{\cot\alpha + \cot\beta} + \{\cos(\alpha-\beta)\sec(\alpha+\beta) + 1\}^{-1} = 1$, then $\tan\alpha\tan\beta$ is equal to

(a) 1 (b) -1 (c) 2 (d) -2

164. The value of

$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}$, is

(a) 1/128 (b) 1/64

(c) 1/16 (d) none of these

[JEE (WB) 200

165. The value of

$\cos 12°\cos 24°\cos 36°\cos 48°\cos 72°\cos 84°$, is

(a) 1/64 (b) 1/32 (c) 1/16 (d) 1/1

166. The value of $\sin\frac{15\pi}{32}\sin\frac{7\pi}{16}\sin\frac{3\pi}{8}$ is

(a) $\frac{1}{8\sqrt{2}\cos\left(\frac{15\pi}{32}\right)}$ (b) $\frac{1}{8\sin\left(\frac{\pi}{32}\right)}$

(c) $\frac{1}{4\sqrt{2}}\operatorname{cosec}\left(\frac{\pi}{16}\right)$ (d) none of these

167. $\sum_{r=1}^{n-1}\cos^2\frac{r\pi}{n}$ is equal to

(a) $\frac{n}{2}$ (b) $\frac{n-1}{2}$

(c) $\frac{n}{2}-1$ (d) none of these

168. The value of

$\cot\theta-\tan\theta-2\tan 2\theta-4\tan 4\theta-8\cot 8\theta$, is

(a) 0 (b) 1

(c) -1 (d) none of these

169. If $\cos(x-y)$, $\cos x$ and $\cos(x+y)$ are in H.P., then $\left|\cos x \sec\frac{y}{2}\right|$ equals

(a) 1 (b) 2 (c) $\sqrt{2}$ (d) none of these

170. The value of $\tan 20°+2\tan 50°-\tan 70°$, is

(a) 1 (b) 0

(c) $\tan 50°$ (d) none of these

171. If $\cos A+\cos B+\cos C=0$, then

$\cos 3A+\cos 3B+\cos 3C$ is equal to

(a) $\cos A\cos B\cos C$ (b) $12\cos A\cos B\cos C$

(c) 0 (d) $8\cos^3 A\cos^3 B\cos^3 C$

172. The minimum value of $\cos 2\theta+\cos\theta$ for real values of θ, is

(a) $-9/8$ (b) 0

(c) -2 (d) none of these

173. The value of $\cos 9°-\sin 9°$ is

(a) $\frac{5+\sqrt{5}}{4}$ (b) $\frac{\sqrt{5-\sqrt{5}}}{2}$

(c) $-\frac{\sqrt{5-\sqrt{5}}}{2}$ (d) none of these

174. If $y=\frac{\sin 3\theta}{\sin\theta}$, $\theta\neq n\pi$, then

(a) $y\in[-1,3]$ (b) $y\in(-\infty,-1]$

(c) $y\in(3,\infty)$ (d) $y\in[-1,3)$

175. If $\tan(\pi\cos\theta)=\cot(\pi\sin\theta)$, $0<\theta<\frac{3\pi}{4}$, then $\sin\left(\theta+\frac{\pi}{4}\right)$ equals

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{2\sqrt{2}}$ (d) $\sqrt{2}$

176. If $\sin(\pi\cos\theta)=\cos(\pi\sin\theta)$, then the value of $\cos\left(\theta+\frac{\pi}{4}\right)$ equals

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2\sqrt{2}}$ (c) $-\frac{1}{2\sqrt{2}}$ (d) $-\frac{1}{\sqrt{2}}$

177. $1+\sin x+\sin^2 x+\ldots$ to $\infty=4+2\sqrt{3}$, if

(a) $x=\frac{2\pi}{3}$ or, $\frac{\pi}{3}$ (b) $x=\frac{7\pi}{6}$

(c) $x=\frac{\pi}{6}$ (d) $x=\frac{\pi}{4}$

178. If $\sin(x-y)=\cos(x+y)=\frac{1}{2}$, the values of x and y lying between 0° and 90° are given by

(a) $x=15°, y=25°$ (b) $x=65°, y=15°$

(c) $x=45°, y=45°$ (d) $x=45°, y=15°$

179. If α and β be between 0 and $\frac{\pi}{2}$ and if $\cos(\alpha+\beta)=\frac{12}{13}$ and $\sin(\alpha-\beta)=\frac{3}{5}$, then $\sin 2\alpha$ is equal to

(a) 64/65 (b) 56/65 (c) 0 (d) 16/15

180. If $\tan\frac{\alpha}{2}$ and $\tan\frac{\beta}{2}$ are the roots of the equation $8x^2-26x+15=0$, then $\cos(\alpha+\beta)=$

(a) $-\frac{627}{725}$ (b) $\frac{627}{725}$ (c) -1 (d) none of these

181. In a ΔPQR, $\angle R=\frac{\pi}{2}$. If $\tan\frac{P}{2}$ and $\tan\frac{Q}{2}$ are the roots of the equation $ax^2+bx+c=0$ $(a\neq 0)$, then

(a) $a+b=c$ (b) $b+c=a$ (c) $c+a=b$ (d) $b=c$

[PET (MP) 2000]

182. If $A+B+C=0$, then the value of $\Sigma\cot(B+C-A)\cot(C+A-B)$ is equal to

(a) 0 (b) 1 (c) -1 (d) 2

183. In $(0,\pi/2)$, $\tan^m x+\cot^m x$ attains

(a) a minimum value which is independent of m

(b) a minimum value which is a function of m

(c) the minimum value of 2

(d) the minimum value at the some point independent of m.

Which one of the aobve statements is correct?

184. For $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, $\frac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta}$ lies in the interval

(a) $(-\infty,\infty)$ (b) $(-2,2)$ (c) $(0,\infty)$ (d) $(-1,1$

185. If $y\tan(A+B+C)=x\tan(A+B-C)=\lambda$, then $\tan 2C=$

(a) $\frac{\lambda(x+y)}{\lambda^2-xy}$ (b) $\frac{\lambda(x+y)}{\lambda^2+xy}$

(c) $\frac{\lambda(x-y)}{xy-\lambda^2}$ (d) $\frac{\lambda(x-y)}{xy+\lambda^2}$

186. The quadratic equation whose roots are $\sec^2\theta$ an $\mathrm{cosec}^2\theta$ can be

(a) $x^2-2x+2=0$ (b) $x^2+5x+5=0$

(c) $x^2-4x+4=0$ (d) none of these

187. $\cos\alpha\sin(\beta-\gamma)+\cos\beta\sin(\gamma-\alpha)+\cos\gamma\sin(\alpha-\beta)=$

(a) 0 (b) 1/2 (c) 1 (d) $4\cos\alpha\cos\beta$ c

[EAMCET 2

188. $\sin 47°-\sin 25°+\sin 61°-\sin 11°=$

(a) $\cos 7°$ (b) $\sin 7°$ (c) $2\cos 7°$ (d) 2

[EAMCET 2

189. $\cos 1°+\cos 2°+\cos 3°+\ldots+\cos 180°=$

(a) 1 (b) 0 (c) 2 (d) -

190. The value of $\frac{\tan 70°-\tan 20°}{\tan 50°}=$

(a) 2 (b) 1 (c) 0 (d)

Answers

1. (a) 2. (b) 3. (a) 4. (a) 5. (b) 6. (b) 7. (a)
8. (b) 9. (b) 10. (a) 11. (a) 12. (c) 13. (d) 14. (a)
15. (b) 16. (a) 17. (b) 18. (b) 19. (a) 20. (c) 21. (a)
22. (a) 23. (a) 24. (a) 25. (c) 26. (a) 27. (a) 28. (b)
29. (a) 30. (d) 31. (c) 32. (c) 33. (b) 34. (a) 35. (d)
36. (b) 37. (b) 38. (a) 39. (a) 40. (c) 41. (a) 42. (d)
43. (c) 44. (c) 45. (a) 46. (d) 47. (a) 48. (b) 49. (c)
50. (d) 51. (b) 52. (a) 53. (b) 54. (b) 55. (b) 56. (b)
57. (b) 58. (c) 59. (c) 60. (c) 61. (b) 62. (a) 63. (b)
64. (d) 65. (c) 66. (c) 67. (c) 68. (d) 69. (a) 70. (d)
71. (c) 72. (c) 73. (a) 74. (a) 75. (b) 76. (b) 77. (d)
78. (a) 79. (b) 80. (d) 81. (b) 82. (b) 83. (c) 84. (b)
85. (c) 86. (b) 87. (b) 88. (b) 89. (c) 90. (d) 91. (c)
92. (b) 93. (a) 94. (c) 95. (b) 96. (d) 97. (b) 98. (c)
99. (a) 100. (b) 101. (d) 102. (c) 103. (b) 104. (c) 105. (d)
106. (a) 107. (b) 108. (c) 109. (b) 110. (d) 111. (b) 112. (c)
113. (a) 114. (a) 115. (a) 116. (a) 117. (d) 118. (b) 119. (c)
120. (d) 121. (a) 122. (a) 123. (a) 124. (d) 125. (b) 126. (b)
127. (b) 128. (a) 129. (d) 130. (d) 131. (a) 132. (c) 133. (c)
134. (c) 135. (d) 136. (d) 137. (b) 138. (d) 139. (d) 140. (c)
141. (d) 142. (a) 143. (b) 144. (a) 145. (a) 146. (b) 147. (d)
148. (b) 149. (c) 150. (b) 151. (a) 152. (d) 153. (b) 154. (c)
155. (a) 156. (c) 157. (c) 158. (b) 159. (a) 160. (a) 161. (a)
162. (b) 163. (a) 164. (a) 165. (a) 166. (a) 167. (c) 168. (a)
169. (c) 170. (b) 171. (b) 172. (a) 173. (b) 174. (d) 175. (c)
176. (b) 177. (a) 178. (d) 179. (b) 180. (a) 181. (a) 182. (b)
183. (b) 184. (a) 185. (d) 186. (c) 187. (a) 188. (a) 189. (d)
190. (a)

CHAPTER TEST

Each of the following questions has four alternatives (a), (b), (c) and (d), out of which only one is correct. Mark the correct alternative.

1. If $\cos\alpha + \cos\beta = 0 = \sin\alpha + \sin\beta$, then $\cos 2\alpha + \cos 2\beta =$
   (a) $-2\sin(\alpha+\beta)$ (b) $-2\cos(\alpha+\beta)$
   (c) $2\sin(\alpha+\beta)$ (d) $2\cos(\alpha+\beta)$

2. If $\sin\beta$ is the GM between $\sin\alpha$ and $\cos\alpha$, then $\cos 2\beta =$
   (a) $2\sin^2\left(\frac{3\pi}{4}-\alpha\right)$ (b) $2\cos^2\left(\frac{\pi}{4}-\alpha\right)$
   (c) $\cos^2\left(\frac{\pi}{4}+\alpha\right)$ (d) $2\sin^2\left(\frac{\pi}{4}+\alpha\right)$

3. $\tan\frac{2\pi}{5} - \tan\frac{\pi}{15} - \sqrt{3}\tan\frac{2\pi}{5}\tan\frac{\pi}{15}$ is equal to
   (a) $-\sqrt{3}$ (b) $1/\sqrt{3}$ (c) 1 (d) $\sqrt{3}$

4. If $\sin B = \frac{1}{5}\sin(2A+B)$, then $\frac{\tan(A+B)}{\tan A}$ is equal to
   (a) 5/3 (b) 2/3 (c) 3/2 (d) 3/5

5. $\frac{\sin 7\theta + 6\sin 5\theta + 17\sin 3\theta + 12\sin\theta}{\sin 6\theta + 5\sin 4\theta + 12\sin 2\theta}$ is equal to
   (a) $2\cos\theta$ (b) $\cos\theta$ (c) $2\sin\theta$ (d) $\sin\theta$

6. If $\frac{\cos(\theta_1-\theta_2)}{\cos(\theta_1+\theta_2)} + \frac{\cos(\theta_3+\theta_4)}{\cos(\theta_3-\theta_4)} = 0$, then $\tan\theta_1\tan\theta_2\tan\theta_3\tan\theta_4 =$
   (a) 1 (b) 2 (c) -1 (d) none of these

7. If $1+\cos 56° + \cos 58° - \cos 66° = \lambda\cos 28°\cos 29°\sin 33°$, then $\lambda =$
   (a) 2 (b) 3 (c) 4 (d) none of these

8. If α and β are acute angles $\cos 2\alpha = \frac{3\cos 2\beta - 1}{3 - \cos 2\beta}$, then $\tan\alpha\cot\beta =$
   (a) $\sqrt{3}$ (b) $\sqrt{2}$ (c) 1 (d) none of these

9. If $\operatorname{cosec}\theta = \frac{p+q}{p-q}$, then $\cot(\pi/4 + \theta/2) =$
   (a) $\sqrt{\frac{p}{q}}$ (b) $\sqrt{\frac{q}{p}}$ (c) $\sqrt{pq}$ (d) pq

10. If $\sin\alpha + \sin\beta = a$ and $\cos\alpha + \cos\beta = b$, then $\sin(\alpha+\beta) =$
    (a) ab (b) $a+b$ (c) $\frac{2ab}{a^2-b^2}$ (d) $\frac{2ab}{a^2+b^2}$

11. If $\cos(\alpha+\beta) = \frac{4}{5}$, $\sin(\alpha-\beta) = \frac{5}{13}$ and α, β lie between 0 and $\frac{\pi}{4}$, then $\tan 2\alpha =$
    (a) $\frac{56}{33}$ (b) $\frac{33}{56}$ (c) $\frac{16}{65}$ (d) $\frac{60}{61}$

12. The value of $\cos^2 76° + \cos^2 16 - \cos 76°\cos 16°$, is
    (a) 1/2 (b) 0 (c) $-1/4$ (d) 3/4

13. The value of $\sum_{k=1}^{3}\cos^2(2k-1)\frac{\pi}{12}$, is
    (a) 0 (b) 1/2 (c) $-1/2$ (d) 3/2

14. If $\frac{a^2+1}{2a} = \cos\theta$, then $\frac{a^6+1}{2a^3} =$
    (a) $\cos^2\theta$ (b) $\cos^3\theta$ (c) $\cos 2\theta$ (d) $\cos 3\theta$

15. The value of $\frac{1}{\cos 290°} + \frac{1}{\sqrt{3}\sin 250°}$ is equal to
    (a) $\sqrt{3}/4$ (b) 4/3 (c) 3/4 (d) $4/\sqrt{3}$

16. If $\tan\alpha = (1+2^{-x})^{-1}$, $\tan\beta = (1+2^{x+1})^{-1}$, then $\alpha+\beta$ equals
    (a) $\pi/6$ (b) $\pi/4$ (c) $\pi/3$ (d) $\pi/2$

17. A and B are positive acute angles satisfying the equations $3\cos^2 A + 2\cos^2 B = 4$ and $\frac{3\sin A}{\sin B} = \frac{2\cos B}{\cos A}$, then $A + 2B$ is equal to
(a) $\pi/4$ (b) $\pi/3$ (c) $\pi/6$ (d) $\pi/2$

18. If $T_n = \cos^n\theta + \sin^n\theta$, then $2T_6 - 3T_4 + 1 =$
(a) 2 (b) 3 (c) 0 (d) 1

19. The maximum value of $1 + 8\sin^2 x^2 \cos^2 x^2$, is
(a) 3 (b) -1 (c) -8 (d) 9

20. Let $\theta \in (0, \pi/4)$ and $t_1 = (\tan\theta)^{\tan\theta}$, $t_2 = (\tan\theta)^{\cot\theta}$, $t_3 = (\cot\theta)^{\tan\theta}$ and $t_4 = (\cot\theta)^{\cot\theta}$. Then,
(a) $t_1 > t_2 > t_3 > t_4$ (b) $t_4 > t_3 > t_1 > t_2$
(c) $t_3 > t_1 > t_2 > t_4$ (d) $t_2 > t_3 > t_1 > t_4$

21. The expression $3\left\{\sin^4\left(\frac{3\pi}{2} - \alpha\right) + \sin^4(3\pi - \alpha)\right\}$ $-2\left\{\sin^6\left(\frac{\pi}{2} + \alpha\right) + \sin^6(5\pi - \alpha)\right\}$ is equal to
(a) 0 (b) 1 (c) 3 (d) $\sin 4\alpha + \cos 6\alpha$

22. The minimum value of $\frac{1}{3\sin\theta - 4\cos\theta + 7}$, is
(a) $\frac{1}{12}$ (b) $\frac{5}{12}$ (c) $\frac{7}{12}$ (d) $\frac{1}{6}$

23. The maximum value of $\cos^2 A + \cos^2 B - \cos^2 C$, is
(a) 0 (b) 1 (c) 3 (d) 2

24. If $\cos(\alpha + \beta)\sin(\gamma + \delta) = \cos(\alpha - \beta)\sin(\gamma - \delta)$, then the value of $\cot\alpha\cot\beta\cot\gamma$ is
(a) $\cot\alpha$ (b) $\cot\beta$
(c) $\cot\delta$ (d) $\cot(\alpha + \beta + \gamma + \delta)$

25. The maximum value of $\cos x\left\{\frac{\cos x}{1 - \sin x} + \frac{1 - \sin x}{\cos x}\right\}$, is
(a) 1 (b) 3 (c) 2 (d) 4

26. If $\tan x = \frac{b}{a}$, then $\sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}} =$
(a) $\frac{2\sin x}{\sqrt{\sin 2x}}$ (b) $\frac{2\cos x}{\sqrt{\cos 2x}}$
(c) $\frac{2\cos x}{\sqrt{\sin 2x}}$ (d) $\frac{2\sin x}{\sqrt{\cos 2x}}$

27. In $\tan\theta + \sec\theta = \sqrt{3}$, $0 < \theta < \pi$, then θ is equal to
(a) $5\pi/6$ (b) $2\pi/3$ (c) $\pi/6$ (d) $\pi/3$

28. If $\sqrt{3}\sin\theta + \cos\theta > 0$, then θ lies in the interval
(a) $(-\pi/3, \pi/2)$ (b) $(-\pi/6, 5\pi/6)$
(c) $(\pi/4, \pi/3)$ (d) none of these

29. Let $0 < x \le \pi/4$, then $(\sec 2x - \tan 2x)$ equals
(a) $\tan^2(x + \pi/4)$ (b) $\tan(x + \pi/4)$
(c) $\tan(\pi/4 - x)$ (d) $\tan(x - \pi/4)$

30. If n is an odd positive integer, then $\left(\frac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\frac{\sin A + \sin B}{\cos A - \cos B}\right)^n =$
(a) -1 (b) 1 (c) 0 (d) none of these

31. If $3\tan(\theta - 15°) = \tan(\theta + 15°)$, $0 < \theta < \pi$, then $\theta =$
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{3\pi}{4}$ (d) $\frac{\pi}{6}$

32. If $\frac{\cos\theta}{a} = \frac{\sin\theta}{b}$, then $\frac{a}{\sec 2\theta} + \frac{b}{\operatorname{cosec} 2\theta}$ is equal to
(a) a (b) b (c) $\frac{a}{b}$ (d) $a + b$

33. If $a = \tan 27\theta - \tan\theta$ and $b = \frac{\sin\theta}{\cos 3\theta} + \frac{\sin 3\theta}{\cos 9\theta} + \frac{\sin 9\theta}{\cos 27\theta}$, then
(a) $a = b$ (b) $a = 2b$ (c) $b = 2a$ (d) $a + b = 2$

34. The number of integral values of k for which the equation $7\cos\theta + 5\sin\theta = 2k + 1$ has a solution is
(a) 4 (b) 8 (c) 10 (d) 12

35. If the equation $\sin^4\theta + \cos^4\theta = a$ has a real solution then
(a) $a \le \frac{1}{2}$ (b) $a \ge \frac{1}{2}$ (c) $\frac{1}{2} \le a \le 1$ (d) $a \ge 0$

36. The minimum value of $f(x) = \sin^4 x + \cos^4 x$, $0 \le x \le \frac{\pi}{2}$ is
(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{4}$ (c) $-\frac{1}{2}$ (d) $\frac{1}{2}$

37. The value of $\sin\frac{\pi}{16}\sin\frac{3\pi}{16}\sin\frac{5\pi}{16}\sin\frac{7\pi}{16}$, is
(a) $\frac{\sqrt{2}}{16}$ (b) $\frac{1}{8}$ (c) $\frac{1}{16}$ (d) $\frac{\sqrt{2}}{32}$

38. If $A + B + C = \pi$, then $\sin 2A + \sin 2B + \sin 2C =$
(a) $4\sin A\sin B\sin C$ (b) $4\cos A\cos B\cos C$
(c) $4\cos A\cos B\sin C$ (d) $2\sin A\sin B\sin C$

39. The expression $\tan^2\alpha + \cot^2\alpha$, is
(a) ≥ 2 (b) ≤ 2
(c) ≥ -2 (d) none of these

40. If $\alpha + \beta - \gamma = \pi$, then $\sin^2\alpha + \sin^2\beta - \sin^2\gamma$ is equal to
(a) $2\sin\alpha\sin\beta\cos\gamma$ (b) $2\cos\alpha\cos\beta\cos\gamma$
(c) $2\sin\alpha\sin\beta\sin\gamma$ (d) none of these

41. If $\tan\left(\frac{\alpha\pi}{4}\right) = \cot\left(\frac{\beta\pi}{4}\right)$, then
(a) $\alpha + \beta = 0$ (b) $\alpha + \beta = 2n$
(c) $\alpha + \beta = 2n + 1$ (d) $\alpha + \beta = 2(2n + 1)$

42. The roots of the equation $4x^2 - 2\sqrt{5}x + 1 = 0$ are
(a) $\cos 18°, \cos 36°$ (b) $\sin 36°, \cos 18°$
(c) $\sin 18°, \cos 36°$ (d) $\sin 18°, \sin 36°$

43. The radius of the circle whose arc of length 15π cm makes an angle of $\frac{3\pi}{4}$ radian at the centre is
(a) 10 cm (b) 20 cm (c) $11\frac{1}{4}$ cm (d) $22\frac{1}{2}$ cm

44. If $\frac{\cos A}{\cos B}=n$ and $\frac{\sin A}{\sin B}=m$, then $(m^2-n^2)\sin^2 B=$
(a) $1-n^2$ (b) $1+n^2$ (c) $1-n$ (d) $1+n$

45. If $\tan\theta\tan\left(\frac{\pi}{3}+\theta\right)\tan\left(-\frac{\pi}{3}+\theta\right)=k\tan 3\theta$, then the value of k is
(a) 1 (b) 1/3 (c) 3 (d) none of these

46. If $\cos(\theta+\phi)=m\cos(\theta-\phi)$, then $\tan\theta$ is equal to
(a) $\frac{1+m}{1-m}\tan\phi$ (b) $\frac{1-m}{1+m}\tan\phi$
(c) $\frac{1-m}{1+m}\cot\phi$ (d) $\frac{1+m}{1-m}\sec\phi$

47. $\alpha, \beta\ (\alpha\neq\beta)$ satisfy the equation $a\cos\theta+b\sin\theta=c$, then the value of $\tan\left(\frac{\alpha+\beta}{2}\right)$ is
(a) b/a (b) c/a (c) a/b (d) c/b

48. Let n be a positive integer such that $\sin\frac{\pi}{2^n}+\cos\frac{\pi}{2^n}=\frac{\sqrt{n}}{2}$.
Then,
(a) $6\le n\le 8$ (b) $4\le n\le 8$ (c) $4<n\le 8$ (d) $4\le n<8$

49. $\cos^4\theta-\sin^4\theta$ is equal to
(a) $1+2\sin^2\frac{\theta}{2}$ (b) $2\cos^2\theta-1$
(c) $1-2\sin^2\frac{\theta}{2}$ (d) $1+2\cos^2\theta$

50. If $\tan\alpha=\frac{m}{m+1}$ and $\tan\beta=\frac{1}{2m+1}$, then $\alpha+\beta$ is equal to
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) 0 (d) $\frac{\pi}{2}$

51. $\tan\alpha+2\tan 2\alpha+4\tan 4\alpha+8\cot 8\alpha$ is equal to
(a) $\tan 16\alpha$ (b) 0
(c) $\cot\alpha$ (d) none of these

52. If $\cos\theta-4\sin\theta=1$, then $\sin\theta+4\cos\theta=$
(a) ± 1 (b) 0 (c) ± 2 (d) ± 4

53. If $A+C=2B$, then $\frac{\cos C-\cos A}{\sin A-\sin C}=$
(a) $\cot B$ (b) $\cot 2B$ (c) $\tan 2B$ (d) $\tan B$

54. If $A+B=C$, then
$\cos^2 A+\cos^2 B+\cos^2 C-2\cos A\cos B\cos C=$
(a) 1 (b) 2 (c) 0 (d) 3

55. If $5\cos x+12\cos y=13$, then the maximum value of $5\sin x+12\sin y$ is
(a) 12 (b) $\sqrt{120}$ (c) $\sqrt{20}$ (d) 13

56. If $x=\tan 15°$, $y=\operatorname{cosec} 75°$, $z=4\sin 18°$
(a) $x<y<z$ (b) $y<z<x$
(c) $z<x<y$ (d) $x<z<y$

57. For all values of θ, $3-\cos\theta+\cos\left(\theta+\frac{\pi}{3}\right)$ lie in the interval
(a) $[-2, 3]$ (b) $[-2, 1]$
(c) $[2, 4]$ (d) $[1, 5]$

58. $\frac{\tan 80°-\tan 10°}{\tan 70°}=$
(a) 0 (b) 1 (c) 2 (d) 3

59. If $\sin A+\sin B=\sqrt{3}(\cos B\cos A)$, then $\sin 3A+\sin 3B=$
(a) 0 (b) 2 (c) 1 (d) -1

60. If $\alpha+\beta+\gamma=2\theta$, then
$\cos\theta+\cos(\theta-\alpha)+\cos(\theta-\beta)+\cos(\theta-\gamma)$ is equal to
(a) $4\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}$ (b) $4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$
(c) $4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$ (d) $4\sin\alpha\sin\beta\sin\gamma$

Answers

1. (b) 2. (c) 3. (d) 4. (c) 5. (a) 6. (c) 7. (c)
8. (b) 9. (b) 10. (d) 11. (a) 12. (d) 13. (d) 14. (d)
15. (d) 16. (b) 17. (d) 18. (c) 19. (a) 20. (b) 21. (b)
22. (b) 23. (d) 24. (c) 25. (c) 26. (b) 27 (c) 28. (b)
29. (c) 30. (c) 31. (b) 32. (a) 33. (b) 34. (b) 35. (c)
36. (d) 37. (a) 38. (a) 39. (a) 40. (a) 41. (d) 42. (c)
43. (b) 44. (a) 45. (d) 46. (c) 47. (a) 48. (b) 49. (b)
50. (b) 51. (a) 52. (d) 53. (d) 54. (a) 55. (b) 56. (a)
57. (c) 58. (c) 59. (a) 60. (b)

Solutions of **Exercises** and **Chapter-tests** are available in a separate book on **"Solutions of Objective Mathematics".**

PROPERTIES OF TRIANGLES AND CIRCLES CONNECTED WITH THEM

1. PROPERTIES OF TRIANGLES

In any triangle *ABC*, the side *BC*, opposite to the angle *A* is denoted by *a*; the sides *CA* and *AB*, opposite to the angles *B* and *C* respectively are denoted by *b* and *c* respectively. Semi-perimeter of the triangle is denoted by *s* and its area by Δ or *S*. In this chapter, we shall discuss various relations between the sides *a*, *b*, *c* and the angles *A*, *B*, *C* of ΔABC.

THEOREM 1 *(Sine formula) In any* ΔABC,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

i.e. the sines of the angles are proportional to the lengths of the opposite sides.

PROOF Let *AD* be perpendicular from *A* on *BC*.
In ΔABD, we have

$$\sin B = \frac{AD}{AB} \Rightarrow AD = c \sin B.$$

In ΔACD, we have

$$\sin C = \frac{AD}{AC} \Rightarrow AD = b \sin C.$$

$\therefore \quad AD = b \sin C = c \sin B$

$$\Rightarrow \quad \frac{\sin B}{b} = \frac{\sin C}{c} \qquad \text{...(i)}$$

Similarly, we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} \qquad \text{...(ii)}$$

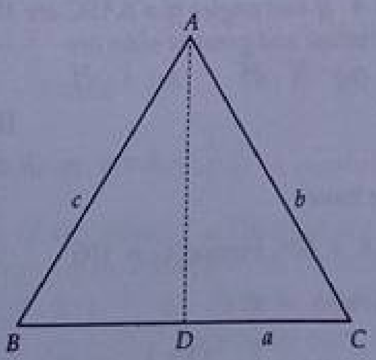

Fig. 1

From (i) and (ii), we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

NOTE 1 *The above rule can also be written as*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

NOTE 2 *The sine rule is generally used to express sides of the triangle in terms of sines of angles and vice-versa as discussed below.*

Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ (say)

$$a = k \sin A, \; b = k \sin B, \; c = k \sin C$$

or $\quad \sin A = \frac{1}{k} a, \; \sin B = \frac{1}{k} b, \; \sin C = \frac{1}{k} c$

ILLUSTRATION 1 *In any* $\Delta ABC, \sum a(\sin B - \sin C) =$

(a) $2s$ (b) $a^2 + b^2 + c^2$ (c) 0 (d) *none of these*

Ans. (c)

SOLUTION Using Sine rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$, we have

$\sum a(\sin B - \sin C)$

$= k \sum \sin A(\sin B - \sin C)$

$= k\,|\sin B(\sin A - \sin C) + \sin B(\sin C - \sin A) + \sin C(\sin A - \sin B$

$= k \times 0 = 0$

ILLUSTRATION 2 *In any* $\Delta ABC, \sum a \sin(B - C) =$

(a) $2s$ (b) $a + b + c$ (c) $a^2 + b^2 + c^2$ (d) 0

Ans. (d)

SOLUTION Using Sine rule $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$, w

have

$$\sum a \sin(B - C)$$

$$= k \sum \sin A \sin(B - C)$$

$$= k \sum \sin(B + C) \sin(B - C)$$

EXAMPLE 13 *In ΔABC it is given that $a:b:c = \cos A:\cos B:\cos C$*

Statement–1: ΔABC is equilateral.

Statement–2: $\cos A = \frac{b^2+c^2-a^2}{2bc}$, $\cos B = \frac{c^2+a^2-b^2}{2ac}$,

$\cos C = \frac{a^2+b^2-c^2}{2ab}$

(a) 1 (b) 2 (c) 3 (d) 4

Ans. (a)

SOLUTION Clearly, statement–2 is true.

Now, $a:b:c = \cos A : \cos B : \cos C$

$\Rightarrow \frac{a}{\cos A} = \frac{b}{\cos B} = \frac{c}{\cos C}$

$\Rightarrow \frac{2abc}{b^2+c^2-a^2} = \frac{2abc}{c^2+a^2-b^2} = \frac{2abc}{a^2+b^2+c^2}$

$\Rightarrow b^2+c^2-a^2 = c^2+a^2-b^2 = a^2+b^2-c^2$

$\Rightarrow b^2-a^2 = a^2-b^2$ and $c^2-b^2 = b^2-c^2$

$\Rightarrow a^2 = b^2 = c^2 \Rightarrow a = b = c \Rightarrow \Delta ABC$ is equilateral.

So, statement-1 is also true and statement-2 is a correct explanation for statement-1.

EXERCISE

This exercise contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) for its answer, out of which only one is correct.

1. In a triangle ABC, $b=\sqrt{3}$, $c=1$ and $\angle A = 30°$, then the measure of the largest angle of the triangle is
   (a) 60° (b) 135° (c) 90° (d) 120°
   [JEE (WB) 2006]
2. The area of the triangle ABC, in which $a=1, b=2$, $\angle C = 60°$, is
   (a) 4 sq. units (b) $\frac{1}{2}$ sq. unit
   (c) $\frac{\sqrt{3}}{2}$ sq. unit (d) $\sqrt{3}$ sq. units
   [JEE (WB) 2006]
3. In a triangle ABC, vertex angles A, B, C and side BC are given. The area of ΔABC is
   (a) $\frac{s(s-a)(s-b)(s-c)}{2}$ (b) $\frac{b^2 \sin C \sin A}{\sin B}$
   (c) $ab \sin C$ (d) $\frac{1}{2}\frac{a^2 \sin B \sin C}{\sin^2 A}$
4. The area of the circle and the area of a regular polygon of n sides and of perimeter equal to that of the circle are in the ratio of
   (a) $\tan\left(\frac{\pi}{n}\right):\frac{\pi}{n}$ (b) $\cos\left(\frac{\pi}{n}\right):\frac{\pi}{n}$
   (c) $\sin\frac{\pi}{n}:\frac{\pi}{n}$ (d) $\cot\left(\frac{\pi}{n}\right):\frac{\pi}{n}$
5. If $\cot\frac{A}{2} = \frac{b+c}{a}$, then the ΔABC is
   (a) isosceles (b) equilateral
   (c) right angled (d) none of these
   [IP (Delhi) 2003]
6. In a ΔABC, $\tan\frac{A}{2} = \frac{5}{6}$, $\tan\frac{C}{2} = \frac{2}{5}$, then
   (a) a, c, b are in A.P. (b) a, b, c are in A.P.
   (c) b, a, c are in A.P. (d) a, b, c are in G.P.
7. In a triangle ABC, the line joining the circumcentre to the incentre is parallel to BC, then $\cos B + \cos C =$
   (a) 3/2 (b) 1 (c) 3/4 (d) 1/2
8. In a triangle ABC, $r =$
   (a) $(s-a)\tan\frac{B}{2}$ (b) $(s-b)\tan\frac{B}{2}$
   (c) $(s-b)\tan\frac{C}{2}$ (d) $(s-a)\tan\frac{C}{2}$
9. The ex-radii of a triangle r_1, r_2, r_3 are in harmonic progression, then the sides a, b, c are
   (a) in H.P. (b) in A.P. (c) in G.P. (d) none of these
10. $\Sigma a^3 \cos(B-C) =$
    (a) $3abc$ (b) $3(a+b+c)$ (c) $abc(a+b+c)$ (d)
11. If $c^2 = a^2+b^2$, $2s = a+b+c$, then $4s(s-a)(s-b)(s-c) =$
    (a) s^4 (b) b^2c^2 (c) c^2a^2 (d) a^2b^2
12. The sides of a triangle are 13, 14, 15, then the radius of its in-circle is
    (a) 67/8 (b) 65/4 (c) 4 (d) 24
13. If $a\cos A = b\cos B$, then the triangle is
    (a) equilateral (b) right angled
    (c) isosceles (d) isosceles or right an
14. The in-radius of the triangle whose sides are 3, 5, 6 is
    (a) $\sqrt{8/7}$ (b) $\sqrt{8}$ (c) $\sqrt{7}$ (d) $\sqrt{7}$
15. In an equilateral triangle of side $2\sqrt{3}$ cms, the circu radius is
    (a) 1 cm (b) $\sqrt{3}$ cm (c) 2 cm (d) $2\sqrt{3}$
16. If the angles of a triangle are in the ratio 1 : 2 : 3, corresponding sides are in the ratio
    (a) 2 : 3 : 1 (b) $\sqrt{3}:2:1$ (c) $2:\sqrt{3}:1$ (d) $1:\sqrt{}$
17. In any triangle ABC, $\Sigma\frac{\sin^2 A+\sin A+1}{\sin A}$ is always gre than
    (a) 9 (b) 3 (c) 27 (d) none of t
18. In any ΔABC, $\Pi\left(\frac{\sin^2 A+\sin A+1}{\sin A}\right)$ is always greater
    (a) 9 (b) 3 (c) 27 (d) none of
19. In a right angled ΔABC $\sin^2 A+\sin^2 B+\sin^2 C =$
    (a) 0 (b) 1 (c) −1 (d) none of
20. In any ΔABC if $2\cos B = \frac{a}{c}$, then the triangle is
    (a) right angled (b) equilateral
    (c) isosceles (d) none of these
21. If in a ΔABC, $a\sin A = b\sin B$, then the triangle is

(a) isosceles (b) right angled
(c) equilateral (d) none of these

22. In any ΔABC, if $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in A.P., then a, b, c are in
(a) A.P. (b) G.P. (c) H.P. (d) none of these

23. In any ΔABC, $b^2 \sin 2C + c^2 \sin 2B =$
(a) Δ (b) 2Δ (c) 3Δ (d) 4Δ

24. If in a triangle ABC, $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$, then the triangle is
(a) right angled (b) obtuse angled
(c) equilateral (d) isosceles

25. If in a ΔABC, $\Delta = a^2 - (b-c)^2$, then $\tan A =$
(a) 15/16 (b) 8/15 (c) 8/17 (d) 1/2

26. If the angles A, B, C of a triangle are in A.P. and sides a, b, c are in G.P, then a^2, b^2, c^2 are in
(a) A.P. (b) H.P. (c) G.P. (d) none of these

27. In a triangle, the lengths of the two larger sides are 10 and 9. If the angles are in A.P., then the length of the third side can be
(a) $5 \pm \sqrt{6}$ (b) $3\sqrt{3}$ (c) 5 (d) $\sqrt{5} \pm 6$

28. There exists a triangle ABC satisfying the conditions
(a) $b \sin A = a$, $A < \frac{\pi}{2}$ (b) $b \sin A > a$, $A > \frac{\pi}{2}$
(c) $b \sin A > a$, $A < \frac{\pi}{2}$ (d) $b \sin A > a$, $A > \frac{\pi}{2}, b > a$

29. In a triangle the length of the two larger sides are 24 and 22, respectively. If the angles are in AP, then the third side is
(a) $12 + 2\sqrt{13}$ (b) $12 - 2\sqrt{13}$
(c) $2\sqrt{13} + 2$ (d) $2\sqrt{13} - 2$

30. If in a triangle $a\cos^2\left(\frac{C}{2}\right) + c\cos^2\left(\frac{A}{2}\right) = \frac{3b}{2}$, then the sides of the triangle are in
(a) AP (b) GP (c) HP (d) none of these **[AIEEE 2003]**

31. If twice the square of the diameter of a circle is equal to half the sum of the squares of the sides of incribed triangle ABC, then $\sin^2 A + \sin^2 C$ is equal to
(a) 1 (b) 2 (c) 4 (d) 8

32. In a triangle ABC, angle A is greater than B. If the measures of angles A and B satisfy the equation $3\sin x - 4\sin^3 x - k = 0$, $0 < k < 1$, then the measure of angle C, is
(a) $\pi/3$ (b) $\pi/2$ (c) $2\pi/3$ (d) $5\pi/6$

33. If in a triangle ABC
$$2\frac{\cos A}{a} + \frac{\cos B}{b} + 2\frac{\cos C}{c} = \frac{a}{bc} + \frac{b}{ca},$$
then the value of the angle A is
(a) $\pi/3$ (b) $\pi/4$ (c) $\pi/2$ (d) $\pi/6$

34. If $A > 0$, $B > 0$ and $A + B = \frac{\pi}{3}$, then the maximum value of $\tan A \tan B$, is
(a) 1/3 (b) 1 (c) ∞ (d) $1/\sqrt{3}$

35. If $\cos(\theta - \alpha)$, $\cos\theta$, $\cos(\theta + \alpha)$ are in H.P., then $\cos\theta \sec(\alpha/2)$ is equal to
(a) -1 (b) $\pm\sqrt{2}$ (c) ± 2 (d) ± 3

36. If $\sin\beta$ is the GM between $\sin\alpha$ and $\cos\alpha$, then $\cos 2\beta$ is equal to
(a) $2\sin^2\left(\frac{\pi}{4} - \alpha\right)$ (b) $2\cos^2\left(\frac{\pi}{4} - \alpha\right)$
(c) $2\cos^2\left(\frac{3\pi}{4} + 2\alpha\right)$ (d) $2\sin^2\left(\frac{\pi}{4} + \alpha\right)$

37. If in a triangle ABC, $\sin A = \sin^2 B$ and $2\cos^2 A = 3\cos^2 B$, then the ΔABC is
(a) right angled (b) obtuse angled
(c) isosceles (d) equilateral

38. If in a ΔABC,
$(\sin A + \sin B + \sin C)(\sin A + \sin B - \sin C) = 3\sin A \sin B$, then
(a) $A = 60°$ (b) $B = 60°$ (c) $C = 60°$ (d) none of these

39. In a ΔABC, $\sin A + \sin B + \sin C = 1 + \sqrt{2}$ and , $\cos A + \cos B + \cos C = \sqrt{2}$
if, the triangle is
(a) equilateral (b) isosceles
(c) right angled (d) right angled isosceles

40. Points D, E are taken on the side BC of a triangle ABC, such that $BD = DE = EC$. If $\angle BAD = x$, $\angle DAE = y$, $\angle EAC = z$, then the value of $\frac{\sin(x+y)\sin(y+z)}{\sin x \sin z}$ is equal to
(a) 1 (b) 2 (c) 4 (d) none of these

41. If in a ΔABC, $3a = b + c$, then the value of $\cot\frac{B}{2}\cot\frac{C}{2}$ is
(a) 1 (b) $\sqrt{3}$ (c) 2 (d) none of these

42. If $A + B + C = \pi$, $n \in Z$, then $\tan nA + \tan nB + \tan nC$ is equal to
(a) 0 (b) 1
(c) $\tan nA \tan nB \tan nC$ (d) none of these

43. If A, B, C are angles of a triangle, then the minimum value of $\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2}$, is
(a) 0 (b) 1 (c) 1/2 (d) none of these

44. In a triangle ABC, $\cos A + \cos B + \cos C = \frac{3}{2}$, then the triangle is
(a) isosceles (b) right angled
(c) equilateral (d) none of these

45. If in a triangle ABC, $\cos A \cos B + \sin A \sin B \sin C = 1$, then the triangle is
(a) isosceles (b) right angled
(c) isosceles right angled (d) equilateral

46. If in a triangle ABC, $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$ then $\cos A$ equal to
(a) 1/5 (b) 5/7 (c) 19/35 (d) none of the

47. If p_1, p_2, p_3 are altitudes of a triangle ABC from the vertic A, B, C and Δ, the area of the triangle, th $p_1^{-2} + p_2^{-2} + p_3^{-2}$ is equal to

(a) $\frac{a+b+c}{\Delta}$ (b) $\frac{a^2+b^2+c^2}{4\Delta^2}$
(c) $\frac{a^2+b^2+c^2}{\Delta^2}$ (d) none of these

48. If p_1, p_2, p_3 are altitudes of a triangle ABC from the vertices A, B, C and Δ, the area of the triangle, then $p_1 p_2 p_3$ is equal to
(a) abc (b) $8R$ (c) $a^2b^2c^2$ (d) $\frac{a^2b^2c^2}{8R^3}$
[CEE (Delhi) 1997]

49. If p_1, p_2, p_3 are altitudes of a triangle ABC from the vertices A, B, C and Δ, the area of the triangle, then $p_1^{-1}+p_2^{-1}-p_3^{-1}$ is equal to
(a) $\frac{s-a}{\Delta}$ (b) $\frac{s-b}{\Delta}$ (c) $\frac{s-c}{\Delta}$ (d) $\frac{s}{\Delta}$
[IP (Delhi) 2003]

50. If the median of ΔABC through A is perpendicular to AB, then
(a) $\tan A + \tan B = 0$ (b) $2\tan A + \tan B = 0$
(c) $\tan A + 2\tan B = 0$ (d) none of these

51. If in a triangle ABC, $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, then
(a) a, b, c are in A.P. (b) a^2, b^2, c^2 are in A.P.
(c) a, b, c are in H.P. (d) a^2, b^2, c^2 are in H.P.

52. If in a ΔABC, $a\tan A + b\tan B = (a+b)\tan\left(\frac{A+B}{2}\right)$, then
(a) $A = B$ (b) $A = -B$ (c) $A = 2B$ (d) $B = 2A$

53. If in a ΔABC, $\cos A = \frac{\sin B}{2\sin C}$, then the ΔABC is
(a) equilateral (b) isosceles
(c) right angled (d) none of these
[JEE (WB) 2006]

54. If in a triangle ABC, $\frac{a^2-b^2}{a^2+b^2} = \frac{\sin(A-B)}{\sin(A+B)}$, then the triangle is
(a) right angled or isosceles
(b) right angled and isosceles
(c) equilateral
(d) none of these

55. If in a triangle ABC, $b + c = 3a$, then $\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)$ is equal to
(a) 1 (b) -1 (c) 2 (d) None of these

56. Let ABC be a triangle such that $\angle A = 45°$, $\angle B = 75°$, then $a + c\sqrt{2}$ is equal to
(a) 0 (b) b (c) $2b$ (d) $-b$

57. If in a ΔABC, $\cos A + 2\cos B + \cos C = 2$, then a, b, c are in
(a) A.P. (b) H.P. (c) G.P. (d) none of these

58. If the altitudes of a triangle are in AP, then the sides of the triangle are in
(a) A.P. (b) G.P. (c) H.P. (d) none of these
[EAMCET 2002]

59. In any ΔABC, the distance of the orthocentre from the vertices A, B, C are in the ratio
(a) $\sin A : \sin B : \sin C$ (b) $\cos A : \cos B : \cos C$
(c) $\tan A : \tan B : \tan C$ (d) none of these

60. If R is the radius of circumscribing circle of a regular polygon of n-sides, then $R =$
(a) $\frac{a}{2}\sin\left(\frac{\pi}{n}\right)$ (b) $\frac{a}{2}\cos\left(\frac{\pi}{n}\right)$
(c) $\frac{a}{2}\operatorname{cosec}\left(\frac{\pi}{n}\right)$ (d) $\frac{a}{2}\operatorname{cosec}\left(\frac{\pi}{2n}\right)$

61. If r is the radius of inscribed circle of a regular polygon of n-sides, then r is equal to
(a) $\frac{a}{2}\cot\left(\frac{\pi}{2n}\right)$ (b) $\frac{a}{2}\cot\left(\frac{\pi}{n}\right)$
(c) $\frac{a}{2}\tan\left(\frac{\pi}{n}\right)$ (d) $\frac{a}{2}\cos\left(\frac{\pi}{n}\right)$

62. The area of a regular polygon of n sides is
(a) $\frac{nR^2}{2}\sin\left(\frac{2\pi}{n}\right)$ (b) $nr^2\tan\left(\frac{2\pi}{2n}\right)$
(c) $\frac{nr^2}{2}\sin\left(\frac{2\pi}{n}\right)$ (d) $nR^2\tan\left(\frac{\pi}{n}\right)$

63. If r, r_1, r_2, r_3 have their usual meanings, the value of $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$, is
(a) 1 (b) 0 (c) $1/r$ (d) none of these

64. If p_1, p_2, p_3 are respectively the perpendiculars from the vertices of a triangle to the opposite sides, then $p_1 p_2 p_3 =$
(a) $\frac{a^2b^2c^2}{R^2}$ (b) $\frac{a^2b^2c^2}{4R^2}$ (c) $\frac{4a^2b^2c^3}{R^2}$ (d) $\frac{a^2b^2c^2}{8R^2}$
[CEE (Delhi) 1997]

65. If p_1, p_2, p_3 are respectively the perpendiculars from the vertices of a triangle to the opposite sides, then $\frac{\cos A}{p_1}+\frac{\cos B}{p_2}+\frac{\cos C}{p_3}$ is equal to
(a) $1/r$ (b) $1/R$ (c) $1/\Delta$ (d) none of these

66. If in a ΔABC, $8R^2 = a^2+b^2+c^2$, then the triangle ABC is
(a) right angled (b) isosceles
(c) equilateral (d) none of these

67. If A_1, A_2, A_3 denote respectively the areas of an inscribed polygon of $2n$ sides, inscribed polygon of n sides and circumscribed polygon of n sides, then A_2, A_1, A_3 are in
(a) A.P. (b) G.P. (c) H.P. (d) none of these

68. In a triangle ABC, $\frac{a\cos A + b\cos B + c\cos C}{a+b+c}$ is equal to
(a) $\frac{r}{R}$ (b) $\frac{R}{r}$ (c) $\frac{2r}{R}$ (d) $\frac{R}{2r}$

69. The sides of an equilateral triangle, a square and a regular hexagon circumscribed in a circle are in
(a) A.P. (b) G.P. (c) H.P. (d) none of these

70. If the sides of a triangle are proportional to $2, \sqrt{6}$ and $\sqrt{3}-1$, the greatest and the least angles of the triangle
(a) $120°, 15°$ (b) $90°, 15°$ (c) $75°, 45°$ (d) $150°$

71. If the angles of a triangle are in A.P. with common difference equal 1/3 of the least angle, then the sides are in the ratio
(a) $\sqrt{2}:2\sqrt{3}:\sqrt{6}+\sqrt{2}$ (b) $2\sqrt{2}:\sqrt{3}:\sqrt{6}-\sqrt{2}$
(c) $2\sqrt{2}:2\sqrt{3}:\sqrt{6}-\sqrt{2}$ (d) $2\sqrt{2}:2\sqrt{3}:\sqrt{6}+\sqrt{2}$

72. In a ΔABC, if $a=8$, $b=10$ and $c=12$, then C is equal to
(a) $A/2$ (b) $2A$ (c) $3A$ (d) none of these

73. If the sides a, b, c of a triangle ABC are the roots of the equation $x^3-13x^2+54x-72=0$, then the value of $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}$ is equal to
(a) $\frac{169}{144}$ (b) $\frac{61}{72}$ (c) $\frac{61}{144}$ (d) $\frac{169}{72}$

74. The area of a ΔABC is $b^2-(c-a)^2$. Then, $\tan B=$
(a) $\frac{4}{3}$ (b) $\frac{3}{4}$ (c) $\frac{8}{15}$ (d) none of these

75. If in a triangle ABC,
$\sin A:\sin C=\sin(A-B):\sin(B-C)$ *then,* $a^2:b^2:c^2$ *are in*
(a) A.P. (b) G.P. (c) H.P. (d) none of these

76. If in a ΔABC, $3\sin A=6\sin B=2\sqrt{3}\sin C$, then angle A is
(a) 0° (b) 30° (c) 60° (d) 90°

77. The sides of a triangle are in A.P. and its area is 3/5 times the area of an equilateral triangle of the same perimeter. Then, the ratio of the sides is
(a) 1 : 2 : 3 (b) 3 : 5 : 7 (c) 1 : 3 : 5 (d) none of these

78. If in a ΔABC,
$\sin^4 A+\sin^4 B+\sin^4 C$
$=\sin^2 B\sin^2 C+2\sin^2 C\sin^2 A+2\sin^2 A\sin^2 B$, then $A=$
(a) $\frac{\pi}{6},\frac{5\pi}{6}$ (b) $\frac{\pi}{3},\frac{5\pi}{6}$ (c) $\frac{5\pi}{6},\frac{2\pi}{3}$ (d) none of these

79. In any triangle ABC, $\dfrac{\tan\frac{A}{2}-\tan\frac{B}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}}$ is equal to
(a) $\frac{a-b}{a+b}$ (b) $\frac{a-b}{c}$ (c) $\frac{a-b}{a+b+c}$ (d) $\frac{c}{a+b}$
[CEE (Delhi) 2008]

80. If the sides a, b and c of a ΔABC are in A.P., then
$\left(\tan\frac{A}{2}+\tan\frac{C}{2}\right):\cot\frac{B}{2}$, is
(a) 3 : 2 (b) 1 : 2 (c) 3 : 4 (d) none of these

81. If the sides of a triangle are the roots of the equation $x^3-2x^2-x-16=0$, then the product of the in-radius and circum-radius of the triangle is
(a) 3 (b) 6 (c) 4 (d) 2

82. If AD, BE and CF are the medians of a ΔABC, then $(AD^2+BE^2+CF^2):(BC^2+CA^2+AB^2)$ is equal to
(a) 4 : 3 (b) 3 : 2 (c) 3 : 4 (d) 2 : 3

83. In a ΔABC, if the diameter of the incircle is $a+c-b$, then $\angle B=$
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) none of these

84. If a^2, b^2, c^2 are in A.P., then which of the following is also in A.P.?
(a) $\sin A, \sin B, \sin C$ (b) $\tan A, \tan B, \tan C$
(c) $\cot A, \cot B, \cot C$ (d) none of these
[CEE (Delhi) 2007]

85. If in a ΔABC,
$\sin^3 A+\sin^3 B+\sin^3 C=3\sin A\sin B\sin C$, *then*
$\begin{vmatrix} a & b & c\\ b & c & a\\ c & a & b\end{vmatrix}=$
(a) 0 (b) $(a+b+c)^3$
(c) $(a+b+c)(ab+bc+ca)$ (d) none of these

86. If the ex-radii of a triangle are in H.P., then the corresponding sides are in
(a) A.P. (b) G.P. (c) H.P. (d) none of these

87. If I is the incentre of a ΔABC, then $IA:IB:IC$ is equal to
(a) $\operatorname{cosec}\frac{A}{2}:\operatorname{cosec}\frac{B}{2}:\operatorname{cosec}\frac{C}{2}$ (b) $\sin\frac{A}{2}:\sin\frac{B}{2}:\sin\frac{C}{2}$
(c) $\sec\frac{A}{2}:\sec\frac{B}{2}:\sec\frac{C}{2}$ (d) none of these

88. In a ΔABC, the HM of the ex-radii is equal to
(a) $3r$ (b) $2R$ (c) $R+r$ (d) none of these

89. In a ΔABC if $r_1:r_2:r_3=2:4:6$, then $a:b:c=$
(a) 3 : 5 : 7 (b) 1 : 2 : 3
(c) 5 : 8 : 9 (d) none of these

90. If in a ΔABC, $\angle A=\pi/3$ and AD is a median, then
(a) $2AD^2=b^2+c^2+bc$ (b) $4AD^2=b^2+c^2+bc$
(c) $6AD^2=b^2+c^2+bc$ (d) none of thses

91. In a ΔABC, $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=$
(a) $2-\frac{r}{R}$ (b) $2-\frac{r}{2R}$ (c) $2+\frac{r}{2R}$ (d) none of these

92. The base of a triangle is 80 cm and one of the base angles is 60°. If the sum of the lengths of the other two sides is 90 cm, then the length of the shortest side is
(a) 15 cm (b) 19 cm (c) 21 cm (d) 17 cm

93. In a ΔABC if $r_1=16$, $r_2=48$ *and* $r_3=24$, then its in-radius is
(a) 7 (b) 8 (c) 6 (d) none of these

94. In a ΔABC, $\dfrac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}{\cot A+\cot B+\cot C}=$
(a) $\frac{(a+b+c)^2}{a^2+b^2+c^2}$ (b) $\frac{a^2+b^2+c^2}{(a+b+c)^2}$
(c) s (d) Δ

95. In a ΔABC if $a=26$, $b=30$ and $\cos C=\frac{63}{65}$, then $r_2=$

(a) 84 (b) 45 (c) 48 (d) 24

96. In a ΔABC if $a = 13, b = 14$ and $c = 15$, then reciprocals of r_1, r_2 and r_3 are in the ratio
(a) $6:7:8$ (b) $6:8:7$ (c) $8:7:6$ (d) none of these

97. In a ΔABC, $\sin A$ and $\sin B$ are the roots of the equation $c^2 x^2 - c(a+b)x + ab = 0$, then $\sin C =$
(a) $1/\sqrt{2}$ (b) $1/2$ (c) 1 (d) 0

98. If a, b, c denote the sides of a ΔABC and the equations $ax^2 + bx + c = 0$ and $x^2 + \sqrt{2}\,x + 1 = 0$ have a common root, then $\angle C =$
(a) $30°$ (b) $45°$ (c) $90°$ (d) $60°$

99. In a ΔABC, if $b + c = 2a$ and $\angle A = 60°$, then ΔABC is
(a) equilateral (b) right angled
(c) isosceles (d) scalene
[JEE (WB) 2006]

100. In a ΔABC, if $b = 20, c = 21$ and $\sin A = \frac{3}{5}$, then $a =$
(a) 12 (b) 13 (c) 14 (d) 15
[EAMCET 2003]

101. Let A, B and C be the angles of a plain triangle and $\tan\frac{A}{2} = \frac{1}{3}$, $\tan\frac{B}{2} = \frac{2}{3}$. Then, $\tan\frac{C}{2}$ is equal to
(a) 7/9 (b) 2/9 (c) 1/3 (d) 2/3
[JEE (Orissa) 2003]

Answers

1. (d) 2. (c) 3. (d) 4. (a) 5. (c) 6. (b) 7. (b)
8. (b) 9. (b) 10. (a) 11. (d) 12. (c) 13. (d) 14. (a)
15. (c) 16. (d) 17. (a) 18. (c) 19. (d) 20. (c) 21. (a)
22. (a) 23. (d) 24. (c) 25. (b) 26. (a) 27. (a) 28. (a)
29. (a) 30. (a) 31. (c) 32. (c) 33. (c) 34. (a) 35. (b)
36. (a) 37. (b) 38. (c) 39. (d) 40. (c) 41. (c) 42. (c)
43. (b) 44. (c) 45. (c) 46. (a) 47. (c) 48. (d) 49. (c)
50. (c) 51. (b) 52. (a) 53. (b) 54. (a) 55. (d) 56. (c)
57. (a) 58. (c) 59. (b) 60. (c) 61. (b) 62. (a) 63. (c)
64. (d) 65. (b) 66. (a) 67. (b) 68. (a) 69. (c) 70. (a)
71. (d) 72. (b) 73. (c) 74. (c) 75. (a) 76. (d) 77. (b)
78. (a), (c) 79. (b) 80. (d) 81. (c) 82. (c) 83. (c)
84. (c) 85. (a) 86. (a) 87. (a) 88. (a) 89. (c) 90. (b)
91. (c) 92. (d) 93. (b) 94. (a) 95. (c) 96. (c) 97. (c)
98. (b) 99. (a) 100. (b) 101. (a)

CHAPTER TEST

Each of the following questions has four choices (a), (b), (c) and (d) out of which only one is correct. Mark the correct choice in each case.

1. If the sides of a triangle are in the ratio $3:7:8$, then $R:r$ is equal to
(a) $2:7$ (b) $7:2$ (c) $3:7$ (d) $7:3$

2. The area of the regular polygon of n sides is (where R is the radius of the circumpolygon),
(a) $\frac{1}{2}R^2\sin\left(\frac{2\pi}{n}\right)$ (b) $\frac{n}{2}R^2\sin\left(\frac{\pi}{n}\right)$
(c) $\frac{n}{2}R\sin\left(\frac{2\pi}{n}\right)$ (d) $\frac{nR^2}{2}\sin\left(\frac{2\pi}{n}\right)$

3. If the angles of a triangle are 30° and 45° and the included side is $(\sqrt{3}+1)$ cms, then the area of the triangle is
(a) $\frac{1}{\sqrt{3}-1}$ (b) $\sqrt{3}+1$ (c) $\frac{1}{\sqrt{3}+1}$ (d) none of these

4. In a triangle ABC, $\angle B = \frac{\pi}{3}$ and $\angle C = \frac{\pi}{4}$. Let D divide BC internally in the ratio $1:3$. Then, $\frac{\sin\angle BAD}{\sin\angle CAD}$ equals
(a) $\frac{1}{\sqrt{6}}$ (b) $\frac{1}{3}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{\frac{2}{3}}$

5. If A is the area and $2s$ the sum of 3 sides of a triangle, then
(a) $A \le \frac{s^2}{3\sqrt{3}}$ (b) $A \le \frac{s^2}{2}$
(c) $A > \frac{s^2}{\sqrt{3}}$ (d) none of these

6. If in a triangle ABC, right angled at B, $s - a = 3$, $s - c = 2$, then the values of a and c are respectively
(a) 2, 3 (b) 3, 4 (c) 4, 3 (d) 6, 8

7. In triangles ABC and DEF, $AB = DE$, $AC = EF$ and $\angle A = 2\angle E$. Two triangles will have the same area if angle A is equal to
(a) $\pi/3$ (b) $\pi/2$ (c) $2\pi/3$ (d) $5\pi/6$

8. If p is the product of the sines of angles of a triangle, and q the product of their cosines, then tangents of the angles are roots of the equation
(a) $qx^3 - px^3 + (1+q)x - p = 0$
(b) $px^3 - qx^2 + (1+p)x - q = 0$
(c) $(1+q)x^3 - px^2 + qx - p = 0$
(d) none of these

9. Angles A, B and C of a triangle ABC are in A.P. If $\frac{b}{c} = \frac{\sqrt{3}}{\sqrt{2}}$, then angle A is equal to
(a) $\pi/6$ (b) $\pi/4$ (c) $5\pi/12$ (d) $\pi/2$

10. The two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60°. If the third side is 3, the remaining fourth side is
(a) 2 (b) 3 (c) 4 (d) 5

11. If a circle is inscribed in an equilateral triangle of side a, then area of the square inscribed in the circle is

(a) $\frac{a^2}{6}$ (b) $\frac{a^2}{3}$ (c) $\frac{2a^2}{5}$ (d) $\frac{2a^2}{3}$

12. If the radius of the incircle of a triangle with its sides $5k$, $6k$, and $5k$ is 6, then k is equal to
(a) 3 (b) 4 (c) 5 (d) 6

13. Two sides of a triangle are $2\sqrt{2}$ cm and $2\sqrt{3}$ cm and the angle opposite to the shorter side of the two is $\frac{\pi}{4}$. The largest possible length of the third side is
(a) $(\sqrt{6}+\sqrt{2})$ cm (b) $(6+\sqrt{2})$ cm
(c) $(\sqrt{6}-\sqrt{2})$ cm (d) none of these

14. In a ΔABC, $a = 13$ cm, $b = 12$ cm and $c = 5$ cm. The distance of A from BC is
(a) $\frac{144}{13}$ (b) $\frac{65}{12}$ (c) $\frac{60}{13}$ (d) $\frac{25}{13}$

15. In a ΔABC, $B = \frac{\pi}{8}$ and $C = \frac{5\pi}{8}$. The altitude from A to the side BC, is
(a) $\frac{a}{2}$ (b) $2a$ (c) $\frac{1}{2}(b+c)$ (d) $b+c$

16. In a ΔABC, $A = \frac{2\pi}{3}$, $b - c = 3\sqrt{3}$ cm and $\Delta = \frac{9\sqrt{3}}{2}$ cm^2. Then, a =
(a) $6\sqrt{3}$ cm (b) 9 cm (c) 18 cm (d) 12 cm

17. In a ΔABC, if $a = (b-c)\sec\theta$, then $\frac{2\sqrt{bc}}{b-c}\sin\frac{A}{2}$ =
(a) $\cos\theta$ (b) $\cot\theta$ (c) $\tan\theta$ (d) $\sin\theta$

18. If, in a ΔABC, $(a+b+c)(b+c-a) = \lambda\, bc$, then
(a) $\lambda < 0$ (b) $\lambda > 4$ (c) $\lambda > 0$ (d) $0 < \lambda < 4$

19. In a ΔABC, $a = 2b$ and $A = 3B$, then A =
(a) 90° (b) 60° (c) 30° (d) 45°

20. Let the angles A, B, C of ΔABC be in A.P. and let
(a) 75° (b) 45° (c) 60° (d) 15°

21. If in a ΔABC, AD, BE and CF are the altitudes and R is the circum-radius, then radius of the circumcircle DEF is
(a) $\frac{R}{2}$ (b) $2R$ (c) R (d) $\frac{3}{2}R$

22. If in a ΔABC, $\frac{a}{\cos A} = \frac{b}{\cos B}$, then
(a) $2\sin A\sin B\sin C = 1$ (b) $\sin^2 A + \sin^2 B = \sin^2 C$
(c) $2\sin A\cos B = \sin C$ (d) none of these

23. In a ΔABC, $\frac{s}{R}$ =
(a) $\sin A + \sin B + \sin C$ (b) $\cos A + \cos B + \cos C$
(c) $\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}$ (d) none of these

24. If in a ΔABC, $A = \frac{\pi}{3}$ and AD is the median, then
(a) $2AD^2 = b^2 + c^2 + bc$ (b) $4AD^2 = b^2 + c^2 + bc$
(c) $6AD^2 = b^2 + c^2 + bc$ (d) none of these

25. In any ΔABC, the value of
$a(b^2+c^2)\cos A + b(c^2+a^2)\cos B + c(a^2+b^2)\cos C$ =
(a) $3abc^2$ (b) $3a^2bc$ (c) $3abc$ (d) $3ab^2c$

26. The angle of a right angled triangle are in A.P. The ratio of the in-radius and the perimeter is
(a) $(2-\sqrt{3}) : 2\sqrt{3}$ (b) $1 : 8\sqrt{3}(2+\sqrt{3})$
(c) $(2+\sqrt{3}) : 4\sqrt{3}$ (d) none of these

27. The sum of the radii of inscribed and circumscribed circles for an n sided regular polygon of side a, is
(a) $\frac{a}{4}\cot\frac{\pi}{2n}$ (b) $a\cot\frac{\pi}{n}$ (c) $\frac{a}{2}\cot\frac{\pi}{2n}$ (d) $a\cot\frac{\pi}{2n}$ [AIEEE 2003]

28. If $0 < x < \frac{\pi}{2}$, then the largest angle of a triangle whose sides are 1, $\sin x$, $\cos x$ is
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) x (d) $\frac{\pi}{2} - x$

29. The sides of a triangle are $3x+4y$, $4x+3y$ and $5x+5y$, where, $x, y > 0$ then the triangle is
(a) right angled (b) obtuse angled
(c) equilateral (d) none of these

30. The perimeter of a triangle is 16 cm. One of the sides is of length 6 cm. If the area of the triangle is 12 cm^2, then the triangle is
(a) right angled (b) isosceles (c) equilateral (d) scalene

31. In a ΔABC, if $\frac{a}{b^2-c^2} + \frac{c}{b^2-a^2} = 0$, then $\angle B$ =
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{3}$

32. In a ΔABC, $a^2\sin 2C + c^2\sin 2A$ =
(a) Δ (b) 2Δ (c) 3Δ (d) 4Δ

33. In a ΔABC, $\frac{\cos C + \cos A}{c+a} + \frac{\cos B}{b}$ =
(a) $\frac{1}{a}$ (b) $\frac{1}{b}$ (c) $\frac{1}{c}$ (d) $\frac{c+a}{b}$

34. If in a ΔABC, sides a, b, c are in A.P., then $\tan\frac{A}{2}\tan\frac{C}{2}$ =
(a) 1/4 (b) 1/3 (c) 3 (d) 4

35. In a triangle ABC, $\cos A + \cos B + \cos C$ =
(a) $1+\frac{r}{R}$ (b) $1-\frac{r}{R}$ (c) $1-\frac{R}{r}$ (d) $1+\frac{R}{r}$

36. In a ΔABC, $\cos A = \cos B\cos C$, then $\cot B\cot C$ is equal to
(a) 2 (b) 3 (c) 1/2 (d) 5

37. In a ΔABC,
$a(b^2+c^2)\cos A + b(c^2+a^2)\cos B + c(a^2+b^2)\cos C$
is equal to
(a) abc (b) $2abc$ (c) $3abc$ (d) $4abc$

38. If the sides of a triangle are x^2+x+1, x^2-1, $2x+1$, where $x > 1$, then the largest angle is
(a) 120° (b) 60° (c) 40° (d) 30°

39. In a ΔABC, if $C = 60°$, then $\frac{a}{b+c} + \frac{b}{c+a}$ =

(a) 2 (b) 1 (c) 4 (d) none of these

40. In a ΔABC, if a, c, b are in A.P., then the value of $\frac{a \cos B - b \cos A}{a-b}$, is

(a) 3 (b) 2 (c) 1 (d) none of these

41. If a triangle is right angled at B, then the diameter of the incircle of the triangle is

(a) $c+a-b$ (b) $2(c+a-b)$
(c) $c+a-2b$ (d) $c+a+2b$

42. If the angles of a right angled triangle are in A.P., then the ratio of the in-radius and the perimeter is

(a) $(2+\sqrt{3}):2\sqrt{3}$ (b) $(2+\sqrt{3}):\sqrt{3}$
(c) $(2-\sqrt{3}):2\sqrt{3}$ (d) $(2-\sqrt{3}):4\sqrt{3}$

43. If the angles of a triangle are in the ratio $1:2:7$, then the ratio of the greatest side to the least side is

(a) $(\sqrt{5}-1):(\sqrt{5}+1)$ (b) $(\sqrt{5}+1):(\sqrt{5}-1)$
(c) $(\sqrt{5}+2):(\sqrt{5}-2)$ (d) $(\sqrt{5}-2):(\sqrt{5}+2)$

44. In a ΔABC, if $a = 5$ cm, $b = 4$ cm and $\cos(A-B) = \frac{31}{32}$, then $\cos C =$

(a) 1/4 (b) 1/8 (c) 1/6 (d) 1/2

45. In a ΔABC if $c = (a+b)\sin\theta$ and $\cos\theta = \frac{k\sqrt{ab}}{a+b}$, then $k =$

(a) $2\cos\frac{C}{2}$ (b) $2\cos\frac{B}{2}$ (c) $2\cos\frac{A}{2}$ (d) $\cos\frac{C}{2}$

46. In ΔABC, if $\frac{s-a}{\Delta} = \frac{1}{8}$, $\frac{s-b}{\Delta} = \frac{1}{12}$ and $\frac{s-c}{\Delta} = \frac{1}{24}$, then $b =$

(a) 16 (b) 20 (c) 24 (d) 28

47. If in a ΔABC, $2a = \sqrt{3}\,b + c$, then

(a) $c^2 = a^2 + b^2 - ab$ (b) $a^2 = b^2 + c^2$
(c) $b^2 = a^2 + c^2 - \sqrt{3}\,ac$ (d) none of these

48. In a ΔABC, if $a\cos^2\frac{C}{2} + c\cos^2\frac{A}{2} = \frac{3b}{2}$, then a, b, c are in

(a) A.P. (b) G.P. (c) H.P. (d) none of these

49. In a right-angled triangle if the sides are in A.P., then their ratio is

(a) $3:4:5$ (b) $4:5:6$ (c) $3:4:6$ (d) none of these

50. In a ΔABC, if $B = 90°$, then the value of $\tan\frac{A}{2}$ in terms of the sides is

(a) $\sqrt{\frac{b+c}{b-c}}$ (b) $\sqrt{\frac{b-c}{b+c}}$ (c) $\sqrt{\frac{a+c}{a-c}}$ (d) $\sqrt{\frac{a-c}{a+c}}$

51. If in a ΔABC, we define $x = \tan\frac{B-C}{2}\tan\frac{A}{2}$, $y = \tan\frac{C-A}{2}\tan\frac{B}{2}$ and $z = \tan\frac{A-B}{2}\tan\frac{C}{2}$, then $x+y+z =$

(a) xyz (b) x^2yz (c) $x^2y^2z^2$ (d) none of these

52. In a ΔABC if $a = 5$, $b = 4$ and $\tan\frac{C}{2} = \frac{\sqrt{7}}{3}$, then $c =$

(a) $\sqrt{6}$ (b) $\sqrt{5}$ (c) 6 (d) 5

53. In a ΔABC if $C = 60°$, then $\frac{a}{b+c} + \frac{b}{c+a} =$

(a) 2 (b) 4 (c) 3 (d) 1

54. If p_1, p_2, p_3 are altitude of a triangle ABC from the vertices A, B, C and Δ, the area of the triangle, then $\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} =$

(a) $\frac{\cot A + \cos B + \cot C}{\Delta}$ (b) $\frac{\Delta}{\cot A + \cot B + \cot C}$
(c) $\Delta(\cot A + \cot B + \cot C)$ (d) none of these

55. In a ΔABC, $\frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab}$ is equal to

(a) $\frac{1}{2R} - \frac{1}{r}$ (b) $2R - r$ (c) $r - 2R$ (d) ...

56. In a ΔABC, angles A, B, C are in A.P., then $\lim_{A \to C} \frac{\sqrt{3 - 4\sin A \sin C}}{|A-C|}$ is equal to

(a) 1 (b) 2 (c) 3 (d) ...

57. In a ΔABC, AD is the altitude from A. Given $\angle C = 23°$ and $AD = \frac{abc}{b^2 - c^2}$, then $\angle B$ is equal to

(a) 53° (b) 113° (c) 87° (d) none o...

58. In ΔABC, $\angle A = \frac{\pi}{3}$ and $b : c = 2 : 3$, $\tan\theta = \frac{\sqrt{3}}{5}$, $0 <$ ... then

(a) $B = 60° + \theta$ (b) $C = 60° + \theta$
(c) $B = 60° - \theta$ (d) $C = 60° - \theta$

59. In a ΔABC, $\sum (b+c)\tan\frac{A}{2}\tan\left(\frac{B-C}{2}\right) =$

(a) a (b) b (c) c (d) ...

60. The sides of a triangle are $a, b, \sqrt{a^2+b^2+ab}$, th... greatest angle is

(a) 60° (b) 90° (c) 120° (d) ...

Answers

1. (b) 2. (a) 3. (a) 4. (a) 5. (a) 6. (c) 7. (c)
8. (a) 9. (c) 10. (a) 11. (a) 12. (b) 13. (a) 14. (c)
15. (a) 16. (b) 17. (c) 18. (d) 19. (a) 20. (a) 21. (a)
22. (c) 23. (a) 24. (b) 25. (c) 26. (a) 27. (d) 28. (b)
29. (b) 30. (b) 31. (d) 32. (d) 33. (b) 34. (b) 35. (a)
36. (c) 37. (c) 38. (a) 39. (b) 40. (b) 41. (a) 4...
43. (b) 44. (b) 45. (a) 46. (a) 47. (b) 48. (a) 4...
50. (b) 51. (d) 52. (c) 53. (d) 54. (a) 55. (d) 5...
57. (b) 58. (b) 59. (d) 60. (c)

Solutions of Exercises and Chapter-tests are available in a separate book on "Solutions of Objective Mathematics".

SOLUTIONS OF TRIANGLES

1. INTRODUCTION

In a triangle there are six elements viz. three sides and three angles. In plane geometry we have learned that if three of the elements are given, at least one of which must be a side, then the other three elements can be uniquely determined. The procedure of determining unknown elements from the known elements is called *solving a triangle.*

2. SOLUTION OF A RIGHT ANGLED TRIANGLE

CASE I *When two sides are given:*

Let the triangle be right angled at C. Then we can determine the remaining elements as given in the following table:

Given	Required
(i) a, b	$\tan A = \frac{a}{b}, B = 90^\circ - A, c = \frac{a}{\sin A}$
(ii) a, c	$\sin A = \frac{a}{c}, b = c\cos A, B = 90^\circ - A$

ILLUSTRATION 1 *In a right triangle ABC, right angled at C, if $a = 7$ cm and $b = 7\sqrt{3}$ cm, then $\angle A =$*

(a) 30° (b) 60° (c) 45° (d) *none of these*

Ans. (a)

SOLUTION We have,

$$\tan A = \frac{a}{b} = \frac{7}{7\sqrt{3}} = \frac{1}{\sqrt{3}} = \tan 30^\circ \Rightarrow \angle A = 30^\circ$$

ILLUSTRATION 2 *In a ΔABC, if $B = 90^\circ$, then $\tan^2\frac{A}{2} =$*

(a) $\frac{a-b}{a+b}$ (b) $\frac{b-c}{b+c}$ (c) $\frac{c-a}{c+a}$ (d) $\frac{b+c}{b-c}$

Ans. (b)

SOLUTION In ΔABC right angled at B, we have

$$\cos A = \frac{c}{b}$$

$$\therefore \quad \tan^2\frac{A}{2} = \frac{1-\cos A}{1+\cos A} = \frac{b-c}{b+c}$$

CASE II *When a side and an acute angle are given:*

Let ABC be a right triangle right angled at C.

In this case we can determine the remaining elements as given in the following table:

Given	Required
(i) a, A	$B = 90^\circ - A, b = a\cot A, c = \frac{a}{\sin A}$
(ii) c, A	$B = 90^\circ - A, a = c\sin A, b = c\cos A$

ILLUSTRATION 3 *If $A = 30^\circ$, $c = 7\sqrt{3}$ and $C = 90^\circ$ in ΔABC, then $a =$*

(a) $7\sqrt{3}$ (b) $7\sqrt{3}/2$ (c) $7/2$ (d) *none of these*

Ans. (b)

SOLUTION We have, $C = 90^\circ$.

$$\therefore \quad \sin A = \frac{a}{c} \Rightarrow a = c\sin A = 7\sqrt{3}\sin 30^\circ = \frac{7\sqrt{3}}{2}$$

3. SOLUTION OF A TRIANGLE IN GENERAL

CASE I *When three sides a, b and c are given:*

In this case, the remaining elements are determined by using the following formulae:

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}, \quad \text{where } 2s = a+b+c$$

$$\sin A = \frac{2\Delta}{bc}, \ \sin B = \frac{2\Delta}{ac}, \ \sin C = \frac{2\Delta}{ab}$$

$$\tan\frac{A}{2} = \frac{\Delta}{s(s-a)}, \ \tan\frac{B}{2} = \frac{\Delta}{s(s-b)}, \ \tan\frac{C}{2} = \frac{\Delta}{s(s-c)}$$

$$A + B + C = 180^\circ$$

CASE II *When two sides a, b and the included angle C are given:*

In this case, we use the following formulae:

$$\Delta = \frac{1}{2}ab\sin C, \ \tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2},$$

$$\frac{A+B}{2} = 90^\circ - \frac{C}{2}, \ c = \frac{a\sin C}{\sin A}$$

ILLUSTRATION *If two sides and included angle of a triangle are respectively $3+\sqrt{3}$, $3-\sqrt{3}$ and 60°, then the third sides is*

(a) $2\sqrt{2}$ (b) $4\sqrt{2}$ (c) $3\sqrt{2}$ (d) *none of these*

Ans. (c)

SOLUTION Let ABC be a triangle such that $a = 3+\sqrt{3}$, $b = 3-\sqrt{3}$ and $C = 60^\circ$.

$$\therefore \quad \tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2} \Rightarrow \tan\frac{A-B}{2} = 1$$

$\Rightarrow \dfrac{k\sin A}{k^2(\sin^2 B-\sin^2 c)}+\dfrac{k\sin C}{k^2(\sin^2 B-\sin^2 A)}=0$

$\Rightarrow \dfrac{\sin A}{\sin(B+C)\sin(B-C)}+\dfrac{\sin C}{\sin(B+A)\sin(B-A)}=0$

$\Rightarrow \dfrac{1}{\sin(B-C)}+\dfrac{1}{\sin(B-A)}=0$

$\Rightarrow \sin(B-A)+\sin(B-C)=0$

$\Rightarrow \sin(A-B)=\sin(B-C)$

$\Rightarrow A-B=B-C \Rightarrow A+C=2B \Rightarrow B=60°$

EXAMPLE 16 *In the ambiguous case, if a, b and A are given and c_1, c_2 are the two values of the third side, then $(c_1-c_2)^2+(c_1+c_2)^2\tan^2 A$ is equal to*

(a) 4 (b) $4a^2$ (c) $4b^2$ (d) $4c^2$

Ans. (b)

SOLUTION We have,

$$\cos A=\frac{b^2+c^2-a^2}{2bc}$$

$\Rightarrow \quad c^2-(2b\cos A)c+(b^2-a^2)=0$

Since c_1 and c_2 are the roots of this equation.

$\therefore \quad c_1+c_2=2b\cos A$ and $c_1c_2=b^2-a^2$

Now,

$(c_1-c_2)^2+(c_1+c_2)^2\tan^2 A$

$=(c_1+c_2)^2-4c_1c_2+(c_1+c_2)^2\tan^2 A$

$=(c_1+c_2)^2\sec^2 A-4c_1c_2$

$=4b^2\cos^2 A\times\sec^2 A-4(b^2-a^2)=4a^2$

EXERCISE

This exercise contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) for its answer, out of which only one is correct.

1. If $b=3, c=4$ and $B=\pi/3$, then the number of triangles that can be constructed is
   (a) Infinite (b) two (c) one (d) nil
2. If the data given to construct a triangle ABC are $a=5$, $b=7$, $\sin A=3/4$, then it is possible to construct
   (a) only one triangle (b) two triangles
   (c) infinitely many triangles (d) no triangles
3. We are given b, c and $\sin B$ such that B is acute and $b<c\sin B$. Then,
   (a) no triangle is possible
   (b) one triangle is possible
   (c) two triangles are possible
   (d) a right-angled triangle is possible
4. In a triangle ABC, if $a=2$, $B=60°$ and $C=75°$, then $b=$
   (a) $\sqrt{3}$ (b) $\sqrt{6}$
   (c) $\sqrt{9}$ (d) $1+\sqrt{2}$
5. In triangle ABC, $A=30°$, $b=8$, $a=6$, then $B=\sin^{-1}x$, where $x=$
   (a) 1/2 (b) 1/3 (c) 2/3 (d) 1
6. If $a=2, b=3, c=5$ in ΔABC, then $C=$
   (a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{3}$
   (c) $\dfrac{\pi}{2}$ (d) none of these
7. If the sides of a triangle are 7 cm, $4\sqrt{3}$cm and $\sqrt{13}$cm, then the smallest angle of the triangle is
   (a) 15° (b) 45°
   (c) 30° (d) none of these
8. In a ΔABC, if $c=2$, $A=120°$, $a=\sqrt{6}$, then $C=$
   (a) 30° (b) 60°
   (c) 45° (d) none of these
9. If $A=30°, a=7, b=8$ in ΔABC, then B has
   (a) one solution (b) two solutions
   (c) no solution (d) none of these
10. In a ΔABC, $b=2$, $C=60°$, $c=\sqrt{6}$, then $a=$
    (a) $\sqrt{3}-1$ (b) $\sqrt{3}$
    (c) $\sqrt{3}+1$ (d) none of these
11. In a ΔABC, $a=5, b=4$ and $\cos(A-B)=\dfrac{31}{32}$, then side c is
    (a) 6 (b) 7
    (c) 9 (d) none of these
12. In a ΔABC, if $A=30°$, $b=2$, $c=\sqrt{3}+1$, then $\dfrac{C-B}{2}=$
    (a) 15° (b) 30°
    (c) 45° (d) none of these
13. In a ΔABC if $a=2$, $b=\sqrt{6}$, $c=\sqrt{3}+1$, then $\cos A=$
    (a) 30° (b) 45°
    (c) 60° (d) none of these
14. In a ΔABC, if $A=45°$, $b=\sqrt{6}$, $a=2$, then $B=$
    (a) 30° or 150° (b) 60° or 120°
    (c) 45° or 135° (d) none of these
15. In a triangle the angles are in A.P. and the lengths of the two larger sides are 10 and 9 respectively, then the length of the third side can be
    (a) $5\pm\sqrt{6}$ (b) 0.7
    (c) $\sqrt{5}+6$ (d) none of these
16. The sides of a triangle are $3x+4y$, $4x+3y$ and $5x+5y$ units, where $x, y>0$. The triangle is
    (a) right angled (b) equilateral
    (c) obtuse angled (d) none of these

    [AIEEE 2002]

17. In a ΔABC, a, b, A are given and c_1, c_2 are two values of the third side c. The sum of the areas of two triangles with sides a, b, c_1 and a, b, c_2 is

(a) $(1/2)\, b^2 \sin 2A$ (b) $(1/2)\, a^2 \sin 2A$

(c) $b^2 \sin 2A$ (d) none of these

18. In the ambiguous case, given a, b and A. The difference between the two values of C is

(a) $2\sqrt{a^2 - b^2}$ (b) $\sqrt{a^2 - b^2 \sin^2 A}$

(c) $2\sqrt{a^2 - b^2 \sin^2 A}$ (d) $\sqrt{a^2 - b^2}$

19. In the ambiguous case, if a, b and A are given and c_1, c_2 are two values of the third side c, then $c_1^2 - 2c_1 c_2 \cos 2A + c_2^2 =$

(a) $4a^2 \cos^2 A$ (b) $4a^2 \cos A$

(c) $4a \cos^2 A$ (d) none of these

20. The smallest angle of the triangle whose sides are $6 + \sqrt{12}, \sqrt{48}, \sqrt{24}$ is

(a) $\pi/3$ (b) $\pi/4$

(c) $\pi/6$ (d) none of these

Answers

1. (d) 2. (d) 3. (a) 4. (b) 5. (c) 6. (d) 7. (c) 8. (c) 9. (b) 10. (c) 11. (a) 12. (b) 13. (b) 14. (b) 15. (a) 16. (c) 17. (a) 18. (c) 19. (a) 20. (c)

Solutions of Exercises and Chapter-tests are available in a separate book on **"Solutions of Objective Mathematics"**.

30

INVERSE TRIGONOMETRIC FUNCTIONS

1. INVERSES OF TRIGONOMETRIC FUNCTIONS

We know that corresponding to every bijection (one-one-onto function) $f: A \to B$ there exists a bijection $g: B \to A$ defined by

$$g(y) = x \text{ if and only if } f(x) = y$$

The function $g: B \to A$ is called the inverse of function $f: A \to B$ and is denoted by f^{-1}.

Thus, we have $f(x) = y \Leftrightarrow f^{-1}(y) = x$.

We have also learnt that

$$(f^{-1} of)(x) = f^{-1}\left(f(x)\right) = f^{-1}(y) = x, \text{ for all } x \in A.$$

and, $$(fo f^{-1})(y) = f\left(f^{-1}(y)\right) = f(x) = y, \text{ for all } y \in B.$$

We know that trigonometric functions are periodic functions, and hence, in general, all trigonometric functions are not bijections. Consequently, their inverses do not exist. However, if we restrict their domains and co-domains, they can be made bijections and we can obtain their inverses.

Consider the function $f: R \to R$ given by $f(x) = \sin x$. The graph of this function is shown in Fig. 1. Clearly, it is a many-one into function as it attains same value at infinitely many points and its range $[-1, 1]$ is not same as its co-domain. We know that any function can be made an onto function, if we replace its co-domain by its range. Therefore, $f: R \to [-1, 1]$ is a many-one onto functions. In order to make f a one-one function, we will have to restrict its domain in such a way that in that domain there is no turn in the graph of the function and the function takes every value between -1 and 1. It is evident from the graph of $f(x) = \sin x$ that if we take the domain as $[-\pi/2, \pi/2]$, then $f(x)$ becomes one one. Thus,

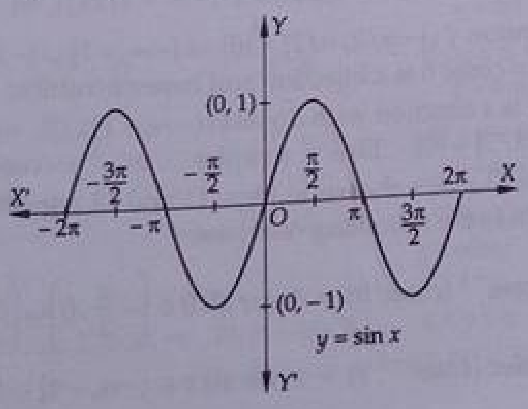

Fig. 1

$$f: [-\pi/2, \pi/2] \to [-1, 1] \text{ given by } f(\theta) = \sin\theta$$

is a bijection and hence invertible.

The inverse of the Sine function is denoted by Sin^{-1}. Thus, Sin^{-1} is a function with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$ such that

$$\text{Sin}^{-1} x = \theta \Leftrightarrow Sin\,\theta = x.$$

Also,

$$\text{Sin}^{-1}(\text{Sin}\,\theta) = \theta \qquad \left[\because f^{-1} of(x) = f^{-1}(f(x)) = x\right]$$

and, $$Sin\,(Sin^{-1} x) = x \qquad \left[fof^{-1}(y) = f(f^{-1}(y)) = y\right]$$

for all $\theta \in [-\pi/2, \pi/2]$ and, for all $x \in [-1, 1]$

The graph of the function $f: [-\pi/2, \pi/2] \to [-1, 1]$ given by $f(x) = \sin x$ is shown in Fig. 2. In order to obtain the graph of $\sin^{-1}: [-1, 1] \to [-\pi/2, \pi/2]$ we interchange x and y axes as shown in Fig. 3.

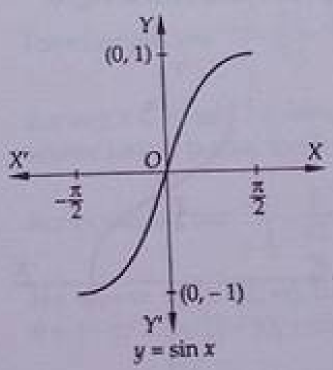

Fig. 2

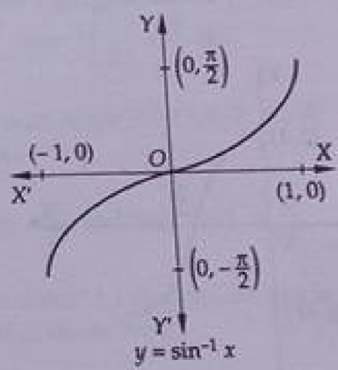

Fig. 3

REMARK 1 *In the above discussion, we have restricted the domain of sine function to the interval $[-\pi/2, \pi/2]$ to make it a bijection. In fact, if we restrict its domain to any of the intervals $[-\pi/2, \pi/2]$, $[\pi/2, 3\pi/2]$, $[3\pi/2, 5\pi/2]$, $[-3\pi/2, -\pi/2]$, $[-5\pi/2, -3\pi/2]$ in general $[n\pi - \pi/2, n\pi + \pi/2]$, $n \in Z$, then it becomes a bijection. We can, therefore, define the inverse of the sine function in each of these intervals. Thus, $\sin^{-1} x$ is a function with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$ or, $[-3\pi/2, -\pi/2]$ or, $[\pi/2, 3\pi/2]$ and so on. Corresponding to each such interval, we get a branch of the function $\sin^{-1} x$. The branch of the function $\sin^{-1}: [-1, 1] \to [-\pi/2, \pi/2]$ called the principal branch as shown in Fig. 3.*

$\Rightarrow \left(\tan^{-1}x - \frac{\pi}{6}\right)\left(\tan^{-1}x - \frac{\pi}{3}\right) = 0$

$\Rightarrow \tan^{-1}x = \frac{\pi}{6}, \frac{\pi}{3} \Rightarrow \tan^{-1}\alpha = \frac{\pi}{6}$ and $\tan^{-1}\beta = \frac{\pi}{3}$

$\Rightarrow \alpha = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ and $\beta = \tan\frac{\pi}{3} = \sqrt{3} \Rightarrow \alpha + \beta = \frac{4}{\sqrt{3}}$

So, statement–1 is true.

$$\sec^2\left(\cos^{-1}\frac{1}{4}\right) + \operatorname{cosec}^2\left(\sin^{-1}\frac{1}{5}\right)$$

$$= \{\sec(\sec^{-1}4)\}^2 + \{\operatorname{cosec}(\operatorname{cosec}^{-1}5)\}^2 = 16 + 25 = 41.$$

So, statement–2 is true.

EXAMPLE 10 *Statement–1:* $\sin^{-1}\left\{x - \frac{x^2}{2} + \frac{x^3}{4} - \dots\right\} = \frac{\pi}{2} - \cos^{-1}\left\{x^2 - \frac{x^4}{2} + \frac{x^6}{4} \dots\right\}$ *for* $0 < |x| < \sqrt{2}$ *has a unique solution.*

Statement–2: $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$ *has no solution for* $-\sqrt{2} < x < 0$.

(a) 1 (b) 2 (c) 3 (d) 4

Ans. (c)

SOLUTION Using $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ in statement–1, we get

$$x - \frac{x^2}{2} + \frac{x^3}{4} - \dots = x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \dots$$

$$\Rightarrow \frac{x}{1 + \frac{x}{2}} = \frac{x^2}{1 + \frac{x^2}{2}}$$

$\Rightarrow x(2 + x^2) = x^2(2 + x)$

$\Rightarrow x = 0, x = 1 \Rightarrow x = 1$ $[\because 0 < |x| < \dots$

So, statement–1 is true.

LHS of statement–2 is meaningful, if

$x^2 + x \geq 0$, $x^2 + x + 1 \geq 0$ and $0 \leq \sqrt{x^2 + x + 1} \leq 1$

$\Rightarrow x^2 + x \geq 0$ and $x^2 + x \leq 0 \Rightarrow x^2 + x = 0 \Rightarrow x = 0, -1$

$\Rightarrow x = -1$ $[\because -\sqrt{2} < x \dots$

Clearly, $x = -1$ satisfies the statement–1.

So, statement–2 is not true.

EXAMPLE 11 *Statement–1:* $\sin^{-1}\left[\tan\{\tan^{-1}x + \tan^{-1}(1-x)\}\right]$ … *has no non-zero integral solution.*

Statement–2: The greatest and least values of $(\sin^{-1}x)^3 + (\cos^{-1}x)^3$ *are* $\frac{7\pi^3}{8}$ *and* $\frac{\pi^3}{32}$ *respectively.*

(a) 1 (b) 2 (c) 3 (d) 4

Ans. (d)

SOLUTION $\sin^{-1}\left[\tan\{\tan^{-1}x + \tan^{-1}(1-x)\}\right] = \frac{\pi}{2}$

$$\Rightarrow \sin^{-1}\left[\tan\left\{\tan^{-1}\left(\frac{x + 1 - x}{1 - x(1-x)}\right)\right\}\right] = \frac{\pi}{2}$$

$$\Rightarrow \tan\left\{\tan^{-1}\left(\frac{1}{1 - x + x^2}\right)\right\} = 1$$

$$\Rightarrow \frac{1}{1 - x + x^2} = 1 \Rightarrow x^2 - x + 1 = 1 \Rightarrow x = 0, 1$$

So, statement–1 is not true.

Statement–2 is true (See Q. No. 35 on page 30.27)

EXERCISE

This exercise contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) for its answer, out of which only one is correct.

1. If $\theta \in [4\pi, 5\pi]$, then $\cos^{-1}(\cos\theta)$ equals
   (a) $-4\pi + \theta$ (b) $5\pi - \theta$ (c) $4\pi - \theta$ (d) $\theta - 5\pi$

2. If $x < 0$, then $\tan^{-1}\left(\frac{1}{x}\right)$ equals
   (a) $\cot^{-1}x$ (b) $-\cot^{-1}x$
   (c) $-\pi + \cot^{-1}x$ (d) $-\pi - \cot^{-1}x$

3. If $\sin^{-1}(2x\sqrt{1-x^2}) - 2\sin^{-1}x = 0$, then x belongs to the interval
   (a) $[-1, 1]$ (b) $[-1/\sqrt{2}, 1/\sqrt{2}]$
   (c) $[-1, -1/\sqrt{2}]$ (d) $[1/\sqrt{2}, 1]$

4. $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$ is equal to
   (a) π (b) $\pi/2$ (c) $\pi/3$ (d) $\pi/4$

5. If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$, then x is equal to
   (a) 1 (b) 0 (c) 4/5 (d) 1/5

6. If $A = \tan^{-1}x, x \in R$, then the value of $\sin 2A$ is
   (a) $\frac{2x}{1-x^2}$ (b) $\frac{2x}{\sqrt{1-x^2}}$ (c) $\frac{2x}{1+x^2}$ (d) $\frac{1-x^2}{1+x^2}$

7. The value of $\sin\left(2\sin^{-1}(0.8)\right)$ is equal to
   (a) $\sin 1.2^\circ$ (b) $\sin 1.6^\circ$ (c) 0.48 (d) 0.96

8. $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right) =$
   (a) $\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$ (b) $\frac{1}{2}\sin^{-1}\left(\frac{3}{5}\right)$
   (c) $\frac{1}{2}\tan^{-1}\left(\frac{3}{5}\right)$ (d) $\tan^{-1}\left(\frac{1}{2}\right)$

9. If $\sin^{-1}\frac{x}{5} + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$, then $x =$
   (a) 4 (b) 5 (c) 1 (d) 3
   [AIEEE 200…

10. $2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) =$

(a) $\tan^{-1}\left(\frac{49}{29}\right)$ (b) $\frac{\pi}{2}$ (c) 0 (d) $\frac{\pi}{4}$

11. $\cos^{-1}\left(\frac{15}{17}\right)+2\tan^{-1}\left(\frac{1}{5}\right)=$

(a) $\frac{\pi}{2}$ (b) $\cos^{-1}\left(\frac{171}{221}\right)$ (c) $\frac{\pi}{4}$ (d) none of these

12. $\cot\left\{\cos^{-1}\left(\frac{7}{25}\right)\right\}=$

(a) $\frac{25}{24}$ (b) $\frac{25}{7}$ (c) $\frac{24}{25}$ (d) none of these

13. $\sin^{-1}\left(\frac{3}{5}\right)+\tan^{-1}\left(\frac{1}{7}\right)=$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\cos^{-1}\left(\frac{4}{5}\right)$ (d) π

14. A solution of the equation $\tan^{-1}(1+x)+\tan^{-1}(1-x)=\frac{\pi}{2}$, is

(a) $x=1$ (b) $x=-1$ (c) $x=0$ (d) $x=\pi$

15. If $x^2+y^2+z^2=r^2$, then

$$\tan^{-1}\left(\frac{xy}{zr}\right)+\tan^{-1}\left(\frac{yz}{xr}\right)+\tan^{-1}\left(\frac{xz}{yr}\right)=$$

(a) π (b) $\frac{\pi}{2}$ (c) 0 (d) none of these

16. If $x+y+z=xyz$, then $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=$

(a) 0 (b) $\pi/2$ (c) 1 (d) none of these

17. If $xy+yz+zx=1$, then $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=$

(a) π (b) $\pi/2$ (c) 1 (d) none of these

18. If x_1, x_2, x_3, x_4 are roots of the equation

$$x^4-x^3\sin 2\beta+x^2\cos 2\beta-x\cos\beta-\sin\beta=0,$$

then $\tan^{-1}x_1+\tan^{-1}x_2+\tan^{-1}x_3+\tan^{-1}x_4=$

(a) β (b) $\pi/2-\beta$ (c) $\pi-\beta$ (d) $-\beta$

19. The value of $\cos\left(2\cos^{-1}0.8\right)$, is

(a) 0.48 (b) 0.96 (c) 0.6 (d) none of these

20. If $0\le x\le 1$, then $\cos^{-1}(2x^2-1)$ equals

(a) $2\cos^{-1}x$ (b) $\pi-2\cos^{-1}x$
(c) $2\pi-2\cos^{-1}x$ (d) none of these

21. The value of $\tan\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right)$, is

(a) $\frac{6}{17}$ (b) $\frac{7}{16}$ (c) $\frac{17}{6}$ (d) none of these

[CEE (Delhi) 2000]

22. The value of $\tan\left\{\frac{1}{2}\cos^{-1}\left(\frac{\sqrt{5}}{3}\right)\right\}$, is

(a) $\frac{3+\sqrt{5}}{2}$ (b) $3+\sqrt{5}$ (c) $\frac{1}{2}(3-\sqrt{5})$ (d) none of these

23. If $\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\sin^{-1}\left(\frac{2b}{1+b^2}\right)=2\tan^{-1}x$, then x is equal to

(a) $\frac{a-b}{1+ab}$ (b) $\frac{b}{1+ab}$ (c) $\frac{b}{1-ab}$ (d) $\frac{a+b}{1-ab}$

24. The value of $\cot^{-1}\left\{\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}\right\}$, is $\left(0<x<\frac{\pi}{2}\right)$

(a) $\pi-\frac{x}{2}$ (b) $2\pi-x$ (c) $\frac{x}{2}$ (d) $2\pi-$

25. The value of $\sin\left[\cot^{-1}\{\cos(\tan^{-1}x)\}\right]$, is

(a) $\sqrt{\frac{x^2+2}{x^2+1}}$ (b) $\sqrt{\frac{x^2+1}{x^2+2}}$ (c) $\frac{x}{\sqrt{x^2+2}}$ (d) $\frac{1}{\sqrt{x^2}}$

26. If $x\ge 1$, then $2\tan^{-1}x+\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ is equal to

(a) $4\tan^{-1}x$ (b) 0 (c) $\pi/2$ (d) π

27. If $A=\tan^{-1}\left(\frac{x\sqrt{3}}{2k-x}\right)$ and $B=\tan^{-1}\left(\frac{2x-k}{k\sqrt{3}}\right)$, then th value of $A-B$, is

(a) 0° (b) 45° (c) 60° (d) 30°

28. If $\sin^{-1}x+\sin^{-1}(1-x)=\cos^{-1}x$, then x equals

(a) $1,-1$ (b) $1, 0$ (c) $0, \frac{1}{2}$ (d) none of th

29. If $-1\le x\le 0$, then $\cos^{-1}(2x^2-1)$ equals

(a) $2\cos^{-1}xx$ (b) $\pi-2\cos^{-1}x$
(c) $2\pi-2\cos^{-1}x$ (d) $-2\cos^{-1}x$

30. If $-\frac{1}{2}\le x\le\frac{1}{2}$, then $\sin^{-1}(3x-4x^3)$ equals

(a) $3\sin^{-1}x$ (b) $\pi-3\sin^{-1}x$
(c) $-\pi-3\sin^{-1}x$ (d) none of these

31. The value of $\sin^{-1}(\sin 10)$, is

(a) 10 (b) $10-3\pi$ (c) $3\pi-10$ (d) none of t

32. The value of $\tan^{-1}1+\tan^{-1}2+\tan^{-1}3$, is

(a) 0 (b) 1 (c) π (d) –

33. The value of $\sin^{-1}\left\{\cos\left(\frac{33\pi}{5}\right)\right\}$, is

(a) $\frac{3\pi}{5}$ (b) $\frac{7\pi}{5}$ (c) $\frac{\pi}{10}$ (d)

34. The smallest and the largest values of

$$\tan^{-1}\left(\frac{1-x}{1+x}\right),\ 0\le x\le 1 \text{ are}$$

(a) $0, \pi$ (b) $0, \frac{\pi}{4}$ (c) $-\frac{\pi}{4}, \frac{\pi}{4}$ (d)

35. The greatest and least values of $(\sin^{-1}x)^3+(\cos^{-1}$

(a) $-\frac{\pi}{2}, \frac{\pi}{2}$ (b) $-\frac{\pi^3}{8}, \frac{\pi^3}{8}$
(c) $\frac{\pi^3}{32}, \frac{7\pi^3}{8}$ (d) none of these

36. If $a<\frac{1}{32}$, then the number of solutions of

$$(\sin^{-1}x)^3+(\cos^{-1}x)^3=a\pi^3, \text{ is}$$

(a) 0 (b) 1 (c) 2 (d

37. If x takes negative permissible value, then $\sin^{-1}$

(a) $\cos^{-1}\sqrt{1-x^2}$ (b) $-\cos^{-1}\sqrt{1-x^2}$
(c) $\cos^{-1}\sqrt{x^2-1}$ (d) $\pi-\cos^{-1}\sqrt{1-x^2}$

38. If $-1 \le x \le -\frac{1}{\sqrt{2}}$, then $\sin^{-1}(2x\sqrt{1-x^2})$ equals
(a) $2\sin^{-1}x$ (b) $\pi-2\sin^{-1}x$
(c) $-\pi-2\sin^{-1}x$ (d) none of these

39. If $\frac{1}{\sqrt{2}} \le x \le 1$, then $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$ equals
(a) $2\sin^{-1}x$ (b) $\pi-2\sin^{-1}x$
(c) $-\pi-2\sin^{-1}x$ (d) none of these

40. If $0 \le x \le 1$, then $\cos^{-1}(2x^2-1)$ equals
(a) $2\cos^{-1}x$ (b) $\pi-2\cos^{-1}x$
(c) $2\pi-2\cos^{-1}x$ (d) none of these

41. If $-1 \le x \le 0$, then $\cos^{-1}(2x^2-1)$ equals
(a) $2\cos^{-1}x$ (b) $\pi-2\cos^{-1}x$
(c) $2\pi-2\cos^{-1}x$ (d) $-2\cos^{-1}x$

42. If $-\frac{1}{2} \le x \le \frac{1}{2}$, then $\sin^{-1}(3x-4x^3)$ equals
(a) $3\sin^{-1}x$ (b) $\pi-3\sin^{-1}x$
(c) $-\pi-3\sin^{-1}x$ (d) none of these

43. If $\frac{1}{2} \le x \le 1$, then $\sin^{-1}(3x-4x^3)$ equals
(a) $3\sin^{-1}x$ (b) $\pi-3\sin^{-1}x$
(c) $-\pi-3\sin^{-1}x$ (d) none of these

44. If $-1 \le x \le -\frac{1}{2}$, then $\sin^{-1}(3x-4x^3)$ equals
(a) $3\sin^{-1}x$ (b) $\pi-3\sin^{-1}x$
(c) $-\pi-3\sin^{-1}x$ (d) none of these

45. If $\frac{1}{2} \le x \le 1$, then $\cos^{-1}(4x^3-3x)$ equals
(a) $3\cos^{-1}x$ (b) $2\pi-3\cos^{-1}x$
(c) $-2\pi+3\cos^{-1}x$ (d) none of these

46. If $-\frac{1}{2} \le x \le \frac{1}{2}$, then $\cos^{-1}(4x^3-3x)$ equals
(a) $3\cos^{-1}x$ (b) $2\pi-3\cos^{-1}x$
(c) $-2\pi+3\cos^{-1}x$ (d) none of these

47. If $-1 \le x \le -\frac{1}{2}$, then $\cos^{-1}(4x^3-3x)$ equals
(a) $3\cos^{-1}x$ (b) $2\pi-3\cos^{-1}x$
(c) $-2\pi+3\cos^{-1}x$ (d) none of these

48. If $-1 < x < 1$, then $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ equals
(a) $2\tan^{-1}x$ (b) $-\pi+2\tan^{-1}x$
(c) $\pi+2\tan^{-1}x$ (d) none of these

49. If $x \in (1, \infty)$, then $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ equals
(a) $2\tan^{-1}x$ (b) $-\pi+2\tan^{-1}x$
(c) $\pi+2\tan^{-1}x$ (d) none of these

50. If $x \in (-\infty, -1)$, then $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ equals
(a) $2\tan^{-1}x$ (b) $-\pi+2\tan^{-1}x$
(c) $\pi+2\tan^{-1}x$ (d) none of these

51. If $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$, then $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ equals
(a) $3\tan^{-1}x$ (b) $-\pi+3\tan^{-1}x$
(c) $\pi+3\tan^{-1}x$ (d) none of these

52. If $x > \frac{1}{\sqrt{3}}$, then $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ equals
(a) $3\tan^{-1}x$ (b) $-\pi+3\tan^{-1}x$
(c) $\pi+3\tan^{-1}x$ (d) none of these

53. If $x > -\frac{1}{\sqrt{3}}$, then $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ equals
(a) $3\tan^{-1}x$ (b) $-\pi+3\tan^{-1}x$
(c) $\pi+3\tan^{-1}x$ (d) none of these

54. If $0 \le x < \infty$, then $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ equals
(a) $2\tan^{-1}x$ (b) $-2\tan^{-1}x$
(c) $\pi-2\tan^{-1}x$ (d) $\pi+2\tan^{-1}x$

55. If $-\infty < x \le 0$, then $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ equals
(a) $2\tan^{-1}x$ (b) $-2\tan^{-1}x$
(c) $\pi-2\tan^{-1}x$ (d) $\pi+2\tan^{-1}x$

56. If $x \in [-1, 1]$, then $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ equals
(a) $2\tan^{-1}x$ (b) $\pi-2\tan^{-1}x$
(c) $-\pi-2\tan^{-1}x$ (d) none of these

57. If $x \in (1, \infty)$, then $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ equals
(a) $2\tan^{-1}x$ (b) $\pi-2\tan^{-1}x$
(c) $-\pi-2\tan^{-1}x$ (d) none of these

58. If $x \in (-\infty, -1)$ then $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ equals
(a) $2\tan^{-1}x$ (b) $\pi-2\tan^{-1}x$
(c) $-\pi-2\tan^{-1}x$ (d) none of these

59. If $\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=4\tan^{-1}x$, then
(a) $x \in -(-\infty, -1)$ (b) $x \in (1, \infty)$
(c) $x \in [0, 1]$ (d) $x \in [-1, 0)$

60. If $2\tan^{-1}x+\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ is independent of x, th
(a) $x \in [1, \infty) \cup (-\infty, -1)$ (b) $x \in [-1, 1]$
(c) $x \in (-\infty, 1]$ (d) none of thes

61. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, then $x+y+z$ is
(a) xyz (b) 0 (c) 1

62. The value of $\cos\{\tan^{-1}(\tan 2)\}$, is

(a) $1/\sqrt{5}$ (b) $-1/\sqrt{5}$ (c) $\cos 2$ (d) $-\cos 2$ [JEE (Orissa) 2002]

63. If $\sec^{-1} x = \operatorname{cosec}^{-1} y$, then $\cos^{-1}\frac{1}{x} + \cos^{-1}\frac{1}{y} =$

(a) π (b) $\frac{\pi}{4}$ (c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$

64. Let $\cos(2\tan^{-1} x) = \frac{1}{2}$, then the value of x is

(a) $\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $1-\sqrt{3}$ (d) $1-\frac{1}{\sqrt{3}}$

Answers

1. (a)	2. (c)	3. (b)	4. (d)	5. (d)	6. (c)	7. (d)
8. (d)	9. (d)	10. (d)	11. (d)	12. (d)	13. (a)	14. (c)
15. (b)	16. (a)	17. (b)	18. (b)	19. (d)	20. (b)	21. (c)
22. (c)	23. (d)	24. (a)	25. (b)	26. (d)	27. (d)	28. (c)
29. (d)	30. (d)	31. (c)	32. (c)	33. (d)	34. (b)	35. (c)
36. (a)	37. (b)	38. (c)	39. (b)	40. (a)	41. (c)	42. (a)
43. (b)	44. (c)	45. (a)	46. (b)	47. (c)	48. (a)	49. (b)
50. (c)	51. (a)	52. (b)	53. (c)	54. (a)	55. (b)	56. (a)
57. (b)	58. (c)	59. (c)	60. (a)	61. (a)	62. (d)	63. (d)
64. (b)						

CHAPTER TEST

Each of the following questions has four choices (a), (b), (c) and (d) out of which only one is correct. Mark the correct choice.

1. If $\sin^{-1}(1-x) - 2\sin^{-1} x = \frac{\pi}{2}$, then x equals

(a) $0, -\frac{1}{2}$ (b) $0, \frac{1}{2}$

(c) 0 (d) none of these

2. If $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$, then x equals

(a) -1 (b) 1

(c) 0 (d) none of these

3. If $\tan\theta + \tan\left(\frac{\pi}{3}+\theta\right) + \tan\left(\frac{-\pi}{3}+\theta\right) = K\tan 3\theta$, then the value of K is

(a) 1 (b) $1/3$ (c) 3 (d) none of these

4. If $\frac{1}{2} \le x \le 1$, then $\sin^{-1}(3x-4x^3)$ equals

(a) $3\sin^{-1} x$ (b) $\pi - 3\sin^{-1} x$

(c) $-\pi - 3\sin^{-1} x$ (d) none of these

5. The numerical value of $\tan\left(2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right)$ is

(a) 1 (b) 0 (c) $\frac{7}{17}$ (d) $-\frac{7}{17}$

6. If $\tan(x+y) = 33$ and $x = \tan^{-1} 3$, then y will be

(a) 0.3 (b) $\tan^{-1}(1.3)$

(c) $\tan^{-1}(0.3)$ (d) $\tan^{-1}\left(\frac{1}{18}\right)$

7. Two angles of a triangle are $\cot^{-1} 2$ and $\cot^{-1} 3$. Then, the third angle is

(a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$

8. If $A = 2\tan^{-1}(2\sqrt{2}-1)$ and $B = 3\sin^{-1}\frac{1}{3} + \sin^{-1}\frac{3}{5}$, then

(a) $A = B$ (b) $A < B$ (c) $A > B$ (d) none of thes

9. $\tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$

(a) $\pi/4$ (b) $\pi/2$ (c) π (d) 0

10. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$, the value of $x^{100}+y^{100}+z^{100} - \frac{9}{x^{101}+y^{101}+z^{101}}$, is

(a) 0 (b) 1 (c) 2 (d) 3

11. $\frac{\alpha^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right) + \frac{\beta^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right)$ is equal to

(a) $(\alpha-\beta)(\alpha^2+\beta^2)$ (b) $(\alpha+\beta)(\alpha^2-\beta^2)$

(c) $(\alpha+\beta)(\alpha^2+\beta^2)$ (d) none of these

12. If a, b are positive quantities and if $a_1 = \frac{a+b}{2}$, b_1 … $a_2 = \frac{a_1+b_1}{2}$, $b_2 = \sqrt{a_2 b_1}$ and so on, then

(a) $a_\infty = \frac{\sqrt{b^2-a^2}}{\cos^{-1}\left(\frac{a}{b}\right)}$ (b) $b_\infty = \frac{\sqrt{b^2-…}}{\cos^{-1}(…}$

(c) $b_\infty = \frac{\sqrt{a^2+b^2}}{\cos^{-1}\left(\frac{b}{a}\right)}$ (d) none of the…

13. $\tan\frac{2\pi}{5} - \tan\frac{\pi}{15} - \sqrt{3}\tan\frac{2\pi}{5}\tan\frac{\pi}{15}$ is equal t…

(a) $-\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) 1

14. If $a_1, a_2, a_3, \ldots, a_n$ is an A.P. with common difference d, then

$$\tan\left\{\tan^{-1}\left(\frac{d}{1+a_1 a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2 a_3}\right)+\ldots+\tan^{-1}\left(\frac{d}{1+a_{n-1} a_n}\right)\right\}$$ is equal to

(a) $\frac{(n-1)d}{a_1+a_n}$ (b) $\frac{(n-1)d}{1+a_1 a_n}$

(c) $\frac{nd}{1+a_1 a_n}$ (d) $\frac{a_n-a_1}{a_n+a_1}$

15. If $x=\sin(2\tan^{-1}2)$ and $y=\sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right)$, then

(a) $x=y^2$ (b) $y^2=1-x$

(c) $x^2=\frac{y}{2}$ (d) $y^2=1+x$

16. If $\theta_1=\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{1}{3}$ and $\theta_2=\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{1}{3}$, then

(a) $\theta_1>\theta_2$ (b) $\theta_1=\theta_2$

(c) $\theta_1<\theta_2$ (d) none of these

17. The value of $\cos\left[\frac{1}{2}\cos^{-1}\left\{\cos\left(\sin^{-1}\frac{\sqrt{63}}{8}\right)\right\}\right]$, is

(a) $\frac{3}{16}$ (b) $\frac{3}{8}$ (c) $\frac{3}{4}$ (d) $\frac{3}{2}$

18. The solutions of the equation

$\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31}$ are

(a) $-\frac{1}{4}, 8$ (b) $\frac{1}{4}, -8$ (c) $-4, \frac{1}{8}$ (d) $4, -\frac{1}{8}$

19. If $\alpha=\sin^{-1}\frac{\sqrt{3}}{2}+\sin^{-1}\frac{1}{3}$, $\beta=\cos^{-1}\frac{\sqrt{3}}{2}+\cos^{-1}\frac{1}{3}$, then

(a) $\alpha>\beta$ (b) $\alpha=\beta$

(c) $\alpha<\beta$ (d) $\alpha+\beta=2\pi$

20. The sum of the two angles $\cot^{-1}3$ and $\operatorname{cosec}^{-1}\sqrt{5}$, is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

21. The value of $\sin\left(4\tan^{-1}\frac{1}{3}\right)-\cos\left(2\tan^{-1}\frac{1}{7}\right)$, is

(a) $\frac{3}{7}$ (b) $\frac{7}{8}$

(c) $\frac{8}{21}$ (d) none of these

22. The number of solutions of the equation

$\sin^{-1}x+\sin^{-1}2x=\frac{\pi}{3}$, is

(a) 0 (b) 1

(c) 2 (d) infinite

23. $\cos\left\{\cos^{-1}\left(-\frac{1}{7}\right)+\sin^{-1}\left(-\frac{1}{7}\right)\right\}=$

(a) $-\frac{1}{3}$ (b) 0 (c) $\frac{1}{3}$ (d) $\frac{4}{9}$

24. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=3\pi$, then $xy+yz+zx=$

(a) 1 (b) 0 (c) -3 (d) 3

25. $\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right)=$

(a) $-\frac{1}{\sqrt{10}}$ (b) $\frac{1}{\sqrt{10}}$ (c) $-\frac{1}{10}$ (d) $\frac{1}{10}$

26. If $\theta=\sin^{-1}x+\cos^{-1}x-\tan^{-1}x$, $x\geq 0$, then the smallest interval in which θ lies is

(a) $\frac{\pi}{2}\leq\theta\leq\frac{3\pi}{4}$ (b) $0\leq\theta\leq\frac{\pi}{4}$

(c) $-\frac{\pi}{4}\leq\theta\leq 0$ (d) $\frac{\pi}{4}\leq\theta\leq\frac{\pi}{2}$

27. If $\tan^{-1}a+\tan^{-1}b=\sin^{-1}1-\tan^{-1}c$, then

(a) $a+b+c=abc$ (b) $ab+bc+ca=abc$

(c) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{1}{abc}=0$ (d) $ab+bc+ca=a+b+c$

28. The value of $\cot\left(\operatorname{cosec}^{-1}\frac{5}{3}+\tan^{-1}\frac{2}{3}\right)$, is

(a) $\frac{4}{17}$ (b) $\frac{5}{17}$ (c) $\frac{6}{17}$ (d) $\frac{3}{17}$

29. $\sin^{-1}\frac{4}{5}+2\tan^{-1}\frac{1}{3}=$

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) 0

30. The equation $\sin^{-1}x-\cos^{-1}x=\cos^{-1}\frac{\sqrt{3}}{2}$ has

(a) no solution (b) unique solution

(c) infinite number of solutions (d) none of these

Answers

1. (c) 2. (a) 3. (c) 4. (b) 5. (d) 6. (c) 7. (b)
8. (c) 9. (c) 10. (a) 11. (c) 12. (b) 13. (d) 14. (b)
15. (b) 16. (c) 17. (c) 18. (b) 19. (c) 20. (c) 21. (d)
22. (b) 23. (b) 24. (d) 25. (b) 26. (d) 27. (c) 28. (c)
29. (c) 30. (b)

Solutions of **Exercises** and **Chapter-tests** are available in a separate book on **"Solutions of Objective Mathematics"**.

TRIGONOMETRIC EQUATIONS AND INEQUATIONS

1. SOLUTIONS OF TRIGONOMETRIC EQUATIONS

SOLUTION OF A TRIGONOMETRIC EQUATION *A solution of a trigonometric equation is the value of the unknown angle that satisfies the equation.*

Consider the equation $\sin\theta = 1/2$.

This equation is, clearly, satisfied by $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ etc. so, these are its solutions.

Solving an equation means to find the set of all values of the unknown angle which satisfy the given equation.

Consider now, the equation

$$2\cos\theta + 1 = 0 \text{ or, } \cos\theta = -1/2.$$

Clearly, this equation is satisfied by $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ etc.

Since the trigonometric functions are periodic, therefore, if a trigonometric equation has a solution, it will have infinitely many solutions.

For example, $\theta = \frac{2\pi}{3},\ 2\pi \pm \frac{2\pi}{3}, 4\pi \pm \frac{2\pi}{3}$, ...

are solutions of $2\cos\theta + 1 = 0$. These solutions can be put together in compact form as

$$\theta = 2n\pi \pm 2\pi/3, \text{ where } n \text{ is an integer.}$$

This solution is known as the general solution.

Thus, a solution generalised by means of periodicity is known as the general solution.

It also follows from the above discussion that solving an equation means to find its general solution.

Following are the general solutions of some trignometric equations:

	Equaiton	*General solution*
(i)	$\sin\theta = 0$	$\theta = n\pi,\ n \in Z$
(ii)	$\cos\theta = 0$	$\theta = (2n+1)\frac{\pi}{2},\ n \in Z$
(iii)	$\tan\theta = 0$	$\theta = n\pi,\ n \in Z$
(iv)	$\cot\theta = 0$	$\theta = (2n+1)\frac{\pi}{2},\ n \in Z$
(v)	$\sin\theta = \sin\alpha$	$\theta = n\pi + (-1)^n\alpha, n \in Z$
(vi)	$\cos\theta = \cos\alpha$	$\theta = 2n\pi \pm \alpha, n \in Z$
(vii)	$\tan\theta = \tan\alpha$	$\theta = n\pi + \alpha, n \in Z$
(viii)	$\left.\begin{array}{l}\sin^2\theta = \sin^2\alpha\\ \cos^2\theta = \cos^2\alpha\\ \tan^2\theta = \tan^2\alpha\end{array}\right\}$	$\theta = n\pi \pm \alpha, n \in Z$

SECTION - I

SOLVED MCQs

This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) for its answer, out of which only one is correct

EXAMPLE 1 *The general solution of the equation* $\sin 2x + 2\sin x + 2\cos x + 1 = 0$, *is*

(a) $3n\pi - \frac{\pi}{4}, n \in Z$

(b) $2n\pi + \frac{\pi}{4}, n \in Z$

(c) $2n\pi + (-1)^n \sin^{-1}\left(\frac{1}{\sqrt{3}}\right), n \in Z$

(d) $n\pi - \frac{\pi}{4}, n \in Z$

Ans. (d)

SOLUTION We have,

$$\sin 2x + 2\sin x + 2\cos x + 1 = 0$$

$$\Rightarrow (1 + 2\sin x\cos x) + 2(\sin x + \cos x) = 0$$

$$\Rightarrow (\sin x + \cos x)^2 + 2(\sin x + \cos x) = 0$$

$$\Rightarrow (\sin x + \cos x)\{\sin x + \cos x + 2\} = 0$$

$$\Rightarrow \sin x + \cos x = 0 \quad \left[\begin{array}{l}\because -\sqrt{2} \le \sin x + \cos x \le \\ \therefore \sin x + \cos x \ne -2\end{array}\right.$$

$$\Rightarrow \tan x = -1$$

$$\Rightarrow x = n\pi - \frac{\pi}{4}, n \in Z.$$

$$\exp\left\{\left(\sin^2 x + \sin^4 x + \sin^6 x + \ldots\right)\log_e 2\right\}$$
$$= e^{(\tan^2 x)\log_e 2} = 2^{\tan^2 x}$$

It is given that $2^{\tan^2 x}$ satisfies the equation $x^2 - 9x + 8 = 0$.

$\therefore \quad 2^{\tan^2 x} = 2^3$ or $2^{\tan^2 x} = 1$

$\Rightarrow \quad \tan^2 x = 3$ or $\tan^2 x = 0$

$\Rightarrow \quad \tan x = \sqrt{3} \Rightarrow x = \frac{\pi}{3} \qquad \left[\because 0 < x < \frac{\pi}{2}\right]$

$\therefore \quad \frac{\cos x}{\cos x + \sin x} = \frac{1}{1 + \tan x} = \frac{1}{1+\sqrt{3}} = \frac{\sqrt{3}-1}{2}$

So, statement–1 is true. Also, statement–2 is a correct explanation for statement–1.

EXAMPLE 2 *Statement–1: If* $2\sin 2x - \cos 2x = 1$, $x \neq (2n+1)\frac{\pi}{2}, n \in Z$, *then* $\sin 2x + \cos 2x = 5$.

Statement–2: $\sin 2x + \cos 2x = \frac{1 + 2\tan x - \tan^2 x}{1 + \tan^2 x}$

(a) 1 (b) 2 (c) 3 (d) 4

Ans. (d)

SOLUTION We have,

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x} \text{ and } \cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$$

$\therefore \quad \sin 2x + \cos 2x = \frac{1 + 2\tan x - \tan^2 x}{1 + \tan^2 x}$

So, statement–2 is true.

Let us now consider statement–1.

We have, $2\sin 2x - \cos 2x = 1$

$\Rightarrow \quad 2\sin 2x = 2\cos^2 x$

$\Rightarrow \quad 2\sin x\cos x = \cos^2 x$

$\Rightarrow \quad \tan x = \frac{1}{2} \qquad \left[\because x \neq (2n+1)\frac{\pi}{2} \therefore \cos x \neq 0\right]$

$\therefore \quad \sin 2x + \cos 2x = \frac{1 + 2\tan x - \tan^2 x}{1+\tan^2 x}$

$\Rightarrow \quad \sin 2x + \cos 2x = \frac{2 - \frac{1}{4}}{1 + \frac{1}{4}} = \frac{7}{5}$

So, statement–1 is not correct.

EXAMPLE 3 *Statement–1:* $\cos^7 x + \sin^4 x = 1$ *has only two non zero solutions in the interval* $(-\pi, \pi)$.

Statement–2: $\cos^5 x + \cos^2 x - 2 = 0$ *is possible only when* $\cos x = 1$.

(a) 1 (b) 2 (c) 3 (d) 4

Ans. (b)

SOLUTION $\cos^5 x + \cos^2 x - 2 = 0$

$\Leftrightarrow \quad \cos^5 x + \cos^2 x = 2$

$\Leftrightarrow \quad \cos^5 x = 1$ and $\cos^2 x = 1 \Leftrightarrow \cos x = 1$

So, statement–2 is true.

Now, $\cos^7 x + \sin^4 x = 1$

$\Rightarrow \quad \cos^7 x + (1 - \cos^2 x)^2 = 1$

$\Rightarrow \quad \cos^7 x + \cos^4 x - 2\cos^2 x = 0$

$\Rightarrow \quad \cos^2 x(\cos^5 x + \cos^2 x - 2) = 0$

$\Rightarrow \quad \cos^2 x = 0$ or, $\cos x = 1$

$\Rightarrow \quad x = \pm\frac{\pi}{2}$ or, $x = 0 \qquad [\because -\pi < x < \pi]$

So, $\cos^7 x + \sin^4 x = 1$ has only two non-zero solutions in the interval $(-\pi, \pi)$.

EXAMPLE 4 *Statement–1: The number of solutions of the simultaneous system of equations*

$2\sin^2\theta - \cos 2\theta = 0$

$2\cos^2\theta - 3\sin\theta = 0$ *in the interval* $[0, 2\pi]$ *is two.*

Statement–2: If $2\cos^2\theta - 3\sin\theta = 0$, *then* θ *does not lie in III or IV quadrant.*

(a) 1 (b) 2 (c) 3 (d) 4

Ans. (a)

SOLUTION If θ lies in III or IV quadrant, then $\sin\theta < 0$.

$\therefore \quad 2\cos^2\theta - 3\sin\theta > 0$

So, statement–2 is correct.

Now, $2\sin^2\theta - \cos 2\theta = 0$

$\Rightarrow \quad 4\sin^2\theta = 1 \Rightarrow \sin\theta = 1/2$ [Using statement-2]

$\Rightarrow \quad \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

So, statement–1 is also true.

EXERCISE

This exercise contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) for its answer, out of which only one is correct.

1. Number of solutions of the equation $\tan x + \sec x = 2\cos x$, lying in the interval $[0, 2\pi]$ is

(a) 0 (b) 1 (c) 2 (d) 3

2. In a triangle ABC, the angle A is greater than angle B. If the values of the angles A and B satisfy the equation $3\sin x - 4\sin^3 x - k = 0, 0 < k < 1$, then the measure of angle C is

(a) $\pi/3$ (b) $\pi/2$ (c) $2\pi/3$ (d) $5\pi/6$

3. The general solution of $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$ is

(a) $n\pi + \frac{\pi}{8}$ (b) $\frac{n\pi}{2} + \frac{\pi}{8}$

(c) $(-1)^n\left(\frac{n\pi}{2} + \frac{\pi}{8}\right)$ (d) $2n\pi + \cos^{-1}\left(\frac{3}{2}\right)$

4. The equation $(\cos p - 1)x^2 + (\cos p)x + \sin p = 0$, where x is a variable, has real roots. Then, the interval of p may be any one of the following :
(a) $(0, 2\pi)$ (b) $(-\pi, 0)$ (c) $(-\pi/2, \pi/2)$ (d) $(0, \pi)$

5. The solution of the equation $\cos^2\theta + \sin\theta + 1 = 0$, lies in the interval
(a) $(-\pi/4, \pi/4)$ (b) $(\pi/4, 3\pi/4)$
(c) $(3\pi/4, 5\pi/4)$ (d) $(5\pi/4, 7\pi/4)$

6. If $\tan\theta + \tan 4\theta + \tan 7\theta = \tan\theta \tan 4\theta \tan 7\theta$, then $\theta =$
(a) $\frac{n\pi}{4}, n \in Z$ (b) $\frac{n\pi}{7}, n \in Z$
(c) $\frac{n\pi}{12}, n \in Z$ (d) $n\pi, n \in Z$

7. The general value of θ satisfying the equation $2\sin^2\theta - 3\sin\theta - 2 = 0$ is
(a) $n\pi + (-1)^n \frac{\pi}{6}$ (b) $n\pi + (-1)^n \frac{\pi}{2}$
(c) $n\pi + (-1)^n \frac{5\pi}{6}$ (d) $n\pi + (-1)^n \frac{7\pi}{6}$

8. General solution of the equation $(\sqrt{3} - 1)\sin\theta + (\sqrt{3} + 1)\cos\theta = 2$ is
(a) $2n\pi \pm \frac{\pi}{4} + \frac{\pi}{12}$ (b) $n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{12}$
(c) $2n\pi \pm \frac{\pi}{4} - \frac{\pi}{12}$ (d) $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{12}$

9. The most general value of θ, satisfying the two equations, $\cos\theta = -\frac{1}{\sqrt{2}}$, $\tan\theta = 1$ is
(a) $2n\pi \pm \frac{5\pi}{4}$ (b) $2n\pi + \frac{\pi}{4}$
(c) $n\pi + \frac{5\pi}{4}$ (d) $(2n+1)\pi + \frac{\pi}{4}$

10. In a right angled triangle the hypotenuse is $2\sqrt{2}$ times the length of perpendicular drawn from the opposite vertex on the hypotenuse, then the other two anlges are
(a) $\frac{\pi}{3}, \frac{\pi}{6}$ (b) $\frac{\pi}{4}, \frac{\pi}{4}$ (c) $\frac{\pi}{8}, \frac{3\pi}{8}$ (d) $\frac{\pi}{12}, \frac{5\pi}{12}$

11. The set of values of x for which $\frac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x} = 1$ is
(a) ϕ (b) $\left\{\frac{\pi}{4}\right\}$
(c) $\left\{n\pi + \frac{\pi}{4}, n = 1, 2, 3, \ldots\right\}$ (d) $\left\{2n\pi + \frac{\pi}{4}, n = 1, 2, 3, \ldots\right\}$

12. The values of θ lying between $\theta = 0$ and $\theta = \frac{\pi}{2}$ and satisfying the equation

$$\begin{vmatrix} 1+\cos^2\theta & \sin^2\theta & 4\sin 4\theta \\ \cos^2\theta & 1+\sin^2\theta & 4\sin 4\theta \\ \cos^2\theta & \sin^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0, \text{ is}$$

(a) $\frac{11\pi}{24}, \frac{7\pi}{24}$ (b) $\frac{7\pi}{24}, \frac{5\pi}{24}$ (c) $\frac{5\pi}{24}, \frac{\pi}{24}$ (d) $\frac{\pi}{24}, \frac{11\pi}{24}$

13. The solution set of $(2\cos x - 1)(3 + 2\cos x) = 0$ in the interval $0 \le x \le 2\pi$, is
(a) $\left\{\frac{\pi}{3}\right\}$ (b) $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$
(c) $\left\{\frac{\pi}{3}, \frac{5\pi}{3}, \cos^{-1}\left(-\frac{3}{2}\right)\right\}$ (d) none of these

14. If $\tan 2\theta \tan\theta = 1$, then $\theta =$
(a) $n\pi + \frac{\pi}{6}, n \in Z$ (b) $n\pi \pm \frac{\pi}{6}, n \in Z$
(c) $2n\pi \pm \frac{\pi}{6}, n \in Z$ (d) none of these

15. The general solution of the trigonometrical equation $\sin x + \cos x = 1$ for $n = 0, \pm 1, \ldots$ is given by
(a) $x = 2n\pi$ (b) $x = 2n\pi + \frac{\pi}{2}$
(c) $x = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}$ (d) none of these

16. If $\sin 5x + \sin 3x + \sin x = 0$, then the value of x other than zero, lying between $0 \le x \le \frac{\pi}{2}$ is
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{12}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

17. The most general value of θ which satisfies both the equations $\tan\theta = -1$ and $\cos\theta = 1/\sqrt{2}$ will be
(a) $n\pi + \frac{7\pi}{4}$ (b) $n\pi + (-1)^n \frac{7\pi}{4}$
(c) $2n\pi + \frac{7\pi}{4}$ (d) none of these

18. The values of θ satisfying $\sin 7\theta = \sin 4\theta - \sin\theta$ and $0 < \theta < \frac{\pi}{2}$ are
(a) $\frac{\pi}{9}, \frac{\pi}{4}$ (b) $\frac{\pi}{3}, \frac{\pi}{9}$ (c) $\frac{\pi}{6}, \frac{\pi}{9}$ (d) $\frac{\pi}{3}, \frac{\pi}{4}$

19. If α, β are different values of x satisfying $a\cos x + b\sin x = c$, then $\tan\left(\frac{\alpha+\beta}{2}\right) =$
(a) $a + b$ (b) $a - b$ (c) b/a (d) a/b
[JEE (Orissa) 2003]

20. The equation $a\sin x + b\cos x = c$, where $|c| > \sqrt{a^2 + b^2}$ has
(a) a unique solution (b) infinite no. of solutions
(c) no solution (d) none of these

21. If α is a root of $25\cos^2\theta + 5\cos\theta - 12 = 0$, $\frac{\pi}{2} < \alpha < \pi$, then $\sin 2\alpha$ is equal to
(a) $\frac{24}{25}$ (b) $-\frac{24}{25}$ (c) $\frac{13}{18}$ (d) $-\frac{13}{18}$

22. If $\cos 3x + \sin\left(2x - \frac{7\pi}{6}\right) = -2$, then $x =$
(a) $\frac{\pi}{3}(6k + 1), k \in Z$ (b) $\frac{\pi}{3}(6k - 1), k \in Z$
(c) $\frac{\pi}{3}(2k + 1), k \in Z$ (d) none of these

23. The number of solutions of $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + \frac{1}{x^2}$, $0 \le x \le \frac{\pi}{2}$ is
(a) 0 (b) 1
(c) infinite (d) none of these

24. If A and B are acute positive angles satisfying the equations $3\sin^2 A + 2\sin^2 B = 1$ and $3\sin 2A - 2\sin 2B = 0$, then $A + 2B =$
(a) 0 (b) $\pi/2$ (c) $\pi/4$ (d) $\pi/3$

25. The equation $\sin\theta = x + \frac{p}{x}$ for real values of x is possible when
(a) $p \ge 0$ (b) $p \le 0$ (c) $p \le \frac{1}{4}$ (d) $p \ge \frac{1}{2}$

26. If $\sin A = \sin B$, $\cos A = \cos B$, then the value of A in terms of B is
(a) $n\pi + B$ (b) $n\pi + (-1)^n B$
(c) $2n\pi + B$ (d) $2n\pi - B$

27. If $5\cos 2\theta + 2\cos^2\frac{\theta}{2} + 1 = 0, -\pi < \theta < \pi$, then $\theta =$
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{3}, \cos^{-1}(3/5)$
(c) $\cos^{-1}(3/5)$ (d) $\frac{\pi}{3}, \pi - \cos^{-1}(3/5)$

28. If $(1+\tan\theta)(1+\tan\phi) = 2$, then $\theta + \phi =$
(a) 30° (b) 45° (c) 60° (d) 75°

29. The general solution of $\tan 3x = 1$, is
(a) $n\pi + \frac{\pi}{4}$ (b) $\frac{n\pi}{3} + \frac{\pi}{12}$
(c) $n\pi$ (d) $n\pi \pm \frac{\pi}{4}$

30. If $1 + \sin\theta + \sin^2\theta + \ldots$ to $\infty = 4 + 2\sqrt{3}, 0 < \theta < \pi, \theta \ne \frac{\pi}{2}$, then $\theta =$
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$
(c) $\frac{\pi}{3}$ or, $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$ or, $\frac{2\pi}{3}$

31. If α, β are the solutions of $a\tan\theta + b\sec\theta = c$, then $\tan(\alpha+\beta) =$
(a) $\frac{2ac}{a^2-c^2}$ (b) $\frac{2ac}{c^2-a^2}$ (c) $\frac{2ac}{a^2+c^2}$ (d) $\frac{ac}{a^2+c^2}$

32. The number of pairs (x, y) satisfying the equations $\sin x + \sin y = \sin(x+y)$ and $|x| + |y| = 1$ is
(a) 2 (b) 4 (c) 6 (d) infinite

33. The expression $(1 + \tan x + \tan^2 x)(1 - \cot x + \cot^2 x)$ has the positive values for x, given by
(a) $0 \le x \le \frac{\pi}{2}$ (b) $0 \le x \le \pi$
(c) for all $x \in R - \{0, \pi/2\}$ (d) $x \ge 0$

34. The equation $k\sin x + \cos 2x = 2k - 7$ possesses a solution, if
(a) $k > 6$ (b) $2 \le k \le 6$
(c) $k > 2$ (d) none of these

35. The equation $\sin^6 x + \cos^6 x = \lambda$, has a solution if
(a) $\lambda \in [1/2, 1]$ (b) $\lambda \in [1/4, 1]$
(c) $\lambda \in [-1, 1]$ (d) $\lambda \in [0, 1/2]$

36. If $y + \cos\theta = \sin\theta$ has a real solution, then
(a) $-\sqrt{2} \le y \le \sqrt{2}$ (b) $y > \sqrt{2}$
(c) $y \le -\sqrt{2}$ (d) none of these

37. The solution set of the equation $4\sin\theta\cos\theta - 2\cos\theta - 2\sqrt{3}\sin\theta + \sqrt{3} = 0$ in the interval $(0, 2\pi)$ is
(a) $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$ (b) $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$
(c) $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{3}\right\}$ (d) $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}\right\}$

38. The most general solution of $\tan\theta = -1, \cos\theta = \frac{1}{\sqrt{2}}$ is
(a) $n\pi + \frac{7\pi}{4}, n \in Z$ (b) $n\pi + (-1)^n\frac{7\pi}{4}, n \in$
(c) $2n\pi + \frac{7\pi}{4}, n \in Z$ (d) none of these

39. If the complex numbers $\sin x + i\cos 2x$ an $\cos x - i\sin 2x$ are conjugate to each other, then x is equ to
(a) $n\pi$ (b) $\left(n + \frac{1}{2}\right)\pi, n \in Z$
(c) 0 (d) none of these

40. The smallest positive root of the equation, $\tan x - x = 0$ l in
(a) $(0, \pi/2)$ (b) $(\pi/2, \pi)$
(c) $(\pi, 3\pi/2)$ (d) $(3\pi/2, 2\pi)$

41. The number of solutions of the equati $\sin x = \cos 3x$ in $[0, \pi]$, is
(a) 1 (b) 2 (c) 3 (d) 4

42. The most general value of θ satisfying
$$\tan\theta + \tan\left(\frac{3\pi}{4} + \theta\right) = 2 \text{ are}$$
(a) $n\pi \pm \frac{\pi}{3}, n \in Z$ (b) $2n\pi + \frac{\pi}{3}, n \in Z$
(c) $2n\pi \pm \frac{\pi}{3}, n \in Z$ (d) $n\pi + (-1)^n\frac{\pi}{3}, n \in$

43. If $\sec\theta\tan\theta = \sqrt{2}$, then $\theta =$

(a) $n\pi+(-1)^n\frac{\pi}{4}, n\in Z$ (b) $2n\pi\pm\frac{\pi}{3}, n\in Z$

(c) $n\pi\pm\frac{2\pi}{3}, n\in Z$ (d) $n\pi-\frac{\pi}{4}, n\in Z$

44. The number of solutions of the equation $\tan x+\sec x = 2\cos x$ lying in the interval $[0, 2\pi]$ is

(a) 0 (b) 1 (c) 2 (d) 3

45. If $\cot\theta\cot 7\theta+\cot\theta\cot 4\theta+\cot 4\theta\cot 7\theta = 1$, then $\theta =$

(a) $n\pi, n\in Z$ (b) $(2n+1)\frac{\pi}{2}, n\in Z$

(c) $n\pi+(-1)^n\frac{\pi}{2}, n\in Z$ (d) $\frac{n\pi}{12}, n\in Z$

46. The number of values of x in $[0, 5\pi]$ satisfying the equation $3\cos 2x-10\cos x+7 = 0$, is

(a) 5 (b) 6 (c) 8 (d) 10

47. The number of values of $x\in[0, 2\pi]$ that satisfy $\cot x-\operatorname{cosec} x = 2\sin x$, is

(a) 3 (b) 2 (c) 1 (d) 0

48. $\cot\theta = \sin 2\theta, \theta\neq n\pi, n\in Z$, if θ equals

(a) 45° or 90° (b) 45° or 60°

(c) 90° only (d) 45° only

49. The solution of the equation

$\cos^2 x-2\cos x = 4\sin x-\sin 2x \ (0\leq x\leq\pi)$ is

(a) $\pi-\cot^{-1}\frac{1}{2}$ (b) $\pi-\tan^{-1}2$

(c) $\pi+\tan^{-1}\left(-\frac{1}{2}\right)$ (d) none of these

50. If $\tan\theta, \cos\theta, \frac{1}{6}\sin\theta$ are in G.P., then general value of θ is

(a) $2n\pi\pm\frac{\pi}{3}, n\in Z$ (b) $2n\pi\pm\frac{\pi}{6}, n\in Z$

(c) $n\pi+(-1)^n\frac{\pi}{3}, n\in Z$ (d) $n\pi+\frac{\pi}{3}, n\in Z$

51. Number of solutions of the equation $\sin 2\theta+2 = 4\sin\theta+\cos\theta$ lying in the interval $[\pi, 5\pi]$, is

(a) 0 (b) 2 (c) 4 (d) 5

52. If $\sin 2x, \frac{1}{2}$ and $\cos 2x$ are in A.P., then the general values of x are given by

(a) $n\pi, n\pi+\frac{\pi}{2}, n\in Z$ (b) $n\pi, n\pi+\frac{\pi}{4}, n\in Z$

(c) $n\pi+\frac{\pi}{4}, n\in Z$ (d) $n\pi, n\in Z$

53. The number of points of intersection of the curves $2y = 1$ and $y = \sin x, -2\pi\leq x\leq 2\pi$, is

(a) 2 (b) 3 (c) 4 (d) 1

54. For $m\neq n$, if $\tan m\theta = \tan n\theta$, then different values of θ are in

(a) A.P. (b) H.P.

(c) G.P. (d) no particular sequence

55. If $\cos p\theta = \cos q\theta, p\neq q$, then

(a) $\theta = 2n\pi, n\in Z$ (b) $\theta = \frac{2n\pi}{p\pm q}, n\in Z$

(c) $\theta = \frac{n\pi}{p+q}, n\in Z$ (d) none of these

56. Solutions of the equation $\cos^2\left(\frac{1}{2}px\right)+\cos^2\left(\frac{1}{2}qx\right) = 1$ form an arithmetic progression with common difference

(a) $\frac{2}{p+q}$ (b) $\frac{2}{p-q}$ (c) $\frac{\pi}{p+q}$ (d) none of these

57. If $(\sec\theta-1) = (\sqrt{2}-1)\tan\theta$, then $\theta =$

(a) $n\pi+\frac{\pi}{8}, n\in Z$ (b) $2n\pi, 2n\pi+\frac{\pi}{4}, n\in Z$

(c) $2n\pi, n\in Z$ (d) none of these

58. If $\sec^2\theta = \sqrt{2}(1-\tan^2\theta)$, then $\theta =$

(a) $n\pi+\frac{\pi}{8}, n\in Z$ (b) $n\pi\pm\frac{\pi}{4}, n\in Z$

(c) $n\pi\pm\frac{\pi}{8}, n\in Z$ (d) none of these

59. The most general solution of the equation

$$8\tan^2\frac{\theta}{2} = 1+\sec\theta, \text{ is}$$

(a) $\theta = 2n\pi\pm\cos^{-1}\left(\frac{1}{3}\right)$ (b) $\theta = 2n\pi\pm\frac{\pi}{6}$

(c) $\theta = 2n\pi\pm\cos^{-1}\left(\frac{-1}{3}\right)$ (d) none of these

60. The number of values of x for which $\sin 2x+\cos 4x = 2$, is

(a) 0 (b) 1 (c) 2 (d) infinite

61. If $2\sec 2\alpha = \tan\beta+\cot\beta$, then one of the values of $\alpha+\beta$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) none of these

62. The equation $8\sec^2\theta-6\sec\theta+1 = 0$ has

(a) exactly two roots (b) exactly four roots

(c) infinitely many roots (d) no roots

63. The equation $\sin x+\sin y+\sin z = -3$ for $0\leq x\leq 2\pi$, $0\leq y\leq 2\pi, 0\leq z\leq 2\pi$, has

(a) one solution (b) two sets of solution

(c) four sets of solution (d) no solution

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64. The solution set of $(5+4\cos\theta)(2\cos\theta+1) = 0$ in the interval $[0, 2\pi]$, is

(a) $\{\pi/3, 2\pi/3\}$ (b) $\{\pi/3, \pi\}$

(c) $\{2\pi/3, 4\pi/3\}$ (d) $\{2\pi/3, 5\pi/3\}$

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65. The solution of the equation $1-\cos\theta=\sin\theta\sin\frac{\theta}{2}$ is

(a) $n\pi, n\in Z$ (b) $2n\pi, n\in Z$

(c) $\frac{n\pi}{2}, n\in Z$ (d) none of these

66. $\{x\in R:\cos 2x+2\cos^2 x=2\}$ is equal to

(a) $\left\{2n\pi+\frac{\pi}{3}:n\in Z\right\}$ (b) $\left\{n\pi\pm\frac{\pi}{6}:n\in Z\right\}$

(c) $\left\{n\pi+\frac{\pi}{3}:n\in Z\right\}$ (d) $\left\{2n\pi-\frac{\pi}{3}:n\in Z\right\}$

[EAMCET 2008]

Answers

1. (c) 2. (c) 3. (b) 4. (d) 5. (d) 6. (c) 7. (d)
8. (a) 9. (d) 10. (c) 11. (a) 12. (a) 13. (b) 14. (b)
15. (c) 16. (c) 17. (c) 18. (a) 19. (c) 20. (c) 21. (b)
22. (a) 23. (a) 24. (b) 25. (c) 26. (c) 27. (d) 28. (b)
29. (b) 30. (d) 31. (a) 32. (c) 33. (c) 34. (b) 35. (b)
36. (a) 37. (d) 38. (c) 39. (d) 40. (c) 41. (c) 42. (a)
43. (a) 44. (c) 45. (d) 46. (c) 47. (d) 48. (a) 49. (c)
50. (a) 51. (c) 52. (b) 53. (c) 54. (s) 55. (b) 56. (d)
57. (b) 58. (c) 59. (a) 60. (a) 61. (a) 62. (d) 63. (a)
64. (c) 65. (b) 66. (b)

CHAPTER TEST

Each of the following questions has four choices (a), (b), (c) and (d) out of which only one is correct. Mark the correct choice.

1. If $|k|=5$ and $0°\le\theta\le 360°$, then the number of different solutions of $3\cos\theta+4\sin\theta=k$ is

(a) zero (b) two (c) one (d) infinite

2. The number of all possible triplets (a_1, a_2, a_3) such that

$$a_1+a_2\cos 2x+a_3\sin^2 x=0 \text{ for all } x, \text{ is}$$

(a) zero (b) 1 (c) 2 (d) infinite

3. The number of all possible 5-tuples $(a_1, a_2, a_3, a_4, a_5)$ such that $a_1+a_2\sin x+a_3\cos x+a_4\sin 2x+a_5\cos 2x=0$ holds for all x is

(a) zero (b) 1 (c) 2 (d) infinite

4. The general solution of the equation $\cos x\cos 6x=-1$ is

(a) $x=(2n+1)\pi,\ n\in Z$ (b) $x=2n\pi,\ n\in Z$

(c) $x=(2n-1)\pi,\ n\in Z$ (d) none of these

5. The values of x satisfying the system of equations

$$2^{\sin x+\cos y}=1,\quad 16^{\sin^2 x+\cos^2 y}=4$$

are given by

(a) $x=n\pi+(-1)^n\frac{\pi}{6}$ and $y=2n\pi\pm\frac{\pi}{3}, n\in Z$

(b) $x=n\pi+(-1)^{n+1}\frac{\pi}{6}$ and $y=2n\pi\pm\frac{2\pi}{3}, n\in Z$

(c) $x=n\pi+(-1)^n\frac{\pi}{6}$ and $y=2n\pi\pm\frac{2\pi}{3}, n\in Z$

(d) $x=n\pi+(-1)^{n+1}\frac{\pi}{6}$ and $y=2n\pi\pm\frac{2\pi}{3}, n\in Z$

The general solution of the equation $\tan 3x=\tan 5x$ is

(a) $x=\frac{n\pi}{2},\ n\in Z$ (b) $x=n\pi,\ n\in Z$

(c) $x=(2n+1)\pi,\ n\in Z$ (d) none of these

7. The number of all possible ordered pairs (x, y), $x, y\in R$ satisfying the system of equations

$$x+y=\frac{2\pi}{3},\quad \cos x+\cos y=\frac{3}{2}, \text{ is}$$

(a) 0 (b) 1 (c) infinite (d) none of these

8. If the expression $\dfrac{\sin\frac{x}{2}+\cos\frac{x}{2}-i\tan x}{1+2i\sin\frac{x}{2}}$ is real, then x is equal to

(a) $2n\pi+2\tan^{-1}k,\ k\in R,\ n\in Z$

(b) $2n\pi+2\tan^{-1}k$, where $k\in(0,1),\ n\in Z$

(c) $2n\pi+2\tan^{-1}k$, where $k\in(1,2),\ n\in Z$

(d) $2n\pi+2\tan^{-1}k,\ k\in(2,3),\ n\in Z$

9. If the equation $\sec\theta+\operatorname{cosec}\theta=c$ has real roots between 0 and 2π, then

(a) $c^2<8$ (b) $c^2>8$

(c) $c^2=8$ (d) none of these

10. If the equation $\sec\theta+\operatorname{cosec}\theta=c$ has four real roots between 0 and 2π, then

(a) $c^2<8$ (b) $c^2>8$

(c) $c^2=8$ (d) none of these

11. If $\theta_1, \theta_2, \theta_3, \theta_4$ are roots of the equation $\sin(\theta+\alpha)=k\sin 2\theta$ no two of which differ by a multiple of 2π, then $\theta_1+\theta_2+\theta_3+\theta_4$ is equal to

(a) $2n\pi,\ n\in Z$ (b) $(2n+1)\pi,\ n\in Z$

(c) $n\pi,\ n\in Z$ (d) none of these

12. If $\sin(\pi\cos\theta)=\cos(\pi\sin\theta)$, then $\cos\left(\theta\pm\frac{\pi}{4}\right)$ is equal to

(a) $\cos\frac{\pi}{4}$ (b) $\frac{1}{2}\cos\frac{\pi}{4}$

(c) $\cos\frac{\pi}{8}$ (d) none of these

13. If $\tan(\pi\cos\theta)=\cot(\pi\sin\theta)$, then the value(s) of $\cos\left(\theta-\frac{\pi}{4}\right)$ is, (are)

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\pm\frac{1}{2\sqrt{2}}$ (d) none of these

14. The general solution of $\tan\left(\frac{\pi}{2}\sin\theta\right)=\cot\left(\frac{\pi}{2}\cos\theta\right)$ is

(a) $\theta=2r\pi+\frac{\pi}{2}$, $r\in Z$

(b) $\theta=2r\pi$, $r\in Z$

(c) $\theta=2r\pi+\frac{\pi}{2}$ and $\theta=2r\pi$, $r\in Z$

(d) none of these

15. The most general value of θ which satisfy both the equations $\cos\theta=-\frac{1}{\sqrt{2}}$ and $\tan\theta=1$, is

(a) $2n\pi+\frac{5\pi}{4}$, $n\in Z$ (b) $2n\pi+\frac{\pi}{4}$, $n\in Z$

(c) $2n\pi+\frac{3\pi}{4}$, $n\in Z$ (d) none of these

16. The number of roots of the equation $x+2\tan x=\frac{\pi}{2}$ in the interval $[0, 2\pi]$, is

(a) 1 (b) 2 (c) 3 (d) infinite

17. If $\sin(\pi\cot\theta)=\cos(\pi\tan\theta)$, then $\cot 2\theta$ is equal to

(a) $n-\frac{1}{4}$ (b) $n+\frac{1}{4}$ (c) $4n+1$ (d) $4n-1$,

where $n\in Z$.

18. The number of distinct roots of the equation $A\sin^3 x+B\cos^3 x+C=0$ no two of which differ by 2π is

(a) 3 (b) 4 (c) infinite (d) 6

19. The value of x between 0 and 2π which satisfy the equation $\sin x\sqrt{8\cos^2 x}=1$ are in A.P. with common difference

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{8}$ (c) $\frac{3\pi}{8}$ (d) $\frac{5\pi}{8}$

20. If $\cos 20^\circ=k$ and $\cos x=2k^2-1$, then the possible values of x between 0° and 360° are

(a) 140° (b) 40° and 140°

(c) 40° and 320° (d) 50° and 130°

21. The general solution of the trigonometric equation $\sin x+\cos x=1$ is given by

(a) $x=2n\pi$, $n\in Z$

(b) $x=2n\pi+\frac{\pi}{2}$, $n\in Z$

(c) $x=n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{4}$, $n\in Z$

(d) none of these

22. The general solution of : $\sin^2\theta\sec\theta+\sqrt{3}\tan\theta=0$ is

(a) $\theta=n\pi+(-1)^{n+1}\frac{\pi}{3}$, $\theta=n\pi$, $n\in Z$

(b) $\theta=n\pi$, $n\in Z$

(c) $\theta=n\pi+(-1)^{n+1}\frac{\pi}{3}$, $n\in Z$

(d) $\theta=\frac{n\pi}{2}$, $n\in Z$.

23. If $x=X\cos\theta-Y\sin\theta$, $y=X\sin\theta+Y\cos\theta$ and $x^2+4xy+y^2=AX^2+BY^2$, $0\le\theta\le\frac{\pi}{2}$, $n\in Z$, then

(a) $\theta=\frac{\pi}{6}$, $A=3, B=1$ (b) $\theta=\frac{\pi}{2}$, $A=3, B=1$

(c) $A=3, B=-1, \theta=\frac{\pi}{4}$ (d) $A=-3, B=1, \theta=\frac{\pi}{4}$

24. The equation $3^{\sin 2x+2\cos^2 x}+3^{1-\sin 2x+2\sin^2 x}=28$ is satisfied for the values of x given by

(a) $\cos x=0, \tan x=-1$ (b) $\tan x=-1, \cos x=1$

(c) $\tan x=1, \cos x=0$ (d) none of these

25. If $0\le x\le\pi/2$ and $81^{\sin^2 x}+81^{\cos^2 x}=30$, then x is equal to

(a) $\frac{\pi}{6}, \frac{\pi}{3}$ (b) $\frac{\pi}{3}, \frac{5\pi}{2}$ (c) $\frac{5\pi}{6}, \frac{\pi}{6}$ (d) $\frac{2\pi}{3}, \frac{\pi}{3}$

26. The smallest positive vaues of x and y which satisfy $\tan(x-y)=1$, $\sec(x+y)=\frac{2}{\sqrt{3}}$ are

(a) $x=\frac{25\pi}{24}, y=\frac{7\pi}{24}$ (b) $x=\frac{37\pi}{24}, y=\frac{19\pi}{24}$

(c) $x=\frac{\pi}{4}, y=\frac{\pi}{2}$ (d) $x=\frac{\pi}{3}, y=\frac{7\pi}{12}$

27. The solution set of the inequality $\cos^2\theta<\frac{1}{2}$, is

(a) $\left\{\theta:(8n+1)\frac{\pi}{4}<\theta<(8n+3)\frac{\pi}{4}, n\in Z\right\}$

(b) $\left\{\theta:(8n-3)\frac{\pi}{4}<\theta<(8n-1)\frac{\pi}{4}, n\in Z\right\}$

(c) $\left\{\theta:(4n+1)\frac{\pi}{4}<\theta<(4n+3)\frac{\pi}{4}, n\in Z\right\}$.

(d) none of these

28. The equation $\sin^4 x+\cos^4 x+\sin 2x+\alpha=0$ is solval for

(a) $-\frac{1}{2}\le\alpha\le\frac{1}{2}$ (b) $-3\le\alpha\le1$

(c) $-\frac{3}{2}\le\alpha\le\frac{1}{2}$ (d) $-1\le\alpha\le1$

29. The equation $\sin^4 x-2\cos^2 x+a^2=0$ is solvable if

(a) $-\sqrt{3}\le a\le\sqrt{3}$ (b) $-\sqrt{2}\le a\le\sqrt{2}$

(c) $-1\le a\le1$ (d) none of these

30. If $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, then the two curves $y = \cos x$ and $y = \sin 3x$ intersect at

(a) $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{\pi}{8}, \cos\frac{\pi}{8}\right)$

(b) $\left(\frac{-\pi}{4}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{-\pi}{8}, \cos\frac{\pi}{8}\right)$

(c) $\left(\frac{\pi}{4}, \frac{-1}{\sqrt{2}}\right)$ and $\left(\frac{\pi}{8}, -\cos\frac{\pi}{8}\right)$

(d) $\left(\frac{-\pi}{4}, \frac{1}{\sqrt{2}}\right)$

31. If $\frac{1}{6}\sin x, \cos x, \tan x$ are in G.P., then x is equal to

(a) $n\pi \pm \frac{\pi}{3},\ n \in Z$ (b) $2n\pi \pm \frac{\pi}{3},\ n \in Z$

(c) $n\pi + (-1)^n \frac{\pi}{3},\ n \in Z$ (d) none of these

32. The minimum value of $2^{\sin x} + 2^{\cos x}$, is

(a) 1 (b) 2 (c) $2^{-\frac{1}{\sqrt{2}}}$ (d) $2^{1-\frac{1}{\sqrt{2}}}$

33. From the identity $\sin 3x = 3\sin x - 4\sin^3 x$ it follows that if x is real and $|x| < 1$, then

(a) $(3x - 4x^3) > 1$ (b) $(3x - 4x^3) \le 1$

(c) $(3x - 4x^3) < 1$ (d) none of these

34. The most general solution of

$$2^{1 + |\cos x| + \cos^2 x + |\cos^3 x| + \dots \infty} = 4$$

is given by

(a) $x = n\pi \pm \frac{\pi}{3}, n \in Z$ (b) $x = 2n\pi \pm \frac{\pi}{3}, n \in Z$

(c) $x = 2n\pi \pm \frac{2\pi}{3}, n \in Z$ (d) none of these

35. Let α, β be any two positive values of x for which $2\cos x$, $|\cos x|$ and $1 - 3\cos^2 x$ are in G.P. The minimum value of $|\alpha - \beta|$, is

(a) $\pi/3$ (b) $\pi/4$

(c) $\pi/2$ (d) none of these

36. If $\underset{x \in R}{\text{Max}} \{5\sin x + 3\sin(x - \theta)\} = 7$, then $\theta =$

(a) $2n\pi \pm \frac{\pi}{3}, n \in Z$ (b) $2n\pi \pm \frac{2\pi}{3}, n \in Z$

(c) $\frac{\pi}{3}, \frac{2\pi}{3}$ (d) none of these

37. The most general value of θ for which

$$\sin\theta - \cos\theta = \min_{x \in R} \{1, x^2 - 4x + 6\}$$

are given by

(a) $\theta = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}, n \in Z$

(b) $\theta = n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{4}, n \in Z$

(c) $\theta = 2n\pi + \frac{\pi}{4}, n \in Z$

(d) none of these

38. The number of points of intersection of the two curves $y = 2\sin x$ and $y = 5x^2 + 2x + 3$, is

(a) 0 (b) 1 (c) 2 (d) ∞

39. Let $2\sin^2 x + 3\sin x - 2 > 0$ and $x^2 - x - 2 < 0$ (x is measured in radians). Then, x lies in the interval.

(a) $(\pi/6, 5\pi/6)$ (b) $(-1, 5\pi/6)$

(c) $(-1, 2)$ (d) $(\pi/6, 2)$

40. The largest positive solution of $1 + \sin^4 x = \cos^2 3x$ in $[-5\pi/2, 5\pi/2]$ is

(a) π (b) 2π

(c) $\frac{5\pi}{2}$ (d) none of these

41. The set of values of x in $(-\pi, \pi)$ satisfying the inequation $|4\sin x - 1| < \sqrt{5}$ is

(a) $(-\pi/10, 3\pi/10)$ (b) $(-\pi/10, \pi)$

(c) $(-\pi, \pi)$ (d) $(-\pi, 3\pi/10)$

42. If $\theta \in [0, 5\pi]$ and $r \in R$ such that $2\sin\theta = r^4 - 2r^2 + 3$, then the maximum number of values of the pair (r, θ) is

(a) 6 (b) 8

(c) 10 (d) none of these

43. The total number of ordered pairs (r, θ) satisfying $r\sin\theta = 3$, $r = 4(1 + \sin\theta)$, where $r > 0$ and $\theta \in [-\pi, \pi]$ is

(a) 0 (b) 2 (c) 4 (d) none of these

44. The solution set of the inequation $\log_{1/2}\sin x > \log_{1/2}\cos x$ in $[0, 2\pi]$, is

(a) $(0, \pi/2)$ (b) $(-\pi/4, \pi/4)$

(c) $(0, \pi/4)$ (d) none of these

45. If the equation $\sin\theta(\sin\theta + 2\cos\theta) = a$ has a real solution, then the shortest interval containing 'a' is

(a) $\left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]$ (b) $\left(\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}+1}{2}\right)$

(c) $(-1/2, 1/2)$ (d) none of these

46. The equation $\sin^4\theta + \cos^4\theta = a$ has a real solution if

(a) $a \in [1/2, 1]$ (b) $a \in [1/4, 1/2]$

(c) $a \in [1/3, 1]$ (d) none of these

47. If $32\tan^8\theta = 2\cos^2\alpha - 3\cos\alpha$ and $3\cos 2\theta = 1$, then $\alpha =$

(a) $2n\pi, n \in Z$ (b) $2n\pi \pm \frac{2\pi}{3}, n \in Z$

(c) $2n\pi \pm \frac{\pi}{3}, n \in Z$ (d) $n\pi \pm \frac{\pi}{3}, n \in Z$

48. If $\tan\theta \tan(120° - \theta) \tan(120° + \theta) = \frac{1}{\sqrt{3}}$, then $\theta =$

(a) $\frac{n\pi}{3} - \frac{\pi}{2}, n \in Z$ (b) $\frac{n\pi}{3} - \frac{\pi}{18}, n \in Z$

(c) $\frac{n\pi}{3} + \frac{\pi}{18}, n \in Z$ (d) $\frac{n\pi}{3} + \frac{\pi}{12}, n \in Z$

49. The solution of the equation

$\log_{\cos x} \sin x + \log_{\sin x} \cos x = 2$ is given by

(a) $x = 2n\pi + \frac{\pi}{4}, n \in Z$ (b) $x = n\pi + \frac{\pi}{2}, n \in Z$

(c) $x = n\pi + \frac{\pi}{8}, n \in Z$ (d) $x = 2n\pi + \frac{\pi}{6}, n \in Z$

50. The number of solutions of the equation $\tan\theta + \sec\theta = 2\cos\theta$ lying in the interval $[0, 2\pi]$, is

(a) 0 (b) 1 (c) 2 (d) 3

51. One root of the equation $\cos\theta - \theta + \frac{1}{2} = 0$ lies in the interval

(a) $(0, \pi/2)$ (b) $(-\pi/2, 0)$

(c) $(\pi/2, \pi)$ (d) $(\pi, 3\pi/2)$

52. If $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$, then which of the following is corrcet?

(a) $\cos\theta = \frac{3}{2\sqrt{2}}$ (b) $\cos\left(\theta - \frac{\pi}{2}\right) = \frac{1}{2\sqrt{2}}$

(c) $\cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$ (d) $\cos\left(\theta + \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$

53. If $2\sec(2\alpha) = \tan\beta + \cot\beta$, then one of the values of $\alpha + \beta$, is

(a) π (b) $n\pi - \frac{\pi}{4}, n \in Z$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

54. The values of α for which the equation $\sin^4 x + \cos^4 x + \sin 2x + \alpha = 0$ may be valid, are

(a) $-\frac{3}{2} \le \alpha \le 1$ (b) $0 \le \alpha \le \frac{1}{2}$

(c) $-\frac{3}{2} \le \alpha \le \frac{1}{2}$ (d) none of these

55. $\tan|x| = |\tan x|$, if

(a) $x \in \left(-k\pi, (2k-1)\frac{\pi}{2}\right), k \in Z$

(b) $x \in \left((2k-1)\frac{\pi}{2}, k\pi\right), k \in Z$

(c) $x \in \left(-(2k+1)\frac{\pi}{2}, -k\pi\right) \cup \left(k\pi, (2k+1)\frac{\pi}{2}\right), k \in Z$

(d) none of these

56. The number of solutions of the equation $2^{\cos x} = |\sin x|$ in $[-2\pi, 2\pi]$, is

(a) 1 (b) 2 (c) 3 (d) 4

57. If $\sin x \cos x \cos 2x = \lambda$ has a solution, then λ lies in the interval

(a) $[-1/4, 1/4]$ (b) $[-1/2, 1/2]$

(c) $(-\infty, -1/4] \cup [1/4, \infty)$ (d) $(-\infty, -1/2] \cup [1/2, \infty)$

58. If $\sin 3\theta = 4\sin\theta(\sin^2 x - \sin^2\theta)$, $\theta \ne n\pi, n \in Z$. Then, the set of values of x is

(a) $\left\{n\pi \pm \frac{\pi}{3} : n \in Z\right\}$ (b) $\left\{n\pi \pm \frac{2\pi}{3} : n \in Z\right\}$

(c) $\left\{n\pi \pm \frac{\pi}{2} : n \in Z\right\}$ (d) $\left\{n\pi \pm \frac{\pi}{4} : n \in Z\right\}$

59. If $\sin 2x \cos 2x \cos 4x = \lambda$ has a solution, then λ lies in the interval

(a) $[-1/2, 1/2]$ (b) $[-1/4, 1/4]$

(c) $[-1/3, 1/3]$ (d) none of these

60. If the equation $\cos(\lambda\sin\theta) = \sin(\lambda\cos\theta)$ has a solution in $[0, 2\pi]$, then the smallest positive value of λ is

(a) $\frac{\pi}{\sqrt{2}}$ (b) $\sqrt{2}\pi$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{2\sqrt{2}}$

Answers

1. (b) 2. (d) 3. (b) 4. (a) 5. (c) 6. (b) 7. (a)
8. (c) 9. (a) 10. (b) 11. (b) 12. (b) 13. (c) 14. (b)
15. (a) 16. (c) 17. (b) 18. (d) 19. (a) 20. (c) 21. (c)
22. (b) 23. (b) 24. (a) 25. (a) 26. (a) 27. (c) 28. (c)
29. (b) 30. (a) 31. (b) 32. (d) 33. (d) 34. (a) 35. (d)
36. (a) 37. (b) 38. (a) 39. (d) 40. (b) 41. (a) 42. (a)
43. (b) 44. (c) 45. (a) 46. (a) 47. (b) 48. (c) 49. (a)
50. (c) 51. (a) 52. (c) 53. (c) 54. (a) 55. (c) 56. (d)
57. (a) 58. (a) 59. (b) 60. (d)

Solutions of Exercises and **Chapter-tests** are available in a separate book on **"Solutions of Objective Mathematics".**

HEIGHTS AND DISTANCES

1. ANGLES OF ELEVATION AND DEPRESSION

Let O and P be two points such that the point P is at higher level. Let OA and PB be horizontal lines through O and P respectively.

If an observer is at O and the point P is the object under consideration, then the line OP is called the line of sight of the point P and the angle $\angle AOP$, between the line of sight and the horizontal line OA, is known as the angle of elevation of point P as seen from O.

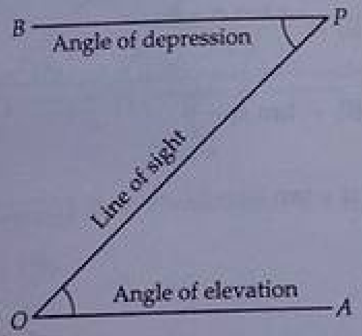

Fig. 1

If an observer is at P and the object under consideration is at O, then the angle $\angle BPO$ is known as the angle of depression of O as seen from P.

Obviously, the angle of elevation of a point P as seen from a point O is equal to the angle of depression of O as seen from P.

SOME USEFUL RESULTS

(i) Any line perpendicular to a plane is perpendicular to every line lying in the plane.

(ii) In a triangle the internal bisector of an angle divides the opposite side in the ratio of the arms of the angle.

(iii) In an isosceles triangle the median is perpendicular to the base.

(iv) Angles in the same segment of a circle are equal.

(v) **m – n THEOREM** In Fig. 2, we have

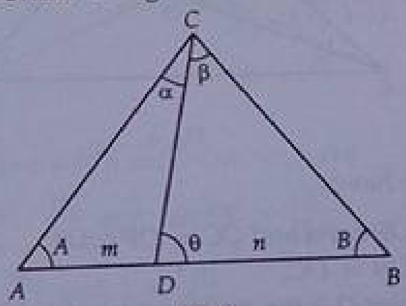

Fig. 2

(a) $(m+n)\cot\theta = m\cot\alpha - n\cot B$

(b) $(m+n)\cot\theta = n\cot A - m\cot B$

SECTION - I

SOLVED MCQs

EXAMPLE 1 *A man from the top of a 100 metres high tower sees a car moving towards the tower at an angle of depression of 30°. After some time, the angle of depression becomes 60°. The distance (in metres) travelled by the car during this time is*

(a) $100\sqrt{3}$ (b) $\frac{200\sqrt{3}}{3}$ (c) $\frac{100\sqrt{3}}{3}$ (d) $200\sqrt{3}$

[IIT (S) 2001]

Ans. (b)

SOLUTION Let OT be the tower and A and B be the positions of the car.

In Δ'S AOT and BOT, we have

$$\tan 30° = \frac{OT}{OA} \text{ and } \tan 60° = \frac{OT}{OB}$$

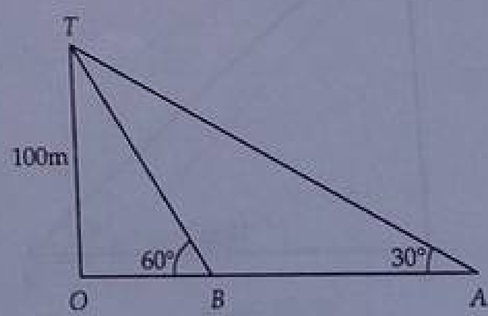

Fig. 3

Ans. (a)

SOLUTION Let AB be the tower of height h metre and C be a point on the ground such that the angle of elevation of B from C is 30°.

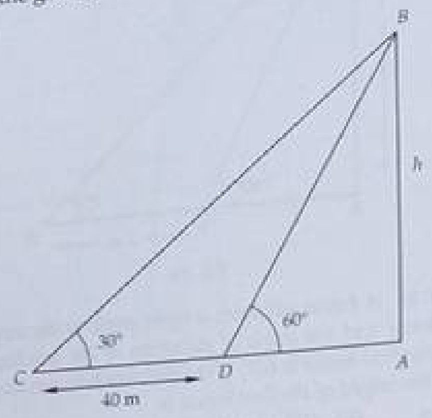

Fig. 21

In ΔCAB, we have

$$\tan 30° = \frac{h}{AC} \Rightarrow AC = h\sqrt{3}\text{ m}$$

In ΔDAB, we have

$$\tan 60° = \frac{h}{AD} \Rightarrow AD = \frac{h}{\sqrt{3}}\text{ m}$$

Now, $AC = h\sqrt{3}$ m

$$\Rightarrow \quad AD + DC = h\sqrt{3}$$

$$\Rightarrow \quad \frac{h}{\sqrt{3}} + 40 = h\sqrt{3} \Rightarrow h\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) = 40 \Rightarrow h = 20\sqrt{3}\text{ m}$$

EXAMPLE 21 *From the top of a hill h metres high the angles of depression of the top and the bottom of a pillar are α and β respectively. The height (in metres) of the pillar is*

(a) $\dfrac{h(\tan\beta - \tan\alpha)}{\tan\beta}$ (b) $\dfrac{h(\tan\alpha - \tan\beta)}{\tan\alpha}$

(c) $\dfrac{h(\tan\beta + \tan\alpha)}{\tan\beta}$ (d) $\dfrac{h(\tan\beta + \tan\alpha)}{\tan\alpha}$

Ans. (a) **[EAMCET 2009]**

SOLUTION Let AB be the pole of height H metres and Q be the top of the pillar. In Δ's APQ and BCQ, we have

$$\tan\beta = \frac{PQ}{AP} \text{ and, } \tan\alpha = \frac{QC}{BC}$$

$$\Rightarrow \quad \tan\beta = \frac{h}{x} \text{ and } \tan\alpha = \frac{h-H}{x}$$

$$\Rightarrow \quad \frac{\tan\alpha}{\tan\beta} = \frac{h-H}{h}$$

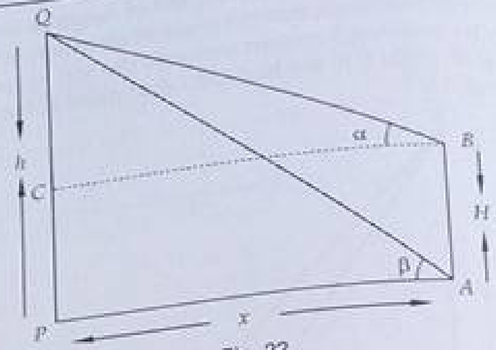

Fig. 22

$$\Rightarrow \quad \frac{H}{h} = \frac{\tan\beta - \tan\alpha}{\tan\beta} \Rightarrow H = \frac{h(\tan\beta - \tan\alpha)}{\tan\beta}$$

EXAMPLE 22 *A bird is sitting on the top of a vertical pole 20 m high and its elevation from a point O on the ground is 45°. It flies off horizontally straight away from the point O. After one second, the elevation of the bird from O is reduced to 30°. Then the speed (in m/sec) of the bird is*

(a) $20\sqrt{2}$ (b) $20(\sqrt{3}-1)$ (c) $40(\sqrt{2}-1)$ (d) $40(\sqrt{3}-\sqrt{2})$

[JEE (Main) 2014]

Ans. (b)

SOLUTION Let AB be the vertical pole of height 20 m and let Q be the position of bird after one second. It is given that $\angle AOB = 45°$ and $\angle POQ = 30°$.

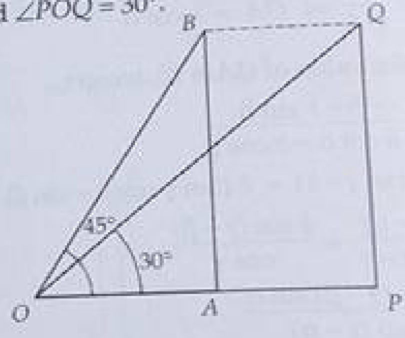

Fig. 23

In ΔOAB and OPQ, we have

$$\tan 45° = \frac{AB}{OA} \text{ and } \tan 30° = \frac{PQ}{OP}$$

$$\Rightarrow \quad OA = 20 \text{ and } \sqrt{3}\,PQ = OP$$

$$\Rightarrow \quad OP = 20\sqrt{3} \qquad [\because PQ = OA = 20]$$

$$\Rightarrow \quad OA + AP = 20\sqrt{3}$$

$$\Rightarrow \quad AP = 20\sqrt{3} - 20 = 20(\sqrt{3}-1)$$

EXERCISE

This exercise contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) for its answer, out of which only one is correct. Mark the correct choice in each question.

1. The angle of elevation of the top of the tower observed from each of the three points A, B, C on the ground, forming a triangle is the same angle α. If R is the circum-radius of the triangle ABC, then the height of the tower is

(a) $R \sin\alpha$ (b) $R\cos\alpha$ (c) $R\cot\alpha$ (d) $R\tan\alpha$ **[CEE (Delhi) 2006]**

2. A flag staff of 5 m high stands on a building of 25 m high. At an observer at a height of 30 m. The flag staff and the building subtend equal angles. The distance of the observer from the top of the flag staff is
(a) $\frac{5\sqrt{3}}{2}$ (b) $5\sqrt{\frac{3}{2}}$ (c) $5\sqrt{\frac{2}{3}}$ (d) none of these

3. ABC is a triangular park with $AB = AC = 100$ metres. A clock tower is situated at the mid-point of BC. The angles of elevation of the top of the tower at A and B are $\cot^{-1} 3.2$ and $\operatorname{cosec}^{-1} 2.6$ respectively. The height of the tower is
(a) 16 m (b) 25 m (c) 50 m (d) none of these

4. If a flag-staff of 6 metres high placed on the top of a tower throws a shadow of $2\sqrt{3}$ metres along the ground then the angle (in degrees) that the sun makes with the ground is
(a) 60° (b) 30° (c) 45° (d) none of these

5. The angle of elevation of the top of an incomplete vertical pillar at a horizontal distance of 100 m from its base is 45°. If the angle of elevation of the top of the complete pillar at the same point is to be 60°, then the height of the incomplete pillar is to be increased by
(a) $50\sqrt{2}$ m (b) 100 m
(c) $100(\sqrt{3}-1)$ m (d) $100(\sqrt{3}+1)$ m

6. The top of a hill observed from the top and bottom of a building of height h is at angles of elevation p and q respectively. The height of the hill is
(a) $\frac{h\cot q}{\cot q - \cot p}$ (b) $\frac{h\cot p}{\cot p - \cot q}$
(c) $\frac{h\tan p}{\tan p - \tan q}$ (d) none of these

7. The angle of elevation of a cliff at a point A on the ground and a point B, 100 m vertically at A are α and β respectively. The height of the cliff is
(a) $\frac{100\cot\alpha}{\cot\alpha - \cot\beta}$ (b) $\frac{100\cot\beta}{\cot\alpha - \cot\beta}$
(c) $\frac{100\cot\beta}{\cot\beta - \cot\alpha}$ (d) $\frac{100\cot\beta}{\cot\beta + \cot\alpha}$

8. The angle of elevation of a cloud from a point h mt. above is $\theta°$ and the angle of depression of its reflection in the lake is ϕ. Then, the height is
(a) $\frac{h\sin(\phi-\theta)}{\sin(\phi+\theta)}$ (b) $\frac{h\sin(\phi+\theta)}{\sin(\phi-\theta)}$
(c) $\frac{h\sin(\theta+\phi)}{\sin(\theta-\phi)}$ (d) none of these

9. On the level ground the angle of elevation of the top of a tower is 30°. On moving 20 m nearer the tower, the angle of elevation is found to be 60°. The height of the tower is
(a) 10 m (b) 20 m (c) $10\sqrt{3}$ m (d) none of these

10. Each side of a square subtends an angle of 60° at the top of a tower h metres high standing in the centre of the square. If a is the length of each side of the square, then
(a) $2a^2 = h^2$ (b) $2h^2 = a^2$
(c) $3a^2 = 2h^2$ (d) $2h^2 = 3a^2$

11. The angle of elevation of the top of a tower at any point on the ground is 30° and moving 20 metres towards the tower it becomes 60°. The height of the tower is
(a) 10 m (b) $10\sqrt{3}$ m
(c) $\frac{10}{\sqrt{3}}$ m (d) none of these

12. From the top of a light house 60 metres high with its base at the sea level, the angle of depression of a boat is 15°. The distance of the boat from the foot of the light house is
(a) $\frac{\sqrt{3}-1}{\sqrt{3}+1}\cdot 60$ metres (b) $\frac{\sqrt{3}+1}{\sqrt{3}-1}\cdot 60$ metres
(c) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ metres (d) none of these

13. A person standing on the bank of a river observes that the angle subtended by a tree on the opposite bank is 60°, when he retires 40 metres from the bank he finds the angle to be 30°. Then, the breadth of the river is
(a) 40 m (b) 60 m (c) 20 m (d) 30 m

14. AB is a vertical pole. The end A is on the level ground. C is the middle point of AB. P is a point on the level ground. The portion BC subtends an angle β at P. If $AP = n\,AB$, then $\tan\beta =$
(a) $\frac{n}{2n^2+1}$ (b) $\frac{n}{n^2-1}$
(c) $\frac{n}{n^2+1}$ (d) none of these

15. A tree is broken by wind, its upper part touches the ground at a point 10 metres from the foot of the tree and makes an angle of 45° with the ground. The entire length of the tree is
(a) 15 metres (b) 20 metres
(c) $10(1+\sqrt{2})$ metres (d) $10\left(1+\frac{\sqrt{3}}{2}\right)$ metres

16. An aeroplane flying at a height of 300 metres above the ground passes vertically above another plane at an instant when the angles of elevation of the two planes from the same point on the ground are 60° and 45° respectively. The height of the lower plane from the ground (in metres) is
(a) $100\sqrt{3}$ (b) $\frac{100}{\sqrt{3}}$ (c) 50 (d) $150(\sqrt{3}+1)$

17. A tower subtends an angle α at a point in the plane of its base and the angle of depression of the foot of the tower at a point b ft. just above A is β. Then, height of the tower is
(a) $b\tan\alpha\cot\beta$ (b) $b\cot\alpha\tan\beta$
(c) $b\tan\alpha\tan\beta$ (d) $b\cot\alpha\cot\beta$

18. The angle of elevation of the top of a tower standing on a horizontal plane from a point A is α. After walking a distance a towards the foot of the tower the angle of elevation is found to be β. The height of the tower is
(a) $\frac{a\sin\alpha\sin\beta}{\sin(\beta-\alpha)}$ (b) $\frac{a\sin\alpha\sin\beta}{\sin(\alpha-\beta)}$

(c) $\frac{a \sin(\beta - \alpha)}{\sin\alpha \sin\beta}$ (d) $\frac{a \sin(\alpha - \beta)}{\sin\alpha \sin\beta}$

[EAMCET 2007]

19. From an aeroplane vertically over a straight horizontal road, the angles of depression of two consecutive mile-stones on opposite sides of the aeroplane are observed to be α and β. The height of the aeroplane above the road is

(a) $\frac{\tan\alpha + \tan\beta}{\tan\alpha \tan\beta}$ (b) $\frac{\tan\alpha \tan\beta}{\tan\alpha + \tan\beta}$

(c) $\frac{\cot\alpha \cot\beta}{\cot\alpha + \cot\beta}$ (d) none of these

20. A vertical tower stands on a declivity which is inclined at 15° to the horizon. From the foot of the tower a man ascends the declivity for 80 feet and then finds that the tower subtends an angle of 30°. The height of the tower is

(a) $20(\sqrt{6} - \sqrt{2})$ (b) $40(\sqrt{6} - \sqrt{2})$

(c) $40(\sqrt{6} + \sqrt{2})$ (d) none of these

21. The angle of elevation of an object on a hill from a point on the ground is 30°. After walking 120 metres the elevation of the object is 60°. The height of the hill is

(a) 120 m (b) $60\sqrt{3}$ m (d) $120\sqrt{3}$ m (d) 60 m

[EAMCET 2006]

22. A tower of x metres height has flag staff at its top. The tower and the flag staff subtend equal angles at a point distant y metres from the foot of the tower. Then, the length of the flag staff in metres is

(a) $y\left(\frac{x^2 - y^2}{x^2 + y^2}\right)$ (b) $x\left(\frac{x^2 + y^2}{y^2 - x^2}\right)$

(c) $x\left(\frac{x^2 + y^2}{x^2 - y^2}\right)$ (d) $x\left(\frac{x^2 - y^2}{x^2 + y^2}\right)$

[EAMCET 2005, JEE (WB) 2006]

23. A house of height 100 m subtends a right angle at the window of an opposite house. If the height of the window be 64 m, then the distance between two houses is

(a) 48 m (b) 36 m (c) 54 m (d) 72 m

[JEE (WB) 2007]

24. A tower of height b subtends an angle at a point O on the level of the foot of the tower and at a distance 'a' from the foot of the tower. If the pole mounted on the tower also subtends an equal angle at O, the height of the pole is

(a) $b\left(\frac{a^2 - b^2}{a^2 + b^2}\right)$ (b) $b\left(\frac{a^2 + b^2}{a^2 - b^2}\right)$

(c) $a\left(\frac{a^2 - b^2}{a^2 + b^2}\right)$ (d) $a\left(\frac{a^2 + b^2}{a^2 - b^2}\right)$

25. A man of height 6 ft. observes the top of a tower and the foot of the tower at angles of 45° and 30° of elevation and depression respectively. The height of the tower is

(a) 13.79 m (b) 14.59 m

(c) 14.29 m (d) none of these

26. If the elevation of the sun is 30°, then the length of the shadow cast by a tower of 150 ft. height is

(a) $75\sqrt{3}$ ft. (b) $200\sqrt{3}$ ft.

(c) $150\sqrt{3}$ ft. (d) none of these

27. A ladder rests against a vertical wall at angle α to the horizontal. If its foot is pulled away from the wall through a distance 'a' so that it slides a distance 'b' down the wall making an angle β with the horizontal, then $a =$

(a) $b\tan\left(\frac{\alpha - \beta}{2}\right)$ (b) $b\tan\left(\frac{\alpha + \beta}{2}\right)$

(c) $b\cot\left(\frac{\alpha - \beta}{2}\right)$ (d) none of these

28. From the top of a cliff 300 metres high, the top of a tower was observed at an angle of depression 30° and from the foot of the tower the top of the cliff was observed at an angle of elevation 45°, The height of the tower is

(a) $50(3 - \sqrt{3})$ m (b) $200(3 - \sqrt{3})$ m

(c) $100(3 - \sqrt{3})$ m (d) none of these

29. The angles of elevation of the top of a tower at the top and the foot of a pole of height 10 m are 30° and 60° respectively. The height of the tower is

(a) 10 m (b) 15 m (c) 20 m (d) none of these

30. A person standing on the bank of a river finds that the angle of elevation of the top of a tower on the opposite bank is 45°, then which of the following statements is correct?

(a) Breadth of the river is twice the height of the tower.

(b) Breadth of the river and the height of the tower are the same.

(c) Breadth of the river is half of the height of the tower

(d) none of these

31. A tower subtends an angle of 30° at a point distant d from the foot of the tower and on the same level as the foot of the tower. At a second point, h vertically above the first, the angle of depression of the foot of the tower is 60°. The height of the tower is

(a) $\frac{h}{3}$ (b) $\frac{h}{3d}$ (c) $3h$ (d) $\frac{3h}{d}$

32. AB is a vertical pole and C is its middle point. The end A is on the level ground and P is any point on the level ground other than A the portion CB subtends an angle β at P. If $AP : AB = 2 : 1$, then $\beta =$

(a) $\tan^{-1}\frac{4}{9}$ (b) $\tan^{-1}\frac{1}{9}$

(c) $\tan^{-1}\frac{5}{9}$ (d) $\tan^{-1}\frac{2}{9}$

33. The angle of depression of a point situated at a distance of 70 metres from the base of a tower is 45°. The height of the tower is

(a) 70 m (b) $70\sqrt{2}$ m (c) $\frac{70}{\sqrt{2}}$ m (d) 35 m

34. The angle of elevation of the top of a vertical tower from two points distance a and b from the base and in the same line with it, are complimentary. If θ is the angle subtended at the top of the tower by the line joining these points then $\sin\theta =$

(a) $\frac{a - b}{\sqrt{2}(a + b)}$ (b) $\frac{a + b}{a - b}$

(c) $\frac{a-b}{a+b}$ (d) none of these

35. An aeroplane flying horizontally 1 km above the ground is observed at an elevation of 60°. If after 10 seconds the elevation is observed to be 30°, the uniform speed per hour of the aeroplane is
(a) $120\sqrt{3}$ km/hour (b) $240\sqrt{3}$ km/hour
(c) $250\sqrt{3}$ km/hour (d) none of these

36. At the foot of the mountain the elevation of its summit is 45°, after ascending 100 m towards the mountain up a slope of 30° inclination, the elevation is found to be 60°. The height of the mountain is
(a) $\frac{\sqrt{3}+1}{2}$ m (b) $\frac{\sqrt{3}-1}{2}$ m
(c) $\frac{\sqrt{3}+1}{2\sqrt{3}}$ m (d) none of these

37. At a distance 12 metres from the foot A of a tower AB of height 5 metres, a flagstaff BC on top of AB and the tower subtend the same angle. Then, the height of flagstaff is
(a) $\frac{1440}{119}$ metres (b) $\frac{475}{119}$ metres
(c) $\frac{845}{119}$ metres (d) none of these

38. A tower 50 m high, stands on top of a mount, from a point on the ground the angles of elevation of the top and bottom of the tower are found to be 75° and 60° respectively. The height of the mount is
(a) 25 m (b) $25(\sqrt{3}-1)$ m
(c) $25\sqrt{3}$ m (d) $25(\sqrt{3}+1)$ m

39. A person on a ship sailing north sees two lighthouses which are 6 km apart, in a line due west. After an hour's tailing one of them bears south west and the other southern south west. The ship is travelling at a rate of
(a) 12 km/hr (b) 6 km/hr
(c) $3\sqrt{2}$ km/hr (d) $(6+3\sqrt{2})$ km/hr

40. An observer finds that the elevation of the top of a tower is $22\frac{1}{2}^{\circ}$ and after walking 150 metres towards the foot of the tower he finds that the elevation of the top has increased to $67\frac{1}{2}^{\circ}$. The height of the tower in metres is
(a) 50 (b) 75 (c) 125 (d) 175

41. A vertical lamp-post, 6 m high, stands at a distance of 2 m from a wall, 4 m high. A 1.5 m tall man starts to walk away from the wall on the other side of the wall, in line with the lamp-post the maximum distance to which the man can walk remaining in the shadow is
(a) $\frac{5}{2}$ m (b) $\frac{3}{2}$ m
(c) 4 m (d) none of these

42. The angle of elevation of the top of a vertical pole when observed from each vertex of a regular hexagon is $\frac{\pi}{3}$. If the area of the circle circumscribing the hexagon be A metre^2, then the area of the hexagon is
(a) $\frac{3\sqrt{3}A}{8}$ m^2 (b) $\frac{\sqrt{3}A}{\pi}$ m^2
(c) $\frac{3\sqrt{3}A}{4\pi}$ m^2 (d) $\frac{3\sqrt{3}A}{2\pi}$ m^2

43. The upper $\left(\frac{3}{4}\right)^{\text{th}}$ portion of a vertical pole subtends an angle $\tan^{-1}\frac{3}{5}$ at a point in the horizontal plane through its foot at a distance 40 m from the foot. A possible height of the vertical pole is
(a) 80 m (b) 20 m (c) 40 m (d) 60 m
[AIEEE 2003]

44. A person standing on the bank of a river observes that the angle of elevation of the top of a tree on the opposite bank of the river is 60° and when he retires 40 metres away from the tree the angle of elevation becomes 30°. The breadth of the river is
(a) 60 m (b) 30 m (c) 40 m (d) 20 m
[AIEEE 2004]

45. A tower subtends angles $\alpha, 2\alpha, 3\alpha$ respectively at points A, B and C all lying in a horizontal line through the foot of the tower. Then, $\frac{AB}{BC}=$
(a) $\frac{\sin 3\alpha}{\sin 2\alpha}$ (b) $1+2\cos 2\alpha$
(c) $2+\cos 3\alpha$ (d) $\frac{\sin 2\alpha}{\sin \alpha}$
[EAMCET 2003]

Answers

1. (d) **2.** (b) **3.** (b) **4.** (a) **5.** (c) **6.** (b) **7.** (c)
8. (b) **9.** (c) **10.** (b) **11.** (b) **12.** (b) **13.** (c) **14.** (a)
15. (c) **16.** (a) **17.** (a) **18.** (a) **19.** (b) **20.** (b) **21.** (b)
22. (b) **23.** (a) **24.** (b) **25.** (a) **26.** (d) **27.** (b) **28.** (a)
29. (c) **30.** (a) **31.** (b) **32.** (b) **33.** (d) **34.** (c) **35.** (b)
36. (a) **37.** (c) **38.** (c) **39.** (d) **40.** (b) **41.** (a) **42.** (d)
43. (c) **44.** (d) **45.** (b)

Solutions of **Exercises** and **Chapter-tests** are available in a separate book on **"Solutions of Objective Mathematics".**