Mecânica dos Fluidos Computacional

Métodos de Diferenças Finitas para Problemas Bidimensionais

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Conteúdo

Problema Modelo - Difusão

Seja o domínio espacial $\Omega \subset \mathbb{R}^2$ *com contorno* $\partial\Omega$, *dada as funções* $f(x,y) : \Omega \to \mathbb{R}$, $\overline{u}(x,y) : \partial\Omega \to \mathbb{R}$ *e* $g(x,y)$, *encontrar* $u(x,y) : \Omega \to \mathbb{R}$, *tal que:*

$$-\Delta u(x,y) = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f(x,y) \quad \text{em} \quad \Omega$$

suplementado por condições de contorno do tipo Dirichlet

$$u(x,y) = \overline{u}(x,y) \quad \text{sobre} \quad \partial\Omega_D$$

ou Neumann

$$\nabla u(x,y) \cdot \mathbf{n} = g(x,y) \quad \text{sobre} \quad \partial\Omega_N$$

onde $\mathbf{n}$ é o vetor unitário normal com orientação exterior a $\partial\Omega_N$.

Discretização

Seja $u(x, y)$ e o domínio $\Omega = [a_x, b_x] \times [a_y, b_y]$ onde $x \in [a_x, b_x]$ e $y \in [a_y, b_y]$. Neste domínio definimos discretizações uniformes nas direções x e y, como segue

- $x_i = x_0 + i\Delta x, \quad i = 0, 1, ..., I$, com $\Delta x = \dfrac{b_x - a_x}{I - 1}$
- $y_j = y_0 + j\Delta y, \quad j = 0, 1, ..., J$, com $\Delta y = \dfrac{b_y - a_y}{J - 1}$

Neste contexto, definimos a seguinte discretização por diferença central para as derivadas de segunda ordem nas direções x e y:

$$
\begin{aligned}
\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \approx & \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2} \\
& + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\Delta y^2}
\end{aligned}
$$

Discretização

Escolhendo $\Delta x = \Delta y = h$, temos que

$$\Delta u \approx \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2}$$

Logo podemos enunciar o seguinte problema discreto para difusão em 2 dimensões:
Encontrar $u_{i,j}$*, com* $i = 1, 2, ..., I-1$ *e* $j = 1, 2, ..., J-1$*, satisfazendo:*

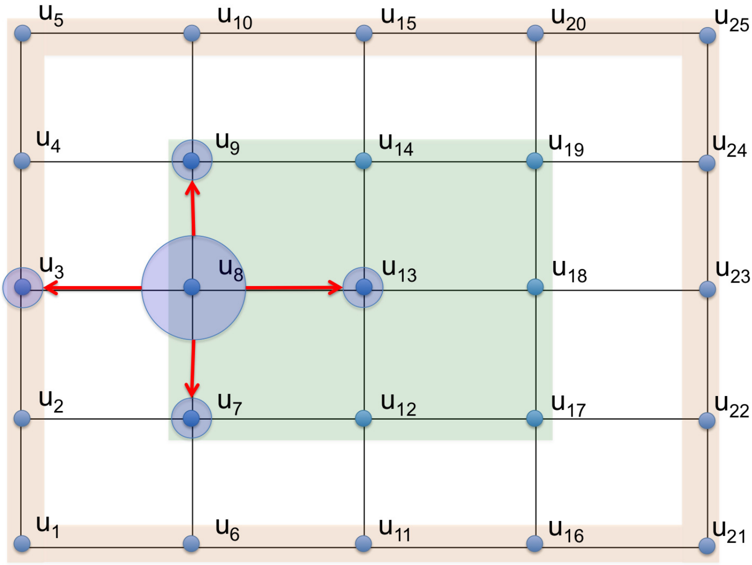
$$-\left(\frac{u_{i-1,j} + u_{i,j-1} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1}}{h^2}\right) = f(x_i, y_j)$$

com condição de contorno

$$u_{0,j} = u_{i,0} = u_{I,j} = u_{i,J} = \overline{u}$$

Metodologia de Resolução

Supondo 4×4 elementos, ou seja, $I = J = 5$, temos:



$\mathbf{i = 1}$ e $\mathbf{j = 1}$

$$-\left(\frac{u_{i-1,j} + u_{i,j-1} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1}}{h^2}\right) = f_{i,j}$$

logo

$$-\left(\frac{u_{0,1} + u_{1,0} - 4u_{1,1} + u_{2,1} + u_{1,2}}{h^2}\right) = f_{1,1}$$

ou ainda

$$-\left(\frac{-4u_{1,1} + u_{2,1} + u_{1,2}}{h^2}\right) = f(x_1, y_1) + \frac{u_{0,1} + u_{1,0}}{h^2}$$

$$-\frac{1}{h^2}\begin{bmatrix} \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{bmatrix} = \begin{bmatrix} f_{1,1} \\ f_{2,1} \\ f_{3,1} \\ f_{1,2} \\ f_{2,2} \\ f_{2,3} \\ f_{1,3} \\ f_{2,3} \\ f_{3,3} \end{bmatrix} + \frac{1}{h^2}\begin{bmatrix} u_{0,1} + u_{1,0} \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix}$$

$\mathbf{i = 2}$ e $\mathbf{j = 1}$

$$-\left(\frac{u_{i-1,j} + u_{i,j-1} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1}}{h^2}\right) = f_{i,j}$$

logo

$$-\left(\frac{u_{1,1} + u_{2,0} - 4u_{2,1} + u_{3,1} + u_{2,2}}{h^2}\right) = f_{2,1}$$

ou ainda

$$-\left(\frac{u_{1,1} - 4u_{2,1} + u_{3,1} + u_{2,2}}{h^2}\right) = f(x_2, y_1) + \frac{u_{2,0}}{h^2}$$

$$-\frac{1}{h^2}\begin{bmatrix} \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{bmatrix} = \begin{bmatrix} f_{1,1} \\ f_{2,1} \\ f_{3,1} \\ f_{1,2} \\ f_{2,2} \\ f_{2,3} \\ f_{1,3} \\ f_{2,3} \\ f_{3,3} \end{bmatrix} + \frac{1}{h^2}\begin{bmatrix} u_{0,1} + u_{1,0} \\ u_{2,0} \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix}$$

$\mathbf{i = 3}$ e $\mathbf{j = 1}$

$$-\left(\frac{u_{i-1,j} + u_{i,j-1} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1}}{h^2}\right) = f_{i,j}$$

logo

$$-\left(\frac{u_{2,1} + u_{3,0} - 4u_{3,1} + u_{4,1} + u_{3,2}}{h^2}\right) = f_{3,1}$$

ou ainda

$$-\left(\frac{u_{2,1} - 4u_{3,1} + u_{3,2}}{h^2}\right) = f(x_3, y_1) + \frac{u_{2,0} + u_{4,1}}{h^2}$$

$$-\frac{1}{h^2}\begin{bmatrix} \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{-4} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{bmatrix} = \begin{bmatrix} f_{1,1} \\ f_{2,1} \\ f_{3,1} \\ f_{1,2} \\ f_{2,2} \\ f_{2,3} \\ f_{1,3} \\ f_{2,3} \\ f_{3,3} \end{bmatrix} + \frac{1}{h^2}\begin{bmatrix} u_{0,1} + u_{1,0} \\ u_{2,0} \\ u_{2,0} + u_{4,1} \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix}$$

$\mathbf{i = 1}$ e $\mathbf{j = 2}$

$$-\left(\frac{u_{i-1,j} + u_{i,j-1} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1}}{h^2}\right) = f_{i,j}$$

logo

$$-\left(\frac{u_{0,2} + u_{1,1} - 4u_{1,2} + u_{2,2} + u_{1,3}}{h^2}\right) = f_{1,2}$$

ou ainda

$$-\left(\frac{u_{1,1} - 4u_{1,2} + u_{2,2} + u_{1,3}}{h^2}\right) = f(x_1, y_2) + \frac{u_{0,2}}{h^2}$$

$$-\frac{1}{h^2}\begin{bmatrix} \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{-4} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{bmatrix} = \begin{bmatrix} f_{1,1} \\ f_{2,1} \\ f_{3,1} \\ f_{1,2} \\ f_{2,2} \\ f_{2,3} \\ f_{1,3} \\ f_{2,3} \\ f_{3,3} \end{bmatrix} + \frac{1}{h^2}\begin{bmatrix} u_{0,1} + u_{1,0} \\ u_{2,0} \\ u_{2,0} + u_{4,1} \\ u_{0,2} \\ \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix}$$

$\mathbf{i = 2}$ e $\mathbf{j = 2}$

$$-\left(\frac{u_{i-1,j} + u_{i,j-1} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1}}{h^2}\right) = f_{i,j}$$

logo

$$-\left(\frac{u_{1,2} + u_{2,1} - 4u_{2,2} + u_{3,2} + u_{2,3}}{h^2}\right) = f_{2,2}$$

$$-\frac{1}{h^2}\begin{bmatrix} \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{-4} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{bmatrix} = \begin{bmatrix} f_{1,1} \\ f_{2,1} \\ f_{3,1} \\ f_{1,2} \\ f_{2,2} \\ f_{2,3} \\ f_{1,3} \\ f_{2,3} \\ f_{3,3} \end{bmatrix} + \frac{1}{h^2}\begin{bmatrix} u_{0,1} + u_{1,0} \\ u_{2,0} \\ u_{2,0} + u_{4,1} \\ u_{0,2} \\ 0 \\ \times \\ \times \\ \times \\ \times \end{bmatrix}$$

$\mathbf{i = 3}$ e $\mathbf{j = 2}$

$$-\left(\frac{u_{i-1,j} + u_{i,j-1} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1}}{h^2}\right) = f_{i,j}$$

logo

$$-\left(\frac{u_{2,2} + u_{3,1} - 4u_{3,2} + u_{4,2} + u_{3,3}}{h^2}\right) = f_{3,2}$$

ou ainda

$$-\left(\frac{u_{2,2} + u_{3,1} - 4u_{3,2} + u_{3,3}}{h^2}\right) = f_{3,2} + \frac{u_{4,2}}{h^2}$$

$$-\frac{1}{h^2}\begin{bmatrix} \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{-4} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{-4} & 0 & 0 & \mathbf{1} \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{bmatrix} = \begin{bmatrix} f_{1,1} \\ f_{2,1} \\ f_{3,1} \\ f_{1,2} \\ f_{2,2} \\ f_{2,3} \\ f_{1,3} \\ f_{2,3} \\ f_{3,3} \end{bmatrix} + \frac{1}{h^2}\begin{bmatrix} u_{0,1} + u_{1,0} \\ u_{2,0} \\ u_{2,0} + u_{4,1} \\ u_{0,2} \\ 0 \\ u_{4,2} \\ \times \\ \times \\ \times \end{bmatrix}$$

$\mathbf{i = 1}$ e $\mathbf{j = 3}$

$$-\left(\frac{u_{i-1,j} + u_{i,j-1} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1}}{h^2}\right) = f_{i,j}$$

logo

$$-\left(\frac{u_{0,3} + u_{1,2} - 4u_{1,3} + u_{2,3} + u_{1,4}}{h^2}\right) = f_{1,3}$$

ou ainda

$$-\left(\frac{u_{1,2} - 4u_{1,3} + u_{2,3}}{h^2}\right) = f_{1,3} + \frac{u_{0,3} + u_{1,4}}{h^2}$$

$$-\frac{1}{h^2}\begin{bmatrix} \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{-4} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{-4} & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{-4} & \mathbf{1} & 0 \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{bmatrix} = \begin{bmatrix} f_{1,1} \\ f_{2,1} \\ f_{3,1} \\ f_{1,2} \\ f_{2,2} \\ f_{2,3} \\ f_{1,3} \\ f_{2,3} \\ f_{3,3} \end{bmatrix} + \frac{1}{h^2}\begin{bmatrix} u_{0,1} + u_{1,0} \\ u_{2,0} \\ u_{2,0} + u_{4,1} \\ u_{0,2} \\ 0 \\ u_{4,2} \\ u_{0,3} + u_{1,4} \\ \times \\ \times \end{bmatrix}$$

$\mathbf{i = 2}$ e $\mathbf{j = 3}$

$$-\left(\frac{u_{i-1,j} + u_{i,j-1} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1}}{h^2}\right) = f_{i,j}$$

logo

$$-\left(\frac{u_{1,3} + u_{2,2} - 4u_{2,3} + u_{3,3} + u_{2,4}}{h^2}\right) = f_{2,3}$$

ou ainda

$$-\left(\frac{u_{1,3} + u_{2,2} - 4u_{2,3} + u_{3,3}}{h^2}\right) = f_{2,3} + \frac{u_{2,4}}{h^2}$$

$$-\frac{1}{h^2}\begin{bmatrix} \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{-4} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{-4} & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{-4} & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{-4} & \mathbf{1} \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{bmatrix} = \begin{bmatrix} f_{1,1} \\ f_{2,1} \\ f_{3,1} \\ f_{1,2} \\ f_{2,2} \\ f_{2,3} \\ f_{1,3} \\ f_{2,3} \\ f_{3,3} \end{bmatrix} + \frac{1}{h^2}\begin{bmatrix} u_{0,1} + u_{1,0} \\ u_{2,0} \\ u_{2,0} + u_{4,1} \\ u_{0,2} \\ 0 \\ u_{4,2} \\ u_{0,3} + u_{1,4} \\ u_{2,4} \\ \times \end{bmatrix}$$

$\mathbf{i = 3}$ e $\mathbf{j = 3}$

$$-\left(\frac{u_{i-1,j} + u_{i,j-1} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1}}{h^2}\right) = f_{i,j}$$

logo

$$-\left(\frac{u_{2,3} + u_{3,2} - 4u_{3,3} + u_{4,3} + u_{3,4}}{h^2}\right) = f_{3,3}$$

ou ainda

$$-\left(\frac{u_{2,3} + u_{3,2} - 4u_{3,3}}{h^2}\right) = f_{3,3} + \frac{u_{4,3} + u_{3,4}}{h^2}$$

$$-\frac{1}{h^2}\begin{bmatrix} \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{-4} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{-4} & \mathbf{1} & 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{-4} & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{-4} & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{-4} & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{-4} \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{bmatrix} = \begin{bmatrix} f_{1,1} \\ f_{2,1} \\ f_{3,1} \\ f_{1,2} \\ f_{2,2} \\ f_{2,3} \\ f_{1,3} \\ f_{2,3} \\ f_{3,3} \end{bmatrix} + \frac{1}{h^2}\begin{bmatrix} u_{0,1} + u_{1,0} \\ u_{2,0} \\ u_{2,0} + u_{4,1} \\ u_{0,2} \\ 0 \\ u_{4,2} \\ u_{0,3} + u_{1,4} \\ u_{2,4} \\ u_{4,3} + u_{3,4} \end{bmatrix}$$

Caso Geral

Dado um problema com $K \times K$ elementos, temos que a matriz gerada é da ordem de $(K-1)^2$, ou seja, é uma matriz quadrada $(K-1)^2 \times (K-1)^2$, que pode ser generalizada como segue

$$\frac{1}{h^2}\begin{bmatrix} T & I & & & & \\ I & T & I & & & \\ & I & T & I & & \\ & & \ddots & \ddots & \ddots & \\ & & & I & T & I \\ & & & & I & T \end{bmatrix}, \qquad T = \begin{bmatrix} -4 & 1 & & & & \\ 1 & -4 & 1 & & & \\ & 1 & -4 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -4 & 1 \\ & & & & 1 & -4 \end{bmatrix}$$

onde I é a matriz identidade.

Para resolver pode-se empregar a decomposição LU que necessita de aproximadamente $\dfrac{2}{3}k^3$ operações, onde k é a dimensão da matriz.

Exemplo: Dada uma malha de 64×64 elementos, a matriz gerada é da ordem de $k = (K-1)^2 = (64-1)^2 = 3969$. Assim, aplicando a decomposição LU, necessitamos de
$\dfrac{2}{3}k^3 = \dfrac{2}{3}3969^3 = 41.682.334.806$ operações.

Validação Numérica

Encontrar $u(x,y)$ satisfazendo a seguinte equação:

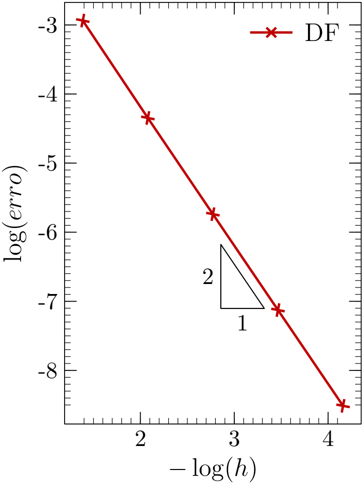
$$\begin{aligned} -\Delta u(x,y) &= 2\pi^2 \sin(\pi x)\sin(\pi y) && \text{em} \quad \Omega \\ u(x,y) &= \overline{u}, && \text{sobre} \quad \partial\Omega \end{aligned}$$

onde $\overline{u}$ é definido pela solução exata

$$u(x,y) = \sin(\pi x)\sin(\pi y).$$

Estudo de Convergência

Para este estudo foram adotadas malhas de
$4 \times 4, 8 \times 8, 16 \times 16, 32 \times 32, 64 \times 64$ elementos.



Problema Modelo - Difusão-Conveção

Seja o domínio espacial $\Omega \subset \mathbb{R}^2$ *com contorno* $\partial\Omega$*, dado* $\varepsilon > 0$ *e as funções* $f(x, y) : \Omega \to \mathbb{R}$ *e* $\overline{u}(x, y) : \partial\Omega \to \mathbb{R}$ *e o vetor* $\mathbf{b} = (b_1, b_2)$*, encontrar* $u(x, y) : \Omega \to \mathbb{R}$*, tal que:*

$$-\varepsilon \Delta u(x, y) + \mathbf{b} \cdot \nabla u(x, y) = f(x, y) \quad \text{em} \quad \Omega$$

ou ainda

$$-\varepsilon \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + b_1 \frac{\partial u}{\partial x} + b_2 \frac{\partial u}{\partial y} = f(x, y)$$

suplementado por condições de contorno do tipo Dirichlet

$$u(x, y) = \overline{u}(x, y) \quad \text{sobre} \quad \partial\Omega$$

Discretização Upwind

Adotando discretização central para a difusão

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \approx \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\Delta y^2}$$

e

$$\mathbf{b} \cdot \nabla u = b_1 \frac{\partial u}{\partial x} + b_2 \frac{\partial u}{\partial y} \approx \begin{cases} b_1 \dfrac{u_{i,j} - u_{i-1,j}}{\Delta x} + b_2 \dfrac{u_{i,j} - u_{i,j-1}}{\Delta y}, & b_1, b_2 > 0 \\ b_1 \dfrac{u_{i+1,j} - u_{i,j}}{\Delta x} + b_2 \dfrac{u_{i,j+1} - u_{i,j}}{\Delta y}, & b_1, b_2 < 0 \end{cases}$$

para o termo convectivo.

Discretização Upwind

Adotando $\Delta x = \Delta y = h$ e supondo $b_1, b_2 > 0$, temos:

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2}$$
$$+ b_1 \frac{u_{i,j} - u_{i-1,j}}{h} + b_2 \frac{u_{i,j} - u_{i,j-1}}{h} = f(x_i, y_j)$$

Logo uma discretização para o problema modelo é dada por: Encontrar $u_{i,j}$, com $i = 1, 2, ..., I-1$ e $j = 1, 2, ..., J-1$, satisfazendo:

$$-\varepsilon \left(\frac{u_{i-1,j} + u_{i,j-1} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1}}{h^2} \right)$$
$$+ b_1 \frac{u_{i,j} - u_{i-1,j}}{h} + b_2 \frac{u_{i,j} - u_{i,j-1}}{h} = f(x_i, y_j)$$

Agrupando os termos, podemos reescrever como:

$$-\frac{1}{h^2}\Big[(\varepsilon + b_1 h)u_{i-1,j} + (\varepsilon + b_2 h)u_{i,j-1}$$
$$- (4\varepsilon + b_1 h + b_2 h)u_{i,j} + \varepsilon u_{i+1,j} + \varepsilon u_{i,j+1}\Big] = f(x_i, y_j)$$

Discretização Esquema Central

Adotando discretização central para a difusão

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \approx \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\Delta y^2}$$

e central para a convecção

$$\mathbf{b} \cdot \nabla u = b_1 \frac{\partial u}{\partial x} + b_2 \frac{\partial u}{\partial y} \approx b_1 \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + b_2 \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}$$

geramos a seguinte aproximação de segunda ordem:
Encontrar $u_{i,j}$, com $i = 1, 2, ..., I-1$ e $j = 1, 2, ..., J-1$, satisfazendo:

$$-\varepsilon\left(\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\Delta y^2}\right) + b_1 \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + b_2 \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y} = f(x_i, y_j)$$

Discretização Esquema Central

Adotando $\Delta x = \Delta y = h$, obtemos:

$$-\varepsilon\left(\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2}\right)$$
$$+b_1\frac{u_{i+1,j} - u_{i-1,j}}{2h} + b_2\frac{u_{i,j+1} - u_{i,j-1}}{2h} = f(x_i, y_j)$$

Logo, temos que

$$-\varepsilon\left(\frac{u_{i-1,j} + u_{i,j-1} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1}}{h^2}\right)$$
$$+ b_1\frac{u_{i+1,j} - u_{i-1,j}}{2h} + b_2\frac{u_{i,j+1} - u_{i,j-1}}{2h} = f(x_i, y_j)$$

Agrupando os termos, podemos reescrever como:

$$-\frac{1}{h^2}\Big[(\varepsilon + b_1 h/2)u_{i-1,j} + (\varepsilon + b_2 h/2)u_{i,j-1}$$
$$- 4\varepsilon u_{i,j} + (\varepsilon - b_1 h/2)u_{i+1,j} + (\varepsilon - b_2 h/2)u_{i,j+1}\Big] = f(x_i, y_j)$$

Resultados Numéricos

Encontrar $u(x,y)$ satisfazendo a seguinte equação:

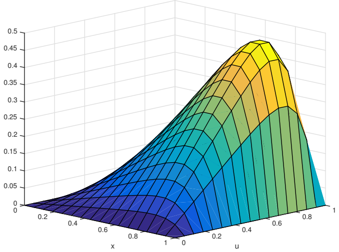
$$\begin{aligned} -\varepsilon \Delta u(x,y) + \mathbf{b} \cdot \nabla u(x,y) &= f(x,y) \quad \text{em} \quad \Omega \\ u(x,y) &= \overline{u} \qquad \text{sobre} \quad \partial\Omega \end{aligned}$$

onde $\overline{u}(x,y)$ e $f(x,y)$ são derivados da seguinte solução exata

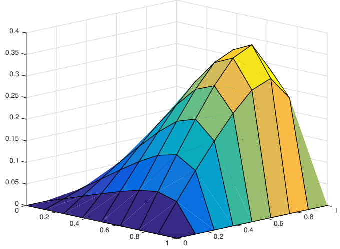
$$u(x,y) = xy\frac{(1-e^{(x-1)b_1})(1-e^{(y-1)b_2})}{(1-e^{-b_1})(1-e^{-b_2})}$$

onde $\mathbf{b} = (b_1, b_2)$, $\varepsilon = 1$ e o domínio $\Omega = [0,1] \times [0,1]$.

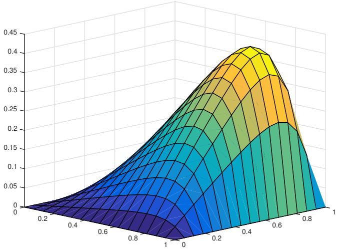
Upwind – $\mathbf{b} = (10, 10)$



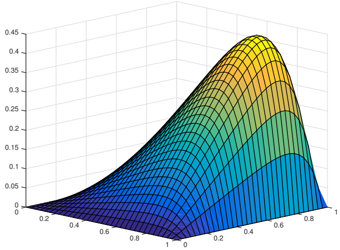
Exata



8×8

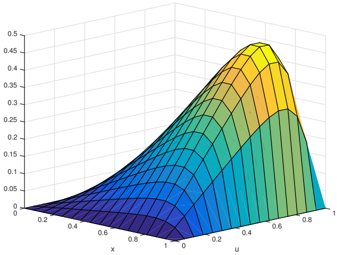


16×16

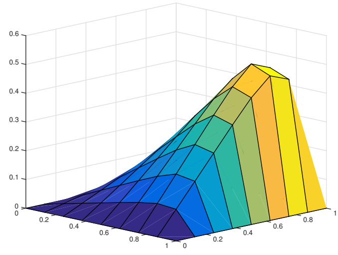


32×32

Esquema Central – $\mathbf{b} = (10, 10)$



Exata



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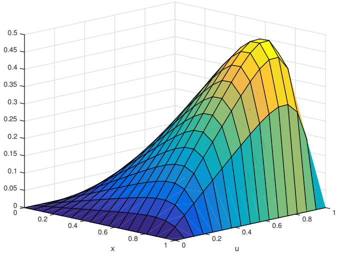
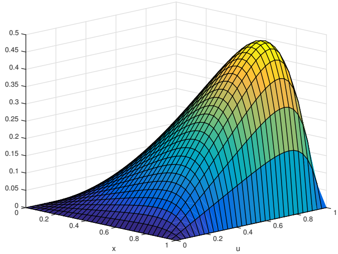
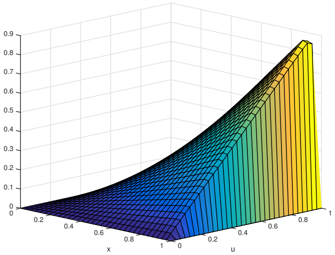


16×16

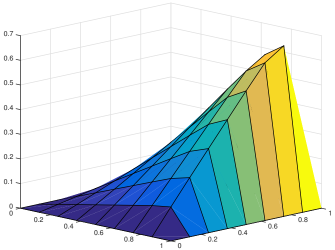


32×32

Upwind – $\mathbf{b} = (100, 100)$



Exata



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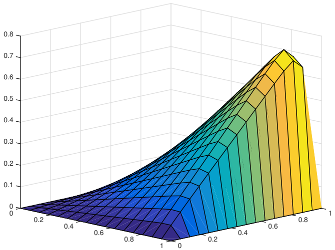

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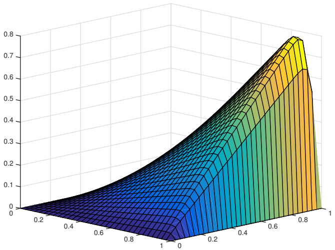
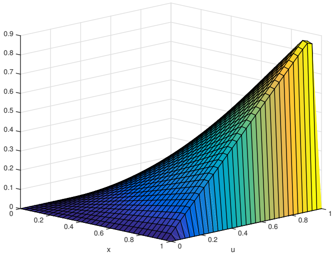


32×32

Esquema Central – $\mathbf{b} = (100, 100)$



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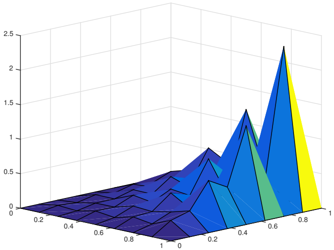

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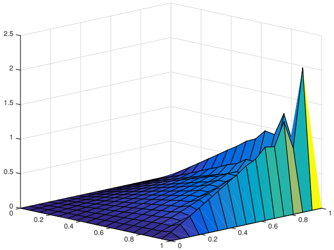
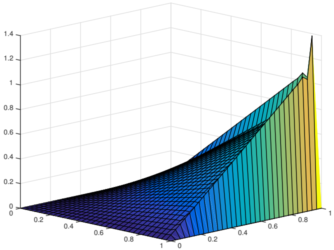
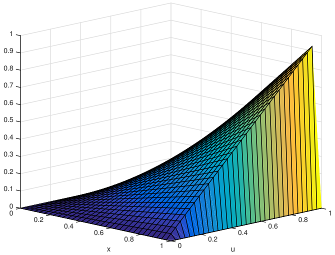


16×16

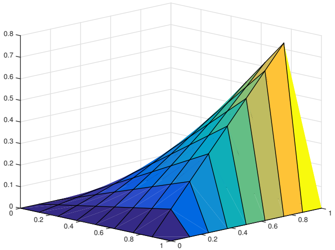


32×32

Upwind – $\mathbf{b} = (10^6, 10^6)$



Exata



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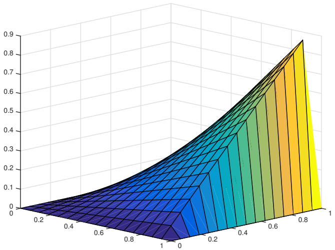
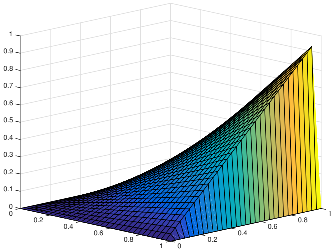


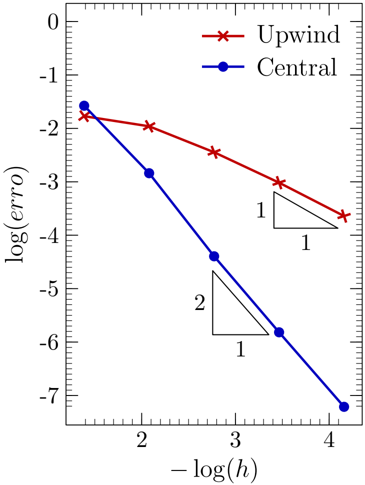
16×16



32×32

Estudo de Convergência

Para este estudo foi adotado $4 \times 4, 8 \times 8, 16 \times 16, 32 \times 32, 64 \times 64$ elementos e $\mathbf{b} = (10, 10)$.



Problema Modelo - Difusão Transiente

Seja o domínio espacial $\Omega \subset \mathbb{R}^d$ *com contorno* $\partial\Omega$ *e o domínio temporal* $\Theta = [0, T]$, *dada as funções* $f(\mathbf{x}, t) : \Omega \times \Theta \rightarrow \mathbb{R}$, $\overline{u}(\mathbf{x}, t) : \partial\Omega \times \Theta \rightarrow \mathbb{R}$, $\varphi(\mathbf{x}) : \Omega \rightarrow \mathbb{R}$ *e* $\varepsilon > 0$, *encontrar* $u(\mathbf{x}, t) : \Omega \times \Theta \rightarrow \mathbb{R}$, *tal que:*

$$\frac{\partial u}{\partial t} - \varepsilon \Delta u = f(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times \Theta$$

suplementado por condições de Dirichlet

$$u(\mathbf{x}, t) = \overline{u}(\mathbf{x}, t),$$

e a condição inicial

$$u(\mathbf{x}, 0) = \varphi(\mathbf{x}).$$

onde $\mathbf{x} = (x, y)$

Discretização

Definindo $\Delta t = T/(N-1)$ e $t_n = n\Delta t$, com $n = 0, 1, 2, ..., N$, onde N é um inteiro positivo. Para o termo da derivada segunda no espaço adotamos um esquema de diferença central, com $\Delta x = \Delta y = h$,

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} - \varepsilon \left(\frac{u_{i-1,j}^{n+1} + u_{i,j-1}^{n+1} - 4u_{i,j}^{n+1} + u_{i+1,j}^{n+1} + u_{i,j+1}^{n+1}}{h^2} \right) = f(x_i, y_j, t_n + \Delta t),$$

ordem de convergência $\mathcal{O}(h^2, \Delta t)$

Comentários

- Matriz gerada seguindo os mesmos passos apresentados anteriormente incluindo o termo temporal no passo de tempo $n+1$.
- O termo fonte inclui a solução no passo de tempo n.
- O custo computacional aplicando a estratégia de decomposição LU para resolver o sistema é da ordem de $\frac{2}{3}k^3n$, onde k é a dimensão da matriz e n o número de passos temporais

Exemplo: Já vimos que para uma malha de 64×64 elementos, aplicando a decomposição LU, necessitamos de $\frac{2}{3}k^3 = \frac{2}{3}3969^3 = 41.682.334.806$ operações. Se for necessário $n = 1000$ iterações no tempo, esse número de operações aumenta mil vezes.

Método ADI

Uma alternativa para reduzir o custo computacional é o método implícito de direções alternadas ADI[1] (*Alternating Direction Implicit*). Esta metodologia é feita em dois passos, como segue:

- **Primeiro Passo:** (Implícito em x)

$$\frac{\hat{u}_{i,j}^{n+1/2} - u_{i,j}^{n}}{\Delta t/2} = \varepsilon \left[\frac{\hat{u}_{i+1,j}^{n+1/2} - 2\hat{u}_{i,j}^{n+1/2} + \hat{u}_{i-1,j}^{n+1/2}}{\Delta x^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{\Delta y^2} \right]$$

- **Segundo Passo:** (Implícito em y)

$$\frac{u_{i,j}^{n+1} - \hat{u}_{i,j}^{n+1/2}}{\Delta t/2} = \varepsilon \left[\frac{\hat{u}_{i+1,j}^{n+1/2} - 2\hat{u}_{i,j}^{n+1/2} + \hat{u}_{i-1,j}^{n+1/2}}{\Delta x^2} + \frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{\Delta y^2} \right]$$

- Método incondicionalmente estável por von Neumann
- Ordem de convergência $\mathcal{O}(h^2, \Delta t^2)$
- Solução de matrizes tridiagonais em cada passo.

[1] D. W. Peaceman and H. H. Rachford, Jr., The Numerical Solution of Parabolic and Elliptic Differential Equations. J Soc. Indust. Appl. Math., 3(1), 28–41. https://doi.org/10.1137/0103003

Método ADI

Escolhendo $\Delta x = \Delta y = h$, podemos reescrever a discretização da seguinte forma:

$$(1+\sigma)\hat{u}_{i,j}^{n+1/2} - \frac{\sigma}{2}\left(\hat{u}_{i+1,j}^{n+1/2} + \hat{u}_{i-1,j}^{n+1/2}\right) = (1-\sigma)u_{i,j}^{n} + \frac{\sigma}{2}\left(u_{i,j+1}^{n} + u_{i,j-1}^{n}\right)$$

$$(1+\sigma)u_{i,j}^{n+1} - \frac{\sigma}{2}\left(u_{i,j+1}^{n+1} + u_{i,j-1}^{n+1}\right) = (1-\sigma)\hat{u}_{i,j}^{n+1/2} + \frac{\sigma}{2}\left(\hat{u}_{i+1,j}^{n+1/2} + \hat{u}_{i-1,j}^{n+1/2}\right)$$

- Primeiro passo: resolve a difusão na direção x com a condição inicial. Neste passo, a difusão em x é incógnita e a difusão em y conhecida.
- Segundo passo: resolve a difusão na direção y, que agora é incógnita, utilizando a solução do primeiro passo para difusão em x, que entra na segunda expressão de forma conhecida.

Método ADI – First step

Supondo uma malha de 4×4 elementos, primeiramente resolvemos a difusão em x implícita e em y explícita para cada plano x. Assim:

- Para $n = 0$, $i = 1$ e $j = 1$

$$(1+\sigma)\hat{u}_{1,1}^{1/2} - \frac{\sigma}{2}\left(\hat{u}_{2,1}^{1/2} + \hat{u}_{0,1}^{1/2}\right) = (1-\sigma)u_{1,1}^{0} + \frac{\sigma}{2}\left(u_{1,2}^{0} + u_{1,0}^{0}\right)$$

$$\begin{bmatrix} (1+\sigma) & -\sigma/2 & 0 \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix} \begin{bmatrix} \hat{u}_{1,1}^{1/2} \\ \hat{u}_{2,1}^{1/2} \\ \hat{u}_{3,1}^{1/2} \end{bmatrix} = \begin{bmatrix} (1-\sigma)u_{1,1}^{0} + \frac{\sigma}{2}(u_{1,2}^{0} + u_{1,0}^{0} + \hat{u}_{0,1}^{1/2}) \\ \times \\ \times \end{bmatrix}$$

Método ADI – First step

- Para $n = 0$, $i = 2$ e $j = 1$

$$(1+\sigma)\hat{u}_{2,1}^{1/2} - \frac{\sigma}{2}\left(\hat{u}_{3,1}^{1/2} + \hat{u}_{1,1}^{1/2}\right) = (1-\sigma)u_{2,1}^{0} + \frac{\sigma}{2}\left(u_{2,2}^{0} + u_{2,0}^{0}\right)$$

$$\begin{bmatrix} (1+\sigma) & -\sigma/2 & 0 \\ -\sigma/2 & (1+\sigma) & -\sigma/2 \\ \times & \times & \times \end{bmatrix} \begin{bmatrix} \hat{u}_{1,1}^{1/2} \\ \hat{u}_{2,1}^{1/2} \\ \hat{u}_{3,1}^{1/2} \end{bmatrix} = \begin{bmatrix} (1-\sigma)u_{1,1}^{0} + \frac{\sigma}{2}(u_{1,2}^{0} + u_{1,0}^{0} + \hat{u}_{0,1}^{1/2}) \\ (1-\sigma)u_{2,1}^{0} + \frac{\sigma}{2}(u_{2,2}^{0} + u_{2,0}^{0}) \\ \times \end{bmatrix}$$

Método ADI – First step

- Para $n = 0$, $i = 3$ e $j = 1$

$$(1+\sigma)\hat{u}_{3,1}^{1/2} - \frac{\sigma}{2}\left(\hat{u}_{4,1}^{1/2} + \hat{u}_{2,1}^{1/2}\right) = (1-\sigma)u_{3,1}^{0} + \frac{\sigma}{2}\left(u_{3,2}^{0} + u_{3,0}^{0}\right)$$

$$\begin{bmatrix} (1+\sigma) & -\sigma/2 & 0 \\ -\sigma/2 & (1+\sigma) & -\sigma/2 \\ 0 & -\sigma/2 & (1+\sigma) \end{bmatrix} \begin{bmatrix} \hat{u}_{1,1}^{1/2} \\ \hat{u}_{2,1}^{1/2} \\ \hat{u}_{3,1}^{1/2} \end{bmatrix} = \begin{bmatrix} (1-\sigma)u_{1,1}^{0} + \frac{\sigma}{2}(u_{1,2}^{0} + u_{1,0}^{0} + \hat{u}_{0,1}^{1/2}) \\ (1-\sigma)u_{2,1}^{0} + \frac{\sigma}{2}(u_{2,2}^{0} + u_{2,0}^{0}) \\ (1-\sigma)u_{3,1}^{0} + \frac{\sigma}{2}(u_{3,2}^{0} + u_{3,0}^{0} + \hat{u}_{4,1}^{1/2}) \end{bmatrix}$$

Método ADI – First step

- Para $n=0$, $i=1$ e $j=2$

$$(1+\sigma)\hat{u}_{1,2}^{1/2}-\frac{\sigma}{2}\left(\hat{u}_{2,2}^{1/2}+\hat{u}_{0,2}^{1/2}\right)=(1-\sigma)u_{1,2}^{0}+\frac{\sigma}{2}\left(u_{1,3}^{0}+u_{1,1}^{0}\right)$$

- Para $n=0$, $i=2$ e $j=2$

$$(1+\sigma)\hat{u}_{2,2}^{1/2}-\frac{\sigma}{2}\left(\hat{u}_{3,2}^{1/2}+\hat{u}_{1,2}^{1/2}\right)=(1-\sigma)u_{2,2}^{0}+\frac{\sigma}{2}\left(u_{2,3}^{0}+u_{2,1}^{0}\right)$$

- Para $n=0$, $i=3$ e $j=2$

$$(1+\sigma)\hat{u}_{3,2}^{1/2}-\frac{\sigma}{2}\left(\hat{u}_{4,2}^{1/2}+\hat{u}_{2,2}^{1/2}\right)=(1-\sigma)u_{3,2}^{0}+\frac{\sigma}{2}\left(u_{3,3}^{0}+u_{3,1}^{0}\right)$$

$$\begin{bmatrix}(1+\sigma) & -\sigma/2 & 0\\ -\sigma/2 & (1+\sigma) & -\sigma/2\\ 0 & -\sigma/2 & (1+\sigma)\end{bmatrix}\begin{bmatrix}\hat{u}_{1,2}^{1/2}\\ \hat{u}_{2,2}^{1/2}\\ \hat{u}_{3,2}^{1/2}\end{bmatrix}=\begin{bmatrix}(1-\sigma)u_{1,2}^{0}+\frac{\sigma}{2}(u_{1,3}^{0}+u_{1,1}^{0}+\hat{u}_{0,2}^{1/2})\\ (1-\sigma)u_{2,2}^{0}+\frac{\sigma}{2}(u_{2,3}^{0}+u_{2,1}^{0})\\ (1-\sigma)u_{3,2}^{0}+\frac{\sigma}{2}(u_{3,3}^{0}+u_{3,1}^{0}+\hat{u}_{4,2}^{1/2})\end{bmatrix}$$

Método ADI – First step

- Para $n = 0$, $i = 1$ e $j = 3$

$$(1+\sigma)\hat{u}_{1,3}^{1/2} - \frac{\sigma}{2}\left(\hat{u}_{2,3}^{1/2} + \hat{u}_{0,3}^{1/2}\right) = (1-\sigma)u_{1,3}^{0} + \frac{\sigma}{2}\left(u_{1,4}^{0} + u_{1,2}^{0}\right)$$

- Para $n = 0$, $i = 2$ e $j = 3$

$$(1+\sigma)\hat{u}_{2,3}^{1/2} - \frac{\sigma}{2}\left(\hat{u}_{3,3}^{1/2} + \hat{u}_{1,3}^{1/2}\right) = (1-\sigma)u_{2,3}^{0} + \frac{\sigma}{2}\left(u_{2,4}^{0} + u_{2,2}^{0}\right)$$

- Para $n = 0$, $i = 3$ e $j = 3$

$$(1+\sigma)\hat{u}_{3,3}^{1/2} - \frac{\sigma}{2}\left(\hat{u}_{4,3}^{1/2} + \hat{u}_{2,3}^{1/2}\right) = (1-\sigma)u_{3,3}^{0} + \frac{\sigma}{2}\left(u_{3,4}^{0} + u_{3,2}^{0}\right)$$

$$\begin{bmatrix} (1+\sigma) & -\sigma/2 & 0 \\ -\sigma/2 & (1+\sigma) & -\sigma/2 \\ 0 & -\sigma/2 & (1+\sigma) \end{bmatrix} \begin{bmatrix} \hat{u}_{1,3}^{1/2} \\ \hat{u}_{2,3}^{1/2} \\ \hat{u}_{3,3}^{1/2} \end{bmatrix} = \begin{bmatrix} (1-\sigma)u_{1,3}^{0} + \frac{\sigma}{2}(u_{1,4}^{0} + u_{1,2}^{0} + \hat{u}_{0,3}^{1/2}) \\ (1-\sigma)u_{2,3}^{0} + \frac{\sigma}{2}(u_{2,4}^{0} + u_{2,2}^{0}) \\ (1-\sigma)u_{3,3}^{0} + \frac{\sigma}{2}(u_{3,4}^{0} + u_{3,2}^{0} + \hat{u}_{4,3}^{1/2}) \end{bmatrix}$$

Método ADI – Second step

Na segunda etapa, a solução $\hat{u}_{i,j}^{n+1/2}$ do termo difusivo em x é usada para obter a solução da variável de interesse $u_{i,j}^{n+1}$ referente a componente difusiva y que é calculada implicitamente.

- Para $n = 0$, $i = 1$ e $j = 1$

$$(1+\sigma)u_{1,1}^{1} - \frac{\sigma}{2}\left(u_{1,2}^{1} + u_{1,0}^{1}\right) = (1-\sigma)\hat{u}_{1,1}^{1/2} + \frac{\sigma}{2}\left(\hat{u}_{2,1}^{1/2} + \hat{u}_{0,1}^{1/2}\right)$$

- Para $n = 0$, $i = 1$ e $j = 2$

$$(1+\sigma)u_{1,2}^{1} - \frac{\sigma}{2}\left(u_{1,3}^{1} + u_{1,1}^{1}\right) = (1-\sigma)\hat{u}_{1,2}^{1/2} + \frac{\sigma}{2}\left(\hat{u}_{2,2}^{1/2} + \hat{u}_{0,2}^{1/2}\right)$$

- Para $n = 0$, $i = 1$ e $j = 3$

$$(1+\sigma)u_{1,3}^{1} - \frac{\sigma}{2}\left(u_{1,4}^{1} + u_{1,2}^{1}\right) = (1-\sigma)\hat{u}_{1,3}^{1/2} + \frac{\sigma}{2}\left(\hat{u}_{2,3}^{1/2} + \hat{u}_{0,3}^{1/2}\right)$$

$$\begin{bmatrix} (1+\sigma) & -\sigma/2 & 0 \\ -\sigma/2 & (1+\sigma) & -\sigma/2 \\ 0 & -\sigma/2 & (1+\sigma) \end{bmatrix} \begin{bmatrix} u_{1,1}^{1} \\ u_{1,2}^{1} \\ u_{1,3}^{1} \end{bmatrix} = \begin{bmatrix} (1-\sigma)\hat{u}_{1,1}^{1/2} + \frac{\sigma}{2}(\hat{u}_{2,1}^{1/2} + \hat{u}_{0,1}^{1/2} + u_{1,0}^{1}) \\ (1-\sigma)\hat{u}_{1,2}^{1/2} + \frac{\sigma}{2}(\hat{u}_{2,2}^{1/2} + \hat{u}_{0,2}^{1/2}) \\ (1-\sigma)\hat{u}_{1,3}^{1/2} + \frac{\sigma}{2}(\hat{u}_{2,3}^{1/2} + \hat{u}_{0,3}^{1/2} + u_{1,4}^{1}) \end{bmatrix}$$

Método ADI – Second step

- Para $n = 0$, $i = 2$ e $j = 1$

$$(1+\sigma)u_{2,1}^{1} - \frac{\sigma}{2}\left(u_{2,2}^{1} + u_{2,0}^{1}\right) = (1-\sigma)\hat{u}_{2,1}^{1/2} + \frac{\sigma}{2}\left(\hat{u}_{3,1}^{1/2} + \hat{u}_{1,1}^{1/2}\right)$$

- Para $n = 0$, $i = 2$ e $j = 2$

$$(1+\sigma)u_{2,2}^{1} - \frac{\sigma}{2}\left(u_{2,3}^{1} + u_{2,1}^{1}\right) = (1-\sigma)\hat{u}_{2,2}^{1/2} + \frac{\sigma}{2}\left(\hat{u}_{3,2}^{1/2} + \hat{u}_{1,2}^{1/2}\right)$$

- Para $n = 0$, $i = 2$ e $j = 3$

$$(1+\sigma)u_{2,3}^{1} - \frac{\sigma}{2}\left(u_{2,4}^{1} + u_{2,2}^{1}\right) = (1-\sigma)\hat{u}_{2,3}^{1/2} + \frac{\sigma}{2}\left(\hat{u}_{3,3}^{1/2} + \hat{u}_{1,3}^{1/2}\right)$$

$$\begin{bmatrix} (1+\sigma) & -\sigma/2 & 0 \\ -\sigma/2 & (1+\sigma) & -\sigma/2 \\ 0 & -\sigma/2 & (1+\sigma) \end{bmatrix} \begin{bmatrix} u_{2,1}^{1} \\ u_{2,2}^{1} \\ u_{2,3}^{1} \end{bmatrix} = \begin{bmatrix} (1-\sigma)\hat{u}_{2,1}^{1/2} + \frac{\sigma}{2}(\hat{u}_{3,1}^{1/2} + \hat{u}_{1,1}^{1/2} + u_{2,0}^{1}) \\ (1-\sigma)\hat{u}_{2,2}^{1/2} + \frac{\sigma}{2}(\hat{u}_{3,2}^{1/2} + \hat{u}_{1,2}^{1/2}) \\ (1-\sigma)\hat{u}_{2,3}^{1/2} + \frac{\sigma}{2}(\hat{u}_{3,3}^{1/2} + \hat{u}_{1,3}^{1/2} + u_{2,4}^{1}) \end{bmatrix}$$

Método ADI – Second step

- Para $n = 0$, $i = 3$ e $j = 1$

$$(1+\sigma)u_{3,1}^{1} - \frac{\sigma}{2}\left(u_{3,2}^{1} + u_{3,0}^{1}\right) = (1-\sigma)\hat{u}_{3,1}^{1/2} + \frac{\sigma}{2}\left(\hat{u}_{4,1}^{1/2} + \hat{u}_{2,1}^{1/2}\right)$$

- Para $n = 0$, $i = 3$ e $j = 2$

$$(1+\sigma)u_{3,2}^{1} - \frac{\sigma}{2}\left(u_{3,3}^{1} + u_{3,1}^{1}\right) = (1-\sigma)\hat{u}_{3,2}^{1/2} + \frac{\sigma}{2}\left(\hat{u}_{4,2}^{1/2} + \hat{u}_{2,2}^{1/2}\right)$$

- Para $n = 0$, $i = 3$ e $j = 3$

$$(1+\sigma)u_{3,3}^{1} - \frac{\sigma}{2}\left(u_{3,4}^{1} + u_{3,2}^{1}\right) = (1-\sigma)\hat{u}_{3,3}^{1/2} + \frac{\sigma}{2}\left(\hat{u}_{4,3}^{1/2} + \hat{u}_{2,3}^{1/2}\right)$$

$$\begin{bmatrix} (1+\sigma) & -\sigma/2 & 0 \\ -\sigma/2 & (1+\sigma) & -\sigma/2 \\ 0 & -\sigma/2 & (1+\sigma) \end{bmatrix} \begin{bmatrix} u_{3,1}^{1} \\ u_{3,2}^{1} \\ u_{3,3}^{1} \end{bmatrix} = \begin{bmatrix} (1-\sigma)\hat{u}_{3,1}^{1/2} + \frac{\sigma}{2}(\hat{u}_{4,1}^{1/2} + \hat{u}_{2,1}^{1/2} + u_{3,0}^{1}) \\ (1-\sigma)\hat{u}_{3,2}^{1/2} + \frac{\sigma}{2}(\hat{u}_{4,2}^{1/2} + \hat{u}_{2,2}^{1/2}) \\ (1-\sigma)\hat{u}_{3,3}^{1/2} + \frac{\sigma}{2}(\hat{u}_{4,3}^{1/2} + \hat{u}_{2,3}^{1/2} + u_{3,4}^{1}) \end{bmatrix}$$

Testes Numéricos

$$\frac{\partial u}{\partial t} - \varepsilon \Delta u = 0 \quad \text{em} \quad \Omega \times \Theta$$

suplementado por condições de Dirichlet

$$u(x, y, t) = 0, \quad \text{sobre} \quad \partial\Omega \times \Theta$$

e a condição inicial

$$u(x, y, 0) = \varphi(x, y) = \sin(\pi x/2)\sin(\pi y/2).$$

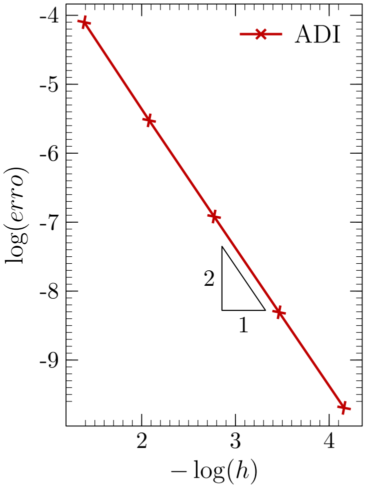
Uma solução exata para este problema é dada por:

$$u(x, y, t) = e^{-\pi^2 t/2}\sin(\pi x/2)\sin(\pi y/2)$$

no domínio espacial $\Omega = [0, 2] \times [0, 2]$ e temporal $\Theta = [0, 0.5]$ com $\varepsilon = 1.0$.

Estudo de Convergência

Para este estudo foi adotado $4 \times 4, 8 \times 8, 16 \times 16, 32 \times 32, 64 \times 64$ elementos e $\Delta t = h$.



Problema Modelo – Difusão-Convecção Transiente

Seja o domínio espacial $\Omega \subset \mathbb{R}^d$ *com contorno* $\partial\Omega$ *e o domínio temporal* $\Theta = [0, T]$, *dada as funções* $f(\mathbf{x}, t) : \Omega \times \Theta \to \mathbb{R}$, $\overline{u}(\mathbf{x}, t) : \partial\Omega \times \Theta \to \mathbb{R}$, $\varphi(\mathbf{x}) : \Omega \to \mathbb{R}$, $\varepsilon > 0$ *e* $\mathbf{b} = (b_1, b_2)$, *encontrar* $u(\mathbf{x}, t) : \Omega \times \Theta \to \mathbb{R}$, *tal que:*

$$\frac{\partial u}{\partial t} - \varepsilon \Delta u + \mathbf{b} \cdot \nabla u = f(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times \Theta$$

suplementado por condições de Dirichlet

$$u(\mathbf{x}, t) = \overline{u}(\mathbf{x}, t),$$

e a condição inicial

$$u(\mathbf{x}, 0) = \varphi(\mathbf{x}),$$

onde $\mathbf{x} = (x, y)$.

Método ADI – Esquema Upwind

Combinando a estratégia ADI com uma discretização upwind para o termo convectivo, obtemos:

$$\frac{\hat{u}_{i,j}^{n+1/2} - u_{i,j}^{n}}{\Delta t/2} = \varepsilon \left[\frac{\hat{u}_{i+1,j}^{n+1/2} - 2\hat{u}_{i,j}^{n+1/2} + \hat{u}_{i-1,j}^{n+1/2}}{\Delta x^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{\Delta y^2} \right]$$

$$- \begin{cases} b1 \dfrac{\hat{u}_{i,j}^{n+1/2} - \hat{u}_{i-1,j}^{n+1/2}}{\Delta x} + b_2 \dfrac{u_{i,j}^{n} - u_{i,j-1}^{n}}{\Delta y} & b_1, b_2 > 0 \\ b1 \dfrac{\hat{u}_{i+1,j}^{n+1/2} - \hat{u}_{i,j}^{n+1/2}}{\Delta x} + b_2 \dfrac{u_{i,j+1}^{n} - u_{i,j}^{n}}{\Delta y} & b_1, b_2 < 0 \end{cases}$$

$$\frac{u_{i,j}^{n+1} - \hat{u}_{i,j}^{n+1/2}}{\Delta t/2} = \varepsilon \left[\frac{\hat{u}_{i+1,j}^{n+1/2} - 2\hat{u}_{i,j}^{n+1/2} + \hat{u}_{i-1,j}^{n+1/2}}{\Delta x^2} + \frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{\Delta y^2} \right]$$

$$- \begin{cases} b1 \dfrac{\hat{u}_{i,j}^{n+1/2} - \hat{u}_{i-1,j}^{n+1/2}}{\Delta x} + b_2 \dfrac{u_{i,j}^{n+1} - u_{i,j-1}^{n+1}}{\Delta y} & b_1, b_2 > 0 \\ b1 \dfrac{\hat{u}_{i+1,j}^{n+1/2} - \hat{u}_{i,j}^{n+1/2}}{\Delta x} + b_2 \dfrac{u_{i,j+1}^{n+1} - u_{i,j}^{n+1}}{\Delta y} & b_1, b_2 < 0 \end{cases}$$

Método ADI – Esquema Upwind

Escolhendo $\Delta x = \Delta y = h$ e supondo $b_1, b_2 > 0$, podemos reescrever a discretização da seguinte forma:

$$\left(1 + \sigma + \frac{\rho_1}{2}\right) \hat{u}_{i,j}^{n+1/2} - \frac{\sigma}{2} \hat{u}_{i+1,j}^{n+1/2} - \left(\frac{\rho_1 + \sigma}{2}\right) \hat{u}_{i-1,j}^{n+1/2} = \left(1 - \sigma - \frac{\rho_2}{2}\right) u_{i,j}^{n} + \frac{\sigma}{2} u_{i,j+1}^{n} + \left(\frac{\sigma + \rho_2}{2}\right) u_{i,j-1}^{n}$$

$$\left(1 + \sigma + \frac{\rho_2}{2}\right) u_{i,j}^{n+1} - \frac{\sigma}{2} u_{i,j+1}^{n+1} - \left(\frac{\rho_2 + \sigma}{2}\right) u_{i,j-1}^{n+1} = \left(1 - \sigma - \frac{\rho_1}{2}\right) \hat{u}_{i,j}^{n+1/2} + \frac{\sigma}{2} \hat{u}_{i+1,j}^{n+1/2} + \left(\frac{\sigma + \rho_1}{2}\right) \hat{u}_{i-1,j}^{n+1/2}$$

onde

$$\sigma = \frac{\varepsilon \Delta t}{h^2}; \qquad \rho_1 = \frac{b_1 \Delta t}{h}; \qquad \rho_2 = \frac{b_2 \Delta t}{h}.$$

Método ADI – Esquema Central

Combinando a estratégia ADI com uma discretização upwind para o termo convectivo, obtemos:

$$\frac{\hat{u}_{i,j}^{n+1/2} - u_{i,j}^{n}}{\Delta t/2} = \varepsilon \left[\frac{\hat{u}_{i+1,j}^{n+1/2} - 2\hat{u}_{i,j}^{n+1/2} + \hat{u}_{i-1,j}^{n+1/2}}{\Delta x^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{\Delta y^2} \right] - \left[b1 \frac{\hat{u}_{i+1,j}^{n+1/2} - \hat{u}_{i-1,j}^{n+1/2}}{2\Delta x} + b_2 \frac{u_{i,j+1}^{n} - u_{i,j-1}^{n}}{2\Delta y} \right]$$

$$\frac{u_{i,j}^{n+1} - \hat{u}_{i,j}^{n+1/2}}{\Delta t/2} = \varepsilon \left[\frac{\hat{u}_{i+1,j}^{n+1/2} - 2\hat{u}_{i,j}^{n+1/2} + \hat{u}_{i-1,j}^{n+1/2}}{\Delta x^2} + \frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{\Delta y^2} \right] - \left[b1 \frac{\hat{u}_{i+1,j}^{n+1/2} - \hat{u}_{i-1,j}^{n+1/2}}{2\Delta x} + b_2 \frac{u_{i,j+1}^{n+1} - u_{i,j-1}^{n+1}}{2\Delta y} \right]$$

Método ADI – Esquema Central

Escolhendo $\Delta x = \Delta y = h$, podemos reescrever a discretização da seguinte forma:

$$
\begin{aligned}
(1+\sigma)\,\hat{u}_{i,j}^{n+1/2} + \left(\frac{\rho_1}{4} - \frac{\sigma}{2}\right)\hat{u}_{i+1,j}^{n+1/2} - \left(\frac{\rho_1}{4} + \frac{\sigma}{2}\right)\hat{u}_{i-1,j}^{n+1/2} &= (1-\sigma)\,u_{i,j}^{n} \\
&+ \left(\frac{\sigma}{2} - \frac{\rho_2}{4}\right)u_{i,j+1}^{n} + \left(\frac{\sigma}{2} + \frac{\rho_2}{4}\right)u_{i,j-1}^{n}
\end{aligned}
$$

$$
\begin{aligned}
(1+\sigma)\,u_{i,j}^{n+1} + \left(\frac{\rho_2}{4} - \frac{\sigma}{2}\right)u_{i,j+1}^{n+1} - \left(\frac{\rho_2}{4} + \frac{\sigma}{2}\right)u_{i,j-1}^{n+1} &= (1-\sigma)\,\hat{u}_{i,j}^{n+1/2} \\
&+ \left(\frac{\sigma}{2} - \frac{\rho_1}{4}\right)\hat{u}_{i+1,j}^{n+1/2} + \left(\frac{\sigma}{2} + \frac{\rho_1}{4}\right)\hat{u}_{i-1,j}^{n+1/2}
\end{aligned}
$$

Resultados Numéricos

Encontrar $u(x,y,t) \in \Omega \times \Theta$ satisfazendo a seguinte equação:

$$\begin{aligned} \frac{\partial u}{\partial t} - \varepsilon \Delta u + \mathbf{b} \cdot \nabla u = f(x,y,t) \quad & \text{em} \quad \Omega \times \Theta \\ u(x,y,t) = \overline{u} \qquad\quad & \text{sobre} \quad \partial\Omega \times \Theta \\ u(x,y,0) = \varphi(x,y) \qquad & \text{em} \quad \Omega \end{aligned}$$

Uma solução exata para este problema é dada por

$$u(x,y,t) = e^{-2\pi^2 t} \sin(\pi x) \sin(\pi y)$$

para esta solução no domínio $\Omega = [0,1] \times [0,1]$, temos $\overline{u} = 0$,

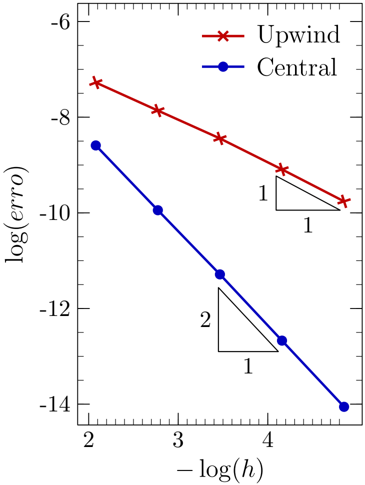
$$\varphi(x,y) = \sin(\pi x) \sin(\pi y)$$

e

$$f(x,y,t) = \pi e^{-2\pi^2 t} \left(b_1 \cos(\pi x) \sin(\pi y) + b_2 \sin(\pi x) \cos(\pi y) \right)$$

Estudo de Convergência

Para este estudo foi adotado $8 \times 8, 16 \times 16, 32 \times 32, 64 \times 64$, 128×128 elementos, $\varepsilon = 1$, $\mathbf{b} = (1, 1)$ e $\Delta t = h$.



Custo Computacional

Supondo os dados:

- malha de 64×64 elementos
- $n = 1000$ iterações no tempo
- k número de equações do sistema

AS metodologias requerem a seguinte quantidade de operações

- Decomposição LU (complexidade – $2k^3/3$)
  - $\frac{2}{3}k^3 = \frac{2}{3}3969^3 = 41.682.334.806$ (a cada passo de tempo)
  - $\frac{2}{3}k^3 n = \frac{2}{3}3969^3 1000 = 41.682.334.806.000$ (total)
- Algoritmo de Thomas (complexidade – $8k$)
  - 64 equações calculadas 64 vezes em cada direção (x e y)
  - $8 \times 64 \times 64 \times 64 = 2.097.152$ (a cada passo de tempo)
  - $8 \times 64 \times 64 \times 64 \times 1000 = 2.097.152.000$ (total)