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# HCR Gearing and Optimization of Its Geometry

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**Abstract.** A special kind of the basic involute profile of non-standard gearing is called high contact ratio (HCR) gearing, where the contact ratio is higher, there are always at least two pairs of teeth in contact and the unit addendum height is not equal to one like for standard gearing. Thus, the tooth height is increased. When HCR gearing is used, it is not necessary to achieve a greater gear load capacity, but nevertheless there is a greater risk of interference due to the greater tooth height. The advantages of HCR gearing is higher resistance (load is distributed on more pairs of teeth at the same time) and a lower relative noise level of gearing, which can be significantly reduced by using an integer HCR factor. HCR profiles are more complicated than standard involute profiles, they have a greater predisposition for interference, pointed tip thickness and undercut of teeth during production (primary production interference). Due to increased addendum height, there is a larger possibility of some interference or pointed tooth tip occurring. Therefore, these issues need to be prevented in the design phase, and ensured that all relevant equations and constraints are satisfied. The described method of finding optimal gear parameter values uses a Generalized Particle Swarm (GPS) optimization algorithm and MATLAB. The GPS optimization is shown to be a very fast and reliable method.

## Introduction

High contact ratio (HCR) gearing is contact between gears with at least two pairs of teeth in contact. A high contact ratio is obtained with increased (larger than standard) addendum height. Geometry of HCR gearing is much more complicated due to the fact there is a higher possibility of interference occurring, which is much greater than interference involving standard involute profiles. There is also a higher risk of insufficient tooth tip thickness and significantly less favorable values of specific slips on the flanks [1].

It is well known that increasing the average number of teeth in contact leads to exclusion or reduction of the vibration amplitude. Firstly, it was established experimentally that dynamic loads decrease by increasing the contact ratio in spur gearing [2]. Moreover, in order to get a further vibration reduction, HCR gear profiles can be optimized. Sato *et al.* [3] found that HCR gears are less sensitive with respect to manufacturing errors. In particular, these gears allow larger tolerances in the tip relief length. Furthermore, they found that in the absence of pressure angle error, the optimal contact ratio is approximately 2; otherwise, it is better to have a contact ratio of approximately 1.7 or higher than 2.3. Kahraman and Blankenship [4] published an experimental work on HCR gear vibration. They found that the best behavior is obtained with an integer contact ratio, even though other specific non integer (rational) contact ratios can minimize the amplitude of specific harmonics of the static transmission error. It is important to note that in their work, HCR gears were obtained by modifying the outside diameter whilst the other macro-geometric parameters, e.g. the number of teeth, were left unchanged.

The contact ratio is increased by increasing tooth height. Dynamic loads and noise are reduced by using high contact ratio gears. According to results of different measurements of gear pairs, reduction of noise has proved to be best using HCR gearing with a contact ratio of $\varepsilon_\alpha = 2$. A decrease in noise results because there are always two pairs of teeth in contact, which means that when one pair of teeth is removed from contact, another pair of teeth is coming into contact. Furthermore, the applied force is considerably smaller as it is divided between two pairs of teeth. Therefore, gearing in the automotive industry should be done with $\varepsilon_\alpha = 2$ in order to reduce noise and dynamic loads [5].

**Contact Ratio in Standard Involute Gearing**

Fig. 1 shows the driving pinion tooth is just coming into contact at point A with a tooth on the driven gear. The zone of action of the meshing gear teeth is shown in the figure, where tooth contact begins and ends at the intersections of the two addendum circles and the line of action (points A and E).

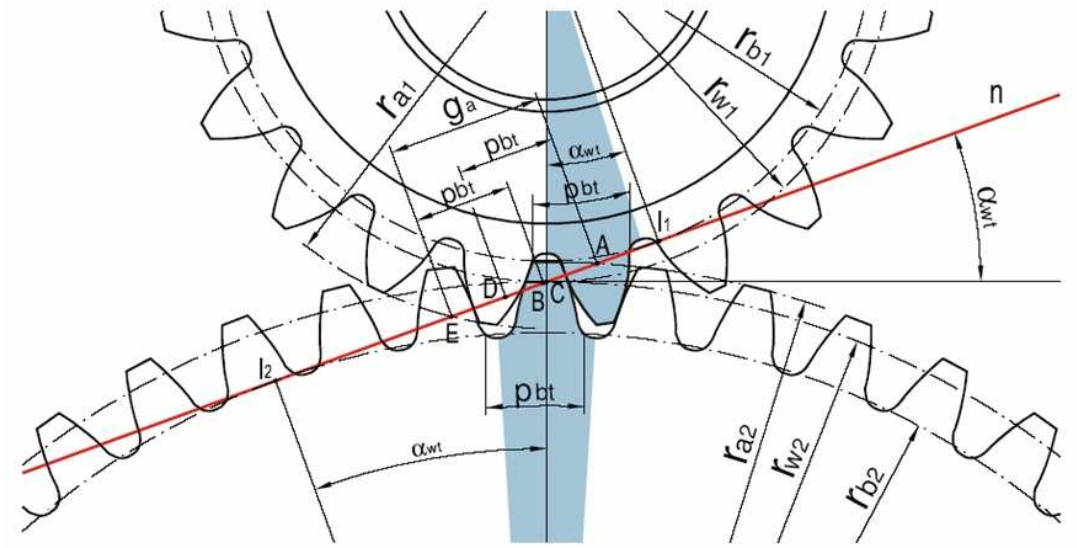


Fig. 1. Contact ratio [6]

The driving gear tip circle cuts the line of action at point E, whilst the tip circle of the driven gear cuts this line at point A. Before point A or after point E, contact between the two teeth cannot take place as the driven gear does not exist. The straight line segment AE is called the path of contact. The segment AC is called the approach path while segment CE is called the recess path.

The distance between two corresponding flanks is called the base pitch. If the value of the base pitch is higher than the value of the path of contact, contact between the two following teeth does not occur. Thus when two mating teeth release, continuity in meshing is not guaranteed. The correct working of the gear will be assured if the value of the path of contact is higher than that of the base pitch. The ratio between the length of contact $g_a$, and pitch on the base cylinder $p_{bt}$ is given by the following formula [7]:

$$\varepsilon_\alpha = \frac{\text{length of contact}}{\text{base pitch}} = \frac{g_a}{p_{bt}} \tag{1}$$

Which is called the transverse contact ratio. Therefore, contact ratio can be defined as the average number of teeth in contact at one time.

It also results in the following equations:

$$\varepsilon_\alpha = \varepsilon_1 + \varepsilon_2 = \frac{g_f + g_a}{\pi\ m \cos\alpha} = \frac{z_1}{2\pi}\left(\tan\alpha_{a1} - \tan\alpha_w\right) + \frac{z_2}{2\pi}\left(\tan\alpha_{a2} - \tan\alpha_w\right) \qquad (2)$$

Or according to Fig. 1:

$$\varepsilon_\alpha = \frac{g_a}{p_{bt}} = \frac{\sqrt{r_{a1}^2 - r_{b1}^2} + \sqrt{r_{a2}^2 - r_{b2}^2} - a_w \sin\alpha_{wt}}{p_{bt}} \qquad (3)$$

Tooth contact begins at A and terminates at E during the course of tooth action. In a pair of meshing spur gears, the line of contact along the width of the gears is parallel to the gear axes and shifts its position along the tooth profile curve from the top to the bottom region of the tooth height or vice versa as the engagement proceeds during the course of tooth action. Contact begins when the line of action intersects the tip circle of the driven gear. At the beginning of contact, the tooth of the driving pinion comes into contact with the top of the driven gear. The previous teeth pair is already in mesh so that the load is shared by these two pairs. This condition continues for a short time until the previous pair goes out of mesh. From this point onwards, the sole pair takes the full load and continues to do so till a new pair comes into mating position. Thereafter, the load is again shared by the former pair and the new pair for a short time until the former pair goes out of mesh. Contact ends when the line of action intersects the tip circle of the driver. At this point the tip of the driver just leaves the flank of the driven gear. This portion of the line of contact AE is called the length of contact or the length of action.

At larger contact ratios than 1, there is the possibility of load sharing amongst the teeth. For contact ratios between 1 and 2, which are common for standard spur gears, there will still be times during the mesh when one pair of teeth takes the entire load. However, these situations will occur towards the center of the mesh region where the load is applied at a lower position on the tooth, rather than at its tip. This means that when one pair of teeth is just entering into contact at point A, another pair, already in contact, will not yet have reached point E. The minimum acceptable contact ratio for smooth operation is 1.2. Gears should not generally be designed having a contact ratio lower than about 1.2, because inaccuracies in mounting might reduce the contact ratio further, increasing the possibility of impact between the teeth as well as an increase in the noise level. To ensure smooth and continuous operation, the contact ratio must be as high as possible. A minimum contact ratio of 1.4 is preferred, but higher is better. Contact ratios for conventional gearing are generally in the range of 1.4 to 1.6 [8, 9]; so the number of tooth engagements is either one or two. For example, a contact ratio of 1.6 means two pairs of teeth are in contact 60% of the time and one pair carries the load 40% of the time.

### HCR Involute Gear Profile

A special form of the basic involute profile of non-standard gearing is called high contact ratio (HCR) gearing, where the contact ratio is higher, there are always at least two pairs of teeth in contact ($\varepsilon_\alpha \geq 2$), and where unit addendum height is not equal to one as for standard gearing. Therefore, the tooth height is increased and is larger than one $h_a^* > 1$.

When HCR gearing is used, it is not necessary to achieve a greater gear load capacity, but nevertheless there is a greater risk of interference due to a larger tooth height. The advantage of HCR gearing is also a higher resistance (the load distribution is shared over more pairs of teeth at the same time) and lower relative noise level of gearing, which can be significantly reduced by using an integer HCR factor $\varepsilon_\alpha$.

HCR profiles are more complicated than standard involute profiles. They have a greater predisposition to interference and pointed tip thickness, but also an undercut of teeth during production (primary production interference).

The coefficient of contact ratio is the main indicator of HCR gearing, which differs from the commonly used standard profiles. The geometry of involute HCR gearing is presented in Fig. 2.

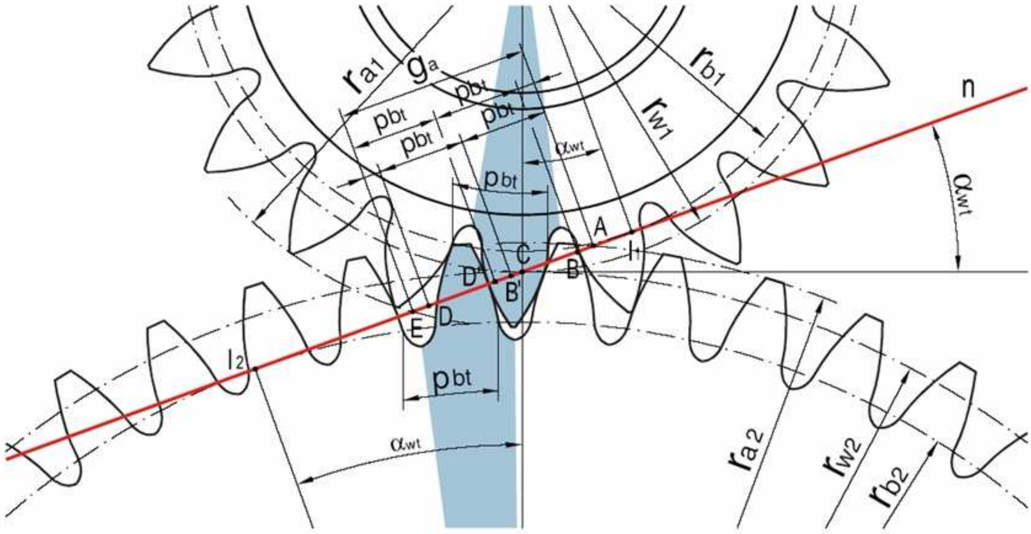


Fig. 2. Geometry of involute HCR gearing [10]

The general distribution of applied force at the characteristic points of the HCR gear tooth flank is shown in Fig. 3. When comparing both types of gearing it is clear that in the case of involute gearing, maximum force is applied when one pair of teeth is in contact (between points B and D, Fig. 1), where 100% of the force F is applied. The largest applied force in HCR gearing (along BB' and DD') is approximately 50%, when two pairs of teeth are in contact. Consequently, the magnitude of the applied force decreases when three pairs of teeth are in contact. Therefore, the load is reduced by a factor of two moving from involute to HCR gearing, which is more favourable.

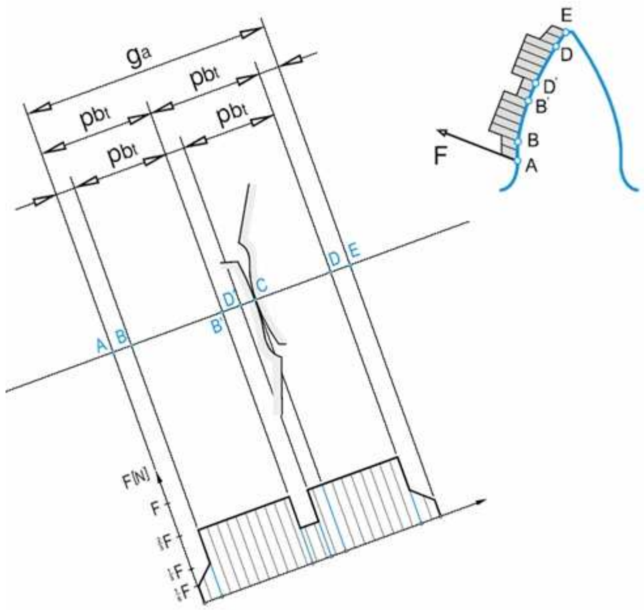


Fig. 3. Distribution of tooth load in high contact ratio (HCR) gearing

The term high contact ratio (HCR) applies to gearing that has at least two teeth pairs in contact at all times, i.e. a contact ratio of two or more. As the percentage change in mesh stiffness for HCR meshes is lower than for normal contact ratio (NCR) meshes, one can expect high-quality HCR gear meshes to have lower mesh-induced vibration and noise than NCR gear meshes [11]. A ratio between 2 and 3 means 2 or 3 pairs of teeth are always in contact.

HCR gearing with a line of action length $g_a \geq 2 \cdot p_{bt}$ is assumed to be more favorable compared to standard involute gearing. When $\varepsilon_\alpha > 2$, three pairs of teeth are in contact along the line of action. Triple tooth contact generally has a higher load capacity, lower gearing noise and increased resistance to fatigue damage.

In high precision and heavily loaded spur gears, the effect of gear errors is negligible, so the periodic variation of tooth stiffness is the principal cause of noise and vibration. High contact ratio spur gears could be used to exclude or reduce the effects of variations in teeth stiffness.

## Geometric Parameter Optimization in HCR Gearing

The main characteristic of HCR gearing, which differs from the commonly used standard involute profiles, is a higher contact ratio.

According to results of different gear pair measurements, reduction of noise was optimal in HCR gearing with a contact ratio $\varepsilon_\alpha = 2$. A decrease in noise results when $\varepsilon_\alpha = 2$ as there are always two pairs of teeth in contact. Therefore when one pair of teeth goes out of contact, another pair of teeth comes into contact and the applied force is considerably smaller as it is divided over two pairs of teeth. However other parameters can influence operational noise levels, such as the rack shift factor of gearing, the gear ratio, gear manufacturing deviations, and gear lubrication.

It is advantageous for the contact ratio to be as large as possible as HCR gears are less sensitive to manufacturing errors, the vibration and gear noise are lower, the load capacity is higher and the load distribution is more favorable.

The increase in contact ratio can be implemented in two ways: by decreasing the pressure angle and by increasing tooth height:

$$\varepsilon_\alpha = f(h_a, \alpha) \tag{4}$$

Obviously, the use of a standard pressure angle and standard tools is preferable [12]. Therefore, the most favorable solution is obtained by increasing addendum height, however there are a lot of geometric and manufacturing constraints that have to be satisfied, which limits the desired increase in contact ratio.

HCR gearing with a path of contact length $g_a \geq 2 \cdot p_{bt}$ is more favorable than in standard involute gearing. When $\varepsilon_\alpha > 2$ the gearing functions in triple tooth contact, which means that traction results from contact of three pairs of teeth. Triple tooth contact generally has a higher capacity and lower noise. These favorable properties of HCR gearing are therefore also influenced by increasing the resistance to fatigue contact damage - pitting, which is one of the main design requirements for gears used in automotive transmissions.

The incidence of pitting fatigue damage in tooth flanks directly depends on the load transfer. The workload during triple tooth contact is spread over two or three pairs of tooth flanks instead of one or two pairs (standard gearing), so its magnitude proportionally decreases (Fig. 2).

Eq. 1 shows that $\varepsilon_\alpha = f(g_a, p_{bt})$. The tooth pitch on the base circle in standard involute gearing is equal to the base pitch in HCR gearing, and it is considered a constant. This means that the maximum contact ratio $\varepsilon_\alpha$ is achieved by maximizing the line of action length $g_a$. This line of action length is calculated using Eq. 5:

$$g_a = \sqrt{r_{a1}^2 - r_{b1}^2} + \sqrt{r_{a2}^2 - r_{b2}^2} - a_w \sin\alpha_{wt} \tag{5}$$

Where the tip diameters of the pinion and wheel are calculated as follows:

$$r_{a1} = r_1 + \left(h_a + x_1 \cdot m_n\right) \tag{6}$$

$$r_{a2} = r_2 + \left(h_a + x_2 \cdot m_n\right) \tag{7}$$

From Eq. 6 and Eq. 7 it is clear that the line of action length $g_a$ is directly dependent on the addendum heights $h_{a1}$, $h_{a2}$ and the correction factors $x_1$, $x_2$. Geometric parameter optimization in HCR gearing can be based on maximizing the contact ratio $\varepsilon_\alpha$ for a given centre distance $a_w$. The main optimization parameters are these addendum heights and correction factors. For a given distance between wheel centers $x_c$, a relationship between $x_1$ and $x_2$ can be formulated.

The addendum heights $h_{a1}$ and $h_{a2}$ can be found from following equations:

$$h_{a1} = h_{a1}^{*} \cdot m_n \tag{8}$$

$$h_{a2} = h_{a2}^{*} \cdot m_n \tag{9}$$

Where $h_{a1}^{*}$ and $h_{a2}^{*}$ are addendum heights for the module equal to one.

Further, it follows:

$$x_2 = x_c - x_1 \tag{10}$$

Therefore, contact ratio is a function of both addendum heights and the correction factor of pinion $\varepsilon_\alpha = f\,(h_{a1}^{*}, h_{a2}^{*}, x_1)$. To maximise this function, a three variable nonlinear optimization is hence required, with the following limitations [6, 13]:

- removal of meshing interference
- minimum arc thickness of the tooth tip $s_{a1,2}$
- distribution of $x_1$, $x_2$ has to be done through balancing specific slips, strength, or a particular condition.

The optimization is directly influenced by the resistance of teeth against pitting damage, which is one of the critical factors in the design of automotive transmission gears.

**Interference During Production**

This interference occurs during the production process of gear forming when the tooth of the rack tool overlaps a transition curve of the wheel gear, resulting in a so-called undercut tooth.

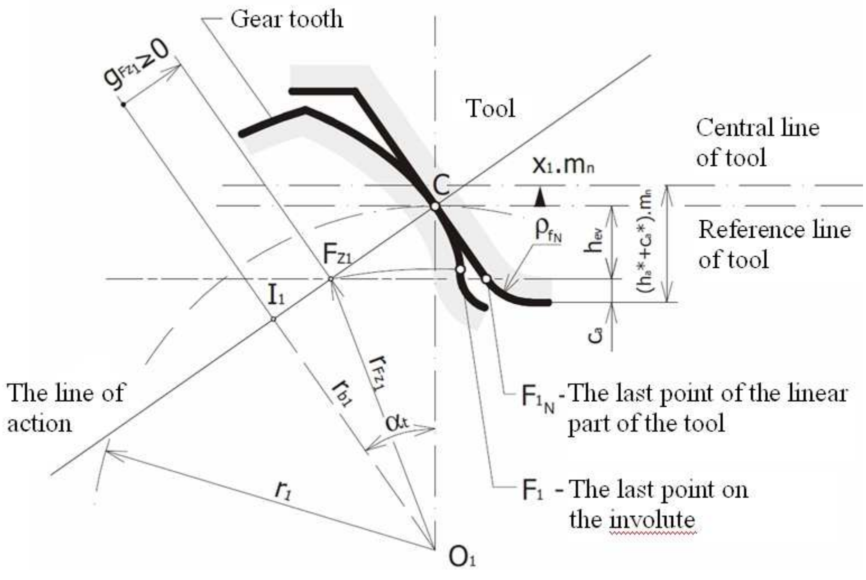


Fig. 4. Interference during production

This incidence of this phenomenon largely depends on the manufacturing process. Unfavorable conditions arise during manufacture by tool rack, so if the production method is unknown in advance, the interference level should always be checked.

Interference during production will not occur if following condition is satisfied [14]:

$$g_{F_{Z1}} \geq 0 \tag{11}$$

Where the expression is also valid for length $g_{F2}$ (Fig. 4):

$$g_{F_{Z1}} = I_1C - F_{z1}C = r_{b1} \cdot \tan\alpha_t - \frac{h_{ev}}{\sin\alpha_t} \tag{12}$$

Where:

$$h_{ev1} = m_n \cdot \left(h_{a1}^* - x_1\right) \tag{13}$$

Substituting Eq. 13 into Eq. 12, it follows:

$$g_{F_{Z1}} = r_{b1} \cdot \tan\alpha_t - \frac{m_n}{\sin\alpha_t}\left(h_{a1}^* - x_1\right) \tag{14}$$

For the boundary condition $g_{F_{Z1}} = 0$, maximum values of the parameters $h_{a1}^*$ and $h_{a2}^*$ can be determined to ensure interference during production doesn't occur:

$$0 \leq r_{b1} \cdot \tan\alpha_t - \frac{m_n}{\sin\alpha_t}\left(h_{a1}^* - x_1\right) \tag{15}$$

Re-arranging Eq. 15 to obtain an expression for $h_{a1}^*$:

$$h_{a1}^* \leq \frac{r_{b1} \cdot \sin^2\alpha_t}{m_n \cdot \cos\alpha_t} + x_1 \tag{16}$$

It is very important that the unit value of addendum height $h_{a1}^*$ for the pinion satisfies Eq. 16, to ensure interference during production does not occur. Similarly, the corresponding condition for $h_{a2}^*$ should also be satisfied:

$$h_{a2}^* \leq \frac{r_{b2} \cdot \sin^2\alpha_t}{m_n \cdot \cos\alpha_t} + x_2 \tag{17}$$

**Meshing Interference**

Meshing interference refers to the case of overlap between teeth profile curves. It may occur as a result of interference between the gear head and the transition curve of the pinion (Fig. 5), and/or the head of the pinion and the transition curve of the wheel [14].

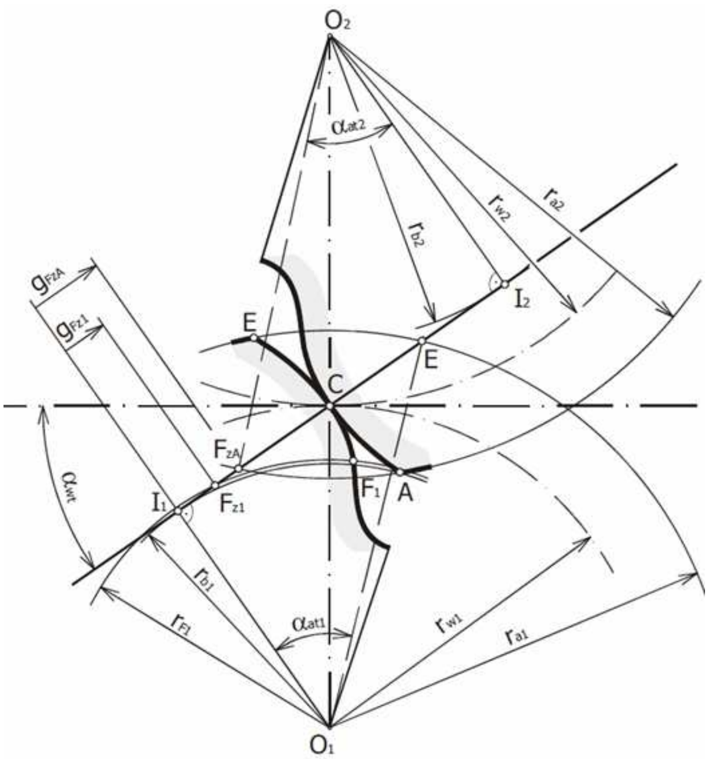


Fig. 5. Meshing interference between wheel head and transition curve of the pinion

As per Fig. 5, it is possible to assess the interference between the meshed wheel gear head and the transition curve of the pinion, where the correct operation in this case occurs only if the wheel head engages the involute part of the pinion, which is bordered by the meshing line and point $F_{z1}$ (last point on the pinion involute).

Referring to Fig. 5, meshing interference will not occur only if distance $g_{FzA}$ is bigger than distance $g_{Fz1}$, or if engagement takes place on the root of pinion tooth and on the addendum of the wheel tooth. The position of the point $F_{z1}$ was described by Eq. 14:

$$g_{F_{ZA}} = a_w \cdot \sin\alpha_{wt} - r_{a2} \cdot \sin\alpha_{at2} \tag{18}$$

The distance $g_{FzA}$ should be bigger than distance $g_{Fz1}$, so:

$$g_{F_{ZA}} \geq g_{F_{Z1}} \tag{19}$$

Then, when substituting the positions of point A and $F_{z1}$ in Eq. 19:

$$a_w \cdot \sin\alpha_{wt} - r_{a2} \cdot \sin\alpha_{at2} \geq r_{b1} \cdot \tan\alpha_t - \frac{m_n}{\sin\alpha_t}\left(h_{a1}^* - x_1\right) \tag{20}$$

The expression for $r_{a2}$ applies:

$$r_{a2} \leq a_w \frac{\sin\alpha_{wt}}{\sin\alpha_{at2}} - r_{b1}\frac{\tan\alpha_t}{\sin\alpha_{at2}} + \frac{m_n}{\sin\alpha_t \cdot \sin\alpha_{at2}}\left(h_{a1}^* - x_1\right) \tag{21}$$

And after substituting in the relation for $r_{a2}$, the expression becomes:

$$r_2 + \left(h_{a2}^* + x_2\right)\cdot m_n \leq a_w \frac{\sin\alpha_{wt}}{\sin\alpha_{at2}} - r_{b1}\frac{\tan\alpha_t}{\sin\alpha_{at2}} + \frac{m_n}{\sin\alpha_t \cdot \sin\alpha_{at2}}\left(h_{a1}^* - x_1\right) \tag{22}$$

On the basis of Eq. 22, addendum height $h^*_{a2}$ can be expressed as follows:

$$h^*_{a2} \le \frac{1}{m_n}\left[a_w \frac{\sin\alpha_{wt}}{\sin\alpha_{at2}} - r_{b1}\frac{\tan\alpha_t}{\sin\alpha_{at2}} + \frac{m_n}{\sin\alpha_t \cdot \sin\alpha_{at2}}\left(h^*_{a1} - x_1\right) - r_2\right] - x_2 \tag{23}$$

Where:

$$\cos\alpha_{at2} = \frac{r_{b2}}{r_{a2}} \quad , \quad \alpha_{at2} = \arccos\frac{r_{b2}}{r_{a2}} \quad , \quad \alpha_{at2} = \arccos\frac{r_{b2}}{r_2 + \left(h^*_{a2} + x_2\right)\cdot m_n} \tag{24a,b,c}$$

Since the magnitude of angle $\alpha_{at2}$ is a function of the addendum unit coefficient, i.e. $\alpha_{at2} = f(h^*_{a2})$, to determine the maximum value of $h^*_{a2}$ for which meshing interference will not occur (in which the tooth will not be shortened due to meshing interference), it is necessary to solve the transcendental Eq. 23. A similar expression is also developed for the parameter $h^*_{a1}$:

$$h^*_{a1} \le \frac{1}{m_n}\left[a_w \frac{\sin\alpha_{wt}}{\sin\alpha_{at1}} - r_{b2}\frac{\tan\alpha_t}{\sin\alpha_{at1}} + \frac{m_n}{\sin\alpha_t \cdot \sin\alpha_{at1}}\left(h^*_{a2} - x_2\right) - r_1\right] - x_1 \tag{25}$$

A further commonly used condition to avoid occurrence of these interferences is $g_{E2} \ge g_{F2} \ge 0$.

**Minimum Thickness of the Tooth Head Circle**

Changing the addendum height $h^*_{a1}$ will certainly influence the total thickness of the tooth on the tip circle. A greater tooth height, as well as a positive correction factor, may cause the thickness of the tooth on the tip circle to be below permissible values. Referring to Fig. 6, tooth tip thickness can be expressed [10, 13] as follows:

$$s_{a1} = 2\,\psi_{a1} \cdot r_{a1} \tag{26}$$

$$2\,\psi_{a1} = 2\,\psi_{b1} - 2\,\mathrm{inv}\alpha_{at1} \tag{27}$$

$$2\,\psi_{b1} = 2\,\psi_1 + 2\,\mathrm{inv}\alpha_t \tag{28}$$

$$2\,\psi_1 = \frac{2\,s_{kt1}}{d_1} \tag{29}$$

Combining Eqs. 26 – 29, tooth thickness on the tip circle is given by:

$$s_{a1} = d_{a1}\left(\frac{s_{kt1}}{d_1} + \mathrm{inv}\alpha_t - \mathrm{inv}\alpha_{at1}\right) \tag{30}$$

Whilst tooth thickness on the pitch circle $s_{kt1}$ can be calculated using:

$$s_{kt1} = \frac{p_t}{2} + 2\cdot x_1 \cdot m_n \cdot \tan\alpha_t \tag{31}$$

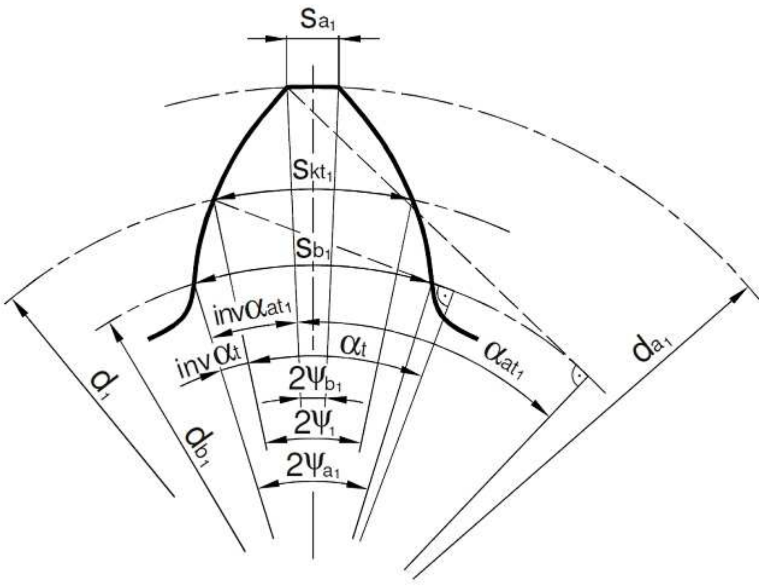


Fig. 6. Determination of tooth thickness on the tip circle $d_a$

After modifying and substituting Eq. 31 into Eq. 30, the expression becomes:

$$s_{a1} = d_{a1}\left(\frac{p_t + 4x_1\, m_n\, \tan\alpha_t}{2d_1} + \text{inv}\alpha_t - \text{inv}\alpha_{at1}\right) \quad (32)$$

As for all conventional involute gearing, the thickness of HCR teeth on the tip circle is governed by the condition $s_a \geq (0.25\text{-}0.4)\ m_n$. For case-hardened teeth, the expression should be $s_a \geq 0.4\ m_n$, due to a lower risk of tip abruption.

As stated in Eq. 24c, the size of angle $\alpha_{at1}$ is also a function of the unit addendum height: $\alpha_{at1} = f\ (h^*_{a1})$. Then for case-hardened teeth, the inequality becomes:

$$0.4 \cdot m_n \leq \left(d_1 + 2\cdot\left(h^*_{a1} + x_1\right)\cdot m_n\right)\cdot\left(\frac{p_t + 4x_1\, m_n\, \tan\alpha_t}{2d_1} + \text{inv}\alpha_t - \text{inv}\alpha_{at1}\right) \quad (33)$$

And, after rearranging:

$$h^*_{a1} \leq \frac{0.2}{\dfrac{p_t + 4x_1\, m_n\, \tan\alpha_t}{d_1} + 2\cdot\left(\text{inv}\alpha_t - \text{inv}\alpha_{at1}\right)} - \frac{d_1}{2m_n} - x_1 \quad (34)$$

A similar expression is valid for the unit addendum height $h^*_{a2}$:

$$h^*_{a2} \leq \frac{0.2}{\dfrac{p_t + 4x_2\, m_n\, \tan\alpha_t}{d_2} + 2\cdot\left(\text{inv}\alpha_t - \text{inv}\alpha_{at2}\right)} - \frac{d_2}{2m_n} - x_2 \quad (35)$$

**Optimization of HCR Gear Geometry**

For optimization of the HCR gear geometry, the following data was used - number of pinion teeth: $z_1 = 21$, number of wheel teeth: $z_2 = 51$, module: $m_n = 4$ mm, modified centre distance: $a_w = 144$ mm, pressure angle: $\alpha_n = 20° = \pi/9$ rad, and helix angle: $\beta = 0°$.

The goal was to achieve a contact ratio $\varepsilon_\alpha = 2$, so that two pairs of gears are always engaged. Implementing this contact ratio is expected to reduce the vibration and gear noise. In order to achieve a high contact ratio, the addendum height was increased in order to obtain longer line of action.

The variable parameters were:

- addendum height for the pinion teeth: $h_{a1}^* \in <1, 1.5>$
- addendum height for the wheel teeth: $h_{a2}^* \in <1, 1.5>$
- rack shift factor of pinion – $x_1 \in <-1, 1>$,

These parameter values were selected to ensure a contact ratio $\varepsilon_\alpha = 2$. However, there were several constraints that also had to be satisfied (interference during the production, meshing interference, minimum thickness of the tooth head circle, HCR involute gearing slide-conditions).

Before these limitations were considered, the tooth parameters were calculated:

- transverse pressure angle: $\alpha_t = \text{arc tan}\left(\dfrac{\tan\alpha_n}{\cos\beta}\right)$ (36)

- reference radius: $r_1 = \dfrac{z_1 \cdot m_n}{2 \cdot \cos\beta}$ , $r_2 = \dfrac{z_2 \cdot m_n}{2 \cdot \cos\beta}$ (37)

- base radius: $r_{b1} = r_1 \cdot \cos\alpha_t$ , $r_{b2} = r_2 \cdot \cos\alpha_t$ (38)

- working pressure angle: $\alpha_{wt} = \arccos\left(\dfrac{m_n}{\cos\beta} \cdot \dfrac{z_1 + z_2}{2} \cdot \dfrac{\cos\alpha_t}{a_w}\right)$ (39)

- involute value of angle $\alpha_t$: $inv\alpha_t = \tan\alpha_t - \alpha_t$ (40)

- involute value of angle $\alpha_{wt}$: $inv\alpha_{wt} = \tan\alpha_{wt} - \alpha_{wt}$ (41)

- sum of rack shift factors: $x_c = \dfrac{z_1 + z_2}{2\tan\alpha_n}\left(inv\alpha_{wt} - inv\alpha_t\right) = 0$ (42)

- rack shift factor of the wheel: $x_2 = x_c - x_1 = 0 - x_1 = -x_1$ (43)

- gear ratio: $i = -\dfrac{z_2}{z_1}$ (44)

- tip radius: $r_{a1} = r_1 + m_n \cdot (h_{a1}^* + x_1)$ , $r_{a2} = r_2 + m_n \cdot (h_{a2}^* + x_c - x_1)$ (45)

- the value of the tip thickness: $s_{kt1} = \dfrac{\pi \cdot m_n}{2\cos\beta} + 2\, x_1\, m_n \tan\alpha_t$ ,

$$s_{kt2} = \frac{\pi \cdot m_n}{2\cos\beta} + 2\left(x_c - x_1\right) m_n \tan\alpha_t \qquad (46)$$

- the angle corresponding to tip diameter: $\alpha_{at1} = \arc\cos\frac{r_{b1}}{r_{a1}}$ , $\alpha_{at2} = \arc\cos\frac{r_{b2}}{r_{a2}}$ (47)

- involute value of angle $\alpha_{at}$: $inv\alpha_{at1} = \tan\alpha_{at1} - \alpha_{at1}$ , $inv\alpha_{at2} = \tan\alpha_{at2} - \alpha_{at2}$ (48)

After calculating these parameters, several constraints were applied. These constraints came from the following limitation conditions:

- interference during the production: $g_{F1} = r_{b1}\tan\alpha_t - \frac{\left(h_{a1}^* - x_1\right)\cdot m_n}{\sin\alpha_t} \geq 0$

$$g_{F2} = r_{b2}\tan\alpha_t - \frac{\left(h_{a2}^* - x_c + x_1\right)\cdot m_n}{\sin\alpha_t} \geq 0 \qquad (49)$$

- meshing interference: $B_1 = \sqrt{r_{b1}^2 + \left[a_w \sin\alpha_{wt} - g_{F2}\right]^2} \geq r_{a1}$,

$$B_2 = \sqrt{r_{b2}^2 + \left[a_w \sin\alpha_{wt} - g_{F1}\right]^2} \geq r_{a2} \qquad (50)$$

- minimum thickness of the tooth head circle: $s_{a1} = 2r_{a1}\left(\frac{s_{kt1}}{2r_1} + inv\alpha_t - inv\alpha_{at1}\right) \geq 0.4\, m_n$

$$s_{a2} = 2r_{a2}\left(\frac{s_{kt2}}{2r_2} + inv\alpha_t - inv\alpha_{at2}\right) \geq 0.4\, m_n \qquad (51)$$

Finally, the following equation describes the overall optimisation goal function to ensure a contact ratio $\varepsilon_\alpha = 2$.

$$\varepsilon_\alpha = \frac{z_1}{2\pi}\left[\tan\alpha_{at1} - \tan\alpha_{wt} - i\cdot\tan\alpha_{at2} + i\cdot\tan\alpha_{wt}\right] = 2 \qquad (52)$$

**Numerical Methods for Optimization of HCR Gear Geometry**

Optimization of HCR gear geometry can be done using different methods. The first possible method is using MS Excel. The advantage of this method is that programming knowledge is not required. Unfortunately, this method is very slow and inaccurate. Its overall accuracy depends on the input data, and the user has to spend a significant amount of time to obtain the required contact ratio. This method is not user-friendly as there are numerous parameters which must be calculated before the contact ratio.

The second optimization method is the use of MS Visual Basic. The advantage of this method is that it is relatively simple to calculate the required contact ratio. The downside to the method is the required programming knowledge. However, the results are much more accurate than the previous method (accuracy about $10^{-3}$), with calculation time approximately 1 minute.

The numerical method used in the present study is described in the following section. It requires MATLAB and a new optimizing method, the Generalized Particle Swarm Optimization (GPSO) Algorithm.

The described GPSO algorithm was implemented in MATLAB, a high-performance language for technical computing, which integrates computation, visualization, and programming in an easy-to-use environment where problems and solutions are expressed in familiar mathematical notation.

The algorithm was implemented as a function which allows the user to specify the values of all adjustable parameters, and to define several algorithm options. The input parameters are the name of the optimization function, the number of variables and selection of the algorithm options, whilst the output parameters are the optimal function value, its coordinates and other optional parameters.

**Generalized Particle Swarm Optimization Algorithm**

The Particle Swarm Optimization (PSO) algorithm is relatively novel, yet well studied and a proven optimizer based on the social behavior of animals moving in large groups (particularly birds) [15]. Compared to other evolutionary techniques, PSO only has a few adjustable parameters, plus it is computationally inexpensive and very easy to implement [16, 17].

PSO uses a set of particles called a swarm to investigate the search space. Each particle is described by its position ($x$) and velocity ($v$). The position of each particle is a potential solution, and the best position that each particle achieved during the entire optimization process is memorized ($p$). The best position achieved by any of the particles within the swarm is also memorized ($g$). The position and the velocity of each particle are updated as follows:

$$\begin{aligned} v[k+1] &= w \cdot v[k] + cp \cdot rp[k] \cdot \left(p[k] - x[k]\right) + cg \cdot rg[k] \cdot \left(g[k] - x[k]\right) \\ x[k+1] &= x[k] + v[k+1] \end{aligned} \tag{53}$$

The acceleration factors $cp$ and $cg$ control the relative impact of the personal (local) and common (global) knowledge on the movement of each particle. The inertia factor $w$, which was introduced for the first time in the work of Shi & Eberhart [18], keeps the swarm together and prevents it from diversifying excessively, thus reducing PSO to a purely random search. Random numbers $rp$ and $rg$ are mutually independent and uniformly distributed in the range [0, 1].

Particle swarm however, also has some disadvantages. The key disadvantage is its inability to independently control various aspects of the search, such as the stability, oscillation frequency and the impact of personal and global knowledge [19]. The new algorithm, named Generalized PSO (GPSO), which is described and analyzed in detail by Kanović *et al.* [20], overcomes the above mentioned flaw. It considers each particle within the swarm as a second-order linear stochastic system with two inputs and one output. The output of such a system is the current position of the particle ($x$), while its inputs are the optimal personal and global positions ($p$ and $g$, respectively). Such systems have been extensively studied in engineering literature [21]. The stability and response properties of such a system can be directly related to its performance as an optimizer, i.e., its explorative and exploitative properties.

GPSO employs a canonical equation often used in control theory [21]:

$$x[k+1] - 2\zeta\,\rho\, x[k] + \rho^2 x[k-1] = \left(1 - 2\zeta\rho + \rho^2\right)\left(c \cdot p[k] + (1-c) \cdot g[k]\right) \tag{54}$$

Where $\rho$ is the eigenvalues module, and $\zeta$ is the cosine of their arguments. The parameter $c$ is introduced to replace both $b_p$ and $b_g$. The parameters in this equation allow more direct and independent control of the various aspects of the search procedure.

Based on many analyses of PSO, two parameter adjustment schemes, GPSO1 and GPSO2 were proposed [20]. In both schemes, it was recommended to linearly decrease $\rho$ from aproximately 0.95 to 0.6, and $c$ from 0.8 to approximately 0.2. In the first scheme (GPSO1), $\zeta$ was adopted as a stochastic parameter with a uniform distribution ranging from -0.9 to 0.2, whereas in the second scheme (GPSO2), $\zeta$ was uniformly distributed in the range [-0.9, 0.6].

Using the GPSO algorithm, a solution for HCR gearing is obtained in a very short time (less than one second). This solution is also very accurate ($10^{-15}$). The results obtained using this method are as follows:

*Given data:*
z1 = 21
z2 = 51
mn = 4 mm
aw = 144 mm
alfan = 0.349065850398866
beta = 0

*Calculated optimal values for 100 iterations:*
ha1dot = 1.184548908194918
ha2dot = 1.313253162560083
x1 = 0.174169864574093
Main function:
epsilonalfa = 2.000000052991015

*Calculated optimal values for 200 iterations:*
ha1dot = 1.18219030491075
ha2dot = 1.31311869844519
x1 = 0.16471064649626
Main function:
epsilonalfa = 2.00000000000000

*Calculated optimal values for 500 iterations:*
ha1dot = 1.17779276410901
ha2dot = 1.31951635807650
x1 = 0.17860257195356
Main function:
epsilonalfa = 2. 00000000000000

**Summary**

High contact ratio gearing is contact between gears with at least two pairs of teeth in contact. The high contact ratio is obtained with increased addendum height. Geometry of HCR gears is much more complicated due to the fact there is larger possibility of meshing and production interference occurring, compared to standard involute profiles. Also, there is a higher risk of a smaller than permissible tooth tip, and significantly less favorable values of specific slips in the flanks.

Dynamic loads and noise are reduced by using high contact ratio gears. According to the results of different gear pair measurements, noise reductions proved to be best using HCR gearing with a contact ratio of $\varepsilon_\alpha = 2$. This reduced noise is obtained because there are always two pairs of teeth in contact, which means when one pair of teeth leave contact, another pair of teeth is coming into contact. The applied force is also considerably smaller since it is distributed over two pairs of teeth. Therefore, gearing in the automotive industry should be based around $\varepsilon_\alpha = 2$ in order to reduce noise and dynamic forces.

Due to the increased addendum height, there is a larger possibility of interference or pointed tooth tip occurring. Therefore, to ensure these errors are prevented, all relevant equations and constraints must be satisfied. Such constraints include conditions for the number of teeth on both the pinion and wheel to ensure production interference does not occur, conditions so that meshing interference does not occur, and threshold conditions for thickness of the tooth head circle of both gears.

The described method of finding optimal solutions for $h_{a1}^*$, $h_{a2}^*$ and $x_1$ uses Generalized Particle Swarm Optimization Algorithm and MATLAB. GPSO is a very fast and reliable method. There are infinite highly precise solutions of the governing transcendental equation using this method, all supplying a solution for contact ratio $\varepsilon_\alpha = 2$.

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