Unit – III
2D, 3D Transformations and Projections

1. 2–D transformations

1.1 Introduction

Transformation means changing some graphics into something else by applying rules. We can have various types of transformations such as translation, scaling up or down, rotation, shearing, etc. When a transformation takes place on a 2D plane, it is called 2D transformation.

Transformations play an important role in computer graphics to reposition the graphics on the screen and change their size or orientation.

1.2 Homogeneous Coordinates

To perform a sequence of transformation such as translation followed by rotation and scaling, we need to follow a sequential process:

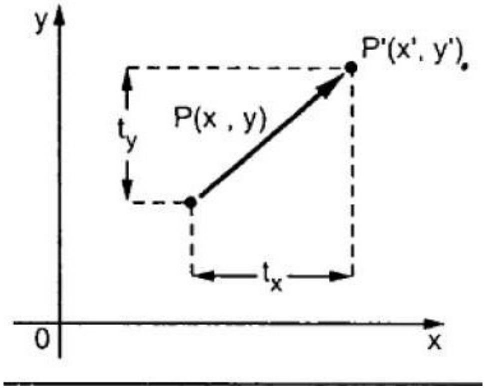
- Translate the coordinates.
- Rotate the translated coordinates.
- Scale the rotated coordinates to complete the composite transformation.

To shorten this process, we have to use 3×3 transformation matrix instead of 2×2 transformation matrix. To convert a 2×2 matrix to 3×3 matrix, we have to add an extra dummy coordinate W.

In this way, we can represent the point by 3 numbers instead of 2 numbers, which is called **Homogenous Coordinate** system. In this system, we can represent all the transformation equations in matrix multiplication. Any Cartesian point $P(X,Y)$ can be converted to homogenous coordinates by $P'(X_h, Y_h, h)$.

1.3 Translation

A translation moves an object to a different position on the screen. You can translate a point in 2D by adding translation coordinate (t_x, t_y) to the original coordinate X,Y to get the new coordinate X', Y'.



From the above figure, you can write that −

$$X' = X + t_x$$

$$Y' = Y + t_y$$

The pair (t_x, t_y) is called the translation vector or shift vector. The above equations can also be represented using the column vectors.

$P = \frac{[X]}{[Y]}$ $\quad P' = \frac{[X']}{[Y']}$ $\quad T = \frac{[t_x]}{[t_y]}$

We can write it as −

$$P' = P + T$$

The translation matrix moves a point by t_x units in the x–direction and ty units in the y–direction. This transformation is represented as:

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$$

1.4 Scaling

To change the size of an object, scaling transformation is used. In the scaling process, you either expand or compress the dimensions of the object. Scaling can be achieved by multiplying the original coordinates of the object with the scaling factor to get the desired result.

Let us assume that the original coordinates are X, Y, the scaling factors are (S_x, S_y), and the produced coordinates are X', Y'. This can be mathematically represented as shown below −

$X' = x \cdot S_x$ **and** $Y' = y \cdot S_y$

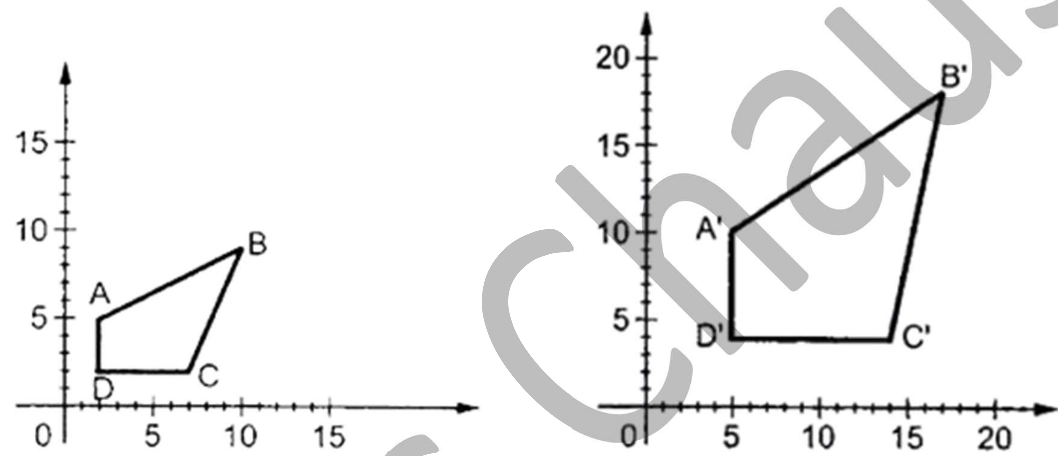
The scaling factor S_X, S_Y scales the object in X and Y direction respectively. The above equations can also be represented in matrix form as below −

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$$

OR

P' = P · S

Where S is the scaling matrix. The scaling process is shown in the following figures.



If we provide values less than 1 to the scaling factor S, then we can reduce the size of the object. If we provide values greater than 1, then we can increase the size of the object.

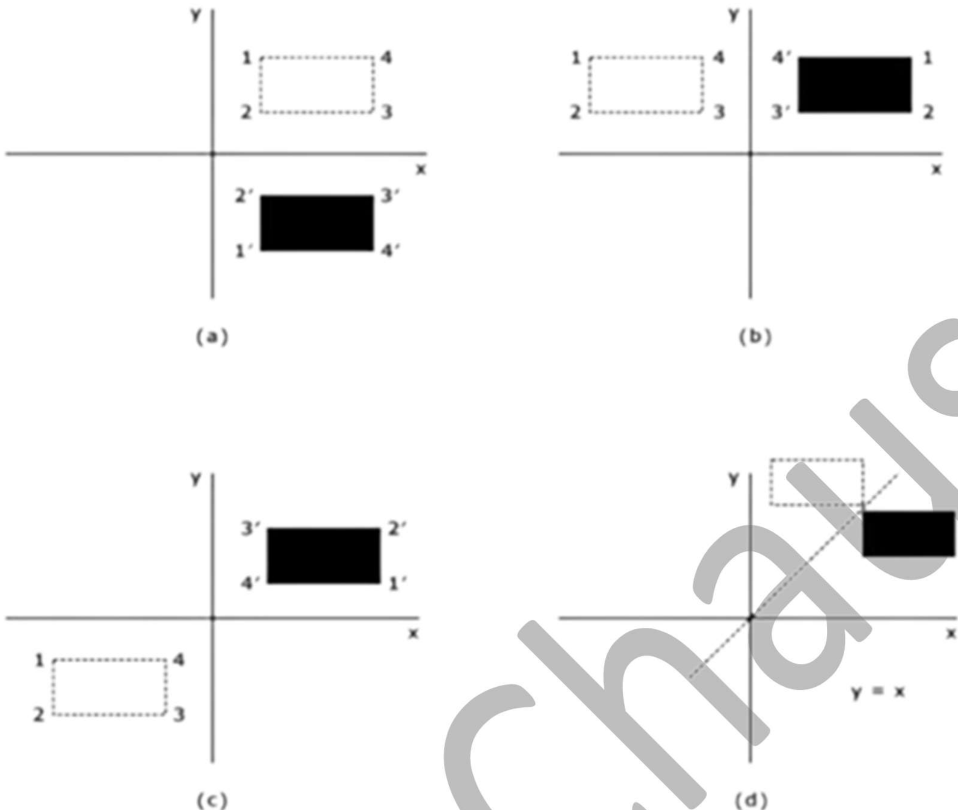
The scaling matrix, where S_x is the scaling factor along the x–axis and S_y is the scaling factor along the y–axis, is:

$$S = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$$

1.5 Reflection

Reflection is the mirror image of original object. In other words, we can say that it is a rotation operation with 180°. In reflection transformation, the size of the object does not change.

The following figures show reflections with respect to x and y axes, and about the origin respectively.



a) The reflection matrix about the x–axis is:

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) The reflection matrix about the y–axis is:

$$R_y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c) The reflection matrix about the origin is:

$$R_o = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

d) The reflection matrix about the line $y = mx$ is:

$$R_l = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1.6 Shear

A transformation that slants the shape of an object is called the shear transformation. There are two shear transformations **X–Shear** and **Y–Shear**. One shifts X coordinates values and other shifts Y coordinate values. However; in both the cases only one coordinate changes its coordinates and other preserves its values. Shearing is also termed as **Skewing**.

X–Shear

The X–Shear preserves the *y* coordinate and changes are made to *x* coordinates, which causes the vertical lines to tilt right or left as shown in below figure.

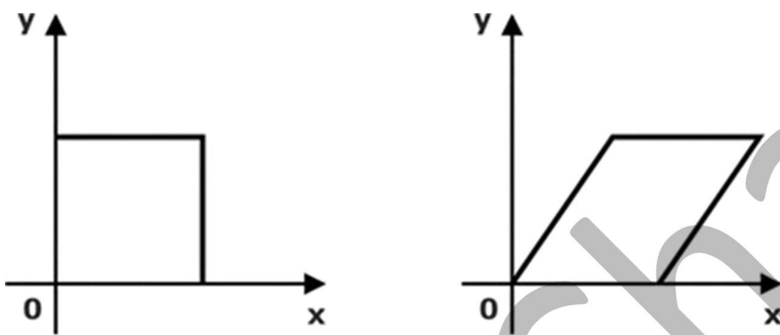


fig. shearing about the x-axis

$$X_{sh} = \begin{bmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$X' = X + sh_x \cdot Y$

$Y' = Y$

Y–Shear

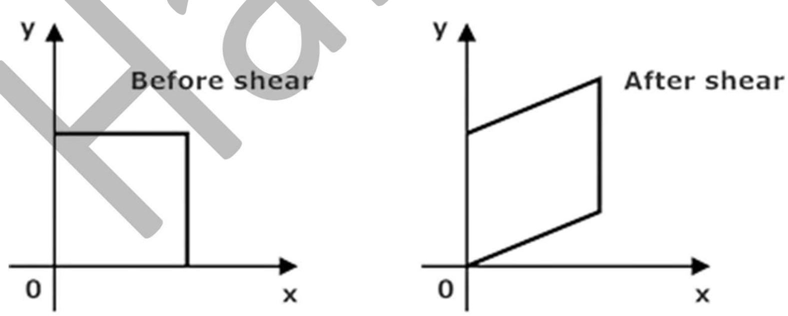


fig. shear in y direction

The Y–Shear preserves the *x* coordinates and changes the *y* coordinates which causes the horizontal lines to transform into lines which slopes up or down as shown in the following figure.

The Y–Shear can be represented in matrix from as −

$$Y_{sh} = \begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

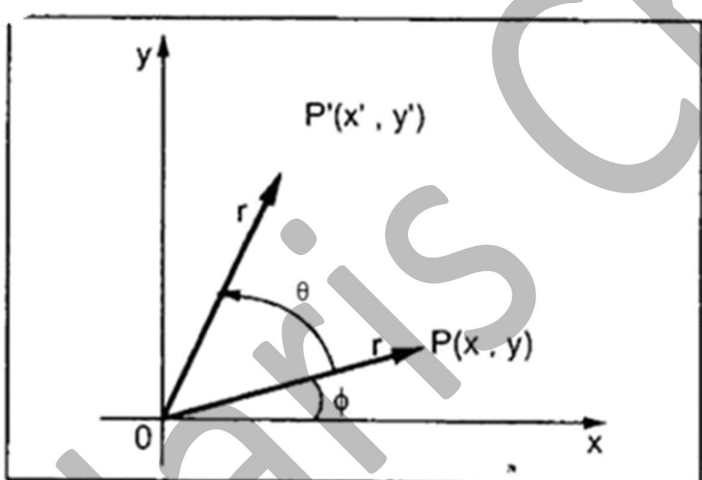
$Y' = Y + sh_y \cdot X$

$X' = X$

1.7 Rotation

In rotation, we rotate the object at particular angle θ from its origin. From the following figure, we can see that the point $P (X,Y)$ is located at angle φ from the horizontal x coordinate with distance r from the origin.

Let us suppose you want to rotate it at the angle θ. After rotating it to a new location, you will get a new point $P' (X',Y')$.



Using standard trigonometric the original coordinate of point $P (X,Y)$ can be represented as −

$X = r\cos\phi$ …(1)

$Y = r\sin\phi$ …(2)

Same way we can represent the point $P'(X',Y')$ as −

$X' = r\cos(\phi+\theta) = r\cos\phi\cos\theta - r\sin\phi\sin\theta$ ...(3)

$Y' = r\sin(\phi+\theta) = r\cos\phi\sin\theta + r\sin\phi\cos\theta$ ...(4)

Substituting equation 1 & 2 in 3 & 4 respectively, we will get

$X' = xcos\theta - ysin\theta x' = xcos\theta - ysin\theta$

$Y' = xsin\theta + ycos\theta y' = xsin\theta + ycos\theta$

Representing the above equation in matrix form,

$$[X'Y'] = [XY]\begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$

or **$P' = P \cdot R$**

Where R is the rotation matrix

$$R = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$

The rotation angle can be positive and negative.

For positive rotation angle (clockwise direction), we can use the above rotation matrix. However, for negative angle rotation (anticlockwise direction), the matrix will change as shown below:

$$R = \begin{bmatrix} cos(-\theta) & sin(-\theta) \\ -sin(-\theta) & cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} (\because cos(-\theta) = cos\theta \ and \ sin(-\theta) = -sin\theta)$$

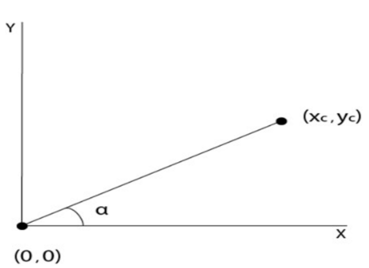
1.8 Rotation about an arbitrary point

If we want to rotate an object or point about an arbitrary point, first of all, we translate the point about which we want to rotate to the origin. Then rotate point or object about the origin, and at the end, we again translate it to the original place. We get rotation about an arbitrary point.

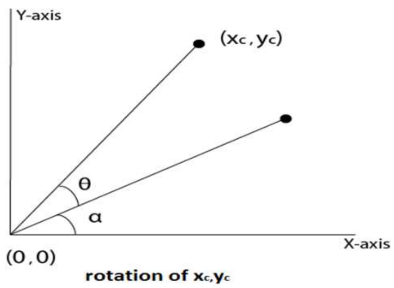
Example: The point (x, y) is to be rotated

The (x_c y_c) is a point about which counterclockwise rotation is done

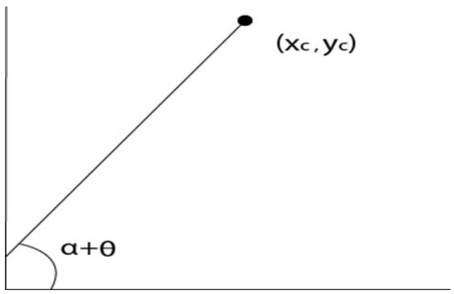
Step 1: Translate point (x_c y_c) to origin

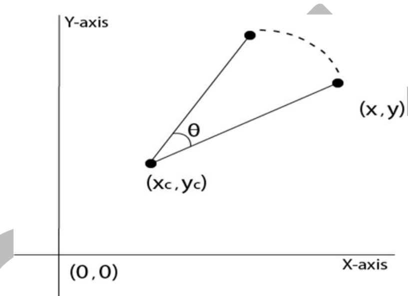


Step 2: Rotation of (x, y) about the origin



Step 3: Translation of center of rotation back to its original position.





2. 3–D transformations

2.1 Introduction

In very general terms a 3D model is a mathematical representation of a physical entity that occupies space. In more practical terms, a 3D model is made of a description of its shape and a description of its color appearance. 3–D Transformation is the process of manipulating the view of a 3D object with respect to its original position by modifying its physical attributes through various methods of transformation like Translation, Scaling, Rotation, Shear, etc.

Properties of 3–D Transformation:

- Lines are preserved,
- Parallelism is preserved,
- Proportional distances are preserved.

2.2 Translation

It is the movement of an object from one position to another position. Translation is done using translation vectors. There are three vectors in 3D instead of two. These vectors are in *X*, *Y*, and

Z directions. Translation in the x–direction is represented using T_x. The translation is y–direction is represented using T_y. The translation in the z–direction is represented using T_z.

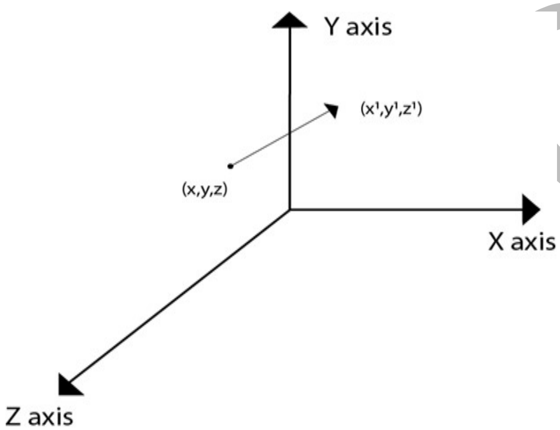
If P, a point having co–ordinates in three directions $P(X, Y, Z)$ is translated, then after translation its coordinates will be $P'(X', Y', Z')$ after translation. T_x T_y T_z are translation vectors in x, y, and z directions respectively.

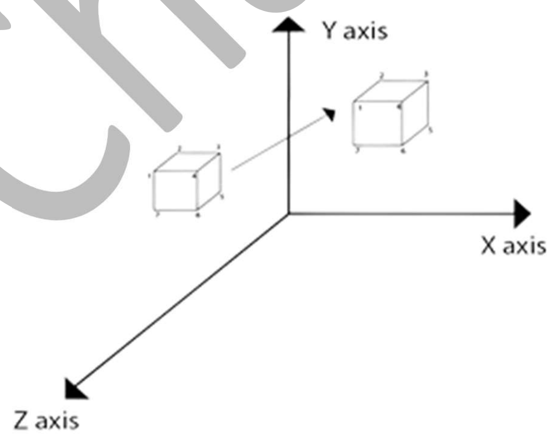
$X' = X + T_x$

$Y' = Y + T_y$

$Z' = Z + T_z$

Three–dimensional transformations are performed by transforming each vertex of the object. If an object has five corners, then the translation will be accomplished by translating all five points to new locations. Following figure 1 shows the translation of point figure 2 shows the translation of the cube.





Matrix for Translation:

$$T[X, Y, Z] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ T_x & T_y & T_z & 1 \end{bmatrix}$$

Matrix representation of point translation

Point shown in fig is (X, Y, Z). It becomes (X', Y', Z') after translation. T_x T_y T_z are translation vectors.

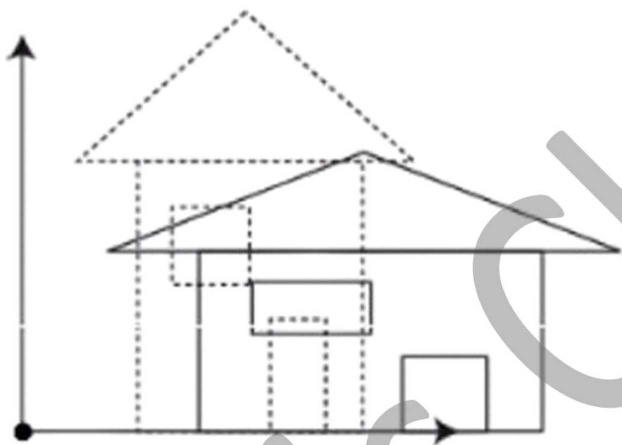
$P'[X', Y', Z', 1] = P[X, Y, Z, 1] \cdot T[X, Y, Z]$

$$[X' \ Y' \ Z' \ 1] = [X \ \ Y \ \ Z \ \ 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ T_x & T_y & T_z & 1 \end{bmatrix}$$

2.3 Scaling

You can change the size of an object using scaling transformation. In the scaling process, you either expand or compress the dimensions of the object. Scaling can be achieved by multiplying the original coordinates of the object with the scaling factor to get the desired result.

The following figure shows the effect of 3D scaling –



In 3D scaling operation, three coordinates are used. Let us assume that the original coordinates are $X, Y, Z,$ scaling factors are (S_x, S_y, S_z) respectively, and the produced coordinates are X', Y', Z'. This can be mathematically represented as shown below −

$$S = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P' = P \cdot S$$

$$[X' \ \ Y' \ \ Z' \ \ 1] = [X \ \ Y \ \ Z \ \ 1] \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= [X.S_x \ \ Y.S_y \ \ Z.S_z \ \ 1]$$

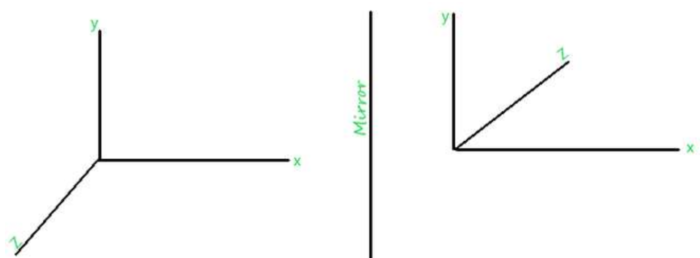
2.4 Reflection

Reflection in 3D space is quite similar to the reflection in 2D space, but there is a single difference in 3D and that is we have to deal with **three axes (x, y, z).** Reflection is nothing but a mirror image of an object.

Three kinds of Reflections are possible in 3D space:

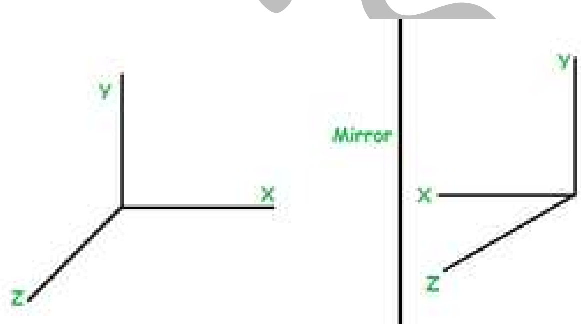
- Reflection along the X–Y plane.
- Reflection along Y–Z plane.
- Reflection along X–Z plane.

1. **Reflection relative to XY plane**



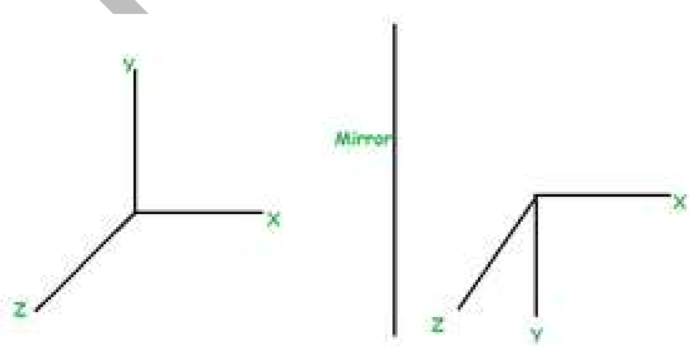
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2. **Reflection relative to YZ plane**



$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

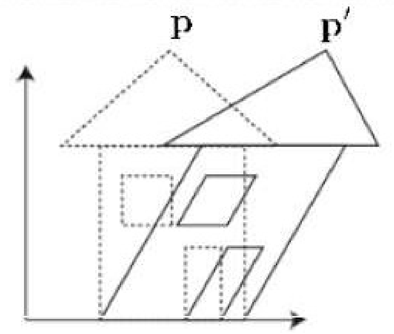
3. **Reflection relative to ZX plane**



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2.5 Shear

A transformation that slants the shape of an object is called the **shear transformation**. Like in 2D shear, we can shear an object along the X–axis, Y–axis, or Z–axis in 3D.



As shown in the above figure, there is a coordinate *P*. You can shear it to get a new coordinate *P'*, which can be represented in 3D matrix form as below −

$$Sh = \begin{bmatrix} 1 & sh_x^y & sh_x^z & 0 \\ sh_y^x & 1 & sh_y^z & 0 \\ sh_z^x & sh_z^y & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

P' = P · Sh

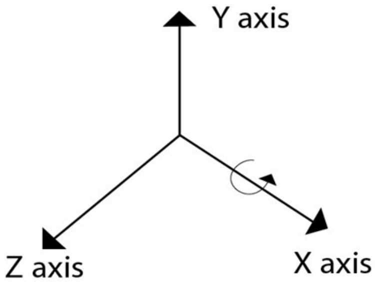
$$X' = X + Sh_x^y Y + Sh_x^z Z$$

$$Y' = Sh_y^x X + Y + sh_y^z Z$$

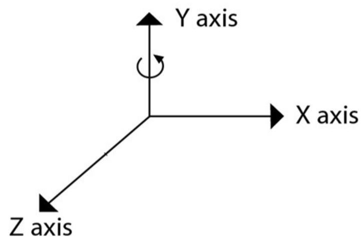
$$Z' = Sh_z^x X + Sh_z^y Y + Z$$

2.6 Rotation

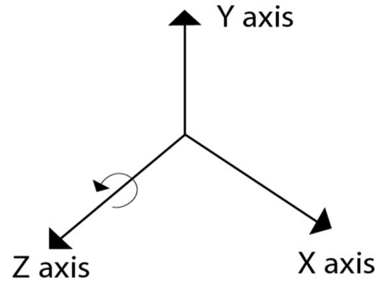
Rotation is moving of an object about an angle. Movement can be anticlockwise or clockwise. 3D rotation is complex as compared to the 2D rotation. For 2D we describe the angle of rotation, but for 3D, angle of rotation and axis of rotation are required. The axis can be either *x* or *y* or *z*.



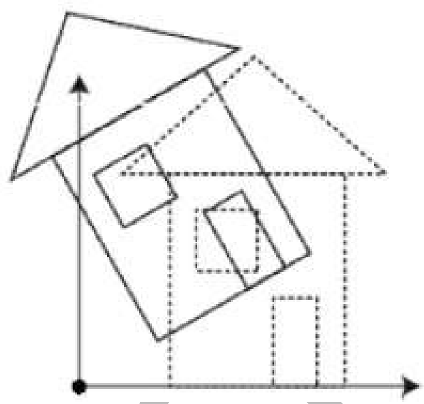
rotation object X axis anticlockwise



rotation object Y axis anticlockwise



rotation object Z axis anticlockwise



Derivation of 3D transformation matrix for rotation about a principal axis:

To derive the 3D transformation matrices for rotation about the **principal axes** (*x, y,* and *z* axes), we will use the right-hand rule convention. The general approach involves rotating a point in 3D space by an angle θ around one of the coordinate axes.

Coordinates before rotation:

$$P = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

1. Rotation about the x-axis:

When rotating around the ***x*-axis**, the *y* and *z* coordinates are transformed, while the *x* coordinate remains unchanged.

Rotation matrix for *x*-axis:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) & 0 \\ 0 & -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Explanation:

- The x-coordinate is unchanged.
- The transformation affects the y and z coordinates:
  - New y coordinate: $y' = y\cos(\theta) + z\sin(\theta)$
  - New z coordinate: $z' = -y\sin(\theta) + z\cos(\theta)$

2. Rotation about the *y*-axis:

When rotating around the ***y*-axis**, the x and z coordinates are transformed, while the y coordinate remains unchanged.

Rotation matrix for *y*-axis:

$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Explanation:

- The y-coordinate is unchanged.
- The transformation affects the x and z coordinates:
  - New x coordinate: $x' = x\cos(\theta) - z\sin(\theta)$
  - New z coordinate: $z' = x\sin(\theta) + z\cos(\theta)$

3. Rotation about the *z*-axis:

When rotating around the ***z*-axis**, the x and y coordinates are transformed, while the z coordinate remains unchanged.

Rotation matrix for z-axis:

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Explanation:

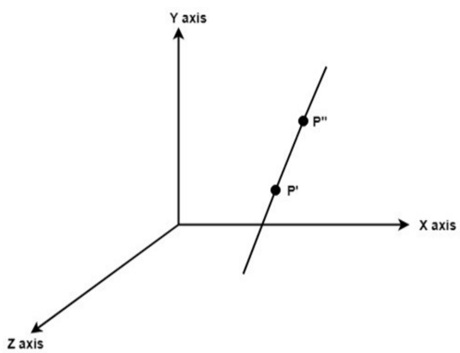
- The z-coordinate is unchanged.

- The transformation affects the *x* and *y* coordinates:
  - New *x* coordinate: $x'=x\cos(\theta)+y\sin(\theta)$
  - New *y* coordinate: $y'=-x\sin(\theta)+y\cos(\theta)$

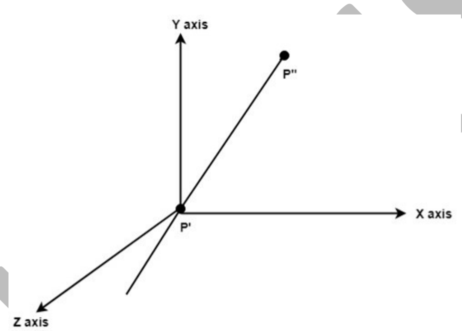
2.7 Rotation about an arbitrary axis

When the object is rotated about an axis that is not parallel to any one of co-ordinate axis, i.e., *x, y, z*. Then additional transformations are required. First of all, alignment is needed, and then the object is being back to the original position. Following steps are required

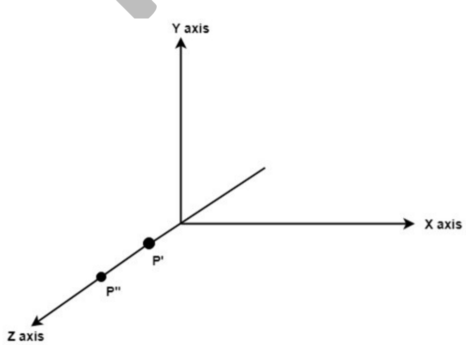
Step 1: Initial position of P' and P'' is shown



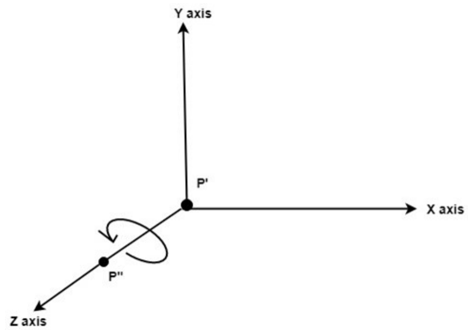
Step 2: Translate object P' to origin



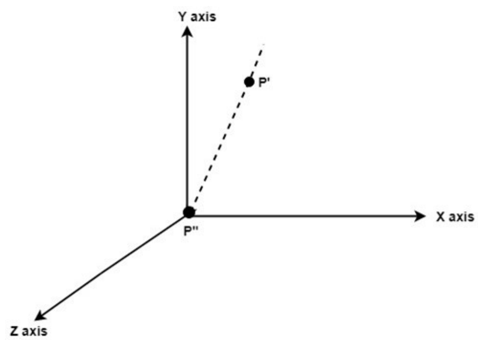
Step 3: Rotate P'' to z axis so that it aligns along the z-axis



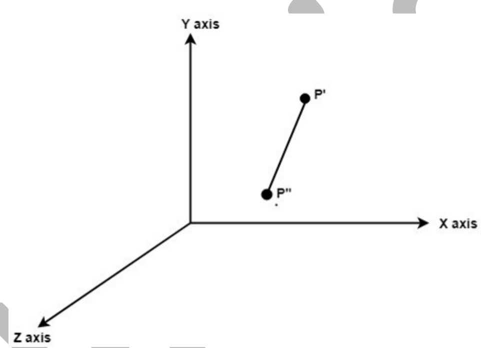
Step 4: Rotate about around z- axis



Step 5: Rotate axis to the original position

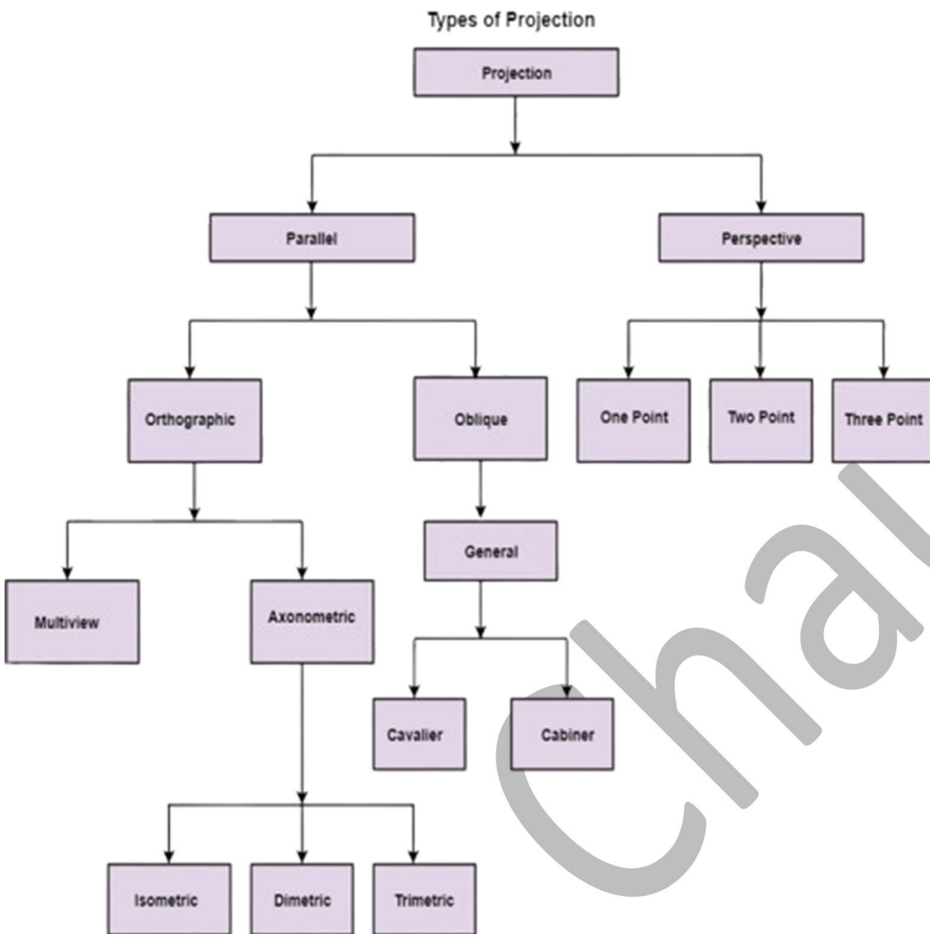


Step 6: Translate axis to the original position.



3. Projections

Projection is a kind of phenomena that is used in computer graphics to map the view of a 3D object onto the projecting display panel where the viewing volume is specified by the world coordinate and then map these world coordinate over the view port.



3.1 Parallel Projection

Parallel projection is a kind of projection where the projecting lines emerge parallelly from the polygon surface and then incident parallelly on the plane. In parallel projection, the centre of the projection lies at infinity. In parallel projection, the view of the object obtained at the plane is less-realistic as there is no for-shortcoming and the relative dimension of the object remains preserves.

3.1.1 Orthographic Projection

It is a kind of parallel projection where the projecting lines emerge parallelly from the object surface and incident perpendicularly at the projecting plane.

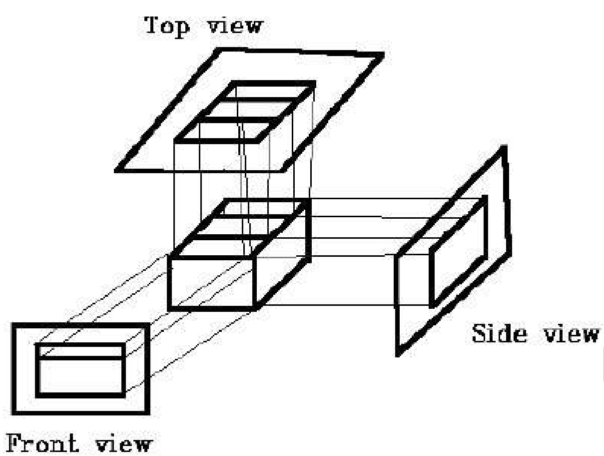
Orthographic Projection is of two categories:

1. Multiview Projection: It is further divided into three categories

a. Top-View: In this projection, the rays that emerge from the top of the polygon surface are observed.

b. Side-View: It is another type of projection orthographic projection where the side view of the polygon surface is observed.

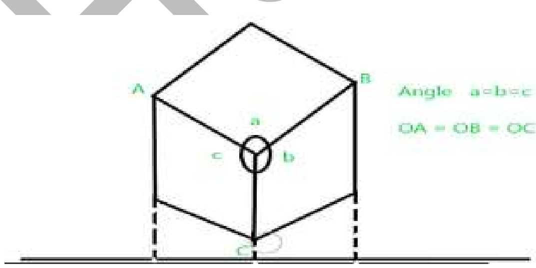
c. Front-View: In this orthographic projection front face view of the object is observed.



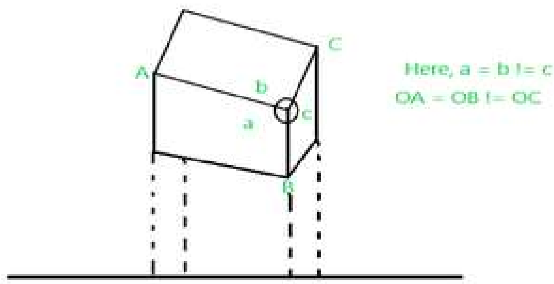
2. Axonometric Projection: Axonometric projection is an orthographic projection, where the projection lines are perpendicular to the plane of projection, and the object is rotated around one or more of its axes to show multiple sides.

It is further divided into three categories:

a. Isometric Projection: It is a method for visually representing three-dimensional objects in two-dimensional display in technical and engineering drawings. Here in this projection, the three coordinate axes appear equally foreshortened and the angle between any two of them is 120 degrees.

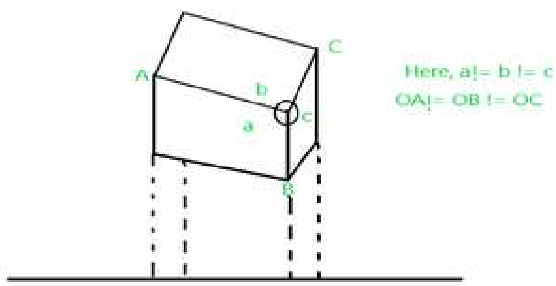


b. Dimetric Projection: It is a kind of orthographic projection where the visualized object appears to have only two adjacent sides and angles are equal.



Dimetric Projection

c. Trimetric Projection: It is a kind of orthographic projection where the visualized object appears to have all the adjacent sides and angles unequal.

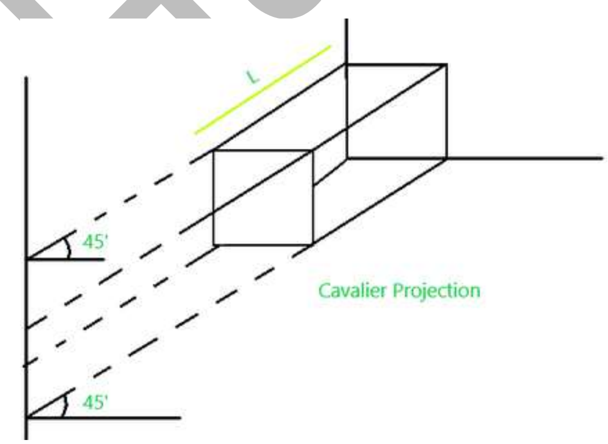


Trimetric Projection

3.1.2 Oblique Projection

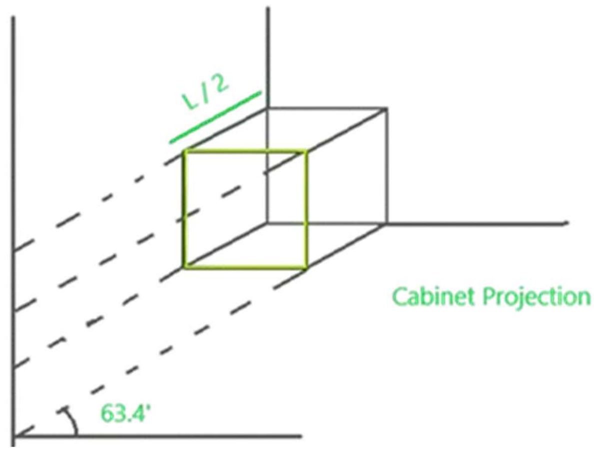
It is a kind of parallel projection where projecting rays emerges parallelly from the surface of the polygon and incident at an angle other than 90 degrees on the plane. It is of two kinds:

1. Cavalier Projection: It is a kind of oblique projection where the projecting lines emerge parallelly from the object surface and incident at 45° rather than 90° at the projecting plane. In this projection, the length of the reading axis is larger than the cabinet projection.



Cavalier Projection

2. Cabinet Projection: It is similar to the cavalier projection but here the length of reading axes is half than the cavalier projection and the incident angle at the projecting plane is 63.4° rather 45°.



3.2 Perspective Projection

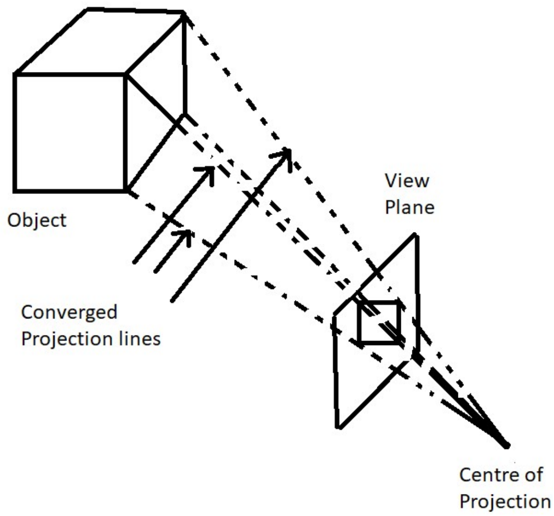
In Perspective Projection the **center of projection** is at finite distance from **projection plane**. This projection produces realistic views but does not preserve relative proportions of an object dimensions. Projections of distant object are smaller than projections of objects of same size that are closer to projection plane.

1. **Center of Projection –** It is a point where lines or projection that are not parallel to projection plane appear to meet.
2. **View Plane or Projection Plane –** The view plane is determined by :
   - View reference point $R_0(x_0, y_0, z_0)$
   - View plane normal.
3. **Location of an Object –** It is specified by a point P that is located in world coordinates at (x, y, z) location. The objective of perspective projection is to determine the image point P' whose coordinates are (x', y', z')

The perspective projection, on the other hand, produces realistic views but does not preserve relative proportions.

In perspective projection, the lines of projection are not parallel. Instead, they all converge at a single point called the center of projection or projection reference point.

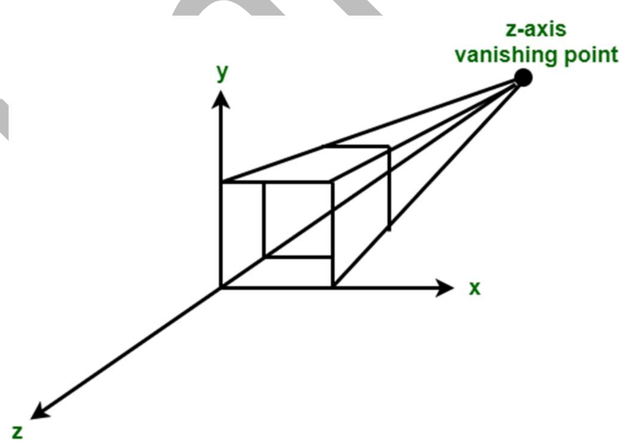
The object positions are transformed to the view plane along these converged projection lines and the projected view of an object is determined by calculating the intersection of the converged projection lines with the view plane, as shown in the figure below:



Types of Perspective Projection: Classification of perspective projection is on basis of vanishing points (It is a point in image where a parallel line through center of projection intersects view plane.). We can say that a vanishing point is a point where projection line intersects view plane.

3.2.1 One-Point Perspective Projection

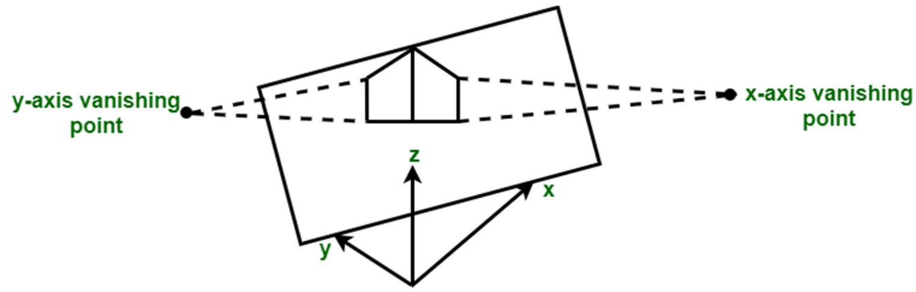
One point perspective projection occurs when any of principal axes intersects with projection plane or we can say when projection plane is perpendicular to principal axis.



In the above figure, *z* axis intersects projection plane whereas *x* and *y* axis remain parallel to projection plane.

3.2.2 Two-Point Perspective Projection

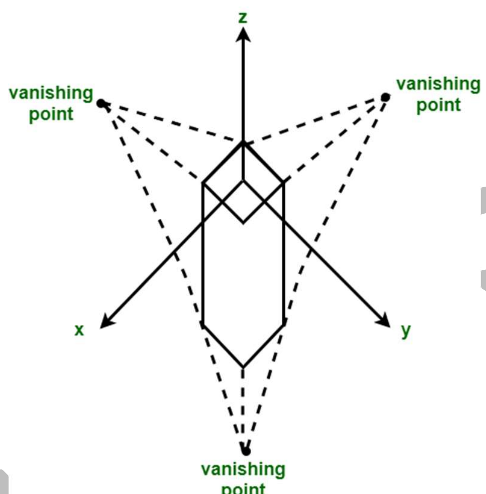
Two-point perspective projection occurs when projection plane intersects two of principal axis.



In the above figure, projection plane intersects x and y axis whereas z axis remains parallel to projection plane.

3.2.3 Three-Point Perspective Projection

Three-point perspective projection occurs when all three axis intersects with projection plane. There isn't any principal axis which is parallel to projection plane.



3.3 Difference between Orthographic and Isometric Projection

Orthographic Projection	Isometric Projection
Provides a 2D view of the object.	Provides a 3D view of the object.
Each view of orthographic projection shows only one side of the object.	Isometric projection displays at least three sides of the object.

In orthographic projection, the projection plane is parallel to one of the principal planes.	In isometric projection, the projection plane is not parallel to any of the principal planes.
It does not preserve depth.	It does include depth.
The true shape and size of an object are preserved.	The projected object is foreshortened equally in all three directions.

3.4 Difference between Parallel and Perspective Projection

Parallel Projection	Perspective projection
Parallel projection represents the object in a different way like telescope.	Perspective projection represents the object in three-dimensional way.
In parallel projection, such effects are not created.	In perspective projection, objects that are far away appear smaller, and objects that are near appear bigger.
The distance of the object from the center of projection is infinite.	The distance of the object from the center of projection is finite.
Parallel projection can give the accurate view of object.	Perspective projection cannot give the accurate view of object.
It does not form realistic view of object.	It forms a realistic view of object.
The lines of parallel projection are parallel.	The lines of perspective projection are not parallel.

Numerical Problems:

Q1. Find transformation of a triangle *A(1, 0) B(0, 1) C(1, 1)* by performing translation by one unit in *x* and *y* directions and then rotating 45° about the origin.

Solution: Given: *A(1, 0) B(0, 1) C(1, 1)*

Representing the triangle by Homogeneous Matrix Form:

$$P(X,Y) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

For translation of 1 unit in X and Y directions, the translation matrix will be:

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

To find the translation of 1 unit:

$$M(X', Y') = M(X, Y) \cdot T$$

$$P(X', Y') = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 2 & 1 \end{bmatrix}$$

Now, we need to rotate this triangle by 45° about origin.

$$R = \begin{bmatrix} cos\,\theta & sin\,\theta & 0 \\ -sin\,\theta & cos\,\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P(X'', Y'') = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 2 & 1 \end{bmatrix} * \begin{bmatrix} cos\,45 & sin\,45 & 0 \\ -sin\,45 & cos\,45 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.7 & 2.1 & 1 \\ -0.7 & 2.1 & 1 \\ 0 & 2.8 & 1 \end{bmatrix}$$

Therefore, the final coordinates are *A(0.7, 2.1), B(–0.7, 2.1), C(0, 2.8)*.

Q2. Given a circle C with radius 5 and center coordinates (1, 4). Apply the translation with distance 5 towards X axis and 1 towards Y axis. Obtain the new coordinates of C without changing its radius.

Solution: Initial center of the circle: *(x,y) = (1,4)*

Translation: 5 units towards the ***x*-axis** and 1 unit towards the ***y*-axis**

To find the new coordinates of the center, add the translation values to the current coordinates:

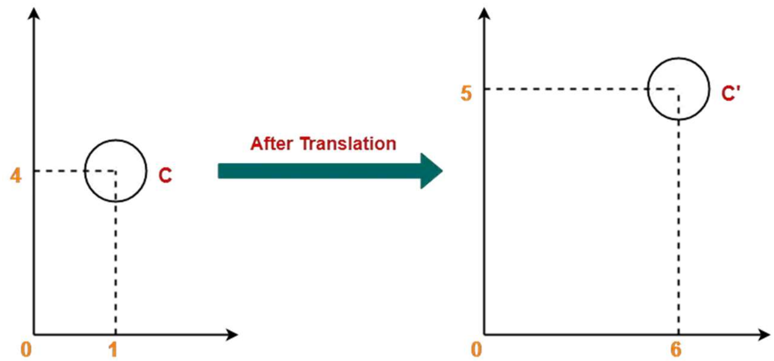
New center = $(x + t_x, y + t_y)$

Where:

- $x=1$ *(initial x-coordinate)*
- $y=4$ *(initial y-coordinate)*
- $t_x=5$ *(translation along the x-axis)*
- $t_y=1$ *(translation along the y-axis)*

New center = $(1+5, 4+1) = (6,5)$

- The **new coordinates** of the circle after translation are *(6,5)*.



Q3. Given a line segment with starting point as (0, 0) and ending point as (4, 4). Apply 30° rotation in the anticlockwise direction on the line segment and find out the new coordinates of the line.

Solution: The 2D rotation matrix for a counterclockwise rotation by an angle θ is:

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

For a rotation of $\theta = 30°$, we know

$$\cos(30^\circ) = \frac{\sqrt{3}}{2}, \quad \sin(30^\circ) = \frac{1}{2}$$

So, the rotation matrix begins:

$$R(30°) = \begin{bmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{bmatrix}$$

We will apply this rotation matrix to both points of the line segment.

Rotation of Starting Point (0,0)

The starting point (0,0) remains the same after the rotation because rotating the origin results in no change.

Rotation of Ending Point (4,4)

Now apply the rotation matrix to the point (4,4):

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = R(30°) \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1.464 \\ 5.464 \end{bmatrix}$$

Thus, the new coordinates of the line segment after a 30° anticlockwise rotation are *(0,0)* for the starting point and approximately *(1.464, 5.464)* for the ending point.

Q4. Consider the square *A (1, 0), B(0, 0), C(0, 1), D (1, 1)*. Rotate the square *ABCD* by 45º anticlockwise about point *A (1,0)*.

Solution: Given: *A (1, 0), B(0, 0), C(0, 1), D (1, 1)*

Representing the square by Homogeneous Matrix Form:

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

For translation, $t_x = -1$, $t_y = 0$, as we have to bring the point *A(1,0)* to origin.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

So, the square after translation has the vertices, *A′ (0, 0), B′ (−1, 0), C′ (−1, 1), D′ (0, 1)*

We will now rotate the translated square 45º anticlockwise using the 2D rotation matrix. The rotation matrix for a 45º anticlockwise rotation is:

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(45°) = \begin{bmatrix} \cos 45° & \sin 45° & 0 \\ -\sin 45° & \cos 45° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -0.707 & -0.707 & 1 \\ -1.414 & 0 & 1 \\ -0.707 & 0.707 & 1 \end{bmatrix}$$

We will apply this rotation matrix to the new coordinates A′, B′, C′, D′.

The new coordinates are *A″ (0, 0)*, *B″ (–0.707, –0.707)*, *C″ (–1.414, 0)*, *D″ (–0.707, 0.707)*.

Now we translate the points back to their original positions with $t_x = 1$ and $t_y = 0$:

$$\begin{bmatrix} 0 & 0 & 1 \\ -0.707 & -0.707 & 1 \\ -1.414 & 0 & 1 \\ -0.707 & 0.707 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0.293 & -0.707 & 1 \\ -0.414 & 0 & 1 \\ 0.293 & 0.707 & 1 \end{bmatrix}$$

After rotating the square ABCD by 45º anticlockwise about point A(1,0)A(1, 0)A(1,0), the new coordinates of the vertices are *A (1,0), B (0.293,−0.707), C (−0.414,0), D (0.293,0.707).*

Q5. A triangle is defined by $\begin{bmatrix} 2 & 4 & 4 \\ 2 & 2 & 4 \end{bmatrix}$. Find transformed coordinates after the following transformation.

i) 90° rotation about the origin.

ii) Reflection about line *X* = *Y*

Solution:

Given a triangle with vertices defined by the matrix:

$$\text{Triangle} = \begin{bmatrix} 2 & 4 & 4 \\ 2 & 2 & 4 \end{bmatrix}$$

Here, the first row represents the x-coordinates, and the second row represents the y-coordinates of the three vertices of the triangle: *(2,2), (4,2),* and *(4,4)*.

i) 90º Rotation about the Origin

The transformation matrix for a **90º counterclockwise rotation** about the origin is:

$$R(90^\circ) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

We multiply this matrix by the original coordinates of the triangle:

$$\text{New coordinates} = R(90^\circ) \times \begin{bmatrix} 2 & 4 & 4 \\ 2 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 4 & 4 \\ 2 & 2 & 4 \end{bmatrix}$$

Performing the matrix multiplication:

$$= \begin{bmatrix} (0 \cdot 2) + (-1 \cdot 2) & (0 \cdot 4) + (-1 \cdot 2) & (0 \cdot 4) + (-1 \cdot 4) \\ (1 \cdot 2) + (0 \cdot 2) & (1 \cdot 4) + (0 \cdot 2) & (1 \cdot 4) + (0 \cdot 4) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -2 & -4 \\ 2 & 4 & 4 \end{bmatrix}$$

So, the new coordinates of the triangle after a 90º rotation are: *(−2,2), (−2,4), (−4,4)*

ii) Reflection about the Line *X=Y*

The transformation matrix for reflecting a point about the line ***X=Y*** is

$$R_{X=Y} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

We multiply this reflection matrix by the new coordinates of the triangle:

New coordinates = $R_{X=Y} \times \begin{bmatrix} -2 & -2 & -4 \\ 2 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} -2 & -2 & -4 \\ 2 & 4 & 4 \end{bmatrix}$

Performing the matrix multiplication:

$$= \begin{bmatrix} (0\cdot -2) + (1\cdot 2) & (0\cdot -2) + (1\cdot 4) & (0\cdot -4) + (1\cdot 4) \\ (1\cdot -2) + (0\cdot 2) & (1\cdot -2) + (0\cdot 4) & (1\cdot -4) + (0\cdot 4) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 4 \\ -2 & -2 & -4 \end{bmatrix}$$

So, the new coordinates of the triangle after reflection about the line *X=Y* are *(2, –2), (4, –2)* and *(4, –4)*

Q6. Perform 45° rotation of a triangle *A(0, 0), B(1, 1) and C(5, 2)*. Find transformed coordinates after rotation, (i) About origin, (ii) About P (–1, 1).

Solution: Given: *A(0, 0), B(1, 1) and C(5, 2)*

$$P(X,Y) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 5 & 2 & 1 \end{bmatrix}$$

$$R(45°) = \begin{bmatrix} \cos 45° & \sin 45° & 0 \\ -\sin 45° & \cos 45° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i) 45º Rotation About the Origin

$$P(X',Y') = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 5 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1.414 & 1 \\ 2.121 & 4.949 & 1 \end{bmatrix}$$

So, the new coordinates are *A(0, 0), B(0, 1.414), C(2.121, 4.95).*

ii) 45º Rotation About Point P(−1,1)

Given: *A(0, 0), B(1, 1) and C(5, 2)*

To rotate about a point other than the origin, we first translate the point *P (−1,1)* to the origin, perform the rotation, and then translate back.

Therefore, $t_x = 1$ and $t_y = -1$

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

$$P(X',Y') = P(X,Y) \times T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 5 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 6 & 1 & 1 \end{bmatrix}$$

The new coordinates are *A'(1, –1), B'(2, 0)* and *C'(6, 1).*

Now we apply the rotation matrix to the translated coordinates.

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 6 & 1 & 1 \end{bmatrix} \times R(45°) = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 6 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} cos\,45° & sin\,45° & 0 \\ -sin\,45° & cos\,45° & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 6 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.414 & 0 & 1 \\ 1.414 & 1.414 & 1 \\ 3.535 & 4.949 & 1 \end{bmatrix}$$

The new coordinates are *A"(1.414, 0), B"(1.414, 1.414)* and *C"(3.535, 4.949).*

Now we translate the rotated triangle back by adding the coordinates of point *P(−1,1)*. Therefore, $t_x = -1$ *and* $t_y = 1$

$$\begin{bmatrix} 1.414 & 0 & 1 \\ 1.414 & 1.414 & 1 \\ 3.535 & 4.949 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.414 & 1 & 1 \\ 0.414 & 2.414 & 0 \\ 2.535 & 5.949 & 1 \end{bmatrix}$$

The new coordinates are *A(0.414, 1), B(0.414, 2.414)* and *C"(2.535, 5.949).*

Q7. Consider a polygon with 4 sides as follow *P1(1, 1), P2(3, 1), P3(3, 3), P4(1, 3)*. Scale this polygon to half of its size.

Solution: Given: ***P1(1, 1), P2(3, 1), P3(3, 3), P4(1, 3)***

To scale the polygon with vertices *P1(1,1), P2(3,1), P3(3,3)* and *P4(1,3)* to **half its size**, we'll apply a **scaling transformation**.

Scaling Transformation Formula:

The scaling transformation for 2D coordinates is given by:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

S_x and S_y are the scaling factors for the x and y axes, respectively.

In this case, since we want to scale the polygon to half its size, $S_x = S_y = 0.5$

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \times \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 1 \\ 1.5 & 0.5 & 1 \\ 1.5 & 1.5 & 1 \\ 0.5 & 1.5 & 1 \end{bmatrix}$$

The new coordinates are *P1(0.5,0.5), P2(1.5,0.5), P3(1.5,1.5)* and *P4(0.5,1.5)*.