An Intuitive Introduction To Two-dimensional Topological Optical Quantum Emitter

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Stable and precise excitation and control of a two-dimensional quantum emitter as a single-photon source is crucial in the research and application of integrated quantum photons. The robustness inherent in topological photonic materials makes topological edge state immunise to photons scattering caused by impurities, defects, or minor perturbations, and also ensures unidirectional photons transport. This paper will introduce the theoretical calculation and prediction of topological properties in a two-dimensional atomic emitter arrays system. Under the symmetry breaking caused by the applied magnetic field, the Zeeman splitting changes the structure of the energy level, which makes the system have the stability property brought by topological protection. Such a model at a single photon level yields quantum non-linearity, thus we can explore quantum optical as analogs of topological phonic system, and it allows us to better control the excitation of individual atoms.

I. WHAT IS TOPOLOGY?

Topology originally studied the invariant properties and quantities of geometry under continuous deformation. For example, a donut can be continuously transformed into a cup with a handle, but not a basketball. The doughnuts and cups here have the same topological properties despite their different shapes, and the topological invariant here is the number of holes.

Topology in physics is different from mathematics. Physicist does not concerned with how open subsets are used to define intuitive concepts or to prove why basketball cannot be obtained by continuous deformation. Physics is more concerned with using mathematical conclusions to integrate topological invariants and analyze the physical properties and meanings of this invariants based on the first principles of physics.

II. TOPOLOGY IN ELECTRONIC SYSTEM

For a long time, the understanding of phase transitions of matter was that because the lowest energy state of the Lagrangian was not a invariants under symmetry group transformations[1]. The application of topology to condensed matter physics has revealed a magical phase transition of states of matter that cannot be described by the Landau symmetry breaking theory. That is the topological state of matter, in the electronic system it was originally applied to explain the KT phase transition in two-dimensional superfluids and superconductors in real space[2], then the next most critical is the phenomenon [3] and interpretation [4] of integer quantum Hall effect in momentum space. One of the most interesting topological physical properties as shown in Figure 1, is that the band structure of integer quantum Hall has an edge state between the conduction band and the valence band gap when we applied strong magnetic field,

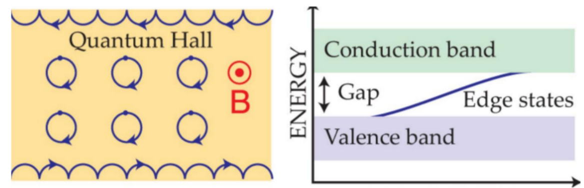

FIG. 1. The bulk electrons move circularly under applied magnetic field, opens a band gap between the valence and conduction band, so that current can not flows in side. Amazingly, on the surface there's an edge state crossing the band gap, so that the currents can be flow though without dissipated and unidirectionality.[8]

which enables electrons to have integer multiplier conductance on the surface of the material, but is an insulator inside the material. Most particularly, the integer multiplier of conductance can be well explained and calculated by the topological invariant-Chern number, and the electrons conducted on the surface have unidirectionality (chiral edge states) and the ability to resist scattering from impurities or defects. This will bring very broad and valuable application prospects for quantum information transmission [5] and computing [6]. Beside the application, the studies of theoretical topological physics is also a broad new field, and even more there's some new mathematics potentially will be come in to physics [7].

Before deducing the Hall conductance and Chen number in detail, let's first try to intuitively understand and feel the topological invariant from the Brillouin zone(BZ) in Figure 2, Due to the periodical of the BZ, we may reshape the k_x axis into a circle and thus form a cylindrical BZ, we can further make the k_y into circle to form a donut-like BZ. The phase $\theta(k_y)$ vary along the k_y from $-\pi$ to π with an integer multiple of 2π, in order to keep periodical property of the head been equal to the tail. Thus the phase $\theta(k_y)$ on donut-like BZ will been change as an integer winding numbers C - Chern number [9].

The Chen invariant is derived from the fiber bundle in mathematics, however the key to its understanding and application in physics is to consider the Berry phase of

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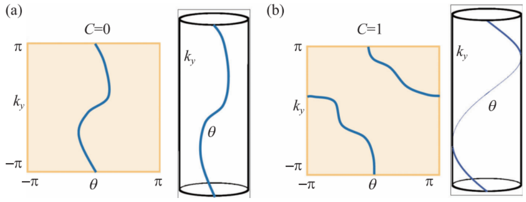

FIG. 2. The phase $\theta(k_y)$ on the cylindrical BZ. Chern numbers for (a.) is 0 and (b.) is 1.[11]

the Bloch function$|u_n(\mathbf{k})\rangle$. The Barry connection is given by $A_n(\mathbf{k}) = i\langle u_n(\mathbf{k})|\nabla_{\mathbf{k}}|u_n(\mathbf{k})\rangle$, and the Barry curvature $\Omega_n(\mathbf{k}) = \nabla_k \times A_n(k)$ can be expand as,

$$\Omega_n = i\left(\left\langle \partial_{k_x} u_{n,\mathbf{k}} \mid \partial_{k_y} u_{n,\mathbf{k}}\right\rangle - \left\langle \partial_{k_y} u_{n,\mathbf{k}} \mid \partial_{k_x} u_{n,\mathbf{k}}\right\rangle\right) \quad (1)$$

In case for all N bands under Fermi surface are full, the gauge invariant $\Omega_n(\mathbf{k})$ under the gauge transformation will remains as the same, however the Barry connection will be transformed into,

$$A'_n(\mathbf{k}) \rightarrow A_n(\mathbf{k}) - \nabla_{\mathbf{k}} \sum_N \theta_n(\mathbf{k}) \quad (2)$$

Were the gauge-invariant $\Omega_n(\mathbf{k})$ and the gauge-dependent $A_n(\mathbf{k})$ can be analog to the the magnetic flux density through a closed surface and electromagnetic vector potential, respectively. Given the BZ of this system is a 2D closed surface, by stock theorem, the total number of "Burry fluxes" is an integer over the BZ [10],

$$C_n = \frac{1}{2\pi} \int_{\mathrm{BZ}} d^2k \Omega_n(k_x, k_y) \quad (3)$$

Where the "Burry fluxes" C_n is the Chern number, and the physics meaning of the C_n is the quantumtized multiples of quantum hall conductance,

$$\sigma_{xy} = -\frac{e^2}{h} \sum_n C_n \quad (4)$$

$$= \frac{ie^2}{4\pi h} \sum \oint dk_j \int d^2r \left(u \frac{\partial u}{\partial k_j} - \frac{\partial u}{\partial k_j} u\right) \quad (5)$$

In general, the topological phase transition theoretical explanation of this phenomenon is that the Bloch function on the momentum space without degeneracy (applied magnetic field) can give the Barry connection, and its closed loop integration in k space can give the Berry connection phase. The most important thing is that Barry curvature can be analogized to the Stoke integral of Gaussian flux in electromagnetism. Finally, the Chern number we can get by integrating the whole Brillouin zone is the factor of integer Hall conductance.

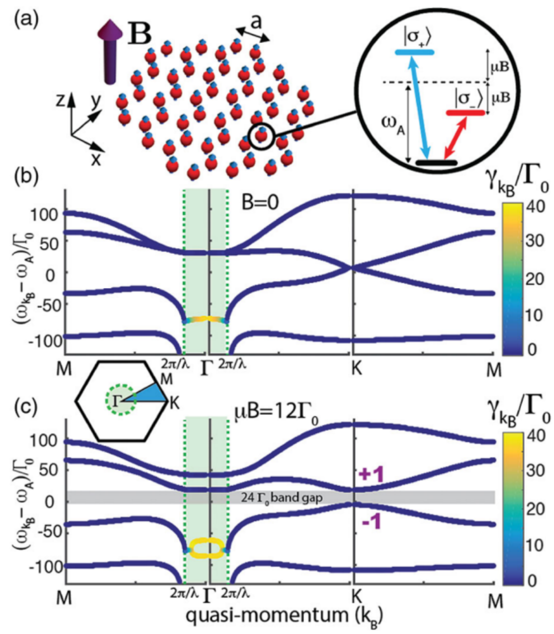

FIG. 3. (a) The subwavelength spacing atomic emitter array honeycomb lattice with V-type energy level for each atom.(b) Band structure without magnetic field and (c) with applied transverse magnetic field. Given parameters are $\Gamma = 2\pi \times 6\mathrm{MHz}, \lambda = 790\mathrm{nm}, \mathrm{a} = 0.05\lambda$.[12]

III. TOPOLOGY IN OPTICAL SYSTEMS

The magical topological properties of the integer quantum Hall effect in the electronic system can similarly bring a whole new field and perspective to the application and development of robust devices with immunity to disturbance and scattering in the photonic system.

A. Topological Photonic

Similar to the quantum Hall, light can not propagated though the bulk of the topological photonic lattice, however its surface can support a edge wave propagation mode. Surprisingly, the photons on the edge propagates restrictively also in unidirectional, thus it will not be reflected when it encounters obstacles, impurities, and defects. Although photons are dissipative systems, we can also construct stable energy bands and topological phases via a dissipatively stable Mott photonic insulator in the non-Hermitian system[13]. That allows the waveguides that made by topological photonic materials are stable under impurities and defects. The light in photonic crystals, similar to electrons in crystals, have the periodic potentials, thus the phtonic band structure can be constructed [14]. Analogically as in quantum Hall, by calculation of the Barry curvature over the photonic lattice within the formulation of Maxwell's equations, we thus can obtain the Chern number and properties of edge state for photonic crystals[15]. Here the electromagnetic fields analogs current, the permeability and variations of permittivity within the photonic crystal is similar to the periodic potential, and the gradients of the gyrotropic components of the permeability tensor breaks the time-reversal symmetry is analogs to the applied magnetic field in quantum Hall. This analogy is also confirmed experimentally, and the observation that the number of chiral edge states turns out to be equal to the sum of all Chern number of the low energy volume bands[16].

B. Topological Quantum Optics

Due to the instability of quantum light sources and the dissipative nature of optical systems, optical quantum information is easily lost during transmission. Quantum emitters can generate and control ideal single-photon sources, so if a system with the combination of quantum optics and topological photonics, it may be able to design and construct a photon propagation immune to quantum decoherence effects that bring dissipation into the system[14]. Thus photons in this system not only possess topological robustness such as unidirectionality of chiral edge states and impurity immunity, but also maintain polarization, amplitude, correlation, and entanglement during long-distance propagation. Therefore, we will focus on the introduction and analysis of a topological photonics property of a two-dimensional atomic emitter array under an external magnetic field.

The subwavelength spacing two-dimensional atomic emitter array with honeycomb lattice and the time-reversal symmetry is broken by magnetic field, shows the topologically protected properties. As shown in Figure 5, the probability rate of photonic emission "propagation" in edge state modes is robust, unidirectional, immune to the impurities, and the free-space emission that scatter to other directions and the emission rate inside the bulk is strongly suppressed[12].

In Figure 3, (a) Each atomic emitter has a V-type level structure corresponding to the polarization of light, with the transitions from ground state to the excited states $|\sigma_+\rangle$ and $|\sigma_-\rangle$. As we deduced that the Barry connection in the electronic system needs to eliminate the degeneracy on the band structure of the Bloch function by an external magnetic field, we similarly use an magnetic field to generate Zeeman splitting that breaks the degeneracy of ground state wave function into two non-degenerate excite states and obtains band gap in the excitation spectrum. Next, we will calculate and simulate the time-dependent evolution of the excitation probability of atomic emitters in this system based on first-principles calculation. Therefore, we first need to construct the Hamiltonian of this system, then solve the Bloch wave function as an eigenmode, and finally calculate the band structure and the evolution of edge states[12].

By the adiabatic elimination, the single excitation is

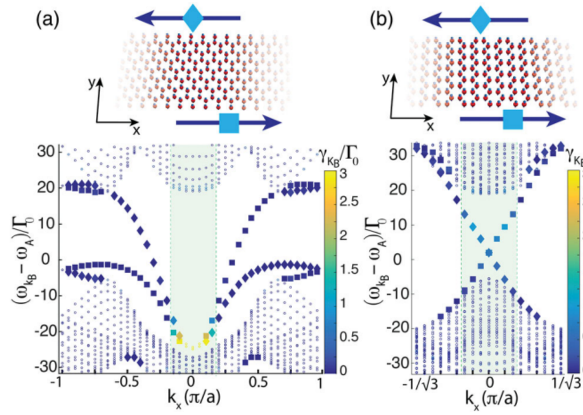

FIG. 4. Topological edge states.[12]

described by the non-Hermitian spin Hamiltonian that given by,

$$
\begin{aligned}
H =& \hbar \sum_{i=1}^{N} \sum_{\alpha=\sigma_+,\sigma_-} \left(\omega_A + \mathrm{sgn}\left(\alpha_i\right)\mu B - \mathrm{i}\frac{\Gamma_0}{2}\right) |\alpha_i\rangle \langle\alpha_i| \\
&+ \frac{3\pi\hbar\Gamma_0 c}{\omega_A} \sum_{i\neq j} \sum_{\alpha,\beta=\sigma_+,\sigma_-} G_{\alpha\beta}\left(\mathbf{r}_i - \mathbf{r}_j\right) |\alpha_i\rangle \langle\beta_j| ,
\end{aligned}
\tag{6}
$$

where N is the number of emitters, $\omega_A = 2\pi c/\lambda$ is the atomic transition frequency, $\mathrm{sgn}(\sigma_\pm) = \pm$, and $\Gamma_0 = d^2\omega_A^3/(3\pi\epsilon_0\hbar c^3)$ is the radiate line width of each atom in free space, and here d is transition dipole moment, μB is Zeeman shift with the magnetic moment μ that created by the magnetic field $\mathbf{B} = B\hat{z}$. $G_\alpha\beta(\mathbf{r})$ is dyadic Green's function for dipolar spin-spin interaction[12].

The single excitation eigenmodes of the Hamiltonian in an infinite periodic honeycomb lattice are Bloch modes that is given by,

$$
|\psi_{\mathbf{k}_B}\rangle = \sum_n \sum_{b=1,2} e^{i\mathbf{k}_B\cdot\mathbf{R}_n} \left[c^b_{+,\mathbf{k}_B}\left|\sigma^b_{+,n}\right\rangle + c^b_{-,\mathbf{k}_B}\left|\sigma^b_{-,n}\right\rangle\right]
\tag{7}
$$

where the sum is over lattice vectors $\mathbf{R}$, and $\mathbf{k_B}$ is the Bloch wave vector. The modes decay rate is given by imaginary part of the eigenvalues [12].

Then the calculation of band structure along the symmetry points $\mathbf{M}, \mathbf{\Gamma}, \mathbf{K}$ of the Brillouin zone, as shown in Figure 3. The Green shaded region is short lived modes with quasimomentum $k_B < \omega_{K_B}/c$ couple to free space modes. Green dashed lines $k_B = 2\pi/\lambda$ is the edges of the free space light cone for dispersion $k_B = \omega_{k_B}/c$. The decay rates of the modes in this region shown in yellow color, which shows it decays rapidly.The grey shaded region in (c) is the band gap which is opened by a transverse magnetic field $\mu B = 12\Gamma_0$, the sum of nontrivial-Chern numbers below and above is $+1$and -1, respectively. A linear Dirac cone is formed at the $\mathbf{K}$ point by the mag-

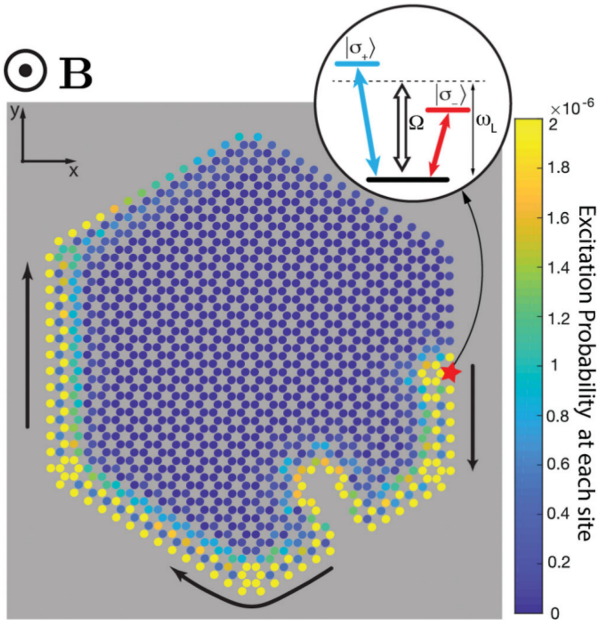

FIG. 5. Snapshot simulation of the time evolution at $t = 5.7\Gamma_0$ of the atomic arrays with $N = 1243$.[12]

netic field which breaks the degeneracy and opened a topological band gap.

There are two types of the edge states as shown in Figure 4. (a) bearded edge states, each atom on the edge separated in same spacing, this edge has higher decay rate when crossing band gap making it live long, (b) armchair edge state has a pair of two atoms that closer to each other when crossing result in a short lived.

Finaly, a time evolution demo as shown in figure 5, started at $t = 5.7\Gamma_0$ of the atomic arrays with $N = 1243$. The red star is where the atomic emitters on the edge was derived by a laser, the excitation of emitters propagating unidirectional on the boundary with clock wisely. The color bar here is the excitation probability $\left|\left\langle\psi(t) \mid \sigma_{+}^{i}\right\rangle\right|^{2}+\left|\left\langle\psi(t) \mid \sigma_{-}^{i}\right\rangle\right|^{2}$. As we expect the backward, free-space, and bulk excitation is strongly suppressed, and e immune to the large scale defect.

IV. CONCLUSIONS

Topological photonic crystals are topologically nontrivial photonic crystals created by simulating the electron wave function in the crystal under a periodic potential field. Using a similar approach to electrons, it is possible to characterize the topological properties of photons and then construct topologically protected edge states on their surfaces, producing unidirectionally propagating waveguides that remain stable under defects. Topological quantum optics is the studies of the topological photonic in quantum optics, that is base on the quantum atomic emitter with adiabatic elimination find the Hamiltonian and Bloch wave equation for the calculation of the non-trivial Chern numbers and band gaps induced by the Zeeman splitting under magnetic fields, and obtain the interesting topologically protected edge states on the surfaces. For the future prospects, perhaps it's also possible to imitate the method of constructing topological insulators, sandwiching structure of atomic arrays, and construct topological systems with time-reversal symmetry.

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